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ON THE IMBEDDING OF DERIVATIONS OF FINITE RANK INTO DERIVATIONS OF INFINITE RANK

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Introduction. Throughout this paper, we shall let $A=k[x_1,\cdots,x_n]$ denote a finitely generated integral domain over a perfect field k. Let p be a maximal ideal of A and set $R=A_p$, the local ring at p. By a k-derivation δ of rank m on R, we shall mean a set $\delta=\{\delta_0,\delta_1,\cdots,\delta_m\}$ of mappings $\delta_i\in\mathrm{Hom}_k(R,R)$ such that δ_0 is the identity map on R and for all $a,b\in R$, $q=1,\cdots,m$, we have

$$(1)\qquad \delta_q(ab)=\sum_{i+j=q}\delta_i(a)\delta_j(b)\,.$$

By a k-derivation D of infinite rank on R, we shall mean an infinite sequence $D=\{D_0,D_1,D_2,\cdots\}$ of k-endomorphisms D_i of R such that for each m, $\{D_0,D_1,\cdots,D_m\}$ is a k-derivation of rank m on R. We shall say that a k-derivation $\delta=\{\delta_0,\delta_1,\cdots,\delta_m\}$ of rank m on R (or A) is integrable on $R(A)$ if there exists a k-derivation $D=\{D_0,D_1,\cdots\}$ of infinite rank on $R(A)$ such that $\delta_i=D_i$, $i=0,1,\cdots,m$.

The problem of finding conditions on R such that every k-derivation of rank m is integrable was to the author's knowlege first suggested by Y. Nakai in [7]. Some work on this problem has been done by several authors. In particular, it follows from [8; (q) p. 33] that if the characteristic of the field k is zero, then every k-derivation of rank m on R is integrable. For this reason, we can assume throughout the rest of this paper that char $k=\rho\neq 0$.

The main results of this paper are the following two theorems: A global results:

Theorem 1. *Let $A=k[x_1,\cdots,x_n]$ be a finitely generated integral domain over a perfect field k. Suppose that for each maximal ideal $p\subset A$, the local ring A_p is regular. Then any k-derivation δ of finite rank on A is integrable on A.*

A complete characterization of regularity on the local level:

Theorem 2. *Let $A=k[x_1,\cdots,x_n]$ be a finitely generated integral domain over a perfect field k. Let p be a maximal ideal of A and set $R=A_p$ (A localized at p). Assume A has dimension r. Then R is a regular local ring if and only if the following two conditions are satisfied:*

(a) *Every k-derivation of finite rink on R is integrable on R.*

(b) *There exist r derivations $\delta_1, \cdots, \delta_r \in Der_k^1(R)$ and elements $z_1, \cdots, z_r$ in the maximal ideal of R such that the matrix $(\delta_i(z_j))$ is invertible. Here $Der_k^1(R)$ denotes the R-module of rank one k-derivations on R.*

We shall complete this paper with a few examples which show that these theorems are about as good as can be expected.

Main results. We begin by studying field extensions.

Proposition 1. *Let K be a field containing a perfect field k. Then any k-derivation $\delta = \{\delta_0, \cdots, \delta_n\}$ of rank m on K is integrable.*

Proof. Let S be a p-basis of K. Define a sequence of set mappings $\psi_i: S \to K$ as follows: $\psi_i = \delta_i$ if $i=0, \cdots, m$, ψ_i is arbitrary if $i > m$. Then by [3:thm.9] there exists a unique derivation D of infinite rank on K such that $D_i|_S = \psi_i$. Since k is a perfect subfield of K, [3; prop 7] implies D is a k-derivation on K. Since $D_i|_S = \delta_i$ for $i=0, \cdots, m$ one easily checks that $D_i = \delta_i$ for $i=0, \cdots, m$.

Proposition 2. *Let R be a local ring containing the field k. Let p denote the maximal ideal of R, and let $\hat{R}$ denote the completion of R in its p-adic topology. Then if $\delta = \{\delta_0, \cdots, \delta_m\}$ is a k-derivation of rank m on R, δ has a unique extension $\bar{\delta} = \{\bar{\delta}_0, \cdots, \bar{\delta}_m\}$ to a k-derivation of rank m on $\hat{R}$.*

Proof. On can easily check that each δ_i is a continuous mapping in the p-adic topology on R. Thus, the result follows from [2; prop 2].

Proposition 3. *Let K be a field containing a perfect field k. Let $R = K[[X_1, \cdots, X_n]]$, the formal power series ring over K in the indeterminates $X_1, \cdots, X_n$. Then if $\delta = \{\delta_0, \cdots, \delta_m\}$ is a k-derivation of rank m on R, δ is integrable.*

Proof. This result follows from Proposition 1 and known results in [2]. δ may be viewed as k-derivation of rank m on K to R. Following the procedure of Proposition 1, we may extend δ to $D': K \to R$, a k-derivation of infinite rank. Since $X_1, \cdots, X_n$ are algebraically independent over K, we may extend D' to a k-derivation $D'': K[X_1, \cdots, X_n] \to R$ (of infinite rank) such that $D_i''(X_j) = \delta_i(X_j)$ for $i=0, \cdots, m$. We may now use [2; prop. 2] to extend D'' to $D: R \to R$ a k-derivation of infinite rank such that $D_i = \delta_i$ for $i=0, \cdots, m$.

Now let $X_1, \cdots, X_n$ be indeterminates over the field k. Let $\sum$ denote the quotient field of $k[X_1, \cdots, X_n]$. Let $\{u_{ij} | j=1, \cdots, n; i=1, \cdots, \infty\}$ and T be indeterminates over $\sum$. For each $i=1, \cdots, \infty$, we define a k-linear mapping $q_i: k[X_1, \cdots, X_n] \to k[X_1, \cdots, X_n][u_{ij}]$ as follows: Given any monomial $X_1^{t_1} \cdots X_n^{t_n} \in k[X_1, \cdots, X_n]$, we define $q_i(X_1^{t_1} \cdots X_n^{t_n})$ to be the coefficient of T^i in the following power series in $(k[X_1, \cdots, X_n]\,[u_{ij}])\,[[T]]$:

$$(2)\qquad \{X_1+\sum_{l=1}^{\infty} u_{l1}T^l\}^{t_1} \cdots \{X_n+\sum_{l=1}^{\infty} u_{ln}T^l\}^{t_n}$$

We then extend the definition of each q_i by linearity to all of $k[X_1, \cdots, X_n]$.

Now suppose $\sum'=k(x_1, \cdots, x_n)$ is a finitely generated field extension of k. Let $\{\bar{u}_{ij} \mid j=1, \cdots, n, i=1, \cdots, \infty\}$ be a collection of elements in $\sum'$. Then we have a natural k-algebra homomorphism $\prod$: $k[X_1, \cdots, X_n]\,[u_{ij}] \to k[x_1, \cdots, x_n]\,[\bar{u}_{ij}] \subset \sum'$ given by $\prod(X_i)=x_i$, and $\prod(u_{ij})=\bar{u}_{ij}$. If $f(X_1, \cdots, X_n) \in k[X_1, \cdots, X_n]$, we shall say that the $\{\bar{u}_{ij}\}$ solve $q_i(f)=0, i=1, \cdots, \infty$ if $\prod(q_i(f))=0$ for $i=1, \cdots, \infty$.

We need the following lemma:

Lemma 1. *Let $\sum'=k(x_1, \cdots, x_n)$ be a finitely generated field extension of k with relations $f_1, \cdots, f_r \in k[X_1, \cdots, X_n]$. If $D=\{D_i\}$ is a k-derivation of infinite rank on $\sum'$, then $\{\bar{u}_{ij}=D_i(x_j) \mid j=1, \cdots, n, i=1, \cdots, \infty\}$ is a system of elements of $\sum'$ which solve the equations*

$$(3)\qquad q_i(f_l) = 0 \quad (l = 1, \cdots, r,\ i = 1, \cdots, \infty)$$

Conversely, if $\{\bar{u}_{ij} \mid j=1, \cdots, n,\ i = 1, \cdots, \infty\}$ is a collection of elements of $\sum'$ which solve (3), *then there exists a k-derivation $D=\{D_i\}$ of infinite rank on $\sum'$ such that $D_i(x_j)=\bar{u}_{ij}$ $(j=1, \cdots, n,\ i = 1, \cdots, \infty)$.*

Proof. See [1; lemma 3].

We note that the equations which appear in (3) have the following form: For fixed $i=1, \cdots, \infty, q_i(f_l)=0$ $(l=1, \cdots, r)$ can be written as

$$(4)\qquad \sum_{j=1}^{n} A_{ilj}u_{ij}+B_l = 0 \quad (l = 1, \cdots, r)$$

where $A_{ilj}, B_l \in k[X_1, \cdots, X_n]\,[u_{tj} \mid t=1, \cdots, i-1; j=1, \cdots, n]$. Hence, for each i the equations in (3) are linear in $u_{ij} (j=1, \cdots, n)$. We can now prove the following important result:

Proposition 4. *Let $A=k[x_1, \cdots, x_n]$ be a finitely generated integral domain over a perfect field k. Let p be a maximal ideal of A such that that the local ring $R=A_p$ is regular. Then if $\delta=\{\delta_0, \cdots, \delta_m\}$ is a k-derivation of rank m on R, δ is integrable.*

Proof. Let $\sum'=k(x_1, \cdots, x_n)$ be the quotient field of A. Let $f_1, \cdots, f_r \in k[X_1, \cdots, X_n]$ be the relations on A. That is $k[X_1, \cdots, X_n]/(f_1, \cdots, f_r) \cong A$. To construct a k-derivation of infinite rank on R which extends δ, it suffices by Lemma 1 to find elements $\{\bar{u}_{ij} \in A_p \mid j=1, \cdots, n; i=1, \cdots, \infty\}$ which solve the equations $q_i(f_l)=0$ $(l=1, \cdots, r; i=1, \cdots, \infty)$, and such that $\bar{u}_{ij}=\delta_i(x_j)$ for $i=$

$1,\cdots,m,j=1,\cdots,n$.

Set $\bar{u}_{ij}=\delta_i(x_j)$ for $i=1,\cdots,m\ j=1,\cdots,n$. Then since δ is a k-derivation of rank m, the $\{\bar{u}_{ij}|i=1,\cdots,m\ j=1,\cdots,n\}$ solve the equations $q_i(f_l)=0$ when $i=1,\cdots,m$. Consider the system

$$(5)\qquad q_i(f_l)=0\quad l=1,\cdots,r,i=1,\cdots,m+1$$

If we substitute $\bar{u}_{ij}=\delta_i(x_j)$ for u_{ij} in (5), we obtain a system of linear equations:

$$(6)\qquad \sum_{j=1}^{n} A_{m+1,l,j}u_{m+1,j}+B_l=0\quad (l=1,\cdots,r)$$

where $A_{m+1,l,j}, B_l\in A_p$.

Now by Proposition 2, δ extends uniquely to a k-derivation $\bar{\delta}$ of rank m on the completion $\hat{R}$ of R. Since R is a regular local ring of equal characteristic, $\hat{R}$ has the form $K[[X_1,\cdots,X_s]]$. That is $\hat{R}$ is the formal power series ring in s indeterminates over the field $K\cong A/P$[1]. Thus by Proposition 3, $\bar{\delta}$ can be imbedded in a k-derivation D of infinite rank on $\hat{R}$. But this implies that the equations (6) have a solution $\bar{u}_{m+1,j}=D_{m+1}(x_j)$ in $\hat{R}$. It now follows from [10; lemma p.39] that (6) has a solution $\{\bar{u}_{m+1,j}\}\in R$. Thus, there exists a δ_{m+1}: $R\to R$ such that $\{\delta_0,\cdots,\delta_m,\delta_{m+1}\}$ is a k-derivation of rank $m+1$ on R. We may now repeat this same argument as often as we please. Thus, we have proven the following: For all integers $\alpha\geq m$, the equations $q_i(f_l)=0$ ($l=1,\cdots,r$, $i=1,\cdots,\alpha$) have a solution $\{\bar{u}_{ij}|\,j=1,\cdots,n,\ i=1,\cdots,\alpha\}\subset R$ such that $\bar{u}_{ij}=\delta_i(x_j)$ if $i=1,\cdots,m,j=1,\cdots,n$. Furthermore, if $\{\bar{u}_{ij}|\,j=1,\cdots,n,\ i=1,\cdots,\alpha\}\subset R$ is a solution to $q_i(f_l)=0$ ($l=1,\cdots,r$, $i=1,\cdots,\alpha$), we can find elements $\{\bar{u}_{\alpha+1,j}\}\subset R$ such that $\{\bar{u}_{ij}|\,j=1,\cdots,n,\ i=1,\cdots,\alpha\}\cup\{\bar{u}_{\alpha+1,j}\}$ is a solution to the equations $q_i(f_l)=0$ ($l=1,\cdots,r$, $i=1,\cdots,\alpha+1$).

A simple application of Zorn's lemma now shows that there exists a solution $\{\bar{u}_{ij}|\,j=1,\cdots,n,\ i=1,\cdots,\infty\}\subset R$ to $q_i(f_l)=0$ ($l=1,\cdots,r$, $i=1,\cdots,\infty$) such that $\bar{u}_{ij}=\delta_i(x_j)$ for $i=1,\cdots,m$. Thus, there exists a k-derivation D of infinite rank on R such that $D_i=\delta_i$ if $i=0,\cdots,m$.

Before proving the main result of this paper, we need the following lemma:

Lemma 2. *Let $A=k[x_1,\cdots,x_n]$ be a finitely generated integral domain. Let $\sum_{j=1}^{m} a_{ij}z_j+b_i=0$ ($i=1,\cdots,r$) be a system of linear equations with coefficients $a_{ij},b_i\in A$. For each maximal ideal $p\subset A$, assume this system has a solution in A_p. Then the system has a solution in A.*

Proof. Suppose $r=1$. Then we have one equation $a_1z_1+\cdots+a_mz_m=b$ in A. Let $I=(a_1,\cdots,a_m)$, the ideal in A generated by $a_1,\cdots,a_m$. If $I=(0)$ or A, clearly the equation has a solution in A. Hence, we may assume I is a proper

1) We can assu $K\supset$ by using [10: lemma p. 29] and the fact that k is perfect.

ideal of A. Set $J=(I:(b))=\{x\in A \mid xb\in I\}$. We wish to show $J=A$. Since (b) is finitely generated, we have $JA_p\cong J\otimes_A A_p\cong(I\otimes_A A_p:(b)\otimes_A A_p)\cong(IA_p:bA_p)$ for all maximal ideals $p\subset A$. By hypothesis, $(IA_p:bA_p)=A_p$. Thus, $JA_p=A_p$ for all maximal ideals p. This implies $J=A$.

Suppose $r>1$. For each equation present, adjoin an indeterminate u_i to A. Set $R=A[u_1,\cdots,u_r]$. Then

$$(7)\qquad \sum_{j=1}^{m}\left(\sum_{i=1}^{r} a_{ij}u_i\right)z_j+\sum_{i=1}^{r} b_iu_i = 0$$

is an equation in the unknowns $z_1,\cdots,z_m$ with coefficients in R.

Since A is a homomorphic image of $k[X_1,\cdots,X_n]$, A is a Hilbert domain. Hence, $A[u_1,\cdots,u_q]$ $(1\leq q\leq r)$ is also a Hilbert domain. Now if M is any maximal ideal of R, then by [5; thm. 27], $M\cap A[u_1,\cdots,u_{r-1}]$ is a G-ideal in $A[u_1,\cdots,u_{r-1}]$. But all G-ideals are maximal in a Hilbert ring. Therefore, $M\cap A[u_1,\cdots,u_{r-1}]$ is a maximal ideal in $A[u_1,\cdots,u_{r-1}]$. Repeating this argument, we get $M\cap A$ is a maximal ideal of A. We note that $R_M\supset A_{M\cap A}$.

Now by hypothesis, there exist elements $\bar{z}_1,\cdots,\bar{z}_m\in A_{M\cap A}$ such that $\sum_{j=1}^{m} a_{ij}\bar{z}_j+b_i=0$ for $i=1,\cdots,r$. Thus, the equation (7) has a solution $\bar{z}_1,\cdots,\bar{z}_m$ in R_M. From the proof of the case $r=1$, we get (7) has a solution $\hat{z}_1,\cdots,\hat{z}_m$ in R. Write each $\hat{z}_i=\alpha_i+$terms of degree bigger than or equal to one in the u_i. Here $\alpha_i\in A$. Then one easily sees that $\alpha_1,\cdots,\alpha_m$ is a solution to $\sum a_{ij}z_j+b_i=0$ in A.

We can now state the main result.

Theorem 1. *Let $A=k[x_1,\cdots,x_n]$ bea finitely generated integral domain over a perfect field k. Suppose for each maximal ideal $p\subset A$, the local ring $R=A_p$ is regular. Then any k-derivation $\delta=\{\delta_0,\cdots,\delta_m\}$ of rank m on A is integrable on A.*

Proof. For every maximal ideal $p\subset A$, δ induces a k-derivation of rank m on A_p. Thus, by Proposition 4, δ can be imbedded in a k-derivation D_p of infinite rank on A_p. Let $f_1,\cdots,f_r$ denote the relations on A. Set $\bar{u}_{ij}=\delta_i(x_j)$ $(i=1,\cdots,m;j=1,\cdots,n)$ in the equations $q_i(f_l)=0$ $(l=1,\cdots,r;\ i=1,\cdots,m+1)$. We then get a system of linear equations like (6) with $A_{m+1,l,j}$, $B_l\in A$. Since δ imbeds in a k-derivation of infinite rank on A_p, (6) has a solution $\{\bar{u}_{m+1,j}\}\subset A_p$ for every maximal ideal $p\subset A$. Thus, it follows from Lemma 2, that (6) has a solution in A. Hence, there exists a k-linear map $\delta_{m+1}\colon A\to A$ such that $\{\delta_0,\cdots,\delta_m,\delta_{m+1}\}$ is a k-derivation on A of rank $m+1$. We now proceed as in the proof of Proposition 4 to construct a k-derivation D of infinite rank on A such that $D_i=\delta_i$ if $i=0,\cdots,m$.

We note that the regularity condition in Theorem 1 on A cannot be omitted (see [1; Example 1]). The regularity hypothesis on A implies that A is integrally closed. One might ask if Theorem 1 is true for finitely generated integral domains which are integrally closed. The following example shows that Theorem 1 is not true if we only assume A is integrally closed.

EXAMPLE 1. Let Z_2 denote the integers modulo two. Set $Z_2[X, Y, Z]/(X^3+Y^2+Z^3)=Z_2[x, y, z]=A$. One can easily check that A is an integral domain of transcendence degree two over Z_2. Set $A_0=Z_2[y, z]$. Then A_0 is isomorphic to a polynomial ring in two indeterminates over Z_2. Thus, A_0 is an integrally closed domain contained in A.

Let K_0 and K denote the quotient fields of A_0 and A respectively. Then K is a three dimensional separable algebraic extension of K_0. Let $\bar{A}_0$ denote the integral closure of A_0 in K. Since A is integral over $A_0, A\subset\bar{A}_0$. We shall show $A=\bar{A}_0$. An integral basis of K/K_0 is given by $\{1, x, x^2\}\subset\bar{A}_0$. The discriminant of this basis is $(y^2+z^3)^2$. Hence, it follows from the proof of Theorem 7 in [11; p. 264] that $(y^2+z^3)^2\bar{A}_0\subset A_0+A_0x+A_0x^2$. Thus, if $w\in\bar{A}_0$, there exist elements $a_0, a_1, a_2\in A_0$ such that

$$(y^2+z^3)^2w = a_0+a_1x+a_2x^2 . \tag{8}$$

A routine calculation shows that (8) implies $(y^2+z^3)^2 | a_i$. Thus, $w=a_0'+a_1'x+a_2'x^2$ for $a_i'\in A_0$. Hence $\bar{A}_0=A$, and we have proven A is an integrally closed domain.

Now define a Z_2-derivation δ_1 of rank one on A by setting $\delta_1(x)=\delta_1(z)=0$ and $\delta_1(y)=1$. Then $\delta_1(x^3+y^2+z^3)=x^2\delta_1(x)+z^2\delta_1(z)=0$. So δ_1 is well defined. Now if δ_1 was imbeddable in a Z_2-derivation $D=\{D_i\}$ of infinite rank on A, then we would have $0=D_2(x^3+y^2+z^3)=x^2D_2(x)+z^2D_2(z)+1$. Thus, $1\in(x, z)$ which is impossible.

Since the Jacobian of $x^3+y^2+z^3$ is $(x^2, 0, z^2)$, we see that $p=(x, y, z)$ is a maximal ideal in A for which A_p is not a regular local ring.

We now proceed with the proof of Theorem 2. We need the following result of Y. Ishibashi:

Proposition 5. *Let A be a ring and let M be an ideal of A such that A is a complete Hausdorff space in its M-adic topology. Assume there exist integrable derivations $\delta_1, \cdots, \delta_r$ of rank one on A and elements $z_1, \cdots, z_r\in M$ such that the matrix $(\delta_i(z_j))$ is invertible. Then there exists a subring B of A such that $z_1, \cdots, z_r$ are analytically independent over B, and $B[[z_1, \cdots, z_r]]=A$.*

Proof. See [4]

Theorem 2. *Let $A=k[x_1, \cdots, x_n]$ be a finitely generated integral domain over a perfect field k. Let p be a maximal ideal of A and set $R=A_p$ (A localized at p). Assume A has dimension r. Then R is a regular local ring if and only if the following two conditions are satisfied:*

(a) *Every k-derivation of finite rank on R is integrable on R.*
(b) *There exist r derivations* $\delta_1, \cdots, \delta_r \in Der_k^1(R)$ *and elements* $z_1, \cdots, z_r$ *in the maximal ideal of R such that the matrix* $(\delta_i(z_j))$ *is invertible.*

Proof. Assume R is a regular local ring. Then condition (a) is Proposition 4 of this paper. Condition (b) is well known. A proof easily follows from [6; Theorem 3].

Now suppose conditions (a) and (b) are satisfied. We consider $\hat{R}$ the completion of R. By condition (a), $\delta_1, \cdots, \delta_r$ are integrable on R. Thus, from Proposition 2, we conclude that $\delta_1, \cdots, \delta_r$ are integrable k-derivations of rank one on $\hat{R}$. It now follows from Proposition 5, that $\hat{R}$ has the form $\hat{R}=B[[z_1, \cdots, z_r]]$ where $\{z_1, \cdots, z_r\}$ are analytically independent over B. Since the dimension of $\hat{R}$ is r, we conclude that B is a field. Therefore, $\hat{R}$ and, consequently, R itself are regular local rings. Thus, Theorem 2 is proven.

We complete this paper with a few remarks concerning the hypotheses in Theorem 2. First, the assumption that k be a perfect field is essential for condition (a) even when R is a field. Consider the following example:

EXAMPLE 2. Let P denote a perfect field of characteristic $\rho \neq 0$. Let X be an indeterminate over P and set $k=P(X^\rho)$. Then k is a field which is not perfect. Let $A=k[X]=k(X)$, a finitely generated integral domain over k. Set $R=A$ (a localization of A at the maximal ideal (0)). Certainly R is a regular local ring. We can define a k-derivation $\delta \in Der_k^1(R)$ by $\delta(X)=1$. If δ were integrable, then there would exist a k-derivation $D=\{D_0, D_1, \cdots\}$ of infinite rank on R such that $\delta=D_1$. But then we would have

$$0 = D_\rho(X^\rho) = \{D_1(X)\}^\rho = \{\delta(X)\}^\rho = 1^\rho = 1 . \tag{9}$$

Thus, δ is not integrable and conclusion (a) of Theorem 2 fails when k is not perfect.

We next note that in proving Theorem 2, we only used the fact that every k-derivation of rank one (i.e., the ordinary derivations on R) was integrable on R. Thus, a corollary to Theorem 2 is the following:

Corollary. *Let R be as in Theorem 2. Then R is a regular local ring if and only if the following two conditions are satisfied:*
(a) *Every k-derivation of rank one on R is integrable.*
(b) *There exist r derivations* $\delta_1, \cdots, \delta_r \in Der_k^1(R)$ *and elements* $z_1, \cdots, z_r$ *in the maximal ideal of R such that the matrix* $(\delta_i(z_j))$ *is invertible.*

We give examples which show that conditions (a) and (b) in this corollary are independent of each other, and that neither condition by itself is in general strong enough to imply regularity. Both examples come from looking at the curve $\mathcal{C}$: $X^2=Y^3$ defined over a perfect field k.

EXAMPLE 3. Consider the curve $\mathcal{C}$ when k has characteristic two. Let $A=k[X,Y]/(X^2-Y^3)=k[x,y]$ be the coordinate ring of $\mathcal{C}$. One easily checks that A is a (finitely generated) domain no matter what characteristic k has. Also one easily checks that the integral closure of A is given by $\bar{A}=A[t]$ where $t=x/y$. Let R denote the local ring at the origin on $\mathcal{C}$. Thus, $R=k[x,y]_p$ where $p=(x,y)$. Since $\mathcal{C}$ has a singularity at the origin, R is not a regular local ring.

If k has characteristic two, then $\delta(x)=1$, $\delta(y)=0$ is a well defined k-derivation on R. Thus, R satisfies condition (b) in the corollary. Suppose δ was integrable. Then by [9; p.173], $\delta(\bar{A})\subset\bar{A}$. But, $\delta(t)=1/y\notin\bar{A}$. Consequently, δ is not integrable. Thus, if k has characteristic two, R is a nonregular local ring satisfying condition (b) but not condition (a).

EXAMPLE 4. Again consider the curve $\mathcal{C}$ when k has characteristic not equal to two or three. Thus, 2 and 3 are units in k. In this case, condition (b) can never be satisfied. This follows from the observation that $p=(x,y)$ is a differential prime in A. That is, $\delta(p)\subset p$ for all $\delta\in Der_k^1(A)$. To show this, we need only argue that $\delta(x)$ and $\delta(y)$ are in p. If $\delta(x)\notin p$, then we would have $X\in(X^2,XY,Y^2)$ in $k[X,Y]$. Thus, $\delta(x)\in p$. Similarly, if $\delta(y)\notin p$, then $Y^2\in(X,Y^3)$. Thus, p is differential under $Der_k^1(A)$. Since $Der_k^1(R)\cong R\otimes_A Der_k^1(A)$, we conclude that condition (b) is impossible in R when the characteristic of of k is not two or three.

To complete this example, we need the following lemma:

Lemma 3. *Let $\mathcal{C}$ denote the curve $X^2=Y^3$ defined over any perfect field k of characteristic $\rho\neq 2$ or 3. Let R denote the local ring at the origin of $\mathcal{C}$. Then every k-derivation of rank one on R is integrable.*

Proof. We use the same notation as in Examples 3 and 4. In particular, the integral closure $\bar{A}$ of A is given by $k[t]$ where $t=x/y$. Since $t^2=y$, and $t^3=x$, $A=k[x,y]=k[t^2,t^3]$. Thus, $tA\subset A$.

Let $\frac{\partial}{\partial t}$ denote the canonical k-derivation of $\bar{A}$ given by $\frac{\partial}{\partial t}(t)=1$. Then one easily checks that $\lambda=t\frac{\partial}{\partial t}$ is a k-derivation of A.

We now claim that $Der_k^1(A)=\bar{A}\lambda$. If $a\in\bar{A}$, then $a\lambda(x)=3ax$ and $a\lambda(y)=2ay$. Since $tA\subset A$, we conclude that $a\lambda\in Der_k^1(A)$. Thus, $\bar{A}\lambda\subset Der_k^1(A)$. Let $k(t)$ denote the quotient field of A. Then λ generates $Der_k^1(k(t))$ as a $k(t)$-module. In particular, if $\delta\in Der_k^1(A)$, then $\delta=a\lambda$ for some $a\in k(t)$. From the computation in Example 4, we know p is differential under δ. Thus, $\delta(x)=cx+dy$, and $\delta(y)=ex+fy$ for some elements $c,d,e,f\in A=k[t^2,t^3]$. So in $k[t]$, we have

$$(10)\qquad \begin{aligned} 2at^2 &= et^3+ft^2 \\ 3at^3 &= ct^3+dt^2\,. \end{aligned}$$

Now equation (10) immediately implies that $a \in k[t]=\bar{A}$. Thus, $\delta \in \bar{A}\lambda$, and $Der_k^1(A)=\bar{A}\lambda$.

We can now argue that every derivation on A is integrable. Let $\delta \in Der_k^1(A)$. From the last paragraph, we conclude that $\delta=g(t)\dfrac{\partial}{\partial t}$ where $g(t) \in k[t]$ and has no constant term. Since t is transcendental over k, we can define a k-derivation $D=\{D_0, D_1, \cdots\}$ of infinite rank on $\bar{A}$ by the formulas:

$$\begin{aligned} D_1(t) &= g(t) \\ D_n(t) &= t \quad \text{if} \quad n \geq 2 . \end{aligned} \tag{11}$$

We note that $D_1=g(t)\dfrac{\partial}{\partial t}=\delta$. Using equation (1) together with that fact that $tA \subset A$, we easily see that $D_n(A) \subset A$ for all n. Thus, D is a k-derivation of infinite rank on A. Consequently, δ is integrable.

Now since $Der_k^1(R) \cong R \otimes_A Der_k^1(A)$, we conclude that every derivation on R is integrable. This completes the proof of Lemma 3. Thus, if k has characteristic not equal to 2 or 3, then R in Example 4 satisfies condition (a) but not condition (b).

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References

[1] W.C. Brown and W.E. Kuan: *Ideals and higher derivations in commutative rings*, Canad. J. Math. **24** (1972), 400–415.

[2] N. Heerema: *Convergent higher derivations on local rings*, Trans. Amer. Math. Soc. **132** (1968), 31–44.

[3] N. Heerema: *Higher derivations and automorpisms of complete local rings*, Bull. Amer. Math. Soc. **76** (1972), 1212–1225.

[4] Y. Ishibashi: *A note on a Lemma of Zariski*, Bull. Fukuoka Univ. Ed. III **27** (1978), 1-3.

[5] I. Kaplansky: Commutative rings, Allyn and Bacon, Boston, 1970.

[6] Y. Nakai: *On the theory of differentials in commutative rings*, J. Math. Soc. Japan **13** (1961), 63–84, MR 23 #A 2437.

[7] ————: *High order derivations* I, Osaka J. Math. **7** (1970), 1–27.

[8] P. Ribenboim: *Higher derivations of rings* I, Rev. Roum. Math. Pures Appl. **16** (1971), 77–110.

[9] A. Seidenberg: *Derivations and integral colosure*, Pacific J. Math. **16** (1966), 167–173, MR 32 #5686.

[10] ————: *Differential ideals in rings of finitely generated type*, Amer. J. Math. **89** (1967), 22–42.

[11] O. Zariski and P. Samuel: Commutative algebra I, Van Nostrand, New Yrok.