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ON LIFTING PROPERTY ON DIRECT SUMS OF HOLLOW MODULES

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Following E. Mares [12] and H. Bass [2] we shall first consider a semi-perfect module P over a ring R. One of the important properties of P is the lifting property as follows: Let $P/J(P)=\sum_I \oplus K_\alpha$ be a decomposition of $P/J(P)$, then there exists a decomposition of P: $P=\sum_I \oplus P_\alpha$ such that $\varphi(P_\alpha)=K_\alpha$ for all $\alpha \in I$, where $J(P)$ is the Jacobson radical of P and φ is the natural epimorphism of P onto $P/J(P)$. In case the module is injective, we have studied irredundant sum of indecomposable injective modules and the lifting property of decomposition over a perfect ring satisfying a certain condition in [7].

In this note we shall generalize those properties over an arbitrary ring. In order to do so, it is quite natural to take a module M_α such that $M_\alpha/J(M_\alpha)$ is a simple module instead of P_α, namely a hollow module [3]. For a direct sum of hollow modules M we shall give some characterizations of the lifting property of simple module and of decomposition of M (see the definition in §1). Finally, we shall give characterizations of artinian rings with lifting property (namely, generalized uniserial ring and semi-simple ring). We shall study the dual property -the extending property- of simple module in [8].

1. Definitions

Throughout this paper we consider a ring R with identity and we assume every module M is a unitary right R-module. We shall denote the Jacobson radical of M by $J(M)$.

Let $\{M_\alpha\}_I$ be a set of submodules of M. If $M=\sum_I M_\alpha$ and $M \neq \sum_J M_\beta$ for any proper subset J of I, we call $\sum_I M_\alpha$ *be an irredundant sum* [7]. If $\sum_K M_\gamma$ is a direct summand of M for every finite subset K of I, we say $\sum_I M_\alpha$ be a *locally direct summand* of M [9]. We denote the natural epimorphism of M onto $M/J(M)$ by φ. If there exists a direct summand M_α of M such that $\varphi(M_\alpha)=A_\alpha$ for each simple submodule A_α of $M/J(M)$, then we say M have the *lifting property of simple module.*

Now, $A_\alpha \approx M_\alpha/N_\alpha$ and $N_\alpha \supset J(M_\alpha)$. In this paper we are interested in modules such that $N_\alpha = J(M_\alpha)$ and N_α is small in M_α. In this case M_α is cyclic and $M_\alpha \approx R/A'_\alpha$, where A'_α is a right ideal of R and A'_α is contained in a unique maximal right ideal. We call such a module *cyclic hollow module* [3]. Furthermore, we only consider modules M which are direct sums of cyclic hollow modules M_α. Let $M = \sum_I \oplus M_\alpha$. Then $\varphi_M | N = \varphi_N$ for every direct summand N of M and $\varphi | K = \varphi_K$ for a cyclic hollow submodule K with $K \not\subset J(M)$. If $\varphi(M_\alpha) \neq 0$ for all α and $\sum_J \varphi(M_\alpha) = \sum_J \oplus \varphi(M_\alpha)$, we say $\sum_J M_\alpha$ be *a direct sum modulo* J(M). Finally, if for any decomposition $\varphi(M) = \sum_I \oplus A_\alpha$ with A_α simple, there exists a decomposition $M = \sum_I \oplus M_\alpha$ of M such that $\varphi(N_\alpha) = A_\alpha$ for each α, then we say M have the *lifting property of decomposition.* We shall denote $\varphi(M)$ by $\bar{M}$ if there are no confusions.

Here we shall give some remarks on hollow modules. Let N be an R-module. If $\mathrm{End}_R(N)$ is a local ring, we say N *completely indecomposable.* We do not know whether a cyclic hollow module is completely indecomposable or not (cf. [3] and [6]). In this note we are interested in completely indecomposable and cyclic hollow modules. If R is a commutative, every cyclic hollow module N is completely indecomposable, since every epimorphism of N onto itself is isomorphic. We shall consider the above property.

(E-I) *Every epimorphism of N onto itself is isomorphic.*

REMARKS. 1. If N is noetherian, N satisfies (E-I).

2. If R is directly finite i.e. $xy=1$ implies $yx=1$ and R/A is hollow for a two-sided ideal A, R/A satisfies (E-I).

3. Let R be a right perfect ring. Then every indecomposable and quasi-projective module is a hollow module satisfying (E-I) (see §3).

We note that if a hollow module N satisfies (E-I), N is completely indecomposable. Let $\{M_\alpha\}_I$ be a set of hollow modules satisfying (E-I). We define a partial order $\succ$ in $\{M_\alpha\}_I$. If $M_\alpha \approx M_\beta$, we put $M_\alpha \equiv M_\beta$. If there exists an epimorphism f of M_α onto M_β, we put $M_\alpha \succ M_\beta$. We know from (E-I) that $\succ$ and $\equiv$ define a partial order in $\{M_\alpha\}_I$. Let $M_1 \succeq M_2$, then $\bar{M}_1 \approx \bar{M}_2$. If every element in $\mathrm{Hom}_R(\bar{M}_1, \bar{M}_2)$ is induced by some element in $\mathrm{Hom}_R(M_1, M_2)$, then we say $\mathrm{Hom}_R(\bar{M}_1, \bar{M}_2)$ *be induced from* $\mathrm{Hom}_R(M_1, M_2)$.

2. Lifting property

Let R be a ring and $J = J(R)$.

Lemma 1. *Let M be an R-module and $\{M_\alpha\}_I$ a set of cyclic hollow submodules of M such that $M = \sum_I M_\alpha$. Then $M = \sum_I M_\alpha$ is an irredundant sum of M*

if $\varphi(M)=\sum_I\oplus\varphi(M_\alpha)$ *and* $\varphi(M_\alpha)\neq 0$ *for all* α. *If* $J(M)$ *is small in* M, *the converse is valid.*

Proof. It is clear.

First we give a proposition concerning with (E-I).

Proposition 1. *Let* $\{M_i\}_1^n$ *be a finite set of cyclic hollow modules with* (E-I). *We assume if* f: $M_i\to M_j$ *is epimorphic,* f *is isomorphic for any pair* i *and* j. *Then* $M=\sum\oplus M_i$ *satisfies* (E-I).

Proof. We can express any element of $\mathrm{End}_R(M)$ by a matrix (f_{ij}), where $f_{ij}\in\mathrm{Hom}_R(M_j, M_i)$. Let $M=\sum_{i=1}^{t}\sum_{j=1}^{p(i)}\oplus M_{ij}$, where $M_{ij}\approx M_{i1}$ and $M_{i1}\not\approx M_{i'1}$ if $i\neq i'$. Then (f_{ij}) is regarded as a block matrix $(f_{ij,kt})$. Let F be an epimorphism of M and $F=(f_{ij})$. We shall show one f_{ij} among f_{ik}, $k=1, 2, \cdots, n$, is isomorphic. Since F is epimorphic, $M_i=\sum_j f_{ij}(M_j)$. However, M_i is hollow and so $M_i=f_{ij}(M_j)$ for some j. Hence, f_{ij} is isomorphic by the assumption. Since $M_{ij}\approx M_{i1}$, we may assume $M_{ij}=M_{i1}$ for all j and matrix units $e_{ij,ik}$ are elements in $\mathrm{End}_R(M)$. Using those remarks and fundamental transformations of matrices, we know there exist regular matrices P_1, Q_1 such that

$$P_1FQ_1=\begin{pmatrix}1 & 0\\ 0 & F_2\end{pmatrix},\quad F_2\in\mathrm{End}_R(\sum_{i\geqslant 2}\oplus M_i)\,.$$

Noting that F_2 is epimorphic, and repeating those arguments, we get regular matrices P, Q such that $PFQ=I_M$. Hence, F is isomorphic.

Theorem 1. *Let* $\{M_\alpha\}_I$ *be a set of completely indecomposable and cyclic hollow modules and* $M=\sum_I\oplus M_\alpha$. *Then the following conditions are equivalent.*

1) *Every direct sum modulo* $J(M)$ *of indecomposable direct summands of* M *is a direct sum* (*and a locally direct summand of* M).

2) *If there exists an epimorphism* f *of* M_α *to* M_β *for any pair* α *and* β *in* I, *then* f *is isomorphic.*

Proof. 1)→2). We assume there exists an epimorphism f of M_α onto M_β and $\ker f\neq 0$. We put $M'_\alpha=\{x+f(x)\,|\,x\in M_\alpha\}\subset M_\alpha\oplus M_\beta\subset M$. Then $M_\alpha\oplus M_\beta=M'_\alpha\oplus M_\beta$ and $M'_\alpha+M_\alpha$ is a direct sum modulo $J(M)$ and hence, $M'_\alpha+M_\alpha=M'_\alpha\oplus M_\alpha$ by 1). However, $M'_\alpha\cap M_\alpha=\ker f\neq 0$.

2)→1). We note first that if $M_\alpha\approx M_\beta$ for $\alpha\neq\beta$, M_α satisfies (E-I) by the assumption. Let $\sum_J N_\alpha$ be a direct sum modulo $J(M)$ of indecomposable direct summands N_α of M. Let $K=\{1, 2, \cdots, n\}$ be a finite subset of J and put $N(n)=\sum_{i=1}^{n}N_i$. We shall show by the induction on n that $N(n)$ is a direct summand of M and $N(n)=\sum_{i=1}^{n}\oplus N_i$. If $n=1$, it is clear by the assumption. We assume $M=N(n-1)\oplus M'$ and $N(n-1)=\sum_{i=1}^{n-1}\oplus N_i$. Since N_n is a direct summand of M, N_n is isomorphic to some one M_{γ_1} in $\{M_\alpha\}_I$ and $M'=\sum_{I'}\oplus M'_\beta$ by [1], where $I'=I-K$ and M'_β is isomorphic to some $M_{\rho(\beta)}$ in $\{M_\alpha\}_I$. Furtheromre, since N_n has the exchange property in M by [1] and [4], either $M=N_n\oplus\sum_{j\neq k}\oplus N_j\oplus M'$ for some k or $M=N_n\oplus N(n-1)\oplus\sum_{I'-\delta}\oplus M'_\beta$ for some δ. We have proved our assertion in the latter case. In the former case $N_n\approx N_k\approx M_{\eta(k)}$. Let π_β be the projection of $M=N(n-1)\oplus\sum_{I'}\oplus M'_\beta$ onto M'_β. Since $\varphi(N_n)\not\subset\varphi(N(n-1))$, $\pi_\gamma|N_n$ is epimorphic for some γ. If $\eta(k)\neq\rho(\gamma)$, $\pi_\gamma|N_n$ is isomorphic by 2). If $\eta(k)=\rho(\gamma)$, M contains a direct summand $N_k\oplus M'_\gamma$ such that $N_n\approx N_k\approx M'_\gamma$. Hence, N_n satisfies (E-I) by 2) and [1]. In either case $\pi_\gamma|N_n$ is isomorphic. Accordingly, $M=N_n\oplus\ker\pi_\gamma=N_n\oplus N(n-1)\oplus\sum_{I'-\gamma}\oplus M'_\beta=N(n)\oplus\sum_{I'-\gamma}\oplus M'_\beta$.

Theorem 1′. *Let M and $\{M_\alpha\}_I$ be as above. Then the following conditions are equivalent.*

1) *Every direct sum modulo* J(M) *of indecomposable direct summands is a direct sum and a direct summand of M.*

2) *$\{M_\alpha\}_I$ is a semi-T-nilpotent set* [4] *and if there exists an epimorphism f of M_α to M_β for any pair α and β in I, then f is isomorphic.*

Proof. 1)→2). Let $\{f_i\colon M_{\alpha_i}\to M_{\alpha_{i+1}}\}$ be a set of non-isomorphic homomorphisms. Then every f_i is not epimorphic by Theorem 1. Hence, $f_i(M_{\alpha_i})\subset J(M_{\alpha_{i+1}})$. We put $M'_{\alpha_i}=\{x+f_i(x)\,|\,x\in M_{\alpha_i}\}$ and $M'=M'_{\alpha_1}\oplus M'_{\alpha_2}\oplus\cdots\oplus M'_{\alpha_n}\oplus\cdots$. Then $\varphi(M'_{\alpha_i})=\varphi(M_{\alpha_i})$ and so M' is a direct sum modulo $J(M)$. Hence, M' is a direct summand of M by 1). Therefore, $\{M_\alpha\}_I$ is a semi-T-nilpotent set by [4]. The remaining paıts are clear by Theorem 1.
2)→1). It is clear from Theorem 1 and [4], [9] and [10].

REMARK 4. If $J(M)$ is small in M, $\{M_\alpha\}_I$ is a semi-T-nilpotent set by [5] when 2) in Theorem 1 is satisfied.

Theorem 2. *Let M and $\{M_\alpha\}_I$ be as in Theorem* 1. *Then the following conditions are equivalent.*

1) *M has the lifting property of simple module.*

2) *For any pair α, β in I such that $\bar{M}_\alpha\approx\bar{M}_\beta$ and any isomorphism f of $\bar{M}_\alpha$ onto $\bar{M}_\beta$, there exists an epimorphism g of either M_α onto M_β or M_β onto M_α such that $\bar{g}=f$ or $\bar{g}=f^{-1}$.*

Proof. 1)→2). Let M_1, M_2 be two elements in $\{M_\alpha\}_I$. We assume $\bar{M}_1\approx\bar{M}_2$. Let $f\in\mathrm{Hom}_R(\bar{M}_1, \bar{M}_2)$ and we put $K=\{\bar{x}+f(\bar{x})\,|\,x\in M_1\}\subset\bar{M}_1\oplus\bar{M}_2$.

Since M has the lifting property, there exists a decomposition $M=M_1'\oplus M'$ such that $\varphi(M_1')=K$. Let x be in M_1' and $x=\sum x_i$; $x_i\in M_i$. Then since $\bar{x}=\sum \bar{x}_i\in K$, $\bar{x}_2=f(\bar{x}_1)$ and $\bar{x}_1\neq 0$ for some x. Let π_i': $M\to M_i$ be the projection. Then $\pi_i'|M_1'$ is epimorphic for $i=1, 2$. M_1' is isomorphic to some in $\{M_\alpha\}$ and $M=M_1'\oplus(M_1\oplus\cdots\overset{i}{\check{}}\cdots\oplus M_n\oplus\cdots)$ by the exchange property of M_1' by [1], where $\check{i}$ means the i-th component is omited. If i were neither 1 nor 2, $\varphi(M_1')\cap(\bar{M}_1\oplus\bar{M}_2)=0$. Hence, $i=1$ or 2. Thus, we obtain $M=M_1'\oplus M_2\oplus M_3\oplus\cdots$ or $M=M_1'\oplus M_1\oplus M_3\oplus\cdots$. In the former case, let π_i be the projection of M onto M_i (π_1: $M\to M_1'$) and $m_1=\sum\pi_i(m_1)$ for $m_1\in M_1$. Then $\bar{m}_1=\sum\overline{\pi_i(m_1)}=\bar{m}_1'+(f(\bar{m}_1')+\overline{\pi_1(m_2)})+\sum_{i\geqslant 3}\overline{\pi_i(m_i)}$; $m_1'\in M_1$. Hence, $\bar{m}_1=\bar{m}_1'$, $f(\bar{m}_1')=-\overline{\pi_2(m_1)}$ and $\sum_{i\geqslant 3}\overline{\pi_i(m_i)}=0$. Accordingly, $-(\pi_2|M_1)\in\mathrm{Hom}_R(M_2, M_1)$ and f is induced from $-(\pi_2|M_1)$. In the latter case, $K=\{f^{-1}(\bar{y})+\bar{y}\,|\,\bar{y}\in M_2\}$ and we know that $-(\pi_1|M_2)$ induces f^{-1} as above.

2)→1). Let A be a simple submodule of $\bar{M}$. Let n be the minimal integer among m such that $A\subseteq\sum_{i=1}^{m}\oplus\bar{M}_{\alpha_i}$. Put $M_{\alpha_i}=M_i$ and let $\bar{\pi}_i$ be the projection of $\sum_{i=1}^{m}\oplus\bar{M}_i$ onto $\bar{M}_i$. Then $\bar{\pi}_i|A$ is isomorphic and $A=\{\bar{m}_1+f_2(\bar{m}_1)+\cdots+f_n(m_1)\,|\,m_1\in M_1,\ f_i=(\bar{\pi}_i|A)(\bar{\pi}_1|A)^{-1}\}$. We consider a set $\{(\bar{M}_i, \bar{M}_j),\ g_{ji}=f_jf_i^{-1}\}_{i\neq j}$. Then from 2) there exists either $h_{ji}\in\mathrm{Hom}_R(M_i, M_j)$ or $h_{ij}\in\mathrm{Hom}_R(M_j, M_i)$ such that $\bar{h}_{ji}=g_{ji}$ or $\bar{h}_{ij}=g_{ij}$. In the former case (resp. the latter case) we denote $M_i\geqslant M_j$ (resp. $M_i\leqslant M_j$). We can easily see by the induction and the fact $g_{ij}g_{jk}=g_{ik}$ that there exists a maximal one among M_i's with respect to the relation $\geqslant$, say M_t. Then $A=\{g_{1t}(\bar{m}_t)+\cdots+g_{t-1t}(\bar{m}_t)+\bar{m}_t+\cdots+g_{nt}(\bar{m}_t)\,|\,m_t\in M_t\}$. Hence, we may assume $t=1$. Now from the construction above, there exist $g_j\in\mathrm{Hom}_R(M_1, M_j)$ such that $\bar{g}_j=f_j$ for all j. Put $M_1'=\{m_1+g_2(m_1)+\cdots+g_n(m_1)\,|\,m_1\in M_1\}\subset M_1\oplus M_2\oplus\cdots\oplus M_n$. Then $M_1'\oplus M_2\oplus\cdots\oplus M_n=M_1\oplus\cdots\oplus M_n$ and $\bar{M}_1'=\{\bar{m}_1+g_2(\bar{m}_1)+\cdots+g_n(\bar{m}_1)\}=A$.

Corollary. *Let* $\{M_\alpha\}_I$ *and* M *be as above. We assume each* M_α *satisfies* (E-I). *Then the following conditions are equivalent.*

1) *M has the lifting property of simple module.*

2) *In the subset* $\{M_i\}$ *of* $\{M_\alpha\}_I$ *such that* $\bar{M}_j\approx\bar{M}_1$, *the relation* $\geqslant$ *is linear and* $\mathrm{Hom}_R(\bar{M}_\alpha, \bar{M}_\beta)$ *is induced from* $\mathrm{Hom}_R(M_\alpha, M_\beta)$ *for any pair* $M_\alpha\geqslant M_\beta$.

The following theorem is a generalization on the lifting property of perfect modules.

Theorem 3.[1] *Let M and* $\{M_\alpha\}_I$ *be as in Theorem* 1. *Then the following*

1) If each M_α satisfies (E-I), then 1) and 2) are equivalent to the fact that M has the lifting property of decomposition and $\{M_\alpha\}_I$ is semi-T-nilpotent (see [8], corollacy 20).

conditions are equivalent.

1) *M has the lifting property of simple module and for any direct sum modulo $J(M)$ of indecomposable direct summands is a direct sum and a direct summand of M.*

2) *For any pair α and β in I* $\mathrm{Hom}_R(\bar{M}_\alpha, \bar{M}_\beta)$ *is induced from* $Hom_R(M_\alpha, M_\beta)$ *and any epimorphism of M_α onto M_β is isomorphic and $\{M_\alpha\}_I$ is a semi-T-nilpotent set.*

In this case M has the lifting property of decomposition.

Proof. It is clear from Theorems 1, 1′ and 2.

Finally, we shall give some characterizations of artinian rings with lifting property.

Theorem 4. *Let R be a right artinian ring. Then the following conditions* 1), 2) *and* 3), 4) *are equivalent, respectively.*

1) *R is right generalized uniserial* [13].

2) *Every direct sum of hollow modules has the lifting property of simple module.*

3) *R is semi-simple.*

4) *Every direct sum of hollow modules has the lifting property of decomposition.*

Proof. 1)→2). Every hollow module is of forms eR/eJ^t, where e is a primitive idempotent. Hence, M has the lifting property of simple module by Theorem 2.

2)→1). Let e be a primitive idempotent. We take two right ideals eA_i $i=1, 2$ such that $eJ^t \supset eA_i \supsetneqq eJ^{t+1}$ and eA_i/eJ^{t+1} is simple. Since the length of composition series of eR/eA_1 is equal to one of eR/eA_2, $eR/eA_1 \approx eR/eA_2$ by Theorem 2. Let θ be any element in $\mathrm{End}_R(eR/eJ)$. Then θ is given by the left multiplication of a regular element x in eRe. θ is also extended to an element in $\mathrm{Hom}_R(eR/eA_1, eR/eA_2)$ by Theorem 2. This homomorphism is given by the left multiplication of $x+j$, where $j \in eJe$. Hence, $(x+j)eA_1 = eA_2$. Since $jeA_1 \subset eJeJ^t \subset eA_2$, $xeA_1 \subset eA_2$ and so $xeA_1 = eA_2$. e is a regular element in eRe. Hence $eA_1 = eeA_1 = eA_2$. Thus, we have shown eJ^t/eJ^{t+1} is simple and so R is right generalized uniserial.

3)→4). It is clear.

4)→3). Let e be a primitive idempotent. We consider $M = eR/eJ \oplus eR$. Then $M/J(M) = eR/eJ \oplus eR/eJ$ and we put $A_1 = \{\bar{x} + \bar{x} \mid \bar{x} \in eR/eJ\}$ and $A_2 = \{o + \bar{x} \mid \bar{x} \in eR/eJ\}$. Then $\bar{M} = A_1 \oplus A_2$. Since M has the lifting property of decomposition, there exists a decomposition $M = M_1 \oplus M_2$ such that $\bar{M}_i = A_i$. It is clear $M_2 = 0 \oplus eR$ and so $M_1 \approx eR/eJ$ is simple. Hence, $\bar{M}_1 = A_1$ implies that eR is simple. Therefore, R is semi-simple.

Remark 5. If $M = J(M)$, we may understand M has the lifting property of simple module. Then the above theorems are valid for a direct sum of completely indecomposable hollow modules if we put some restrictions on the conditions in the theorems. Let R be a commutative Dedekind domain. Then every hollow module is isomorphic to one of R/p^n, R (if R is local) and $E(R/p^n)$ and Q (if R is local) by [6], where p is prime and Q is the quotient field of R. Since $J(E(R/p^n))=E(R/p^n)$ and $J(Q)=Q$, R satisfies Theorem 4, 2), however R is not generalized uni-serial.

Let M be as in Theorem 1. Then every direct summand N of M with $N/J(N)$ simple is a completely indecomposable (and cyclic hollow) module. Hence, every lifted direct summand from simple module is as above. Let T be an R-module and $T/J(T)$ semi-simple. We assume that for any simple submodule A of $T/J(T)$ there exists a direct summand T_1 of T such that $\bar{T}_1=A$ and T_1 is a completely indecomposable.

Proposition 2. *Let T be the R-module as above. Then every direct summand of T has the same property.*

Proof. Let $T=T_1\oplus T_2$ and $A\subset\bar{T}=T_1/J(T_1)$. Then there exists a completely indecomposable direct summand N_1 of T such that $\bar{N}_1=A$. Since N_1 has the exchange property by [14], $T=N_1\oplus T_1'\oplus T_2$ and $T_1=T_1'\oplus T_1''$ (see the proof of Theorem 2). Now $N_1\approx T_1''=T_1\subset(N_1\oplus T_2)$ and $T_1''/J(T_1'')$ is simple. Let $\bar{t}''$ be a generator of $\bar{T}''$ and $t''=n_1+t_2$; $n_1\in N_1$ and $t_2\in T_2$. Since $\bar{n}_1\in A\subset\bar{T}_1$, $\bar{t}_2=0$. Hence, $A=\bar{n}_1R=\bar{t}''R=\bar{T}_1''$.

Corollary. *Let T be as above. We assume $T/J(T)=\sum_{i=1}^{n}\oplus A_i$; the A_i is simple. Then $T=\sum_{i=1}^{n}\oplus T_i\oplus S$ whith $\bar{T}_i$ simple and $\bar{S}=\bar{0}$.*

Proof. We can prove it by the proposition and the induction on n.

3. Corollaries

We shall study some special cases.

Corollary 1. *Let M and $\{M_\alpha\}$ be as in Theorem 1. If M satisfies the equivalent conditions in Theorems 1, 2 or 3, then every direct summand of M satisfies the same condition.*

Proof. Since each M_α is cyclic, every direct summand of M is a direct sum of indecomposable modules which are isomorphic to some in $\{M_\alpha\}_I$ by [14]. Hence, we have the corollary.

Corollary 2. *Let M and $\{M_\alpha\}_I$ be as above. We assume $J(M)$ is small. Then the following conditions are equivalent.*

1) *Every irredundant sum of indecomposable direct summands of M is a direct sum.*

2) *If there exists an epimorphism f of M_α onto M_β, f is isomorphic for any pair α and β in I.*

Proof. It is clear from Lemma 1, Remark 4 and Theorem 1′ and the proof of Theorem 1.

Corollary 3.[2] *Let R be a right perfect ring. Then every quasi-projective module Q* [11] *is isomorphic to $\sum_{i=1}^{n}\sum_{j}\oplus e_iR/e_iA_{ij}$. Then Q has the lifting property of simple module if and only if $\{e_iA_{ij}\}$ is linear with respect to the inclusion for each i. Q has the lifting property of decomposition if and only if $e_iA_{ij}=e_iA_{i1}$ for each i, where the e_i is primitive, $e_iR \not\approx e_jR$ if $i \neq j$ and the e_iA_{ij} is the right ideal such that $e_iRe_iA_{ij} \subset e_iA_{ij}$.*

Proof. Every quasi-projective module Q is of a form P/K, where P is projective and K a character submodule in P which is contained in PJ by [11]. Since $P \approx \sum_{i=1}^{n}\oplus(e_iR)^{I_i}$, Q is a direct sum of e_iR/e_iA_{ij} with $e_iRe_iA_{ij} \subset e_iA_{ij}$. Hence, noting $\mathrm{Hom}_R(e_iR/e_iA_{ij}, e_iR/e_iA_{ij'})$ is given by some elements in e_iRe_i and $\mathrm{Hom}_R(e_iR/e_iJ, e_iR/e_iJ) \approx e_iRe_i/e_iJe_i$, we have the corollary from Theorems 2 and 3 and Remark 4.

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2) Added in proof. Dr. K. Oshiro informed to the author that Q has always the lifting property of decomposition.

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