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CALIBRATED GEOMETRIES IN QUATERNIONIC GRASSMANNIANS

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Introduction

Let $\boldsymbol{H}$ denote the field of quaternions and $\boldsymbol{H}^n$ the set of all n-column vectors over $\boldsymbol{H}$. We regard $\boldsymbol{H}^n$ as a right $\boldsymbol{H}$-space. The object of this paper is the quaternionic Grassmannian $G_p(\boldsymbol{H}^{p+q})$, that is, the set of all right $\boldsymbol{H}$-subspaces of $\boldsymbol{H}$-dimension p in $\boldsymbol{H}^{p+q}$.

We apply the method of calibrated geometries to the invariant differential forms on the quaternionic Grassmannians and show that certain sub-Grassmannians in the quaetrnionic Grassmannians are uniquely volume minimizing in their homology classes. Strictly speaking, we prove the following theorem.

Theorem 1. *Take a right $\boldsymbol{H}$-subspace E of $\boldsymbol{H}$-dimension $p+r$ in $\boldsymbol{H}^{p+q}$. Then the sub-Grassmannian $G_p(E)$ in $G_p(\boldsymbol{H}^{p+q})$ is a volume minimizing submanifold in its real homology class. Moreover any volume minimizing submanifold in the same homology class is congruent to it.*

Here we comment on earlier results concerning Theorem 1. Gluck-Morgan-Ziller [4] proved that in the real Grassmannian $G_p(\boldsymbol{R}^{p+q})$ each sub-Grassmannian $G_p(\boldsymbol{R}^{p+r})$ for $1\leqslant r\leqslant q-1$ is uniquely volume minimizing in its homology class if p is an even integer greater than or equal to 4. The present paper was inspired by their paper.

Berger [2] proved that the projective subplane $P^r(\boldsymbol{H})=G_1(\boldsymbol{H}^{1+r})$ in the quaternionic projective space $P^q(\boldsymbol{H})$ is volume minimizing in its homology class for $1\leqslant r\leqslant q-1$. His method is applicable to all quaternionic Kähler manifolds and as a result of the application it follows that a compact quaternionic submanifold in a quaternionic Kähler manifold is volume minimizing in its homology class. Moreover Fomenko [3] showed that $G_1(\boldsymbol{H}^{1+r})$ in $G_p(\boldsymbol{H}^{p+q})$ is volume minimizing in its homology class for $1\leqslant r\leqslant q-1$.

There is a homologically volume minimizing sub-Grassmannian whose underlying field is different from that of the ambient Grassmannain. $G_1(\boldsymbol{H}^k)$

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naturally imbedded in $G_2(\boldsymbol{C}^n)$ for $2 \leqslant k \leqslant [n/2]$, $G_1(\boldsymbol{H}^k)$ and $G_2(\boldsymbol{C}^l)$ naturally imbedded in $G_4(\boldsymbol{R}^n)$ for $2 \leqslant k \leqslant [n/4]$ and $3 \leqslant l \leqslant [n/2]$ respectively are such examples. These are all quaternionic submanifolds. See Tasaki [6].

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1. Calibrated geometries in symmetric spaces

We first define calibrations after Harvey-Lawson [5]. Let V be a real vector space of finite dimension with an inner product and ϕ a d-form on V. If ϕ satisfies $\phi(\xi) \leqslant 1$ for each oriented d-plane ξ in V, we call ϕ a *calibration* on V. For a calibration ϕ on V we say that ϕ *calibrates* an oriented d-plane ξ if $\phi(\xi)=1$.

Let X be a Riemannian manifold and ϕ a closed d-form on X. If ϕ is a calibration on each tangent space to X, we call ϕ a *calibration* on X. For a calibration ϕ on X we say that ϕ *calibrates* an oriented submanifold M if ϕ calibrates the tangent space to M at each point.

We consider a Riemannian manifold X with a calibration ϕ on it. Let M be a compact oriented submanifold calibrated by ϕ and M' a compact oriented submanifold contained in the same real homology class as M. Then, using Stokes' theorem, we obtain

$$\mathrm{vol}\,(M) = \int_M \phi = \int_{M'} \phi \leqslant \mathrm{vol}\,(M')\,.$$

Hence M is volume minimizing in its real homology class. If M' is also volume minimizing, then M' is calibrated by ϕ.

Now we consider calibrated geometries in symmetric spaces. Let X be a compact symmetric space and G the identity component of the group of all isometries of X. Take and fix a point x in X. Let K be the isotropy subgroup of G at x. Then K acts linearly on the tangent space $T_x(X)$. We can extend any K-invariant form on $T_x(X)$ to a parallel form on X. So it is important for us to construct K-invariant calibrations on $T_x(X)$. We do so on the tangent space to the quaternionic Grassmannian in Section 3.

2. Quaternionic linear algebra and quaternionic Grassmannians

In this section we review the quaternionic linear algebra and prepare for studying the geometry of the quaternionic Grassmannians. We denote by $Sp(1)$ the group of quaternions with norm 1.

Let X be a right $\boldsymbol{H}$-space of finite dimension with an $Sp(1)$-invariant inner product $\cdot$. Let $Sp(X)$ denote the group of all right $\boldsymbol{H}$-linear isometries of X. For another right $\boldsymbol{H}$-space Y of finite dimension with an $Sp(1)$-invariant inner product $\cdot$, we denote by $\mathrm{Hom}_{\boldsymbol{H}}(X, Y)$ the real vector space of all right $\boldsymbol{H}$-linear maps from X to Y. We can consider the transposed map ${}^tS$ of S in $\mathrm{Hom}_{\boldsymbol{H}}(X,$

Y): $({}^tSy)\cdot x = y\cdot(Sx)$ for $x\in X$ and $y\in Y$. Then ${}^tS$ is a right $\boldsymbol{H}$-linear map from Y to X, because the inner products are $Sp(1)$-invariant. Note that the transposed map ${}^tA$ of A in $Sp(X)$ is equal to A^{-1}.

The canonical $Sp(1)$-invariant inner product $\cdot$ on $\boldsymbol{H}^n$ is defined by

$$x\cdot y = \mathrm{Re}\sum_{s=1}^{n} \bar{x}_s y_s$$

for $x=(x_s)$ and $y=(y_s)$ in $\boldsymbol{H}^n$. The action of $Sp(\boldsymbol{H}^{p+q})$ on $\boldsymbol{H}^{p+q}$ induces that of $Sp(\boldsymbol{H}^{p+q})$ on $G_p(\boldsymbol{H}^{p+q})$, which is transitive. Take an element V in $G_p(\boldsymbol{H}^{p+q})$. The orthogonal complement $V^\perp$ of V in $\boldsymbol{H}^{p+q}$ is a right $\boldsymbol{H}$-subspace of $\boldsymbol{H}$-dimension q in $\boldsymbol{H}^{p+q}$. The isotropy subgroup of $Sp(\boldsymbol{H}^{p+q})$ at V is $Sp(V)\times Sp(V^\perp)$. Hence $G_p(\boldsymbol{H}^{p+q})$ is a homogeneous space of the form $Sp(\boldsymbol{H}^{p+q})/Sp(V)\times Sp(V^\perp)$. We define an action of $Sp(V)\times Sp(V^\perp)$ on $\mathrm{Hom}_{\boldsymbol{H}}(V, V^\perp)$ by

$$(A, B)\, S = BSA^{-1}$$

for $A\in Sp(V)$, $B\in Sp(V^\perp)$ and $S\in \mathrm{Hom}_{\boldsymbol{H}}(V, V^\perp)$.

Now we construct a local parametrization of $G_p(\boldsymbol{H}^{p+q})$ around V:

$$\begin{aligned} c\colon \mathrm{Hom}_{\boldsymbol{H}}(V, V^\perp) &\to G_p(\boldsymbol{H}^{p+q}) \\ S \qquad &\mapsto \text{the graph of } S\,. \end{aligned}$$

Note that the graph of S is the right $\boldsymbol{H}$-subspace of the form $\{v+Sv;\, v\in V\}$ in $\boldsymbol{H}^{p+q}$. Thus the tangent space $T_V(G_p(\boldsymbol{H}^{p+q}))$ is identified with $\mathrm{Hom}_{\boldsymbol{H}}(V, V^\perp)$.

Lemma 2.1. *The local parametrization c is $Sp(V)\times Sp(V^\perp)$-equivariant. In particular, the linear isotropy action of $Sp(V)\times Sp(V^\perp)$ on* $\mathrm{Hom}_{\boldsymbol{H}}(V, V^\perp)$ *is the action defined above.*

Proof. For $(A, B)\in Sp(V)\times Sp(V^\perp)$ and $S\in \mathrm{Hom}_{\boldsymbol{H}}(V, V^\perp)$,

$$\begin{aligned} (A, B)\, c(S) &= \{Av+BSv;\, v\in V\} \\ &= \{v+BSA^{-1}v;\, v\in V\} \\ &= c((A, B)\, S)\,. \end{aligned}$$

We define an inner product $\cdot$ on $\mathrm{Hom}_{\boldsymbol{H}}(V, V^\perp)$ by

$$S\cdot T = \mathrm{tr}_{\boldsymbol{R}}({}^tST)\,,$$

for S and T in $\mathrm{Hom}_{\boldsymbol{H}}(V, V^\perp)$. Then this inner product is $Sp(V)\times Sp(V^\perp)$-invariant. So it induces an $Sp(\boldsymbol{H}^{p+q})$-invariant metric on $G_p(\boldsymbol{H}^{p+q})$, with respect to which $G_p(\boldsymbol{H}^{p+q})$ is a symmetric space.

3. Invariant calibrations on the tangent space

Take and fix an element V in $G_p(\boldsymbol{H}^{p+q})$. We construct $Sp(V)\times Sp(V^\perp)$-

invariant calibrations on $\mathrm{Hom}_{\boldsymbol{H}}(V, V^{\perp})$.

We first consider the set

$$C=\{J\in Sp(V); J^2=-1_V\} .$$

$Sp(V)$ acts on C by $g(J)=gJg^{-1}$ for $g\in Sp(V)$ and $J\in C$. Indeed C is invariant under the action of $Sp(V)$. Moreover we obtain the next lemma.

Lemma 3.1. *The action of $Sp(V)$ on C is transitive.*

Proof. Take an orthonormal $\boldsymbol{H}$-basis $\{e_1, \cdots, e_p\}$ for V. For $(\theta_1, \cdots, \theta_p)\in \boldsymbol{R}^p$, put

$$t(\theta_1, \cdots, \theta_p)\sum_{a=1}^{p} e_a h_a=\sum_{a=1}^{p} e_a(\cos\theta_a+i\sin\theta_a)h_a\,, \quad h_a\in\boldsymbol{H}\,.$$

Then the subgroup $T=\{t(\theta_1, \cdots, \theta_p); \theta_a\in\boldsymbol{R}\}$ of $Sp(V)$ is a maximal torus of $Sp(V)$. So for each J in C there is g in $Sp(V)$ such that $gJg^{-1}\in T$. Since $(gJg^{-1})^2=-1_V$, $gJg^{-1}=t(\pm\pi/2, \cdots, \pm\pi/2)$. We can retake g_1 in $Sp(V)$ such that $g_1Jg_1^{-1}=t(\pi/2, \cdots, \pi/2)$, hence the action of $Sp(V)$ on C is transitive.

Since C is a subset of $Sp(V)$, each element J in C acts in natural way on $\mathrm{Hom}_{\boldsymbol{H}}(V, V^{\perp})$. The action of J on $\mathrm{Hom}_{\boldsymbol{H}}(V, V^{\perp})$ gives an orthogonal complex structure on it. Let ω_J denote the corresponding fundamental 2-form on $\mathrm{Hom}_{\boldsymbol{H}}(V, V^{\perp})$.

Let $\int_{Sp(V)}$ be the invariant measure on $Sp(V)$ with total volume 1. Take an element J_0 in C and consider the form

$$\lambda_r=\frac{1}{(2pr)\,!}\int_{g\in Sp(V)} g^*\omega_{J_0}^{2pr}$$

for $1\leqslant r\leqslant q-1$. Then λ_r is an $Sp(V)\times Sp(V^{\perp})$-invariant $4pr$-form on $\mathrm{Hom}_{\boldsymbol{H}}(V, V^{\perp})$. Since $g^*\omega_J=\omega_{g^{-1}Jg}$, the form λ_r is regarded as the average of $\omega_J^{2pr}/(2pr)!$ over all J in C by Lemma 3.1 and independent of the choice of J_0.

Let R be a right $\boldsymbol{H}$-subspace of $\boldsymbol{H}$-dimension r in $V^{\perp}$. Since $\mathrm{Hom}_{\boldsymbol{H}}(V, R)$ is a J-invariant $4pr$-plane for each J in C, we can consider the canonical orientation of $\mathrm{Hom}_{\boldsymbol{H}}(V, R)$ with respect to each orthogonal complex structure J. These orientations are the same, because C is connected. We call this orientation the *canonical orientation* of $\mathrm{Hom}_{\boldsymbol{H}}(V, R)$.

Theorem 3.2. *The form λ_r is a calibration on $\mathrm{Hom}_{\boldsymbol{H}}(V, V^{\perp})$. For each oriented $4pr$-plane ξ in $\mathrm{Hom}_{\boldsymbol{H}}(V, V^{\perp})$, λ_r calibrates ξ if and only if ξ is of the form $\mathrm{Hom}_{\boldsymbol{H}}(V, R)$ with the canonical orientation for some right $\boldsymbol{H}$-subspace R of $\boldsymbol{H}$-dimension r in $V^{\perp}$.*

Proof. For J in C and an oriented $4pr$-plane ξ in $\mathrm{Hom}_{\boldsymbol{H}}(V, V^{\perp})$, by Wirtinger's inequality we have $\omega_J^{2pr}(\xi)/(2pr)! \leqslant 1$, and the equality holds if and only if ξ is a canonically oriented J-invariant $4pr$-plane in $\mathrm{Hom}_{\boldsymbol{H}}(V, V^{\perp})$. So $\lambda_r(\xi)=1$ if ξ is of the form $\mathrm{Hom}_{\boldsymbol{H}}(V, R)$ with the canonical orientation for some right $\boldsymbol{H}$-subspace R of $\boldsymbol{H}$-dimension r in $V^{\perp}$.

Next we show that ξ is of the form $\mathrm{Hom}_{\boldsymbol{H}}(V, R)$ with the canonical orientation if $\lambda_r(\xi)=1$. It is sufficient to show that a $4pr$-plane P which is J-invariant for each J in C is of the form $\mathrm{Hom}_{\boldsymbol{H}}(V, R)$ for some right $\boldsymbol{H}$-subspace R of $\boldsymbol{H}$-dimension r in $V^{\perp}$.

Let $V_1, \cdots, V_p$ be right $\boldsymbol{H}$-subspaces of $\boldsymbol{H}$-dimension 1 in V such that $V= V_1\oplus\cdots\oplus V_p$ is an orthogonal direct sum decomposition. We can regard in natural way $\mathrm{Hom}_{\boldsymbol{H}}(V_a, V^{\perp})$ as a subspace of $\mathrm{Hom}_{\boldsymbol{H}}(V, V^{\perp})$ for $1\leqslant a\leqslant p$. Then $\mathrm{Hom}_{\boldsymbol{H}}(V, V^{\perp})=\mathrm{Hom}_{\boldsymbol{H}}(V_1, V^{\perp})\oplus\cdots\oplus\mathrm{Hom}_{\boldsymbol{H}}(V_p, V^{\perp})$ is an orthogonal direct sum decomposition. Take a nonzero element S in P. We have a decomposition of S:

$$S = S_1+\cdots+S_p\,, \quad S_a\in\mathrm{Hom}_{\boldsymbol{H}}(V_a, V^{\perp})\,.$$

As S is nonzero, $S_b\neq 0$ for some b. Take a unit vector e_a in V_a for each a. Put

$$J_0\Big(\sum_{a=1}^{p} e_a h_a\Big) = \sum_{a=1}^{p} e_a ih_a\,, \quad h_a\in\boldsymbol{H}\,,$$

$$J_b\Big(\sum_{a=1}^{p} e_a h_a\Big) = -\sum_{a\neq b} e_a ih_a+e_b ih_b\,.$$

J_0 and J_b are contained in C. By the assumption of P

$$S_b = \frac{1}{2}(S-SJ_0J_b)\in P\,.$$

Since S_b is nonzero, the image R_1 of S_b is a right $\boldsymbol{H}$-subspace of $\boldsymbol{H}$-dimension 1 in $V^{\perp}$. The set Ce_b spans V as a real vector space, so CS_b spans $\mathrm{Hom}_{\boldsymbol{H}}(V, R_1)$. Hence $\mathrm{Hom}_{\boldsymbol{H}}(V, R_1)$ is contained in P. The orthogonal complement of $\mathrm{Hom}_{\boldsymbol{H}}(V, R_1)$ in P is also J-invariant for all J in C. Iterating the above argument, we can show that P is of the form $\mathrm{Hom}_{\boldsymbol{H}}(V, R)$ for some right $\boldsymbol{H}$-subspace R of $\boldsymbol{H}$-dimension r in $V^{\perp}$.

REMARK 3.3. The set of all oriented $4pr$-planes ξ in $\mathrm{Hom}_{\boldsymbol{H}}(V, V^{\perp})$ which satisfy $\lambda_r(\xi)=1$ is homeomorphic to $G_r(V^{\perp})$, hence it is compact and connected.

Corollary 3.4. *Regard λ_r as a constant coefficient differential $4pr$-form on* $\mathrm{Hom}_{\boldsymbol{H}}(V, V^{\perp})$. *Then the submanifolds in* $\mathrm{Hom}_{\boldsymbol{H}}(V, V^{\perp})$ *calibrated by λ_r are locally the canonically oriented $4pr$-planes* $\mathrm{Hom}_{\boldsymbol{H}}(V, R)$ *for some right $\boldsymbol{H}$-subspaces R of $\boldsymbol{H}$-dimension r in $V^{\perp}$ and their parallel translates in* $\mathrm{Hom}_{\boldsymbol{H}}(V, V^{\perp})$.

Proof. By Theorem 3.2 the canonically oriented $4pr$-planes $\mathrm{Hom}_{\boldsymbol{H}}(V, R)$ for right $\boldsymbol{H}$-subspaces R of $\boldsymbol{H}$-dimension r in $V^{\perp}$ and their parallel translates in $\mathrm{Hom}_{\boldsymbol{H}}(V, V^{\perp})$ are calibrated by λ_r.

Conversely let M be a submanifold calibrated by λ_r in $\mathrm{Hom}_{\boldsymbol{H}}(V, V^{\perp})$. Take and fix an element g in $Sp(\boldsymbol{H}^{p+q})$ such that $g\boldsymbol{H}^p = V$. For each $h \in \boldsymbol{H}$ and $v \in V$ we define $hv = ghg^{-1}v$. Then we can regard V and hence $\mathrm{Hom}_{\boldsymbol{H}}(V, V^{\perp})$ as left $\boldsymbol{H}$-spaces. By Theorem 3.2 the tangent spaces to M are all left $\boldsymbol{H}$-subspaces in $\mathrm{Hom}_{\boldsymbol{H}}(V, V^{\perp})$. By Assertion 2 in Alekseevskii [1], M is totally geodesic, hence it is locally of the form $\mathrm{Hom}_{\boldsymbol{H}}(V, R)$ for some right $\boldsymbol{H}$-subspace R of $\boldsymbol{H}$-dimension r in $\mathrm{Hom}_{\boldsymbol{H}}(V, V^{\perp})$ or its parallel translate.

4. Proof of Theorem 1

Take an element V_0 in $G_p(\boldsymbol{H}^{p+q})$. The form λ_r on $\mathrm{Hom}_{\boldsymbol{H}}(V_0, V_0^{\perp})$ is $Sp(V_0) \times Sp(V_0^{\perp})$-invariant, so we can extend λ_r to a parallel form on $G_p(\boldsymbol{H}^{p+q})$. The extended form is also denoted by λ_r, which is independent of the choice of V_0.

Lemma 4.1. *The form λ_r is a calibration on $G_p(\boldsymbol{H}^{p+q})$.*

Proof. This lemma follows from Theorem 3.2.

Proof of Theorem 1. Take an element V in $G_p(E)$. Let R be the orthogonal complement of V in E. Then R is a right $\boldsymbol{H}$-subspace of $\boldsymbol{H}$-dimension r in $V^{\perp}$. By the definition of the local parametrization c around V, $\mathrm{Hom}_{\boldsymbol{H}}(V, R)$ in $\mathrm{Hom}_{\boldsymbol{H}}(V, V^{\perp})$ is tangent to $G_p(E)$ at V. So $G_p(E)$ is calibrated by λ_r by Theorem 3.2, hence it is volume minimizing in its real homology class.

Here we give another representation of the local parametrization c in order to characterize submanifolds calibrated by λ_r. For $V \in G_p(\boldsymbol{H}^{p+q})$, $S \in \mathrm{Hom}_{\boldsymbol{H}}(V, V^{\perp})$, $u, v \in V$ and $x \in V^{\perp}$,

$$(v+Sv)\cdot(u+x) = v\cdot(u+{}^tSx)\,.$$

Hence we obtain

$$c(S)^{\perp} = \{v+Sv; v \in V\}^{\perp} = \{-{}^tSx+x; x \in V^{\perp}\}\,.$$

Take a right $\boldsymbol{H}$-subspace R of $\boldsymbol{H}$-dimension r in $V^{\perp}$. Let Q be the orthogonal complement of R in $V^{\perp}$ and $Q' = \{-{}^tSx+x; x \in Q\}$. Let E be the orthogonal complement of Q' in $\boldsymbol{H}^{p+q}$. Now we assert that $c(S+\mathrm{Hom}_{\boldsymbol{H}}(V, R))$ is contained in $G_p(E)$. For $T \in \mathrm{Hom}_{\boldsymbol{H}}(V, R)$, $v \in V$ and $x \in Q$,

$$({}^tTx)\cdot v = x\cdot(Tv) = 0\,,$$

hence we obtain ${}^tTx = 0$.

$$\begin{aligned} c(S+T)^{\perp} &= \{v+(S+T)\,v;\, v\in V\}^{\perp} \\ &= \{-({}^tS+{}^tT)\,x+x;\, x\in V^{\perp}\} \\ &\supset \{-{}^tSx+x;\, x\in Q\} = Q'\,. \end{aligned}$$

Therefore we have $c(S+T)\subset E$, that is, $c(S+\mathrm{Hom}_{\boldsymbol{H}}(V, R))\subset G_p(E)$. Since $\dim(S+\mathrm{Hom}_{\boldsymbol{H}}(V, R))=\dim(G_p(E))=4pr$, $c(S+\mathrm{Hom}_{\boldsymbol{H}}(V, R))$ is open in $G_p(E)$ and the images of the tangent spaces to $S+\mathrm{Hom}_{\boldsymbol{H}}(V, R)$ under the differential of c are the tangent spaces to $G_p(E)$.

Now at each point of $G_p(\boldsymbol{H}^{p+q})$ the set of oriented tangent planes calibrated by λ_r is compact and connected by Remark 3.3, hence for each S in $\mathrm{Hom}_{\boldsymbol{H}}(V, V^{\perp})$ the differential of c gives a one to one correspondence between the set of oriented tangent planes calibrated by λ_r in $T_S(\mathrm{Hom}_{\boldsymbol{H}}(V, V^{\perp}))$ and that in $T_{c(S)}(G_p(\boldsymbol{H}^{p+q}))$. Therefore the inverse image of a submanifold calibrated by λ_r in $G_p(\boldsymbol{H}^{p+q})$ under c is a submanifold calibrated by λ_r in $\mathrm{Hom}_{\boldsymbol{H}}(V, V^{\perp})$, which is locally of the form $S+\mathrm{Hom}_{\boldsymbol{H}}(V, R)$ for some S in $\mathrm{Hom}_{\boldsymbol{H}}(V, V^{\perp})$ and some right $\boldsymbol{H}$-subspace R of $\boldsymbol{H}$-dimension r in $V^{\perp}$ by Corollary 3.4. Hence by the above argument a submanifold calibrated by λ_r in $G_p(\boldsymbol{H}^{p+q})$ is locally a sub-Grassmannian in $G_p(\boldsymbol{H}^{p+q})$.

Now let M be a compact oriented submanifold of $G_p(\boldsymbol{H}^{p+q})$ which minimizes volume in the homology class $[G_p(E)]$. Then it is also calibrated by λ_r. By the above result M is a sub-Grassmannian in $G_p(\boldsymbol{H}^{p+q})$, hence it is congruent to $G_p(E)$.

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