ON A GENERALIZATION OF THE WILCOXON TEST
FOR CENSORED DATA

BY

NARIAKI SUGIURA

1. Introduction

Let two populations \( \Pi_i \) \((i = 1, 2)\) be such that

\[
\Pi_i : P \{ X_i \leq x \} = \begin{cases} 
0 & x < 0, \\
p_{i-1} + \int_0^x f_{i-1}(t) dt & x \geq 0.
\end{cases}
\]

When we wish to test the hypothesis \( H : \Pi_1 = \Pi_2 \) by two random samples \((X_1, X_2, \ldots, X_n)\) and \((Y_1, Y_2, \ldots, Y_n)\) taken from \( \Pi_1 \) and \( \Pi_2 \) respectively, ties occurring at the origin prevent us from using the Wilcoxon statistic. As Kruskal and Wallis [4] and Putter [7] considered, however, the concept of midrank is available in this case and we define the test statistic \( U_m \) as follows:

\[
U_m = \frac{1}{n_1 n_2} \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} m(X_i, Y_j),
\]

where

\[
m(X, Y) = \begin{cases} 
1 & X > Y, \\
2 & X = Y, \\
0 & X < Y.
\end{cases}
\]

If we define \( V_m \) by interchanging \( X \) and \( Y \) in (1.2), we can easily see that \( U_m + V_m = 1 \). So we consider only \( U_m \) in the following.

The mean and variance of \( U_m \) are calculated in section 2 and the consistency as well as unbiasedness of the test based on \( U_m \) are shown in section 3. The asymptotic relative efficiency is calculated with respect to the location alternative in section 4 and the asymptotic efficiency relative to Halperin's \( U_c \) conditional test [2] in section 5. Finally in section 6 we apply these tests to some data of cleft-palate patients kindly provided by Mr. A. Takayori, Dental School, Osaka University.
2. Mean and variance of \( U_m \)

**Proposition 1.** Mean and variance of the statistic \( U_m \) defined in (1.2) are such that

\[
(2.1) \quad \mathbb{E}(U_m) = \frac{1}{2} p_0 p_1 + q_0 p_1 + \int_0^\infty F_1(t) dF_0(t) \quad (= \bar{p}, \text{ say}) ,
\]

\[
(2.2) \quad \text{Var}(U_m) = \frac{1}{n_1 n_2} \left\{ pq - \frac{1}{4} p_0 p_1 + (n_1 - 1)s_1 + (n_2 - 1)s_2 \right\} ,
\]

where

\[
q = 1 - \bar{p} , \quad s_1 = \int_0^\infty (F_0(t) - q + p_0)^2 dF_1(t) + p_1 \left(q - \frac{1}{2} p_0\right)^2 ,
\]

\[
q_0 = 1 - \bar{p}_0 , \quad s_2 = \int_0^\infty (F_0(t) - p + p_0)^2 dF_0(t) + p_0 \left(p - \frac{1}{2} \bar{p}_0\right)^2 .
\]

\((i = 0, 1)\),

Proof. By the definition of \( U_m \) in (1.2), we have

\[
\mathbb{E}(U_m) = \mathbb{E}[m(X, Y)] = \mathbb{P}\{X > Y \geq 0\} + \frac{1}{2} \mathbb{P}\{X = Y = 0\} = \int_{t_1 > t_2 > 0} f_1(t_1) f_1(t_2) dt_1 dt_2 + q_0 p_1 + \frac{1}{2} p_0 p_1,
\]

to get (2.1). Since \( U_m \) is a kind of \( U \) statistic due to Hoeffding [3] and Lehmann [5], Problem 8 in Fraser [1, p. 257] is available to calculate its variance, that is,

\[
\text{Var}(U_m) = \frac{1}{n_1 n_2} \{\xi_{1,1} + (n_1 - 1)\xi_{0,1} + (n_2 - 1)\xi_{1,0}\} ,
\]

where

\[
\xi_{1,1} = \text{Var}[m(X, Y)] = pq - \frac{1}{4} p_0 p_1 ,
\]

\[
\xi_{0,1} = \text{Var}[f_{0,1}(Y)] , \quad f_{0,1}(y) = \mathbb{E}[m(X, Y) | Y = y] ,
\]

\[
\xi_{1,0} = \text{Var}[f_{1,0}(X)] , \quad f_{1,0}(x) = \mathbb{E}[m(X, Y) | X = x] .
\]

In our case

\[
f_{0,1}^*(y) = \begin{cases} 1 & y < 0 , \\ q_0 + \frac{1}{2} p_0 & y = 0 , \\ q_0 - F_0(y) & y > 0 , \end{cases}
\]
and hence
\[ \xi_{o,1} = \left( q_o + \frac{1}{2} p_o \right)^2 + p_i + \int_0^{\infty} (q_o - F_o(t))^2 dF_i(t) - p^2, \]
which, after some calculations, turns out to be equal to \( s_1 \) in (2.2). In the same way, we get \( s_2 \) in (2.2) from \( \xi_{1,0} \).

**Corollary.** Under the null hypothesis \( H \)

\[
(2.3) \quad \text{E}(U_m) = \frac{1}{2},
\]
\[
(2.4) \quad \text{Var}(U_m) = \frac{1}{12n_1n_2} \left\{ 3(1-\rho_0^2) + (n_1+n_2-2) (1-\rho_0^2) \right\}.
\]

Proof. Since under the null hypothesis \( f_o(t) = f_i(t) \) and \( p_o = p_i \), we have from (2.1)
\[
\text{E}(U_m) = \frac{1}{2} p_o^2 + p_o q_o + \frac{1}{2} [F_o(t) t]_o^\infty = \frac{1}{2},
\]
and from (2.2)
\[
\text{Var}(U_m) = \frac{1}{n_1n_2} \left\{ \frac{1}{4} (1-\rho_0^2) + (n_1+n_2-2)s \right\},
\]
where
\[
s = \int_0^{\infty} \left( F_o(t) - \frac{1}{4} + p_o \right)^2 dF_i(t) + \frac{1}{4} p_o q_o^2
\]
\[= \frac{1}{12} (1-\rho_0^2). \]
This proves (2.4).

3. **Consistency and unbiasedness of the \( U_m \) test**

In this section we consider the following alternative,
\[
(3.1) \quad K: p_o + F_0(x) < p_i + F_i(x) \text{ for any } x \geq 0,
\]
that is to say, \( \Pi_1 \) is stochastically larger than \( \Pi_2 \). Let the test function determined by \( U_m \), which will be called the \( U_m \) test, be
\[
(3.2) \quad \phi(X_1, \ldots, X_{n_1}; Y_1, \ldots, Y_{n_2}) = \begin{cases} 1 & U_m > U_a, \\ 0 & U_m < U_a, \end{cases}
\]
where the constant \( U_a \) is determined such that \( \text{E}\phi = \alpha \) under the null hypothesis \( H \).

**Theorem 1.** The \( U_m \) test of the hypothesis \( H: \Pi_1 = \Pi_2 \) against the alternative \( K \) is unbiased and consistent under the limiting condition:
(3.3) \[ n_1 + n_2 = N, \quad n_1 = \alpha_1 N, \quad n_2 = \alpha_2 N, \quad \text{and} \quad N \to \infty, \]
(with \( \alpha_1 \) and \( \alpha_2 \) fixed).

Proof. By the lemma 3.1 in Lehmann [5], it is sufficient, to prove consistency, to show that, under the alternative \( K \), \( E(U_m) \geq 1/2 \) and \( \text{Var}(U_m) \to 0 \) as \( N \to \infty \). The former is derived from (2.1) and (3.1), while the latter from (2.2).

Unbiasedness is proved from the following lemma which assures the validity of Theorem 3.1 in Lehmann [5], even when the population distribution is discontinuous at the origin as is the case with (1.1).

**Lemma.** If the test function satisfies \( \phi(x_1, \ldots, x_n; y_1, \ldots, y_n) \geq \phi(x_1, \ldots, x_n; z_1, \ldots, z_n) \) whenever \( y_i \leq z_i \) (\( i = 1, 2, \ldots, n \)), then the power function against the alternative \( K \) in (3.1) satisfies \( E_{G_0, g_1}(\phi) \geq E_{G_0, g_0}(\phi) \) for \( G_0 \) and \( G_1 \) representing the distribution function of \( I_1 \) and \( I_2 \) respectively in (1.1).

Proof.* Let
\[
(3.4) \quad g(x) = \begin{cases} G_1^{-1}(G_0(x)) & G_0^{-1}(p) \leq x, \\ 0 & G_0^{-1}(p) > x, \end{cases}
\]
then the distribution function of \( g(x) \) under \( I_1 \) is \( G_1(x) \). From (3.4) and (3.1), \( g(x) \leq x \) for all \( x \geq 0 \). Hence
\[
E_{G_0, g_1}[\phi(X_1, \ldots, X_n; Y_1, \ldots, Y_n)] = E_{G_0, g_0}[\phi(X_1, \ldots, X_n; g(Y_1), \ldots, g(Y_n))] \geq E_{G_0, g_0}[\phi(X_1, \ldots, X_n; Y_1, \ldots, Y_n)].
\]

4. Efficiency of the \( U_m \) test for the location alternative

When the hypothesis \( H: I_1 = I_2 \) and the alternative \( K \) in (3.1) differ only in location such that
\[
(4.1) \quad I_1: \quad \int_0^x f(t-\theta_0)dt, \quad \int_0^0 f(t-\theta_0)dt, \quad I_2: \quad \int_0^x f(t-\theta)dt, \quad \int_0^0 f(t-\theta)dt,
\]
then we are concerned with testing \( H: \theta = \theta_0 \) against \( K: \theta < \theta_0 \). Suppose there exist the maximum likelihood estimators of \( \theta_0 \) and \( \theta \) denoted by \( \hat{\theta}_0 = \hat{\theta}_0(X_1, \ldots, X_n) \) and \( \hat{\theta} = \hat{\theta}(Y_1, \ldots, Y_n) \) and let the test function \( \psi \) be
\[
(4.2) \quad \psi(X_1, \ldots, X_n; Y_1, \ldots, Y_n) = \begin{cases} 1 & \hat{\theta} - \hat{\theta}_0 < \epsilon, \\ 0 & \hat{\theta} - \hat{\theta}_0 > \epsilon, \end{cases}
\]

* This lemma is also proved from the lemma 1 of chapter 3 and the lemma 2 of chapter 5 in Lehmann, "Testing Statistical Hypothesis", John Wiley & Sons, Inc. 1959.
where the constant $c_0$ is determined such that $E\psi = \alpha$ under $H$.

**Theorem 2.** If the maximum likelihood estimators of $\theta_0$ and $\theta$ denoted by $\hat{\theta}_0$ and $\hat{\theta}$ exist and are distributed asymptotically normal and efficient, the asymptotic efficiency of the test $U_m$ defined by (3.2) relative to the test $\psi$ defined by (4.2) is

$$e_{\psi, \phi} = \frac{\left\{ p_0 f(-\theta_0) + 2 \int_{-\theta_0}^{\infty} f(t)^2 \, dt \right\}^2}{\left( p_0 \theta_0 + \frac{1}{3} q_0^3 \right) \left\{ -f'(-\theta_0) + \frac{1}{p_0} f(-\theta_0)^2 - E \left( \frac{\partial^2 \log f(X - \theta_0)}{\partial \theta_0^2} \right) \right\} .}$$

Proof. Put $\theta = \theta_0 - k N^{-1/2}$ and consider the limiting condition (3.3). As $U_m$ is distributed asymptotically normal under $K$ by Lehmann [5], the asymptotic power of the test $\phi$ is $\Phi \left[ \left( E \theta_0 (U_m - U_0) / \text{Var}_\theta(U_m)^{1/2} \right) \right]$, where $\Phi$ is the distribution function of standardized normal distribution. From Proposition 1 we have

$$\left. \frac{\partial E_\theta(U_m)}{\partial \theta} \right|_{\theta = \theta_0} = -\frac{1}{2} p_0 f(-\theta_0) \cdot \int_{-\theta_0}^{\infty} f(t)^2 \, dt,$$

and

$$\text{Var}_\theta(U_m) = \frac{1 - p_0^3}{12 \alpha \alpha_2 N} + O(N^{-2}).$$

Hence the asymptotic power of $\phi$ is

$$\Phi \left( \frac{E_\theta(U_m) - U_0}{\text{Var}_\theta(U_m)^{1/2}} \right) = \Phi \left( a + \frac{\partial E_\theta(U_m)}{\partial \theta} \bigg|_{\theta = \theta_0} \text{Var}_\theta(U_m)^{-1/2}(\theta - \theta_0) + O(N^{-1/2}) \right)$$

$$= \Phi(a + kc + O(N^{-1/2})),$$

where

$$c = \sqrt{\frac{\alpha \alpha_z}{\alpha_2}} \frac{p_0 f(-\theta_0) + 2 \int_{-\theta_0}^{\infty} f(t)^2 \, dt}{(p_0 \theta_0 + q_0^3 / 3)^{1/2}} \quad \text{and} \quad \Phi(a) = \alpha.$$ 

Since by assumption $\hat{\theta} - \hat{\theta}_0$ is distributed asymptotically normal with mean $\theta - \theta_0$ and variance

$$-n_1 \left\{ p_0 \frac{d^2 \log p_0}{d \theta_0^2} + E \left( \frac{\partial^2 \log f(X - \theta_0)}{\partial \theta_0^2} \right)^{-1} \right\}$$

$$-n_2 \left\{ p_1 \frac{d^2 \log p_1}{d \theta^2} + E \left( \frac{\partial^2 \log f(X - \theta)}{\partial \theta^2} \right)^{-1} \right\},$$

the asymptotic power of the test $\psi$ at $\theta = \theta_0 - k^* N^{-1/2}$ is

$$\Phi(a + k^* c^* + O(N^{-1/2})).$$
where
\begin{equation}
(4.6) \quad c^* = \sqrt{\alpha_1 \alpha_2} \left\{ -f'(-\theta_0) + \frac{1}{p_0} f(-\theta_0)^2 - \mathbb{E} \left( \frac{\partial^2 \log f(X-\theta_0)}{\partial \theta_0^2} \right) \right\}^{1/2}.
\end{equation}

We can get the asymptotic relative efficiency from (4.4), (4.5), and 
\(e_{\theta, \psi} = (k^*/k)^2 = (c/c^*)^2\).

**EXAMPLE 1. Normal distribution.** When \(f(x) = (2\pi)^{-1/2} e^{-x^2/2}\) in Theorem 2, the efficiency becomes
\begin{equation}
(4.7) \quad e_{\theta, \psi} = \frac{\Phi(-\theta_0) f(\theta_0) + \pi^{-1/2} \Phi(\sqrt{2} \theta_0)^2 \Phi(-\theta_0)}{\Phi(-\theta_0) + \Phi(\theta_0)^2/3 \Phi(-\theta_0) \Phi(\theta_0) - \theta_0 f(\theta_0)^2 + \Phi(\theta_0)^2 \Phi(\theta_0)}.
\end{equation}

Some numerical values of \(e_{\theta, \psi}\) are shown in the following Table 1.

<table>
<thead>
<tr>
<th>(\theta_0)</th>
<th>(-\infty)</th>
<th>(-1)</th>
<th>0</th>
<th>1</th>
<th>(+\infty)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(e_{\theta, \psi})</td>
<td>1</td>
<td>0.970</td>
<td>0.972</td>
<td>0.969</td>
<td>0.955 ((-3/\pi))</td>
</tr>
</tbody>
</table>

As \(\theta_0\) tends to plus infinity, \(e_{\theta, \psi}\) tends to the efficiency \(3/\pi\) for the ordinary Wilcoxon test relative to the Student \(t\)-test (see Mood [6]). It is interesting that the efficiency is nearly equal to 1 irrespective of the value of \(\theta_0\).

**EXAMPLE 2. Exponential distribution.** When \(f(x) = e^{-x}\) for \(x \geq 0\) and zero otherwise, the condition concerning \(\hat{\theta}_0\) and \(\hat{\theta}\) stated in Theorem 2 is not satisfied. Calculating directly, we get \(\hat{\theta}_0 = \log (1-r_1/n_1)\) and \(\hat{\theta} = \log (1-r_2/n_2)\). Using the asymptotic normality of \(r_1/n_1\) and \(r_2/n_2\), we can conclude that \(\hat{\theta}_0\) is distributed asymptotically normal with mean \(\log q_o\) and variance \(p_o/n_o\), and \(\hat{\theta}\) with mean \(\log q_i\) and variance \(p_i/n_iq_i\). From this we can get the asymptotic power (4.5) of the test \(\psi\) in (4.2) with \(c^*\) in (4.6) as follows:
\begin{equation}
(4.8) \quad e_{\theta, \psi} = \frac{3(1-e^{\theta_0})}{3-3e^{\theta_0}+e^{2\theta_0}}.
\end{equation}

This turns out to be equal to the right side of (4.6), and hence the efficiency may be calculated by (4.3), i.e.
\begin{equation}
(4.8) \quad e_{\theta, \psi} = \frac{3(1-e^{\theta_0})}{3-3e^{\theta_0}+e^{2\theta_0}}.
\end{equation}

From (4.8) we can see that the efficiency decreases monotonically from one to zero, as \(\theta_0\) changes from \(-\infty\) to zero. Some numerical values are shown below.
Table 2. Efficiency for the exponential distribution.

<table>
<thead>
<tr>
<th>$\theta_0$</th>
<th>$-\infty$</th>
<th>$-2$</th>
<th>$-1$</th>
<th>$-0.5$</th>
<th>$-0.2$</th>
<th>$-0.1$</th>
<th>$0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$e_{\phi,\psi}$</td>
<td>1</td>
<td>0.99</td>
<td>0.93</td>
<td>0.76</td>
<td>0.45</td>
<td>0.26</td>
<td>0</td>
</tr>
</tbody>
</table>

**Example 3.** Uniform distribution in $[0, 1]$. In this case we take the test function $\psi$ corresponding to (4.2) as follows:

$$
\psi(X_1, \ldots, X_{n_1}, Y_1, \ldots, Y_{n_2}) = \begin{cases} 
1 & \frac{r_1}{n_1} - \frac{r_2}{n_2} < c_a, \\
0 & \frac{r_1}{n_1} - \frac{r_2}{n_2} > c_a.
\end{cases}
$$

Then the asymptotic power of the test $\psi$ is given by (4.5) with

$$
c^* = \sqrt{\alpha_1 \alpha_2 \{-\theta(1 + \theta_0)\}^{-1/2}}.
$$

Hence

$$(4.9) \quad e_{\phi,\psi} = \frac{-3\theta(2 + \theta_0)^2}{1 - \theta_0 + \theta_0^2}.$$

From (4.9) we can see that the curve of efficiency is unimodal with the maximum value $3(4\sqrt{2} - 5)$ at $\theta_0 = 1 - \sqrt{2}$.

Table 3. Efficiency for the uniform distribution.

<table>
<thead>
<tr>
<th>$\theta_0$</th>
<th>$-1$</th>
<th>$-0.8$</th>
<th>$-0.6$</th>
<th>$-0.4$</th>
<th>$-0.2$</th>
<th>$-0.1$</th>
<th>$-0.05$</th>
<th>$0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$e_{\phi,\psi}$</td>
<td>1</td>
<td>1.42</td>
<td>1.80</td>
<td>1.97</td>
<td>1.57</td>
<td>0.98</td>
<td>0.54</td>
<td>0</td>
</tr>
</tbody>
</table>

5. **Efficiency of the $U_m$ test relative to Halperin’s $U_c$ conditional test**

Halperin [2] proposed the following $U_c$ conditional test: Put

$$(5.1) \quad U_c = \frac{1}{n_1 n_2} \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} c(X_i, Y_j),$$

where

$$c(X, Y) = \begin{cases} 
1 & X > Y \geq 0, \\
0 & Y \geq X \geq 0,
\end{cases}
$$

and let $r_1$ and $r_2$ be the number of zeroes appearing in the $X$'s and the $Y$'s respectively, then Halperin [2] showed the conditional asymptotic normality of $U_c$ under the null hypothesis $H$ for given $r (= r_1 + r_2)$ and considered the test (3.2) with $U_m$ replaced by $U_c$, which will be denoted by $\phi'$. The relation between two statistics $U_m$ and $U_c$ is given by

$$(5.2) \quad U_c = U_m - \frac{r_1 r_2}{2n_1 n_2}.$$
Theorem 3. Suppose $p_0$ and $p_1$ in (1.1) are such that $p_0 = p(\theta_0)$ and $p_1 = p(\theta)$ with the function $p(\theta)$ differentiable in some neighbourhood of $\theta = \theta_0$, then $(U - E)V^{-1/2}$ is distributed asymptotically normal with mean zero and variance one, for given $r$, under the alternative $K: \theta < \theta_0$ with $\theta = \theta_0 + o(N^{-1/2})$ and under the limiting condition:

$$n_1 = \alpha_1 N, \quad r = Np_0 + o(N^{1/2}),$$

$$n_2 = \alpha_2 N, \quad \text{and} \quad N \to \infty,$$

$$n_1 + n_2 = N,$$

where

$$E = \frac{(n_1 + n_2 - r)}{(n_1 + n_2)^2} \left\{ (n_1 + n_2 - r) \int_0^\infty F_1^* dF_0^* + r \right\}$$

$$+ \left( \frac{dp}{d\theta} \right)_{\theta=\theta_0} n_1 + n_2 \left\{ (n_2 - n_1) q_0 \int_0^\infty F_1^* dF_0^* + n_2 p_0 \right\} + o(N^{-1}),$$

$$V = \frac{p_0 q_0}{n_1 n_2} \left\{ (n_2 - n_1) q_0 \int_0^\infty F_1^* dF_0^* + n_2 p_0 \right\}^2$$

$$+ \frac{q_0^2}{n_1 n_2} \left\{ n_1 \int_0^\infty \left( F_1^* - \int_0^\infty dF_0^* \right)^2 dF_0^* + n_2 \int_0^\infty \left( F_1^* - \int_0^\infty dF_0^* \right)^2 dF_0^* \right\} + o(N^{-3/2}),$$

and

$$F_1^* = \frac{F_1(t)}{q_i} \quad (i = 0, 1).$$

Proof*. The conditional distribution of $r_1$ for given $r$ is

$$\left( \begin{array}{c} n_1 \cr \theta + r \end{array} \right) \left( \begin{array}{c} n_2 \cr \theta + r \end{array} \right) p_0 q_0^{n_1-r} p_1^{r-r_1} q_1^{n_2-r_1}$$

$$\sum_k \left( \begin{array}{c} n_1 \cr k \end{array} \right) \left( \begin{array}{c} n_2 \cr k \end{array} \right) p_0^k q_0^{n_1-k} p_1^{k-r_1} q_1^{n_2-r_1}.$$ 

Using the normal approximation of $r_1$ and $r_2$ in (5.5), we find that under the condition for $r$'s being given, $w = (r_1 - E(r_1|r))V(r_1|r)^{-1/2}$ is distributed asymptotically normal with mean zero and variance one, where

$$E(r_1|r) = \frac{n_1 n_2 p_0 q_0 + n_2 p_1 q_1}{n_1 p_0 q_0 + n_2 p_1 q_1} \left( \frac{1}{q_0} - \frac{1}{q_1} + \frac{r}{n_2 p_1 q_1} \right)$$

$$= \frac{n_1 r}{n_1 + n_2} - \frac{n_1 n_2}{n_1 + n_2} \left( \frac{dp}{d\theta} \right)_{\theta=\theta_0} (\theta - \theta_0) + o(1),$$

* Prof. M. Okamoto, Osaka University, remarks that this proof is heuristic and seems to be improved and simplified by generalizing the Theorem of Steck [8]. This point will be discussed in another occasion.
(5.7) \[ V(r_1 | r) = \frac{n_1 n_2}{n_1 + n_2} p_o q_o (1 + O(N^{-1/2})). \]

On the other hand,

(5.8) \[ U_c = \frac{(n_1 - r_1)(n_2 - r + r_1)}{n_1 n_2} U^* + \frac{(n_1 - r_1)(r - r_1)}{n_1 n_2} , \]

where \( U^* \) is calculated from \( U_c \) in (5.1) by excluding the \( r_1 + r_2 \) zeroes in two samples. Since \( U^* \) is the ordinary Wilcoxon statistic for given \( r \) and \( r_1, t = (U^* - E^*)V^*^{-1/2} \) is distributed asymptotically normal with mean zero and variance one under the same condition, where

\[ E^* = \int_{0}^{\infty} F^*_t dF^*_t , \]

\[ V^* = \int_{0}^{\infty} \left( F^*_t - \int_{0}^{\infty} F^*_s dF^*_s \right)^2 dF^*_t \]

Rewriting (5.8) in terms of \( w \) and \( t \) instead of \( r_1 \) and \( U^* \) as in Halperin [2] and noticing \( r = (n_1 + n_2) p_0 + 0(N^{1/2}) \), we have

\[ U_c = E + w \{ p_0 q_o / n_1 n_2 (n_1 + n_2) \}^{1/2} \{ (n_1 - n_2) E^* - n_2 q_o - n_2 p_0 \}
+ \int_{0}^{\infty} \left( F^*_t - \int_{0}^{\infty} F^*_s dF^*_s \right)^2 dF^*_t \]

\[ + \frac{1}{n_1} \int_{0}^{\infty} \left( F^*_t - \int_{0}^{\infty} F^*_s dF^*_s \right)^2 dF^*_s \]

whence follows Theorem 3.

In particular we get from Theorem 3,

**Corollary.** Under the null hypothesis \( H \), \( U_c \) is distributed asymptotically normal with mean \( E \) and variance \( V \) for given \( r \), where

\[ E = \frac{(n_1 + n_2 + r)(n_1 + n_2 - r)}{2(n_1 + n_2)^2} , \]

\[ V = \frac{p_o q_o}{4n_1 n_2 (n_1 + n_2)} \{ n_1 + n_2 + (n_2 - n_1) p_o \}^2 + \frac{(n_1 + n_2) q_o}{12 n_1 n_2} . \]

From Theorem 3, we can calculate the asymptotic efficiency of the \( U_m \) test relative to the \( U_c \) conditional test for the location alternative.

**Theorem 4.** Let two populations \( \Pi_1 \) and \( \Pi_2 \) be defined by (4.1), then the asymptotic efficiency of the \( U_m \) test relative to the \( U_c \) conditional test at \( r = (n_1 + n_2) p_0 \) is given by
Proof. The method of proof is the same as in Theorem 2. Corresponding to \(c^*\) in (4.5) and (4.6), we get from Theorem 3,

\[
c^* = -\left(\frac{dE}{d\theta}\right)_{\theta=\theta_0} V^{-1/2} = \sqrt{\alpha_1} \alpha_2 \left(\frac{xf(-\theta_0) + 2 \int_{-\theta_0}^{\infty} f(t)^2 dt}{p_0 f(-\theta_0) + 2 \int_{-\theta_0}^{\infty} f(t)^2 dt} \right)^{1/2}.
\]

With \(c\) in (4.4), we get \(e_{\phi, \phi'} = (c/c^*)\) in (5.12).

Resolving \(e_{\phi, \phi'}\) into partial fractions with respect to \(x\) \((0 \leq x \leq 2)\), we have

\[
e_{\phi, \phi'} = \frac{p_0 f(-\theta_0) + 2 \int_{-\theta_0}^{\infty} f(t)^2 dt}{p_0 f(-\theta_0) + 2 \int_{-\theta_0}^{\infty} f(t)^2 dt} \left\{p_0 + \frac{2p_0 D}{p_0 f(-\theta_0) x + 2 \int_{-\theta_0}^{\infty} f(t)^2 dt} + \frac{p_0 D^2 + \frac{1}{3} f(-\theta_0)^2 q_0^2}{p_0 f(-\theta_0) x + 2 \int_{-\theta_0}^{\infty} f(t)^2 dt} \right\},
\]

where

\[
D = q_0 f(-\theta_0) - 2 \int_{-\theta_0}^{\infty} f(t)^2 dt.
\]

Regarding \(e_{\phi, \phi'}\) as a function of \(x\), we get the following:

(i) When \(D \geq 0\), \(e_{\phi, \phi'}(x)\) is nonincreasing. Hence \(\max e_{\phi, \phi'} = e(0)\) and \(\min e_{\phi, \phi'} = e(2)\).

(ii) When \(D < 0\), we put \(x_0 = -q_0 p_0^{-1}(1 + f(-\theta_0)/3D)\), and

(a) if \(x_0 < 0\), \(\max e_{\phi, \phi'} = e(2)\) and \(\min e_{\phi, \phi'} = e(0)\),

(b) if \(0 \leq x_0 \leq 2\), \(\max e_{\phi, \phi'} = \max(e(0), e(2))\) and \(\min e_{\phi, \phi'} = p_0 q_0^2 \left(p_0 f(-\theta_0) + 2 \int_{-\theta_0}^{\infty} f(t)^2 dt\right) \left(p_0 + \frac{1}{3} q_0^2\right) (3p_0 D^2 + q_0^2 f(-\theta_0)^2)\),

(c) if \(x_0 > 2\), \(\max e_{\phi, \phi'} = e(0)\) and \(\min e_{\phi, \phi'} = e(2)\). From these facts we can get the following:

**Corollary.** The asymptotic relative efficiency \(e_{\phi, \phi'}\) in Theorem 4 is one.
when \( \lim n_1/n_2 = 1 \) and takes both larger values and smaller values than one as \( \lim n_1/n_2 \) changes, except when \( x_0 = 1 \) and \( D < 0 \).

**Example 4.** Normal distribution in Example 1. When \( \theta_0 = 0 \), \( D = \frac{1 - \sqrt{2}}{2} \) and \( x_0 = 0.6 \). Hence this is the case (ii) (b). So we have

\[
\max_{\phi, \phi'} e_{\phi, \phi'} = \frac{15 + 10 \sqrt{2}}{28} = 1.04,
\min_{\phi, \phi'} e_{\phi, \phi'} = \frac{57 + 40 \sqrt{2}}{343} = 0.99.
\]

**Example 5.** Exponential distribution in Example 2. Since \( D \) becomes always zero, this is the case (i). Hence

\[
\max_{\phi, \phi'} e_{\phi, \phi'} = 1 + \frac{1 - e^{\theta_0}}{3(1 - e^{\theta_0}) + e^{\theta_0}},
\min_{\phi, \phi'} e_{\phi, \phi'} = 1 - \frac{e^{\theta_0}(3 - e^{\theta_0})(1 - e^{\theta_0})}{(3 - 3e^{\theta_0} + e^{2\theta_0})(2 - e^{\theta_0})^2}.
\]

**Example 6.** Uniform distribution in Example 3. Since \( D = -(1 + \theta_0) \) < 0, this is the case (ii). We have \( x_0 = 1 + \frac{4}{3\theta_0} + \frac{1}{3\theta_0^2} \). When

\[
\theta_0 = -\frac{1}{4}, \quad x_0 = 1, \quad \text{and} \quad \max_{\phi, \phi'} e_{\phi, \phi'} = \frac{49}{48},
\min_{\phi, \phi'} e_{\phi, \phi'} = 1,
\theta_0 = -\frac{1}{2}, \quad x_0 < 0, \quad \text{and} \quad \max_{\phi, \phi'} e_{\phi, \phi'} = \frac{261}{224},
\min_{\phi, \phi'} e_{\phi, \phi'} = \frac{45}{56}.
\]

It is interesting that in case \( \theta_0 = -1/4 \) the \( U_m \) test is better than the \( U_c \) conditional test irrespective of the value of \( \lim n_1/n_2 \).

6. Application

The following table shows the ratio of nasal to oral leakage at the time of blowing for each one of 38 cleft-palate patients classified according to their ages at operation.

<table>
<thead>
<tr>
<th>age at operation</th>
<th>0, 0, 0, 0, 0, 0, 0, 0.25, 0.46, 0.50, 0.55, 0.62, 0.75, 0.84, 1.00, 1.70</th>
</tr>
</thead>
<tbody>
<tr>
<td>1-3.</td>
<td>0.0, 0.0, 0.11, 0.32, 0.47, 0.58, 0.70, 0.81, 0.83, 0.86, 0.94, 1.01, 1.39, 1.40, 1.44, 1.62, 1.85, 2.01, 2.50</td>
</tr>
</tbody>
</table>

From these data, we want to test whether the ratio is stochastically larger in the group of operation age above 16 than the group of age
1-3. After numerical calculation we get \( U_m = 269/(21 \times 17) \) and \( (U_m - E(U_m))/\text{Var}(U_m)^{1/2} = 2.72 \) from (1.2), (2.3) and (2.4). From (5.2) and (5.11), we also get \( U_c = 257/(21 \times 17) \) and \( (U_c - E)/\text{Var}^{1/2} = 2.87 \). Noticing the asymptotic normality of \( U_m \) and \( U_c \), we can conclude that there is a significant difference between two groups detected either by the \( U_m \) or the \( U_c \) test.

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OSAKA UNIVERSITY

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References