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BELTRAMI DIFFERENTIAL EQUATION AND QUASICONFORMAL MAPPING

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1. Introduction

Let $\varphi(z)$ be a holomorphic function in the disk $D = \{z \mid |z| < 1\}$ and k (<1)a positive constant. Put $\mu(z) = k\overline{\varphi}'(z)/\varphi'(z)$ in D and $\mu(z) = 0$ outside D. Then the Beltrami differential equation $w_{\overline{z}} = \mu(z)w_{\overline{z}}$ is known to have a homeomorphic solution w = f(z) in $|z| \le \infty : f(z)$ is a Teichmüller mapping in D and is meromorphic outside D: further the solution f(z) is unique if normalized by the condition $f(1)=1, \lim_{z\to\infty} f(z)/z=1$. (see [1], p. 91). In this paper we restrict ourselves to the case in which $\varphi(z)$ is rational and investigate the solutions of those Beltrami equations. First we introduce a function $\Phi(z)$ which is defined by means of $\varphi(z)$ and satisfies the relation $g \circ f(z) = \Phi(z)$ for some rational function g(z). Next we find the conditions for $\varphi(z)$ under which f(z) maps D onto itself. These are equivalent to the condition for f(z) to fix the boundary of Dpointwise. From this we shall obtain short proofs of Theorem 6 in [2] and Theorem 2.3 in [3]. Finally we have an example which fixes the boundary of D pointwise for some k but not for k' other than k.

2. $\Phi(z)$ is a branched covering

Let $\varphi(z)$ be a non-constant rational function holomorphic in D. Put with some k, 0 < k < 1,

$$\Phi(z) = \begin{cases} \varphi(z) + k\bar{\varphi}(z) & \text{ for } z \text{ in } D, \\ \varphi(z) + k\bar{\varphi}(1/\bar{z}) & \text{ for } z \text{ outsde } D. \end{cases}$$

Then we have

Lemma. $\Phi(z)$ is a branched covering and has the same number of sheets as $\varphi(z)$.

Proof. By definition $\Phi(z)$ is a branched covering in D and outside \overline{D} . On the boundary of D, $\varphi(z)+k\overline{\varphi}(z)$ and $\varphi(z)+k\overline{\varphi}(1/\overline{z})$ have the same values and the same orientation. Therefore $\Phi(z)$ has the same multiplicities as $\varphi(z)$ on the T. SASAKI

boundary of D, so that it is an unlimited branched covering. Next we count the number of sheets. Writing

$$\varphi(z) = \gamma \frac{(z - \alpha_1) \cdots (z - \alpha_m)}{(z - \beta_1) \cdots (z - \beta_n)}$$

we have

$$\Phi(z) = \gamma \frac{(z-\alpha_1)\cdots(z-\alpha_m)}{(z-\beta_1)\cdots(z-\beta_n)} + k\bar{\gamma} \frac{z^{n-m}(1-\bar{\alpha}_1 z)\cdots(1-\bar{\alpha}_m z)}{(1-\bar{\beta}_1 z)\cdots(1-\bar{\beta}_n z)}$$

on the complement of *D*. Since $\beta_{i,i} = 1, \dots, n$, lie outside *D*, the number of *z* with the multiplicities at which $\Phi(z) = \infty$ is $n + \max(m - n, 0) = \max(m, n)$ which is equal to the number of sheets of $\varphi(z)$. q.e.d.

Let f(z) be the normalized solution of $f_{\overline{z}}(z) = \mu(z)f_z(z)$ with $\mu(z) = k\overline{\varphi}'(z)/\varphi'(z)$ in D and =0 outside D. Then $\Phi \circ f^{-1}$ is a branched covering with the same number of sheets as $\varphi(z)$. f(z) is meromorphic outside \overline{D} with a simple pole at ∞ so that $\Phi \circ f^{-1}$ is meromorphic outside $f(\overline{D})$. It will be shown as follows that $\Phi \circ f^{-1}$ is holomorphic in f(D). The differentiation of $f \circ f^{-1}(w) = w$ with respect to \overline{w} gives

$$(f_z \circ f^{-1}(w))f^{-1}(w)_{\overline{w}} + (f_{\overline{z}} \circ f^{-1}(w))\overline{f^{-1}}(w)_{\overline{w}} = 0 \quad a.e.$$

or

$$f^{-1}(w)_{\overline{w}} = -\frac{f_{\overline{z}}(z)}{f_{z}(z)}\overline{f^{-1}}(w)_{\overline{w}} = -k\frac{\overline{\varphi'}(z)}{\varphi'(z)}\overline{f^{-1}}(w)_{\overline{w}} \quad a.e..$$

So, we have

$$\begin{split} (\Phi \circ f^{-1}(w))_{\overline{w}} &= (\Phi_z \circ f^{-1}(w))f^{-1}(w)_{\overline{w}} + (\Phi_{\overline{z}} \circ f^{-1}(w))\overline{f^{-1}}(w)_{\overline{w}} \\ &= (\varphi' \circ f^{-1}(w))(-k)\frac{\overline{\varphi'}(z)}{\varphi'(z)}\overline{f^{-1}}(w)_{\overline{w}} + k(\overline{\varphi'} \circ f^{-1}(w))\overline{f^{-1}}(w)_{\overline{w}} = 0 \quad \text{a.e.}. \end{split}$$

This shows the holomorphy of $\Phi \circ f^{-1}$ in f(D). Except for a finite number of points which are *f*-images of the critical points of $\varphi(z)$, $\Phi \circ f^{-1}(w)$ is holomorphic on the boundary of f(D) because it is locally a composition of the quasiconformal mappings and 1-quasiconformal. By the finite multivalency of $\Phi \circ f^{-1}$ it is meromrophic at the excepted points so that it is a rational function. We formulate this as

Theorem 1. f(z) and $\Phi(z)$ are related with a rational function g(z) such that $g \circ f(z) = \Phi(z)$.

Application. We consider the expansion of f(z) outside \overline{D} . Under the normalization f(0)=0, instead of f(1)=1, f(z) has an expansion

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$$f(z) = z + P(\mu(h+1)) = z + P\mu + P\mu T\mu + P\mu T\mu T\mu + \cdots$$
,

where

$$Ph(\zeta) = -\frac{1}{\pi} \iint h(z) \left(\frac{1}{z-\zeta} - \frac{1}{z} \right) dx dy \text{ and } Th(\zeta) = \lim_{z \to 0} -\frac{1}{\pi} \iint_{|z-\zeta| > z} \frac{h(z)}{(z-\zeta)^2} dx dy.$$

(see [1]). If g(z) is determined explicitly we shall be able to see P_{μ} , $P_{\mu}T_{\mu}$, $P_{\mu}T_{\mu}T_{\mu}$, \cdots successively. For example consider the case of

$$p(z) = z^{n} + a_{n-1}z^{n-1} + \dots + a_{1}z^{n}$$

where a_i are real and $\{z | \varphi'(z)=0\}$ lies on the segment [-1, 1]. Then we find $g(w)=(1+k)\varphi((1+k)^{-1/n}w)$ and $\Phi(z)=\varphi(z)+k\varphi(1/z)$ outside D. Substituting these into $g \circ f(z)=\Phi(z)$ we have

$$\sum_{i=1}^{n} a_{i}(1+k)^{1-i/n} (z+kP\mu'+k^{2}P\mu'T\mu'+k^{3}P\mu'T\mu'T\mu'+\cdots)^{i}$$

= $\varphi(z)+k\varphi(1/z), a_{n}=1.$

Here we put $\mu' = \mu/k = \overline{\varphi}'(z)/\varphi'(z)$. Comparing the coefficients of *j*-th power of $k, j=0, 1, 2, \cdots$, of both sides we have

$$P\mu' = (\varphi(1/z) - \varphi(z))/\varphi'(z) + z/n ,$$

$$P\mu'T\mu' = -\frac{1}{2\varphi'(z)} (\varphi''(z)(P\mu')^2 + 2P\mu'\sum_{i=1}^n i(1-i/n)a_i z^{i-1} + \sum_{i=1}^n (1-i/n)(-i/n)a_i z^i) ,$$

...

3. The case in which f(z) keeps every boundary point of D fixed

Let F(z) be a quasiconformal mapping of D onto D which satisfies $F_{\overline{z}}(z) = \mu(z)F_z(z)$. If F(z) fixes the boundary of D pointwisely, then we have a normalized solution of $f_{\overline{z}}(z) = \mu(z)f_z(z)$ by setting f(z) = F(z) in D and f(z) = z outside D. This implies that f(z) maps D onto D. Conversely if a normalized solution f(z) maps D onto D then f(z) = z outside D and therefore it fixes the boundary of D pointwisely. The restriction of f(z) to D is a solution of $F_{\overline{z}}(z) = \mu(z)F_z(z)$ which fixes the boundary of D pointwisely. Therefore we can say that F(z) fixes the boundary of D pointwisely if and only if the normalized solution f(z) maps D onto D. If f(z) maps D onto D, then we have $g(z) = \varphi(z) + k\overline{\varphi}(1/\overline{z})$ outside D and therefore everywhere. In this case all poles of $\varphi(z)$ lie on the boundary of D. More precisely, $m \leq n$ and $|\beta_i| = 1, i = 1, 2, \dots, n$. Proof is as follows;

First we observe that the number of sheets of g(z) is equal to max (m, n). This follows readily from Lemma and Theorem 1. g(z) has poles at $\beta_i, \overline{\beta_i}^{-1}, i =$ T. SASAKI

1, 2, ..., *n*, and at 0, ∞ if m > n. If m > n then the number of *z* with the multiplicities at which $g(z) = \infty$ is not less than 2(m-n)+n=m+(m-n), which is a contradiction. Therefore $m \le n$. If there is a β_i , $|\beta_i| \ne 1$, then the number of *z* with the multiplicities at which $g(z) = \infty$ is greater than *n*, a contradiction. The assertion follows.

The identity $g(z) = \varphi(z) + k\overline{\varphi}(1/\overline{z})$ implies that $\varphi(z) + k\overline{\varphi}(1/\overline{z})$ has the branch points at w_i , i=1, 2, ..., l, and only there in D, where w_i is the f-image of the branch point z_i , i=1, 2, ..., l, of $\varphi(z)$ in D with the same order as $\varphi(z)$ has at z_i , and that $\varphi(w_i) + k\overline{\varphi}(1/\overline{w}_i) = \varphi(z_i) + k\overline{\varphi}(z_i)$, i=1, 2, ..., l. Conversely if $g(z) = \varphi(z) + k\overline{\varphi}(1/\overline{z})$, this is true when the above conditions on $\varphi(z)$ are satisfied, then f(z) maps D onto D. We summurize those as

Theorem 2. The followings are all equivalent.

- a) F(z) fixes the boundary of D pointwisely,
- b) f(z) maps D onto D,
- c) $g(z) = \varphi(z) + k\overline{\varphi}(1/\overline{z}),$

d) $\varphi(z)$ has poles only on the boundary of D, $\varphi(z)+k\overline{\varphi}(1/\overline{z})$ has the branch points at w_i , $i=1, 2, \dots, l$, and only there in D, where w_i is the f-image of the branch point z_i , $i=1, 2, \dots, l$, of $\varphi(z)$ in D with the same order as $\varphi(z)$ has at z_i , and $\varphi(w_i)+k\overline{\varphi}(1/\overline{w}_i)=\varphi(z_i)+k\overline{\varphi}(z_i)$, $i=1, 2, \dots, l$.

4. Short proofs

If $\varphi(z)$ has no branch point in D, then d) implies that $\varphi(z)$ has poles only on the boundary of D. This is Theorem 6 in [2]. On the other hand if $\varphi(z)$ has the branch points in D and if a) is true for all k, 0 < k < 1, then we can show $w_i = z_i, i = 1, 2, \dots, l$. In this case d) implies that $\varphi(z)$ has poles only on the boundary of D and $\overline{\varphi}(1/\overline{z})$ has the branch points at $z_i, i = 1, 2, \dots, l$, and only there in D with the same order as $\varphi(z)$ has at z_i and that $\varphi(1/\overline{z}_i) = \varphi(z_i), i =$ $1, 2, \dots, l$. Conversely if the above conditions on $\varphi(z)$ are satisfied then d) is satisfied with $w_i = z_i, i = 1, 2, \dots, l$, and hence a) is true for all k, 0 < k < 1. This is Theorem 2.3 in [3].

Proof of $w_i = z_i$, i = 1, 2, ..., l. By the well known fact that $|w_i - z_i| < 2k$ for all k, 0 < k < 1, d) implies that for all k

$$k(\overline{\varphi}(z_i) - \overline{\varphi}(1/\overline{w}_i)) = \varphi(w_i) - \varphi(z_i) = \varphi'(z_i)(w_i - z_i) + O((w_i - z_i)^2).$$

Dividing both sides by k and letting $k \to 0$, we have $\varphi(1/\bar{z}_i) = \varphi(z_i)$. Therefore $\varphi(z) + k\bar{\varphi}(1/\bar{z}) = \varphi(z_i) + k\bar{\varphi}(z_i) = \varphi(z_i) + k\bar{\varphi}(1/\bar{z}_i)$ is satisfied by z_i and w_i . We set $E_i = \{z | \varphi(z) = \varphi(z_i)\} \cap D$ and $E_i' = \{z \in E_i | z \neq z_i\}$. Then for sufficiently small k, w_i lies near z_i , and $f(E_i')$ and w_i have a positive distance which tends to the distance between E_i' and z_i as $k \to 0$, hence we have $w_i = z_i$ for small k. By the continuity of f(z) in k we have $w_i = z_i$ for all k, 0 < k < 1, because all w_i , $i = z_i$.

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1, 2, \cdots , *l*, are fixed for small *k* and they do not change with each other without a jump.

5. Special solution

In general c) in Theorem 2 does not imply that $w_i = z_i$, $i=1, 2, \dots, l$, for there are $\varphi(z)$ and k, that is $\mu(z) = k\overline{\varphi}'(z)/\varphi'(z)$, such that f(z) maps D onto D and z_1 to $w_1 \neq z_1$. This gives an example of $\mu(z)$ for which F(z) fixes the boundary of D pointwisely but not for $k' \neq k$, 0 < k' < 1. The existence of such $\mu(z)$ is known in [2] where $\mu(z)$ is not restricted to be the Teichmüller type.

Let $\varphi(z) = z^2(z - (5 + \sqrt{13})/2)/(z - 1)^3$, $k = (3 + \sqrt{13})/8$. Then we have $\varphi'(z) = 0$ at z = 0 and $\varphi(0) + k\overline{\varphi}(0) = 0$. On the other hand $\varphi(z) + k\overline{\varphi}(1/\overline{z}) = (z - 1/2)^2(z - (3 + \sqrt{13})/2)/(z - 1)^3$, hence $(\varphi(z) + k\overline{\varphi}(1/\overline{z}))' = 0$ at z = 1/2 and $\varphi(1/2) + k\overline{\varphi}(2) = 0$. Therefore we have $g(z) = \varphi(z) + k\overline{\varphi}(1/\overline{z})$ with $k = (3 + \sqrt{13})/8$ and $z_1 = 0$, $w_1 = 1/2$.

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