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BELTRAMI DIFFERENTIAL EQUATION AND QUASICONFORMAL MAPPING

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1. Introduction

Let $\varphi(z)$ be a holomorphic function in the disk $D=\{z\mid |z|<1\}$ and k (<1) a positive constant. Put $\mu(z)=k\overline{\varphi}'(z)/\varphi'(z)$ in D and $\mu(z)=0$ outside D. Then the Beltrami differential equation $w_{\bar{z}}=\mu(z)w_z$ is known to have a homeomorphic solution $w=f(z)$ in $|z|\leq\infty$: $f(z)$ is a Teichmüller mapping in D and is meromorphic outside D: further the solution $f(z)$ is unique if normalized by the condition $f(1)=1$, $\lim_{z\to\infty} f(z)/z=1$. (see [1], p. 91). In this paper we restrict ourselves to the case in which $\varphi(z)$ is rational and investigate the solutions of those Beltrami equations. First we introduce a function $\Phi(z)$ which is defined by means of $\varphi(z)$ and satisfies the relation $g\circ f(z)=\Phi(z)$ for some rational function $g(z)$. Next we find the conditions for $\varphi(z)$ under which $f(z)$ maps D onto itself. These are equivalent to the condition for $f(z)$ to fix the boundary of D pointwise. From this we shall obtain short proofs of Theorem 6 in [2] and Theorem 2.3 in [3]. Finally we have an example which fixes the boundary of D pointwise for some k but not for k' other than k.

2. $\Phi(z)$ is a branched covering

Let $\varphi(z)$ be a non-constant rational function holomorphic in D. Put with some k, $0<k<1$,

$$\Phi(z)=\begin{cases}\varphi(z)+k\overline{\varphi}(z) & \text{for } z \text{ in } D,\\ \varphi(z)+k\overline{\varphi}(1/\bar{z}) & \text{for } z \text{ outsde } D.\end{cases}$$

Then we have

Lemma. *$\Phi(z)$ is a branched covering and has the same number of sheets as $\varphi(z)$.*

Proof. By definition $\Phi(z)$ is a branched covering in D and outside $\bar{D}$. On the boundary of D, $\varphi(z)+k\overline{\varphi}(z)$ and $\varphi(z)+k\overline{\varphi}(1/\bar{z})$ have the same values and the same orientation. Therefore $\Phi(z)$ has the same multiplicities as $\varphi(z)$ on the boundary of D, so that it is an unlimited branched covering. Next we count the number of sheets. Writing

$$\varphi(z)=\gamma\frac{(z-\alpha_1)\cdots(z-\alpha_m)}{(z-\beta_1)\cdots(z-\beta_n)}$$

we have

$$\Phi(z)=\gamma\frac{(z-\alpha_1)\cdots(z-\alpha_m)}{(z-\beta_1)\cdots(z-\beta_n)}+k\bar{\gamma}\frac{z^{n-m}(1-\bar{\alpha}_1 z)\cdots(1-\bar{\alpha}_m z)}{(1-\bar{\beta}_1 z)\cdots(1-\bar{\beta}_n z)}$$

on the complement of D. Since β_i, $i=1, \cdots, n$, lie outside D, the number of z with the multiplicities at which $\Phi(z)=\infty$ is $n+\max(m-n, 0)=\max(m, n)$ which is equal to the number of sheets of $\varphi(z)$. q.e.d.

Let $f(z)$ be the normalized solution of $f_{\bar{z}}(z)=\mu(z)f_z(z)$ with $\mu(z)=k\bar{\varphi}'(z)/\varphi'(z)$ in D and $=0$ outside D. Then $\Phi\circ f^{-1}$ is a branched covering with the same number of sheets as $\varphi(z)$. $f(z)$ is meromorphic outside $\bar{D}$ with a simple pole at ∞ so that $\Phi\circ f^{-1}$ is meromorphic outside $f(\bar{D})$. It will be shown as follows that $\Phi\circ f^{-1}$ is holomorphic in $f(D)$. The differentiation of $f\circ f^{-1}(w)=w$ with respect to $\bar{w}$ gives

$$(f_z\circ f^{-1}(w))f^{-1}(w)_{\bar{w}}+(f_{\bar{z}}\circ f^{-1}(w))\overline{f^{-1}}(w)_{\bar{w}}=0 \quad a.e.$$

or

$$f^{-1}(w)_{\bar{w}}=-\frac{f_{\bar{z}}(z)}{f_z(z)}\overline{f^{-1}}(w)_{\bar{w}}=-k\frac{\bar{\varphi}'(z)}{\varphi'(z)}\overline{f^{-1}}(w)_{\bar{w}} \quad a.e..$$

So, we have

$$\begin{aligned}(\Phi\circ f^{-1}(w))_{\bar{w}}&=(\Phi_z\circ f^{-1}(w))f^{-1}(w)_{\bar{w}}+(\Phi_{\bar{z}}\circ f^{-1}(w))\overline{f^{-1}}(w)_{\bar{w}}\\&=(\varphi'\circ f^{-1}(w))(-k)\frac{\bar{\varphi}'(z)}{\varphi'(z)}\overline{f^{-1}}(w)_{\bar{w}}+k(\bar{\varphi}'\circ f^{-1}(w))\overline{f^{-1}}(w)_{\bar{w}}=0 \quad \text{a.e..}\end{aligned}$$

This shows the holomorphy of $\Phi\circ f^{-1}$ in $f(D)$. Except for a finite number of points which are f-images of the critical points of $\varphi(z)$, $\Phi\circ f^{-1}(w)$ is holomorphic on the boundary of $f(D)$ because it is locally a composition of the quasiconformal mappings and 1-quasiconformal. By the finite multivalency of $\Phi\circ f^{-1}$ it is meromrophic at the excepted points so that it is a rational function. We formulate this as

Theorem 1. *$f(z)$ and $\Phi(z)$ are related with a rational function $g(z)$ such that $g\circ f(z)=\Phi(z)$.*

Application. We consider the expansion of $f(z)$ outside $\bar{D}$. Under the normalization $f(0)=0$, instead of $f(1)=1$, $f(z)$ has an expansion

$$f(z) = z+P(\mu(h+1)) = z+P\mu+P\mu T\mu+P\mu T\mu T\mu+\cdots ,$$

where

$$Ph(\zeta) = -\frac{1}{\pi}\iint h(z)\left(\frac{1}{z-\zeta}-\frac{1}{z}\right)dx\,dy \text{ and } Th(\zeta) = \lim_{\varepsilon\to 0} -\frac{1}{\pi}\iint_{|z-\zeta|>\varepsilon}\frac{h(z)}{(z-\zeta)^2}dx\,dy .$$

(see [1]). If $g(z)$ is determined explicitly we shall be able to see $P\mu$, $P\mu T\mu$, $P\mu T\mu T\mu$, $\cdots$ succesively. For example consider the case of

$$\varphi(z) = z^n+a_{n-1}z^{n-1}+\cdots+a_1 z$$

where a_i are real and $\{z\,|\,\varphi'(z)=0\}$ lies on the segment $[-1, 1]$. Then we find $g(w)=(1+k)\varphi((1+k)^{-1/n}w)$ and $\Phi(z)=\varphi(z)+k\varphi(1/z)$ outside D. Substituting these into $g\circ f(z)=\Phi(z)$ we have

$$\sum_{i=1}^{n} a_i(1+k)^{1-i/n}(z+kP\mu'+k^2P\mu'T\mu'+k^3P\mu'T\mu'T\mu'+\cdots)^i$$
$$= \varphi(z)+k\varphi(1/z),\ a_n = 1 .$$

Here we put $\mu'=\mu/k=\bar{\varphi}'(z)/\varphi'(z)$. Comparing the coefficients of j-th power of k, $j=0, 1, 2, \cdots$, of both sides we have

$$P\mu' = (\varphi(1/z)-\varphi(z))/\varphi'(z)+z/n ,$$
$$P\mu'T\mu' = -\frac{1}{2\varphi'(z)}(\varphi''(z)(P\mu')^2+2P\mu'\sum_{i=1}^{n} i(1-i/n)a_i z^{i-1}$$
$$+\sum_{i=1}^{n}(1-i/n)(-i/n)a_i z^i) ,$$
$$\cdots .$$

3. The case in which $f(z)$ keeps every boundary point of D fixed

Let $F(z)$ be a quasiconformal mapping of D onto D which satisfies $F_{\bar{z}}(z)=\mu(z)F_z(z)$. If $F(z)$ fixes the boundary of D pointwisely, then we have a normalized solution of $f_{\bar{z}}(z)=\mu(z)f_z(z)$ by setting $f(z)=F(z)$ in D and $f(z)=z$ outside D. This implies that $f(z)$ maps D onto D. Conversely if a normalized solution $f(z)$ maps D onto D then $f(z)=z$ outside D and therefore it fixes the boundary of D pointwisely. The restriction of $f(z)$ to D is a solution of $F_{\bar{z}}(z)=\mu(z)F_z(z)$ which fixes the boundary of D pointwisely. Therefore we can say that $F(z)$ fixes the boundary of D pointwisely if and only if the normalized solution $f(z)$ maps D onto D. If $f(z)$ maps D onto D, then we have $g(z)=\varphi(z)+k\bar{\varphi}(1/z)$ outside D and therefore everywhere. In this case all poles of $\varphi(z)$ lie on the boundary of D. More precisely, $m\leqq n$ and $|\beta_i|=1$, $i=1, 2, \cdots, n$. Proof is as follows;

First we observe that the number of sheets of $g(z)$ is equal to max (m, n). This follows readily from Lemma and Theorem 1. $g(z)$ has poles at β_i, $\bar{\beta}_i^{-1}$, $i=$

$1, 2, \cdots, n$, and at $0, \infty$ if $m>n$. If $m>n$ then the number of z with the multiplicities at which $g(z)=\infty$ is not less than $2(m-n)+n=m+(m-n)$, which is a contradiction. Therefore $m\leqq n$. If there is a β_i, $|\beta_i|\neq 1$, then the number of z with the multiplicities at which $g(z)=\infty$ is greater than n, a contradiction. The assertion follows.

The identity $g(z)=\varphi(z)+k\bar{\varphi}(1/\bar{z})$ implies that $\varphi(z)+k\bar{\varphi}(1/\bar{z})$ has the branch points at w_i, $i=1, 2, \cdots, l$, and only there in D, where w_i is the f-image of the branch point z_i, $i=1, 2, \cdots, l$, of $\varphi(z)$ in D with the same order as $\varphi(z)$ has at z_i, and that $\varphi(w_i)+k\bar{\varphi}(1/\bar{w}_i)=\varphi(z_i)+k\bar{\varphi}(z_i)$, $i=1, 2, \cdots, l$. Conversely if $g(z)=\varphi(z)+k\bar{\varphi}(1/\bar{z})$, this is true when the above conditions on $\varphi(z)$ are satisfied, then $f(z)$ maps D onto D. We summurize those as

Theorem 2. *The followings are all equivalent.*

a) *$F(z)$ fixes the boundary of D pointwisely,*

b) *$f(z)$ maps D onto D,*

c) $g(z)=\varphi(z)+k\bar{\varphi}(1/\bar{z})$,

d) *$\varphi(z)$ has poles only on the boundary of D, $\varphi(z)+k\bar{\varphi}(1/\bar{z})$ has the branch points at w_i, $i=1, 2, \cdots, l$, and only there in D, where w_i is the f-image of the branch point z_i, $i=1, 2, \cdots, l$, of $\varphi(z)$ in D with the same order as $\varphi(z)$ has at z_i, and $\varphi(w_i)+k\bar{\varphi}(1/\bar{w}_i)=\varphi(z_i)+k\bar{\varphi}(z_i)$, $i=1, 2, \cdots, l$.*

4. Short proofs

If $\varphi(z)$ has no branch point in D, then d) implies that $\varphi(z)$ has poles only on the boundary of D. This is Theorem 6 in [2]. On the other hand if $\varphi(z)$ has the branch points in D and if a) is true for all k, $0<k<1$, then we can show $w_i=z_i$, $i=1, 2, \cdots, l$. In this case d) implies that $\varphi(z)$ has poles only on the boundary of D and $\bar{\varphi}(1/\bar{z})$ has the branch points at z_i, $i=1, 2, \cdots, l$, and only there in D with the same order as $\varphi(z)$ has at z_i and that $\varphi(1/\bar{z}_i)=\varphi(z_i)$, $i=1, 2, \cdots, l$. Conversely if the above conditions on $\varphi(z)$ are satisfied then d) is satisfied with $w_i=z_i$, $i=1, 2, \cdots, l$, and hence a) is true for all k, $0<k<1$. This is Theorem 2.3 in [3].

Proof of $w_i=z_i$, $i=1, 2, \cdots, l$. By the well known fact that $|w_i-z_i|<2k$ for all k, $0<k<1$, d) implies that for all k

$$k(\bar{\varphi}(z_i)-\bar{\varphi}(1/\bar{w}_i)) = \varphi(w_i)-\varphi(z_i) = \varphi'(z_i)(w_i-z_i)+O((w_i-z_i)^2)\,.$$

Dividing both sides by k and letting $k\to 0$, we have $\varphi(1/\bar{z}_i)=\varphi(z_i)$. Therefore $\varphi(z)+k\bar{\varphi}(1/\bar{z})=\varphi(z_i)+k\bar{\varphi}(z_i)=\varphi(z_i)+k\bar{\varphi}(1/\bar{z}_i)$ is satisfied by z_i and w_i. We set $E_i=\{z\,|\,\varphi(z)=\varphi(z_i)\}\cap D$ and $E_i'=\{z\in E_i\,|\,z\neq z_i\}$. Then for sufficiently small k, w_i lies near z_i, and $f(E_i')$ and w_i have a positive distance which tends to the distance between E_i' and z_i as $k\to 0$, hence we have $w_i=z_i$ for small k. By the continuity of $f(z)$ in k we have $w_i=z_i$ for all k, $0<k<1$, because all w_i, $i=$

$1, 2, \cdots, l$, are fixed for small k and they do not change with each other without a jump.

5. Special solution

In general c) in Theorem 2 does not imply that $w_i = z_i$, $i = 1, 2, \cdots, l$, for there are $\varphi(z)$ and k, that is $\mu(z) = k\bar{\varphi}'(z)/\varphi'(z)$, such that $f(z)$ maps D onto D and z_1 to $w_1 \neq z_1$. This gives an example of $\mu(z)$ for which $F(z)$ fixes the boundary of D pointwisely but not for $k' \neq k$, $0 < k' < 1$. The existence of such $\mu(z)$ is known in [2] where $\mu(z)$ is not restricted to be the Teichmüller type.

Let $\varphi(z) = z^2(z - (5 + \sqrt{13})/2)/(z-1)^3$, $k = (3 + \sqrt{13})/8$. Then we have $\varphi'(z) = 0$ at $z = 0$ and $\varphi(0) + k\bar{\varphi}(0) = 0$. On the other hand $\varphi(z) + k\bar{\varphi}(1/\bar{z}) = (z - 1/2)^2(z - (3 + \sqrt{13})/2)/(z-1)^3$, hence $(\varphi(z) + k\bar{\varphi}(1/\bar{z}))' = 0$ at $z = 1/2$ and $\varphi(1/2) + k\bar{\varphi}(2) = 0$. Therefore we have $g(z) = \varphi(z) + k\bar{\varphi}(1/\bar{z})$ with $k = (3 + \sqrt{13})/8$ and $z_1 = 0$, $w_1 = 1/2$.

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