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THE SLICE DETERMINED BY MODULI EQUATION $xy=2z$ IN THE DEFORMATION SPACE OF ONCE PUNCTURED TORI

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1. Introduction and statement of results

We recall some terminology and results in [2], [3] and [6]. Let A and B be loxodromic elements of $PSL(2, \mathbb{C})$, that is, A and B are 2×2 complex matrices with determinant 1 and their traces do not lie on the closed interval $[-2, 2]$. By an obvious isomorphism we identify $PSL(2, \mathbb{C})$ with the Möbius transformations group. Denote by $G = \langle A, B \rangle$ the Möbius subgroup generated by A and B . Let x , y and z be the traces of A , B and AB , respectively. The triple (x, y, z) is called a moduli triple of G . We restrict ourselves to the case in which triple (x, y, z) satisfies the moduli equation

$$(*) \quad x^2 + y^2 + z^2 = xyz.$$

Let $A_0 = \begin{pmatrix} \sqrt{2}+1 & 0 \\ 0 & \sqrt{2}-1 \end{pmatrix}$ and $B_0 = \begin{pmatrix} \sqrt{2} & 1 \\ 1 & \sqrt{2} \end{pmatrix}$. Put $G_0 = \langle A_0, B_0 \rangle$. Then $(\sqrt{8}, \sqrt{8}, 4)$ is a moduli triple of G_0 satisfying $(*)$, G_0 is a Fuchsian group of the first kind and $\Omega(G_0)/G_0$ is a pair of once punctured tori, where $\Omega(G_0)$ denotes the region of discontinuity of G_0 . For each quasi-Fuchsian group $G = \langle A, B \rangle$ such that $\Omega(G)/G$ is a pair of once punctured tori, there is a quasiconformal mapping f of the extended plane such that $A = fA_0f^{-1}$ and $B = fB_0f^{-1}$. Hence G is a quasiconformal deformation of G_0 . The set of all such quasi-Fuchsian groups is called a deformation space of once punctured tori and denoted by $D(G_0)$. Under a normalization each triple satisfying $(*)$ determines A and B uniquely so that a group $G = \langle A, B \rangle$ of $D(G_0)$, too. We identify $D(G_0)$ with the subset of \mathbb{C}^3 consisting of triples each of which satisfies $(*)$ and determines a quasi-Fuchsian group. We put

$$T^* = \{(x, y, z) | x^2 + y^2 + z^2 = xyz\} \subset \mathbb{C}^3.$$

Then, by stability of quasi-Fuchsian groups, $D(G_0)$ is an open subset of T^* . In [3] the following is shown.

Theorem 1 ([3]). *Let $G = \langle A, B \rangle$ be a group generated by loxodromic elements A and B satisfying (*). If triple (x, y, z) of G also satisfies*

$$(1) \quad x > 2 \quad \text{and} \quad y > 2,$$

then $G \in \mathcal{D}(G_0)$.

We put

$$S = \{(x, y, z) | x > 2, y > 2\} \cap T^*.$$

Then Theorem 1 implies $S \subset \mathcal{D}(G_0)$. We denote by \bar{S} the closure of S in T^* . Then $\bar{S} \setminus S$ consists of the following three sets:

$$b_1 = \{(2, y, z) | y^2 + z^2 + 4 = 2yz, y > 2\}$$

$$b_2 = \{(x, 2, z) | x^2 + z^2 + 4 = 2xz, x > 2\}$$

$$b_3 = \{(2, 2, z) | z^2 + 8 = 4z\}.$$

For these sets the following are noted in [6].

Theorem 2 ([6]). *Let G be a group whose moduli triple lies on $b_1 \cup b_2$. Then G is a boundary group of $\mathcal{D}(G_0)$. More precisely, G is a regular b -group.*

Theorem 3 ([6]). *Let G be a group whose moduli triple lies on b_3 . Then G is a boundary group of $\mathcal{D}(G_0)$. More precisely, G is a web group.*

A regular b -group is a Kleinian group G having only one simply connected invariant component Δ such that

$$2\text{Area}(\Delta/G) = \text{Area}(\Omega(G)/G),$$

where Area implies the hyperbolic area. A web group is a Kleinian group such that each component subgroup is a quasi-Fuchsian group of the first kind. We remark that the web group in Theorem 3 is not quasi-Fuchsian, but it has an infinite number of components. By Theorems 2 and 3 we see that

$$S = \{(x, y, z) | x > 2, y > 2\} \cap \mathcal{D}(G_0),$$

so we shall call S the slice determined by (1).

Comparing with Theorems 1, 2 and 3, we shall investigate another slice S' of $\mathcal{D}(G_0)$ determined by moduli equation

$$(2) \quad xy = 2z.$$

Explicitly,

$$S' = \{(x,y,z) | xy = 2z\} \cap D(G_0).$$

In comparison with Theorem 1 we prove the following.

Theorem 4. *Let*

$$E = \{(x,y,z) | xy = 2z, |x| > 2, |y| > 2\} \cap T^*.$$

Then $E \subset S'$.

The set $\bar{E} \setminus E$ consists of the following three sets:

$$e_1 = \{(x,y,z) | xy = 2z, |x| = 2, |y| > 2\} \cap T^*$$

$$e_2 = \{(x,y,z) | xy = 2z, |x| > 2, |y| = 2\} \cap T^*$$

$$e_3 = \{(x,y,z) | xy = 2z, |x| = |y| = 2\} \cap T^*.$$

In contrast to Theorem 2 the following holds.

Theorem 5. $e_1 \cup e_2 \subset S'$.

We also prove the following.

Theorem 6. *Let G be a group whose moduli triple lies on e_3 . Then G is a boundary group of $D(G_0)$.*

In §2 we make a normalization of generators and then show as Theorems 7 and 8 that each group of the slice S' has a symmetric region of discontinuity and a symmetric fundamental domain with respect to the origin and that none of the boundary groups of S' is a b -group. In §3 we modify a criterion for discontinuity of [4] and then prove Theorems 4 and 5. Lastly, we prove Theorem 6 in §4.

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2. Normalization and symmetry

Let A and B be loxodromic elements of $PSL(2, \mathbb{C})$ and put $G = \langle A, B \rangle$. We assume that the moduli triple (x, y, z) of G satisfies (*) and (2). Firstly, we shall normalize A and B . Conjugating by a Möbius transformation, we normalize A such that

$$A = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}, \quad \alpha = re^{i\theta}, \quad r > 1 \quad \text{and} \quad -\frac{\pi}{2} < \theta \leq \frac{\pi}{2}.$$

Note that $\alpha\beta=1$ and $\beta=1/\alpha$. We write $B=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then $x=\alpha+\beta$, $y=a+d$ and $z=\alpha a+\beta d$.

Proposition. *Equality (2) is equivalent to $a=d$.*

Proof. By the substitution of $x=\alpha+\beta$, $y=a+d$ and $z=\alpha a+\beta d$ into (2), equality (2) reduces to $(\alpha-\beta)(a-d)=0$. Since $\alpha\neq\beta$, we have $a=d$. \square

Substituting $x=\alpha+\beta$, $y=2a$ and $z=(\alpha+\beta)a$ into (*), we have $(\alpha-\beta)^2a^2=(\alpha+\beta)^2$. We shall choose a sign of a such that $a=(\alpha+\beta)/(\alpha-\beta)$. Conjugating by a Möbius transformation having the same fixed points with that of A , which leaves A invariant, we normalize B such that $a=c$. Then we have

$$(3) \quad B = \begin{pmatrix} a & b \\ a & a \end{pmatrix}, \quad a = \frac{\alpha + \beta}{\alpha - \beta} \quad \text{and} \quad y = \frac{2(\alpha + \beta)}{\alpha - \beta}.$$

Under this normalization we remark the following.

Theorem 7. *If G lies on the slice S' determined by (2), then, under the normalization above, $\Omega(G)$ is symmetric with respect to the origin. Furthermore, there is a fundamental domain for G which is symmetric with respect to the origin.*

Proof. By (3) we have $B^{-1}(-z) = -B(z)$. This implies that $B^{-n}(-z) = -B^n(z)$ for any z and any integer n . We also have $A^n(-z) = -A^n(z)$. Let M be an element of G not being the identity. Then

$$M = A^{m_1} B^{n_1} A^{m_2} B^{n_2} \dots A^{m_i} B^{n_i}$$

for some integers m_j, n_j ($j=1,2,\dots,i$). Set

$$M^- = A^{m_1} B^{-n_1} A^{m_2} B^{-n_2} \dots A^{m_i} B^{-n_i}.$$

By the successive use of the identities shown just above we obtain

$$M^-(-z) = -M(z).$$

Let M be a loxodromic element of G and let z_1 and z_2 be the fixed points of M . Then M^- is also a loxodromic element of G and equation $M^-(-z) = -M(z)$ implies that $-z_1$ and $-z_2$ are the fixed points of M^- . Since the limit set, $\Lambda(G)$, of G is the closure of the fixed points of loxodromic elements of G , we obtain that $\Lambda(G)$ is symmetric with respect to the origin, so is the region of discontinuity, too. Since G is a quasi-Fuchsian group of the first kind, there is a fundamental domain consisting of two pieces. Let D^+ be one of the two components of a

fundamental domain for G lying in a component of G . We put $D^- = \{z | -z \in D^+\}$. Now it is not difficult to see that $D^+ \cup D^-$ is a fundamental domain for G . By construction, it is symmetric with respect to the origin. \square

It is shown in [1] that each boundary group of any Bers slice has just one simply connected invariant component. Such a group is called a b -group. In contrast to the Bers slices we have the following.

Theorem 8. *None of the boundary groups of S' is a b -group.*

Proof. Assume that G is a group of $\bar{S}' \setminus S'$ and is a b -group, that is, G has just one simply connected invariant component. Let Δ be the simply connected invariant component of G . Let D^+ be the piece of a fundamental domain for G lying in Δ . By Theorem 7 we see that the set $D^- = \{z | -z \in D^+\}$ is a subset of $\Omega(G)$. If D^- lies in Δ , then there is a curve $C \subset \Delta$ connecting a point $p \in D^+$ to $-p \in D^-$. Then the curve symmetric to C with respect to the origin also lies in Δ and connects p to $-p$. Then the closed curve $C \cup C^- \subset \Delta$ separates 0 from ∞ . Since 0 and ∞ lie on $\Lambda(G)$, this contradicts our assumption that Δ is simply connected. Hence D^- lies in another component, say Δ' . By symmetry, Δ' is also an invariant component of G , a contradiction. \square

3. Proof of Theorems 4 and 5

We recall a sufficient condition for $G = \langle A, B \rangle$ to be Kleinian.

Theorem 9 ([4]). *Let $G = \langle A, B \rangle$ be a subgroup of $PSL(2, C)$ generated by loxodromic elements $A = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$ and $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $bc \neq 0$ and satisfying (*). If, for each integer n , the inequality*

$$(4) \quad \frac{|\alpha^n a| + |\beta^n d|}{|\alpha^n a + \beta^n d|} < \frac{|\alpha| + |\beta|}{|\alpha - \beta|}$$

holds, then G is quasi-Fuchsian and represents a pair of once punctured tori.

REMARK. The moduli equation (*) is equivalent to the trace equation $tr(ABA^{-1}B^{-1}) = -2$ (see, for example, [2] or Lemma 3 in [5]). Since (*) is symmetric with respect to x and y or since $tr(ABA^{-1}B^{-1}) = tr(BAB^{-1}A^{-1})$, Theorem 9 is true even if we interchange the normalizations of A and B .

Under the moduli equation (2), Theorem 9 reduces to the following.

Theorem 10. *Let $G = \langle A, B \rangle$ be as in Theorem 9 and assume that G also*

satisfies (2). If, for each positive integer n , the inequality

$$(5) \quad \frac{|\alpha^n| + |\beta^n|}{|\alpha^n + \beta^n|} < \frac{|\alpha| + |\beta|}{|\alpha - \beta|}$$

holds, then G is quasi-Fuchsian.

Proof. By Proposition in §2 we see that (4) reduces to (5). We shall show that (5) holds for each integer n whenever (5) holds for each positive integer n . Since $\beta = 1/\alpha$, we have $|\alpha^n| + |\beta^n| = |\alpha^{-n}| + |\beta^{-n}|$ and $|\alpha^n + \beta^n| = |\alpha^{-n} + \beta^{-n}|$. Hence (5) holds for each negative integer n whenever it holds for each positive integer n . To show (5) holds for $n=0$ observe that, for $n=0$ and $n=1$, (5) is equivalent to $|\alpha - \beta| < |\alpha| + |\beta|$ and to $|\alpha - \beta| < |\alpha + \beta|$, respectively. Since $|\alpha + \beta| \leq |\alpha| + |\beta|$, we see that (5) holds for $n=0$ whenever it holds for $n=1$. Thus we have Theorem 10 by Theorem 9. \square

Now, we shall check (5) of Theorem 10 for each positive integer n , that will give us proofs of Theorems 4 and 5. By Remark under Theorem 9 we may assume that $(x, y, z) \in E \cup e_1$, so $|x| \geq 2$ and $|y| > 2$.

For $n=1$, (5) reduces to

$$|\alpha - \beta| < |\alpha + \beta|.$$

By (3) we see that $|y| = |2(\alpha + \beta)/(\alpha - \beta)|$ and so condition $|y| > 2$ implies $|\alpha - \beta| < |\alpha + \beta|$. Hence (5) holds for $n=1$.

In order to treat the case $n \geq 2$, we shall put

$$u = r^2 + r^{-2} \quad \text{and} \quad v = \cos 2\theta,$$

where $\alpha = re^{i\theta}$. By the inequalities $|\alpha - \beta| < |\alpha + \beta|$, which is shown just above, and $|x| = |\alpha + \beta| \geq 2$ we have

$$(6) \quad 0 < v \leq 1 \quad \text{and} \quad u \geq 4 - 2v.$$

For $n=2$, in polar coordinate $\alpha = re^{i\theta}$, (5) reduces to

$$\frac{r^4 + r^{-4} + 2}{r^4 + r^{-4} + 2 \cos 4\theta} < \frac{r^2 + r^{-2} + 2}{r^2 + r^{-2} - 2 \cos 2\theta}.$$

Using the equalities $r^4 + r^{-4} = u^2 - 2$ and $\cos 4\theta = 2v^2 - 1$, one shows that it is equivalent to

$$(u - 1 + v)^2 - (1 - v)^2 - 4(1 - v) > 0.$$

Inequalities (6) imply that the left hand side is not smaller than

$$(3+v)^2 - (1-v)^2 - 4(1-v)$$

and that it is greater than 4, so (5) holds for $n=2$.

For $n=3$, in polar coordinate, (5) reduces to

$$\frac{r^6 + r^{-6} + 2}{r^6 + r^{-6} + 2 \cos 6\theta} < \frac{r^2 + r^{-2} + 2}{r^2 + r^{-2} - 2 \cos 2\theta}.$$

Making use of the equalities $r^6 + r^{-6} = u^3 - 3u$ and $\cos 6\theta = 4v^3 - 3v$, we have

$$\frac{u^3 - 3u + 2}{u^3 - 3u + 8v^3 - 6v} < \frac{u + 2}{u - 2v}.$$

A calculation shows that this is equivalent to

$$(7) \quad f(u, v) = (1+v)u^3 - 2(2+3v-2v^3)u + 8v^3 - 4v > 0.$$

Since

$$\frac{\partial f(u, v)}{\partial u} = 3(1+v)u^2 - 2(2+3v-2v^3) > 3u^2 - 10 > 0,$$

$f(u, v)$ is an increasing function of u . Hence by (6), in order to show (7), it suffices to show that $f(4-2v, v) > 0$. A calculation shows that

$$f(4-2v, v) = 4(2 + (1-v)(10 - 3v - 12v^2 + 4v^3)).$$

We put $g(v) = 10 - 3v - 12v^2 + 4v^3$. If $g(v) \geq 0$, then $f(4-2v, v) > 0$. If $g(v) < 0$, then we have

$$2 + (1-v)g(v) \geq 2 + g(v) = 3(1-v^2)(4-v) + v^3 > 0.$$

Thus, in both cases we have $f(4-2v, v) > 0$ so that (7) holds. Therefore we have shown that (5) holds for $n=3$.

For $n \geq 4$, in polar coordinate, (5) reduces to

$$\frac{r^{2n} + r^{-2n} + 2}{r^{2n} + r^{-2n} + 2 \cos 2n\theta} < \frac{r^2 + r^{-2} + 2}{r^2 + r^{-2} - 2 \cos 2\theta} = \frac{u + 2}{u - 2v}.$$

Since

$$\frac{r^{2n} + r^{-2n} + 2}{r^{2n} + r^{-2n} + 2 \cos 2n\theta} \leq \frac{r^8 + r^{-8} + 2}{r^8 + r^{-8} - 2} = \frac{u^4 - 4u^2 + 4}{u^4 - 4u^2},$$

to show (5) it suffices to show

$$\frac{u^4 - 4u^2 + 4}{u^4 - 4u^2} < \frac{u + 2}{u - 2v}$$

or, equivalently,

$$(8) \quad h(u,v) = (1+v)u^4 - 4(1+v)u^2 - 2u + 4v > 0.$$

Making use of inequalities $u > 2$ and $v > 0$, one obtains

$$\frac{\partial h(u,v)}{\partial u} = 4(1+v)(u^2 - 2)u - 2 > 14.$$

Hence $h(u,v)$ is an increasing function of u . There are two cases to consider.

Case I: $v < 1$. By (6) we see that, in order to show (8), it suffices to show $h(4-2v,v) > 0$. A calculation shows that

$$h(4-2v,v) = 8(1-v)(2(1+v)(2-v)(3-v) - 1) > 0.$$

Case II: $v = 1$. Since $r > 1$, we have $u > 2$. Hence

$$h(u,1) = 2(u-2)(u^3 + 2u^2 - 1) > 0.$$

Thus we have shown (8). Hence (5) holds for $n \geq 4$.

Therefore we have shown that (5) holds for all positive integer n . Then Theorem 10 implies Theorems 4 and 5. \square

4. Proof of Theorem 6

We shall prove the theorem in a sequence of lemmas. Let $(x,y,z) \in e_3$.

Lemma 1. $x = \bar{y} = \sqrt{3} \pm i$.

Proof. By (3) and $|x| = |y| = 2$ we have $|\alpha - \beta| = 2$. Hence we have $|\alpha + \beta| = |\alpha - \beta| = 2$ or, in polar coordinate $\alpha = re^{i\theta}$,

$$r^2 + r^{-2} + 2 \cos 2\theta = r^2 + r^{-2} - 2 \cos 2\theta = 4.$$

It follows that $\cos 2\theta = 0$ and $r^2 + r^{-2} = 4$. We obtain $\theta = \pm \pi/4$ and $r = (\sqrt{3} + 1)/\sqrt{2}$. Hence

$$\alpha = \frac{\sqrt{3}+1}{2}(1 \pm i) \quad \text{and} \quad \beta = \frac{\sqrt{3}-1}{2}(1 \mp i).$$

Therefore we have $x = \alpha + \beta = \sqrt{3} \pm i$ and, by (3), $y = \sqrt{3} \mp i$. \square

We choose the sign such that $x = \bar{y} = \sqrt{3} + i$. The proof for the case $x = \bar{y} = \sqrt{3} - i$ is similar. By (3) we see that $a = (\sqrt{3} - i)/2$ and $b = -i$. Hence we have

$$A = \begin{pmatrix} \frac{(\sqrt{3}+1)(1+i)}{2} & 0 \\ 0 & \frac{(\sqrt{3}-1)(1-i)}{2} \end{pmatrix} \text{ and } B = \begin{pmatrix} \frac{\sqrt{3}-i}{2} & -i \\ \frac{\sqrt{3}-i}{2} & \frac{\sqrt{3}-i}{2} \end{pmatrix}.$$

Lemma 2. Let $F = \langle AB, BA \rangle$ and $C = \{z \in \mathbf{C} \mid |z - (\sqrt{3} + 3i)/2| = \sqrt{2}\}$. Then F is a Fuchsian group of the first kind with C as the invariant circle.

Proof. Since

$$AB = \begin{pmatrix} \frac{2 + \sqrt{3} + i}{2} & \frac{(\sqrt{3}+1)(1-i)}{2} \\ \frac{2 - \sqrt{3} - i}{2} & \frac{2 - \sqrt{3} - i}{2} \end{pmatrix}$$

and

$$BA = \begin{pmatrix} \frac{2 + \sqrt{3} + i}{2} & \frac{(\sqrt{3}-1)(1+i)}{2} \\ \frac{2 + \sqrt{3} + i}{2} & \frac{2 - \sqrt{3} - i}{2} \end{pmatrix},$$

putting

$$T = \begin{pmatrix} \frac{(\sqrt{3}+1)(1-i)}{4} & -\sqrt{3} + i \\ 4 & 2(1 - (2 + \sqrt{3})i) \end{pmatrix},$$

we calculate and obtain

$$TABT^{-1} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } TBAT^{-1} = \begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix}.$$

It is well known and is also easy to see that the group TFT^{-1} generated by $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix}$ is a Fuchsian group of the first kind with the extended real axis as the invariant circle. A calculation shows that $T(C)$ is identical with the extended real axis. \square

Lemma 3. G is not quasi-Fuchsian.

Proof. Since F is a subgroup of G so that $C \subset \Lambda(G)$, since 0 is a fixed point of the loxodromic element A so that $0 \in \Lambda(G)$, and since 0 does not lie on C , it is clear that G is not quasi-Fuchsian. \square

This lemma tells us that G does not lie in $D(G_0)$. Theorem 4 tells us that $G \in \bar{E} \subset \bar{S}' \subset \bar{D}(G_0)$ so that G lies on $\bar{D}(G_0)$. These two facts imply Theorem 6.

REMARK. An argument similar to one in [6] will show that G is a web group with two symmetric non-equivalent components; one is bounded by C and the other is symmetric to it with respect to the origin.

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