Tomassini, A. and Vezzoni, L.
Osaka J. Math.
45 (2008), 127–147

CONTACT CALABI-YAU MANIFOLDS AND SPECIAL LEGENDRIAN SUBMANIFOLDS

ADRIANO TOMASSINI and LUIGI VEZZONI

(Received February 16, 2007)

Abstract

We consider a generalization of Calabi-Yau structures in the context of Sasakian manifolds. We study deformations of a special class of Legendrian submanifolds and classify invariant contact Calabi-Yau structures on 5-dimensional nilmanifolds. Finally we generalize to codimension r.

1. Introduction

In their celebrated paper [9] Harvey and Lawson introduced the concept of calibration and calibrated geometry. Namely, a *calibration* on an n-dimensional oriented Riemannian manifold (M, g) is a closed r-form ϕ such that for any $x \in M$

$$\phi_{x|V} \leq \mathrm{Vol}(V),$$

where V is an arbitrary oriented r-plane in $T_x M$. An oriented submanifold $p\colon L \hookrightarrow M$ is said to be *calibrated* by ϕ if $p^*(\phi) = \mathrm{Vol}(L)$. Compact calibrated submanifolds have the important property of minimizing volume in their homology class. As a typical example, the real part of holomorphic volume form of a Calabi-Yau manifold is a calibration; the corresponding calibrated submanifolds are said to be *special Lagrangian*. In [13] McLean studied special Lagrangian submanifolds (and other special calibrated geometries) showing that the Moduli space of deformations of special Lagrangian manifolds of a fixed compact one L is a smooth manifold of dimension equal to the first Betti number of L.

In this paper we consider a generalization of Calabi-Yau structures in the context of Sasakian manifolds. Recall that a *Sasakian structure* on a $2n+1$-dimensional manifold M is a pair (α, J), where α is a contact form on M and J is an integrable complex structure on $\xi = \ker\alpha$ calibrated by $\kappa = (1/2)\, d\alpha$. This is equivalent to require the following data: a quadruple (α, g, R, J), where α is a 1-form, g is a Riemannian

2000 Mathematics Subject Classification. 53C10, 53D10, 53C25, 53C38.

This work was supported by the Projects M.I.U.R. “Geometric Properties of Real and Complex Manifolds”, “Riemannian Metrics and Differentiable Manifolds” and by G.N.S.A.G.A. of I.N.d.A.M.

metric, R is a unitary Killing vector field, $J \in \mathrm{End}(TM)$ satisfying

$$J^2 = -I + \alpha \otimes R, \quad g(J\,\cdot\,,\,J\,\cdot\,) = g(\,\cdot\,,\,\cdot\,) - \alpha \otimes \alpha, \quad \alpha(R) = 1$$

and such that the metric cone $(M \times \mathbb{R}^+, r^2 g + dr \otimes dr)$ endowed with the almost complex structure $\tilde{J} = J - r\alpha \otimes \partial_r + (1/r)\,dr \otimes R$ is Kähler, where we extend J by $J(\partial_r) = 0$ (see e.g. [1], [2], [12]). These manifolds have been studied by many authors (see e.g. [1], [3], [8], [11], [12] and the references included).

We consider contact Calabi-Yau manifolds which are a special class of Sasakian manifolds: namely a *contact Calabi-Yau manifold* is a $2n+1$-dimensional Sasakian manifold (M, α, J) endowed with a closed basic complex volume form ϵ. It turns out that these manifolds are a special class of null-Sasakian α-Einstein manifolds. As a direct consequence of the above definition, in a contact Calabi-Yau manifold (M, α, J, ϵ) the real part of ϵ is a calibration. Furthermore, we have that an n-dimensional submanifold $p\colon L \hookrightarrow M$ of a contact Calabi-Yau manifold admits an orientation making it a calibrated submanifold by $\mathfrak{Re}\,\epsilon$ if and only if

$$p^*(\alpha) = 0, \quad p^*(\mathfrak{Im}\,\epsilon) = 0.$$

In such a case L is said to be a *special Legendrian submanifold*. We prove that:

The moduli space of deformations of special Legendrian submanifolds near a fixed compact one L is a smooth 1-dimensional manifold.

Moreover we get the following extension theorem:

Let $(M, \alpha_t, J_t, \epsilon_t)$ be a smooth family of contact Calabi-Yau manifolds and let $p\colon L \hookrightarrow (M, \alpha_0, J_0, \epsilon_0)$ be a compact special Legendrian submanifold. Then there exists a smooth family of special Legendrian submanifolds $p_t\colon L \hookrightarrow (M, \alpha_t, J_t, \epsilon_t)$ that extends $p\colon L \hookrightarrow M$ if and only if the cohomology class $[p^*(\mathfrak{Im}\,\epsilon)]$ vanishes.

This can be considered a contact version of a theorem of Lu Peng (see [10]) in Calabi-Yau manifolds (see also [14]).

In Section 2 we fix some notation on contact and Sasakian geometry. In Section 3 we define contact Calabi-Yau manifolds and we obtain some simple topological obstructions to the existence of contact Calabi-Yau structures on odd-dimensional manifolds. As a corollary, we get that there are no contact Calabi-Yau structures on odd-dimensional spheres. In Section 4 we study the moduli space of special Legendrian submanifolds, proving the theorems stated above. In Section 5 we classify the 5-dimensional nilmanifolds carrying an invariant contact Calabi-Yau structure. The proof is based on Theorems 21 and 23 of [5]. In the last section we generalize the previous definition to the case of codimension r proving an extension theorem. Some examples of contact Calabi-Yau manifolds and special Legendrian submanifolds are carefully described.

2. Preliminaries

Let M be a manifold of dimension $2n+1$. A *contact structure* on M is a distribution $\xi \subset TM$ of dimension $2n$, such that the defining 1-form α satisfies

$$\alpha \wedge (d\alpha)^n \neq 0. \tag{1}$$

A 1-form α satisfying (1) is said to be a *contact form* on M. Let α be a contact form on M; then there exists a unique vector field R_α on M such that

$$\alpha(R_\alpha) = 1, \quad \iota_{R_\alpha} d\alpha = 0,$$

where $\iota_{R_\alpha} d\alpha$ denotes the contraction of $d\alpha$ along R_α. By definition R_α is called the *Reeb vector field* of the contact form α. A *contact manifold* is a pair (M, ξ) where M is a $2n+1$-dimensional manifold and ξ is a contact structure. Let (M, ξ) be a contact manifold and fix a defining (contact) form α. Then the 2-form $\kappa = (1/2)\, d\alpha$ defines a symplectic form on the contact structure ξ; therefore the pair (ξ, κ) is a symplectic vector bundle over M. A *complex structure* on ξ is the datum of $J \in \mathrm{End}(\xi)$ such that $J^2 = -I_\xi$.

DEFINITION 2.1. Let α be a contact form on M, with $\xi = \ker\alpha$ and let $\kappa = (1/2)\, d\alpha$. A complex structure J on ξ is said to be *κ-calibrated* if

$$g_J[x](\cdot\,,\ \cdot) := \kappa[x](\cdot\,,\ J_x\,\cdot)$$

is a J_x-Hermitian inner product on ξ_x for any $x \in M$.

The set of κ-calibrated complex structures on ξ will be denoted by $\mathfrak{C}_\alpha(M)$. If J is a complex structure on $\xi = \ker\alpha$, then we extend it to an endomorphism of TM by setting

$$J(R_\alpha) = 0.$$

Note that such a J satisfies

$$J^2 = -I + \alpha \otimes R_\alpha.$$

If J is κ-calibrated, then it induces a Riemannian metric g on M given by

$$g := g_J + \alpha \otimes \alpha. \tag{2}$$

Furthermore the Nijenhuis tensor of J is defined by

$$N_J(X, Y) = [JX, JY] - J[X, JY] - J[Y, JX] + J^2[X, Y]$$

for any $X, Y \in TM$. We recall the following

DEFINITION 2.2. A Sasakian structure on a $2n+1$-dimensional manifold M is a pair (α, J), where

- α is a contact form;
- $J \in \mathfrak{C}_\alpha(M)$ satisfies $N_J = -d\alpha \otimes R_\alpha$.

The triple (M, α, J) is said to be a *Sasakian manifold*.

For other characterizations of Sasakian structure see e.g. [1] and [2].

We recall now the definition of basic r-forms.

DEFINITION 2.3. Let (M, ξ) be a contact manifold. A differential r-form γ on M is said to be *basic* if

$$\iota_{R_\alpha}\gamma = 0, \quad \mathcal{L}_{R_\alpha}\gamma = 0,$$

where $\mathcal{L}$ denotes the Lie derivative and R_α is the Reeb vector field of an arbitrary contact form defining ξ.

We will denote by $\Lambda^r_B(M)$ the set of basic r-forms on (M, ξ). Note that

$$d\Lambda^r_B(M) \subset \Lambda^{r+1}_B(M).$$

The cohomology $H^\bullet_B(M)$ of this complex is called the *basic cohomology* of (M, ξ).

If (M, α, J) is a Sasakian manifold, then

$$J(\Lambda^r_B(M)) = \Lambda^r_B(M),$$

where, as usual, the action of J on r-forms is defined by

$$J\phi(X_1, \dots, X_r) = \phi(JX_1, \dots, JX_r).$$

Consequently $\Lambda^r_B(M) \otimes \mathbb{C}$ splits as

$$\Lambda^r_B(M) \otimes \mathbb{C} = \bigoplus_{p+q=r} \Lambda^{p,q}_J(\xi)$$

and, according with this gradation, it is possible to define the cohomology groups $H^{p,q}_B(M)$. The r-forms belonging to $\Lambda^{p,q}_J(\xi)$ are said to be of *type* (p, q) *with respect* *to* J. Note that $\kappa = (1/2)\, d\alpha \in \Lambda^{1,1}_J(\xi)$ and it determines a non-vanishing cohomology class in $H^{1,1}_B(M)$. The Sasakian structure (α, J) also induces a natural connection ∇^ξ on ξ given by

$$\nabla^\xi_X Y = \begin{cases} (\nabla_X Y)^\xi & \text{if } X \in \xi \\ [R_\alpha, Y] & \text{if } X = R_\alpha, \end{cases}$$

where the subscript ξ denotes the projection onto ξ. One easily gets

$$\nabla^\xi_X J = 0, \quad \nabla^\xi_X g_J = 0, \quad \nabla^\xi_X d\alpha = 0, \quad \nabla^\xi_X Y - \nabla^\xi_Y X = [X, Y]^\xi,$$

for any $X, Y \in TM$. Consequently we have

$$\mathrm{Hol}(\nabla^\xi) \subseteq \mathrm{U}(n).$$

Moreover the *transverse Ricci tensor* Ric^T is defined as

$$\mathrm{Ric}^T(X, Y) = \sum_{i=1}^{2n} g\big(\nabla^\xi_X \nabla^\xi_{e_i} e_i - \nabla^\xi_{e_i} \nabla^\xi_X e_i - \nabla^\xi_{[X,e_i]} e_i, Y\big),$$

for any $X, Y \in \xi$, where $\{e_1, \dots, e_{2n}\}$ is an arbitrary orthonormal frame of ξ. It is known that Ric^T satisfies

$$\mathrm{Ric}^T(X, Y) = \mathrm{Ric}(X, Y) + 2g(X, Y),$$

for any $X, Y \in \xi$, where Ric denotes the Ricci tensor of the Riemannian metric $g = g_J + \alpha \otimes \alpha$. Let us denote by ρ^T the Ricci form of Ric^T, i.e.

$$\rho^T(X, Y) = \mathrm{Ric}^T(JX, Y) = \mathrm{Ric}(JX, Y) + 2\kappa(X, Y),$$

for any $X, Y \in \xi$. We recall that ρ^T is a closed form such that $(1/(2\pi))\rho$ represents the first Chern class of (ξ, J) (see e.g. [7]); this form is called the *transverse Ricci form* of (α, J).

DEFINITION 2.4. The basic cohomology class

$$c_1^B(M) = \frac{1}{2\pi}[\rho^T] \in H^{1,1}_B(M)$$

is called the *first basic Chern class* of (M, α, J) and, if it vanishes, then (M, α, J) is said to be *null-Sasakian*.

Furthermore we recall that a Sasakian manifold is called *α-Einstein* if there exist $\lambda, \nu \in C^\infty(M, \mathbb{R})$ such that

$$\mathrm{Ric} = \lambda g + \nu \alpha \otimes \alpha.$$

For general references on these topics see e.g. [4] and [3].

Finally, recall that a submanifold $p\colon L \hookrightarrow M$ of a $2n+1$-dimensional contact manifold (M, ξ) is said to be *Legendrian* if:

1) $\dim_{\mathbb{R}} L = n$,
2) $p_*(TL) \subset \xi$.

Observe that, if α is a defining form of the contact structure ξ, then condition 2) is equivalent to say that $p^*(\alpha) = 0$. Hence Legendrian submanifolds are the analogue of Lagrangian submanifolds in contact geometry.

3. Contact Calabi-Yau manifolds

In this section we study contact Calabi-Yau manifolds. As already explained in the introduction, these manifolds are a natural generalization of the Calabi-Yau ones in the context of contact geometry. Roughly speaking a contact Calabi-Yau manifold is a Sasakian manifold endowed with a basic closed complex volume form. We can give now the following

DEFINITION 3.1. A *contact Calabi-Yau manifold* is a quadruple (M, α, J, ϵ), where

- (M, α, J) is a $2n+1$-dimensional Sasakian manifold;
- $\epsilon \in \Lambda^{n,0}_J(\xi)$ is a nowhere vanishing basic form on $\xi = \ker \alpha$ such that

$$\begin{cases} \epsilon \wedge \overline{\epsilon} = c_n \kappa^n \\ d\epsilon = 0, \end{cases}$$

where $c_n = (-1)^{n(n+1)/2}(2i)^n$ and $\kappa = (1/2)\, d\alpha$.

Now we will describe a couple of examples.

EXAMPLE 3.2. Consider $\mathbb{R}^{2n+1}$ endowed with the standard Euclidean coordinates $\{x_1, \dots, x_n, y_1, \dots, y_n, t\}$. Let

$$\alpha_0 = 2\, dt - 2 \sum_{i=1}^{n} y_i\, dx_i$$

be the *standard contact form* on $\mathbb{R}^{2n+1}$ and let $\xi_0 = \ker \alpha_0$. Then ξ_0 is spanned by

$$\{y_1 \partial_t + \partial_{x_1}, \dots, y_n \partial_t + \partial_{x_n}, \partial_{y_1}, \dots, \partial_{y_n}\}.$$

For simplicity, set $V_i = y_i \partial_t + \partial_{x_i}$, $W_j = \partial_{y_j}$, $i, j = 1, \dots, n$ and

$$\begin{cases} J_0(V_r) = W_r \\ J_0(W_r) = -V_r \end{cases} \quad r = 1, \dots, n.$$

Then J_0 defines a complex structure in $\mathfrak{C}_\alpha(M)$. Since the space of transverse 1-forms is spanned by $\{dx_1, \dots, dx_n, dy_1, \dots, dy_n\}$, then the complex valued form

$$\epsilon_0 := (dx_1 + i\,dy_1) \wedge \cdots \wedge (dx_n + i\,dy_n)$$

is of type $(n, 0)$ with respect to J_0 and it satisfies

$$\begin{cases} \epsilon_0 \wedge \overline{\epsilon_0} = c_n \kappa_0^n \\ d\epsilon_0 = 0, \end{cases}$$

where $\kappa_0 = (1/2)\,d\alpha_0$. Therefore $(\mathbb{R}^{2n+1}, \alpha_0, J_0, \epsilon_0)$ is a contact Calabi-Yau manifold.

The following will describe a compact contact Calabi-Yau manifold.

EXAMPLE 3.3. Let

$$H(3) := \left\{ A = \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} \middle|\, x, y, z \in \mathbb{R} \right\}$$

be the 3-dimensional Heisenberg group and let $M = H(3)/\Gamma$, where Γ denotes the subgroup of $H(3)$ given by the matrices with integral entries. The 1-forms $\alpha_1 = dx$, $\alpha_2 = dy$, $\alpha_3 = x\,dy - dz$ are $H(3)$-invariant and therefore they define a global coframe on M. Then $\alpha = 2\alpha_3$ is a contact form whose contact distribution ξ is spanned by $V = \partial_x$, $W = \partial_y + x\partial_z$. Again

$$\begin{cases} J(V) = W \\ J(W) = -V \end{cases}$$

defines a κ-calibrated complex structure on ξ and $\epsilon = \alpha_1 + i\alpha_2$ is a $(1, 0)$-form on ξ such that (M, α, J, ϵ) is a contact Calabi-Yau manifold.

The last example gives an invariant contact Calabi-Yau structure on a nilmanifold. It can be generalized to the dimension $2n + 1$ in this way: let $\mathfrak{g}$ be the Lie algebra spanned by $\{X_1, \dots, X_{2n+1}\}$ with

$$[X_{2k-1}, X_{2k}] = -X_{2n+1}$$

for $k = 1, \dots, n$ and the other brackets are zero. Then $\mathfrak{g}$ is a $2n + 1$-dimensional nilpotent Lie algebra with rational constant structures and, by Malcev theorem, it follows that if G is the simply connected Lie group with Lie algebra $\mathfrak{g}$, then G has a compact quotient. Let $\{\alpha_1, \dots, \alpha_{2n+1}\}$ be the dual basis of $\{X_1, \dots, X_{2n+1}\}$. Then we immediately get

$$d\alpha_1 = 0, \dots, d\alpha_{2n} = 0, \quad d\alpha_{2n+1} = \sum_{k=1}^{n} \alpha_{2k-1} \wedge \alpha_{2k}.$$

Hence

$$\alpha = 2\alpha_{2n+1},$$

the endomorphism J of $\xi = \ker \alpha$ defined by

$$\begin{cases} J(X_{2k-1}) = X_{2k} \\ J(X_{2k}) = -X_{2k-1} \end{cases}$$

for $k = 1, \dots, n$ and the complex form

$$\epsilon = (\alpha_1 + i\alpha_2) \wedge \cdots \wedge (\alpha_{2n-1} + i\alpha_{2n})$$

define a contact Calabi-Yau structure on any compact nilmanifold associated with $\mathfrak{g}$.

The following proposition gives simple topological obstructions in order that a compact $2n+1$-dimensional manifold M carries a contact Calabi-Yau structure.

Proposition 3.4. *Let M be a $2n+1$-dimensional compact manifold. Assume that M admits a contact Calabi-Yau structure; then the following hold*
1. *if n is* even, *then $b_{n+1}(M) > 0$;*
2. *if n is* odd, *then*

$$\begin{cases} b_n(M) \geq 2 \\ b_{n+1}(M) \geq 2, \end{cases}$$

where $b_j(M)$ denotes the j^{th} Betti number of M.

Proof. Let (α, J, ϵ) be a contact Calabi-Yau structure on M and let $\xi = \ker\alpha$. Set $\Omega = \mathfrak{Re}\,\epsilon$; then, since $\epsilon \in \Lambda_J^{n,0}(\xi)$, we have $\epsilon = \Omega + iJ\Omega$. In view of the assumption $d\epsilon = 0$, we obtain $d\Omega = dJ\Omega = 0$ and since $d\alpha \in \Lambda_J^{1,1}(\xi)$ it follows that

$$\Omega \wedge d\alpha = J\Omega \wedge d\alpha = 0.$$

Hence

$$d(\Omega \wedge \alpha) = d(J\Omega \wedge \alpha) = 0.$$

Furthermore we have

$$\begin{aligned} \epsilon \wedge \overline{\epsilon} &= \Omega \wedge \Omega + J\Omega \wedge J\Omega && \text{if } n \text{ is even;} \\ \epsilon \wedge \overline{\epsilon} &= -2i\Omega \wedge J\Omega && \text{if } n \text{ is odd.} \end{aligned}$$

1. If n is even, then $\alpha\wedge(\Omega\wedge\Omega+J\Omega\wedge J\Omega)$ is a volume form on M. Assume that the cohomology classes $[\Omega\wedge\alpha]$, $[J\Omega\wedge\alpha]$ vanish; then there exist $\beta,\gamma\in\Lambda^n(M)$ such that

$$\alpha\wedge\Omega=d\beta,\quad \alpha\wedge J\Omega=d\gamma.$$

By Stokes theorem we have

$$\begin{aligned}0\neq\int_M \alpha\wedge\Omega\wedge\Omega+\alpha\wedge J\Omega\wedge J\Omega&=\int_M d\beta\wedge\Omega+d\gamma\wedge J\Omega\\&=\int_M d(\beta\wedge\Omega)+d(\gamma\wedge J\Omega)=0,\end{aligned}$$

which is absurd. Therefore one of $[\Omega\wedge\alpha]$, $[J\Omega\wedge\alpha]$ does not vanish. Consequently $b_{n+1}(M)>0$.
2. Let n be odd. We prove that the cohomology classes $[\Omega]$ and $[J\Omega]$ are $\mathbb{R}$-independent. Assume that there exist $a,b\in\mathbb{R}$ such that $a[\Omega]+b[J\Omega]=0$, $(a,b)\neq(0,0)$. Then there exists $\beta\in\Lambda^{n-1}(M)$ such that

$$a\Omega+bJ\Omega=d\beta.$$

We may assume that $a=1$, so that $\Omega=d\beta-bJ\Omega$. Stokes theorem implies

$$0\neq\int_M\alpha\wedge\Omega\wedge J\Omega=\int_M\alpha\wedge d\beta\wedge J\Omega=-\int_M d(\alpha\wedge\beta\wedge J\Omega)=0$$

which is a contradiction. Hence $b_n(M)\geq 2$. With the same argument, it is possible to prove that $b_{n+1}(M)\geq 2$ by showing that $[\Omega\wedge\alpha]$ and $[J\Omega\wedge\alpha]$ are $\mathbb{R}$-independent in $H^{n+1}(M,\mathbb{R})$. □

The following is an immediate consequence of Proposition 3.4.

Corollary 3.5. *A 3-dimensional compact manifold M admitting contact Calabi-Yau structure has $b_1(M)\geq 2$. In particular, there are no compact 3-dimensional simply connected contact Calabi-Yau manifolds. Moreover, the $2n+1$-dimensional sphere has no contact Calabi-Yau structures.*

The following proposition implies that the transverse Ricci tensor of a contact Calabi-Yau manifold vanishes

Proposition 3.6. *Let (M,α,J) be a $2n+1$-dimensional Sasakian manifold and $\xi=\ker\alpha$. The following facts are equivalent:*
1. $\mathrm{Hol}^0(\nabla^\xi)\subseteq \mathrm{SU}(n)$
2. $\mathrm{Ric}^T=0$.

Proof. The connection ∇^{ξ} induces a connection ∇^K on $\Lambda^{n,0}_J(\xi)$ which has $\mathrm{Hol}(\nabla^K) \subseteq \mathrm{U}(1)$. Since $\mathrm{Hol}^0(\nabla^K)$ and $\mathrm{Hol}^0(\nabla^{\xi})$ are related by

$$\mathrm{Hol}^0(\nabla^K) = \det(\mathrm{Hol}^0(\nabla^{\xi})),$$

where det is the map induced by the determinant $\mathrm{U}(n) \to \mathrm{U}(1)$, then it follows that $\mathrm{Hol}^0(\nabla^{\xi}) \subseteq \mathrm{SU}(n)$ if and only if $\mathrm{Hol}^0(\nabla^K) = \{1\}$ and in this case ∇^K is flat. As in the Kähler case it can be showed using transverse holomorphic coordinates (see e.g. [7], [8]) that the curvature form of ∇^K coincides with the transverse Ricci form of (α, J). Hence $\mathrm{Hol}^0(\nabla^{\xi}) \subseteq \mathrm{SU}(n)$ if and only if $\mathrm{Ric}^T = 0$. □

As a consequence of the last proposition we have the following

Corollary 3.7. *Let (M, α, J, ϵ) be a contact Calabi-Yau manifold. Then (M, α, J) is null-Sasakian and the metric g induced by (α, J) is α-Einstein with $\lambda = -2$ and $\nu = 2n + 2$. In particular the scalar curvature of the metric g associated to (α, J) is equal to $-2n$.*

4. Deformations of special Legendrian submanifolds

In this section we are going to study the geometry of Legendrian submanifolds in a contact Calabi-Yau ambient. We will prove a contact version of McLean and Lu Peng theorems (see [13] and [10]).

Let (M, α, J, ϵ) be a contact Calabi-Yau manifold of dimension $2n+1$. It easy to see that for any oriented n-plane $V \subset T_x M$

$$\mathfrak{Re}\, \epsilon_{|V} \leq \mathrm{Vol}(V),$$

where $\mathrm{Vol}(V)$ is computed with respect to the metric g induced by (α, J) on M. Hence $\mathfrak{Re}\, \epsilon$ is a calibration on (M, g) (see [9]). We have the following

Proposition 4.1. *Let $p\colon L \hookrightarrow M$ be an n-dimensional submanifold. The following facts are equivalent*

1. *the submanifold satisfies*

$$\begin{cases} p^*(\alpha) = 0 \\ p^*(\mathfrak{Im}\, \epsilon) = 0, \end{cases}$$

2. *there exists an orientation on L making it calibrated by $\mathfrak{Re}\, \epsilon$.*

We can give the following

DEFINITION 4.2. An n-dimensional submanifold $p\colon L \hookrightarrow M$ is said to be *special Legendrian* if

$$\begin{cases} p^*(\alpha) = 0 \\ p^*(\Im\mathfrak{m}\,\epsilon) = 0. \end{cases}$$

It follows that compact special Legendrian submanifolds minimize volume in their homology class and that there are no compact special Legendrian submanifolds in $(\mathbb{R}^{2n+1}, \alpha_0, J_0, \epsilon_0)$.

EXAMPLE 4.3. Let $(M = H(3)/\Gamma, \alpha, J, \epsilon)$ be the contact Calabi-Yau manifold considered in the Example 3.3. Then the submanifold

$$L := \left\{ [A] \in M \;\middle|\; A = \begin{pmatrix} 1 & x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\} \simeq S^1$$

is a compact special Legendrian submanifold.

Now we define the moduli space of special Legendrian submanifolds.

DEFINITION 4.4. Let (M, α, J, ϵ) be a contact Calabi-Yau manifold and let $p_0\colon L \hookrightarrow M$, $p_1\colon L \hookrightarrow M$ be two special Legendrian submanifolds. Then $p_1\colon L \hookrightarrow M$ is said to be a *deformation* of $p_0\colon L \hookrightarrow M$ if there exists a smooth map $F\colon L \times [0,1] \to M$ such that

- $F(\,\cdot\,, t)\colon L \times \{t\} \to M$ is a special Legendrian embedding for any $t \in [0, 1]$;
- $F(\,\cdot\,, 0) = p_0$, $F(\,\cdot\,, 1) = p_1$.

Let (M, α, J, ϵ) be a contact Calabi-Yau manifold and let $p\colon L \hookrightarrow M$ be a fixed compact special Legendrian submanifold. Set

$$\begin{aligned} \mathfrak{M}(L) := \{&\text{special Legendrian submanifolds of } (M, \alpha, J, \epsilon) \\ &\text{which are deformations of } p\colon L \hookrightarrow M\}/\!\sim, \end{aligned}$$

where two embeddings are considered equivalent if they differ by a diffeomorphism of L; then by definition $\mathfrak{M}(L)$ is the *moduli space of special Legendrian submanifolds* which are deformations of $p\colon L \hookrightarrow M$. We have the following

Theorem 4.5. *Let (M, α, J, ϵ) be a contact Calabi-Yau manifold and let $p\colon L \hookrightarrow M$ be a compact special Legendrian submanifold. Then the moduli space $\mathfrak{M}(L)$ is a 1-dimensional manifold.*

The next lemma will be useful in the proof of Theorem 4.5:

Lemma 4.6 ([13], [6]). *Let* (V, κ) *be a symplectic vector space and let* $i\colon W \hookrightarrow V$ *be a Lagrangian subspace. Then*

1. $\tau\colon V/W \to W^*$ *defined as* $\tau([v]) = i^*(\iota_v \kappa)$ *is an isomorphism*;
2. *let* J *be a* κ*-calibrated complex structure on* V *and let* $\epsilon \in \Lambda_J^{n,0}(V^*)$ *satisfy*

$$i^*(\mathfrak{Im}\,\epsilon) = 0, \quad \epsilon \wedge \overline{\epsilon} = c_n \frac{\kappa^n}{n!}.$$

Then $\theta\colon V/W \to \Lambda^{n-1}(W^*)$ *defined as* $\theta([v]) := i^*(\iota_v\, \mathfrak{Im}\,\epsilon)$ *is an isomorphism. Moreover for any* $v \in V$*, we have*

$$\theta([v]) = - * \tau([v]),$$

where $*$ *is computed with respect to* $i^*(g_J(\,\cdot\,,\,\cdot\,)) := i^*(\kappa(\,\cdot\,, J\,\cdot\,))$ *and the volume form* $\mathrm{Vol}(W) := i^*(\mathfrak{Re}\,\epsilon)$.

For the proof of Lemma 4.6 we refer to [13] and [6].

Proof of Theorem 4.5. Let $\mathcal{N}(L)$ be the normal bundle to L. Then

$$\mathcal{N}(L) = \langle R_\alpha \rangle \oplus J(p_*(TL)),$$

where R_α is the Reeb vector field of α. Let Z be a vector field normal to L and let $\exp_Z\colon L \to M$ be defined as

$$\exp_Z(x) := \exp_x(Z(x)).$$

Let U be a neighborhood of 0 in $C^{2,\alpha}(\langle R_\alpha \rangle) \oplus C^{1,\alpha}(J(p_*(TL)))$ and let

$$F\colon U \to C^{1,\alpha}(\Lambda^1(L)) \oplus C^{0,\alpha}(\Lambda^n(L)),$$

be defined as

$$F(Z) = (\exp_Z^*(\alpha),\, 2\exp_Z^*(\mathfrak{Im}\,\epsilon)).$$

We obviously have

$$Z \in F^{-1}((0,0)) \cap C^\infty(\mathcal{N}(L)) \iff \exp_Z(L) \text{ is a special Legendrian submanifold.}$$

Note that since $\exp_Z$ and p are homotopic via $\exp_{tZ}$, we have

$$[\exp_Z^*(\mathfrak{Im}\,\epsilon)] = [p^*(\mathfrak{Im}\,\epsilon)] = 0.$$

Therefore

$$F\colon U \to C^{1,\alpha}(\Lambda^1(L)) \oplus dC^{1,\alpha}(\Lambda^{n-1}(L)).$$

Let us compute the differential of the map F.

$$F_*[0](Z) = \frac{d}{dt}(\exp^*_{tZ}(\alpha), 2\exp^*_{tZ}(\mathfrak{Im}\,\epsilon))_{|t=0} = (p^*(\mathcal{L}_Z\alpha), 2p^*(\mathcal{L}_Z\,\mathfrak{Im}\,\epsilon)),$$

where $\mathcal{L}$ denotes the Lie derivative. We may write $Z = JX + fR_\alpha$; then applying Cartan formula we obtain

$$\begin{aligned} F_*[0](Z) &= (p^*(\mathcal{L}_Z\alpha), 2p^*(\mathcal{L}_Z\,\mathfrak{Im}\,\epsilon)) \\ &= (p^*(d\iota_Z\alpha + \iota_Z\,d\alpha), 2p^*(d\iota_Z\,\mathfrak{Im}\,\epsilon)) \\ &= (p^*(d\iota_{JX+fR_\alpha}\alpha + \iota_{JX+fR_\alpha}\,d\alpha), 2p^*(d\iota_{JX+fR_\alpha}\,\mathfrak{Im}\,\epsilon)) \\ &= (p^*(d\iota_{fR_\alpha}\alpha + \iota_{JX}\,d\alpha), 2p^*(d\iota_{JX}\,\mathfrak{Im}\,\epsilon)) \\ &= (p^*(df + \iota_{JX}\,d\alpha), 2dp^*(\iota_{JX}\,\mathfrak{Im}\,\epsilon)). \end{aligned}$$

By applying Lemma 4.6 we get

$$F_*[0](Z) = (d(f\circ p) + p^*(\iota_{JX}\,d\alpha), -d * p^*(\iota_{JX}\,d\alpha)), \tag{3}$$

where $*$ is the Hodge star operator with respect to the metric $p^*(g_J)$ and the volume form $p^*(\mathfrak{Re}\,\epsilon)$. Now we show that $F_*[0]$ is surjective. Let $(\eta, d\gamma) \in C^{1,\alpha}(\Lambda^1(L)) \oplus dC^{1,\alpha}(\Lambda^{n-1}(L))$. By the Hodge decomposition theorem we may assume

$$d\gamma = -d * du \quad \text{with} \quad u \in C^{3,\alpha}(L)$$

and we have

$$\eta = dv + d^*\beta + h(\eta)$$

where $v \in C^{2,\alpha}(L)$, $\beta \in C^{2,\alpha}(\Lambda^2(L))$ and $h(\eta)$ denotes the harmonic component of η. Then we get

$$\begin{aligned} (\eta, d\gamma) &= (du - du + dv + d^*\beta + h(\eta), -d * du) \\ &= (dv - du + du + d^*\beta + h(\eta), -d * (du + d^*\beta + h(\eta))). \end{aligned}$$

We can find $f \in C^{2,\alpha}(p(L))$ and $X \in C^{1,\alpha}(p_*(TL))$ such that

$$\begin{aligned} f\circ p &= v - u \\ p^*(\iota_{JX}d\alpha) &= du + d^*\beta + h(\eta). \end{aligned}$$

Hence

$$(\eta, d\gamma) = (d(f\circ p) + p^*(\iota_{JX}\,d\alpha), -d * p^*(\iota_{JX}\,d\alpha))$$

and $F_*[0]$ is surjective. Therefore $(0,0)$ is a regular value of F. Now we compute $\ker F_*[0]$. Formula (3) implies that $Z \in \ker F_*[0]$ if and only if

$$d(f \circ p) + p^*(\iota_{JX}\, d\alpha) = 0 \tag{4}$$

$$d^* p^*(\iota_{JX}\, d\alpha) = 0. \tag{5}$$

By applying d^* to both sides of (4) and taking into account (5) we get

$$0 = d^* d(f \circ p) + d^* p^*(\iota_{JX}\, d\alpha) = d^* d(f \circ p),$$

i.e.

$$\Delta(f \circ p) = 0.$$

Since L is compact f is constant. Hence (4) reduces to

$$p^*(\iota_{JX}\, d\alpha) = 0. \tag{6}$$

The map

$$\Theta\colon p_*(TL) \to \Lambda^1(L)$$

defined by

$$\Theta(X) = p^*(\iota_{JX}\, d\alpha)$$

is an isomorphism; hence equation (6) implies $X = 0$. Therefore $Z = W + f R_\alpha$ belongs to $\ker F_*[0]$ if and only if

$$\begin{cases} W = 0 \\ f = \text{constant.} \end{cases}$$

It follows that $\ker F_*[0] = \mathrm{Span}_{\mathbb{R}}(R_\alpha) \subset C^\infty(\mathcal{N}(L))$. The implicit function theorem between Banach spaces implies that the moduli space $\mathfrak{M}(L)$ is a 1-dimensional smooth manifold. □

REMARK 4.7. Note that the dimension of $\mathfrak{M}(L)$ does not depend on that of L. This is quite different from the Calabi-Yau case, where the dimension of the moduli space of deformations of special Lagrangian submanifolds near a fixed compact L is equal to the first Betti number of L. This difference can be explained in the following way: the deformations parameterized by curves tangent to the contact structure are trivial, while those one along the Reeb vector field R_α parameterize the moduli space.

Now we study the following

Extension problem. Let $(M, \alpha_t, J_t, \epsilon_t)$, $t \in (-\delta, \delta)$, be a smooth family of contact Calabi-Yau manifolds. Given a compact special Legendrian submanifold $p\colon L \hookrightarrow M$ of $(M, \alpha_0, J_0, \epsilon_0)$ does it exist a family $p_t\colon L \hookrightarrow M$ of special Legendrian submanifolds of $(M, \alpha_t, J_t, \epsilon_t)$ such that $p_0\colon L \hookrightarrow M$ coincides with p?

This is a contact version of the extension problem in the Calabi-Yau case (see [10] and [14]). We can state the following

Theorem 4.8. *Let $(M, \alpha_t, J_t, \epsilon_t)_{t\in(-\delta,\delta)}$ be a smooth family of contact Calabi-Yau manifolds. Let $p\colon L \hookrightarrow M$ be a compact special Legendrian submanifold of $(M, \alpha_0, J_0, \epsilon_0)$. Then there exists, for small t, a family of compact special Legendrian submanifolds $p_t\colon L \hookrightarrow (M, \alpha_t, J_t, \epsilon_t)$ such that $p_0 = p$ if and only if the condition*

$$[p^*(\mathfrak{Im}\,\epsilon_t)] = 0 \tag{7}$$

holds for t small enough.

Proof. The condition (7) is necessary. Indeed if we can extend L, then $\mathfrak{Im}\,\epsilon_t$ is a closed form such that $p_t^*(\mathfrak{Im}\,\epsilon_t) = 0$. Since p_t is homotopic to p_0 we have

$$[p_0^*(\mathfrak{Im}\,\epsilon_t)] = [p_t^*(\mathfrak{Im}\,\epsilon_t)] = 0.$$

In order to prove that condition (7) is sufficient, we can consider the map

$$G\colon (-\sigma, \sigma) \times C^{1,\alpha}(J(p_*TL)) \to C^{0,\alpha}(\Lambda^2(L)) \oplus C^{0,\alpha}(\Lambda^n(L))$$

defined as

$$G(t, Z) = (\exp_Z^*(d\alpha_t), 2\exp_Z^*(\mathfrak{Im}\,\epsilon_t)).$$

By our assumption it follows that

$$\mathrm{Im}(G) \subset dC^{1,\alpha}(\Lambda^1(L)) \oplus dC^{(1,\alpha)}(\Lambda^{n-1}(L)).$$

Let $X \in C^{1,\alpha}(p_*(TL))$; a direct computation and Lemma 4.6 give

$$\begin{aligned} G_*[(0,0)](0, JX) &= (dp^*(\iota_{JX}\, d\alpha_0), 2dp^*(\iota_{JX}\,\mathfrak{Im}\,\epsilon)) \\ &= (dp^*(\iota_{JX}\, d\alpha_0), -d * p^*(\iota_{JX}\, d\alpha_0)), \end{aligned}$$

where $*$ is the Hodge operator of the metric $p^*(g_J)$ with respect to the volume form $p^*(\mathfrak{Re}\,\epsilon)$. It follows that $G_*[(0,0)](0, \cdot)$ is surjective and that

$$\ker G_*[(0,0)]_{\{0\}\times C^{1,\alpha}(p_*(J(TL)))} \equiv \mathcal{H}^1(L),$$

where $\mathcal{H}^1(L)$ denotes the space of harmonic 1-forms on L. Let

$$A = \{X \in C^{1,\alpha}(p_*(TL)) \mid p^*(\iota_{JX} d\alpha) \in dC^{1,\alpha}(L) \oplus d^* C^{1,\alpha}(\Lambda^2(L))\}$$

and

$$\hat{G} = G_{|(-\delta,\delta)\times A}.$$

Then by the Hodge decomposition of $\Lambda(L)$ it follows that

$$G_*[(0,0)]_{\{0\}\times A} \colon A \to dC^{1,\alpha}(L) \oplus d^* C^{1,\alpha}(\Lambda^2(L))$$

is an isomorphism. Again by the implicit function theorem and the elliptic regularity there exists a local smooth solution of the equation

$$\hat{G}(t, \psi(t)) = 0.$$

The extension of $p \colon L \hookrightarrow M$ is obtained by considering

$$p_t := \exp_{\psi(t)}.$$

□

5. The 5-dimensional nilpotent case

In this section we study invariant contact Calabi-Yau structures on 5-dimensional nilmanifolds. We will prove that a compact 5-dimensional nilmanifold carrying an invariant Calabi-Yau structure is covered by a Lie group whose Lie algebra is isomorphic to

$$\mathfrak{g} = (0, 0, 0, 0, 12 + 34),$$

just described in Section 2. Notation $\mathfrak{g} = (0, 0, 0, 0, 12 + 34)$ means that there exists a basis $\{\alpha_1, \ldots, \alpha_5\}$ of the dual space of the Lie algebra $\mathfrak{g}$ such that

$$d\alpha_1 = d\alpha_2 = d\alpha_3 = d\alpha_4 = 0, \quad d\alpha_5 = \alpha_1 \wedge \alpha_2 + \alpha_3 \wedge \alpha_4.$$

First of all we note that 5-dimensional contact Calabi-Yau manifolds are in particular hypo. Recall that an *hypo structure* on a 5-dimensional manifold is the datum of $(\alpha, \omega_1, \omega_2, \omega_3)$, where $\alpha \in \Lambda^1(M)$ and $\omega_i \in \Lambda^2(M)$ and

1. $\omega_i \wedge \omega_j = \delta_{ij} v$, for some $v \in \Lambda^4(M)$ satisfying $v \wedge \alpha \neq 0$;
2. $\iota_X \omega_1 = \iota_Y \omega_2 \iff \omega_3(X, Y) \geqslant 0$:
3. $d\omega_1 = 0$, $d(\omega_2 \wedge \alpha) = 0$, $d(\omega_3 \wedge \alpha) = 0$.

These structures have been introduced and studied by D. Conti and S. Salamon in [5]. Let (M, α, J, ϵ) be a contact Calabi-Yau manifold of dimension 5. Then

$$\alpha, \quad \omega_1 = \frac{1}{2} d\alpha, \quad \omega_2 = \mathfrak{Re}\,\epsilon, \quad \omega_3 = \mathfrak{Im}\,\epsilon,$$

define an hypo structure on M.

The following lemma, whose proof is immediate, will be useful in the sequel

Lemma 5.1. *Let $M = G/\Gamma$ be a nilmanifold of dimension* 5*. If M admits an invariant contact form, then the Lie algebra of G is isomorphic to one of the following*

- (0, 0, 12, 13, 14 + 23);
- (0, 0, 0, 12, 13 + 24);
- (0, 0, 0, 0, 12 + 34).

Let $\mathfrak{g}$ be a non-trivial 5-dimensional nilpotent Lie algebra and denote by $V = \mathfrak{g}^*$ the dual vector space of $\mathfrak{g}$. There exists a filtration of V

$$V^1 \subset V^2 \subset V^3 \subset V^4 \subset V^5 = V,$$

with $dV^i \subset \Lambda^2 V^{i-1}$ and $\dim_{\mathbb{R}} V^i = i$. We may choose the filtration V in such a way that $V^2 \subset \ker d \subset V^4$.

Let $(M = G/\Gamma, \alpha, \omega_1, \omega_2, \omega_3)$ be a nilmanifold endowed with an invariant hypo structure $(\alpha, \omega_1, \omega_2, \omega_3)$

1. Assume that $\alpha \in V^4$. Then we have the following (see [5])

Theorem 5.2. *If $\alpha \in V^4$, then $\mathfrak{g}$ is either* (0, 0, 0, 0, 12), (0, 0, 0, 12, 13), *or* (0, 0, 12, 13, 14).

In particular if (M, α, J, ϵ) is contact Calabi-Yau, then $\alpha \in V^4$.

2. Assume that $\alpha \notin V^4$. We have (see again [5])

Lemma 5.3. *If $\alpha \notin V^4$ and all ω_i are closed, then α is orthogonal to V^4.*

Theorem 5.4. *If α is orthogonal to V^4, then $\mathfrak{g}$ is one of*

$$(0, 0, 0, 0, 12), \quad (0, 0, 0, 0, 12 + 34).$$

Let (M, α, J, ϵ) be a contact Calabi-Yau manifold of dimension 5 endowed with an invariant contact Calabi-Yau structure; then by 1. α does not belong to V^4. By Lemma 5.3 α is orthogonal to V^4 and by Theorem 5.4 $\mathfrak{g}$ = (0, 0, 0, 0, 12 + 34). Hence we have proved the following

Theorem 5.5. *Let $M = G/\Gamma$ be a nilmanifold of dimension* 5 *admitting an invariant contact Calabi-Yau structure. Then $\mathfrak{g}$ is isomorphic to*

$$(0, 0, 0, 0, 12 + 34).$$

6. Calabi-Yau manifolds of codimension *r*

In this section we extend the definition of contact Calabi-Yau manifold to codimension r showing the analogous of Theorem 4.8.

Let us consider the following

DEFINITION 6.1. Let M be a $2n+r$-dimensional manifold. An *r-contact structure* on M is the datum $\mathcal{D} = \{\alpha_1, \dots, \alpha_r\}$, where $\alpha_i \in \Lambda^1(M)$, such that

- $d\alpha_1 = d\alpha_2 = \cdots = d\alpha_r$;
- $\alpha_1 \wedge \cdots \wedge \alpha_r \wedge (d\alpha_1)^n \neq 0$.

Note that if $\mathcal{D} = \{\alpha_1, \dots, \alpha_r\}$ is an r-contact structure and $\xi := \bigcap \ker \alpha_i$, then $(\xi, d\alpha_1)$ is a symplectic vector bundle on M and there exists a unique set of vector fields $\{R_1, \dots, R_r\}$ satisfying

$$\alpha_i(R_j) = \delta_{ij}, \quad \iota_{R_i} d\alpha_i = 0 \quad \text{for any} \quad i,\, j = 1, \dots, r.$$

Let us denote by $\mathfrak{C}_\kappa(\xi)$ the set of complex structures on ξ calibrated by the symplectic form $\kappa = (1/2)\, d\alpha_1$ and by $\Lambda^r_0(M)$ the set of r-forms γ on M satisfying

$$\iota_{R_i}\gamma = 0 \quad \text{for any} \quad i = 1, \dots, r.$$

If $J \in \mathfrak{C}_\kappa(\xi)$, then we extend it to TM by defining

$$J(R_i) = 0.$$

Note that such a J satisfies

$$J^2 = -I + \sum_{i=1}^{r} \alpha_i \otimes R_i.$$

Consequently, for any $J \in \mathfrak{C}_\kappa(\xi)$, we have $J(\Lambda^r_0(M)) \subset \Lambda^r_0 M$ and a natural splitting of $\Lambda^r_0(M) \otimes \mathbb{C}$ in

$$\Lambda^r_0(M) \otimes \mathbb{C} = \bigoplus_{p+q=r} \Lambda^{p,q}_J(\xi).$$

We can give the following

DEFINITION 6.2. An r-contact Calabi-Yau manifold is the datum of $(M, \mathcal{D}, J, \epsilon)$, where

- M is a $2n+r$-dimensional manifold;
- $\mathcal{D} = \{\alpha_1, \dots, \alpha_r\}$ is an r-contact structure;

- $J \in \mathfrak{C}_\kappa(\xi)$
- $\epsilon \in \Lambda^{n,0}_J(\xi)$ satisfies

$$\begin{cases} \epsilon \wedge \bar{\epsilon} = c_n \kappa^n \\ d\epsilon = 0. \end{cases}$$

EXAMPLE 6.3. Let $M = H(3)/\Gamma \times S^1$ be the Kodaira-Thurston manifold, where $H(3)$ is the 3-dimensional Heisenberg group and Γ is the lattice of $H(3)$ of matrices with integers entries. Let

$$\alpha_1 = -2\,dz + 2x\,dy,$$
$$\alpha_2 = -2\,dz + 2x\,dy + 2\,dt.$$

One easily gets

$$d\alpha_1 = d\alpha_2 = 2\,dx \wedge dy$$

and that $\mathcal{D} = \{\alpha_1, \alpha_2\}$ is a 2-contact structure on M. Note that $\xi = \ker \alpha_1 \cap \ker \alpha_2$ is spanned by $\{X_1 = \partial_x, X_2 = \partial_y + x\partial_z\}$. Moreover the Reeb fields of $\mathcal{D}$ are

$$R_1 = -\frac{1}{2}\partial_z - \frac{1}{2}\partial_t,$$
$$R_2 = \frac{1}{2}\partial_t.$$

Therefore $\Lambda^1_0(M)$ is generated by $\{dx, dy\}$. Let $J \in \mathrm{End}(\xi)$ be the complex structure given by

$$J(X_1) = X_2, \qquad J(X_2) = -X_1$$

and let $\epsilon \in \Lambda^{1,0}_J(\xi)$ be the form

$$\epsilon = dx + i\,dy.$$

Then $(M, \mathcal{D}, J, \epsilon)$ is a 2-contact Calabi-Yau structure.

As in the contact Calabi-Yau case if $(M, \mathcal{D}, J, \epsilon)$ is an r-contact Calabi-Yau manifold, then the n-form $\Omega = \mathfrak{Re}\,\epsilon$ is a calibration on M. Moreover an n-dimensional submanifold $p\colon L \hookrightarrow M$ admits an orientation making it calibrated by Ω if and only if

$$p^*(\alpha_i) = 0 \quad \text{for any} \quad \alpha_i \in \mathcal{D},$$
$$p^*(\mathfrak{Im}\,\epsilon) = 0.$$

A submanifold satisfying these equations will be called *special Legendrian*.

EXAMPLE 6.4. Let $(M, \mathcal{D}, J, \epsilon)$ be the 2-contact Calabi-Yau structure described in Example 6.3. Then

$$L := \left\{ [A] \in H(3)/\Gamma \;\middle|\; A = \begin{pmatrix} 1 & x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},\ x \in \mathbb{R} \right\} \times \{q\} \simeq S^1$$

is a compact special Legendrian submanifold for any $q \in S^1$.

The proof of the next theorem is very similar to that of Theorem 4.8 and it is omitted.

Theorem 6.5. *Let $(M, \mathcal{D}_t, J_t, \epsilon_t)_{t\in(-\delta,\delta)}$ be a smooth family of r-contact Calabi-Yau manifolds. Let $p\colon L \hookrightarrow M$ be a compact special Legendrian submanifold of $(M, \mathcal{D}_0, J_0, \epsilon_0)$. Then there exists, for small t, a family of compact special Legendrian submanifolds $p_t\colon L \hookrightarrow (M, \mathcal{D}_t, J_t, \epsilon_t)$ extending $p\colon L \hookrightarrow M$ if and only if the condition*

$$[p^*(\mathfrak{Im}\,\epsilon_t)] = 0$$

holds for t small enough.

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Adriano Tomassini
Dipartimento di Matematica
Università di Parma
Viale G.P. Usberti 53/A
43100 Parma
Italy
e-mail: adriano.tomassini@unipr.it

Luigi Vezzoni
Dipartimento di Matematica
Università di Torino
Via Carlo Alberto 10
10123 Torino
Italy
e-mail: luigi.vezzoni@unito.it