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# CONTACT CALABI-YAU MANIFOLDS AND SPECIAL LEGENDRIAN SUBMANIFOLDS

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## Abstract

We consider a generalization of Calabi-Yau structures in the context of Sasakian manifolds. We study deformations of a special class of Legendrian submanifolds and classify invariant contact Calabi-Yau structures on 5-dimensional nilmanifolds. Finally we generalize to codimension  $r$ .

## 1. Introduction

In their celebrated paper [9] Harvey and Lawson introduced the concept of calibration and calibrated geometry. Namely, a *calibration* on an  $n$ -dimensional oriented Riemannian manifold  $(M, g)$  is a closed  $r$ -form  $\phi$  such that for any  $x \in M$

$$\phi_x|_V \leq \text{Vol}(V),$$

where  $V$  is an arbitrary oriented  $r$ -plane in  $T_x M$ . An oriented submanifold  $p: L \hookrightarrow M$  is said to be *calibrated* by  $\phi$  if  $p^*(\phi) = \text{Vol}(L)$ . Compact calibrated submanifolds have the important property of minimizing volume in their homology class. As a typical example, the real part of holomorphic volume form of a Calabi-Yau manifold is a calibration; the corresponding calibrated submanifolds are said to be *special Lagrangian*. In [13] McLean studied special Lagrangian submanifolds (and other special calibrated geometries) showing that the Moduli space of deformations of special Lagrangian manifolds of a fixed compact one  $L$  is a smooth manifold of dimension equal to the first Betti number of  $L$ .

In this paper we consider a generalization of Calabi-Yau structures in the context of Sasakian manifolds. Recall that a *Sasakian structure* on a  $2n+1$ -dimensional manifold  $M$  is a pair  $(\alpha, J)$ , where  $\alpha$  is a contact form on  $M$  and  $J$  is an integrable complex structure on  $\xi = \ker \alpha$  calibrated by  $\kappa = (1/2) d\alpha$ . This is equivalent to require the following data: a quadruple  $(\alpha, g, R, J)$ , where  $\alpha$  is a 1-form,  $g$  is a Riemannian

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metric,  $R$  is a unitary Killing vector field,  $J \in \text{End}(TM)$  satisfying

$$J^2 = -I + \alpha \otimes R, \quad g(J \cdot, J \cdot) = g(\cdot, \cdot) - \alpha \otimes \alpha, \quad \alpha(R) = 1$$

and such that the metric cone  $(M \times \mathbb{R}^+, r^2 g + dr \otimes dr)$  endowed with the almost complex structure  $\tilde{J} = J - r\alpha \otimes \partial_r + (1/r)dr \otimes R$  is Kähler, where we extend  $J$  by  $J(\partial_r) = 0$  (see e.g. [1], [2], [12]). These manifolds have been studied by many authors (see e.g. [1], [3], [8], [11], [12] and the references included).

We consider contact Calabi-Yau manifolds which are a special class of Sasakian manifolds: namely a *contact Calabi-Yau manifold* is a  $2n + 1$ -dimensional Sasakian manifold  $(M, \alpha, J)$  endowed with a closed basic complex volume form  $\epsilon$ . It turns out that these manifolds are a special class of null-Sasakian  $\alpha$ -Einstein manifolds. As a direct consequence of the above definition, in a contact Calabi-Yau manifold  $(M, \alpha, J, \epsilon)$  the real part of  $\epsilon$  is a calibration. Furthermore, we have that an  $n$ -dimensional submanifold  $p: L \hookrightarrow M$  of a contact Calabi-Yau manifold admits an orientation making it a calibrated submanifold by  $\Re \epsilon$  if and only if

$$p^*(\alpha) = 0, \quad p^*(\Im \epsilon) = 0.$$

In such a case  $L$  is said to be a *special Legendrian submanifold*. We prove that:

*The moduli space of deformations of special Legendrian submanifolds near a fixed compact one  $L$  is a smooth 1-dimensional manifold.*

Moreover we get the following extension theorem:

*Let  $(M, \alpha_t, J_t, \epsilon_t)$  be a smooth family of contact Calabi-Yau manifolds and let  $p: L \hookrightarrow (M, \alpha_0, J_0, \epsilon_0)$  be a compact special Legendrian submanifold. Then there exists a smooth family of special Legendrian submanifolds  $p_t: L \hookrightarrow (M, \alpha_t, J_t, \epsilon_t)$  that extends  $p: L \hookrightarrow M$  if and only if the cohomology class  $[p^*(\Im \epsilon)]$  vanishes.*

This can be considered a contact version of a theorem of Lu Peng (see [10]) in Calabi-Yau manifolds (see also [14]).

In Section 2 we fix some notation on contact and Sasakian geometry. In Section 3 we define contact Calabi-Yau manifolds and we obtain some simple topological obstructions to the existence of contact Calabi-Yau structures on odd-dimensional manifolds. As a corollary, we get that there are no contact Calabi-Yau structures on odd-dimensional spheres. In Section 4 we study the moduli space of special Legendrian submanifolds, proving the theorems stated above. In Section 5 we classify the 5-dimensional nilmanifolds carrying an invariant contact Calabi-Yau structure. The proof is based on Theorems 21 and 23 of [5]. In the last section we generalize the previous definition to the case of codimension  $r$  proving an extension theorem. Some examples

of contact Calabi-Yau manifolds and special Legendrian submanifolds are carefully described.

## 2. Preliminaries

Let  $M$  be a manifold of dimension  $2n + 1$ . A *contact structure* on  $M$  is a distribution  $\xi \subset TM$  of dimension  $2n$ , such that the defining 1-form  $\alpha$  satisfies

$$(1) \quad \alpha \wedge (d\alpha)^n \neq 0.$$

A 1-form  $\alpha$  satisfying (1) is said to be a *contact form* on  $M$ . Let  $\alpha$  be a contact form on  $M$ ; then there exists a unique vector field  $R_\alpha$  on  $M$  such that

$$\alpha(R_\alpha) = 1, \quad \iota_{R_\alpha} d\alpha = 0,$$

where  $\iota_{R_\alpha} d\alpha$  denotes the contraction of  $d\alpha$  along  $R_\alpha$ . By definition  $R_\alpha$  is called the *Reeb vector field* of the contact form  $\alpha$ . A *contact manifold* is a pair  $(M, \xi)$  where  $M$  is a  $2n + 1$ -dimensional manifold and  $\xi$  is a contact structure. Let  $(M, \xi)$  be a contact manifold and fix a defining (contact) form  $\alpha$ . Then the 2-form  $\kappa = (1/2)d\alpha$  defines a symplectic form on the contact structure  $\xi$ ; therefore the pair  $(\xi, \kappa)$  is a symplectic vector bundle over  $M$ . A *complex structure* on  $\xi$  is the datum of  $J \in \text{End}(\xi)$  such that  $J^2 = -I_\xi$ .

**DEFINITION 2.1.** Let  $\alpha$  be a contact form on  $M$ , with  $\xi = \ker \alpha$  and let  $\kappa = (1/2)d\alpha$ . A complex structure  $J$  on  $\xi$  is said to be  *$\kappa$ -calibrated* if

$$g_J[x](\cdot, \cdot) := \kappa[x](\cdot, J_x \cdot)$$

is a  $J_x$ -Hermitian inner product on  $\xi_x$  for any  $x \in M$ .

The set of  $\kappa$ -calibrated complex structures on  $\xi$  will be denoted by  $\mathfrak{C}_\alpha(M)$ . If  $J$  is a complex structure on  $\xi = \ker \alpha$ , then we extend it to an endomorphism of  $TM$  by setting

$$J(R_\alpha) = 0.$$

Note that such a  $J$  satisfies

$$J^2 = -I + \alpha \otimes R_\alpha.$$

If  $J$  is  $\kappa$ -calibrated, then it induces a Riemannian metric  $g$  on  $M$  given by

$$(2) \quad g := g_J + \alpha \otimes \alpha.$$

Furthermore the Nijenhuis tensor of  $J$  is defined by

$$N_J(X, Y) = [JX, JY] - J[X, JY] - J[Y, JX] + J^2[X, Y]$$

for any  $X, Y \in TM$ . We recall the following

DEFINITION 2.2. A Sasakian structure on a  $2n + 1$ -dimensional manifold  $M$  is a pair  $(\alpha, J)$ , where

- $\alpha$  is a contact form;
- $J \in \mathfrak{C}_\alpha(M)$  satisfies  $N_J = -d\alpha \otimes R_\alpha$ .

The triple  $(M, \alpha, J)$  is said to be a *Sasakian manifold*.

For other characterizations of Sasakian structure see e.g. [1] and [2].

We recall now the definition of basic  $r$ -forms.

DEFINITION 2.3. Let  $(M, \xi)$  be a contact manifold. A differential  $r$ -form  $\gamma$  on  $M$  is said to be *basic* if

$$\iota_{R_\alpha} \gamma = 0, \quad \mathcal{L}_{R_\alpha} \gamma = 0,$$

where  $\mathcal{L}$  denotes the Lie derivative and  $R_\alpha$  is the Reeb vector field of an arbitrary contact form defining  $\xi$ .

We will denote by  $\Lambda_B^r(M)$  the set of basic  $r$ -forms on  $(M, \xi)$ . Note that

$$d\Lambda_B^r(M) \subset \Lambda_B^{r+1}(M).$$

The cohomology  $H_B^\bullet(M)$  of this complex is called the *basic cohomology* of  $(M, \xi)$ .

If  $(M, \alpha, J)$  is a Sasakian manifold, then

$$J(\Lambda_B^r(M)) = \Lambda_B^r(M),$$

where, as usual, the action of  $J$  on  $r$ -forms is defined by

$$J\phi(X_1, \dots, X_r) = \phi(JX_1, \dots, JX_r).$$

Consequently  $\Lambda_B^r(M) \otimes \mathbb{C}$  splits as

$$\Lambda_B^r(M) \otimes \mathbb{C} = \bigoplus_{p+q=r} \Lambda_J^{p,q}(\xi)$$

and, according with this gradation, it is possible to define the cohomology groups  $H_B^{p,q}(M)$ . The  $r$ -forms belonging to  $\Lambda_J^{p,q}(\xi)$  are said to be of *type*  $(p, q)$  with respect

to  $J$ . Note that  $\kappa = (1/2)d\alpha \in \Lambda_j^{1,1}(\xi)$  and it determines a non-vanishing cohomology class in  $H_B^{1,1}(M)$ . The Sasakian structure  $(\alpha, J)$  also induces a natural connection  $\nabla^\xi$  on  $\xi$  given by

$$\nabla_X^\xi Y = \begin{cases} (\nabla_X Y)^\xi & \text{if } X \in \xi \\ [R_\alpha, Y] & \text{if } X = R_\alpha, \end{cases}$$

where the subscript  $\xi$  denotes the projection onto  $\xi$ . One easily gets

$$\nabla_X^\xi J = 0, \quad \nabla_X^\xi g_J = 0, \quad \nabla_X^\xi d\alpha = 0, \quad \nabla_X^\xi Y - \nabla_Y^\xi X = [X, Y]^\xi,$$

for any  $X, Y \in TM$ . Consequently we have

$$\text{Hol}(\nabla^\xi) \subseteq \text{U}(n).$$

Moreover the *transverse Ricci tensor*  $\text{Ric}^T$  is defined as

$$\text{Ric}^T(X, Y) = \sum_{i=1}^{2n} g(\nabla_X^\xi \nabla_{e_i}^\xi e_i - \nabla_{e_i}^\xi \nabla_X^\xi e_i - \nabla_{[X, e_i]}^\xi e_i, Y),$$

for any  $X, Y \in \xi$ , where  $\{e_1, \dots, e_{2n}\}$  is an arbitrary orthonormal frame of  $\xi$ . It is known that  $\text{Ric}^T$  satisfies

$$\text{Ric}^T(X, Y) = \text{Ric}(X, Y) + 2g(X, Y),$$

for any  $X, Y \in \xi$ , where  $\text{Ric}$  denotes the Ricci tensor of the Riemannian metric  $g = g_J + \alpha \otimes \alpha$ . Let us denote by  $\rho^T$  the Ricci form of  $\text{Ric}^T$ , i.e.

$$\rho^T(X, Y) = \text{Ric}^T(JX, Y) = \text{Ric}(JX, Y) + 2\kappa(X, Y),$$

for any  $X, Y \in \xi$ . We recall that  $\rho^T$  is a closed form such that  $(1/(2\pi))\rho$  represents the first Chern class of  $(\xi, J)$  (see e.g. [7]); this form is called the *transverse Ricci form* of  $(\alpha, J)$ .

**DEFINITION 2.4.** The basic cohomology class

$$c_1^B(M) = \frac{1}{2\pi}[\rho^T] \in H_B^{1,1}(M)$$

is called the *first basic Chern class* of  $(M, \alpha, J)$  and, if it vanishes, then  $(M, \alpha, J)$  is said to be *null-Sasakian*.

Furthermore we recall that a Sasakian manifold is called  $\alpha$ -Einstein if there exist  $\lambda, \nu \in C^\infty(M, \mathbb{R})$  such that

$$\text{Ric} = \lambda g + \nu \alpha \otimes \alpha.$$

For general references on these topics see e.g. [4] and [3].

Finally, recall that a submanifold  $p: L \hookrightarrow M$  of a  $2n+1$ -dimensional contact manifold  $(M, \xi)$  is said to be *Legendrian* if:

- 1)  $\dim_{\mathbb{R}} L = n$ ,
- 2)  $p_*(TL) \subset \xi$ .

Observe that, if  $\alpha$  is a defining form of the contact structure  $\xi$ , then condition 2) is equivalent to say that  $p^*(\alpha) = 0$ . Hence Legendrian submanifolds are the analogue of Lagrangian submanifolds in contact geometry.

### 3. Contact Calabi-Yau manifolds

In this section we study contact Calabi-Yau manifolds. As already explained in the introduction, these manifolds are a natural generalization of the Calabi-Yau ones in the context of contact geometry. Roughly speaking a contact Calabi-Yau manifold is a Sasakian manifold endowed with a basic closed complex volume form. We can give now the following

**DEFINITION 3.1.** A *contact Calabi-Yau manifold* is a quadruple  $(M, \alpha, J, \epsilon)$ , where

- $(M, \alpha, J)$  is a  $2n+1$ -dimensional Sasakian manifold;
- $\epsilon \in \Lambda_J^{n,0}(\xi)$  is a nowhere vanishing basic form on  $\xi = \ker \alpha$  such that

$$\begin{cases} \epsilon \wedge \bar{\epsilon} = c_n \kappa^n \\ d\epsilon = 0, \end{cases}$$

where  $c_n = (-1)^{n(n+1)/2} (2i)^n$  and  $\kappa = (1/2) d\alpha$ .

Now we will describe a couple of examples.

**EXAMPLE 3.2.** Consider  $\mathbb{R}^{2n+1}$  endowed with the standard Euclidean coordinates  $\{x_1, \dots, x_n, y_1, \dots, y_n, t\}$ . Let

$$\alpha_0 = 2 dt - 2 \sum_{i=1}^n y_i dx_i$$

be the *standard contact form* on  $\mathbb{R}^{2n+1}$  and let  $\xi_0 = \ker \alpha_0$ . Then  $\xi_0$  is spanned by

$$\{y_1 \partial_t + \partial_{x_1}, \dots, y_n \partial_t + \partial_{x_n}, \partial_{y_1}, \dots, \partial_{y_n}\}.$$

For simplicity, set  $V_i = y_i \partial_t + \partial_{x_i}$ ,  $W_j = \partial_{y_j}$ ,  $i, j = 1, \dots, n$  and

$$\begin{cases} J_0(V_r) = W_r \\ J_0(W_r) = -V_r \end{cases} \quad r = 1, \dots, n.$$

Then  $J_0$  defines a complex structure in  $\mathfrak{C}_\alpha(M)$ . Since the space of transverse 1-forms is spanned by  $\{dx_1, \dots, dx_n, dy_1, \dots, dy_n\}$ , then the complex valued form

$$\epsilon_0 := (dx_1 + i dy_1) \wedge \cdots \wedge (dx_n + i dy_n)$$

is of type  $(n, 0)$  with respect to  $J_0$  and it satisfies

$$\begin{cases} \epsilon_0 \wedge \bar{\epsilon}_0 = c_n \kappa_0^n \\ d\epsilon_0 = 0, \end{cases}$$

where  $\kappa_0 = (1/2) d\alpha_0$ . Therefore  $(\mathbb{R}^{2n+1}, \alpha_0, J_0, \epsilon_0)$  is a contact Calabi-Yau manifold.

The following will describe a compact contact Calabi-Yau manifold.

EXAMPLE 3.3. Let

$$H(3) := \left\{ A = \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} \middle| x, y, z \in \mathbb{R} \right\}$$

be the 3-dimensional Heisenberg group and let  $M = H(3)/\Gamma$ , where  $\Gamma$  denotes the subgroup of  $H(3)$  given by the matrices with integral entries. The 1-forms  $\alpha_1 = dx$ ,  $\alpha_2 = dy$ ,  $\alpha_3 = x dy - dz$  are  $H(3)$ -invariant and therefore they define a global coframe on  $M$ . Then  $\alpha = 2\alpha_3$  is a contact form whose contact distribution  $\xi$  is spanned by  $V = \partial_x$ ,  $W = \partial_y + x\partial_z$ . Again

$$\begin{cases} J(V) = W \\ J(W) = -V \end{cases}$$

defines a  $\kappa$ -calibrated complex structure on  $\xi$  and  $\epsilon = \alpha_1 + i\alpha_2$  is a  $(1, 0)$ -form on  $\xi$  such that  $(M, \alpha, J, \epsilon)$  is a contact Calabi-Yau manifold.

The last example gives an invariant contact Calabi-Yau structure on a nilmanifold. It can be generalized to the dimension  $2n + 1$  in this way: let  $\mathfrak{g}$  be the Lie algebra spanned by  $\{X_1, \dots, X_{2n+1}\}$  with

$$[X_{2k-1}, X_{2k}] = -X_{2n+1}$$

for  $k = 1, \dots, n$  and the other brackets are zero. Then  $\mathfrak{g}$  is a  $2n + 1$ -dimensional nilpotent Lie algebra with rational constant structures and, by Malcev theorem, it follows that if  $G$  is the simply connected Lie group with Lie algebra  $\mathfrak{g}$ , then  $G$  has a compact



quotient. Let  $\{\alpha_1, \dots, \alpha_{2n+1}\}$  be the dual basis of  $\{X_1, \dots, X_{2n+1}\}$ . Then we immediately get

$$d\alpha_1 = 0, \dots, d\alpha_{2n} = 0, \quad d\alpha_{2n+1} = \sum_{k=1}^n \alpha_{2k-1} \wedge \alpha_{2k}.$$

Hence

$$\alpha = 2\alpha_{2n+1},$$

the endomorphism  $J$  of  $\xi = \ker \alpha$  defined by

$$\begin{cases} J(X_{2k-1}) = X_{2k} \\ J(X_{2k}) = -X_{2k-1} \end{cases}$$

for  $k = 1, \dots, n$  and the complex form

$$\epsilon = (\alpha_1 + i\alpha_2) \wedge \dots \wedge (\alpha_{2n-1} + i\alpha_{2n})$$

define a contact Calabi-Yau structure on any compact nilmanifold associated with  $\mathfrak{g}$ .

The following proposition gives simple topological obstructions in order that a compact  $2n + 1$ -dimensional manifold  $M$  carries a contact Calabi-Yau structure.

**Proposition 3.4.** *Let  $M$  be a  $2n + 1$ -dimensional compact manifold. Assume that  $M$  admits a contact Calabi-Yau structure; then the following hold*

1. *if  $n$  is even, then  $b_{n+1}(M) > 0$ ;*
2. *if  $n$  is odd, then*

$$\begin{cases} b_n(M) \geq 2 \\ b_{n+1}(M) \geq 2, \end{cases}$$

where  $b_j(M)$  denotes the  $j^{\text{th}}$  Betti number of  $M$ .

*Proof.* Let  $(\alpha, J, \epsilon)$  be a contact Calabi-Yau structure on  $M$  and let  $\xi = \ker \alpha$ . Set  $\Omega = \Re \epsilon$ ; then, since  $\epsilon \in \Lambda_J^{n,0}(\xi)$ , we have  $\epsilon = \Omega + iJ\Omega$ . In view of the assumption  $d\epsilon = 0$ , we obtain  $d\Omega = dJ\Omega = 0$  and since  $d\alpha \in \Lambda_J^{1,1}(\xi)$  it follows that

$$\Omega \wedge d\alpha = J\Omega \wedge d\alpha = 0.$$

Hence

$$d(\Omega \wedge \alpha) = d(J\Omega \wedge \alpha) = 0.$$

Furthermore we have

$$\begin{aligned} \epsilon \wedge \bar{\epsilon} &= \Omega \wedge \Omega + J\Omega \wedge J\Omega && \text{if } n \text{ is even;} \\ \epsilon \wedge \bar{\epsilon} &= -2i\Omega \wedge J\Omega && \text{if } n \text{ is odd.} \end{aligned}$$

1. If  $n$  is even, then  $\alpha \wedge (\Omega \wedge \Omega + J\Omega \wedge J\Omega)$  is a volume form on  $M$ . Assume that the cohomology classes  $[\Omega \wedge \alpha]$ ,  $[J\Omega \wedge \alpha]$  vanish; then there exist  $\beta, \gamma \in \Lambda^n(M)$  such that

$$\alpha \wedge \Omega = d\beta, \quad \alpha \wedge J\Omega = d\gamma.$$

By Stokes theorem we have

$$\begin{aligned} 0 \neq \int_M \alpha \wedge \Omega \wedge \Omega + \alpha \wedge J\Omega \wedge J\Omega &= \int_M d\beta \wedge \Omega + d\gamma \wedge J\Omega \\ &= \int_M d(\beta \wedge \Omega) + d(\gamma \wedge J\Omega) = 0, \end{aligned}$$

which is absurd. Therefore one of  $[\Omega \wedge \alpha]$ ,  $[J\Omega \wedge \alpha]$  does not vanish. Consequently  $b_{n+1}(M) > 0$ .

2. Let  $n$  be odd. We prove that the cohomology classes  $[\Omega]$  and  $[J\Omega]$  are  $\mathbb{R}$ -independent. Assume that there exist  $a, b \in \mathbb{R}$  such that  $a[\Omega] + b[J\Omega] = 0$ ,  $(a, b) \neq (0, 0)$ . Then there exists  $\beta \in \Lambda^{n-1}(M)$  such that

$$a\Omega + bJ\Omega = d\beta.$$

We may assume that  $a = 1$ , so that  $\Omega = d\beta - bJ\Omega$ . Stokes theorem implies

$$0 \neq \int_M \alpha \wedge \Omega \wedge J\Omega = \int_M \alpha \wedge d\beta \wedge J\Omega = - \int_M d(\alpha \wedge \beta \wedge J\Omega) = 0$$

which is a contradiction. Hence  $b_n(M) \geq 2$ . With the same argument, it is possible to prove that  $b_{n+1}(M) \geq 2$  by showing that  $[\Omega \wedge \alpha]$  and  $[J\Omega \wedge \alpha]$  are  $\mathbb{R}$ -independent in  $H^{n+1}(M, \mathbb{R})$ .  $\square$

The following is an immediate consequence of Proposition 3.4.

**Corollary 3.5.** *A 3-dimensional compact manifold  $M$  admitting contact Calabi-Yau structure has  $b_1(M) \geq 2$ . In particular, there are no compact 3-dimensional simply connected contact Calabi-Yau manifolds. Moreover, the  $2n + 1$ -dimensional sphere has no contact Calabi-Yau structures.*

The following proposition implies that the transverse Ricci tensor of a contact Calabi-Yau manifold vanishes

**Proposition 3.6.** *Let  $(M, \alpha, J)$  be a  $2n + 1$ -dimensional Sasakian manifold and  $\xi = \ker \alpha$ . The following facts are equivalent:*

1.  $\text{Hol}^0(\nabla^\xi) \subseteq \text{SU}(n)$
2.  $\text{Ric}^T = 0$ .

Proof. The connection  $\nabla^\xi$  induces a connection  $\nabla^K$  on  $\Lambda_J^{n,0}(\xi)$  which has  $\text{Hol}(\nabla^K) \subseteq \text{U}(1)$ . Since  $\text{Hol}^0(\nabla^K)$  and  $\text{Hol}^0(\nabla^\xi)$  are related by

$$\text{Hol}^0(\nabla^K) = \det(\text{Hol}^0(\nabla^\xi)),$$

where  $\det$  is the map induced by the determinant  $\text{U}(n) \rightarrow \text{U}(1)$ , then it follows that  $\text{Hol}^0(\nabla^\xi) \subseteq \text{SU}(n)$  if and only if  $\text{Hol}^0(\nabla^K) = \{1\}$  and in this case  $\nabla^K$  is flat. As in the Kähler case it can be showed using transverse holomorphic coordinates (see e.g. [7], [8]) that the curvature form of  $\nabla^K$  coincides with the transverse Ricci form of  $(\alpha, J)$ . Hence  $\text{Hol}^0(\nabla^\xi) \subseteq \text{SU}(n)$  if and only if  $\text{Ric}^T = 0$ .  $\square$

As a consequence of the last proposition we have the following

**Corollary 3.7.** *Let  $(M, \alpha, J, \epsilon)$  be a contact Calabi-Yau manifold. Then  $(M, \alpha, J)$  is null-Sasakian and the metric  $g$  induced by  $(\alpha, J)$  is  $\alpha$ -Einstein with  $\lambda = -2$  and  $\nu = 2n + 2$ . In particular the scalar curvature of the metric  $g$  associated to  $(\alpha, J)$  is equal to  $-2n$ .*

#### 4. Deformations of special Legendrian submanifolds

In this section we are going to study the geometry of Legendrian submanifolds in a contact Calabi-Yau ambient. We will prove a contact version of McLean and Lu Peng theorems (see [13] and [10]).

Let  $(M, \alpha, J, \epsilon)$  be a contact Calabi-Yau manifold of dimension  $2n + 1$ . It easy to see that for any oriented  $n$ -plane  $V \subset T_x M$

$$\Re \epsilon|_V \leq \text{Vol}(V),$$

where  $\text{Vol}(V)$  is computed with respect to the metric  $g$  induced by  $(\alpha, J)$  on  $M$ . Hence  $\Re \epsilon$  is a calibration on  $(M, g)$  (see [9]). We have the following

**Proposition 4.1.** *Let  $p: L \hookrightarrow M$  be an  $n$ -dimensional submanifold. The following facts are equivalent*

1. *the submanifold satisfies*

$$\begin{cases} p^*(\alpha) = 0 \\ p^*(\Im \epsilon) = 0, \end{cases}$$

2. *there exists an orientation on  $L$  making it calibrated by  $\Re \epsilon$ .*

We can give the following

DEFINITION 4.2. An  $n$ -dimensional submanifold  $p: L \hookrightarrow M$  is said to be *special Legendrian* if

$$\begin{cases} p^*(\alpha) = 0 \\ p^*(\Im \epsilon) = 0. \end{cases}$$

It follows that compact special Legendrian submanifolds minimize volume in their homology class and that there are no compact special Legendrian submanifolds in  $(\mathbb{R}^{2n+1}, \alpha_0, J_0, \epsilon_0)$ .

EXAMPLE 4.3. Let  $(M = H(3)/\Gamma, \alpha, J, \epsilon)$  be the contact Calabi-Yau manifold considered in the Example 3.3. Then the submanifold

$$L := \left\{ [A] \in M \mid A = \begin{pmatrix} 1 & x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\} \simeq S^1$$

is a compact special Legendrian submanifold.

Now we define the moduli space of special Legendrian submanifolds.

DEFINITION 4.4. Let  $(M, \alpha, J, \epsilon)$  be a contact Calabi-Yau manifold and let  $p_0: L \hookrightarrow M$ ,  $p_1: L \hookrightarrow M$  be two special Legendrian submanifolds. Then  $p_1: L \hookrightarrow M$  is said to be a *deformation* of  $p_0: L \hookrightarrow M$  if there exists a smooth map  $F: L \times [0, 1] \rightarrow M$  such that

- $F(\cdot, t): L \times \{t\} \rightarrow M$  is a special Legendrian embedding for any  $t \in [0, 1]$ ;
- $F(\cdot, 0) = p_0$ ,  $F(\cdot, 1) = p_1$ .

Let  $(M, \alpha, J, \epsilon)$  be a contact Calabi-Yau manifold and let  $p: L \hookrightarrow M$  be a fixed compact special Legendrian submanifold. Set

$$\mathfrak{M}(L) := \{\text{special Legendrian submanifolds of } (M, \alpha, J, \epsilon)$$

$$\text{which are deformations of } p: L \hookrightarrow M\} / \sim,$$

where two embeddings are considered equivalent if they differ by a diffeomorphism of  $L$ ; then by definition  $\mathfrak{M}(L)$  is the *moduli space of special Legendrian submanifolds* which are deformations of  $p: L \hookrightarrow M$ . We have the following

**Theorem 4.5.** *Let  $(M, \alpha, J, \epsilon)$  be a contact Calabi-Yau manifold and let  $p: L \hookrightarrow M$  be a compact special Legendrian submanifold. Then the moduli space  $\mathfrak{M}(L)$  is a 1-dimensional manifold.*

The next lemma will be useful in the proof of Theorem 4.5:

**Lemma 4.6** ([13], [6]). *Let  $(V, \kappa)$  be a symplectic vector space and let  $i: W \hookrightarrow V$  be a Lagrangian subspace. Then*

1.  $\tau: V/W \rightarrow W^*$  defined as  $\tau([v]) = i^*(\iota_v \kappa)$  is an isomorphism;
2. let  $J$  be a  $\kappa$ -calibrated complex structure on  $V$  and let  $\epsilon \in \Lambda_J^{n,0}(V^*)$  satisfy

$$i^*(\Im \epsilon) = 0, \quad \epsilon \wedge \bar{\epsilon} = c_n \frac{\kappa^n}{n!}.$$

Then  $\theta: V/W \rightarrow \Lambda^{n-1}(W^*)$  defined as  $\theta([v]) := i^*(\iota_v \Im \epsilon)$  is an isomorphism. Moreover for any  $v \in V$ , we have

$$\theta([v]) = - * \tau([v]),$$

where  $*$  is computed with respect to  $i^*(g_J(\cdot, \cdot)) := i^*(\kappa(\cdot, J \cdot))$  and the volume form  $\text{Vol}(W) := i^*(\Re \epsilon)$ .

For the proof of Lemma 4.6 we refer to [13] and [6].

Proof of Theorem 4.5. Let  $\mathcal{N}(L)$  be the normal bundle to  $L$ . Then

$$\mathcal{N}(L) = \langle R_\alpha \rangle \oplus J(p_*(TL)),$$

where  $R_\alpha$  is the Reeb vector field of  $\alpha$ . Let  $Z$  be a vector field normal to  $L$  and let  $\exp_Z: L \rightarrow M$  be defined as

$$\exp_Z(x) := \exp_x(Z(x)).$$

Let  $U$  be a neighborhood of 0 in  $C^{2,\alpha}(\langle R_\alpha \rangle) \oplus C^{1,\alpha}(J(p_*(TL)))$  and let

$$F: U \rightarrow C^{1,\alpha}(\Lambda^1(L)) \oplus C^{0,\alpha}(\Lambda^n(L)),$$

be defined as

$$F(Z) = (\exp_Z^*(\alpha), 2 \exp_Z^*(\Im \epsilon)).$$

We obviously have

$$Z \in F^{-1}((0, 0)) \cap C^\infty(\mathcal{N}(L)) \iff \exp_Z(L) \text{ is a special Legendrian submanifold.}$$

Note that since  $\exp_Z$  and  $p$  are homotopic via  $\exp_{tZ}$ , we have

$$[\exp_Z^*(\Im \epsilon)] = [p^*(\Im \epsilon)] = 0.$$

Therefore

$$F: U \rightarrow C^{1,\alpha}(\Lambda^1(L)) \oplus dC^{1,\alpha}(\Lambda^{n-1}(L)).$$

Let us compute the differential of the map  $F$ .

$$F_*[0](Z) = \frac{d}{dt}(\exp_{tZ}^*(\alpha), 2 \exp_{tZ}^*(\Im \epsilon))|_{t=0} = (p^*(\mathcal{L}_Z \alpha), 2p^*(\mathcal{L}_Z \Im \epsilon)),$$

where  $\mathcal{L}$  denotes the Lie derivative. We may write  $Z = JX + fR_\alpha$ ; then applying Cartan formula we obtain

$$\begin{aligned} F_*[0](Z) &= (p^*(\mathcal{L}_Z \alpha), 2p^*(\mathcal{L}_Z \Im \epsilon)) \\ &= (p^*(d\iota_Z \alpha + \iota_Z d\alpha), 2p^*(d\iota_Z \Im \epsilon)) \\ &= (p^*(d\iota_{JX+fR_\alpha} \alpha + \iota_{JX+fR_\alpha} d\alpha), 2p^*(d\iota_{JX+fR_\alpha} \Im \epsilon)) \\ &= (p^*(d\iota_{fR_\alpha} \alpha + \iota_{JX} d\alpha), 2p^*(d\iota_{JX} \Im \epsilon)) \\ &= (p^*(df + \iota_{JX} d\alpha), 2dp^*(\iota_{JX} \Im \epsilon)). \end{aligned}$$

By applying Lemma 4.6 we get

$$(3) \quad F_*[0](Z) = (d(f \circ p) + p^*(\iota_{JX} d\alpha), -d * p^*(\iota_{JX} d\alpha)),$$

where  $*$  is the Hodge star operator with respect to the metric  $p^*(g_J)$  and the volume form  $p^*(\Re \epsilon)$ . Now we show that  $F_*[0]$  is surjective. Let  $(\eta, d\gamma) \in C^{1,\alpha}(\Lambda^1(L)) \oplus dC^{1,\alpha}(\Lambda^{n-1}(L))$ . By the Hodge decomposition theorem we may assume

$$d\gamma = -d * du \quad \text{with} \quad u \in C^{3,\alpha}(L)$$

and we have

$$\eta = dv + d^* \beta + h(\eta)$$

where  $v \in C^{2,\alpha}(L)$ ,  $\beta \in C^{2,\alpha}(\Lambda^2(L))$  and  $h(\eta)$  denotes the harmonic component of  $\eta$ . Then we get

$$\begin{aligned} (\eta, d\gamma) &= (du - du + dv + d^* \beta + h(\eta), -d * du) \\ &= (dv - du + du + d^* \beta + h(\eta), -d * (du + d^* \beta + h(\eta))). \end{aligned}$$

We can find  $f \in C^{2,\alpha}(p(L))$  and  $X \in C^{1,\alpha}(p_*(TL))$  such that

$$\begin{aligned} f \circ p &= v - u \\ p^*(\iota_{JX} d\alpha) &= du + d^* \beta + h(\eta). \end{aligned}$$

Hence

$$(\eta, d\gamma) = (d(f \circ p) + p^*(\iota_{JX} d\alpha), -d * p^*(\iota_{JX} d\alpha))$$

and  $F_*[0]$  is surjective. Therefore  $(0, 0)$  is a regular value of  $F$ . Now we compute  $\ker F_*[0]$ . Formula (3) implies that  $Z \in \ker F_*[0]$  if and only if

$$(4) \quad d(f \circ p) + p^*(\iota_{JX} d\alpha) = 0$$

$$(5) \quad d^* p^*(\iota_{JX} d\alpha) = 0.$$

By applying  $d^*$  to both sides of (4) and taking into account (5) we get

$$0 = d^* d(f \circ p) + d^* p^*(\iota_{JX} d\alpha) = d^* d(f \circ p),$$

i.e.

$$\Delta(f \circ p) = 0.$$

Since  $L$  is compact  $f$  is constant. Hence (4) reduces to

$$(6) \quad p^*(\iota_{JX} d\alpha) = 0.$$

The map

$$\Theta: p_*(TL) \rightarrow \Lambda^1(L)$$

defined by

$$\Theta(X) = p^*(\iota_{JX} d\alpha)$$

is an isomorphism; hence equation (6) implies  $X = 0$ . Therefore  $Z = W + fR_\alpha$  belongs to  $\ker F_*[0]$  if and only if

$$\begin{cases} W = 0 \\ f = \text{constant.} \end{cases}$$

It follows that  $\ker F_*[0] = \text{Span}_{\mathbb{R}}(R_\alpha) \subset C^\infty(\mathcal{N}(L))$ . The implicit function theorem between Banach spaces implies that the moduli space  $\mathfrak{M}(L)$  is a 1-dimensional smooth manifold.  $\square$

**REMARK 4.7.** Note that the dimension of  $\mathfrak{M}(L)$  does not depend on that of  $L$ . This is quite different from the Calabi-Yau case, where the dimension of the moduli space of deformations of special Lagrangian submanifolds near a fixed compact  $L$  is equal to the first Betti number of  $L$ . This difference can be explained in the following way: the deformations parameterized by curves tangent to the contact structure are trivial, while those one along the Reeb vector field  $R_\alpha$  parameterize the moduli space.

Now we study the following

**Extension problem.** Let  $(M, \alpha_t, J_t, \epsilon_t)$ ,  $t \in (-\delta, \delta)$ , be a smooth family of contact Calabi-Yau manifolds. Given a compact special Legendrian submanifold  $p: L \hookrightarrow M$  of  $(M, \alpha_0, J_0, \epsilon_0)$  does it exist a family  $p_t: L \hookrightarrow M$  of special Legendrian submanifolds of  $(M, \alpha_t, J_t, \epsilon_t)$  such that  $p_0: L \hookrightarrow M$  coincides with  $p$ ?

This is a contact version of the extension problem in the Calabi-Yau case (see [10] and [14]). We can state the following

**Theorem 4.8.** *Let  $(M, \alpha_t, J_t, \epsilon_t)_{t \in (-\delta, \delta)}$  be a smooth family of contact Calabi-Yau manifolds. Let  $p: L \hookrightarrow M$  be a compact special Legendrian submanifold of  $(M, \alpha_0, J_0, \epsilon_0)$ . Then there exists, for small  $t$ , a family of compact special Legendrian submanifolds  $p_t: L \hookrightarrow (M, \alpha_t, J_t, \epsilon_t)$  such that  $p_0 = p$  if and only if the condition*

$$(7) \quad [p^*(\Im \epsilon_t)] = 0$$

holds for  $t$  small enough.

*Proof.* The condition (7) is necessary. Indeed if we can extend  $L$ , then  $\Im \epsilon_t$  is a closed form such that  $p_t^*(\Im \epsilon_t) = 0$ . Since  $p_t$  is homotopic to  $p_0$  we have

$$[p_0^*(\Im \epsilon_t)] = [p_t^*(\Im \epsilon_t)] = 0.$$

In order to prove that condition (7) is sufficient, we can consider the map

$$G: (-\sigma, \sigma) \times C^{1,\alpha}(J(p_*TL)) \rightarrow C^{0,\alpha}(\Lambda^2(L)) \oplus C^{0,\alpha}(\Lambda^n(L))$$

defined as

$$G(t, Z) = (\exp_Z^*(d\alpha_t), 2 \exp_Z^*(\Im \epsilon_t)).$$

By our assumption it follows that

$$\text{Im}(G) \subset dC^{1,\alpha}(\Lambda^1(L)) \oplus dC^{(1,\alpha)}(\Lambda^{n-1}(L)).$$

Let  $X \in C^{1,\alpha}(p_*(TL))$ ; a direct computation and Lemma 4.6 give

$$\begin{aligned} G_*[(0, 0)](0, JX) &= (dp^*(\iota_{JX} d\alpha_0), 2dp^*(\iota_{JX} \Im \epsilon)) \\ &= (dp^*(\iota_{JX} d\alpha_0), -d * p^*(\iota_{JX} d\alpha_0)), \end{aligned}$$

where  $*$  is the Hodge operator of the metric  $p^*(g_J)$  with respect to the volume form  $p^*(\Re \epsilon)$ . It follows that  $G_*[(0, 0)](0, \cdot)$  is surjective and that

$$\ker G_*[(0, 0)]_{\{0\} \times C^{1,\alpha}(p_*(TL))} \cong \mathcal{H}^1(L),$$



where  $\mathcal{H}^1(L)$  denotes the space of harmonic 1-forms on  $L$ . Let

$$A = \{X \in C^{1,\alpha}(p_*(TL)) \mid p^*(\iota_{JX}d\alpha) \in dC^{1,\alpha}(L) \oplus d^*C^{1,\alpha}(\Lambda^2(L))\}$$

and

$$\hat{G} = G_{|(-\delta,\delta)\times A}.$$

Then by the Hodge decomposition of  $\Lambda(L)$  it follows that

$$G_*[(0, 0)]_{\{0\}\times A} : A \rightarrow dC^{1,\alpha}(L) \oplus d^*C^{1,\alpha}(\Lambda^2(L))$$

is an isomorphism. Again by the implicit function theorem and the elliptic regularity there exists a local smooth solution of the equation

$$\hat{G}(t, \psi(t)) = 0.$$

The extension of  $p : L \hookrightarrow M$  is obtained by considering

$$p_t := \exp_{\psi(t)}.$$

□

### 5. The 5-dimensional nilpotent case

In this section we study invariant contact Calabi-Yau structures on 5-dimensional nilmanifolds. We will prove that a compact 5-dimensional nilmanifold carrying an invariant Calabi-Yau structure is covered by a Lie group whose Lie algebra is isomorphic to

$$\mathfrak{g} = (0, 0, 0, 0, 12 + 34),$$

just described in Section 2. Notation  $\mathfrak{g} = (0, 0, 0, 0, 12 + 34)$  means that there exists a basis  $\{\alpha_1, \dots, \alpha_5\}$  of the dual space of the Lie algebra  $\mathfrak{g}$  such that

$$d\alpha_1 = d\alpha_2 = d\alpha_3 = d\alpha_4 = 0, \quad d\alpha_5 = \alpha_1 \wedge \alpha_2 + \alpha_3 \wedge \alpha_4.$$

First of all we note that 5-dimensional contact Calabi-Yau manifolds are in particular hypo. Recall that an *hypo structure* on a 5-dimensional manifold is the datum of  $(\alpha, \omega_1, \omega_2, \omega_3)$ , where  $\alpha \in \Lambda^1(M)$  and  $\omega_i \in \Lambda^2(M)$  and

1.  $\omega_i \wedge \omega_j = \delta_{ij}v$ , for some  $v \in \Lambda^4(M)$  satisfying  $v \wedge \alpha \neq 0$ ;
2.  $\iota_X \omega_1 = \iota_Y \omega_2 \iff \omega_3(X, Y) \geq 0$ ;
3.  $d\omega_1 = 0, d(\omega_2 \wedge \alpha) = 0, d(\omega_3 \wedge \alpha) = 0$ .

These structures have been introduced and studied by D. Conti and S. Salamon in [5]. Let  $(M, \alpha, J, \epsilon)$  be a contact Calabi-Yau manifold of dimension 5. Then

$$\alpha, \quad \omega_1 = \frac{1}{2}d\alpha, \quad \omega_2 = \Re \epsilon, \quad \omega_3 = \Im \epsilon,$$

define an hypo structure on  $M$ .

The following lemma, whose proof is immediate, will be useful in the sequel

**Lemma 5.1.** *Let  $M = G/\Gamma$  be a nilmanifold of dimension 5. If  $M$  admits an invariant contact form, then the Lie algebra of  $G$  is isomorphic to one of the following*

- $(0, 0, 12, 13, 14 + 23)$ ;
- $(0, 0, 0, 12, 13 + 24)$ ;
- $(0, 0, 0, 0, 12 + 34)$ .

Let  $\mathfrak{g}$  be a non-trivial 5-dimensional nilpotent Lie algebra and denote by  $V = \mathfrak{g}^*$  the dual vector space of  $\mathfrak{g}$ . There exists a filtration of  $V$

$$V^1 \subset V^2 \subset V^3 \subset V^4 \subset V^5 = V,$$

with  $dV^i \subset \Lambda^2 V^{i-1}$  and  $\dim_{\mathbb{R}} V^i = i$ . We may choose the filtration  $V$  in such a way that  $V^2 \subset \ker d \subset V^4$ .

Let  $(M = G/\Gamma, \alpha, \omega_1, \omega_2, \omega_3)$  be a nilmanifold endowed with an invariant hypo structure  $(\alpha, \omega_1, \omega_2, \omega_3)$

1. Assume that  $\alpha \in V^4$ . Then we have the following (see [5])

**Theorem 5.2.** *If  $\alpha \in V^4$ , then  $\mathfrak{g}$  is either  $(0, 0, 0, 0, 12)$ ,  $(0, 0, 0, 12, 13)$ , or  $(0, 0, 12, 13, 14)$ .*

In particular if  $(M, \alpha, J, \epsilon)$  is contact Calabi-Yau, then  $\alpha \in V^4$ .

2. Assume that  $\alpha \notin V^4$ . We have (see again [5])

**Lemma 5.3.** *If  $\alpha \notin V^4$  and all  $\omega_i$  are closed, then  $\alpha$  is orthogonal to  $V^4$ .*

**Theorem 5.4.** *If  $\alpha$  is orthogonal to  $V^4$ , then  $\mathfrak{g}$  is one of*

$$(0, 0, 0, 0, 12), \quad (0, 0, 0, 0, 12 + 34).$$

Let  $(M, \alpha, J, \epsilon)$  be a contact Calabi-Yau manifold of dimension 5 endowed with an invariant contact Calabi-Yau structure; then by 1.  $\alpha$  does not belong to  $V^4$ . By Lemma 5.3  $\alpha$  is orthogonal to  $V^4$  and by Theorem 5.4  $\mathfrak{g} = (0, 0, 0, 0, 12 + 34)$ . Hence we have proved the following

**Theorem 5.5.** *Let  $M = G/\Gamma$  be a nilmanifold of dimension 5 admitting an invariant contact Calabi-Yau structure. Then  $\mathfrak{g}$  is isomorphic to*

$$(0, 0, 0, 0, 12 + 34).$$

## 6. Calabi-Yau manifolds of codimension $r$

In this section we extend the definition of contact Calabi-Yau manifold to codimension  $r$  showing the analogous of Theorem 4.8.

Let us consider the following

**DEFINITION 6.1.** Let  $M$  be a  $2n + r$ -dimensional manifold. An  $r$ -contact structure on  $M$  is the datum  $\mathcal{D} = \{\alpha_1, \dots, \alpha_r\}$ , where  $\alpha_i \in \Lambda^1(M)$ , such that

- $d\alpha_1 = d\alpha_2 = \dots = d\alpha_r$ ;
- $\alpha_1 \wedge \dots \wedge \alpha_r \wedge (d\alpha_1)^n \neq 0$ .

Note that if  $\mathcal{D} = \{\alpha_1, \dots, \alpha_r\}$  is an  $r$ -contact structure and  $\xi := \bigcap \ker \alpha_i$ , then  $(\xi, d\alpha_1)$  is a symplectic vector bundle on  $M$  and there exists a unique set of vector fields  $\{R_1, \dots, R_r\}$  satisfying

$$\alpha_i(R_j) = \delta_{ij}, \quad \iota_{R_i} d\alpha_i = 0 \quad \text{for any } i, j = 1, \dots, r.$$

Let us denote by  $\mathfrak{C}_\kappa(\xi)$  the set of complex structures on  $\xi$  calibrated by the symplectic form  $\kappa = (1/2) d\alpha_1$  and by  $\Lambda_0^r(M)$  the set of  $r$ -forms  $\gamma$  on  $M$  satisfying

$$\iota_{R_i} \gamma = 0 \quad \text{for any } i = 1, \dots, r.$$

If  $J \in \mathfrak{C}_\kappa(\xi)$ , then we extend it to  $TM$  by defining

$$J(R_i) = 0.$$

Note that such a  $J$  satisfies

$$J^2 = -I + \sum_{i=1}^r \alpha_i \otimes R_i.$$

Consequently, for any  $J \in \mathfrak{C}_\kappa(\xi)$ , we have  $J(\Lambda_0^r(M)) \subset \Lambda_0^r M$  and a natural splitting of  $\Lambda_0^r(M) \otimes \mathbb{C}$  in

$$\Lambda_0^r(M) \otimes \mathbb{C} = \bigoplus_{p+q=r} \Lambda_J^{p,q}(\xi).$$

We can give the following

**DEFINITION 6.2.** An  $r$ -contact Calabi-Yau manifold is the datum of  $(M, \mathcal{D}, J, \epsilon)$ , where

- $M$  is a  $2n + r$ -dimensional manifold;
- $\mathcal{D} = \{\alpha_1, \dots, \alpha_r\}$  is an  $r$ -contact structure;

- $J \in \mathfrak{C}_k(\xi)$
- $\epsilon \in \Lambda_J^{n,0}(\xi)$  satisfies

$$\begin{cases} \epsilon \wedge \bar{\epsilon} = c_n \kappa^n \\ d\epsilon = 0. \end{cases}$$

EXAMPLE 6.3. Let  $M = H(3)/\Gamma \times S^1$  be the Kodaira-Thurston manifold, where  $H(3)$  is the 3-dimensional Heisenberg group and  $\Gamma$  is the lattice of  $H(3)$  of matrices with integers entries. Let

$$\alpha_1 = -2 dz + 2x dy,$$

$$\alpha_2 = -2 dz + 2x dy + 2 dt.$$

One easily gets

$$d\alpha_1 = d\alpha_2 = 2 dx \wedge dy$$

and that  $\mathcal{D} = \{\alpha_1, \alpha_2\}$  is a 2-contact structure on  $M$ . Note that  $\xi = \ker \alpha_1 \cap \ker \alpha_2$  is spanned by  $\{X_1 = \partial_x, X_2 = \partial_y + x \partial_z\}$ . Moreover the Reeb fields of  $\mathcal{D}$  are

$$R_1 = -\frac{1}{2} \partial_z - \frac{1}{2} \partial_t,$$

$$R_2 = \frac{1}{2} \partial_t.$$

Therefore  $\Lambda_0^1(M)$  is generated by  $\{dx, dy\}$ . Let  $J \in \text{End}(\xi)$  be the complex structure given by

$$J(X_1) = X_2, \quad J(X_2) = -X_1$$

and let  $\epsilon \in \Lambda_J^{1,0}(\xi)$  be the form

$$\epsilon = dx + i dy.$$

Then  $(M, \mathcal{D}, J, \epsilon)$  is a 2-contact Calabi-Yau structure.

As in the contact Calabi-Yau case if  $(M, \mathcal{D}, J, \epsilon)$  is an  $r$ -contact Calabi-Yau manifold, then the  $n$ -form  $\Omega = \Re \epsilon$  is a calibration on  $M$ . Moreover an  $n$ -dimensional submanifold  $p: L \hookrightarrow M$  admits an orientation making it calibrated by  $\Omega$  if and only if

$$p^*(\alpha_i) = 0 \quad \text{for any } \alpha_i \in \mathcal{D},$$

$$p^*(\Im \epsilon) = 0.$$

A submanifold satisfying these equations will be called *special Legendrian*.

EXAMPLE 6.4. Let  $(M, \mathcal{D}, J, \epsilon)$  be the 2-contact Calabi-Yau structure described in Example 6.3. Then

$$L := \left\{ [A] \in H(3)/\Gamma \mid A = \begin{pmatrix} 1 & x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, x \in \mathbb{R} \right\} \times \{q\} \simeq S^1$$

is a compact special Legendrian submanifold for any  $q \in S^1$ .

The proof of the next theorem is very similar to that of Theorem 4.8 and it is omitted.

**Theorem 6.5.** *Let  $(M, \mathcal{D}_t, J_t, \epsilon_t)_{t \in (-\delta, \delta)}$  be a smooth family of  $r$ -contact Calabi-Yau manifolds. Let  $p: L \hookrightarrow M$  be a compact special Legendrian submanifold of  $(M, \mathcal{D}_0, J_0, \epsilon_0)$ . Then there exists, for small  $t$ , a family of compact special Legendrian submanifolds  $p_t: L \hookrightarrow (M, \mathcal{D}_t, J_t, \epsilon_t)$  extending  $p: L \hookrightarrow M$  if and only if the condition*

$$[p^*(\mathfrak{Im} \epsilon_t)] = 0$$

*holds for  $t$  small enough.*

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