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GENERALIZED RADIX REPRESENTATIONS AND DYNAMICAL SYSTEMS III

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Abstract

For $\mathbf{r} = (r_1, \dots, r_d) \in \mathbb{R}^d$ the map $\tau_{\mathbf{r}}\colon \mathbb{Z}^d \to \mathbb{Z}^d$ given by

$$\tau_{\mathbf{r}}(a_1, \dots, a_d) = (a_2, \dots, a_d, -\lfloor r_1 a_1 + \cdots + r_d a_d \rfloor)$$

is called a shift radix system if for each $\mathbf{a} \in \mathbb{Z}^d$ there exists an integer $k > 0$ with $\tau_{\mathbf{r}}^k(\mathbf{a}) = 0$. As shown in the first two parts of this series of papers shift radix systems are intimately related to certain well-known notions of number systems like β-expansions and canonical number systems.

In the present paper further structural relationships between shift radix systems and canonical number systems are investigated. Among other results we show that canonical number systems related to polynomials

$$\sum_{i=0}^{d} p_i X^i \in \mathbb{Z}[X]$$

of degree d with a large but fixed constant term p_0 approximate the set of $(d-1)$-dimensional shift radix systems. The proofs make extensive use of the following tools: Firstly, vectors $\mathbf{r} \in \mathbb{R}^d$ which define shift radix systems are strongly connected to monic real polynomials all of whose roots lie inside the unit circle. Secondly, geometric considerations which were established in Part I of this series of papers are exploited. The main results establish two conjectures mentioned in Part II of this series of papers.

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1. Introduction

Let $d \geq 1$ be an integer and $\mathbf{r} = (r_1, \ldots, r_d) \in \mathbb{R}^d$. To $\mathbf{r}$ we associate the mapping $\tau_{\mathbf{r}} \colon \mathbb{Z}^d \to \mathbb{Z}^d$ in the following way: For $\mathbf{a} = (a_1, \ldots, a_d) \in \mathbb{Z}^d$ let[1]

$$\tau_{\mathbf{r}}(\mathbf{a}) = (a_2, \ldots, a_d, -\lfloor \mathbf{ra} \rfloor),$$

where $\mathbf{ra} = r_1 a_1 + \cdots + r_d a_d$, i.e., the inner product of the vectors $\mathbf{r}$ and $\mathbf{a}$. We call $\tau_{\mathbf{r}}$ a *shift radix system* (*SRS* for short) if for each $\mathbf{a} \in \mathbb{Z}^d$ we can find some $k > 0$ such that the k-th iterate of $\tau_{\mathbf{r}}$ satisfies $\tau_{\mathbf{r}}^k(\mathbf{a}) = 0$.

For $d \in \mathbb{N}$, $d \geq 1$ let

$$\begin{aligned} \mathcal{D}_d &:= \{\mathbf{r} \in \mathbb{R}^d \colon \forall \mathbf{a} \in \mathbb{Z}^d \text{ the sequence } (\tau_{\mathbf{r}}^k(\mathbf{a}))_{k \geq 0} \text{ is ultimately periodic}\} \quad \text{and} \\ \mathcal{D}_d^0 &:= \{\mathbf{r} \in \mathbb{R}^d \colon \forall \mathbf{a} \in \mathbb{Z}^d \ \exists k > 0 \colon \tau_{\mathbf{r}}^k(\mathbf{a}) = 0\}. \end{aligned} \tag{1.1}$$

$\mathcal{D}_d$ is strongly related to the set of contracting polynomials. In particular, let

$$\begin{aligned} \mathcal{E}_d = \mathcal{E}_d(1) := \{(r_1, \ldots, r_d) \in \mathbb{R}^d \colon & X^d + r_d X^{d-1} + \cdots + r_1 \\ & \text{has only roots } y \in \mathbb{C} \text{ with } |y| < 1\}. \end{aligned}$$

In [2, Lemmas 4.1, 4.2 and 4.3] we proved that

$$\operatorname{int}(\mathcal{D}_d) = \mathcal{E}_d. \tag{1.2}$$

$\mathcal{D}_d^0$ is the set of all parameters $\mathbf{r} \in \mathbb{R}^d$ that give rise to an SRS. The structure of $\mathcal{D}_d^0$ is related to the characterization of bases of well known notions of number systems as β-expansions with a certain finiteness property (F) (cf. [5, 7, 11]) and canonical number systems (cf. [6, 8] and see [2, 4] for the link to SRS). In the present paper we dwell mainly on relations between SRS and canonical number systems.

Let $P(X) = p_d X^d + \cdots + p_0 \in \mathbb{Z}[X]$ with $p_0 \geq 2$ and $p_d = 1$, and set $\mathcal{N} = \{0, 1, \ldots, p_0 - 1\}$. Furthermore, denote the image of X under the canonical epimorphism from $\mathbb{Z}[X]$ to $R := \mathbb{Z}[X]/P(X)\mathbb{Z}[X]$ by x. Since $p_d = 1$ it is clear that each coset of R has a unique element of degree at most $d - 1$, say

$$A(X) = A_{d-1}X^{d-1} + \cdots + A_1 X + A_0 \quad (A_0, \ldots, A_{d-1} \in \mathbb{Z}).$$

Let $\mathcal{G} := \{A(X) \in \mathbb{Z}[X] \colon \deg A < d\}$ and

$$T_P(A) = \sum_{i=0}^{d-1} (A_{i+1} - q p_{i+1}) X^i,$$

[1] $\lfloor \cdots \rfloor$ denotes the floor function.

where $A_d = 0$ and $q = \lfloor A_0/p_0 \rfloor$. Then $T_P : \mathcal{G} \to \mathcal{G}$ and

$$A(x) = (A_0 - qp_0) + xT_P(A), \quad \text{where} \quad A_0 - qp_0 \in \mathcal{N}.$$

If for each $A \in \mathcal{G}$ there is a $k \in \mathbb{N}$ such that $T_P^k(A) = 0$ we call P a *canonical number system polynomial* (*CNS polynomial* for short).

Associated to the notion of CNS we define for each $d \in \mathbb{N}$, $d \geq 1$ the sets

$$\mathcal{C}_d := \{(p_0, \ldots, p_{d-1}) \in \mathbb{Z}^d : |p_0| \geq 2 \text{ and } T_{X^d + p_{d-1}X^{d-1} + \cdots + p_0} \text{ has only finite orbits}\}$$

and

$$\mathcal{C}_d^0 := \{(p_0, \ldots, p_{d-1}) \in \mathbb{Z}^d : |p_0| \geq 2 \text{ and } \forall A \in \mathcal{G}\ \exists l \in \mathbb{N} : T^l_{X^d + p_{d-1}X^{d-1} + \cdots + p_0}(A) = 0\}.$$

For $M \in \mathbb{N}_{>0}$ we set

$$\mathcal{C}_d(M) := \left\{ \left(\frac{p_{d-1}}{M}, \ldots, \frac{p_1}{M} \right) \in \mathbb{R}^{d-1} : (M, p_1, \ldots, p_{d-1}) \in \mathcal{C}_d \right\} \tag{1.3}$$

and

$$\mathcal{C}_d^0(M) := \left\{ \left(\frac{p_{d-1}}{M}, \ldots, \frac{p_1}{M} \right) \in \mathbb{R}^{d-1} : (M, p_1, \ldots, p_{d-1}) \in \mathcal{C}_d^0 \right\}. \tag{1.4}$$

Finally, for $x \in \mathbb{R}$ we need the following "cuts" of $\mathcal{D}_d$ and $\mathcal{D}_d^0$.

$$\begin{aligned} \mathcal{D}_d(x) &:= \{(r_2, \ldots, r_d) \in \mathbb{R}^{d-1} : (x, r_2, \ldots, r_d) \in \mathcal{D}_d\}, \\ \mathcal{D}_d^0(x) &:= \{(r_2, \ldots, r_d) \in \mathbb{R}^{d-1} : (x, r_2, \ldots, r_d) \in \mathcal{D}_d^0\}. \end{aligned} \tag{1.5}$$

In Part I of this series of papers (see [2, Section 3]) we studied the relation between SRS and CNS. In particular, we proved that

$$\begin{aligned} &(p_0, p_1, \ldots, p_{d-1}) \in \mathcal{C}_d \ (\text{resp. } \mathcal{C}_d^0) \\ &\text{if and only if} \quad \left(\frac{1}{p_0}, \frac{p_{d-1}}{p_0}, \ldots, \frac{p_1}{p_0} \right) \in \mathcal{D}_d \ (\text{resp. } \mathcal{D}_d^0). \end{aligned} \tag{1.6}$$

In the present paper we investigate a further relationship between the sets $\mathcal{C}_d$ and $\mathcal{D}_d$ as well as $\mathcal{C}_d^0$ and $\mathcal{D}_d^0$. First we show that the elements of $\mathcal{C}_d$ having a large fixed first coordinate p_0 give a very good approximation of $\mathcal{D}_{d-1}$. We will even prove that the appropriately scaled limit for $p_0 \to \infty$ is equal to $\overline{\mathcal{D}_{d-1}}$. We will also prove that the Lebesgue measure of $\mathcal{D}_{d-1}$ is the limit of the frequency of $\mathcal{C}_d(M)$, i.e., of

$$\frac{|\{(p_1, \ldots, p_{d-1}) \in \mathbb{Z}^{d-1} : (M, p_1, \ldots, p_{d-1}) \in \mathcal{C}_d\}|}{M^{d-1}} \tag{1.7}$$

for $M \to \infty$.

The sets $\mathcal{C}_d^0$ and $\mathcal{D}_d^0$ have a considerably more complicated structure than $\mathcal{C}_d$ and $\mathcal{D}_d$. However, from Figs. 1 and 2 of [2] we see that the elements of $\mathcal{C}_d^0$ with fixed first coordinate p_0 seem to give a very good approximation for $\mathcal{D}_{d-1}^0$. In this paper we make this precise in showing that the appropriately scaled limit of $\mathcal{C}_d^0(M)$ for $M \to \infty$ is equal to $\overline{\mathcal{D}_{d-1}^0}$. Furthermore, we prove that the Lebesgue measure of $\mathcal{D}_{d-1}^0$ is the limit of the frequency of $\mathcal{C}_d^0(M)$, i.e., of

$$\frac{|\{(p_1, \dots, p_{d-1}) \in \mathbb{Z}^{d-1} \colon (M, p_1, \dots, p_{d-1}) \in \mathcal{C}_d^0\}|}{M^{d-1}} \tag{1.8}$$

for $M \to \infty$.

These results enable us to gain precise information about the structure of $\mathcal{C}_d$ as well as $\mathcal{C}_d^0$ by studying $\mathcal{D}_{d-1}$ as well as $\mathcal{D}_{d-1}^0$ and vice versa. Specifically, we show that the number of CNS polynomials of a given constant term is estimated by SRS.

The paper is organized as follows. In Section 2 we prove results on the sets $\mathcal{D}_d^0$ which are needed in the sequel. They contain very general facts about $\mathcal{D}_d^0$ and are of interest in their own right. In Section 3 we review different notions of limits of sets which we will need. Sections 4 and 5 contain our results on $\mathcal{D}_d$ while Sections 6 and 7 are devoted to the results on $\mathcal{D}_d^0$.

2. General properties of the sets $\mathcal{D}_d^0$

In order to prove our main results we need the following theorem which is of interest also in its own right. It is well known that $(p_0, \dots, p_{d-1}) \in \mathcal{C}_d^0$ implies that $p_0 \geq 2$ (cf. e.g. [6, Proposition 6]). Thus the first coordinate of a vector $\mathbf{r} \in \mathcal{D}_d^0$ associated to an element of $\mathcal{C}_d^0$ is non-negative. We show that this is true for all elements of $\mathcal{D}_d^0$.

Theorem 2.1. *If* $(r_1, \dots, r_d) \in \mathcal{D}_d^0$ *then* $r_1 \geq 0$.

Proof. Assume that $\mathbf{r} := (r_1, \dots, r_d) \in \mathcal{D}_d$ has $r_1 < 0$. We will prove that in this case there exists some $\mathbf{z} \in \mathbb{Z}^d$ with

$$\tau_{\mathbf{r}}^k(\mathbf{z}) \neq 0 \quad \text{for all} \quad k \in \mathbb{N}. \tag{2.1}$$

This implies that $\mathbf{r} \notin \mathcal{D}_d^0$ and we are done.

Let $R(\mathbf{r})$ be the matrix associated to the mapping $\tau_{\mathbf{r}}$ (see [2, Section 4]). Its characteristic polynomial is given by

$$\chi(X) := X^d + r_d X^{d-1} + \cdots + r_2 X + r_1.$$

Factorize this polynomial in $\mathbb{R}$:

$$\chi(X) = \prod_{i=1}^{u} (X^2 - \xi_i X - \eta_i)^{m_i} \prod_{j=1}^{v} (X - \alpha_j)^{n_j}$$

where $\xi_i, \eta_i, \alpha_j \in \mathbb{R}$ and m_i, n_j are positive integers with $2\sum_i m_i + \sum_j n_j = d$. Since $r_1 < 0$ the polynomial χ has at least one positive real zero. Assume w.l.o.g. that $\alpha_v > 0$. By the structure theorem of finitely generated modules over principal ideal domains, there exists a real regular matrix $S = (s_{lm})$ which gives a real Jordan block decomposition

$$R(\mathbf{r}) = S^{-1} \operatorname{diag}(B_1, \ldots, B_{u+v}) S.$$

Here B_i $(1 \le i \le u+v)$ are the real Jordan blocks

$$B_i := \begin{pmatrix} \xi_i & 1 & & & & & \\ \eta_i & 0 & 1 & & & & \\ & & \xi_i & 1 & & & \\ & & \eta_i & 0 & 1 & & \\ & & & & \ddots & & \\ & & & & & \xi_i & 1 \\ & & & & & \eta_i & 0 \end{pmatrix}$$

of size $2m_i \times 2m_i$ for $i = 1, \ldots, u$ and

$$B_{u+j} := \begin{pmatrix} \alpha_j & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \alpha_j \end{pmatrix}$$

of size $n_j \times n_j$ for $j = 1, \ldots, v$. Now suppose that for a given $\mathbf{y} \in \mathbb{Z}^d$ there exists a $k \in \mathbb{N}$ such that $\tau_{\mathbf{r}}^k(\mathbf{y}) = 0$. Then (see [2, equation (4.2)]) there exist vectors $\mathbf{v}_i = (0, \ldots, 0, c_i)^t$ with $c_i \in [0, 1)$ such that

$$\mathbf{y} = -\sum_{i=1}^{k} R(\mathbf{r})^{-i} \mathbf{v}_i = -S^{-1} \sum_{i=1}^{k} \operatorname{diag}(B_1, \ldots, B_{u+v})^{-i} (S\mathbf{v}_i).$$

Let $(\mathbf{v})_l$ be the l-th coordinate of a vector $\mathbf{v}$. Then it is easy to see that the d-th coordinate of $-S\mathbf{y}$ satisfies

$$(-S\mathbf{y})_d = \sum_{i=1}^{k} \alpha_v^{-i} s_{dd} c_i.$$

Suppose that $s_{dd} \ge 0$. Then $(-S\mathbf{y})_d$ is always non-negative. Thus if we select $\mathbf{z} \in \mathbb{Z}^d$ with $(-S\mathbf{z})_d < 0$ then $\mathbf{z}$ satisfies (2.1) and we are done (note that we can select $\mathbf{z}$ in this way since S is regular). If $s_{dd} < 0$ we can argue in a similar way. This finishes the proof. $\square$

Note that the same proof shows that $(0,\dots,0,r_i,\dots,r_d)\in\mathcal{D}_d^0$ implies $r_i\geq 0$. The following corollaries follow immediately from Theorem 2.1 by using the correspondence results in [2, Theorems 2.1 and 3.1], respectively.

Corollary 2.2. *Let β be a Pisot number with minimal polynomial $X^d-a_1X^{d-1}-\cdots-a_{d-1}X-a_d$. If β has property (F) then $a_d>0$.*

Corollary 2.3. *If $P(X)=X^d+p_{d-1}X^{d-1}+\cdots+p_1X+p_0\in\mathbb{Z}[X]$ is a CNS polynomial then $p_0\geq 2$.*

The following statement is not used in the sequel, but seems to fit into these surroundings.

Theorem 2.4. *Let $q,m\in\mathbb{N}$, $m>0$, $q>1$, $\mathbf{s}\in\mathbb{R}^m$ and*

$$\mathbf{r}=(\underbrace{s_1,0,\dots,0}_{q},\dots,\underbrace{s_m,0,\dots,0}_{q})\in\mathbb{R}^{mq}.$$

Then we have

$$\mathbf{r}\in\mathcal{D}_{mq}^0 \iff \mathbf{s}\in\mathcal{D}_m^0.$$

Proof. This can easily be checked from the definitions. □

3. Review of several notions of convergence of sets

We first summarize three different kinds of convergence of compact sets. We start with the topological limit of a collection (A_n) $(n\in\mathbb{N})$ of sets in a topological space (cf. [9, p.25] or [10, §29]).

- A point z belongs to the (*topological*) *lower limit* $\underline{\mathrm{Lim}}_{n\to\infty}A_n$ if every neighborhood of z intersects all the A_n for n sufficiently large.
- A point z belongs to the (*topological*) *upper limit* $\overline{\mathrm{Lim}}_{n\to\infty}A_n$ if every neighborhood of z intersects A_n for infinitely many values of n.
- The set A is said to be the (*topological*) *limit* of (A_n), for short $A=\mathrm{Lim}_{n\to\infty}A_n$, if $A=\underline{\mathrm{Lim}}_{n\to\infty}A_n=\overline{\mathrm{Lim}}_{n\to\infty}A_n$.

If (A_n) is compact, then $\mathrm{Lim}\,A_n$ is compact, too.

An analogous notion of limit can be defined also for an uncountable collection $(A_x)_{x\in I}$ for some interval $I\subset\mathbb{R}$. In particular, we have:

- A point z belongs to the (*topological*) *lower limit* $\underline{\mathrm{Lim}}_{x\to x_0}A_x$ if every neighborhood of z intersects all the A_x for $|x-x_0|$ sufficiently small.
- A point z belongs to the (*topological*) *upper limit* $\overline{\mathrm{Lim}}_{x\to x_0}A_x$ if every neighborhood of z intersects A_{x_n} for a sequence $(x_n)_{n\geq 1}$ with $\lim x_n=x_0$.

- The set A is said to be the (*topological*) *limit* of (A_x), for short $A = \mathrm{Lim}_{x \to x_0} A_x$, if $A = \underline{\mathrm{Lim}}_{x \to x_0} A_x = \overline{\mathrm{Lim}}_{x \to x_0} A_x$.

Assume that F is metrizable and let p be its compatible metric. For the collection of compact sets in F, the Hausdorff metric associated to p is defined by

$$p_H(A, B) := \max\left(\max_{x \in A} \min_{y \in B} p(x, y), \max_{x \in B} \min_{y \in A} p(x, y)\right)$$

for two non-empty compact sets A and B. For $\varepsilon \in \mathbb{R}_{\geq 0}$ let

$$A[\varepsilon] := \{x \in F : \exists y \in A,\ p(x, y) \leq \varepsilon\} \tag{3.1}$$

be the ε-body of a subset A of F. Note that the ε-body of A can be written as

$$A[\varepsilon] = \bigcup_{x \in A} \overline{B_\varepsilon(x)}$$

where we set

$$B_\rho(x) := \{x' \in F : p(x, x') < \rho\}.$$

Then one has

$$p_H(A, B) = \max\left(\min_{A[\varepsilon] \supset B} \varepsilon, \min_{B[\varepsilon] \supset A} \varepsilon\right).$$

We say that a sequence (A_n) converges to A by the Hausdorff metric if

$$\lim_n p_H(A_n, A) = 0$$

and write $A_n \xrightarrow{H} A$. It is easily seen from the definition that if $A_n \xrightarrow{H} A$ then $\mathrm{Lim}_n A_n = A$. However the converse is not true. For instance, consider the case $F = \mathbb{R}$, $p(x, y) = |x - y|$ and $A_n = \{0, n\}$: Then $\mathrm{Lim}_n A_n = \{0\}$, but $A_n \overset{H}{\nrightarrow} \{0\}$. If there exists a compact set K in F such that $A_n \subset K$ for all n, then $\mathrm{Lim}\, A_n = A$ implies that $A_n \xrightarrow{H} A$ (cf. [9, p.26]).

The third kind of convergence is defined when (F, p) is equipped with a measure. Let ν be an outer measure of F. Assume that ν is a metric outer measure, i.e., $\nu(A \cup B) = \nu(A) + \nu(B)$ holds for any two subsets A and B with $p(A, B) = \inf_{x \in A, y \in B} p(x, y) > 0$. Then ν gives rise to a Borel measure which is written by the same symbol ν.

We say that a sequence (A_n) of sets converges to a set A by the measure ν if

$$\lim_n \nu(A_n \triangle A) = 0$$

where

$$A \triangle B = (A \setminus B) \cup (B \setminus A)$$

is the symmetric difference of A and B. We denote this convergence by $A_n \xrightarrow{\nu} A$. If $A_n \xrightarrow{H} A$, then for any positive ε we have

$$A_n \triangle A \subset (A_n[\varepsilon] \setminus A_n) \cup (A[\varepsilon] \setminus A).$$

As $A[\varepsilon]$ is measurable and $A[\varepsilon] \setminus A$ is decreasing as $\varepsilon \to 0$, we have

$$\lim_{\varepsilon \to 0} \nu(A[\varepsilon] \setminus A) = \nu(A[0] \setminus A) = 0.$$

Therefore $A_n \xrightarrow{H} A$ implies $A_n \xrightarrow{\nu} A$. Let us summarize these results.

Proposition 3.1. *Let (F, p) be a metric space and (A_n) be a sequence of compact subsets of F. Then $A_n \xrightarrow{H} A$ implies $\mathrm{Lim}_n A_n = A$. If there exists a compact set $K \subseteq F$ such that $A_n \subset K$ for all n, then $\mathrm{Lim}_n A_n = A$ implies $A_n \xrightarrow{H} A$. Assume that ν is a metric outer measure on F. Then $A_n \xrightarrow{H} A$ implies $A_n \xrightarrow{\nu} A$.*

The convergence in the Hausdorff metric as well as the convergence with respect to a measure can be defined also for uncountable classes (A_x) of sets in an obvious way.

Let us come back to the Euclidean space. We denote by $\|\cdot\|_2$ (resp. $\|\cdot\|_\infty$) the Euclidean norm (resp. L^∞ norm) and define the metric by $p(x, y) = \|x - y\|_2$. Define, for a non-negative real number ε,

$$A[-\varepsilon] := \{x \in A\colon\, p(x, \partial A) \geq \varepsilon\}. \tag{3.2}$$

4. Convergence properties of the set $\mathcal{D}_d$

The main result of this section is Theorem 4.11 where we prove that the sets $\mathcal{C}_d(M)$ defined in (1.3) yield a good approximation to the closure of $\mathcal{D}_{d-1}$ for $M \to \infty$. In view of (1.2) we use a characterization of the sets $\mathcal{E}_d$ given by Schur [12]. Therefore we need certain determinants which we define now.

For $\rho \in \mathbb{R}$, $\nu \in \{0, \ldots, d-1\}$ and $\mathbf{r} = (r_1, \ldots, r_d) \in \mathbb{R}^d$ we denote by $\delta_\nu(\mathbf{r}, \rho)$ the determinant of the ν-th Schur-Cohn matrix of the monic polynomial $X^d + r_d X^{d-1} +$

$\cdots + r_2X + r_1 \in \mathbb{R}[X]$ whose roots are bounded by ρ (cf. [12]), i.e.,

$$\delta_\nu(\mathbf{r}, \rho) = \det\begin{pmatrix} \rho^d & 0 & \dots & 0 & r_1 & \rho r_2 & \dots & \rho^\nu r_{\nu+1} \\ \rho^{d-1}r_d & \rho^d & \dots & 0 & 0 & r_1 & \dots & \rho^{\nu-1}r_\nu \\ \vdots & & \ddots & \vdots & & & \ddots & \vdots \\ \rho^{d-\nu}r_{d-\nu+1} & \rho^{d-\nu+1}r_{d-\nu+2} & \cdots & \rho^d & 0 & 0 & \cdots & r_1 \\ r_1 & 0 & \dots & 0 & \rho^d & \rho^{d-1}r_d & \dots & \rho^{d-\nu}r_{d-\nu+1} \\ \rho r_2 & r_1 & \dots & 0 & 0 & \rho^d & \dots & \rho^{d-\nu+1}r_{d-\nu+2} \\ \vdots & & \ddots & \vdots & & & \ddots & \vdots \\ \rho^\nu r_{\nu+1} & \rho^{\nu-1}r_\nu & \cdots & r_1 & 0 & 0 & \cdots & \rho^d \end{pmatrix}.$$

To distinguish values and variables, we introduce indeterminants $R_1, \dots, R_d$.

Lemma 4.1. *For each* $\nu \in \{0, \dots, d-1\}$ *and* $\rho \in (0, 1]$

$$R_1 \nmid \delta_\nu((R_1, \dots, R_d), \rho)$$

holds.

Proof. We prove this assertion by induction on ν. Clearly, the assertion holds for $\nu = 0$ because

$$\delta_0((R_1, \dots, R_d), \rho) = \det\begin{pmatrix} \rho^d & R_1 \\ R_1 & \rho^d \end{pmatrix} = \rho^{2d} - R_1^2.$$

Now assume that $R_1 \nmid \delta_\nu((R_1, \dots, R_k), \rho)$, i.e., $\delta_\nu((0, R_2, \dots, R_k), \rho) \neq 0$ holds for all $0 \leq \nu \leq k < d-1$. Consider $\delta_{\nu+1}((0, R_2, \dots, R_d), \rho)$. By the construction of $\delta_\nu(\mathbf{r}, \rho)$ the $(\nu+1)$-st and the $(\nu+2)$-nd column contain only zeros up to one single ρ^d in the $(\nu+1)$-st and the $(\nu+2)$-nd row, respectively. Applying the Laplace expansion of determinants,

$$\delta_{\nu+1}((0, R_2, \dots, R_d), \rho) = \rho^{2d}\delta_\nu((R_2, \dots, R_d), \rho).$$

As the polynomial on the right hand side is nonzero by the induction hypothesis we get

$$\delta_{\nu+1}((0, R_2, \dots, R_d), \rho) \neq 0, \quad \text{i.e.,} \quad R_1 \nmid \delta_{\nu+1}((R_1, R_2, \dots, R_d), \rho)$$

and we are done. □

An algebraic set in $\mathbb{R}^d$ is the locus of real roots of non-zero polynomials of $\mathbb{R}[R_1, \dots, R_d]$. It is obvious from Fubini's theorem that the d-dimensional Lebesgue measure of an algebraic set is zero. In what follows we need the projection

$$\begin{aligned} \mathrm{proj}\colon \mathbb{R}^d &\to \mathbb{R}^{d-1}, \\ (r_1, \dots, r_d) &\mapsto (r_2, \dots, r_d), \end{aligned}$$

and for $x \in \mathbb{R}$ and $f \in \mathbb{R}[R_1, \dots, R_d]$ we set

$$A_f(x) := \{(x, r_2, \dots, r_d) \in \mathbb{R}^d : f(x, r_2, \dots, r_d) > 0\}. \tag{4.1}$$

Lemma 4.2. *Let $d \geq 2$ and $f \in \mathbb{R}[R_1, \dots, R_d]$ such that*

$$R_1 \nmid f(R_1, \dots, R_d).$$

Then for any compact set W in $\mathbb{R}^{d-1}$,

$$\lim_{x \to 0} \lambda_{d-1}((\operatorname{proj}(A_f(x)) \triangle \operatorname{proj}(A_f(0))) \cap W) = 0.$$

REMARK 4.3. The lemma obviously remains true if we replace "$>$" in the definition of $A_f(x)$ by "$\geq$", "$<$" or "$\leq$".

Proof. By the assumption, there exist $g \in \mathbb{R}[R_1, \dots, R_d]$ and $0 \neq h \in \mathbb{R}[R_2, \dots, R_d]$ such that $f = R_1 g + h$. Since

$$\begin{aligned}
&\operatorname{proj}(A_f(x)) \setminus \operatorname{proj}(A_f(0)) \\
&= \{(r_2, \dots, r_d) \colon f(x, r_2, \dots, r_d) > 0 \text{ and } f(0, r_2 \dots, r_d) \leq 0\} \\
&= \{(r_2, \dots, r_d) \colon -xg(x, r_2, \dots, r_d) < h(r_2, \dots, r_d) \leq 0\}
\end{aligned}$$

and

$$\begin{aligned}
&\operatorname{proj}(A_f(0)) \setminus \operatorname{proj}(A_f(x)) \\
&= \{(r_2, \dots, r_d) \colon f(0, r_2, \dots, r_d) > 0 \text{ and } f(x, r_2 \dots, r_d) \leq 0\} \qquad (4.2)\\
&= \{(r_2, \dots, r_d) \colon 0 < h(r_2, \dots, r_d) \leq -xg(x, r_2, \dots, r_d)\}
\end{aligned}$$

we get

$$\begin{aligned}
&(\operatorname{proj}(A_f(0)) \triangle \operatorname{proj}(A_f(x))) \cap W \\
&\subset \{(r_2, \dots, r_d) \in W : |h(r_2, \dots, r_d)| \leq |xg(x, r_2, \dots, r_d)|\}.
\end{aligned}$$

As W is compact, $|xg(x, r_2, \dots, r_d)| \to 0$ uniformly as $x \to 0$. Noting

$$\{(r_2, \dots, r_d) \in W : |h(r_2, \dots, r_d)| \leq \varepsilon\}$$

is measurable and decreasing as $\varepsilon \to 0$, we have

$$\begin{aligned}
&\lim_{\varepsilon \to 0} \lambda_{d-1}(\{(r_2, \dots, r_d) \in W : |h(r_2, \dots, r_d)| \leq \varepsilon\}) \\
&= \lambda_{d-1}(\{(r_2, \dots, r_d) \in W : h(r_2, \dots, r_d) = 0\}).
\end{aligned}$$

The last measure is 0 since $\{(r_2, \dots, r_d) \colon h(r_2, \dots, r_d) = 0\}$ is an algebraic set defined by $0 \neq h \in \mathbb{R}[R_2, \dots, R_d]$. $\square$

Lemma 4.4. *Let $I \subseteq \mathbb{R}$ be an interval, $x_0 \in I$ and $\{M_i(x) \subset \mathbb{R}^d : x \in I,\ i = 1, \ldots, m\}$ be a family of Lebesgue measurable sets with $\lambda_d(M_i(x))$ finite ($x \in I$, $i = 1, \ldots, m$). Furthermore, assume that*

$$\lim_{x \to x_0} \lambda_d(M_i(x) \triangle M_i(x_0)) = 0 \quad (i = 1, \ldots, m).$$

Then the following assertions hold.
(i) $\lim_{x\to x_0} \lambda_d\left(\bigcap_{i=1}^m M_i(x) \triangle \bigcap_{i=1}^m M_i(x_0)\right) = 0$,
(ii) $\lim_{x\to x_0} \lambda_d\left(\bigcup_{i=1}^m M_i(x) \triangle \bigcup_{i=1}^m M_i(x_0)\right) = 0$,
(iii) $\lim_{x\to x_0} \lambda_d((M_1(x) \setminus M_2(x)) \triangle (M_1(x_0) \setminus M_2(x_0))) = 0$.

Proof. We clearly may assume $m = 2$. Moreover, we only prove the first assertion. The other ones follow similarly. Let $\varepsilon \in \mathbb{R}_{>0}$. By our assumptions we can find some $\delta \in \mathbb{R}_{>0}$ with

$$\lambda_d(M_1(x) \triangle M_1(x')) < \frac{\varepsilon}{2} \quad \text{and} \quad \lambda_d(M_2(x) \triangle M_2(x')) < \frac{\varepsilon}{2}$$

for all $x, x' \in I$ with $|x - x'| < \delta$. Therefore using

$$(M_1(x) \cap M_2(x)) \triangle (M_1(x') \cap M_2(x')) \subseteq (M_1(x) \triangle M_1(x')) \cup (M_2(x) \Delta M_2(x'))$$

we find

$$\begin{aligned}
&\lambda_d((M_1(x) \cap M_2(x)) \triangle (M_1(x') \cap M_2(x')))\\
&\le \lambda_d((M_1(x) \triangle M_1(x')) \cup (M_2(x) \Delta M_2(x')))\\
&\le \lambda_d(M_1(x) \triangle M_1(x')) + \lambda_d(M_2(x) \Delta M_2(x'))\\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$ □

Lemma 4.5. *Let $d \ge 2$ and $f \in \mathbb{R}[R_1, \ldots, R_d]$ such that*

$$R_1 \nmid f(R_1, \ldots, R_d).$$

Then for any compact set W in $\mathbb{R}^{d-1}$,

$$\operatorname*{Lim}_{x\to 0} \operatorname{proj}(A_f(x)) \cap W = \overline{\operatorname{proj}(A_f(0))} \cap W.$$

Proof. We prove two opposite inclusions.

- $\operatorname{proj}(A_f(0)) \cap W \subseteq \underline{\operatorname{Lim}}_{x\to 0} \operatorname{proj}(A_f(x)) \cap W$.

Suppose that $y = (s_2, \ldots, s_d) \in \operatorname{proj}(A_f(0)) \cap W$. Then, using the notation of the proof of Lemma 4.2 we have $h(y) = c > 0$ for some fixed constant c. This implies that there exists an $x_0 > 0$ such that $-xg(x, s_2, \dots, s_d) < c$ holds for $|x| < x_0$. Equation (4.2) now implies that $y \in \operatorname{proj}(A_f(x)) \cap W$ holds for all these x and then clearly

$$\operatorname{proj}(A_f(0)) \cap W \subseteq \underline{\operatorname{Lim}}_{x\to 0} \operatorname{proj}(A_f(x)) \cap W.$$

- $\overline{\operatorname{proj}(A_f(0))} \cap W \supseteq \overline{\operatorname{Lim}}_{x\to 0} \operatorname{proj}(A_f(x)) \cap W$.

Suppose that $y \in \overline{\operatorname{Lim}}_{x\to 0} \operatorname{proj}(A_f(x)) \cap W$. Then for each neighborhood U of y there is (x_n) with $x_n \to 0$ and $\operatorname{proj}(A_f(x_n)) \cap U \neq \emptyset$. We have to prove that

$$y \in \{(r_2, \dots, r_d) \colon f(0, r_2, \dots, r_d) \geq 0\}.$$

Suppose at the contrary that

$$y \in \{(r_2, \dots, r_d) \colon f(0, r_2, \dots, r_d) < 0\}.$$

By the continuity of f this implies that

$$y \in \{(r_2, \dots, r_d) \colon f(x, r_2, \dots, r_d) < 0\}$$

also holds for x small enough. Thus there is a neighborhood U_0 of y such that

$$U_0 \subset \{(r_2, \dots, r_d) \colon f(x, r_2, \dots, r_d) < 0\}$$

for all x that are small enough. This is a contradiction because it implies that

$$\operatorname{proj}(A_f(x_n)) \cap U_0 = \emptyset$$

for n large enough. $\square$

REMARK 4.6. Let $B_i(x)$ ($i \in \{1, \dots, n\}$) be finite unions of finite intersections of $A_f(x)$ for some finite family of f's. Then

$$\operatorname{Lim} \bigcup_i B_i(x) = \bigcup_i \operatorname{Lim} B_i(x),$$
$$\operatorname{Lim} \bigcap_i B_i(x) = \bigcap_i \operatorname{Lim} B_i(x).$$

The first assertion follows from a general property of Lim (see [10, §29, VI]). The second assertion can be proved by slightly modifying the proof of Lemma 4.5 (instead of a set

$$\{(r_2, \dots, r_d) \colon f(0, r_2, \dots, r_d) < 0\}$$

restricted by one strict inequality we get unions of sets restricted by several strict inequalities).

Note that in the previous lemma as well as in this remark we need strict inequalities in the definition of $A_f(x)$. Otherwise the results do not hold.

In the following we denote by $\rho(\mathbf{r})$ the maximum of the absolute values of the roots of the polynomial $X^d + r_d X^{d-1} + \cdots + r_1 \in \mathbb{R}[X]$ for $\mathbf{r} \in \mathbb{R}^d$, and for $x, \varepsilon \in \mathbb{R}$ we let

$$\begin{aligned} \mathcal{D}_{d,\varepsilon} &:= \{\mathbf{r} \in \mathcal{D}_d : \rho(\mathbf{r}) < 1 - \varepsilon\}, \quad \mathcal{D}_{d,\varepsilon}(x) := \mathcal{D}_{d,\varepsilon} \cap W(x), \\ \mathcal{D}^0_{d,\varepsilon} &:= \{\mathbf{r} \in \mathcal{D}^0_d : \rho(\mathbf{r}) < 1 - \varepsilon\}, \quad \mathcal{D}^0_{d,\varepsilon}(x) := \mathcal{D}^0_{d,\varepsilon} \cap W(x), \end{aligned} \tag{4.3}$$

where we fixed some positive $M \in \mathbb{R}$ with

$$\mathcal{E}_d \subseteq [-M, M]^d \tag{4.4}$$

and let

$$W(x) = \{x\} \times [-M, M]^{d-1}. \tag{4.5}$$

Note that by [2, Section 4] we have for $\varepsilon \in (0, 1)$

$$\begin{aligned} \mathcal{D}_{d,\varepsilon} &= \{\mathbf{r} \in \mathbb{R}^d : \rho(\mathbf{r}) < 1 - \varepsilon\} \\ &= \bigcap_{\nu=0}^{d-1} \{\mathbf{r} \in \mathbb{R}^d : \delta_\nu(\mathbf{r}, 1 - \varepsilon) > 0\}. \end{aligned} \tag{4.6}$$

Lemma 4.7. *The following two assertions hold.*

(i) *For all $\delta \in (0, 1)$ there is a $\varepsilon \in \mathbb{R}_{>0}$ such that*

$$\lambda_d(\mathcal{D}_d \setminus \mathcal{D}_{d,\varepsilon}) < \delta.$$

(ii) *We have*

$$\lim_{\varepsilon \to 0} \lambda_d(\mathcal{D}^0_{d,\varepsilon}) = \lambda_d(\mathcal{D}^0_d).$$

Proof. (i) Observe that for $\varepsilon_1, \varepsilon_2 \in [0, 1)$ with $\varepsilon_1 \geq \varepsilon_2$ we have

$$\{\mathbf{r} \in \mathbb{R}^d : \delta_\nu(\mathbf{r}, 1 - \varepsilon_1) > 0\} \subseteq \{\mathbf{r} \in \mathbb{R}^d : \delta_\nu(\mathbf{r}, 1 - \varepsilon_2) > 0\}$$

and use the fact that $\partial\mathcal{E}_d$ is a union of algebraic sets and therefore $\lambda_d(\partial\mathcal{E}_d) = 0$.

(ii) Because of

$$\lambda_d(\mathcal{D}^0_{d,\varepsilon}) \leq \lambda_d(\mathcal{D}^0_d) \leq \lambda_d(\mathcal{D}^0_{d,\varepsilon} \cup (\mathcal{D}_d \setminus \mathcal{D}_{d,\varepsilon})) \leq \lambda_d(\mathcal{D}^0_{d,\varepsilon}) + \lambda_d(\mathcal{D}_d \setminus \mathcal{D}_{d,\varepsilon})$$

the assertion follows from (i). □

Lemma 4.8. *Let $\varepsilon \geq 0$ be arbitrary. Then*

$$\lim_{x\to 0} \lambda_{d-1}(\mathrm{proj}(\mathcal{D}_{d,\varepsilon}(x)) \triangle \overline{\mathcal{D}_{d-1,\varepsilon}}) = 0.$$

Proof. Using the notation defined in (4.1) let

$$D_\nu(x) := A_{\delta_\nu((R_1,\ldots,R_d),1-\varepsilon)}(x) \cap [-M, M]^d \quad (\nu = 0, \ldots, d-1).$$

Then in view of (4.3) we have

$$\mathcal{D}_{d,\varepsilon}(x) = \bigcap_{\nu=0}^{d-1} D_\nu(x). \tag{4.7}$$

From Lemma 4.1 we know that $R_1 \nmid \delta_\nu((R_1, \ldots, R_d), 1-\varepsilon)$. Thus we may apply Lemma 4.2 to conclude that

$$\lim_{x\to 0} \lambda_{d-1}(\mathrm{proj}(D_\nu(x)) \triangle \mathrm{proj}(D_\nu(0))) = 0. \tag{4.8}$$

Now we combine (4.7) and (4.8) to derive from Lemma 4.4 (i) that

$$\lim_{x\to 0} \lambda_{d-1}(\mathrm{proj}(\mathcal{D}_{d,\varepsilon}(x)) \triangle \mathrm{proj}(\mathcal{D}_{d,\varepsilon}(0))) = 0.$$

Since $\mathrm{proj}(\mathcal{D}_{d,\varepsilon}(0)) = \overline{\mathcal{D}_{d-1,\varepsilon}}$ the lemma is proved. □

Theorem 4.9. *For each $d \geq 2$ we have*

$$\lim_{x\to 0} \lambda_{d-1}(\mathcal{D}_d(x) \triangle \mathcal{D}_{d-1}) = 0.$$

Proof. This is an easy consequence of Lemmas 4.7 and 4.8 since ε can be chosen arbitrarily small. □

Theorem 4.10. *Let $\mathcal{D}_d(x)$ and $\mathcal{D}_d$ be defined as in* Section 1. *Then*

$$\operatorname*{Lim}_{x\to 0} \mathcal{D}_d(x) = \overline{\mathcal{D}_{d-1}}.$$

Proof. Using the notation defined in (4.1) let

$$D_\nu(x) := A_{\delta_\nu((R_1,\ldots,R_d),1)}(x) \cap [-M, M]^d \quad (\nu = 0, \ldots, d-1).$$

Then we have

$$\operatorname{int}(\mathcal{D}_d(x)) = \bigcap_{\nu=0}^{d-1} D_\nu(x).$$

From Lemma 4.1 we know that $R_1 \nmid \delta_\nu((R_1, \ldots, R_d), 1-\varepsilon)$. Thus we may apply Lemma 4.5 and Remark 4.6 to obtain the result. □

Theorem 4.11. *Let $\mathcal{C}_d(M)$ and $\mathcal{D}_{d-1}$ be given as in* (1.3) *and* (1.1)*, respectively. Then*

$$\operatorname*{Lim}_{M\to\infty} \mathcal{C}_d(M) = \overline{\mathcal{D}_{d-1}}$$

holds for all $d \geq 2$.

Proof. We first show that $\overline{\mathcal{D}_{d-1}} \subseteq \underline{\mathrm{Lim}}_{M\to\infty}\, \mathcal{C}_d(M)$. Since $\overline{\mathcal{D}_{d-1}}$ is the closure of its interior[2] and $\underline{\mathrm{Lim}}_{M\to\infty}\, \mathcal{C}_d(M)$ is closed it suffices to show that $\operatorname{int}(\mathcal{D}_{d-1}) \subseteq \underline{\mathrm{Lim}}_{M\to\infty}\, \mathcal{C}_d(M)$.

Let

$$\mathbf{y} := (r_2, \ldots, r_d) \in \operatorname{int}(\mathcal{D}_{d-1}). \tag{4.9}$$

We have to show that each neighborhood of $\mathbf{y}$ intersects all but finitely many of the sets $\mathcal{C}_d(M)$. Choose an arbitrary neighborhood U of $\mathbf{y}$. Using [2, Lemmas 4.1 and 4.3] we see that (4.9) implies that the polynomial

$$X^{d-1} + r_d X^{d-2} + \cdots + r_2$$

is contractive. Since the roots of a polynomial vary continuously with respect to its coefficients, there exists a positive constant ε with the following properties:

- The polynomial

$$X^d + t_d X^{d-1} + \cdots + t_2 X + t_1 \tag{4.10}$$

is contractive if $|t_i - r_i| < \varepsilon$ $(i = 2, \ldots, d)$ and $|t_1| < \varepsilon$.
- $B_\varepsilon(\mathbf{y}) \subset U \cap \operatorname{int}(\mathcal{D}_{d-1})$.

Thus for each $M > 1/\varepsilon$ we can choose t_i of the form

$$t_1 = t_1(M) = \frac{1}{M} \quad \text{and} \quad t_i = t_i(M) = \frac{p_i^{(M)}}{M} \quad (i = 2, \ldots, d)$$

[2]This is an easy consequence of the following fact: Suppose that $p(x)$ is a polynomial all of whose roots are contained in the closed unit circle. Then by arbitrarily small modifications of its coefficients we can obtain a polynomial all of whose roots are contained in the open unit circle.

with integers $p_i^{(M)}$. By the choice of ε, $\mathbf{y}_M := (t_1(M),\dots,t_d(M)) \in U$ for each $M > 1/\varepsilon$. On the other hand, since the polynomial (4.10) associated to $\mathbf{y}_M$ is contractive, [2, Lemma 4.2 (1)] implies that $\mathbf{y}_M \in \mathcal{D}_d$ for $M > 1/\varepsilon$. Now (1.3) and (1.6) imply that $\mathbf{y}_M \in \mathcal{C}_d(M)$ for $M > 1/\varepsilon$. Thus U intersects all but finitely many of the sets $\mathcal{C}_d(M)$. Since U was arbitrary this proves $\operatorname{int}(\mathcal{D}_{d-1}) \subseteq \underline{\operatorname{Lim}}_{M\to\infty} \mathcal{C}_d(M)$.

It remains to establish the inclusion $\overline{\mathcal{D}_{d-1}} \supseteq \overline{\operatorname{Lim}}_{M\to\infty} \mathcal{C}_d(M)$. We argue in a similar way as in the proof of [2, Theorem 3.1] and obtain

$$\begin{aligned}
\mathcal{C}_d(M) &= \left\{\left(\frac{p_{d-1}}{M},\dots,\frac{p_1}{M}\right) : (M, p_1, \dots, p_{d-1}) \in \mathcal{C}_d\right\}\\
&= \left\{\left(\frac{p_{d-1}}{M},\dots,\frac{p_1}{M}\right) : \left(\frac{1}{M}, \frac{p_{d-1}}{M},\dots,\frac{p_1}{M}\right) \in \mathcal{D}_d\right\}\\
&\subset \left\{(r_2,\dots,r_d) : \left(\frac{1}{M}, r_2,\dots,r_d\right) \in \mathcal{D}_d\right\} = \mathcal{D}_d\left(\frac{1}{M}\right).
\end{aligned}$$

Thus, using Theorem 4.10 we gain

$$\overline{\operatorname{Lim}}_{M\to\infty} \mathcal{C}_d(M) \subseteq \overline{\operatorname{Lim}}_{M\to\infty} \mathcal{D}_d\left(\frac{1}{M}\right) = \overline{\mathcal{D}_{d-1}}$$

and we are done. □

5. Relations between $\mathcal{D}_{d-1}$ and $\mathcal{C}_d$

In the next theorem we prove that the $(d-1)$-dimensional Lebesgue measure of $\mathcal{D}_{d-1}$ is the limit of the quotient (1.7) for $M \to \infty$. Note that the Lebesgue measurability of $\mathcal{D}_d$ is proved in [2, Theorem 4.10].

We need the following notations. Let

$$W(\mathbf{x}, s) := \left\{\mathbf{x}' \in \mathbb{R}^{d-1} : \|\mathbf{x}' - \mathbf{x}\|_\infty \le \frac{s}{2}\right\} \quad (\mathbf{x} \in \mathbb{R}^{d-1},\ s \in \mathbb{R}) \tag{5.1}$$

and

$$\mathcal{W}_d(M) := \bigcup_{\mathbf{x}\in\mathcal{C}_d(M)} W(\mathbf{x}, M^{-1}).$$

Theorem 5.1. *Let $d \ge 2$, M a positive integer and set*

$$N(d, M) := |\{(p_1,\dots,p_{d-1}) \in \mathbb{Z}^{d-1} : (M, p_1,\dots,p_{d-1}) \in \mathcal{C}_d\}|.$$

Then

$$\lim_{M\to\infty} \frac{N(d, M)}{M^{d-1}} = \lambda_{d-1}(\mathcal{D}_{d-1}).$$

Proof. We obviously have

$$\frac{N(d, M)}{M^{d-1}} = \lambda_{d-1}(\mathcal{W}_d(M)). \tag{5.2}$$

We will compare the latter with the Lebesgue measure of $\mathcal{D}_{d-1}$.

We first claim

$$\overline{\mathcal{D}_{d-1}} \setminus \mathcal{W}_d(M) \subseteq \overline{\mathcal{D}_{d-1}} \setminus \left((\mathcal{D}_d(M^{-1})) \left[-\frac{\sqrt{d}}{M} \right] \right). \tag{5.3}$$

To prove the claim we will show

$$(\mathcal{D}_d(M^{-1})) \left[-\frac{\sqrt{d}}{M} \right] \subset \mathcal{W}_d(M).$$

By the definition of the norm $\|\cdot\|_\infty$, if $\mathbf{y} \in \mathcal{D}_d(M^{-1})[-\sqrt{d}/M]$ then

$$W(\mathbf{y}, M^{-1}) \subset \mathcal{D}_d(M^{-1}). \tag{5.4}$$

Thus we can choose $p_1, \ldots, p_{d-1} \in \mathbb{Z}$ with

$$\mathbf{z} = \left(\frac{p_{d-1}}{M}, \ldots, \frac{p_1}{M} \right) \in W(\mathbf{y}, M^{-1}).$$

Now (1.6) and (5.4) imply that $\mathbf{z} \in \mathcal{C}_d(M) \cap W(\mathbf{y}, M^{-1})$. Thus $\mathbf{y} \in W(\mathbf{z}, M^{-1})$ for a $\mathbf{z} \in \mathcal{C}_d(M)$ which yields $\mathbf{y} \in \mathcal{W}_d(M)$ and the claim is proved.

On the other hand $\mathcal{C}_d(M) \subset \mathcal{D}_d(M^{-1})$ implies

$$\mathcal{W}_d(M) \subseteq (\mathcal{D}_d(M^{-1})) \left[\frac{1}{M} \right]$$

which yields

$$\mathcal{W}_d(M) \setminus \overline{\mathcal{D}_{d-1}} \subseteq (\mathcal{D}_d(M^{-1})) \left[\frac{1}{M} \right] \setminus \overline{\mathcal{D}_{d-1}}. \tag{5.5}$$

Now (5.3) and (5.5) yield that

$$\begin{aligned} \mathcal{W}_d(M) \triangle \overline{\mathcal{D}_{d-1}} &\subseteq \left(\overline{\mathcal{D}_{d-1}} \setminus \left((\mathcal{D}_d(M^{-1})) \left[-\frac{\sqrt{d}}{M} \right] \right) \right) \cup \left((\mathcal{D}_d(M^{-1})) \left[\frac{1}{M} \right] \setminus \overline{\mathcal{D}_{d-1}} \right) \\ &\subseteq (\overline{\mathcal{D}_{d-1}} \triangle \mathcal{D}_d(M^{-1})) \cup \left((\partial \mathcal{D}_d(M^{-1})) \left[\frac{\sqrt{d}}{M} \right] \right). \end{aligned}$$

Note that the second inclusion is an immediate consequence of the definitions (3.1) and (3.2), respectively. From this chain of inclusions we gain

$$\begin{aligned}\int \left|1_{\mathcal{W}_d(M)} - 1_{\overline{\mathcal{D}_{d-1}}}\right| d\lambda_{d-1} &= \int 1_{\mathcal{W}_d(M)\triangle\overline{\mathcal{D}_{d-1}}}\, d\lambda_{d-1} \\ &\le \int 1_{\overline{\mathcal{D}_{d-1}}\triangle\mathcal{D}_d(M^{-1})}\, d\lambda_{d-1} + \int 1_{(\partial\mathcal{D}_d(M^{-1}))[\sqrt{d}/M]}\, d\lambda_{d-1}.\end{aligned}$$

Now we let $M \to \infty$. Then

$$\int 1_{\overline{\mathcal{D}_{d-1}}\triangle\mathcal{D}_d(M^{-1})} d\lambda_{d-1} \to 0$$

by Theorem 4.9. Furthermore,

$$\int 1_{(\partial\mathcal{D}_d(M^{-1}))[\sqrt{d}/M]}\, d\lambda_{d-1} \to 0$$

since $\partial\mathcal{D}_{d-1}(M^{-1})$ is defined by finitely many polynomial equations. Thus

$$\int \left|1_{\mathcal{W}_d(M)} - 1_{\overline{\mathcal{D}_{d-1}}}\right| d\lambda_{d-1} \to 0$$

and the theorem is proved. □

It is worth mentioning the following result which we get as a byproduct of the proof of Theorem 5.1.

Corollary 5.2. *For $d \ge 2$ we have*

$$\lambda_{d-1}(\mathcal{W}_d(M) \triangle \overline{\mathcal{D}_{d-1}}) \to 0$$

for $M \to \infty$.

6. Convergence properties of the set $\mathcal{D}_d^0$

The aim of this section is to describe the convergence of the sets $\mathcal{D}_d^0(x)$ defined in (1.5) to $\mathcal{D}_{d-1}$ for $x \to 0$. To this matter we need a description of the relation between $\mathcal{D}_d$ and $\mathcal{D}_d^0$ where we adopt the following notations and results from [2]. Let $\mathbf{r} \in \mathbb{R}^d$ and let $\mathbf{a} = (a_1, \dots, a_d)$ be a non zero periodic point of period length L for $\tau_{\mathbf{r}}$, i.e., $\mathbf{a} = \tau_{\mathbf{r}}^L(\mathbf{a})$. Suppose this period runs through the points

$$\tau_{\mathbf{r}}^j(\mathbf{a}) = (a_{1+j}, \dots, a_{d+j}) \quad (0 \le j \le L-1),$$

where $a_{1+L} = a_1, \dots, a_{d+L} = a_d$ (note that the structure of the entries follows from the definition of $\tau_{\mathbf{r}}$). Then we will say that

$$a_1, \dots, a_d, a_{d+1}, \dots, a_L$$

is a period of $\tau_{\mathbf{r}}$. If a period occurs for some $\tau_{\mathbf{r}}$ with $\mathbf{r} \in \mathbb{R}^d$, we will call it for short a *period of* $\mathcal{D}_d$.

By the definition of $\tau_{\mathbf{r}}$ the set of all $\mathbf{r} = (r_1, \dots, r_d) \in \mathbb{R}^d$ which admit a given period π is given by the simultaneous inequalities

$$0 \le r_1 a_{1+j} + \cdots + r_d a_{d+j} + a_{d+j+1} < 1 \quad (0 \le j \le L-1). \tag{6.1}$$

As each inequality gives an upper/lower half of hyperplanes in $\mathbb{R}^d$, it is easy to see that (6.1) defines a (possibly degenerated) convex polyhedron. We call this polyhedron a *cutout polyhedron* and denote it by $\mathcal{P}(\pi)$ (cf. [2, Section 4]).

Since $\mathbf{r} \in \mathcal{D}_d^0$ if and only if $\tau_{\mathbf{r}}$ has 0 as its only period we conclude that

$$\mathcal{D}_d^0 = \mathcal{D}_d \setminus \bigcup_{\pi \neq 0} \mathcal{P}(\pi),$$

where the union is extended over all non-zero periods π of $\mathcal{D}_d$. We call the family of (non-empty) polyhedra corresponding to this choice the *family of cutout polyhedra of* $\mathcal{D}_d^0$.

Let now $\varepsilon \in (0, 1)$. We know from [2, Section 7] that there is a finite family $\mathcal{P} := \{P_0, \dots, P_L\}$ of cutout polyhedra such that

$$\mathcal{D}_{d,\varepsilon}^0 = \mathcal{D}_{d,\varepsilon} \setminus \bigcup_{l=0}^{L} P_l,$$

because critical points of $\mathcal{D}_d$ can only occur on the boundary of $\mathcal{D}_d$ (cf. [2, Lemma 7.2]). Because of (6.1) for every $l \in \{0, \dots, L\}$ we can find pairwise disjoint finite sets $I_l, J_l \subset \mathbb{N}$ and linear polynomials $f_{l,i} \in \mathbb{R}[R_1, \dots, R_d]$ with

$$P_l = \bigcap_{i \in I_l} \{\mathbf{r} \in [-M, M]^d : f_{l,i}(\mathbf{r}) > 0\} \cap \bigcap_{i \in J_l} \{\mathbf{r} \in [-M, M]^d : f_{l,i}(\mathbf{r}) \ge 0\}. \tag{6.2}$$

Here we choose M in a way that (4.4) is satisfied. We will subdivide the set $\mathcal{P}$ of cutout polyhedra into three parts. Indeed, set

$$\mathcal{P}_1 := \{P_l \in \mathcal{P} : R_1 \nmid f_{l,i}(R_1, \dots, R_d) \text{ holds for all } i \in I_l \cup J_l\},$$

$$\mathcal{P}_2 := \{P_l \in \mathcal{P} : R_1 \mid f_{l,i}(R_1, \dots, R_d) \text{ holds for at least one } i \in I_l\},$$

$$\mathcal{P}_3 := \{P_l \in \mathcal{P} : R_1 \mid f_{l,i}(R_1, \dots, R_d) \text{ holds for at least one } i \in J_l\}.$$

In what follows we will use the notation $P_l(x) := P_l \cap W(x)$ (see (4.5) for the definition of $W(x)$). We first treat the cutout polyhedra contained in the subfamily $\mathcal{P}_1$.

Lemma 6.1. *For each $P_l \in \mathcal{P}_1$ we have*

$$\lim_{x\to 0} \lambda_{d-1}(\mathrm{proj}(P_l(x)) \triangle \mathrm{proj}(P_l(0))) = 0.$$

Proof. Setting

$$A_{l,i}(x) := \{\mathbf{r} \in [-M, M]^d : f_{l,i}(\mathbf{r}) > 0\} \cap W(x) \quad (i \in I_l)$$

and

$$B_{l,i}(x) := \{\mathbf{r} \in [-M, M]^d : f_{l,i}(\mathbf{r}) \geq 0\} \cap W(x) \quad (i \in J_l)$$

we see from (6.2) that

$$P_l(x) = \bigcap_{i\in I_l} A_{l,i}(x) \cap \bigcap_{i\in J_l} B_{l,i}(x).$$

Because $P_l(x) \in \mathcal{P}_1$ the (linear) polynomials $f_{l,i}$ satisfy the conditions of Lemma 4.2, this lemma together with Remark 4.3 yields

$$\lim_{x\to 0} \lambda_{d-1}(\mathrm{proj}(A_{l,i}(x)) \triangle \mathrm{proj}(A_{l,i}(0))) = 0$$

and

$$\lim_{x\to 0} \lambda_{d-1}(\mathrm{proj}(B_{l,i}(x)) \triangle \mathrm{proj}(B_{l,i}(0))) = 0.$$

The result now follows from Lemma 4.4 (i). □

In order to treat the cutout polyhedra contained in $\mathcal{P}_2$ we need the following auxiliary result.

Lemma 6.2. *Let $d \geq 2$ and $\mathbf{r} = (r_1, \ldots, r_d) \in P(\pi)$ be an element of a cutout polyhedron given by a period π of the form*

$$\cdots a_0 a_1 \underbrace{0 \cdots 0}_{d-1} 1 \cdots.$$

If $r_1 > 0$ then $r_2 \leq -(a_0/a_1)r_1$, and if $r_1 < 0$ then $r_2 \geq -(a_0/a_1)r_1$.

Proof. By the definition we have the inequalities

$$0 \le a_1 r_1 + 1 < 1, \tag{6.3}$$

$$0 \le a_0 r_1 + a_1 r_2 < 1. \tag{6.4}$$

Clearly, $r_1 > 0$ implies $a_1 < 0$ by (6.3) and then $r_2 \le -(a_0/a_1) r_1$ by (6.4). The second assertion is derived analogously. $\square$

For the elements of $\mathcal{P}_2$ we can show the following assertion.

Lemma 6.3. *For all $\varepsilon > 0$ we have*

$$\lim_{x\to 0} \lambda_{d-1}\left(\left(\operatorname{proj}\left(\mathcal{D}_{d,\varepsilon}(x) \setminus \bigcup_{P_l\in\mathcal{P}_2} P_l\right) \cup \operatorname{proj}\left(\mathcal{D}_{d,\varepsilon}(-x) \setminus \bigcup_{P_l\in\mathcal{P}_2} P_l\right)\right) \triangle\, \mathcal{D}_{d,\varepsilon}(0)\right) = 0.$$

Proof. If $P_l \in \mathcal{P}_2$ then for some $i \in I_l$ we have $R_1 \mid f_{l,i}(R_1, \dots, R_d)$. Because $f_{l,i}$ is a linear polynomial this implies that there exists a constant $c_{l,i} \in \mathbb{R} \setminus \{0\}$ such that $f_{l,i}(R_1, \dots, R_d) = c_{l,i} R_1$. This implies that

$$P_l \subset \{(r_1, \dots, r_d) \colon c_{l,i} r_1 > 0\}$$

which means that either

$$P_l \subset \{(r_1, \dots, r_d) \colon r_1 > 0\}$$

or

$$P_l \subset \{(r_1, \dots, r_d) \colon r_1 < 0\}.$$

Applying Lemma 6.2 we even get that there exists $b \in \mathbb{R}$ such that either

$$P_l \subset \{(r_1, \dots, r_d) \colon r_1 > 0 \text{ and } r_2 \le b r_1\}$$

or

$$P_l \subset \{(r_1, \dots, r_d) \colon r_1 < 0 \text{ and } r_2 \ge b r_1\}.$$

Thus

$$\mathcal{D}_{d,\varepsilon}(x) \setminus \bigcup_{P_l\in\mathcal{P}_2} P_l \supset \mathcal{D}_{d,\varepsilon}(x) \cap \{(x, r_2, \dots, r_d) \colon r_2 < bx\} \quad (x > 0),$$

$$\mathcal{D}_{d,\varepsilon}(x) \setminus \bigcup_{P_l\in\mathcal{P}_2} P_l \supset \mathcal{D}_{d,\varepsilon}(x) \cap \{(x, r_2, \dots, r_d) \colon r_2 > bx\} \quad (x < 0).$$

Combining these two inclusions we obtain that

$$\begin{aligned}&\mathcal{D}_{d,\varepsilon}(x) \cap \{(x, r_2, \ldots, r_d)\colon |r_2| > b|x|\}\\ &\subset \left(\mathcal{D}_{d,\varepsilon}(x) \setminus \bigcup_{P_l \in \mathcal{P}_2} P_l\right) \cup \left(\mathcal{D}_{d,\varepsilon}(-x) \setminus \bigcup_{P_l \in \mathcal{P}_2} P_l\right) \subset \mathcal{D}_{d,\varepsilon}(x)\end{aligned}$$

holds for all x. Taking projections and letting x tend to zero yield the result. □

Lemma 6.4. $\mathcal{P}_3 = \emptyset$.

Proof. If $P_l \in \mathcal{P}_3$ then P_l contains an inequality

$$f_{l,i}(r_1, \ldots, r_d) = cr_1 \geq 0$$

for some $i \in J_l$. To get such an inequality the cycle π which generates P_l must contain d consecutive zeros. Thus π is the trivial cycle, a contradiction. □

We are now in a position to prove the following theorem.

Theorem 6.5. *For each $d \geq 2$ we have*

$$\lim_{x \to 0} \lambda_{d-1}(\mathcal{D}_d^0(x) \triangle \mathcal{D}_{d-1}^0) = 0.$$

Proof. Let $\varepsilon > 0$ be arbitrary but fixed. Then for $x > 0$ Theorem 2.1 and Lemma 6.4 yield

$$\begin{aligned}\mathcal{D}_{d,\varepsilon}^0(x) &= \mathcal{D}_{d,\varepsilon}^0(x) \cup \mathcal{D}_{d,\varepsilon}^0(-x) \quad (\text{since } \mathcal{D}_{d,\varepsilon}^0(-x) = \emptyset)\\ &= (\mathcal{D}_{d,\varepsilon}(x) \cup \mathcal{D}_{d,\varepsilon}(-x)) \setminus \bigcup_{P_l \in \mathcal{P}} P_l\\ &= (\mathcal{D}_{d,\varepsilon}(x) \cup \mathcal{D}_{d,\varepsilon}(-x)) \setminus \bigcup_{P_l \in \mathcal{P}_1 \cup \mathcal{P}_2} P_l\\ &= \left((\mathcal{D}_{d,\varepsilon}(x) \cup \mathcal{D}_{d,\varepsilon}(-x)) \setminus \bigcup_{P_l \in \mathcal{P}_1} P_l\right) \cap \left((\mathcal{D}_{d,\varepsilon}(x) \cup \mathcal{D}_{d,\varepsilon}(-x)) \setminus \bigcup_{P_l \in \mathcal{P}_2} P_l\right)\\ &= \bigcap_{i=1,2} \left(\left(\mathcal{D}_{d,\varepsilon}(x) \setminus \bigcup_{P_l \in \mathcal{P}_i} P_l\right) \cup \left(\mathcal{D}_{d,\varepsilon}(-x) \setminus \bigcup_{P_l \in \mathcal{P}_i} P_l\right)\right).\end{aligned}$$

Taking projections this yields

$$
\begin{aligned}
&\operatorname{proj}(\mathcal{D}^0_{d,\varepsilon}(x)) \\
&= \bigcap_{i=1,2}\left(\left(\operatorname{proj}(\mathcal{D}_{d,\varepsilon}(x)) \setminus \bigcup_{P_l\in\mathcal{P}_i}\operatorname{proj}(P_l)\right) \cup \left(\operatorname{proj}(\mathcal{D}_{d,\varepsilon}(-x)) \setminus \bigcup_{P_l\in\mathcal{P}_i}\operatorname{proj}(P_l)\right)\right).
\end{aligned}
\tag{6.5}
$$

Lemmas 4.8 and 6.1 imply together with Lemma 4.4 (ii) and (iii) that

$$
\lim_{x\to 0}\lambda_{d-1}\left(\left(\operatorname{proj}(\mathcal{D}_{d,\varepsilon}(x)) \setminus \bigcup_{P_l\in\mathcal{P}_1}\operatorname{proj}(P_l)\right) \triangle \left(\operatorname{proj}(\mathcal{D}_{d,\varepsilon}(0)) \setminus \bigcup_{P_l\in\mathcal{P}_1}\operatorname{proj}(P_l)\right)\right) = 0.
\tag{6.6}
$$

Here x may approach zero from the left or from the right. For the second part of (6.5) we apply Lemma 6.3 to see that

$$
\lim_{x\to 0}\lambda_{d-1}\left(\left(\operatorname{proj}\left(\mathcal{D}_{d,\varepsilon}(x) \setminus \bigcup_{P_l\in\mathcal{P}_2} P_l\right) \cup \operatorname{proj}\left(\mathcal{D}_{d,\varepsilon}(-x) \setminus \bigcup_{P_l\in\mathcal{P}_2} P_l\right)\right) \triangle \mathcal{D}_{d,\varepsilon}(0)\right) = 0.
\tag{6.7}
$$

Using Lemma 4.4 (i) and (ii) we can collect (6.6) and (6.7) to derive

$$
\begin{aligned}
\lim_{x\to 0}\lambda_{d-1}(\mathcal{D}^0_{d,\varepsilon}(x) \triangle \mathcal{D}^0_{d-1,\varepsilon}) &= \lim_{x\to 0}\lambda_{d-1}(\mathcal{D}^0_{d,\varepsilon}(x) \triangle \mathcal{D}^0_{d,\varepsilon}(0)) \\
&= \lim_{x\to 0}\lambda_{d-1}\left(\mathcal{D}^0_{d,\varepsilon}(x) \triangle \left(\mathcal{D}_{d,\varepsilon}(0) \setminus \bigcup_{P_l\in\mathcal{P}_1} P_l\right)\right) \\
&= 0.
\end{aligned}
$$

Because ε can be chosen arbitrarily small the result follows from Lemma 4.7. □

7. Relations between $\mathcal{D}^0_{d-1}$ and $\mathcal{C}^0_d$

In the next theorem we prove that the $(d-1)$-dimensional Lebesgue measure of $\mathcal{D}^0_{d-1}$ is the limit of the quotient in (1.8) for $M \to \infty$. Note that the Lebesgue measurability of $\mathcal{D}^0_d$ is proved in [2, Theorem 4.10].

Theorem 7.1. *Let $d \ge 2$ and M be a positive integer and set*

$$
N^0(d, M) := |\{(p_1, \dots, p_{d-1}) \in \mathbb{Z}^{d-1} : (M, p_1, \dots, p_{d-1}) \in \mathcal{C}^0_d\}|.
$$

Then

$$
\lim_{M\to\infty}\frac{N^0(d, M)}{M^{d-1}} = \lambda_{d-1}(\mathcal{D}^0_{d-1}).
$$

Proof. We will use the following notation. Let $W(\mathbf{x}, s)$ be defined as in (5.1). Now set for $\varepsilon \in [0, 1)$

$$\mathcal{C}^0_{d,\varepsilon}(M) := \Big\{ \Big(\frac{p_{d-1}}{M}, \ldots, \frac{p_1}{M} \Big) \in \mathbb{R}^{d-1} : (M, p_1, \ldots, p_{d-1}) \in \mathcal{C}^0_d, \\ \rho\Big(\Big(\frac{p_{d-1}}{M}, \ldots, \frac{p_1}{M} \Big)\Big) < 1 - \varepsilon \Big\}$$

and

$$\mathcal{W}^0_{d,\varepsilon}(M) := \bigcup_{\mathbf{x} \in \mathcal{C}^0_{d,\varepsilon}(M)} W(\mathbf{x}, M^{-1}).$$

Furthermore, let

$$N^{0,\varepsilon}(d, M) := |\{(p_1, \ldots, p_{d-1}) \in \mathbb{Z}^{d-1} : (M, p_1, \ldots, p_{d-1}) \in \mathcal{C}^0_{d,\varepsilon}\}|.$$

Then obviously

$$\frac{N^{0,\varepsilon}(d, M)}{M^{d-1}} = \lambda_{d-1}(\mathcal{W}^0_{d,\varepsilon}(M)). \tag{7.1}$$

We will compare the latter with the Lebesgue measure of $\mathcal{D}^0_{d-1,\varepsilon}$.

We first claim

$$\overline{\mathcal{D}^0_{d-1,\varepsilon}} \setminus \mathcal{W}^0_{d,\varepsilon}(M) \subseteq \overline{\mathcal{D}^0_{d-1,\varepsilon}} \setminus \Big((\mathcal{D}^0_{d,\varepsilon}(M^{-1})) \Big[-\frac{\sqrt{d}}{M} \Big] \Big). \tag{7.2}$$

To prove the claim we will show

$$(\mathcal{D}^0_{d,\varepsilon}(M^{-1})) \Big[-\frac{\sqrt{d}}{M} \Big] \subset \mathcal{W}^0_{d,\varepsilon}(M).$$

By the definition of the norm $\| \cdot \|_\infty$, if $\mathbf{y} \in \mathcal{D}^0_{d,\varepsilon}(M^{-1})[-\sqrt{d}/M]$ then

$$W(\mathbf{y}, M^{-1}) \subset \mathcal{D}^0_{d,\varepsilon}(M^{-1}). \tag{7.3}$$

Thus we can choose $p_1, \ldots, p_{d-1} \in \mathbb{Z}$ with

$$\mathbf{z} = \Big(\frac{p_{d-1}}{M}, \ldots, \frac{p_1}{M} \Big) \in W(\mathbf{y}, M^{-1}).$$

Now (1.6) and (7.3) imply that $\mathbf{z} \in \mathcal{C}^0_{d,\varepsilon}(M) \cap W(\mathbf{y}, M^{-1})$. Thus $\mathbf{y} \in W(\mathbf{z}, M^{-1})$ for a $\mathbf{z} \in \mathcal{C}^0_{d,\varepsilon}(M)$ which yields $\mathbf{y} \in \mathcal{W}^0_{d,\varepsilon}(M)$ and the claim is proved.

On the other hand $\mathcal{C}^0_{d,\varepsilon}(M) \subset \mathcal{D}^0_{d,\varepsilon}(M^{-1})$ implies

$$\mathcal{W}^0_{d,\varepsilon}(M) \subseteq (\mathcal{D}^0_{d,\varepsilon}(M^{-1})) \Big[\frac{1}{M} \Big]$$

which yields

$$\mathcal{W}^0_{d,\varepsilon}(M) \setminus \overline{\mathcal{D}^0_{d-1,\varepsilon}} \subseteq (\mathcal{D}^0_{d,\varepsilon}(M^{-1}))\left[\frac{1}{M}\right] \setminus \overline{\mathcal{D}^0_{d-1,\varepsilon}}. \tag{7.4}$$

Now (7.2) and (7.4) yield that

$$\begin{aligned}
&\mathcal{W}^0_{d,\varepsilon}(M) \triangle \overline{\mathcal{D}^0_{d-1,\varepsilon}} \\
&\subseteq \left(\overline{\mathcal{D}^0_{d-1,\varepsilon}} \setminus \left(\left(\mathcal{D}^0_{d,\varepsilon}(M^{-1})\right)\left[-\frac{\sqrt{d}}{M}\right]\right)\right) \cup \left((\mathcal{D}^0_{d,\varepsilon}(M^{-1}))\left[\frac{1}{M}\right] \setminus \overline{\mathcal{D}^0_{d-1,\varepsilon}}\right) \\
&\subseteq \left(\overline{\mathcal{D}^0_{d-1,\varepsilon}} \triangle \mathcal{D}^0_{d,\varepsilon}(M^{-1})\right) \cup \left((\partial\mathcal{D}^0_{d,\varepsilon}(M^{-1}))\left[\frac{\sqrt{d}}{M}\right]\right).
\end{aligned}$$

Note that the second inclusion is an immediate consequence of the definitions (3.1) and (3.2), respectively. From this chain of inclusions we gain

$$\begin{aligned}
\int \left|1_{\mathcal{W}^0_{d,\varepsilon}(M)} - 1_{\overline{\mathcal{D}^0_{d-1,\varepsilon}}}\right| d\lambda_{d-1} &= \int 1_{\mathcal{W}^0_{d,\varepsilon}(M)\triangle\overline{\mathcal{D}^0_{d-1,\varepsilon}}}\, d\lambda_{d-1} \\
&\leq \int 1_{\overline{\mathcal{D}^0_{d-1,\varepsilon}}\triangle\mathcal{D}^0_{d,\varepsilon}(M^{-1})} \lambda_{d-1} + \int 1_{(\partial\mathcal{D}^0_{d,\varepsilon}(M^{-1}))[\sqrt{d}/M]}\, d\lambda_{d-1}.
\end{aligned}$$

Now we let $M \to \infty$. Then

$$\int 1_{\overline{\mathcal{D}^0_{d-1,\varepsilon}}\triangle\mathcal{D}^0_{d,\varepsilon}(M^{-1})} \lambda_{d-1} \to 0$$

by Theorem 6.5. Furthermore,

$$\int 1_{(\partial\mathcal{D}^0_{d,\varepsilon}(M^{-1}))[\sqrt{d}/M]}\, d\lambda_{d-1} \to 0$$

since $\partial\mathcal{D}^0_{d-1,\varepsilon}(M^{-1})$ is defined by finitely many polynomial equations. Thus

$$\int \left|1_{\mathcal{W}^0_{d,\varepsilon}(M)} - 1_{\overline{\mathcal{D}^0_{d-1,\varepsilon}}}\right| d\lambda_{d-1} \to 0$$

and the theorem is proved by letting $\varepsilon \to 0$ and observing Lemma 4.7. (Note that for fixed ε the set $\mathcal{D}^0_{d-1} \setminus \mathcal{D}^0_{d-1,\varepsilon}$ is bounded by finitely many polynomial equations. So for M large enough the number of lattice points $(p_1/M, \ldots, p_{d-1}/M)$ with $p_1, \ldots, p_{d-1} \in \mathbb{Z}$ contained in it is essentially $M^{d-1}\lambda(\mathcal{D}^0_{d-1} \setminus \mathcal{D}^0_{d-1,\varepsilon})$). □

We now give the following result which we get as a byproduct of the proof of Theorem 7.1.

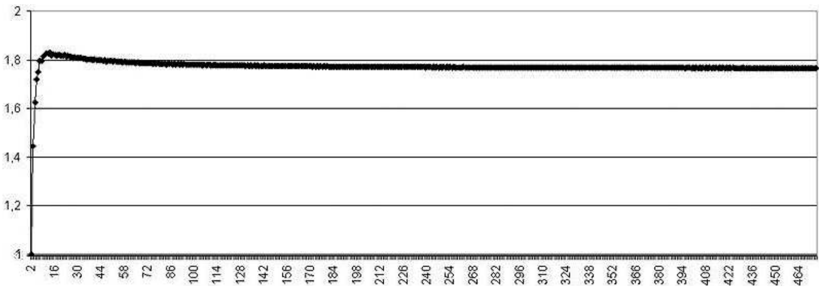

Fig. 1. The behavior of $N^0(3, M)/M^2$ for $2 \leq M \leq 464$.

Corollary 7.2. *Letting* $\mathcal{C}_d^0(M)$ *as in* (1.4) *and setting*

$$\mathcal{W}_d^0(M) := \bigcup_{\mathbf{x} \in \mathcal{C}_d^0(M)} W(\mathbf{x}, M^{-1})$$

we have for $d \geq 2$

$$\lambda_{d-1}\big(\mathcal{W}_d^0(M) \triangle \overline{\mathcal{D}_{d-1}^0}\big) \to 0$$

for $M \to \infty$.

This shows that the set of CNS polynomials

$$\sum_{i=0}^{d} p_i X^i \in \mathbb{Z}[X]$$

with large but fixed constant term p_0 forms a good approximation for the $(d-1)$-dimensional SRS region $\mathcal{D}_{d-1}^0$.

REMARK 7.3. Fig. 1[3] displays $N^0(3, M)/M^2$ for $2 \leq M \leq 464$. It seems that the quotient stabilizes after the first few values at about 1.766. Using known results on the number of cubic CNS polynomials it can easily be seen that for $M \geq 9$ we have

$$\frac{1}{9}(13M^2 - 21M + 51) < N^0(3, M) < 2M^2 - M - 2.$$

As these bounds are quite weak we omit the proof here.

[3]We thank Andrea Huszti for preparing Fig. 1.

8. Open questions

We have shown convergence results with respect to Lebesgue measure in Theorems 4.9 and 6.5. Can we have stronger convergence in the sense of Proposition 3.1?

OPEN QUESTION 1. Is it true that $\mathrm{Lim}_{x\to 0}\,\mathcal{D}_d^0(x) = \overline{\mathcal{D}_{d-1}^0}$?

By Theorem 7.1, the number of CNS polynomials of a given constant term is estimated by SRS.

OPEN QUESTION 2. Can we estimate the number of Pisot polynomials of a given trace having property (F) by SRS?

This question will be explored in a forthcoming paper.

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