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Osaka University
0. Introduction

The aim of this paper is to study the propagation of $C^\infty$-singularities for an hyperbolic pseudodifferential operator whose principal symbol vanishes at order $m \geq 2$ on an involutive manifold, generalizing a well known result obtained by R. Lascar [8] Chapter III, in the case $m = 2$.

Let $X$ be an open subset of $\mathbb{R}^{n+1}$, denote by $T^*X \cong X \times \mathbb{R}^{n+1}$ the cotangent bundle with canonical coordinates $(x, \xi)$ and let $\omega = \sum_{j=0}^n \xi_j \, dx_j$ (resp. $\sigma = d\omega = \sum_{j=0}^n d\xi_j \wedge dx_j$) denote the canonical 1-form (resp. 2-form) on $T^*X$. By $T^*X \setminus 0$ we denote $T^*X$ minus the zero section. Let $P(x, D_x)$ be a classical pseudodifferential operator (pdo) in $X$ of order $m, m \in \mathbb{N}$, with symbol

$$P(x, \xi) \sim \sum_{j=0}^m p_{m-j}(x, \xi)$$

and let $\varphi \in C^\infty(X)$ be a real-valued function, with $d\varphi(x) \neq 0 \ \forall x \in X$.

We shall make the following assumptions:

(H$_1$) $P$ is hyperbolic with respect to the level surfaces of $\varphi$, i.e. $p_m$ is real-valued and

i) $p_m(x, d\varphi(x)) \neq 0 \ \forall x \in X$;

ii) for every $(x, \xi) \in T^*X$, $\xi$ independent of $d\varphi(x)$, the function $p_m(x, \xi + td\varphi(x))$ is a polynomial of degree $m$ in $t$ having only real roots.

(H$_2$) There exists a $C^\infty$-conic, non radial, involutive submanifold $N \subset T^*X \setminus 0$ of codimension $p+1$, such that, for $j \geq 0$, $p_{m-j}$ vanishes at least of order $(m-2j)+$ on $N(t = \max(t, 0))$.

The above conditions on $N$ imply that, for any $\rho \in N$, we have $T_\rho(N)^{\sigma} \subset T_\rho(N)$ ($T_\rho(N)^{\sigma}$ being the orthogonal of $T_\rho(N)$ with respect to $\sigma$) and $\omega(\rho) \in T_\rho(N)^{\sigma}$.

As a consequence, $N$ is foliated by leaves $F_\rho$, $\rho \in N$, which are (immersed) $C^\infty$ submanifold of $N$ of dimension $p+1$ transversal to the radial vector field, with $T_\rho(F_\rho) = T_\rho(N)^{\sigma}$ (note that $p < n$). Moreover, for every $\rho \in N$, the bilinear form $\sigma$ induces an isomorphism $f_\rho: T_\rho(T^*X)/T_\rho(N) \to T^*_\rho(F_\rho)$ (see [6]).
Because of the vanishing conditions on $p$, we can apply the results of [3] and therefore associate to $P$ a family $q_{m-j}, j=0, \ldots, [m/2]$, of $(m-2j)$-multilinear symmetric forms defined on $T(T^*X)/T(N)$, the normal bundle of $N$.

For every $\rho \in N$ and $v \in T_\rho(T^*X)/T_\rho(N)$ we define:

$$q(\rho)(v) = \sum_{j=0}^{[m/2]} q_{m-j}(\rho)(v), \quad q_{m-j}(\rho)(v) = q_{m-j}(\rho)(v, \ldots, v),$$

and observe that

$$q_m(\rho)(v, \ldots, v) = \frac{1}{m!} (d^m q_m)(\rho)(v, \ldots, v).$$

Using the isomorphism $J, q_m$ and $q$ will be considered as $C^\infty$ functions of $\rho \in N$ and $v \in T^*_\rho(P_\pi)$. Thus, fixed a leaf $F$ on $N$, $q_m$ and $q$ will be well defined as $C^\infty$ functions on $T^*(F)$ (see [9]). Let $\varphi = \varphi_0 \pi$ were $\pi: T^*X \rightarrow X$ is the canonical projection.

Since $H_\varphi(\rho)$ is transversal to $T_\rho(N)$, its class modulo $T_\rho(N)$, say $H_\varphi(\rho)$, does not vanish. We shall suppose:

(H3) $q_m(\rho)(v)$ is strictly hyperbolic with respect to $-H_\varphi(\rho)$, $\forall \rho \in N$.

(H4) The polynomial $t \rightarrow q(\rho)(v + t H_\varphi(\rho))$ has $m$ real simple roots, $\forall \rho \in N$ and $\forall v \in T_\rho(T^*X)/T_\rho(N)$.

Some comments on conditions (H3), (H4) are in order.

1— As will be shown in §1, condition (H3) is equivalent to requiring that for $(x, \xi) \in N$ and close to $N$, the real roots of the polynomial $p_m(\xi + t d\varphi(x))$ are simple ($\xi$ independent of $d\varphi(x)$), hence $p_m$ is strictly hyperbolic outside $N$, at least close to $N$.

2— Condition (H4), which is obviously invariant by change of coordinates in $X$, is more technical. In [10] (when $m=2$) and [1] (for $m \geq 2$), the authors consider the case of an operator $P$ satisfying conditions (H1)-(H3), whereas (H4) is replaced by a suitable Levi condition on the lower order terms of $P$, which in particular implies that $\forall \rho \in N, q_m-j(\rho)=0$ for $j=1, \ldots, [m/2]$.

The case (H4), which we will treat here, is, in some sense, on the opposite side.

3— It is easy to see that if $P$ satisfies conditions (H2)-(H4), then the same hypotheses are satisfied by the transposed operator $P^*$, with $N$ replaced by $-N=\{(x, \xi) | (x, -\xi) \in N\}$.

**Examples.** When $m=2$, using standard arguments, we can suppose that $\varphi = x_0$, that the operator $P$ in the form $P = -D_{x_0}^2 + A(x, D), x=(x_0, y), y=(y', y'') \in \mathbb{R}^{n'-2} \times \mathbb{R}^p$, where $A$ is a second order pdo in $\mathbb{R}^n$ depending smoothly on $x_0$, with nonnegative principal symbol $a_\eta(x, \eta) = \sum_{\xi} a_\xi(x, \eta) \xi''\eta$, $\eta=(\xi, \xi'') \in \mathbb{R}^{n'-2} \times \mathbb{R}^p$, and that $N = \{\xi'' = da_\xi = 0\}$.

We have, if $\rho \in N, v \in T_\rho(T^*X)/T_\rho(N),

$$q_2(\rho)(v) = \frac{1}{2} \langle \text{Hess } p_2(\rho) v, v \rangle, \quad q(\rho)(v) = q_2(\rho)(v) + p_1(\rho),$$
where \( p_1(\rho) \) denotes the subprincipal symbol of \( P \).

The hyperbolicity of \( P \) means that \( a_4(x, \eta) \) is non-negative, while condition (H3) is equivalent to require that \( a_2 \) is transversally elliptic with respect to \( \xi'' = 0 \); condition (H4) is then equivalent to \( p_1(\rho) > 0 \), \( \forall \rho \in \Lambda' \). This case was treated in [8].

A typical example in the case \( m=4, \varphi = x_0 \), is represented by an operator \( P \) which is factored as

\[
P = Q^{(1)} Q^{(2)} + A^{(1)} Q^{(3)} + A^{(2)} Q^{(4)} + A_2,
\]
with \( Q^{(1)} = -D_x^2 + \alpha(x, D_x) |D_{x'}|^2 \), \( Q^{(2)} = -D_x^2 + \beta(x, D_x) |D_{x'}|^2 \), where \( \alpha(x, D_x) \), \( \beta(x, D_x) \) are pdo’s in \( y \) of order 0 having real positive principal symbols and, \( \forall i = 1, 2, A^{(i)} \) (resp. \( A_2 \)) are pdo’s of order 1 (resp. of order 2) in \( \mathbb{R}^n \), depending smoothly on \( x_0 \). We have \( \Lambda' = \{ \xi_{\rho} = \xi'' = 0 \} \) and

\[
q_i(\rho)(v) = \frac{1}{4} \langle \text{Hess} q_i^{(1)}(\rho) \rangle \langle \text{Hess} q_i^{(2)}(\rho) \rangle \langle v, v \rangle ,
\]
\[
g_0(\rho)(v) = \frac{1}{2} \langle a_i^{(1)}(\rho) \langle \text{Hess} q_i^{(1)}(\rho) \rangle \rangle + a_i^{(2)}(\rho) \langle \text{Hess} q_i^{(2)}(\rho) \rangle \langle v, v \rangle ,
\]
\[
g_2(\rho)(v) = a_2(\rho) , \quad \rho \in \Lambda', v \in T_x(T^*X)/T_x(\mathbb{R}_+^n).
\]

In this case condition (H3) is equivalent to \( \alpha(\rho) = \beta(\rho) \), \( \forall \rho \in \Lambda' \), while (H4) means that the polynomial

\[
q(\rho)(\xi, \xi'') = (-\xi_0^{(1)} + \alpha(\rho) |\xi||\xi'|^2) (-\xi_0^{(2)} + \beta(\rho) |\xi''|^2) + a_2(\rho) (-\xi_0^{(1)} + \alpha(\rho) |\xi'|^2)
\]
\[
+ a_2^{(2)}(\rho) (-\xi_0^{(2)} + \beta(\rho) |\xi''|^2) + a_2(\rho)
\]

has real simple roots in \( \xi_0, \forall \rho \in \Lambda' \), \( \forall \xi'' \in \mathbb{R}_+^n \).

We now state the main result of this paper, concerning the propagation of singularities for \( P \).

For every \( \rho_0 \in \Lambda' \) consider the following sets:

\[
C'_{\pm}(\rho_0) = \{ \rho \in \Lambda' | \rho \text{ belongs to the leaf } F = F_{\rho_0} \text{ of } \Lambda' \text{ and there exist point } \xi_0 \in T\xi_0^*(F), \xi' \in T\xi_0^*(F) \text{ and a piece of forward (backward) null bicharacteristic of } q \text{ on } T^*(F) \text{ joining } (\rho_0, \xi_0) \text{ and } (\rho, \xi'), \}
\]
\[
C''_{\pm}(\rho_0) = \{ \rho \in \Lambda' | \rho \text{ belongs to the leaf } F = F_{\rho_0} \text{ of } \Lambda' \text{ and there exist points } \xi_0 \in T\xi_0^*(F), \xi' \in T\xi_0^*(F) \text{ and a piece of forward (backward) null bicharacteristic of } q \text{ on } T^*(F) \text{ joining } (\rho_0, \xi_0) \text{ and } (\rho, \xi'), \}
\]

The main result of this paper is the following theorem:

**Theorem.** Let \( P \) satisfy assumptions (H1)-(H4) and let \( f \in \mathcal{D}'(X), \rho_0 \in \Lambda' \setminus WF(f) \). Assume that \( Pu = f, u \in \mathcal{D}'(X) \), and there exists a conic neighborhood \( \omega \) of \( \rho_0 \) and a choice of sign + or — such that

\[
(0.1)_\pm
\]
\[
WF(u) \cap \omega \cap (C'_{\pm}(\rho_0) \cup C''_{\pm}(\rho_0)) \setminus \{ \rho_0 \} = \emptyset.
\]
Then \( p_0 \in WF(u) \).

The above result will be easily obtained by constructing (microlocal) left parametrices for \( P \). We will prove that the methods used in R. Lascar [8] can be suitably adapted to the more general case we are treating here.

1. Reduction to a normal form

Let us first fix some notations. If \( U \) is an open subset of \( \mathbb{R}^r \) and \( \Sigma \subset T^*U \setminus \{0\} \) is a \( C^\infty \) conic submanifold, we denote by \( L^{p,k}(U; \Sigma) \), \( \mu \in \mathbb{R} \), \( k \in \mathbb{Z}_+ \), the class of all classical pdo's with symbols \( p(x, \xi) \sim \sum_{j \geq 0} \rho_{p-j}(x, \xi) \), such that \( \rho_{p-j} \) vanishes at least of order \( (k - 2j) \) on \( \Sigma, j \geq 0 \) (see [2]). With this notation, our operator \( P \) belongs to \( L^{m,m}(X; N) \).

Working microlocally near a given point of \( N \) and using the same kind of arguments as in [1], Sect. 1, we can find a coordinate system \((x, y) = (x_0, y_0, \eta), y = (x', y', \eta) \in \mathbb{R}^{n+p} \times \mathbb{R}^p(\eta = (\xi', \xi''))\) such that, without loss of generality, \( X = \{T, T \} \), \( T \times Y \subset \mathbb{R}_x \times \mathbb{R}_y \) and \( N \), in these coordinates, is given by:

\[ N = \left\{ (x_0, y, \xi_0, \eta) \in T^*X \setminus 0 | \xi_0 = 0, \xi'' = 0 \right\}. \]

By putting \( M = \{ (y, \eta) \in T^*Y \setminus \{ \xi'' = 0 \} \} \) and disregarding elliptic factors, we can suppose that, modulo a smoothing operator, we have:

\[ P = D^m + \sum_{j=1}^m A_j(x_0, y, D_y) D_0^{m-j}, \]

for some \( A_j \in C^\infty(]-T, T[, L^{1,1}(Y; M)), j = 1, \cdots, m \).

Application of Taylor's formula to the \( A_j \)'s easily yields:

\[ P(x, D_x) = \sum_{j=0}^{m} \sum_{k=0}^{m-2j} \sum_{|\alpha| = m-2j-k} A^{(i)}_{a,k}(x_0, y, D_y) D_{x1}^\alpha D_{x2}^\alpha + \sum_{k=0}^{m-1} B_k(x_0, y, D_y) D^k \]

where \( A^{(i)}_{a,k}(x, D_y) \) and \( B_k(x, D_y) \) are suitable pdo's in \( y \) of order \( j \) and \( \left[ \frac{m-k-1}{2} \right] \) respectively, depending smoothly on \( x_0(A^{(i)}_{a,m} = I) \).

Given a point \( \rho = (x_0, y = (x', y'), \xi_0 = 0, \xi', \xi'' = 0) \in N \) the leaf through \( \rho \) is simply:

\[ F_\rho = \{ (x, \xi) \in N | x' = \tilde{x}', \xi' = \tilde{\xi}' \}. \]

Taking \( (x_0, x', \xi_0, \xi') \) as canonical variables in \( T^*_x(F_\rho) \), one can easily see that

\[ q(\rho)(x_0, x', \xi_0, \xi') = \sum_{j=0}^{m} \sum_{k=0}^{m-2j} \sum_{|\alpha| = m-2j-k} a^{(i)}_{a,k}(x_0, \xi', \xi', 0) \xi''^\alpha \xi_0^k, \]

\[ a^{(i)}_{a,k} \] being the principal symbol of \( A^{(i)}_{a,k} \), while

\[ q_0(\rho)(x_0, x', \xi_0, \xi') = \sum_{k=0}^{m} \sum_{|\alpha| = -k} a^{(i)}_{a,k}(x_0, \xi', \xi', 0) \xi''^\alpha \xi_0^k. \]
Condition \((H_3)\) amounts to require that for every \((x_0, x', \xi') \) and \(\xi'' \neq 0\), and for every \(\rho\), the polynomial \(\xi_0 \rightarrow q_m(\rho)(x_0, x', \xi_0, \xi'')\) has \(m\) real simple roots, whereas condition \((H_4)\) means that the polynomial \(\xi_0 \rightarrow q(\rho)(x_0, x', \xi_0, \xi'')\) has \(m\) real simple roots for every \(\rho\) and for every \((x_0, x', \xi')\) (\(\xi''\) is allowed to be zero).

For simplicity, we will use in the following the notation:

\[
q(\rho)(x_0, x', x'', \xi, \xi', \xi'') = q(x_0, x', x', \xi, \xi', \xi''), \\
q_m(\rho)(x_0, x', \xi_0, \xi_0, \xi_0, \xi') = q_m(x_0, x', x', \xi_0, \xi_0, \xi').
\]

**Remarks 1.** Since \(\sum \rho \xi_0 = \sum \rho \xi_0 + \sum \rho \xi_0 + \sum \rho \xi_0 + \sum \rho \xi_0\), by writing \(0 \neq \xi'' = r \omega, r \in \mathbb{R}, \omega \in S^{r-1}\) and \(u = \xi_0 / r\), we get

\[
r^{-m} p_m(x, ru, \xi', \rho) = \sum_{k=0}^{m} \sum_{|\alpha|=m-k} a_{\alpha}^{(0)}(x_0, x', x'', \xi', \xi'') \xi^{m''} \xi_0^{\alpha},
\]

On the other hand, for \(\rho = (x_0, x', x'', \xi_0 = 0, \xi', \xi'' = 0)\), we have

\[
r^{-m} q_m(\rho)(x_0, x', x'', ru, \rho) = \sum_{k=0}^{m} \sum_{|\alpha|=m-k} a_{\alpha}^{(0)}(x_0, x', x'', \xi', 0) \omega^{m''} u^{m-k}.
\]

Using Rouché’s theorem, it is not difficult to verify that the strict hyperbolicity of \(q_m(\rho)\) is equivalent to require that, for \(r\) positive and sufficiently small, \(u \mapsto r^{-m} p_m(x, ru, \xi')\) has \(m\) real simple roots, i.e. \(p_m\) is strictly hyperbolic near \(N\). Moreover, using the arguments of [7], Prop. 0.3 (ii), one can show that the hamiltonian flow of \(H_m\) in \(\text{Char}(P) \setminus N\) has no limit points in \(N\).

2. **Construction of a parametrix**

From now on we will use the notation introduced in Sect. 1. We fix a point \(\rho_0 \in N\) (without loss of generality we will suppose \(\rho_0 = (x_0, \xi_0, \xi'')\), \(\eta=(\xi'=0, 0, \cdots, 0)\)) and try to solve, microlocally near \(\rho_0\), a Cauchy problem of the form:

\[
\begin{cases}
P_x = 0, \\
D_{x_0}^k v(0, x', x'') = \delta_{k, m-1} f(x', x''), \quad k = 0, \cdots, m-1
\end{cases}
\]

Precisely, for every \(t > 0\), if \(\rho = (x_0, \xi_0, \xi'')\), \(\xi_0 = 0, \xi', \xi'' = 0\), we have

\[
q(x_0, x', x'', t, \xi', 0, t \xi', 0) = t^m q(\rho)(x_0, x', \xi_0, \xi''),
\]

i.e., if \(M_t\) denote the dilations \(M_t(\xi_0, \xi', \xi'') = (t \xi_0, t \xi', t \xi'')\), we have

\[
q(\rho)(x_0, x', \xi_0, \xi'') = \frac{1}{t^m} q(M_t(\rho))(x_0, x', M_t(\xi_0, \xi')).
\]
for a given \( f \in C_{c}(Y) \) supported near the origin (\( \delta_{k,m-1} \) denotes the Kronecker symbol). Following an already classical procedure, we will solve the Cauchy problem by using a suitable class of Fourier integral operators. As in [8], we are led to consider operators of the form:

\[
Ef(x_{0}, y) = \int e^{-i(\varphi(x_{0}, y, \eta) - \varphi(0, z, \eta))} e(x_{0}, y, z, \eta) f(z) \, dz \, d\eta,
\]

acting on \( f \in C_{c}^{\infty}(Y) \), having a suitable phase \( \varphi \) and amplitude \( e \).

Since \( \varphi \) and \( e \) will not be classical symbols, we first fix the corresponding notation. Let \( V \subset \mathbb{R}^{n} \) be an open set and let \( \Gamma \subset \mathbb{R}^{n} \backslash 0 \) be a conic neighborhood of \( (\xi' = e_{1}, 0, \cdots, 0, \xi'' = 0) \).

By \( S^{m,k}(V \times \Gamma; M) \), \( \mu, k \in R \), we denote the class of all functions \( a(z, \xi', \xi'') \in C^{\infty}(V \times \Gamma) \) such that the following inequalities hold:

\[
|\varphi_{x}^{\mu} \partial_{y}^{\nu} \partial_{\eta}^{\rho} a(z, \xi', \xi'')| \leq (|\xi'| + |\xi''|)^{m-1} |\xi'|^{k} |\xi''| \left( \frac{|\xi''|^{2} + 1}{|\eta|} \right)^{1/2},
\]

where \( d_{x}(z, \xi, \eta) = \left( \frac{|\xi''|^{2} + 1}{|\eta|} \right)^{1/2} \). The notation \( \lesssim \) means that the left hand side is dominated by a positive constant times the right hand side on every \( V' \times \Gamma' \subset V \times \Gamma \), for \( |\eta| \) large.

When \( \Gamma = \mathbb{R}^{n} \backslash 0 \) we simply write \( S^{m,k}(V; M) \) (cfr. [2] for further details).

We also denote by \( \text{OPS}^{m,k}(V \times \Gamma; M) \) (resp. \( \text{OPS}^{m,k}(V; M) \)) the related class of pdo's.

We will use phase functions \( \varphi \) of the form

\[
(2.1) \quad \varphi(x_{0}, y, \eta) = \langle x', \xi' \rangle + \varphi^{(1)}(x_{0}, y, \eta),
\]

with \( \varphi^{(1)}(x_{0}, y, \eta) \in S^{1,1}(U \times G; M) \), where \( U \) is some neighborhood of the origin in \( X \) and \( G \subset \mathbb{R}^{n} \backslash 0 \) a suitable conic neighborhood of \( (\xi' = e_{1}, \xi'' = 0) \), \( \varphi^{(1)} \) real valued. On \( \varphi^{(1)} \) we will impose the condition

\[
|\det \left( \frac{\partial^{2} \varphi^{(1)}}{\partial x_{j} \partial \xi'_{k}} \right)| \geq c > 0,
\]

when \( (x_{0}, y, \eta) \in U \times G, \) for \( T \) large, \( G^{\tau} = \{ \eta \in G \mid |\eta| \geq T \} \).

For the amplitudes, we will look for symbols \( e(x_{0}, y, z, \eta) \in S^{0,0}(V \times G; M) \) with \( V = \{(x_{0}, y, z) \mid (x_{0}, y) \in U, (0, z) \in U \} \).

Our first task will be the construction of the phase functions. It will be convenient to use the following dilations in \( \mathbb{R}^{n}, \eta = (\xi', \xi'') \):

\[
\sigma_{t}(\eta) = (t^{2} \xi', t \xi''), \quad t > 0.
\]

Accordingly, a function \( g \) will be \( \sigma \)-homogeneous of degree \( k \) iff \( g(\sigma_{t}(\eta)) = t^{k} g(\eta) \) for \( t > 0 \) and \( \eta \neq 0 \). We also put \( \langle \eta \rangle = (|\xi''|^{2} + |\xi'|^{2})^{1/2} \).

2(a). Eikonal equations

As first step we need the asymptotic expansion of
\[ e^{-i\varphi(x, \eta)} P(x, D_x) \left( e^{i\psi(x, \eta)} e(x, \eta) \right), \]

where \( \varphi \) is as in (2.1) and \( e \in S^{0,0} \).

We claim that, modulo terms belonging to \( S^{m-2, m-2} \):

\[ (2.2) \quad e^{-i\varphi(x, \eta)} P(x, D_x) \left( e^{i\psi(x, \eta)} e(x, \eta) \right) = \rho(x, \nabla \varphi) + \frac{1}{i} \sum_{j=1}^{\infty} \frac{\partial \rho}{\partial \xi_j} (x, \nabla \varphi) \frac{\partial e}{\partial x_j} \]

\[ + \frac{1}{i} \sum_{|\alpha| = 2} \frac{\partial^\alpha \rho}{\partial y^\alpha} (x, \nabla \varphi) \frac{\partial^\alpha}{\partial y^\alpha} e. \]

In fact, it is easily verified that \( D_{x_0}^k(e^{i\psi} e) = e^{i\varphi} g_k \), where

\[ g_k(x, \eta) = \left( \frac{\partial \varphi}{\partial x_0} \right)^k e + \frac{1}{i} \left( \frac{k}{2} \right) \left( \frac{\partial \varphi}{\partial x_0} \right)^{k-2} \frac{\partial^2 \varphi}{\partial x_0^2} e + \left( \frac{k}{k-1} \right) \left( \frac{\partial \varphi}{\partial x_0} \right)^{k-1} D_{x_0} e + S^{m-2, m-2}. \]

Moreover:

\[ e^{-i\varphi} A_{x_0}^{(j)}(x, D_x) D_{x_0}^k e^{i\varphi} = e^{-i\varphi} A_{x_0}^{(j)}(x, D_x) D_{x_0}^k(e^{i\varphi} g_k) \sim \]

\[ \sim \sum_{|\alpha| = 0} \frac{\partial \rho}{\partial \eta^\alpha} (a_{l, k}^{(j)}(x, \eta) \eta''^\alpha) (x, \nabla \varphi) D_x^\alpha \rho(x, y, \eta, \zeta, z) e^{i\varphi}. \]

Therefore:

\[ (2.3) \quad e^{-i\varphi} A_{x_0}^{(j)}(x, D_x) D_{x_0}^k e^{i\varphi} = a_{l, k}^{(j)}(x, \nabla \varphi) \left( \frac{\partial \varphi}{\partial x''^\alpha} \right) g_k(x, \eta) + \]

\[ + \frac{1}{i} \sum_{s=1}^{\alpha} \frac{\partial}{\partial \eta_s} (a_{l, k}^{(j)}(x, \eta) \eta''^\alpha) (x, \nabla \varphi) \frac{\partial g_k}{\partial y^s} + \]

\[ + \sum_{|\alpha| = 2} \frac{\partial^\alpha}{\partial \eta^\alpha} (a_{l, k}^{(j)}(x, \eta) \eta''^\alpha) (x, \nabla \varphi) \left( \frac{1}{i} g_k \frac{\partial \varphi}{\partial y^\alpha} \right) + S^{m-2, m-2}. \]

As a consequence, the asymptotic expansion in (2.3) is given (modulo terms in \( S^{m-2, m-2} \)) by:

\[ a_{l, k}^{(j)}(x, \nabla \varphi) \left( \frac{\partial \varphi}{\partial x''^\alpha} \right) \left( \frac{\partial \varphi}{\partial x''^\alpha} \right)^k e + \frac{1}{i} \left( \frac{k}{2} \right) \left( \frac{\partial \varphi}{\partial x_0} \right)^{k-2} \frac{\partial^2 \varphi}{\partial x_0^2} e + k \left( \frac{\partial \varphi}{\partial x_0} \right)^{k-1} D_{x_0} e + \]

\[ + \frac{1}{i} \sum_{|\alpha| = 0} \frac{\partial}{\partial \eta_s} (a_{l, k}^{(j)}(x, \eta) \eta''^\alpha) (x, \nabla \varphi) \frac{\partial}{\partial \eta_s} \left( \frac{\partial \varphi}{\partial x_0} \right)^k e + \]

\[ + \frac{1}{i} \sum_{|\alpha| = 2} \frac{\partial^\alpha}{\partial \eta^\alpha} (a_{l, k}^{(j)}(x, \eta) \eta''^\alpha) (x, \nabla \varphi) \left( \frac{1}{i} \right) \left( \frac{\partial \varphi}{\partial x_0} \right)^k e + \]

\[ = a_{l, k}^{(j)}(x, \nabla \varphi) \left( \frac{\partial \varphi}{\partial x''^\alpha} \right) \left( \frac{\partial \varphi}{\partial x''^\alpha} \right)^k e + \]

\[ + \frac{1}{i} \left( \frac{k}{2} \right) \left( \frac{\partial \varphi}{\partial x_0} \right)^{k-2} \frac{\partial^2 \varphi}{\partial x_0^2} e + \]

\[ + \sum_{|\alpha| = 2} \frac{\partial^\alpha}{\partial \eta^\alpha} (a_{l, k}^{(j)}(x, \eta) \eta''^\alpha) (x, \nabla \varphi) \left( \frac{1}{i} \right) \frac{\partial \varphi}{\partial \eta_s} \left( \frac{\partial \varphi}{\partial x_0} \right)^k e + \]
In the same way we get:

\[ e^{-i\varphi} B_k(x, D_x) (e^{i\varphi} a_k) \sim \sum_{|m|\geq 0} \frac{1}{\beta!} \frac{\partial^{m}}{\partial \varphi^{m}} (b_k(x, \eta)) (x, \nabla_x \varphi) D^m_x (a_k(x_0, z, \eta) e^{i\varphi})_{z=x} \]

\[ = b_k(x, \nabla_x \varphi) \left( \frac{\partial \varphi}{\partial x_0} \right)^k e + S^{m-2, m-2}, \quad k = 0, \ldots, m-1. \]

Hence (2.2) is proved. Furthermore, taking into account that \( S^{m-2, m-2} \subset S^{m-1, m} \), by using the asymptotic expansion of the symbol \( p \) and by applying Taylor's formula in (2.2), we can get rid of the terms which are in \( S^{m-1, m} \) and obtain:

(2.4) \[ e^{-i\varphi} P(x, D_x) (e^{i\varphi} a_k) = \sum_{|m|\geq 0} \frac{1}{\beta!} \frac{\partial^{m}}{\partial \varphi^{m}} (b_k(x, \eta)) (x, \nabla_x \varphi) D^m_x (a_k(x_0, z, \eta) e^{i\varphi})_{z=x} \]

\[ = b_k(x, \nabla_x \varphi) \left( \frac{\partial \varphi}{\partial x_0} \right)^k e + S^{m-1, m}, \]

where \( L_\varphi(e) = \frac{1}{i} \left\{ \sum_{j=0}^{k} \frac{\partial}{\partial x_j} + c \right\} e \), with suitable \( a_j \in S^{m-1, m-1}, \ j = 0, \ldots, p, \ c \in S^{m-1, m-1}. \)

In fact, we have:

(i) \[ \varphi(x, \nabla_x \varphi) = \varphi_m(x, \nabla_x \varphi) + \sum_{j=1}^{m} \frac{1}{\beta!} \frac{\partial^{\beta}}{\partial \varphi^{\beta}} (b_k(x, \eta)) (x, \nabla_x \varphi) \left( \frac{\partial \varphi}{\partial x_j} \right)^k + \]

\[ = \sum_{j=1}^{m} \frac{1}{\beta!} \frac{\partial^{\beta}}{\partial \varphi^{\beta}} (b_k(x, \eta)) (x, \nabla_x \varphi) \left( \frac{\partial \varphi}{\partial x_j} \right)^k + S^{m-1, m}; \]

(ii) \[ \frac{\partial \varphi}{\partial \xi_j} (x, \nabla_x \varphi) \in S^{m-1, m}; \]

(iii) \[ \forall j = 0, \ldots, p: \frac{\partial \varphi}{\partial \xi_j} (x, \nabla_x \varphi) = \frac{\partial q}{\partial \xi_j} (x, \nabla_x \varphi, \xi_j, \varphi) + S^{m-1, m}, \]

(iv) \[ \sum_{|\beta|\geq 2} \frac{1}{\beta!} \frac{\partial^{\beta}}{\partial \varphi^{\beta}} (b_k(x, \eta)) \left( \frac{\partial \varphi}{\partial \xi_j} \right)^k + \]

\[ = \sum_{|\beta|\geq 2} \frac{1}{\beta!} \frac{\partial^{\beta}}{\partial \varphi^{\beta}} (b_k(x, \eta)) \left( \frac{\partial \varphi}{\partial \xi_j} \right)^k + S^{m-1, m}; \]
As a consequence (2.4) holds with

\begin{equation}
L_{\phi}(e) = \frac{1}{i} \left\{ \sum_{j=0}^{m} \sum_{k=0}^{m} \sum_{|\xi|=m-2j-k} \left( \sum_{j=0}^{m} \frac{\partial a_{j}^{(k)}}{\partial \xi_{j}}(x, \xi, 0) \frac{\partial \phi^{(j)}}{\partial x_{k}}(\xi) \frac{\partial \phi^{(j)}}{\partial x_{k}}(\xi) \right) \right\} + \sum_{|\xi|=m-2j-k} b_{k}(x, \xi, 0) \left( \frac{\partial \phi^{(j)}}{\partial x_{k}}(\xi) \right) + \sum_{\|\phi\|=m-2j-k} \frac{\partial \phi^{(j)}}{\partial x_{k}}(\xi) \phi^{(j)}(\xi) + S_{m-1.m}.
\end{equation}
\[ \Omega_{\varepsilon} = \{ (\omega', \omega'', z, \xi) \in S^{n-1} \times \mathbb{R}^p \times \mathbb{R} \mid |\omega'-e_1|<\varepsilon, |\xi|<\varepsilon, 1-\varepsilon<z^2+|\omega''|<1+\varepsilon \} \]

(\varepsilon \text{ small}) and \( \phi^{(1)} \) solves the Cauchy problem:

\[ (2.6) \begin{cases} \sum_{k=0}^{m} \sum_{|\alpha|=m-k} a_{n,k}^{(0)}(x, \omega', \xi) \frac{\partial \phi^{(1)}}{\partial x''} \frac{\partial \phi^{(1)}}{\partial x''} \left( \frac{\partial \phi^{(1)}}{\partial x''} \right)^k + \\
+ \sum_{j=1}^{[m/2]} \sum_{k=0}^{m-2j} \sum_{|\alpha|=m-2j-k} a_{n,k}^{(j)}(x, \omega', 0) z^{2j} \left( \frac{\partial \phi^{(1)}}{\partial x''} \right)^k = 0 \\
\phi^{(1)}|_{x=0} = \langle x', \omega'' \rangle. \end{cases} \]

To prove the existence of \( m \) independent solutions of the Cauchy problem \( (2.6) \) in \( U \times \Omega_{\varepsilon} \), we first observe that for \( x=0, \omega'=e_1, z^2+|\omega''|^2=1 \), equation \( (2.6) \) reduces to

\[ (2.6)' \begin{cases} \sum_{k=0}^{m} \sum_{|\alpha|=m-k} a_{n,k}^{(0)}(0, e_1, \xi, \omega'') \omega'^{\alpha} \tau_0^k + \\
+ \sum_{j=1}^{[m/2]} \sum_{k=0}^{m-2j} \sum_{|\alpha|=m-2j-k} a_{n,k}^{(j)}(0, e_1, 0) z^{2j} \omega'^{\alpha} \tau_0^k = 0 \\
\phi^{(1)}|_{x=0} = \langle x', \omega'' \rangle. \end{cases} \]

where \( \tau_0 = \frac{\partial \phi^{(1)}}{\partial x_0} |_{x=0} \).

If \( \xi=z=0 \), equation \( (2.6)' \) becomes

\[ q_m(0, \tau_0, e_1, \omega'') = \sum_{k=0}^{m} \sum_{|\alpha|=m-k} a_{n,k}^{(0)}(0, e_1, 0) \omega'^{\alpha} \tau_0^k = 0. \]

Since \( |\omega''|=1 \), \( (H_3) \) guarantees that this equation has \( m \) real simple roots in \( \tau_0 \). On the other hand, if \( \xi=0 \) and \( 0<z \leq 1 \), \( (2.6)' \) reduces to

\[ (2.6)'' \begin{cases} \sum_{j=0}^{[m/2]} \sum_{k=0}^{m-2j} \sum_{|\alpha|=m-2j-k} a_{n,k}^{(j)}(0, e_1, 0) z^{2j} \omega'^{\alpha} \tau_0^k = 0 \\
\phi^{(1)}|_{x=0} = \langle x', \omega'' \rangle. \end{cases} \]

which is equivalent to

\[ q(0, \tau_0, e_1, \omega'') = \sum_{j=0}^{[m/2]} \sum_{k=0}^{m-2j} \sum_{|\alpha|=m-2j-k} a_{n,k}^{(j)}(0, e_1, 0) \left( \frac{\omega''}{z} \right)^{\alpha} \left( \frac{\tau_0}{z} \right)^k = 0. \]

By assumption \( (H_4) \) this equation has \( m \) real simple (smooth) roots in \( \frac{\tau_0}{z} \) for any \( \omega'' \), say \( \lambda_j \left( 0, e_1, \omega'' \right), j=1, \ldots, m \), so \( (2.6)'' \) has \( m \) real simple roots in \( \tau_0 \) of the form \( z \lambda_j \left( 0, e_1, \omega'' \right) \).

By using a compactness argument, it follows that \( (2.6) \) has \( m \) real simple roots. Hence, by applying a version with parameter of a classic result (see Th. 6.4.5 in [5]), it is possible to construct \( m \) independent solutions of \( (2.6) \), say
Clearly, for any \( j \), the \( \varphi_j \) corresponding to \( \varphi_j^{(1)} \) solve equation (2.5) in \( U \times G^T \), where

\[
G = \{(\xi', \xi'') \in \mathbb{R}^n \mid \xi'' \leq |\xi|, \frac{\xi'}{|\xi|} - \epsilon_1 \leq \eta \leq \epsilon \}, \quad T = T_\epsilon > 0.
\]

We leave to the reader to verify that \( \varphi_j \), \( j = 1, \ldots, m \), belong to \( \mathcal{S}_{U}(Z \times G; M) \).

We observe that the phases \( \varphi_j \)'s, which are the main technical tool in the construction of the parametrix, are neither homogeneous nor \( \sigma \)-homogeneous.

On the other hand, for a precise description of the singularities of the parametrix we will need other phases which take care of the propagation within \( N \) and on the simple characteristic set of \( P \).

We are led to solve the following Cauchy problems:

\begin{equation}
\begin{aligned}
&\sum_{j=0}^{m-2} \sum_{a=0}^{m-2} \sum_{|\alpha|=m-2-j} \alpha_{a,\beta}(x, \xi', 0) \left( \frac{\partial \varphi_j^{(1)}}{\partial x_0^\alpha} \right) \left( \frac{\partial \varphi_j^{(1)}}{\partial x_0^\beta} \right) = 0 \\
&\varphi_j^{(1)} \mid_{x_0=0} = \langle x'', \xi'' \rangle
\end{aligned}
\end{equation}

\begin{equation}
\begin{aligned}
&\sum_{a=0}^{m} \sum_{|\alpha|=m-a} \alpha_{a,\beta}(x, \xi', \xi'', 0) \left( \frac{\partial \varphi_j^{(1)}}{\partial x_0^\alpha} \right) \left( \frac{\partial \varphi_j^{(1)}}{\partial x_0^\beta} \right) = 0 \\
&\varphi_j^{(1)} \mid_{x_0=0} = \langle x'', \xi'' \rangle
\end{aligned}
\end{equation}

\begin{equation}
\begin{aligned}
&\sum_{a=0}^{m} \sum_{|\alpha|=m-a} \alpha_{a,\beta}(x, \xi', 0) \left( \frac{\partial \varphi_j^{(1)}}{\partial x_0^\alpha} \right) \left( \frac{\partial \varphi_j^{(1)}}{\partial x_0^\beta} \right) = 0 \\
&\varphi_j^{(1)} \mid_{x_0=0} = \langle x'', \xi'' \rangle
\end{aligned}
\end{equation}

By putting as in (2.1)

\[
\begin{aligned}
\varphi(x, \eta) &= \varphi_j^{(1)}(x, \eta) + \langle x', \xi' \rangle, \quad \Phi(x, \eta) = \Phi_j^{(1)}(x, \eta) + \langle x', \xi' \rangle, \\
\Psi(x, \eta) &= \Psi_j^{(1)}(x, \eta) + \langle x', \xi' \rangle,
\end{aligned}
\]

we have the following existence result:

**Proposition 2.2.** If \( U, G \) are as in Prop. 2.1, the equation (2.7) (resp. (2.8), (2.9)) are solvable in \( U \times G^T \) (resp. \( U \times G^T \cap \{\xi'' = 0\} \)), for \( T = T_\epsilon \) large, and each of them has \( m \) independent solutions \( \varphi_j^{(1)}(x, \eta), \Phi_j^{(1)}(x, \eta), \Psi_j^{(1)}(x, \eta), j = 1, \ldots, m \), respectively. Moreover, \( \varphi_j^{(1)}(x, \eta), j = 1, \ldots, m \), are \( \sigma \)-homogeneous symbols of degree 1 in \( S^{1,1}(U \times G; M) \), whereas \( \Phi_j^{(1)}(x, \eta), \Psi_j^{(1)}(x, \eta), j = 1, \ldots, m \), are positively homogeneous symbols of degree 1 in \( S(U \times G \cap \{\xi'' = 0\}) \).

**Proof.** If \( \varphi_j^{(1)}, j = 1, \ldots, m \), are the \( m \) solutions of (2.6) we found in Prop. 2.1, it is easy to verify that...
\[ \psi_j^{(1)}(x, \eta) = \langle \eta \rangle \phi_j^{(1)} \left( x, \frac{\xi'}{|\xi'|}, \frac{\xi''}{|\xi'|}, \frac{|\xi''|^2}{|\xi'|}, 0 \right), \quad j = 1, \ldots, m, \]

solve (2.7) in \( U \times G \), whereas

\[ \Phi_j^{(1)}(x, \eta) = \frac{\xi''}{|\xi'|} \phi_j^{(1)} \left( x, \frac{\xi'}{|\xi'|}, \frac{\xi''}{|\xi'|}, 0, \frac{|\xi''|^2}{|\xi'|} \right), \]

\[ \Psi_j^{(1)}(x, \eta) = \frac{\xi''}{|\xi'|} \phi_j^{(1)} \left( x, \frac{\xi'}{|\xi'|}, \frac{\xi''}{|\xi'|}, 0, 0 \right), \quad j = 1, \ldots, m \]

are defined in \( U \times G \) for \( \xi'' \neq 0 \) and there they are solutions of (2.8) and (2.9) respectively.

It follows from the definition that \( \psi_j^{(1)}(x, \eta) \) are \( \sigma \)-homogeneous symbols of degree 1 belonging to \( S^{1,1}(U \times G; M) \), while \( \Phi_j^{(1)}(x, \eta) \) and \( \Psi_j^{(1)}(x, \eta) \) are homogeneous symbols of degree 1 in \( S^{1}(U \times G \cap \{ \xi'' \neq 0 \}) \).

We now show how the phases \( \psi \) and \( \Phi \) are related to \( \varphi \) on suitable subsets of \( U \times G \).

Precisely, we have the following:

**Corollary 2.3.** Under the same assumption of Proposition 2.2, we have:

\[ \varphi_j(x, \eta) = \psi_j(x, \eta) + \frac{\langle \eta \rangle^2}{|\xi'|} \rho_j(x, \eta) \]

where \( \rho_j(x, \eta) = \frac{\langle \eta \rangle^2}{|\xi'|} \rho_j(x, \eta) \) verify estimates of type \( S^{0,0} \) in any \( \sigma \)-conic set of the form \( \Gamma' = \{ (x, \eta) \in U \times G \times |\xi''|^2 \leq c'|\xi'| \} \);

\[ \varphi_j^{(1)}(x, \eta) = \frac{|\xi'|}{|\xi''|} \sigma_j^{(1)}(x, \eta) \]

where \( \sigma_j(x, \eta) = \frac{|\xi'|}{|\xi''|} \sigma_j(x, \eta) \) verify estimates of type \( S^{0,-1} \) in any \( \sigma \)-conic set of the form \( \Gamma'' = \{ (x, \eta) \in U \times G \times |\xi''|^2 \geq c''|\xi'| \} \).

**Proof.** Using Taylor's formula at \( \xi = 0 \) we get:

\[ \varphi_j^{(1)}(x, \eta) = \psi_j^{(1)}(x, \eta) + \frac{\langle \eta \rangle^2}{|\xi'|} \rho_j(x, \eta) \quad \text{with} \quad \rho_j \in S^{0,0}(U \times G; M). \]

Since \( \frac{\langle \eta \rangle^2}{|\xi'|} \) verify estimates of type \( S^{0,0} \) on every set

\[ \Gamma' = \{ (x, \eta) \in U \times G \times |\xi''|^2 \leq c'|\xi'| \}, \]

we obtain (i).

On the other hand, on any \( \sigma \)-conic set of the form

\[ \Gamma'' = \{ (x, \eta) \in U \times G \times |\xi''|^2 \geq c''|\xi'| \}, \]

by the uniqueness of the solutions of the Cauchy problem (2.6), we can also write

\[ \varphi_j^{(1)}(x, \eta) = \frac{|\xi'|}{|\xi''|} \phi_j^{(1)} \left( x, \frac{\xi'}{|\xi'|}, \frac{\xi''}{|\xi'|}, \frac{|\xi''|^2}{|\xi'|}, 0 \right). \]
Application of Taylor’s formula at $z=0$ yields

$$\varphi_j^{(1)}(x, \eta) = \Phi_j^{(1)}(x, \eta) + \left| \frac{\xi'}{\xi''} \right| \sigma'(x, \eta)$$

for some $\sigma' \in S^{0,0}(U \times G; M)$. Since $\left| \frac{\xi'}{\xi''} \right|$ verifies estimates of type $S^{0,-1}$ on $\Gamma''$, claim (ii) follows.

It will be useful to consider all the $\varphi_j^{(1)}, \Psi_j^{(1)}, \Phi_j^{(1)}, \Psi_j^{(1)}, j=1, \ldots, m$, as smoothly defined on the whole $U \times G$, trivially extending them as 0 in $U \times G$ when $|\eta|<T$.

2(b). Transport equations

If $\varphi_j$ is one of the phases determined in Sect. 2(a) and $\varphi \in S^{0,0}$, from (2.4)

we get:

\begin{equation}
(2.10) 
\Psi_j^{(1)} = L_\varphi^{(1)}(\xi') + R^{(j)}(\xi') 
\end{equation}

where $L_\varphi^{(1)}$ is the first order operator (2.4)' with $\varphi=\varphi_j$ and $R^{(j)}: S^{0,0} \rightarrow S^{m-1,m}$. Let us observe that, possibly after shrinking $U$ and $G$, we can suppose that the coefficient $a_0$ of $\frac{\partial}{\partial x_0}$ in $L_\varphi^{(1)}$ is different from zero on $U \times Gr$, as follows by observing that from (2.4)' we have:

$$<\eta, \eta>^{1-m} a_0(x, \xi', \xi'') = <\eta, \eta>^{1-m} \frac{\partial q}{\partial x_0} (x, \partial \varphi_j^{(1)}, \xi', \partial \varphi_j^{(1)})$$

$$= \sum_{j=0}^{m/2} \sum_{k=1}^{m/2-j} \sum_{|\omega|=m-2j-k} a_{\omega,k}^{(j)}(x, \omega', 0) z^2 \lambda^{k-1} \frac{\partial \varphi_j^{(1)}}{\partial x_0} k$$

which for $x=0, \omega'=e_1, \omega''=e_0$ and $\xi=0$ reduces to

\begin{equation}
(2.11) 
\sum_{j=0}^{m/2} \sum_{k=1}^{m/2-j} \sum_{|\omega|=m-2j-k} a_{\omega,k}^{(j)}(0, e_1, 0) z^{2j} \lambda^{k-1} \frac{\partial \varphi_j^{(1)}}{\partial x_0} |_{x=0}
\end{equation}

with $\lambda=\frac{\partial \varphi_j^{(1)}}{\partial x_0} |_{x=0}$.

Since the roots in $\tau_0$ of equation (2.6)' are simple, (2.11) is different from zero and, as a consequence, $a_0(x, \xi', \xi'') \geq \lambda^{m-1} \varphi_j^{(1)}$ on $U \times G^T$ if $U$ is a small neighborhood of the origin and $G$ is contained in the set described by $(\xi', \xi'')$ when $\lambda=\frac{\xi'}{\xi''} \frac{\xi''}{\xi'} \frac{\xi'}{\xi''} \frac{\xi''}{\xi'} \frac{\xi'}{\xi''} \frac{\xi''}{\xi'} |\xi''| |\xi'|$ belongs to

$$\Omega_\varepsilon = \{ (\omega', \omega'' , z, \xi) \in S^{n-p-1} \times R^2 \times R^2 \times R |$$

$$|\omega' - e_1| < \varepsilon, |\xi| < \varepsilon, 1 - \varepsilon < z^2 + |\omega''|^2 < 1 + \varepsilon \}$$

with a suitable small $\varepsilon$. 
Let us fix some notation. If \( V = \{(x_0, y, z) | (x_0, y) \in U, (0, z) \in \Omega \} \), we put \( \Gamma = V \times G, \partial \Gamma = \{(y, z, \eta)| (0, y, z, \eta) \in \Gamma \} \) and

\[
\Gamma^{c,T} = \Gamma \cap \{ (x = (x_0, y), z, \eta = (\xi', \xi'', \eta')) \in \mathbb{R}^{n+1} \times \mathbb{R}^n \times \mathbb{R}^n \setminus 0 | ||\xi'||^2 \geq c \ | \xi'|, | \xi''| \geq T \}, c, T > 0.
\]

In this section we will look for suitable amplitudes \( e_j(x, y, \eta) \in S^0(\mathbb{R}^{2n+1} \times \mathbb{R}^n \setminus 0; \mathcal{M}) \), with \( \text{supp}(e_j) \subset \Gamma^T \), for any \( j = 1, \ldots, m \). We will construct every \( e_j \) as a sum of two amplitudes.

More precisely we have the following result:

**Proposition 2.3.** If \( \Gamma \) is sufficiently small, \( \omega \) is a small neighborhood of 0 in \( \mathbb{R}^{n+1}, c, T \) are large enough, for any \( k(y, z, \eta) \in S^0 \) supported in a small neighborhood of \( (0, 0, \xi' = \xi'' = 0) = (0, 0, \eta) \) in \( \partial \Gamma^T \), there exist \( \tilde{e}_j \in S^0(\mathbb{R}^{2n+1} \times \mathbb{R}^n \setminus 0; \mathcal{M}) \), \( \text{supp}(\tilde{e}_j) \subset \Gamma^T \) and \( \tilde{p}_j \in S^0(\mathbb{R}^{2n+1} \times \mathbb{R}^n \setminus 0) \), \( \text{supp}(\tilde{p}_j) \subset \Gamma^{c,T}, j = 1, \ldots, m \), such that \( e_j = \tilde{e}_j + \tilde{p}_j \) satisfies

\[
\begin{cases}
\left| e^{-i\psi_j} P(e^{i\psi_j}, e_j) \right|_{\omega \times \mathbb{R} \times \mathbb{R}^n \setminus 0} \in S^{-m}(\omega \times \mathbb{R} \times \mathbb{R}^n \setminus 0) \\
\left| e_j \right|_{x_0 = 0} = k \text{ mod } S^{-m}, \quad j = 1, \ldots, m.
\end{cases}
\]

To prove Prop. 2.3 we need two preliminary results.

**Lemma 2.4.** If \( \Gamma \) is small enough, \( \varepsilon > 0 \) is small, \( j = 1, \ldots, m \) and \( h \in \mathbb{Z}_+ \), then, for any \( \tilde{f} \in S^{m-1, m-1+h}(\mathbb{R}^{2n+1} \times \mathbb{R}^n \setminus 0; \mathcal{M}) \), \( \text{supp}(\tilde{f}) \subset \Gamma^T \), and for any \( \tilde{e} \in S^{0,h}(\mathbb{R}^{2n} \times \mathbb{R}^n \setminus 0; \mathcal{M}) \), \( \text{supp}(\tilde{e}) \subset \partial \Gamma^T \), there exists \( e \in S^{0,h}(\mathbb{R}^{2n+1} \times \mathbb{R}^n \setminus 0; \mathcal{M}) \) with \( \text{supp}(e) \subset \Gamma^T \), such that

\[
\begin{cases}
L_p^{(j)}(e) = \tilde{f} \text{ if } \ |x_0| \leq \varepsilon \\
\left| e \right|_{x_0 = 0} = \tilde{e}.
\end{cases}
\]

**Proof.** By dividing the coefficients \( a_i, i = 0, \ldots, p \) and \( c \) of the operator \( L_p^{(j)} \) for \( \langle \eta \rangle^{-m-1} \), we are led to study a first order equation with respect to \( x \), with coefficients in \( S^{0,0}(U \times G; \mathcal{M}) \). We must verify the possibility of solving this equations globally with respect to \( \xi \).

Let us observe that it is possible to express \( \tilde{a}_i = \langle \eta \rangle^{1-m} a_i, i = 0, \ldots, p, \tilde{c} = \langle \eta \rangle^{1-m} c \), as \( C^\infty \) functions of \( x \) and of the parameter \( \lambda = \left( \frac{\xi'}{\xi}, \frac{\xi''}{\xi}, |\xi'|^2, \frac{\xi'}{\xi}, \frac{\xi''}{\xi}, |\xi'| \right) \); to be more precise, \( \tilde{a}_i(x, \lambda), \tilde{c}(x, \lambda) \) are \( C^\infty(U \times \Omega_\varepsilon, \varepsilon > 0) \), where \( \Omega_\varepsilon \) is the set described by \( \lambda \) when \( \xi \) varies in \( G^T \). As we noted at the beginning of this section, we can also suppose that \( \tilde{a}_0(x, \lambda) \not= 0 \) when \( (x, \lambda) \in U \times \Omega_\varepsilon \).

By integrating the Hamiltonian flow starting from \( x_0 = 0 \), when \( U \) is sufficiently small, we get a diffeomorphism \( \chi: (x, \lambda) \mapsto (x_0, x', \tilde{p}''(x, \lambda)), \lambda \), from \( U \times \Omega_\varepsilon \) onto
its image, such that $\vec{x}_{\ast}'(x, \frac{\xi'}{|\xi'|}, \frac{\xi''}{|\xi'|}, \frac{\eta}{|\xi'|}, \frac{\eta}{|\xi'|})$, $i=1, \ldots, p$ are in $S^{0,0}(U \times G^T; M)$ and verify $|\det(\frac{\partial \vec{x}_{\ast}'}{\partial x_{\ast}'})| \geq c > 0$ for $(x, \eta) \in U \times G^T$. Moreover, in these coordinates, the vector field $\frac{\partial}{\partial x_{\ast}} + \sum_{i=1}^{p} \tilde{a}_{\ast i} \frac{\partial}{\partial x_{\ast}'}$ is transformed into $\frac{\partial}{\partial x_{\ast}}$. In fact, assuming that a cutoff function with respect to $x_{\ast}$ is applied to the coefficients $\tilde{a}_{\ast i}$, $i=1, \ldots, p$ and putting $\sigma = (x', \lambda)$, we obtain the system

$$\begin{align*}
\vec{x}_{\ast}'(t) &= F(t, x_{\ast}'(t), \sigma) \\
\vec{x}_{\ast}'(0) &= x_{\ast}'
\end{align*}$$

with $x_0 = t$, $F = (\tilde{a}_{\ast i} \tilde{a}_i, \ldots, \tilde{a}_{\ast i} \tilde{a}_i)$ and $F(t, x_{\ast}', \sigma) = 0$ when $|x_{\ast}'| \geq C$.

Thus, for $|t| < T$, there exists $C_T > C$ such that $\vec{x}_{\ast}'(t, x_{\ast}', \sigma) = x_{\ast}'$ for $|x_{\ast}'| \geq C_T$.

On the other hand, when $|x_{\ast}'| \leq C_T$, since $\frac{\partial \vec{x}_{\ast}'}{\partial x_{\ast}'}(0, x_{\ast}', \sigma) = I_\mu$, the map $x_{\ast} \mapsto \vec{x}_{\ast}'(t, x_{\ast}', \sigma)$ is locally invertible for $|t| \leq T$ for some $T \leq T$.

Finally, we observe that if $f \in S^{0,0}(\mathbb{R}^{2n+1} \times \mathbb{R}^n \setminus 0; M)$ has sufficiently small support then $f$ defined by $f(x, \eta) = \tilde{f}(x, \eta)$ still belongs to $S^{0,0}(\mathbb{R}^{2n+1} \times \mathbb{R}^n \setminus 0; M)$ and that $\exp(\tilde{a}_{\ast i})$ is in $S^{0,0}(U \times G; M)$, because $\tilde{a}_{\ast i} \in S^{0,0}(U \times G; M)$.

We can thus construct $e \in S^{0,0}(\mathbb{R}^{2n+1} \times \mathbb{R}^n \setminus 0; M)$, supp$(e) \subset \Gamma^T$, satisfying (2.13).

For the next result, we first need a definition.

**Definition.** If $g \in S^0$, we say that $g$ is "flat" on $M$ iff

$$\forall N \geq 0, \quad \left(\frac{|\xi'|}{|\xi'|}\right)^{-N} g \in S^0.$$

We have:

**Lemma 2.5.** If $T$ is sufficiently small, $\epsilon$, $T$ are sufficiently large and $\epsilon > 0$ is small, then for any $h \in S^{m-1}(\mathbb{R}^{2n+1} \times \mathbb{R}^n \setminus 0)$ flat on $M$, supp$(h) \subset \Gamma^T$, there exists $r \in S^{-1}(\mathbb{R}^{2n+1} \times \mathbb{R}^n \setminus 0)$ flat on $M$ such that

$$e^{-i\Phi} P(e^{i\Phi} r) = h \quad \text{modulo a symbol in } S^{m-2} \text{ flat on } M, \text{ if } |x_0| \leq \epsilon$$

(2.14)

for any $t \in \mathbb{Z}_+$, where $\Phi$ is any of the $\Phi_j$'s in Proposition 2.2.

**Proof.** We have to verify that, in spite of the singularities of the function $\Phi$ for $\xi'' \neq 0$, it is possible to perform the classical construction by means of flat symbols. Let $r \in S^{-1}(\mathbb{R}^{2n+1} \times \mathbb{R}^n \setminus 0)$ be flat on $M$. We claim that:

$$e^{-i\Phi} P(x, D_x)(e^{i\Phi} r) = p_n(x, \nabla_x \Phi)r + \tilde{L}(r) \quad \text{modulo a symbol in } S^{m-2} \text{ flat on } M,$$
where

\[ \mathcal{L} = \frac{1}{i} \left\{ \sum_{i=0}^{n} a_i \frac{\partial}{\partial x_i} + c \right\} \]

is the usual transport operator i.e.

\[ a_i = \frac{\partial p_m}{\partial \xi_i}(x, \nabla_x \Phi), \quad i = 0, \ldots, n, \]

\[ c = \sum_{|\beta| = 2} \frac{1}{\beta!} \frac{\partial^\beta p_m}{\partial \xi^\beta}(x, \nabla_x \Phi) \frac{\partial^{\beta(1)}}{\partial y^\beta} + ib_{m-1}(x, \nabla_x \Phi) \left( \frac{\partial^{\Phi(1)}}{\partial x_0} \right)^{m-1} + \]

\[ + i \sum_{\substack{|\alpha| = m-2 \atop \Sigma \neq 0}} \sum_{|\alpha| = m-2 \atop \Sigma \neq 0} a_{\alpha, i}(x, \nabla_x \Phi) \left( \frac{\partial^{\Phi(1)}}{\partial x_i} \right)^{m-1} \left( \frac{\partial^{\Phi(1)}}{\partial x_0} \right)^{k}. \]

In fact, by considering the expansion (2.2) corresponding to \( \Phi \) and proceeding as in Sect. 2(a), we have

(i) \( p(x, \nabla_x \Phi) = p_m(x, \nabla_x \Phi) + \sum_{|\alpha| = m-2} \sum_{|\beta| = m-2 \atop \beta \neq 0} a_{\alpha, i}(x, \nabla_x \Phi) \left( \frac{\partial^{\Phi(1)}}{\partial x_i} \right)^{m-1} \left( \frac{\partial^{\Phi(1)}}{\partial x_0} \right)^{k} + \]

\[ + ib_{m-1}(x, \nabla_x \Phi) \left( \frac{\partial^{\Phi(1)}}{\partial x_0} \right)^{m-1} + S^{m-2}; \]

(ii) \( \frac{\partial p}{\partial \xi_i}(x, \nabla_x \Phi) = \frac{\partial p_m}{\partial \xi_i}(x, \nabla_x \Phi) + S^{m-2}, \quad \forall i = 0, \ldots, n; \]

(iii) \( \sum_{|\alpha| = m-2} \frac{1}{|\beta|} \frac{\partial^\beta p_m}{\partial \xi^\beta}(x, \nabla_x \Phi) \frac{\partial^{\beta(1)}}{\partial y^\beta} = \sum_{|\beta| = 2} \frac{1}{|\beta|} \frac{\partial^\beta p_m}{\partial \xi^\beta}(x, \nabla_x \Phi) \frac{\partial^{\beta(1)}}{\partial y^\beta} + S^{m-2}. \]

It comes out that the \( a_i \), \( i = 0, \ldots, n, \) belong to \( S^{m-1, m-1}(U \times G^T) \), while \( \text{Re} c \in S^{m-1, m-1}(U \times G^T) \) and \( \text{Im} c \in S^{m-1, m-1}(U \times G^T) \).

By the same kind of arguments used in the beginning of this section, we get \( |a_0| \geq |\xi'|^{m-1} \). Hence, since \( |\xi''| \approx |\eta| d_M \) on \( \Gamma^c, \) we get \( |a_0| \geq |\eta| |m-1| d_{m-1} \) on any \( \alpha \)-conic set \( \Gamma^c, \).

Let us point out that \( p_m(x, \nabla_x \Phi) = 0. \)

In order to establish the global solvability with respect to \( \xi \) of the equation \( \tilde{L}(r) = h, \) for \( x \) sufficiently close to 0, we can go on in the same way as in Lemma 2.4. Putting \( \delta_i = |\xi''|^{1-m} a_i, i = 0, \ldots, n, \) \( \delta = |\xi''|^{1-m} c \) and integrating the Hamiltonian flow starting from \( x_0 = 0, \) we obtain the existence of a diffeomorphism transforming the vector field \( \frac{\partial}{\partial x_0} + \sum_{i=1}^{n} \delta_i \frac{\partial}{\partial x_i} \) into \( \frac{\partial}{\partial x_0} \) on

\[ U \times (G \cap \{ \eta = (\xi', \xi'') \}) \in R^n \setminus 0, |\xi''|^2 \geq c |\xi'|, |\xi'| \geq T. \]

for a suitable choice of a neighborhood \( U \) of the origin and of the conic set \( G. \)

Then for any \( t \in Z, \) and for any \( h \in S^{m-1-t}(R^{n+1} \times R^n \setminus 0) \) flat on \( M \) with \( \text{supp}(h) \subset \Gamma^c, \) it is possible to find a solution \( r \in S^{-t} \) flat on \( M \) of the usual transport equation \( \tilde{L}(r) = h, \) with \( r|_{x_0=0} = 0. \)

Proof of Proposition 2.3.
By a well known argument, using (2.10) and Lemma (2.4) we can find a symbol $\tilde{e}_j \in S^{0,0}(\mathbb{R}^{2n+1} \times \mathbb{R}^n \setminus \{0\}; M)$ with $\text{supp}(\tilde{e}_j) \subset \Gamma^\nu$ such that for a suitable neighborhood $\omega$ of the origin

$$\begin{cases} e^{-i\varphi}P(\bar{e}^\varphi \tilde{e}_j)|_{\omega \times \mathbb{R}^n \times \mathbb{R}^n \setminus \{0\}} = f_j \\ \varphi_j|_{x_0=0} = k \mod S^{-\infty} \end{cases}$$

with $f_j \in \bigcap_{k=0}^{\infty}(\omega \times \mathbb{R}^n \times \mathbb{R}^n \setminus \{0\}, M) = S^{n-1,\infty}(\omega \times \mathbb{R}^n \times \mathbb{R}^n \setminus \{0\}, M), \text{supp}(f_j) \subset \Gamma^\nu$.

If $\chi \in C_0^\infty(\mathbb{R}), \chi(t)=1$ when $t \leq c/2$ and $\chi(t)=0$ for $t \geq c$, $c$ large enough, we write

$$f_j = \chi\left(\frac{1}{|\xi'|^2}\right)f_j + g_j,$$

and we observe that the term $\chi\left(\frac{1}{|\xi'|^2}\right)f_j$ belongs to $S^{-\infty}$ since

$$|\chi\left(\frac{1}{|\xi'|^2}\right)f_j| \lesssim |\eta|^{-N}d_{\eta}^N \lesssim \frac{\eta^{-1}(\xi'')^2}{|\eta|^N} \lesssim |\eta|^{-1-N}\xi'|^N \lesssim |\eta|^{-1-N},$$

$\forall N \geq 0$ (being $|\xi'|^2 \leq \frac{c}{2} |\xi'|$ on supp ($\chi$)).

On the other hand, $g_j$ is of class $S^{n-1}(\mathbb{R}^{2n+1} \times \mathbb{R}^n \setminus \{0\})$, flat on $M$, with $\text{supp} (g_j) \subset \Gamma^{n,\tau}$ since

$$\left(\frac{1}{|\xi'|}\right)^{-N}g_j = \left(\frac{1}{|\xi'|}\right)^{-N}\left(1-\chi\left(\frac{1}{|\xi'|}\right)\right)f_j \lesssim \left(\frac{1}{|\xi'|}\right)^{-N} \lesssim \left(\frac{1}{|\xi'|}\right)^{-N} \lesssim \left(\frac{1}{|\xi'|}\right)^{-N} \lesssim |\eta|^{-N}.$$

To conclude the proof of Proposition 2.3 we need to solve

$$\begin{cases} e^{-i\varphi}P(\bar{e}^\varphi \varphi_j) = -g_j \mod S^{-\infty} \\ \varphi_j|_{x_0=0} = 0 \mod S^{-\infty}. \end{cases}$$

We first observe that, given a symbol $g$ of class $S^0(\mathbb{R}^{2n+1} \times \mathbb{R}^n \setminus \{0\}), g \in C_0^\infty(\mathbb{R}^n \times \mathbb{R}^n \setminus \{0\}),$ flat on $M$ with $\text{supp} (g) \subset \Gamma^{n,\tau}$, for $c$ sufficiently large, then by Corollary 2.3 (ii), we have

$$g e^{i\varphi_j} = (g e^{i\varphi_j}) e^{i\varphi_j} \quad \forall j = 1, \cdots, m$$

with $\sigma_j \in S^{0,1}(U \times G; M)$.

Then, by Lemma 4.33 in [8] Chapter III, $h_j = g e^{i\varphi_j}$ is still a symbol of class $S^0(\mathbb{R}^{2n+1} \times \mathbb{R}^n \setminus \{0\})$ flat on $M$.

By applying Lemma 2.5, we can find a symbol $r_j^0 \in S^0$ flat on $M$ such that

$$\begin{cases} e^{-i\varphi_j}P(\bar{e}^\varphi r_j^0) = -e^{i\varphi_j}g_j \mod S^{n-2} \text{ flat on } M \\ r_j^0|_{x_0=0} = 0. \end{cases}$$
Then $\mathcal{P}_0 = e^{-i\sigma_0}r_0$ is still a symbol of class $S^0$ flat on $M$ such that, modulo $S^{-\infty}$, we have
\[
\begin{cases}
  e^{-i\sigma}P(e^{i\sigma}(\overline{e} + \mathcal{P}) \in S^{-2} flat on M \\
  \overline{e} + \mathcal{P} |_{x_0=0} = k.
\end{cases}
\]

By repeating the same argument, we can construct an asymptotic sum $\mathcal{P}_j \sim \sum k \mathcal{P}_j$ with $\mathcal{P}_j \in S^{-k}$ flat on $M$ such that Proposition 2.3 holds.

2(c). Solution of the microlocal Cauchy problem

Consider now the Fourier integral operators
\[
E_j f(x) = \int e^{i\phi_j(x, y, \theta)} e_j(x_0, y, z, \theta) f(x) dx d\theta,
\]
where the phases $\phi_j$ are given by Prop. 2.1 and the amplitudes $e_j$ by Prop. 2.3. It is important to observe that we are still free to choose $e_j |_{x_0=0} = k$ since we only required $k \in S^0$, supp($k$) $\subset \partial \Gamma^\tau$.

It is clear that, since $\phi_j(x_0, y, \theta) |_{x_0=0} = \langle y, \theta \rangle$, $D_0 E_j |_{x_0=0} = 0$, $m-1$ are pseudodifferential operators having principal symbol equal to $(\partial_{x_0}\phi_j(0, y, \theta))^\tau \cdot k(y, z, \theta)$.

Moreover, we can find a conic neighborhood of $(0, \eta)$ in $\mathbb{R}^n \times \mathbb{R}^n \setminus 0$ in which the Vandermonde determinant $\det [\partial_{x_0}\phi_j(x, \theta)]_{x_0=0} = 0^\tau$ is elliptic in the class $S^0 \in S^0(n-1)/2, m(n-1)/2$, because near $(0, \eta)$, taking into account the independence of the $\phi_j$'s, we have
\[
\det [\partial_{x_0}\phi_j(x, \theta)]_{x_0=0} = \prod_{j=1}^{m-1} \det (\partial_x \phi_j - \partial_{x_0}\phi_j) (0, y, \theta) \geq \text{const} \langle \theta \rangle^{m(m-1)/2} d^{m(m-1)/2}.
\]

By using this ellipticity, we can find a combination of the "pure" solutions $E_j$ by means of pdo's on $x_0 = 0$ acting on the right hand side, in order to suitably adjust the traces of the operators $E_j$, as stated in:

**Proposition 2.6.** If $\gamma$ is a sufficiently small conic neighborhood of $(0, \eta)$ in $\mathbb{R}^n \times \mathbb{R}^n \setminus 0$, for a suitable choice of $h(y, z, \theta)$ there exist
\[
\sigma_j(y, D_0) \in \text{OPS}^{1-m(n-1)} \cdot (\mathbb{R}^n \times \mathbb{R}^n \setminus 0; M), j=1, \ldots, m
\]

such that
\[
WF(\sum_{j=1}^{m} D_0 E_j |_{x_0=0} \sigma_j - \delta_{r,m-1} I) \cap (T^* \mathbb{R}^n \setminus 0 \times \gamma) = \emptyset, \quad \forall r = 0, \ldots, m-1.
\]

(see R. Lascar [8], Chapter III, Prop. 4.38).

From Prop. 2.6 it follows that the operator $\mathcal{E} = \sum_{j=1}^{m} \mathcal{E}_j = \sum_{j=1}^{m} E_j \sigma_j$ solves (modulo $C^\infty$-functions) the Cauchy problem:
\[
\begin{cases}
P \mathcal{E} f = 0 \\
D_0 \mathcal{E} f |_{x_0=0} = \delta_{r,m-1} f, \quad r = 0, \ldots, m-1
\end{cases}
\]
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for every \( f \in C_0^\infty(Y) \) (actually for every \( f \in \mathcal{C}'(Y) \) with \( \text{WF}(f) \subset \gamma \)).

We can rewrite the kernel of the operator \( \mathcal{E} \) as:

\[
(2.15) \quad \mathcal{E}(x_0, y, z) = \sum_{j=1}^m \mathcal{E}_j(x_0, y, z) = \sum_{j=1}^m \int e^{i(\varphi_j(x, y) - \varphi_j(0, x, y))} \delta_j(x, y, z, \theta) d\theta,
\]

where \( \delta_j \in S^{1-m,1-m} \) vanish outside a closed conic neighborhood \( \Gamma \) of \((0, 0, \eta)\) in \( \mathbb{R}^{2n+1} \times \mathbb{R}^n \setminus 0 \).

If we want to construct a microlocal right parametrix for the operator \( P \), the usual procedure consists in applying the Duhamel's principle. To this purpose, we first observe that the whole preceding construction which was performed taking \( x_0 = 0 \) as the initial surface, can be actually done for all the initial surfaces \( x_0 = s \) with \(|s|\) small enough.

More precisely, we can construct for \(|s| < X_0 \leq T\) a kernel

\[
(2.16) \quad \mathcal{E}(s, x, y_0, z) = \sum_{j=1}^m \mathcal{E}_j(s, x_0, y_0, z) = \sum_{j=1}^m \int e^{i(\varphi_j(s, x_0, y_0, \theta) - \varphi_j(s, x_0, y_0))} \delta_j(s, x, y, \theta) d\theta,
\]

where \( \varphi_j(s, x_0, y_0, \theta) = \langle x', \theta \rangle + \varphi_j^{(1)}(s, x_0, y_0, \theta) \) and \( \varphi_j^{(1)} \) solve the eikonal equation in (2.5) with \( \varphi_j^{(1)}(s, x_0, y_0, \theta) |_{x_0=0} = \langle x', \theta'' \rangle \), \( \delta_j \in S^{1-m,1-m}([-X_0, X_0] \times \mathbb{R}^{2n+1} \times \mathbb{R}^n \setminus 0; M) \), satisfy equation (2.12) with \( \varphi_j = \varphi_j(s, x, y_0, \theta) \) (and suitable initial condition at \( x_0 = s \), so that the operators \( \mathcal{E}(s) = \sum_{j=1}^m \mathcal{E}_j(s) \) satisfy (modulo \( C^\infty \) functions) the Cauchy problems

\[
\begin{cases}
P \mathcal{E}(s)f = 0 \\
D_0 \mathcal{E}(s)f |_{x_0 = s} = \delta_{r,-m-1}f,
\end{cases}
\]

At this point, by applying the Duhamel's principle, we define (microlocal) forward and backward parametrices for \( P \)

\[
(2.17) \begin{cases}
(E_+ f)(x) = i \int_{x_0}^x \chi(s)(\mathcal{E}(s) \circ \gamma_t \circ A)(f)(x) ds, & f \in C_0^\infty, \\
(E_- f)(x) = -i \int_{x_0}^{x_0} \chi(s)(\mathcal{E}(s) \circ \gamma_t \circ A)(f)(x) ds, & f \in C_0^\infty
\end{cases}
\]

where \( \chi \in C_0^\infty(\mathbb{R}) \), supp \( \chi \subset [-X_0, X_0] \), \( \chi = 1 \) on \(|s| \leq X_0 \) \( \subset X_0 \), \( A \) is a fixed compactly supported pseudodifferential operator with support near \( \rho_0 \) and \( \gamma_t \) is the restriction operator to \( x_0 = s \). Since the normal directions to these surface are not in \( \text{WF}'(A) \), the operators \( \gamma_t \circ A \) are well defined for every \( f \in \mathcal{C}'(X) \) with \( \text{WF}(f) \) concentrated near \( \rho_0 \).

3. **Calculus of the wave front set of the parametrix**

Let us consider the kernel \( \mathcal{E}(x_0, y, z) \) in (2.15) as an element of \( \mathcal{D}'(\mathbb{R}^{2n+1} \times \mathbb{R}^n) \). Then \( \text{WF}(\mathcal{E}) \subset \bigcup_{j=1}^m \text{WF}(\mathcal{E}_j) \) and by the same arguments as in R. Lascar [8], Chap. III, we get.
In the same way, for the forward microlocal right parametrix $E_+$ defined in (2.17), we have $WF'(E_+) \subset \bigcup_{j=1}^{n} WF'(E_{j}^{(f)})$, where

$$(E_{j}^{(f)} f)(x) = i \int_{-\infty}^{t_{0}} \chi(s) (\tilde{E}_{j}(s) \circ \gamma_{s} \circ A)(f)(x) ds.$$ 

By regarding the kernels $\tilde{E}_{j}(s, x_{0}, y, z)$ as elements of $\mathcal{D}'((R \times R^{*+1}) \times R^{n})$, we find:

$$WF'(\tilde{E}_{j}(s)) \subset \{(s, x_{0}, \xi, (x, \eta)|s<x_{0}, \eta'' \neq 0, z = \frac{\partial \Phi_{j}}{\partial \eta}(s, x, \eta),$$

$$\xi = \frac{\partial \Phi_{j}}{\partial x}(s, x, \eta), \sigma_{0} = \frac{\partial \Phi_{j}}{\partial s}(s, x, \eta) = -\xi_{0}\} \cup$$

$$\bigcup \left\{(s, x_{0}, \xi), (x, \eta)|s<x_{0}, \xi_{0} = \sigma_{0} = \xi'' = \eta'' = 0, x' = x', \xi' = \eta' \text{ and } \exists \theta \in R^{n} 0: \theta'' = \eta', z'' = \frac{\partial \psi_{j}}{\partial \theta''}(s, x, \theta)\right\} \cup$$

$$\bigcup \left\{(s, x_{0}, \xi), (x, \eta)|s<x_{0}, \xi_{0} = \sigma_{0} = \xi'' = \eta'' = 0, x' = x', \xi' = \eta' \text{ and } \exists \theta \in R^{n} 0: \theta'' = \eta', z'' = \frac{\partial \psi_{j}}{\partial \theta''}(s, x, \theta)\right\} \cup$$

$$\bigcup \left\{(s, x_{0}, \xi), (x, \eta)|s=x_{0}, \eta'' \neq 0, y = z, \xi' = \eta', \xi'' = \eta'' = 0, y = z, \xi' = \eta'\right\} \cup$$

As a consequence, for the $WF(E_{j}^{(f)})$ we obtain:

$$WF(E_{j}^{(f)}) = \{(x, \xi), (x_{0}, \xi_{0}) \mid |x_{0}| < X_{0} \text{ and}$$

either $x_{0} > x_{0}$ and $(x_{0}, x, \xi_{0} - \xi_{0}, \eta, (y, \eta) \in WF'(\tilde{E}_{j}(x_{0})))$, 

or $x_{0} = x_{0}$ and $\exists \mu \in R$:

$(x_{0}, x, \mu - \xi_{0}, \xi_{0} - \mu, \eta, (y, \eta) \in WF'(\tilde{E}_{j}(x_{0})))$, 

or $x_{0} = x_{0}$, $\eta = \eta = 0, \xi_{0} = \xi_{0}$.
In particular, \((x_0, x, \mu - \xi_0, \xi_0 - \mu, \eta, (\bar{x}, \eta)) \in WF'((\bar{E}_j(x_0)))\) means \(x = \bar{x}, \xi = \xi_0\).

For our choice of the operator \(A\) in (2.17), the terms \(x_0 = \bar{x}_0, \eta = \eta_0 = 0, \xi_0 = \xi_0\) do not give any contribution to \(WF'(E_+)\) and we can conclude that there exists a conic neighborhood \(\Gamma\) of \(\rho_0\) such that

\[WF'(E_+) \subset C_+(\Gamma) \cup C_+^*(\Gamma) \cup C_-'(\Gamma) \cup \Delta^*(\Gamma)\]

with:

\[C_+(\Gamma) = \bigcup_{j=1}^{m} \{(x, \xi), (\bar{x}, \bar{\xi}) \in \Gamma \times \Gamma | x_0 > \bar{x}_0, \xi' \neq 0, \eta = \frac{\partial \Phi_j}{\partial \eta}(\bar{x}_0, x, \eta), \xi_0 = \frac{\partial \Phi_j}{\partial \xi}(\bar{x}_0, x, \eta)\},\]

\[C_+^*(\Gamma) = \bigcup_{j=1}^{m} \{(x, \xi), (\bar{x}, \bar{\xi}) \in \Gamma \times \Gamma | x_0 > \bar{x}_0, \xi_0 = \frac{\partial \Phi_j}{\partial \xi}(\bar{x}_0, x, \eta)\},\]

\[\Delta^*(\Gamma) \text{ being the diagonal in } \Gamma \times \Gamma.\]

The relations \(C_+, C_+^*, C_+^*\) have the following geometrical interpretation:

(i) \( (x, \xi), (\bar{x}, \bar{\xi}) \in C_+ \) if \((x, \xi)\) belongs to the forward null bicharacteristic of \(p\) starting from \((x, \xi)\) (i.e. \(x_0 > \bar{x}_0\));

(ii) \( (x, \xi), (\bar{x}, \bar{\xi}) \in C_+^* \) if \((x, \xi)\) and \((\bar{x}, \bar{\xi})\) belong to the same leaf \(F \subset \mathcal{N}\) and there exist \((\lambda_0, \lambda') \in T^*_{\Sigma, E}(F), (\bar{\lambda}_0, \bar{\lambda}') \in T^*_{\Sigma, E}(F)\) such that \(x, \xi, \lambda_0, \lambda'\) and \(\bar{x}, \bar{\xi}, \bar{\lambda}_0, \bar{\lambda}'\) are connected in \(T^*(F)\) by an integral curve of \(H_+\) (resp. \(H_{\Sigma}^\omega\)) contained in \(q^{-1}(0)\) (resp. \(q_{\Sigma}^{-1}(0)\)) with \(x_0 > \bar{x}_0\).

Clearly, similar arguments give the description of the wave front set for the backward right parametrix \(E_-\) changing the relations \(C_+, C_+^*, C_+^*\) into \(C_-, C_-^*, C_-^*\).

We observe that \(PE_\pm(f) = f, \forall f \in \mathcal{E}'(X)\) with \(WF(f) \subset \Gamma\), modulo smooth functions.

**4. Proof of the theorem**

Let us suppose that \(P\) verifies assumptions \((H_1) - (H_4); u \in \mathcal{G}'(X)\) satisfies \(Pu = f\) with \(f \in \mathcal{G}'(X), \rho_0 \in \mathcal{N} \setminus WF(f)\) and (0.1)+ holds.

As we already observed in remark 3, \(P\) verifies the same assumptions of \(P\) on \(-N = \{(x, \xi) | (x, -\xi) \in \mathcal{N}\}\). Hence we can use the same arguments of the previous Sections to construct microlocal right parametrix \(E_\pm\) for \(P\) near the point \(-\rho_0 = (\bar{x}, -\bar{\xi})\). It is easy to verify that, in some conic neighborhood \(\Gamma\)
of $\rho_0$ we have:

$$WF(E_{\pm}) \cap (-N) \cap \Gamma \subset (-C^*_+{\Gamma}) \cup (-C^*_-{\Gamma}) ,$$

where $-C^*_+$ (resp. $-C^*_-$) is the relation obtained from $C^*_+$ (resp. $C^*_-$) by changing the sign of the fiber variable in both terms.

Passing to the transposed operator $t^* E_{\pm}$, we get microlocal left parametrices for $P$ with

$$WF'(t^* E_{\pm}) = -WF'(E_{\mp}).$$

Now, if $\omega$ is a conic neighborhood of $\rho_0$ in which (0.1)$_+$ holds, by using standard cut off procedures, we can suppose that $WF(u) \subset \omega$ and $WF(t E_{\pm} u - M) \cap \omega = \emptyset$.

Arguing by contradiction, let us suppose that $\rho_0 \in WF(u) \setminus WF(f)$ i.e. $\rho_0 \in WF(E_{\pm} f) \setminus WF(f) \cap \omega$.

Then, since simple bicharacteristics for $P$ do not have limit points in $N$, it would exist $\rho' \in N \cap \omega \cap WF(f)$, $\rho' \neq \rho_0$, such that $(\rho_0, \rho') \in WF'(t E_{\mp})$ i.e.

$$\rho' \in WF(f) \cap \omega \cap ((C^*_+(\rho_0) \cup C^*_-(\rho_0)) \setminus \{\rho_0\}) \subset WF(u) \cap \omega \cap ((C^*_+(\rho_0) \cup C^*_-(\rho_0)) \setminus \{\rho_0\}) = \emptyset,$$

which is impossible.

References


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