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PROPAGATION OF SINGULARITIES FOR HYPERBOLIC OPERATORS WITH MULTIPLE INVOLUTIVE CHARACTERISTICS

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0. Introduction

The aim of this paper is to study the propagation of C^∞-singularities for an hyperbolic pseudodifferential operator whose principal symbol vanishes at order $m\geq 2$ on an involutive manifold, generalizing a well known result obtained by R. Lascar [8] Chapter III, in the case $m=2$.

Let X be an open subset of $\boldsymbol{R}^{n+1}$, denote by $T^*X\cong X\times\boldsymbol{R}^{n+1}$ the cotangent bundle with canonical coordinates (x, ξ) and let $\omega=\sum_{j=0}^{n}\xi_j\,dx_j$ (resp. $\sigma=d\omega=\sum_{j=0}^{n}d\xi_j\wedge dx_j$) denote the canonical 1-form (resp. 2-form) on T^*X. By $T^*X\backslash 0$ we denote T^*X minus the zero section. Let $P(x, D_x)$ be a classical pseudodifferential operator (*pdo*) in X of order m, $m\in\boldsymbol{N}$, with symbol

$$p(x, \xi)\sim\sum_{j\geq 0}p_{m-j}(x, \xi)$$

and let $\varphi\in C^\infty(X)$ be a real-valued function, with $d\varphi(x)\neq 0\ \forall x\in X$.
We shall make the following assumptions:

(H$_1$) P is hyperbolic with respect to the level surfaces of φ, i.e. p_m is real-valued and
  i) $p_m(x, d\varphi(x))\neq 0\ \forall x\in X$;
  ii) for every $(x, \xi)\in T^*X$, ξ independent of $d\varphi(x)$, the function $p_m(x, \xi+td\varphi(x))$ is a polynomial of degree m in t having only real roots.

(H$_2$) There exists a C^∞-conic, non radial, involutive submanifold $N\subset T^*X\backslash 0$ of codimension $p+1$, such that, for $j\geq 0$, p_{m-j} vanishes at least of order $(m-2j)_+$ on $N(t_+=\max(t, 0))$.

The above conditions on N imply that, for any $\rho\in N$, we have $T_\rho(N)^\sigma\subset T_\rho(N)$ ($T_\rho(N)^\sigma$ being the orthogonal of $T_\rho(N)$ with respect to σ) and $\omega(\rho)\notin T_\rho(N)^\sigma$.
As a consequence, N is foliated by leaves F_ρ, $\rho\in N$, which are (immersed) C^∞ submanifold of N of dimension $p+1$ transversal to the radial vector field, with $T_\rho(F_\rho)=T_\rho(N)^\sigma$ (note that $p<n$). Moreover, for every $\rho\in N$, the bilinear form σ induces an isomorphism J_ρ: $T_\rho(T^*X)/T_\rho(N)\rightarrow T_\rho^*(F_\rho)$ (see [6]).

Because of the vanishing conditions on p, we can apply the results of [3] and therefore associate to P a family $q_{m-j}, j=0, \cdots, [m/2]$, of $(m-2j)$-multilinear symmetric forms defined on $T(T^*X)/T(N)$, the normal bundle of N.
For every $\rho \in N$ and $v \in T_\rho(T^*X)/T_\rho(N)$ we define:

$$q(\rho)(v) = \sum_{j=0}^{[m/2]} q_{m-j}(\rho)(v), \quad q_{m-j}(\rho)(v) = q_{m-j}(\rho)(v, \cdots, v),$$

and observe that

$$q_m(\rho)(v, \cdots, v) = \frac{1}{m!}(d^m p_m)(\rho)(v, \cdots, v).$$

Using the isomorphism J_ρ, q_m and q will be considered as C^∞ functions of $\rho \in N$ and $v \in T^*_\rho(F_\rho)$. Thus, fixed a leaf F on N, q_m and q will be well defined as C^∞ functions on $T^*(F)$ (see [9]). Let $\tilde{\varphi} = \varphi \circ \pi$ weere π: $T^*X \to X$ is the canonical projection.

Since $H_{\tilde{\varphi}}(\rho)$ is transversal to $T_\rho(N)$, its class modulo $T_\rho(N)$, say $\hat{H}_{\tilde{\varphi}}(\rho)$, does not vanish. We shall suppose:

(H_3) $q_m(\rho)(v)$ is strictly hyperbolic with respect to $-\hat{H}_{\tilde{\varphi}}(\rho)$, $\forall \rho \in N$.

(H_4) The polynomial $t \to q(\rho)(v + t\hat{H}_{\tilde{\varphi}}(\rho))$ has m real simple roots, $\forall \rho \in N$ and $\forall v \in T_\rho(T^*X)/T_\rho(N)$.

Some comments on conditions (H_3), (H_4) are in order.

1—As will be shown in §1, condition (H_3) is equivalent to requiring that for $(x, \xi) \notin N$ and close to N, the real roots of the polynomial $p_m(x, \xi + td\varphi(x))$ are simple (ξ independent of $d\varphi(x)$), hence p_m is strictly hyperbolic outside N, at least close to N.

2—Condiciton (H_4), which is obviously invariant by change of coordinates in X, is more technical. In [10] (when $m=2$) and [1] (for $m \geq 2$), the authors consider the case of an operator P satisfying conditions (H_1)-(H_3), whereas (H_4) is replaced by a suitable Levi condition on the lower order terms of P, which in particular implies that $\forall \rho \in N$, $q_{m-j}(\rho)=0$ for $j=1, \cdots, [m/2]$.

The case (H_4), which we will treat here, is, in some sense, on the opposite side.

3—It is easy to see that if P satisfies conditions (H_1)-(H_4), then the same hypotheses are satisfied by the transposed operator ${}^tP$, with N replaced by $-N = \{(x, \xi) \mid (x, -\xi) \in N\}$.

EXAMPLES. When $m=2$, using standard arguments, we can suppose that $\varphi = x_0$, that the operator P in the form $P = -D^2_{x_0} + A(x, D)$, $x=(x_0, y)$, $y=(y', y'') \in \boldsymbol{R}^{n-p} \times \boldsymbol{R}^p$, where A is a second order pdo in $\boldsymbol{R}^n$ depending smoothly on x_0, with nonnegative principal symbol $a_2(x, \eta) = \sum_{|\alpha|=2} a_\alpha(x, \eta)\xi''^\alpha$, $\eta = (\xi', \xi'') \in \boldsymbol{R}^{n-p} \times \boldsymbol{R}^p$, and that $N = \{\xi_0 = da_2 = 0\}$.

We have, if $\rho \in N$, $v \in T_\rho(T^*X)/T_\rho(N)$,

$$q_2(\rho)(v) = \frac{1}{2}\langle \text{Hess}\, p_2(\rho)\, v, v\rangle, \quad q(\rho)(v) = q_2(\rho)(v) + p_1^s(\rho),$$

where $p_1^s(\rho)$ denotes the subprincipal symbol of P.
The hyperbolicity of P means that $a_2(x, \eta)$ is non-negative, while condition (H_3) is equivalent to require that a_2 is transversally elliptic with respect to $\xi''=0$; condition (H_4) is then equivalent to $p_1^s(\rho)>0$, $\forall\rho\in N$. This case was treated in [8].
A typical example in the case $m=4$, $\varphi=x_0$, is represented by an operator P which is factored as

$$P = Q^{(1)}Q^{(2)}+A_1^{(1)}Q^{(1)}+A_1^{(2)}Q^{(2)}+A_2\,,$$

with $Q^{(1)}=-D_{x_0}^2+\alpha(x, D_y)|D_{y''}|^2$, $Q^{(2)}=-D_{x_0}^2+\beta(x, D_y)|D_{y''}|^2$, where $\alpha(x, D_y)$, $\beta(x, D_y)$ are pdo's in y of order 0 having real positive principal symbols and, $\forall i=1, 2$, $A_1^{(i)}$(resp. A_2) are pdo's of order 1 (resp. of order 2) in $\boldsymbol{R}^n$, depending smoothly on x_0. We have $N=\{\xi_0=\xi''=0\}$ and

$$q_4(\rho)(v) = \frac{1}{4}\langle\text{Hess}\, q_2^{(1)}(\rho)\,v, v\rangle\langle\text{Hess}\, q_2^{(2)}(\rho)\,v, v\rangle\,,$$

$$q_3(\rho)(v) = \frac{1}{2}\left(a_1^{(1)}(\rho)\langle\text{Hess}\, q_2^{(1)}(\rho)\,v, v\rangle+a_1^{(2)}(\rho)\langle\text{Hess}\, q_2^{(2)}(\rho)\,v, v\rangle\right),$$

$$q_2(\rho)(v) = a_2(\rho)\,,\quad \rho\in N,\, v\in T_\rho(T^*X)/T_\rho(N)\,.$$

In this case condition (H_3) is equivalent to $\alpha(\rho)\neq\beta(\rho)$, $\forall\rho\in N$, while (H_4) means that the polynomial

$$q(\rho)(\xi_0, \xi'') = (-\xi_0^2+\alpha(\rho)|\xi''|^2)(-\xi_0^2+\beta(\rho)|\xi''|^2)+a_1^{(1)}(\rho)(-\xi_0^2+\alpha(\rho)|\xi''|^2)$$
$$+a_1^{(2)}(\rho)(-\xi_0^2+\beta(\rho)|\xi''|^2)+a_2(\rho)$$

has real simple roots in ξ_0, $\forall\rho\in N$, $\forall\xi''\in\boldsymbol{R}^p$.

We now state the main result of this paper, concerning the propagation of singularities for P.
For every $\rho_0\in N$ consider the following sets:

$C'_\pm(\rho_0)=\{\rho\in N|$ ρ belongs to the leaf $F=F_{\rho_0}$ of N and there exist point $\zeta_0\in T^*_{\rho_0}(F)$, $\zeta\in T^*_\rho(F)$ and a piece of forward (backward) null bicharacteristic of q on $T^*(F)$ joining (ρ_0, ζ_0) and $(\rho, \zeta)\}$,

$C''_\pm(\rho_0)=\{\rho\in N|$ ρ belongs to the leaf $F=F_{\rho_0}$ of N and there exist points $\zeta_0\in T^*_{\rho_0}(F)$, $\zeta\in T^*_\rho(F)$ and a piece of forward (backward) null bicharacteristic of q_m on $T^*(F)$ joining (ρ_0, ζ_0) and $(\rho, \zeta)\}$.

The main result of this paper is the following theorem:

Theorem. *Let P satisfy assumptions (H_1)-(H_4) and let $f\in\mathcal{D}'(X)$, $\rho_0\in N\backslash WF(f)$. Assume that $Pu=f$, $u\in\mathcal{D}'(X)$, and there exists a conic neighborhood ω of ρ_0 and a choice of sign $+$ or $-$ such that*

$$(0.1)_\pm \qquad WF(u)\cap\omega\cap((C'_\pm(\rho_0)\cup C''_\pm(\rho_0))\backslash\{\rho_0\} = \emptyset\,.$$

Then $\rho_0 \notin WF(u)$.

The above result will be easily obtained by constructing (microlocal) left parametrices for P. We will prove that the methods used in R. Lascar [8] can be suitably adapted to the more general case we are treating here.

1. Reduction to a normal form

Let us first fix some notations. If U is an open subset of $\boldsymbol{R}^{\nu}$ and $\Sigma \subset T^*U\backslash 0$ is a C^∞ conic submanifold, we denote by $L^{\mu,k}(U;\Sigma)$, $\mu \in \boldsymbol{R}$, $k \in \boldsymbol{Z}_+$, the class of all classical pdo's with symbols $p(x,\xi) \sim \sum_{j\geq 0} p_{\mu-j}(x,\xi)$, such that $p_{\mu-j}$ vanishes at least of order $(k-2j)_+$ on Σ, $j \geq 0$ (see [2]). With this notation, our operator P belongs to $L^{m,m}(X;N)$.
Working microlocally near a given point of N and using the same kind of arguments as in [1], Sect. 1, we can find a coordinate system $(x,\xi)=(x_0,y,\xi_0,\eta)$, $y=(x',x'')\in \boldsymbol{R}^{n-p}\times \boldsymbol{R}^p (\eta=(\xi',\xi''))$ such that, without loss of generality, $X=]-T,T[\times Y \subset \boldsymbol{R}_{x_0}\times \boldsymbol{R}^n_y$ and N, in these coordinates, is given by:

$$N = \{(x_0,y,\xi_0,\eta)\in T^*X\backslash 0 \,|\, \xi_0=0,\ \xi''=0\}\ .$$

By putting $M=\{(y,\eta)\in T^*Y\backslash 0\,|\,\xi''=0\}$ and disregarding elliptic factors, we can suppose that, modulo a smoothing operator, we have:

$$P = D^m_{x_0} + \sum_{j=1}^m A_j(x_0,y,D_y)\, D^{m-j}_{x_0}\ ,$$

for some $A_j \in C^\infty(]-T,T[, L^{j,j}(Y;M))$, $j=1,\cdots,m$.
Application of Taylor's formula to the A_j's easily yields:

$$P(x,D_x) = \sum_{j=0}^{[m/2]} \sum_{k=0}^{m-2j} \sum_{|\alpha|=m-2j-k} A^{(j)}_{\alpha,k}(x_0,y,D_y)\, D^\alpha_{x''} D^k_{x_0} + \sum_{k=0}^{m-1} B_k(x_0,y,D_y)\, D^k_{x_0}$$

where $A^{(j)}_{\alpha,k}(x,D_y)$ and $B_k(x,D_y)$ are suitable pdo's in y of order j and $\left[\dfrac{m-k-1}{2}\right]$ respectively, depending smoothly on $x_0 (A^{(0)}_{0,m}=I)$.
Given a point $\rho=(\bar{x}_0, \bar{y}=(\bar{x}',\bar{x}''), \xi_0=0, \bar{\xi}', \xi''=0)\in N$ the leaf through ρ is simply:

$$F_\rho = \{(x,\xi)\in N\,|\, x'=\bar{x}',\ \xi'=\bar{\xi}'\}\ .$$

Taking (x_0,x'',ξ_0,ξ'') as canonical variables in $T^*_\rho(F_\rho)$, one can easily see that

$$q(\rho)\,(x_0,x'',\xi_0,\xi'') = \sum_{j=0}^{[m/2]} \sum_{k=0}^{m-2j} \sum_{|\alpha|=m-2j-k} a^{(j)}_{\alpha,k}(x_0,\bar{x}',x'',\bar{\xi}',0)\, \xi''^\alpha \xi^k_0\ ,$$

$a^{(j)}_{\alpha,k}$ being the principal symbol of $A^{(j)}_{\alpha,k}$, while

$$q_m(\rho)\,(x_0,x'',\xi_0,\xi'') = \sum_{k=0}^{m} \sum_{|\alpha|=m-k} a^{(0)}_{\alpha,k}(x_0,\bar{x}',x'',\bar{\xi}',0)\, \xi''^\alpha \xi^k_0\ .$$

Condition (H_3) amounts to require that for every (x_0, x'') and $\xi''\neq 0$, and for every ρ, the polynomial $\xi_0\to q_m(\rho)(x_0, x'', \xi_0, \xi'')$ has m real simple roots, whereas condition (H_4) means that the polynomial $\xi_0\to q(\rho)(x_0, x'', \xi_0, \xi'')$ has m real simple roots for every ρ and for every (x_0, x'', ξ'') (ξ'' is allowed to be zero).
For simplicity, we will use in the following the notation:

$$q(\rho)(x_0, x'', \xi_0, \xi'') = q(x_0, \bar{x}', x'', \xi_0, \bar{\xi}', \xi''),$$
$$q_m(\rho)(x_0, x'', \xi_0, \xi'') = q_m(x_0, \bar{x}', x'', \xi_0, \bar{\xi}', \xi'').$$

REMARKS 1. Since $p_m(x, \xi)=\sum_{k=0}^{m}\sum_{|\alpha'|=m-k} a^{(0)}_{\alpha,k}(x_0, x', x'', \xi', \xi'')\,\xi''^{\alpha}\,\xi_0^k$, by writing $0\neq\xi''=r\omega$, $r\in]0, +\infty[$, $\omega\in S^{p-1}$ and $u=\xi_0/r$, we get

$$r^{-m}\,p_m(x, ru, \xi', r\omega) = \sum_{k=0}^{m}\sum_{|\alpha|=m-k} a^{(0)}_{\alpha,k}(x_0, x', x'', \xi', r\omega)\,\omega''^{\alpha}\,u^{m-k}.$$

On the other hand, for $\rho=(x_0, x', x'', \xi_0=0, \xi', \xi''=0)$, we have

$$r^{-m}\,q_m(\rho)(x_0, x'', ru, r\omega) = \sum_{k=0}^{m}\sum_{|\alpha|=m-k} a^{(0)}_{\alpha,k}(x_0, x', x'', \xi', 0)\,\omega''^{\alpha}\,u^{m-k}.$$

Using Rouché's theorem, it is not difficult to verify that the strict hyperbolicity of $q_m(\rho)$ is equivalent to require that, for r positive and sufficiently small, $u\to r^{-m}\,p_m(x, ru, \xi', r\omega)$ has m real simple roots, i.e. p_m is strictly hyperbolic near N. Moreover, using the arguments of [7], Prop. 0.3 (ii), one can show that the hamiltonian flow of H_{p_m} in Char $(P)\backslash N$ has no limit points in N.

2. It will be crucial in the sequel to observe that $q(\rho)(x_0, x'', \xi_0, \xi'')$ has a particular homogeneity property.
Precisely, for every $t>0$, if $\rho=(\bar{x}_0, \bar{y}=(\bar{x}', \bar{x}''), \xi_0=0, \bar{\xi}', \xi''=0)$, we have

$$q(\bar{x}_0, \bar{x}', \bar{x}'', 0, t^2\bar{\xi}', 0)(x_0, x'', t\xi_0, t\xi'') = t^m\,q(\rho)(x_0, x'', \xi_0, \xi''),$$

i.e., if M_t denote the dilations $M_t(\xi_0, \xi', \xi'')=(t\xi_0, t\xi', t\xi'')$, we have

$$q(\rho)(x_0, x'', \xi_0, \xi'') = \frac{1}{t^m}\,q(M_{t^2}(\rho))(x_0, x'', M_t(\xi_0, \xi'')).$$

2. Construction of a parametrix

From now on we will use the notation introduced in Sect. 1. We fix a point $\rho_0\in N$ (without loss of generality we will suppose $\rho_0=(\bar{x}=0, \xi_0=0, \bar{\eta})$, $\bar{\eta}=(\bar{\xi}'=(1, 0, \cdots, 0), \bar{\xi}''=0))$ and try to solve, microlocally near ρ_0, a Cauchy problem of the form:

$$\begin{cases} P_v = 0 \\ D^k_{x_0}\,v(0, x', x'') = \delta_{k,m-1} f(x', x''), \quad k = 0, \cdots, m-1 \end{cases}$$

for a given $f\in C_0^\infty(Y)$ supported near the origin ($\delta_{k,m-1}$ denotes the Kronecker symbol). Following an already classical procedure, we will solve the Cauchy problem by using a suitable class of Fourier integral operators. As in [8], we are led to consider operators of the form:

$$Ef(x_0, y) = \int e^{-i(\varphi(x_0,y,\eta)-\varphi(0,z,\eta))}\, e(x_0, y, z, \eta)\, f(z)\, dz d\eta\,,$$

acting on $f\in C_0^\infty(Y)$, having a suitable phase φ and amplitude e.
Since φ and e will not be classical symbols, we first fix the corresponding notation. Let $V\subset \boldsymbol{R}^v$ be an open set and let $\Gamma\subset \boldsymbol{R}^n\backslash 0$ be a conic nieghborhood of $(\xi'=e_1=(1, 0, \cdots, 0), \xi''=0)$.
By $S^{\mu,k}(V\times\Gamma; M)$, $\mu, k\in \boldsymbol{R}$, we denote the class of all functions $a(z, \xi', \xi'')\in C^\infty(V\times\Gamma)$ such that the following inequalities hold:

$$|\partial_z^\alpha \partial_{\xi'}^{\beta'} \partial_{\xi''}^{\beta''} a(z, \xi', \xi'')| \lesssim (|\xi'|+|\xi''|)^{\mu-|\beta'|-|\beta''|}\, d_M^{k-|\beta''|}(z, \eta)\,, \quad \eta = (\xi', \xi'')\,,$$

where $d_M(z, \eta)=\left(\frac{|\xi''|^2}{|\eta|^2}+\frac{1}{|\eta|}\right)^{1/2}$. The notation $\lesssim$ means that the left hand side is dominated by a positive constant times the right hand side on every $V'\times\Gamma'\subset V\times\Gamma$, for $|\eta|$ large.
When $\Gamma=\boldsymbol{R}^n\backslash 0$ we simply write $S^{\mu,k}(V; M)$ (cfr. [2] for further details).
We also denote by $\mathrm{OP}S^{\mu,k}(V\times\Gamma; M)$ (resp. $\mathrm{OP}S^{\mu,k}(V; M)$) the related class of pdo's. We will use phase functions φ of the form

$$\varphi(x_0, y, \eta) = \langle x', \xi'\rangle+\varphi^{(1)}(x_0, y, \eta)\,, \tag{2.1}$$

with $\varphi^{(1)}(x_0, y, \eta)\in S^{1,1}(U\times G; M)$, where U is some neighborhood of the origin in X and $G\subset \boldsymbol{R}^n\backslash 0$ a suitable conic neighborhood of $(\xi'=e_1, \xi''=0)$, $\varphi^{(1)}$ real valued. On $\varphi^{(1)}$ we will impose the condition

$$\left|\det\left(\frac{\partial^2\varphi^{(1)}}{\partial x_j'' \partial\xi_k''}\right)\right| \geq c>0\,,$$

when $(x_0, y, \eta)\in U\times G^T$, for T large, $G^T=\{\eta\in G \mid |\eta|\geq T\}$.
For the amplitudes, we will look for symbols $e(x_0, y, z, \eta)\in S^{0,0}(V\times G; M)$ with $V=\{(x_0, y, z) \mid (x_0, y)\in U, (0, z)\in U\}$.
Our first task will be the construction of the phase functions. It will be convenient to use the following dilations in $\boldsymbol{R}_\eta^n$, $\eta=(\xi', \xi'')$:

$$\sigma_t(\eta) = (t^2\xi', t\xi''),\ t>0\,.$$

Accordingly, a function g will be σ-homogeneous of degree k iff $g(\sigma_t(\eta))=t^k g(\eta)$ for $t>0$ and $\eta\neq 0$. We also put $\langle\eta\rangle=(|\xi''|^2+|\xi'|)^{1/2}$.

2(a). Eikonal equations

As first step we need the asymptotic expansion of

$$e^{-i\varphi(x,\eta)}P(x,D_x)\,(e^{i\varphi(x,\eta)}e(x,\eta))\,,$$

where φ is as in (2.1) and $e\in S^{0,0}$.
We claim that, modulo terms belonging to $S^{m-2,m-2}$:

$$(2.2)\quad e^{-i\varphi(x,\eta)}P(x,D_x)\,(e^{i\varphi(x,\eta)}e(x,\eta))=p(x,\nabla_x\varphi)+\frac{1}{i}\sum_{j=0}^{n}\frac{\partial p}{\partial\xi_j}(x,\nabla_x\varphi)\frac{\partial e}{\partial x_j}$$
$$+\frac{1}{i}\sum_{|\beta|=2}\frac{1}{\beta!}\frac{\partial^\beta p}{\partial\xi^\beta}(x,\nabla_x\varphi)\frac{\partial^\beta\varphi^{(1)}}{\partial y^\beta}e\,.$$

In fact, it is easily verified that $D^k_{x_0}(e^{i\varphi}e)=e^{i\varphi}g_k$, where

$$g_k(x,\eta)=\left(\frac{\partial\varphi}{\partial x_0}\right)^k e+\frac{1}{i}\binom{k}{2}\left(\frac{\partial\varphi}{\partial x_0}\right)^{k-2}\frac{\partial^2\varphi}{\partial x_0^2}e+\binom{k}{k-1}\left(\frac{\partial\varphi}{\partial x_0}\right)^{k-1}D_{x_0}e+S^{k-2,k-2}\,.$$

Moreover:

$$e^{-i\varphi}A^{(j)}_{\alpha,k}(x,D_y)\,D^\alpha_{x''}D^k_{x_0}(e^{i\varphi}e)=e^{-i\varphi}A^{(j)}_{\alpha,k}(x,D_y)\,D^\alpha_{x''}(e^{i\varphi}g_k)\sim$$
$$\sim\sum_{|\beta|\ge 0}\frac{1}{\beta!}\frac{\partial^\beta}{\partial\eta^\beta}(a^{(j)}_{\alpha,k}(x,\eta)\,\eta''^\alpha)\,(x,\nabla_y\varphi)\,D^\beta_z(g_k(x_0,z,\eta)\,e^{i\rho})\,|_{z=y}$$

with $\rho(x,z,\eta)=\varphi(x_0,z,\eta)-\varphi(x_0,y,\eta)-\langle\nabla_y\varphi(x_0,y,\eta),z-y\rangle$.
Therefore:

$$(2.3)\quad e^{-i\varphi}A^{(j)}_{\alpha,k}(x,D_y)\,D^\alpha_{x''}D^k_{x_0}(e^{i\varphi}e)=a^{(j)}_{\alpha,k}(x,\nabla_y\varphi)\left(\frac{\partial\varphi}{\partial x''}\right)^\alpha g_k(x,\eta)+$$
$$+\frac{1}{i}\sum_{h=1}^{n}\frac{\partial}{\partial\eta_h}(a^{(j)}_{\alpha,k}(x,\eta)\,\eta''^\alpha)\,(x,\nabla_y\varphi)\frac{\partial g_k}{\partial y_h}+$$
$$+\sum_{|\beta|=2}\frac{1}{\beta!}\frac{\partial^\beta}{\partial\eta^\beta}(a^{(j)}_{\alpha,k}(x,\eta)\,\eta''^\alpha)\,(x,\nabla_y\varphi)\left(\frac{1}{i}g_k\frac{\partial^\beta\varphi}{\partial y^\beta}\right)+S^{m-2,m-2}\,.$$

As a consequence, the asymptotic expansion in (2.3) is given (modulo terms in $S^{m-2,m-2}$) by:

$$a^{(j)}_{\alpha,k}(x,\nabla_y\varphi)\left(\frac{\partial\varphi^{(1)}}{\partial x''}\right)^\alpha\left[\left(\frac{\partial\varphi^{(1)}}{\partial x_0}\right)^k e+\frac{1}{i}\binom{k}{2}\left(\frac{\partial\varphi^{(1)}}{\partial x_0}\right)^{k-2}\frac{\partial^2\varphi^{(1)}}{\partial x_0^2}e+k\left(\frac{\partial\varphi^{(1)}}{\partial x_0}\right)^{k-1}D_{x_0}e\right]+$$
$$+\frac{1}{i}\sum_{h=1}^{n}\frac{\partial}{\partial\eta_h}(a^{(j)}_{\alpha,k}(x,\eta)\,\eta''^\alpha)\,(x,\nabla_y\varphi)\frac{\partial}{\partial y_h}\left(\left(\frac{\partial\varphi^{(1)}}{\partial x_0}\right)^k e\right)+$$
$$+\frac{1}{i}\sum_{|\beta|=2}\frac{1}{\beta!}\frac{\partial^\beta}{\partial\eta^\beta}(a^{(j)}_{\alpha,k}(x,\eta)\,\eta''^\alpha)\,(x,\nabla_y\varphi)\left(\frac{\partial\varphi^{(1)}}{\partial x_0}\right)^k\left(\frac{\partial^\beta\varphi^{(1)}}{\partial y^\beta}\right)e$$
$$=a^{(j)}_{\alpha,k}(x,\nabla_y\varphi)\left(\frac{\partial\varphi^{(1)}}{\partial x''}\right)^\alpha\left(\frac{\partial\varphi^{(1)}}{\partial x_0}\right)^k e+$$
$$+\frac{1}{i}\left\{k\,a^{(j)}_{\alpha,k}(x,\nabla_y\varphi)\left(\frac{\partial\varphi^{(1)}}{\partial x''}\right)^\alpha\left(\frac{\partial\varphi^{(1)}}{\partial x_0}\right)^{k-1}\frac{\partial}{\partial x_0}+\right.$$
$$\left.+\sum_{h=1}^{n}\frac{\partial}{\partial\eta_h}(a^{(j)}_{\alpha,k}(x,\eta)\,\eta''^\alpha)\,(x,\nabla_y\varphi)\left(\frac{\partial\varphi^{(1)}}{\partial x_0}\right)^k\frac{\partial}{\partial y_h}\right\}e+$$

$$+\frac{1}{i}\left\{\binom{k}{2} a^{(j)}_{\alpha,k}(x,\nabla_y\varphi)\left(\frac{\partial\varphi^{(1)}}{\partial x''}\right)^{\alpha}\left(\frac{\partial\varphi^{(1)}}{\partial x_0}\right)^{k-2}\frac{\partial^2\varphi^{(1)}}{\partial x_0^2}+\right.$$
$$+k\sum_{h=1}^{n}\frac{\partial}{\partial\eta_h}\left(a^{(j)}_{\alpha,k}(x,\eta)\,\eta''^{\alpha}\right)(x,\nabla_y\varphi)\left(\frac{\partial\varphi^{(1)}}{\partial x_0}\right)^{k-1}\frac{\partial^2\varphi^{(1)}}{\partial x_0\partial y_h}+$$
$$\left.+\sum_{|\beta|=2}\frac{1}{\beta!}\frac{\partial^\beta}{\partial\eta^\beta}\left(a^{(j)}_{\alpha,k}(x,\eta)\,\eta''^{\alpha}\right)(x,\nabla_y\varphi)\left(\frac{\partial\varphi^{(1)}}{\partial x_0}\right)^{k}\frac{\partial^\beta\varphi^{(1)}}{\partial y^\beta}\right\}.$$

In the same way we get:

$$e^{-i\varphi}B_k(x,D_y)(e^{i\varphi}a_k)\sim\sum_{|\beta|\geq 0}\frac{1}{\beta!}\frac{\partial^\beta}{\partial\eta^\beta}(b_k(x,\eta))(x,\nabla_y\varphi)\,D_z^\beta(a_k(x_0,z,\eta)\,e^{i\rho})_{z=y}$$
$$=b_k(x,\nabla_y\varphi)\left(\frac{\partial\varphi}{\partial x_0}\right)^k e+S^{m-2,m-2},\quad k=0,\cdots,m-1.$$

Hence (2.2) is proved. Furthermore, taking into account that $S^{m-2,m-2}\subset S^{m-1,m}$, by using the asymptotic expansion of the symbol p and by applying Taylor's formula in (2.2), we can get rid of the terms which are in $S^{m-1,m}$ and obtain:

$$(2.4)\quad e^{-i\varphi(x,\eta)}P(x,D_x)(e^{i\varphi(x,\eta)}e(x,\eta))=$$
$$=\sum_{k=0}^{m}\sum_{|\alpha|=m-k}a^{(0)}_{\alpha,k}\left(x,\xi'+\frac{\partial\varphi^{(1)}}{\partial x'},\frac{\partial\varphi^{(1)}}{\partial x''}\right)\left(\frac{\partial\varphi^{(1)}}{\partial x''}\right)^{\alpha}\left(\frac{\partial\varphi^{(1)}}{\partial x_0}\right)^{k}+$$
$$+\sum_{j=1}^{[m/2]}\sum_{k=0}^{m-2j}\sum_{|\alpha|=m-2j-k}a^{(j)}_{\alpha,k}(x,\xi',0)\left(\frac{\partial\varphi^{(1)}}{\partial x''}\right)^{\alpha}\left(\frac{\partial\varphi^{(1)}}{\partial x_0}\right)^{k}+L_p(e)+S^{m-1,m},$$

where $L_p(e)=\frac{1}{i}\{\sum_{j=0}^{p}a_j\frac{\partial}{\partial x_j''}+c\}\,e$, with suitable $a_j\in S^{m-1,m-1}$, $j=0,\cdots,p$, $c\in S^{m-1,m-1}$.
In fact, we have:

(i) $$p(x,\nabla_x\varphi)=p_m(x,\nabla_x\varphi)+\sum_{j=1}^{[m/2]}\sum_{k=0}^{m-2j}\sum_{|\alpha|=m-2j-k}a^{(j)}_{\alpha,k}(x,\xi',0)\left(\frac{\partial\varphi^{(1)}}{\partial x''}\right)^{\alpha}\left(\frac{\partial\varphi^{(1)}}{\partial x_0}\right)^{k}+$$
$$+\sum_{j=1}^{[m/2]}\sum_{k=0}^{m-2j}\sum_{|\alpha|=m-2j-k}\left(\sum_{h=1}^{n}\frac{\partial a^{(j)}_{\alpha,k}}{\partial\xi_h}(x,\xi',0)\frac{\partial\varphi^{(1)}}{\partial x_h}\right)\left(\frac{\partial\varphi^{(1)}}{\partial x''}\right)^{\alpha}\left(\frac{\partial\varphi^{(1)}}{\partial x_0}\right)^{k}+$$
$$+\sum_{k=0}^{m-1}b_k(x,\xi',0)\left(\frac{\partial\varphi^{(1)}}{\partial x_0}\right)^{k}+S^{m-1,m};$$

(ii) $$\frac{\partial p}{\partial\xi'}(x,\nabla_x\varphi)\in S^{m-1,m};$$

(iii) $$\forall j=0,\cdots,p:\ \frac{\partial p}{\partial\xi_j''}(x,\nabla_x\varphi)=\frac{\partial q}{\partial\xi_j''}\left(x,\frac{\partial\varphi^{(1)}}{\partial x_0},\xi',\frac{\partial\varphi^{(1)}}{\partial x''}\right)+S^{m-1,m},$$

(iv) $$\sum_{|\beta|=2}\frac{1}{\beta!}\frac{\partial^\beta p}{\partial\xi^\beta}(x,\nabla_x\varphi)\frac{\partial^\beta\varphi^{(1)}}{\partial y^\beta}$$
$$=\sum_{|(\beta_0,\beta'')|=2}\frac{1}{\beta_0!\beta''!}\frac{\partial^{(\beta_0,\beta'')}q}{\partial\xi_0^{\beta_0}\partial\xi''^{\beta''}}\left(x,\frac{\partial\varphi^{(1)}}{\partial x_0},\xi',\frac{\partial\varphi^{(1)}}{\partial x''}\right)\frac{\partial^{(\beta_0,\beta'')}\varphi^{(1)}}{\partial x_0^{\beta_0}\partial x''^{\beta''}}+$$

$$+\sum_{|\beta'|=2}\frac{1}{\beta'!}\frac{\partial^{\beta'}p}{\partial\xi'^{\beta'}}(x,\nabla_x\varphi)\frac{\partial^{\beta'}\varphi^{(1)}}{\partial x'^{\beta'}}+$$

$$+\sum_{\substack{|\beta'|=1\\|(\beta_0,\beta'')|=1}}\frac{\partial^{(\beta_0,\beta',\beta'')}p}{\partial\xi_0^{\beta_0}\partial\xi'^{\beta'}\partial\xi''^{\beta''}}(x,\nabla_x\varphi)\frac{\partial^{(\beta_0,\beta',\beta'')}\varphi^{(1)}}{\partial x_0^{\beta_0}\partial x'^{\beta'}\partial x''^{\beta''}}+S^{m-1,m}$$

$$=\sum_{|(\beta_0,\beta'')|=2}\frac{1}{\beta_0!\,\beta''!}\frac{\partial^{(\beta_0,\beta'')}q}{\partial\xi_0^{\beta_0}\partial\xi''^{\beta''}}\left(x,\frac{\partial\varphi^{(1)}}{\partial x_0},\xi',\frac{\partial\varphi^{(1)}}{\partial x''}\right)\frac{\partial^{(\beta_0,\beta'')}\varphi^{(1)}}{\partial x_0^{\beta_0}\partial x''^{\beta''}}+S^{m-1,m}.$$

As a consequence (2.4) holds with

$$(2.4)'\qquad L_p(e)=\frac{1}{i}\left\{\sum_{j=0}^{p}\frac{\partial q}{\partial\xi_j''}\left(x,\frac{\partial\varphi^{(1)}}{\partial x_0},\xi',\frac{\partial\varphi^{(1)}}{\partial x''}\right)\frac{\partial}{\partial x_j''}+q'_{m-1}\right\}e,$$

where

$$q'_{m-1}=i\left\{\sum_{j=1}^{[m/2]}\sum_{k=0}^{m-2j}\sum_{|\alpha|=m-2j-k}\left(\sum_{h=1}^{n}\frac{\partial a^{(j)}_{\alpha,k}}{\partial\xi_h}(x,\xi',0)\frac{\partial\varphi^{(1)}}{\partial x_h}\right)\left(\frac{\partial\varphi^{(1)}}{\partial x''}\right)^{\alpha}\left(\frac{\partial\varphi^{(1)}}{\partial x_0}\right)^{k}+\right.$$

$$\left.+\sum_{k=0}^{m-1}b_k(x,\xi',0)\left(\frac{\partial\varphi^{(1)}}{\partial x_0}\right)^{k}\right\}+$$

$$+\sum_{|(\beta_0,\beta'')|=2}\frac{1}{\beta_0!\,\beta''!}\frac{\partial^{(\beta_0,\beta'')}q}{\partial\xi_0^{\beta_0}\partial\xi''^{\beta''}}\left(x,\frac{\partial\varphi^{(1)}}{\partial x_0},\xi',\frac{\partial\varphi^{(1)}}{\partial x''}\right)\frac{\partial^{(\beta_0,\beta'')}\varphi^{(1)}}{\partial x_0^{\beta_0}\partial x''^{\beta''}}.$$

From (2.4) we are naturally led to impose that $\varphi^{(1)}$ satisfies the eikonal equation:

$$(2.5)\qquad\begin{cases}\displaystyle\sum_{k=0}^{m}\sum_{|\alpha|=m-k}a^{(0)}_{\alpha,k}\left(x,\xi'+\frac{\partial\varphi^{(1)}}{\partial x'},\frac{\partial\varphi^{(1)}}{\partial x''}\right)\left(\frac{\partial\varphi^{(1)}}{\partial x''}\right)^{\alpha}\left(\frac{\partial\varphi^{(1)}}{\partial x_0}\right)^{k}+\\ \displaystyle\quad+\sum_{j=1}^{[m/2]}\sum_{k=0}^{m-2j}\sum_{|\alpha|=m-2j-k}a^{(j)}_{\alpha,k}(x,\xi',0)\left(\frac{\partial\varphi^{(1)}}{\partial x''}\right)^{\alpha}\left(\frac{\partial\varphi^{(1)}}{\partial x_0}\right)^{k}=0\\ \varphi^{(1)}|_{x_0=0}=\langle x'',\xi''\rangle\end{cases}$$

The following result holds:

Proposition 2.1. *If $U\subset X$ is a sufficiently small neighborhood of the origin and G is a conic neighborhood of $\eta=(\xi'=e_1,\xi''=0)$ in $\boldsymbol{R}^n\setminus 0$ of the form*

$$G=\{(\xi',\xi'')\in\boldsymbol{R}^n\setminus 0\mid |\xi''|<\varepsilon|\xi'|,\ \left|\frac{\xi'}{|\xi'|}-e_1\right|<\varepsilon\},\ \textit{with } \varepsilon>0 \textit{ small enough,}$$

then equation (2.5) *is solvable in $U\times G^T$, for $T=T_\varepsilon$ large, and it has m independent solutions $\varphi_j^{(1)}(x,\eta)\in S^{1,1}(U\times G;M)$, $j=1,\cdots,m$.*

Proof. We look for a solution $\varphi^{(1)}$ in the form

$$\varphi^{(1)}(x,\eta)=\langle\eta\rangle\,\tilde\varphi^{(1)}\left(x,\frac{\xi'}{|\xi'|},\frac{\xi''}{\langle\eta\rangle},\frac{|\xi'|^{1/2}}{\langle\eta\rangle},\frac{\langle\eta\rangle}{|\xi'|}\right)$$

with $\tilde\varphi^{(1)}(x,\omega',\omega'',z,\zeta)\in C^\infty(U\times\Omega_\varepsilon)$, where

$$\Omega_\varepsilon = \{(\omega', \omega'', z, \zeta) \in S^{n-p-1} \times \boldsymbol{R}^p \times \boldsymbol{R} \times \boldsymbol{R} \mid |\omega' - e_1| < \varepsilon,\ |\zeta| < \varepsilon,\ 1 - \varepsilon < z^2 + |\omega''|^2 < 1 + \varepsilon\}$$

(ε small) and $\tilde{\varphi}^{(1)}$ solves the Cauchy problem:

$$(2.6)\quad \begin{cases} \sum_{k=0}^{m} \sum_{|\alpha|=m-k} a^{(0)}_{\alpha,k}\left(x, \omega' + \zeta \frac{\partial \tilde{\varphi}^{(1)}}{\partial x'}, \zeta \frac{\partial \tilde{\varphi}^{(1)}}{\partial x''}\right) \left(\frac{\partial \tilde{\varphi}^{(1)}}{\partial x''}\right)^\alpha \left(\frac{\partial \tilde{\varphi}^{(1)}}{\partial x_0}\right)^k + \\ \quad + \sum_{j=1}^{[m/2]} \sum_{k=0}^{m-2j} \sum_{|\alpha|=m-2j-k} a^{(j)}_{\alpha,k}(x, \omega', 0)\, z^{2j} \left(\frac{\partial \tilde{\varphi}^{(1)}}{\partial x''}\right)^\alpha \left(\frac{\partial \tilde{\varphi}^{(1)}}{\partial x_0}\right)^k = 0 \\ \tilde{\varphi}^{(1)}|_{x_0=0} = \langle x'', \omega'' \rangle . \end{cases}$$

To prove the existence of m independent solutions of the Cauchy problem (2.6) in $U \times \Omega_\varepsilon$, we first observe that for $x=0$, $\omega'=e_1$, $z^2+|\omega''|^2=1$, equation (2.6) reduces to

$$(2.6)' \quad \sum_{k=0}^{m} \sum_{|\alpha|=m-2j-k} a^{(0)}_{\alpha,k}(0, e_1, \zeta\omega'')\, \omega''^\alpha \tau_0^k + \sum_{j=1}^{[m/2]} \sum_{k=0}^{m-2j} \sum_{|\alpha|=m-2j-k} a^{(j)}_{\alpha,k}(0, e_1, 0)\, z^{2j} \omega''^\alpha \tau_0^k = 0$$

where $\tau_0 = \frac{\partial \tilde{\varphi}^{(1)}}{\partial x_0}|_{x=0}$.

If $\zeta = z = 0$, equation (2.6)′ becomes

$$q_m(0, \tau_0, e_1, \omega'') = \sum_{k=0}^{m} \sum_{|\alpha|=m-2j-k} a^{(0)}_{\alpha,k}(0, e_1, 0)\, \omega''^\alpha \tau_0^k = 0 .$$

Since $|\omega''|=1$, (H_3) guarantees that this equation has m real simple roots in τ_0. On the other hand, if $\zeta=0$ and $0<z\leq 1$, (2.6)′ reduces to

$$(2.6)'' \quad \sum_{j=0}^{[m/2]} \sum_{k=0}^{m-2j} \sum_{|\alpha|=m-2j-k} a^{(j)}_{\alpha,k}(0, e_1, 0)\, z^{2j} \omega''^\alpha \tau_0^k = 0$$

which is equivalent to

$$q\left(0, \frac{\tau_0}{z}, e_1, \frac{\omega''}{z}\right) = \sum_{j=0}^{[m/2]} \sum_{k=0}^{m-2j} \sum_{|\alpha|=m-2j-k} a^{(j)}_{\alpha,k}(0, e_1, 0) \left(\frac{\omega''}{z}\right)^\alpha \left(\frac{\tau_0}{z}\right)^k = 0 .$$

By assumption (H_4) this equation has m real simple (smooth) roots in $\frac{\tau_0}{z}$ for any ω'', say $\lambda_j\left(0, e_1, \frac{\omega''}{z}\right)$, $j=1, \cdots, m$, so (2.6)″ has m real simple roots in τ_0 of the form $z\lambda_j\left(0, e_1, \frac{\omega''}{z}\right)$.

By using a compactness argument, it follows that (2.6) has m real simple roots. Hence, by applying a version with parameter of a classic result (see Th. 6.4.5 in [5]), it is possible to construct m independent solutions of (2.6), say

$\tilde{\varphi}_j^{(1)}$, $j=1, \cdots, m$. Clearly, for any j, the $\varphi_j^{(1)}$ corresponding to $\tilde{\varphi}_j^{(1)}$ solve equation (2.5) in $U\times G^T$, where

$$G = \{(\xi', \xi'')\in \boldsymbol{R}^n\backslash 0 \mid |\xi''|<|\varepsilon|\xi'|, \ |\frac{\xi'}{|\xi'|}-e_1|<\varepsilon\}, \ T = T_\varepsilon>0 .$$

We leave to the reader to verify that $\varphi_j^{(1)}$, $j=1, \cdots, m$, belong to $S^{1,1}(U\times G; M)$. Since $\frac{\partial^2\varphi_j^{(1)}(x, \eta)}{\partial x_h''\partial\xi_k''}|_{x_0=0}=I$, we get $\left|\det\left(\frac{\partial^2\varphi_j^{(1)}(x, \eta)}{\partial x_h''\partial\xi_k''}\right)\right|\geq c>0$ for $(x, \eta)\in U\times G^T$, $\forall j=1, \cdots, m$ (by possibly shrinking U).

We observe that the phases φ_j's, which are the main technical tool in the construction of the parametrix, are neither homogeneous nor σ-homogeneous. On the other hand, for a precise description of the singularities of the parametrix we will need other phases which take care of the propagation within N and on the simple characteristic set of P.
We are led to solve the following Cauchy problems:

$$\text{(2.7)} \quad \begin{cases} \sum_{j=0}^{[m/2]}\sum_{k=0}^{m-2j}\sum_{|\alpha|=m-2j-k} a_{\alpha,k}^{(j)}(x, \xi', 0)\left(\frac{\partial\psi^{(1)}}{\partial x''}\right)^\alpha\left(\frac{\partial\psi^{(1)}}{\partial x_0}\right)^k = 0 \\ \psi^{(1)}|_{x_0=0} = \langle x'', \xi''\rangle \end{cases}$$

$$\text{(2.8)} \quad \begin{cases} \sum_{k=0}^{m}\sum_{|\alpha|=m-k} a_{\alpha,k}^{(0)}\left(x, \xi'+\frac{\partial\Phi^{(1)}}{\partial x'}, \frac{\partial\Phi^{(1)}}{\partial x''}\right)\left(\frac{\partial\Phi^{(1)}}{\partial x''}\right)^\alpha\left(\frac{\partial\Phi^{(1)}}{\partial x_0}\right)^k = 0 \\ \Phi^{(1)}|_{x_0=0} = \langle x'', \xi''\rangle \end{cases}$$

$$\text{(2.9)} \quad \begin{cases} \sum_{k=0}^{m}\sum_{|\alpha|=m-k} a_{\alpha,k}^{(0)}(x, \xi', 0)\left(\frac{\partial\Psi^{(1)}}{\partial x''}\right)^\alpha\left(\frac{\partial\Psi^{(1)}}{\partial x_0}\right)^k = 0 \\ \Psi^{(1)}|_{x_0=0} = \langle x'', \xi''\rangle \end{cases}$$

By putting as in (2.1)

$$\psi(x, \eta) = \psi^{(1)}(x, \eta)+\langle x', \xi'\rangle, \ \Phi(x, \eta) = \Phi^{(1)}(x, \eta)+\langle x', \xi'\rangle ,$$
$$\Psi(x, \eta) = \Psi^{(1)}(x, \eta)+\langle x', \xi'\rangle ,$$

we have the following existence result:

Proposition 2.2. *If U, G are as in Prop.* 2.1, *the equation* (2.7) (*resp.* (2.8), (2.9)) *are solvable in $U\times G^T$ (resp. $U\times G^T\cap\{\xi''\neq 0\}$), for $T=T_\varepsilon$ large, and each of them has m independent solutions $\psi_j^{(1)}(x, \eta)$, $\Phi_j^{(1)}(x, \eta)$, $\Psi_j^{(1)}(x, \eta)$, $j=1, \cdots, m$, respectively. Moreover, $\psi_j^{(1)}(x, \eta)$, $j=1, \cdots, m$, are σ-homogeneous symbols of degree* 1 *in $S^{1,1}(U\times G; M)$, whereas $\Phi_j^{(1)}(x, \eta)$, $\Psi_j^{(1)}(x, \eta)$, $j=1, \cdots, m$, are positively homogeneous symbols of degree* 1 *in $S^1(U\times G\cap\{\xi''\neq 0\})$.*

Proof. If $\tilde{\varphi}_j^{(1)}$, $j=1, \cdots, m$, are the m solutions of (2.6) we found in Prop. 2.1, it is easy to verify that

$$\psi_j^{(1)}(x,\eta)=\langle\eta\rangle\tilde{\varphi}_j^{(1)}\Big(x,\frac{\xi'}{|\xi'|},\frac{\xi''}{\langle\eta\rangle},\frac{|\xi'|^{1/2}}{\langle\eta\rangle},0\Big),\ j=1,\cdots,m\,,$$

solve (2.7) in $U\times G^T$, whereas

$$\Phi_j^{(1)}(x,\eta)=|\xi''|\tilde{\varphi}_j^{(1)}\Big(x,\frac{\xi'}{|\xi'|},\frac{\xi''}{|\xi''|},0,\frac{|\xi''|}{|\xi'|}\Big),$$
$$\Psi_j^{(1)}(x,\eta)=|\xi''|\tilde{\varphi}_j^{(1)}\Big(x,\frac{\xi'}{|\xi'|},\frac{\xi''}{|\xi''|},0,0\Big),\ j=1,\cdots,m$$

are defined in $U\times G^T$ for $\xi''\neq 0$ and there they are solutions of (2.8) and (2.9) respectively.

It follows from the definition that $\psi_j^{(1)}(x,\eta)$ are σ-homogeneous symbols of degree 1 belonging to $S^{1,1}(U\times G;M)$, while $\Phi_j^{(1)}(x,\eta)$ and $\Psi_j^{(1)}(x,\eta)$ are homogeneous symbols of degree 1 in $S^1(U\times G\cap\{\xi''\neq 0\})$.

We now show how the phases ψ and Φ are related to φ on suitable subsets of $U\times G^T$.
Precisely, we have the following:

Corollary 2.3. *Under the same assumption of Proposition 2.2, we have:*

(i)
$$\varphi_j(x,\eta)=\psi_j(x,\eta)+\frac{\langle\eta\rangle^2}{|\xi'|}\rho_j'(x,\eta)$$

where $\rho_j(x,\eta)=\frac{\langle\eta\rangle^2}{|\xi'|}\rho_j'(x,\eta)$ *verify estimates of type* $S^{0,0}$ *in any σ-conic set of the form* $\Gamma'=\{(x,\eta)\in U\times G^T\,|\,|\xi''|^2\leq c'|\xi'|\}$;

(ii)
$$\varphi_j^{(1)}(x,\eta)=\Phi_j^{(1)}(x,\eta)+\frac{|\xi'|}{|\xi''|}\sigma_j'(x,\eta)$$

where $\sigma_j(x,\eta)=\frac{|\xi'|}{|\xi''|}\sigma_j'(x,\eta)$ *verify estimates of type* $S^{0,-1}$ *in any σ-conic set of the form* $\Gamma'=\{(x,\eta)\in U\times G^T\,|\ |\xi''|^2\geq c''|\xi'|\}$.

Proof. Using Taylor's formula at $\zeta=0$ we get:

$$\varphi_j^{(1)}(x,\eta)=\psi_j^{(1)}(x,\eta)+\frac{\langle\eta\rangle^2}{|\xi'|}\rho_j'(x,\eta)\quad\text{with}\quad\rho_j'\in S^{0,0}(U\times G;M)\,.$$

Since $\frac{\langle\eta\rangle^2}{|\xi'|}$ verify estimates of type $S^{0,0}$ on every set
$\Gamma'=\{(x,\eta)\in U\times G^T\,|\ |\xi''|^2\leq c'|\xi'|\}$, we obtain (i).
On the other hand, on any σ-conic set of the form
$\Gamma''=\{(x,\eta)\in U\times G^T\,|\,|\xi''|^2\geq c''|\xi'|\}$, by the uniqueness of the solutions of the Cauchy problem (2.6), we can also write

$$\varphi_j^{(1)}(x,\eta)=|\xi''|\tilde{\varphi}_j^{(1)}\Big(x,\frac{\xi'}{|\xi'|},\frac{\xi''}{|\xi''|},\frac{|\xi'|^{1/2}}{|\xi''|},\frac{|\xi''|}{|\xi'|}\Big).$$

Application of Taylor's formula at $z=0$ yields

$$\varphi_j^{(1)}(x, \eta) = \Phi_j^{(1)}(x, \eta)+\frac{|\xi'|}{|\xi''|}\,\sigma_j'(x, \eta)$$

for some $\sigma_j'\in S^{0,0}(U\times G; M)$. Since $\frac{|\xi'|}{|\xi''|}$ verifies estimates of type $S^{0,-1}$ on Γ'', claim (ii) follows.

It will be useful to considerall the $\varphi_j^{(1)}$, $\Psi_j^{(1)}$, $\Phi_j^{(1)}$, $\Psi_j^{(1)}$, $j=1, \cdots, m$, as smoothly defined on the whole $U\times G$, trivially extending them as 0 in $U\times G$ when $|\eta|<T$.

2(b). Transport equations

If φ_j is one of the phases determined in Sect. 2(a) and $e\in S^{0,0}$, from (2.4) we get:

$$e^{-i\varphi_j}\,P(e^{i\varphi_j}\,e) = L_p^{(j)}(e)+R^{(j)}(e) \quad \text{on } U\times G\,, \tag{2.10}$$

where $L_p^{(j)}$ is the first order operator (2.4)$'$ with $\varphi=\varphi_j$ and $R^{(j)}$: $S^{0,0}\mapsto S^{m-1,m}$. Let us observe that, possibly after shrinking U and G, we can suppose that the coefficient a_0 of $\frac{\partial}{\partial x_0}$ in $L_p^{(j)}$ is different from zero on $U\times G^T$, as follows by observing that from (2.4)$'$ we have:

$$\begin{aligned}\langle\eta\rangle^{1-m}\,a_0(x, \xi', \xi'') &= \langle\eta\rangle^{1-m}\frac{\partial q}{\partial \xi_0}\left(x, \frac{\partial\varphi_j^{(1)}}{\partial x_0}, \xi', \frac{\partial\varphi_j^{(1)}}{\partial x''}\right)\\ &= \sum_{j=0}^{[m/2]}\sum_{k=1}^{m-2j}\sum_{|\alpha|=m-2j-k} a_{\alpha,k}^{(j)}(x, \omega', 0)\, z^{2j}\, k\left(\frac{\partial\tilde{\varphi}_j^{(1)}}{\partial x''}\right)^{\alpha}\left(\frac{\partial\tilde{\varphi}_j^{(1)}}{\partial x_0}\right)^{k-1},\end{aligned}$$

which for $x=0$, $\omega'=e_1$, $z^2+|\omega''|^2=1$ and $\zeta=0$ reduces to

$$\sum_{j=0}^{[m/2]}\sum_{k=1}^{m-2j}\sum_{|\alpha|=m-2j-k} a_{\alpha,k}^{(j)}(0, e_1, 0)\, z^{2j}\,\omega''^{\alpha} k\,\tau_0^{k-1} \tag{2.11}$$

with $\tau_0=\frac{\partial\tilde{\varphi}_j^{(1)}}{\partial x_0}|_{x=0}$.

Since the roots in τ_0 of equation (2.6)$''$ are simple, (2.11) is different from zero and, as a consequence, $a_0(x, \xi', \xi'')\geq c\langle\eta\rangle^{m-1}$ on $U\times G^T$ if U is a small neighborhood of the origin and G is contained in the set described by (ξ', ξ'') when $\lambda=\left(\frac{\xi'}{|\xi'|}, \frac{\xi''}{\langle\eta\rangle}, \frac{|\xi'|^{1/2}}{\langle\eta\rangle}, \frac{\langle\eta\rangle}{|\xi'|}\right)$ belongs to

$$\begin{aligned}\Omega_\varepsilon = \{&(\omega', \omega'', z, \zeta)\in S^{n-p-1}\times \boldsymbol{R}^p\times\boldsymbol{R}\times\boldsymbol{R}\,|\\ &|\omega'-e_1|<\varepsilon,\ |\zeta|<\varepsilon,\ 1-\varepsilon<z^2+|\omega''|^2<1+\varepsilon\}\,,\end{aligned}$$

with a suitable small ε.

Let us fix some notation. If $V=\{(x_0, y, z)\,|\,(x_0, y)\in U, (0, z)\in U\}$, we put $\Gamma=V\times G$, $\partial\Gamma=\{(y, z, \eta)\,|\,(0, y, z, \eta)\in\Gamma\}$ and

$$\Gamma^{c,T}=\Gamma\cap\{(x=(x_0, y), z, \eta=(\xi', \xi''))\in \boldsymbol{R}^{n+1}\times\boldsymbol{R}^n\times\boldsymbol{R}^n\backslash 0\,|\\ |\xi''|^2\geq c\,|\xi'|,\ |\xi'|\geq T\},\ c, T>0\,.$$

In this section we will look for suitable amplitudes $e_j(x,z,\eta)\in S^{0,0}(\boldsymbol{R}^{2n+1}\times\boldsymbol{R}^n\backslash 0; M)$, with $\mathrm{supp}(e_j)\subset\Gamma^T$, for any $j=1, \cdots, m$. We will construct every e_j as a sum of two amplitudes.
More presisely we have the following result:

Proposition 2.3. *If Γ is sufficiently small, ω is a small neighborhood of 0 in $\boldsymbol{R}^{n+1}$, c, T are large enough, for any $k(y, z, \eta)\in S^0$ supported in a small neighborhood of $(0, 0, \xi'=e_1, \xi''=0)=(0, 0, \bar\eta)$ in $\partial\Gamma^T$, there exist $\bar e_j\in S^{0,0}(\boldsymbol{R}^{2n+1}\times\boldsymbol{R}^n\backslash 0; M)$, $\mathrm{supp}(\bar e_j)\subset\Gamma^T$ and $\bar r_j\in S^0(\boldsymbol{R}^{2n+1}\times\boldsymbol{R}^n\backslash 0)$, $\mathrm{supp}(\bar r_j)\subset\Gamma^{c,T}, j=1, \cdots, m$, such that $e_j=\bar e_j+\bar r_j$ satisfies*

$$(2.12)\qquad \begin{cases} e^{-i\varphi_j}\,P(e^{i\varphi_j}\,e_j)\,|_{\omega\times \boldsymbol{R}_n\times\boldsymbol{R}^n\backslash 0}\in S^{-\infty}(\omega\times R^n\times R^n\backslash 0)\\ e_j\,|_{x_0=0}=k \mod S^{-\infty}\,, \qquad j=1, \cdots, m\,. \end{cases}$$

To prove Prop. 2.3 we need two preliminary results.

Lemma 2.4. *If Γ is small enough, $\varepsilon>0$ is small, $j\in\{1, \cdots, m\}$ and $h\in\boldsymbol{Z}_+$, then, for any $\tilde f\in S^{m-1,m-1+h}(\boldsymbol{R}^{2n+1}\times\boldsymbol{R}^n\backslash 0; M)$, $\mathrm{supp}(\tilde f)\subset\Gamma^T$, and for any $\tilde e\in S^{0,h}(\boldsymbol{R}^{2n}\times\boldsymbol{R}^n\backslash 0; M)$, $\mathrm{supp}(\tilde e)\subset\partial\Gamma^T$, there exists $e\in S^{0,h}(\boldsymbol{R}^{2n+1}\times\boldsymbol{R}^n\backslash 0; M)$ with $\mathrm{supp}(e)\subset\Gamma^T$, such that*

$$(2.13)\qquad \begin{cases} L_p^{(j)}(e)=\tilde f \quad \text{if} \quad |x_0|\leq\varepsilon\\ e\,|_{x_0=0}=\tilde e\,. \end{cases}$$

Proof. By dividing the coefficients a_i, $i=0, \cdots, p$ and c of the operator $L_p^{(j)}$ for $\langle\eta\rangle^{m-1}$, we are led to study a first order equation with respect to x, with coefficients in $S^{0,0}(U\times G; M)$. We must verify the possibility of solving this equations globally with respect to ξ.
Let us observe that it is possible to express $\tilde a_i=\langle\eta\rangle^{1-m}\,a_i$, $i=0, \cdots, p$, $\tilde c=\langle\eta\rangle^{1-m}\,c$, as C^∞ functions of x and of the parameter $\lambda=\left(\frac{\xi'}{|\xi'|}, \frac{\xi''}{\langle\eta\rangle}, \frac{|\xi'|^{1/2}}{\langle\eta\rangle}, \frac{\langle\eta\rangle}{|\xi'|}\right)$; to be more precise, $\tilde a_i(x, \lambda)$, $\tilde c(x, \lambda)$ are $C^\infty(U\times\Omega_\varepsilon)$, $\varepsilon>0$, where Ω_ε is the set described by λ when ξ varies in G^T. As we noted at the beginning of this section, we can also suppose that $\tilde a_0(x, \lambda)\neq 0$ when $(x, \lambda)\in U\times\Omega_\varepsilon$.
By integrating the Hamiltonian flow starting from $x_0=0$, when U is sufficiently small, we get a diffeomorphism χ: $(x, \lambda)\mapsto(x_0, x', \bar x''(x, \lambda), \lambda)$, from $U\times\Omega_\varepsilon$ onto

its image, such that $\bar{x}_i''\left(x, \frac{\xi'}{|\xi'|}, \frac{\xi''}{\langle\eta\rangle}, \frac{|\xi'|^{1/2}}{\langle\eta\rangle}, \frac{\langle\eta\rangle}{|\xi'|}\right)$, $i=1, \cdots, p$ are in $S^{0,0}(U\times G^T; M)$ and verify $|\det\left(\frac{\partial \bar{x}_i''}{\partial x_i''}\right)|\geq c>0$ for $(x, \eta)\in U\times G^T$. Moreover, in these coordinates, the vector field $\frac{\partial}{\partial x_0}+\sum_{i=1}^{p}\tilde{a}_0^{-1}\tilde{a}_i\frac{\partial}{\partial x_i''}$ is transformed into $\frac{\partial}{\partial x_0}$.
In fact, assuming that a cutoff function with respect to x'' is applied to the coefficients $\tilde{a}_0^{-1}\tilde{a}_i$, $i=1, \cdots, p$ and putting $\sigma=(x', \lambda)$, we obtain the system

$$\begin{cases} \dot{\bar{x}}''(t) = F(t, \bar{x}''(t), \sigma) \\ \bar{x}''(0) = x'' \end{cases}$$

with $x_0=t$, $F=(\tilde{a}_0^{-1}\tilde{a}_1, \cdots, \tilde{a}_0^{-1}\tilde{a}_p)$ and $F(t, x'', \sigma)=0$ when $|x''|\geq C$.
Thus, for $|t|<T$, there exists $C_T\geq C$ such that $\bar{x}''(t, x'', \sigma)=x''$ for $|x''|\geq C_T$.
On the other hand, when $|x''|\leq C_T$, since $\frac{\partial \bar{x}''}{\partial x''}(0, x'', \sigma)=I_p$, the map $x''\to\bar{x}''(t, x'', \sigma)$ is locally invertible for $|t|\leq T$ for some $T\leq T$.
Finally, we observe that if $\tilde{f}\in S^{0,h}(\boldsymbol{R}^{2n+1}\times\boldsymbol{R}^n\backslash 0; M)$ has sufficiently small support then f defined by $f(\chi(x, \eta))=\tilde{f}(x, \eta)$ still belongs to $S^{0,h}(\boldsymbol{R}^{2n+1}\times\boldsymbol{R}^n\backslash 0; M)$ and that $\exp(\tilde{a}_0^{-1}\tilde{c})$ is in $S^{0,0}(U\times G; M)$, because $\tilde{a}_0^{-1}\tilde{c}\in S^{0,0}(U\times G; M)$.
We can thus construct $e\in S^{0,h}(\boldsymbol{R}^{2n+1}\times\boldsymbol{R}^n\backslash 0; M)$, $\operatorname{supp}(e)\subset\Gamma^T$, satisfying (2.13).

For the next result, we first need a definition.

DEFINITION. If $g\in S^v$, we say that g is "flat" on M iff

$$\forall N\geq 0, \quad \left(\frac{|\xi''|}{|\xi'|}\right)^{-N} g\in S^v .$$

We have:

Lemma 2.5. *If Γ is sufficiently small, c, T are sufficiently large and $\varepsilon>0$ is small, then for any $h\in S^{m-1-t}(\boldsymbol{R}^{2n+1}\times\boldsymbol{R}^n\backslash 0)$ flat on M, $\operatorname{supp}(h)\subset\Gamma^{c,T}$, there exists $r\in S^{-t}(\boldsymbol{R}^{2n+1}\times\boldsymbol{R}^n\backslash 0)$ flat on M such that*

$$\text{(2.14)}\qquad \begin{cases} e^{-i\Phi}P(e^{i\Phi}r) = h \quad \text{modulo a symbol in } S^{m-2-t} \text{ flat on } M, \text{ if } |x_0|\leq\varepsilon \\ r|_{x_0=0} = 0\,, \end{cases}$$

for any $t\in\boldsymbol{Z}_+$, where Φ is any of the Φ_j' s in Proposition 2.2.

Proof. We have to verify that, in spite of the singularities of the function Φ for $\xi''\neq 0$, it is possible to perform the classical construction by means of flat symbols. Let $r\in S^{-t}(\boldsymbol{R}^{2n+1}\times R^n\backslash 0)$ be flat on M. We claim that:

$$e^{-i\Phi}P(x, D_x)(e^{i\Phi}r) = p_m(x, \nabla_x\Phi)r+\tilde{L}(r) \text{ modulo a symbol in } S^{m-2-t} \text{ flat on } M,$$

where $$\tilde{L}=\frac{1}{i}\left\{\sum_{i=0}^{n} a_i\frac{\partial}{\partial x_i}+c\right\}$$

is the usual transport operator i.e.

$$a_i=\frac{\partial p_m}{\partial \xi_i}(x,\nabla_x\Phi), \qquad i=0,\cdots,n,$$

$$c=\sum_{|\beta|=2}\frac{1}{\beta!}\frac{\partial^\beta p_m}{\partial \xi^\beta}(x,\nabla_x\Phi)\frac{\partial^\beta\Phi^{(1)}}{\partial y^\beta}+ib_{m-1}(x,\nabla_y\Phi)\left(\frac{\partial\Phi^{(1)}}{\partial x_0}\right)^{m-1}+$$
$$+i\sum_{k=0}^{m-2}\sum_{|\alpha|=m-2-k}a^{(1)}_{\alpha,k}(x,\nabla_y\Phi)\left(\frac{\partial\Phi^{(1)}}{\partial x''}\right)^{\alpha}\left(\frac{\partial\Phi^{(1)}}{\partial x_0}\right)^{k}.$$

In fact, by considering the expansion (2.2) corresponding to Φ and proceeding as in Sect. 2(a), we have

(i) $$p(x,\nabla_x\Phi)=p_m(x,\nabla_x\Phi)+\sum_{k=0}^{m-2}\sum_{|\alpha|=m-2-k}a^{(1)}_{\alpha,k}(x,\nabla_y\Phi)\left(\frac{\partial\Phi^{(1)}}{\partial x''}\right)^{\alpha}\left(\frac{\partial\Phi^{(1)}}{\partial x_0}\right)^{k}+$$
$$+b_{m-1}(x,\nabla_y\Phi)\left(\frac{\partial\Phi^{(1)}}{\partial x_0}\right)^{m-1}+S^{m-2};$$

(ii) $$\frac{\partial p}{\partial \xi_i}(x,\nabla_x\Phi)=\frac{\partial p_m}{\partial \xi_i}(x,\nabla_x\Phi)+S^{m-2}, \qquad \forall i=0,\cdots,n;$$

(iii) $$\sum_{|\beta|=2}\frac{1}{\beta!}\frac{\partial^\beta p}{\partial \xi^\beta}(x,\nabla_x\Phi)\frac{\partial^\beta\Phi^{(1)}}{\partial y^\beta}=\sum_{|\beta|=2}\frac{1}{\beta!}\frac{\partial^\beta p_m}{\partial \xi^\beta}(x,\nabla_x\Phi)\frac{\partial^\beta\Phi^{(1)}}{\partial y^\beta}+S^{m-2}.$$

It comes out that the a_i's, $i=0,\cdots,n$, belong to $S^{m-1,m-1}(U\times G^T)$, while $\mathrm{Re}\, c\in S^{m-1,m-1}(U\times G^T)$ and $\mathrm{Im}\, c\in S^{m-1,m-2}(U\times G^T)$.
By the same kind of arguments used in the beginning of this section, we get $|a_0|\gtrsim|\xi''|^{m-1}$. Hence, since $|\xi''|\approx|\eta|d_M$ on $\Gamma^{c,T}$, we get $|a_0|\gtrsim|\eta|^{m-1}d_M^{m-1}$ on any σ-conic set $\Gamma^{c,T}$.
Let us point out thta $p_m(x,\nabla_x\Phi)=0$.
In order to establish the global sovability with respect to ξ of the equation $\tilde{L}(r)=h$, for x sufficiently close to 0, we can go on in the same way as in Lemma 2.4. Putting $\tilde{a}_i=|\xi''|^{1-m}a_i$, $i=0,\cdots,n$, $\tilde{c}=|\xi''|^{1-m}c$ and integrating the Hamiltonian flow starting from $x_0=0$, we obtain the existence of a diffeomorfism transforming the vector field $\frac{\partial}{\partial x_0}+\sum_{j=1}^{n}\tilde{a}_0^{-1}\tilde{a}_i\frac{\partial}{\partial x_i}$ into $\frac{\partial}{\partial x_0}$ on

$$U\times(G\cap\{\eta=(\xi',\xi'')\in \boldsymbol{R}^n\backslash 0\,|\,|\xi''|^2\geq c|\xi'|,\ |\xi'|\geq T\})$$

for a suitable choice of a neighborhood U of the origin and of the conic set G. Then for any $t\in \boldsymbol{Z}_+$ and for any $h\in S^{m-1-t}(R^{2n+1}\times R^n\backslash 0)$ flat on M with $\mathrm{supp}(h)\subset\Gamma^{c,T}$, it is possible to find a solution $r\in S^{-t}$ flat on M of the usual transport equation $\tilde{L}(r)=h$, with $r|_{x_0=0}=0$.

Proof of Proposition 2.3.

By a well known argument, using (2.10) and Lemma (2.4) we can find a symbol $\bar{e}_j \in S^{0,0}(\boldsymbol{R}^{2n+1}\times\boldsymbol{R}^n\backslash 0; M)$ with $\operatorname{supp}(\bar{e}_j)\subset\Gamma^T$ such that for a suitable neighborhood ω of the origin

$$\begin{cases} e^{-i\varphi_j}P(e^{i\varphi_j}\bar{e}_j)\,|_{\omega\times\boldsymbol{R}^n\times\boldsymbol{R}^n\backslash 0} = f_j \\ \bar{e}_j\,|_{x_0=0} = k \bmod S^{-\infty} \end{cases}$$

with $f_j \in \bigcap_{h\geq 0} S^{m-1,h}(\omega\times R^n\times R^n\backslash 0, M) = S^{m-1,\infty}(\omega\times R^n\times R^n\backslash 0, M)$, $\operatorname{supp}(f_j)\subset\Gamma^T$. If $\chi\in C_0^\infty(\boldsymbol{R})$, $\chi(t)=1$ when $t\leq c/2$ and $\chi(t)=0$ for $t\geq c$, c large enough, we write

$$f_j = \chi\Big(\frac{|\xi''|^2}{|\xi'|}\Big) f_j + g_j,$$

and we observe that the term $\chi\Big(\frac{|\xi''|^2}{|\xi'|}\Big) f_j$ belongs to $S^{-\infty}$ since

$$\Big|\chi\Big(\frac{|\xi''|^2}{|\xi'|}\Big)f_j\Big| \lesssim |\eta|^{m-1} d_M^N \approx |\eta|^{m-1}\frac{(|\xi''|^2+|\xi|)^{N/2}}{|\eta|^N} \lesssim |\eta|^{m-1-N}|\xi'|^{N/2} \lesssim |\eta|^{m-1-N/2},$$

$\forall N\geq 0$ (being $|\xi''|^2\leq\frac{c}{2}|\xi'|$ on supp (χ)).

On the other hand, g_j is of class $S^{m-1}(\boldsymbol{R}^{2n+1}\times\boldsymbol{R}^n\backslash 0))$, flat on M, with supp $(g_j)\subset\Gamma^{c,T}$ since

$$\Big(\frac{|\xi''|}{|\xi'|}\Big)^{-N} g_j = \Big(\frac{|\xi''|}{|\xi'|}\Big)^{-N}\Big(1-\chi\Big(\frac{|\xi''|}{|\xi'|}\Big)\Big)f_j \leq \Big(\frac{|\xi''|}{|\xi'|}\Big)^{-N}|\eta|^{m-1}\frac{(|\xi''|^2+|\xi'|)^{N/2}}{|\eta|^N}$$
$$\leq |\eta|^{m-1}.$$

To conclude the proof of Proposition 2.3 we need to solve

$$\begin{cases} e^{-i\varphi_j}P(e^{i\varphi_j}\bar{r}_j) = -g_j \quad \bmod S^{-\infty} \\ \bar{r}_j\,|_{x_0=0} = 0 \quad \bmod S^{-\infty}. \end{cases}$$

We first observe that, given a symbol g of class $S^\nu(\boldsymbol{R}^{2n+1}\times\boldsymbol{R}^n\backslash 0)$, $\nu\in\boldsymbol{Z}$, flat on M with supp $(g)\subset\Gamma^{c,T}$, for c sufficiently large, then by Corollary 2.3 (ii), we have

$$ge^{i\varphi_j} = (ge^{i\sigma_j})e^{i\Phi_j} \qquad \forall j=1,\cdots,m$$

with $\sigma_j\in S^{0,-1}(U\times G; M)$.
Then, by Lemma 4.33 in [8] Chapter III, $h_j = ge^{i\sigma_j}$ is still a symbol of class $S^\nu(\boldsymbol{R}^{2n+1}\times\boldsymbol{R}^n\backslash 0))$ flat on M.
By applying Lemma 2.5, we can find a symbol $r_0^{(j)}\in S^0$ flat on M such that

$$\begin{cases} e^{-i\Phi_j}P(e^{i\Phi_j}r_0^{(j)}) = -e^{i\sigma_j}g_j \quad \bmod S^{m-2} \text{ flat on } M \\ r_0^{(j)}\,|_{x_0=0} = 0\,. \end{cases}$$

Then $\bar{r}_0^{(j)}=e^{-i\sigma_j}r_0^{(j)}$ is still a symbol of calss S^0 flat on M such that, modulo $S^{-\infty}$, we have

$$\begin{cases} e^{-i\varphi_j}P(e^{i\varphi_j}(\bar{e}_j+\bar{r}_0^{(j)}))\in S^{m-2} \text{ flat on } M \\ \bar{e}_j+\bar{r}_0^{(j)}|_{x_0=0} = k\,. \end{cases}$$

By repeating the same argument, we can construct an asymptotic sum $\bar{r}_j\sim\sum_h \bar{r}_h^{(j)}$ with $\bar{r}_h^{(j)}\in S^{-h}$ flat on M such that Proposition 2.3 holds.

2(c). Solution of the microlocal Cauchy problem

Consider now the Fourier integral operators

$$E_jf(x) = \int e^{i(\varphi_j(x_0,y,\theta)-\varphi_j(0,z,\theta))}e_j(x_0, y, z, \theta)f(z)dzd\theta\,, \qquad j=1,\cdots,m\,,$$

where the phases φ_j are given by Prop. 2.1 and the amplitudes e_j by Prop. 2.3. It is important to observe that we are still free to choose $e_j|_{x_0=0}=k$ since we only required $k\in S^0$, $\text{supp}(k)\subset\partial\Gamma^T$.

It is clear that, since $\varphi_j(x_0, y, \theta)|_{x_0=0}=\langle y, \theta\rangle$, $D_0^rE_j|_{x_0=0}(r=0,\cdots,m-1)$ are pseudodifferential operators having principal symbol equal to $(\partial_{x_0}\varphi_j(0,y,\theta))^r\cdot k(y,z,\theta)$. Moreover, we can find a conic neighborhood of $(0,\bar{\eta})$ in $\boldsymbol{R}^n\times\boldsymbol{R}^n\backslash 0$ in which the Vandermonde determinant det $[(\partial_{x_0}\varphi_j(x,\theta)|_{x_0=0})^r]_{\substack{r=0,\cdots,m-1\\ j=1,\cdots,m}}$ is elliptic in the class $S^{m(m-1)/2,m(m-1)/2}$, because near $(0,\bar{\eta})$, taking into account the independence of the φ_j''s, we have

$$\begin{aligned}&|\det[\partial_{x_0}\varphi_j(x,\theta)|_{x_0=0})^r]_{\substack{r=0,\cdots,m-1\\ j=1,\cdots,m}}| = \\ &=|\prod_{k>i}(\partial_{x_0}\varphi_k-\partial_{x_0}\varphi_i)\,(0,y,\theta)|\geq \text{const}\langle\theta\rangle^{m(m-1)/2}d_M^{m(m-1)/2}\,.\end{aligned}$$

By using this ellipticity, we can find a combination of the "pure" solutions E_j by means of pdo's on $x_0=0$ acting on the right hand side, in order to suitably adjust the traces of the operators E_j, as stated in:

Proposition 2.6. *If γ is a sufficiently small conic neighborhood of $(0,\bar{\eta})$ in $\boldsymbol{R}^n\times\boldsymbol{R}^n\backslash 0$, for a suitable choice of $k(y,z,\theta)$ there exist $\sigma_j(y, D_y)\in \mathrm{OP}S^{1-m,1-m}\cdot(\boldsymbol{R}^n\times\boldsymbol{R}^n\backslash 0;\, M)$, $j=1,\cdots,m$ such that*

$$WF'(\sum_{j=1}^m D_0^rE_{j|x_0=0}\sigma_j-\delta_{r,m-1}I)\cap(T^*\boldsymbol{R}^n\backslash 0)\times\gamma)=\emptyset\,, \qquad \forall r=0,\cdots,m-1\,.$$

(*see R. Lascar* [8], *Chapter III, Prop.* 4.38).

From Prop. 2.6 it follows that the operator $\tilde{E}=\sum_{j=1}^m\tilde{E}_j=\sum_{j=1}^m E_j\sigma_j$ solves (modulo C^∞-functions) the Cauchy problem:

$$\begin{cases} P\tilde{E}f=0 \\ D_0^r\tilde{E}f|_{x_0=0}=\delta_{r,m-1}f\,, \qquad r=0,\cdots,m-1 \end{cases}$$

for every $f\in C_0^\infty(Y)$ (actually for every $f\in\mathcal{E}'(Y)$ with $WF(f)\subset\gamma$).
We can rewrite the kernel of the operator $\tilde{E}$ as:

$$(2.15)\qquad \tilde{E}(x_0, y, z)=\sum_{j=1}^{m}\tilde{E}_j(x_0, y, z)=\sum_{j=1}^{m}\int e^{i(\varphi_j(x,\theta)-\varphi_j(0,z,\theta))}\tilde{e}_j(x, z, \theta)d\theta\,,$$

where $\tilde{e}_j\in S^{1-m,1-m}$ vanish outside a closed conic neighborhood Γ of $(0, 0, \bar{\eta})$ in $\boldsymbol{R}^{2n+1}\times\boldsymbol{R}^n\backslash 0$.
If we want to construct a microlocal right parametrix for the operator P, the usual procedure consists in applying the Duhamel's principle. To this purpose, we first observe that the whole preceding construction which was performed taking $x_0=0$ as the initial surface, can be actually done for all the initial surfaces $x_0=s$ with $|s|$ small enough.
More precisely, we can construct for $|s|<X_0\leq T$ a kernel

$$(2.16)\ \tilde{E}(s, x, y_0, z)=\sum_{j=1}^{m}\tilde{E}_j(s, x_0, y, z)=\sum_{j=1}^{m}\int e^{i(\varphi_j(s,x_0,y,\theta)-\varphi_j(s,s,z,\theta))}\tilde{e}_j(s, x, z, \theta)d\theta,$$

where $\varphi_j(s, x_0, y, \theta)=\langle x', \theta'\rangle+\varphi_j^{(1)}(s, x_0, y, \theta)$ and $\varphi_j^{(1)}$solve the eikonal equation in (2.5) with $\varphi_j^{(1)}(s, x_0, y, \theta)|_{x_0=s}=\langle x'', \theta''\rangle$, $\tilde{e}_j\in S^{1-m,1-m}(]-X_0, X_0[\times\boldsymbol{R}^{2n+1}\times\boldsymbol{R}^n\backslash 0; M)$, satisfy equation (2.12) with $\varphi_j=\varphi_j(s, x, y_0, \theta)$ (and suitable initial condition at $x_0=s$), so that the operators $\tilde{E}(s)=\sum_{j=1}^{m}\tilde{E}_j(s)$ satisfy (modulo C^∞ functions) the Cauchy problems

$$\begin{cases} P\tilde{E}(s)f=0 \\ D_0^r\tilde{E}(s)f\,|_{x_0=s}=\delta_{r,m-1}f\,, & r=0, \cdots, m-1\,. \end{cases}$$

At this point, by applying the Duhamel's principle, we define (microlocal) forward and backward parametrices for P

$$(2.17)\qquad \begin{cases} (E_+f)(x)=i\int_{-\infty}^{x_0}\chi(s)(\tilde{E}(s)\circ\gamma_s\circ A)(f)(x)ds\,, & f\in C_0^\infty, \\ (E_-f)(x)=-i\int_{x_0}^{+\infty}\chi(s)(\tilde{E}(s)\circ\gamma_s\circ A)(f)(x)ds\,, & f\in C_0^\infty \end{cases}$$

where $\chi\in C_0^\infty(\boldsymbol{R})$, $\operatorname{supp}\chi\subset]-X_0, X_0[$, $\chi=1$ on $|s|\leq X_0'<X_0$, A is a fixed compactly supported pseudodifferential operator with support near ρ_0 and γ_s is the restriction operator to $x_0=s$. Since the normal directions to these surface are not in $WF'(A)$, the operators $\gamma_s\circ A$ are well defined for every $f\in\mathcal{E}'(X)$ with $WF(f)$ concentrated near ρ_0.

3. Calculus of the wave front set of the parametrix

Let us consider the kernel $\tilde{E}(x_0, y, z)$ in (2.15) as an element of $\tilde{\mathcal{D}}'(\boldsymbol{R}^{n+1}\times\boldsymbol{R}^n)$. Then $WF'(\tilde{E})\subset\bigcup_{j=1}^{m}WF(\tilde{E}_j)$ and by the same arguments as in R. Lascar [8], Chap. III, we get:

$$WF'(\tilde{E}_j)\subset\Big\{(x,\xi,z,\eta)\in T^*\boldsymbol{R}^{n+1}\backslash 0\times T^*\boldsymbol{R}^n\backslash 0\,|\,\eta''\neq 0,\ z=\frac{\partial\Phi_j}{\partial\eta}(x,\eta),$$

$$\xi=\frac{\partial\Phi_j}{\partial x}(x,\eta)\Big\}\cup\Big\{(x,\xi,z,\eta)\in T^*\boldsymbol{R}^{n+1}\backslash 0\times T^*\boldsymbol{R}^n\backslash 0\,|\,\xi_0=\xi''=\eta''=0,$$

$$x'=z',\ \xi'=\eta' \text{ and } \exists\theta\in\boldsymbol{R}^n\backslash 0,\ \theta'=\eta',\ z''=\frac{\partial\psi_j}{\partial\theta''}(x,\theta)\Big\}\cup$$

$$\cup\Big\{(x,\xi,z,\eta)\in T^*\boldsymbol{R}^{n+1}\backslash 0\times T^*\boldsymbol{R}^n\backslash 0\,|\,\xi_0=\xi''=\eta''=0,\ x'=z',\ \xi'=\eta'$$

$$\text{and } \exists\theta\in\boldsymbol{R}^n\backslash 0,\ \theta'=\eta',\ \theta''\neq 0\ z''=\frac{\partial\Psi_j}{\partial\theta''}(x,\theta)\Big\}.$$

In the same way, for the forward microlocal right parametrix E_+ defined in (2.17), we have $WF'(E_+)\subset\bigcup_{j=1}^{m}WF'(E_+^{(j)})$, where

$$(E_+^{(j)}f)(x)=i\int_{-\infty}^{x_0}\chi(s)(\tilde{E}_j(s)\circ\gamma_s\circ A)(f)(x)ds\,.$$

By regarding the kernels $\tilde{E}_j(s,x_0,y,z)$ as elements of $\mathcal{D}'((\boldsymbol{R}\times\boldsymbol{R}^{n+1})\times\boldsymbol{R}^n)$, we find:

$$WF'(\tilde{E}_j(s))\subset\Big\{(s,x,\sigma_0,\xi),(z,\eta)\,|\,s<x_0,\ \eta''\neq 0,\ z=\frac{\partial\Phi_j}{\partial\eta}(s,x,\eta),$$

$$\xi=\frac{\partial\Phi_j}{\partial x}(s,x,\eta),\ \sigma_0=\frac{\partial\Phi_j}{\partial s}(s,x,\eta)=-\xi_0\Big\}\cup$$

$$\cup\Big\{(s,x,\sigma_0,\xi),(z,\eta)\,|\,s<x_0,\ \xi_0=\sigma_0=\xi''=\eta''=0,\ x'=z',$$

$$\xi'=\eta' \text{ and } \exists\theta\in\boldsymbol{R}^n\backslash 0\colon\ \theta'=\eta',\ z''=\frac{\partial\Psi_j}{\partial\theta''}(s,x,\theta)\Big\}\cup$$

$$\cup\Big\{(s,x,\sigma_0,\xi),(z,\eta)\,|\,s<x_0,\ \xi_0=\sigma_0=\xi''=\eta''=0,\ x'=z',$$

$$\xi'=\eta' \text{ and } \exists\theta\in\boldsymbol{R}^n\backslash 0\colon\ \theta'=\eta',\ \theta''\neq 0,\ z''=\frac{\partial\Psi_j}{\partial\theta''}(s,x,\theta)\Big\}\cup$$

$$\cup\,\{(s,x,\sigma_0,\xi),(z,\eta)\,|\,s=x_0,\ \eta''\neq 0,\ y=z,\ \xi'=\eta',\ \xi''=\eta'',\ \xi_0=-\sigma_0\}\cup$$

$$\cup\,\{(s,x,\sigma_0,\xi),(z,\eta)\,|\,s=x_0,\ \xi_0=\sigma_0=\xi''=\eta''=0,\ y=z,\ \xi'=\eta'\}.$$

As a consequence, for the $WF(E_+^{(j)})$ we obtain:

$$\begin{aligned}WF(E_+^{(j)})=\{(x,\xi),(\bar{x},\bar{\xi})\,|\quad &|\bar{x}_0|<X_0' \text{ and}\\ &\text{either } x_0>\bar{x}_0 \text{ and } (\bar{x}_0,x,\bar{\xi}_0-\xi_0,\eta),(\bar{y},\bar{\eta})\in WF'(\tilde{E}_j(\bar{x}_0)),\\ &\text{or}\quad x_0=\bar{x}_0 \text{ and } \exists\mu\in\boldsymbol{R}\colon\\ &\qquad (x_0,x,\mu-\bar{\xi}_0,\xi_0-\mu,\eta),(\bar{y},\bar{\eta})\in WF'(\tilde{E}_j(x_0)),\\ &\text{or}\quad x_0=\bar{x}_0,\ \eta=\bar{\eta}=0,\ \xi_0=\bar{\xi}_0\}.\end{aligned}$$

In particular, $(x_0, x, \mu-\bar{\xi}_0, \xi_0-\mu, \eta), (\bar{z}, \bar{\eta}) \in WF'(\tilde{E}_j(x_0))$ means $x=\bar{x}$, $\xi=\bar{\xi}$. For our choice of the operator A in (2.17), the terms $x_0=\bar{x}_0$ $\eta=\bar{\eta}=0$, $\xi_0=\bar{\xi}_0$ do not give any contribution to $WF'(E_+)$ and we can conclude that there exists a conic neighborhood Γ of ρ_0 such that

$$WF'(E_+) \subset C_+(\Gamma) \cup C'_+(\Gamma) \cup C''_+(\Gamma) \cup \Delta^*(\Gamma)$$

with:

$$C_+(\Gamma) = \bigcup_{j=1}^{m} \Big\{(x, \xi), (\bar{x}, \bar{\xi}) \in \Gamma \times \Gamma \,|\, x_0 > \bar{x}_0, \xi' \neq 0, \bar{y} = \frac{\partial \Phi_j}{\partial \bar{\eta}}(\bar{x}_0, x, \bar{\eta}),$$
$$\eta = \frac{\partial \Phi_j}{\partial y}(\bar{x}_0, x, \bar{\eta}), \xi_0 = \bar{\xi}_0 = \frac{\partial \Phi_j}{\partial \bar{x}_0}(\bar{x}_0, x, \bar{\eta})\Big\},$$

$$C'_+(\Gamma) = \bigcup_{j=1}^{m} \Big\{(x, \xi), (\bar{x}, \bar{\xi}) \in \Gamma \times \Gamma \,|\, x_0 > \bar{x}_0, \xi_0 = \bar{\xi}_0 = \xi'' = \bar{\xi}'' = 0, x' = \bar{x}',$$
$$\xi' = \bar{\xi}' \text{ and } \exists \theta \in \mathbf{R}^n \setminus 0: \theta' = \bar{\xi}', \bar{x}'' = \frac{\partial \psi_j}{\partial \theta''}(\bar{x}_0, x, \theta)\Big\},$$

$$C''_+(\Gamma) = \bigcup_{j=1}^{m} \Big\{(x, \xi), (\bar{x}, \bar{\xi}) \in \Gamma \times \Gamma \,|\, x_0 > \bar{x}_0, \xi_0 = \bar{\xi}_0 = \xi'' = \bar{\xi}'' = 0, x' = \bar{x}',$$
$$\xi' = \bar{\xi}' \text{ and } \exists \theta \in \mathbf{R}^n \setminus 0: \theta' = \bar{\xi}', \theta'' \neq 0, \bar{x}'' = \frac{\partial \Psi_j}{\partial \theta''}(\bar{x}_0, x, \theta)\Big\},$$

$\Delta^*(\Gamma)$ being the diagonal in $\Gamma \times \Gamma$.
The relations C_+, C'_+, C''_+ have the following geometrical interpretation:

(i) $(x, \xi), (\bar{x}, \bar{\xi}) \in C_+$ if $(\bar{x}, \bar{\xi})$ belongs to the forward null bicharacteristic of p starting from (x, ξ) (i.e. $x_0 > \bar{x}_0$);

(ii) $(x, \xi), (\bar{x}, \bar{\xi}) \in C'_+$(resp. C''_+) if (x, ξ) and $(\bar{x}, \bar{\xi})$ belong to the same leaf $F \subset N$ and there exist $(\lambda_0, \lambda'') \in T^*_{(x,\xi)}(F)$, $(\bar{\lambda}_0, \bar{\lambda}'') \in T^*_{(\bar{x},\bar{\xi})}(F)$ such that $(x, \xi, \lambda_0, \lambda'')$ and $(\bar{x}, \bar{\xi}, \bar{\lambda}_0, \bar{\lambda})$ are connected in $T^*(F)$ by an integral curve of H_q(resp. H_{q_m}) contained in $q^{-1}(0)$ (resp. $q_m^{-1}(0)$) with $x_0 > \bar{x}_0$.

Clearly, similar arguments give the description of the wave front set for the backward right parametrix E_- changing the relations C_+, C'_+, C''_+ into C_-, C'_-, C''_-.
We observe that $PE_\pm(f)=f$, $\forall f \in \mathcal{E}'(X)$ with $WF(f) \subset \Gamma$, modulo smooth functions.

4. Proof of the theorem

Let us suppose that P verifies assumptions $(H_1)-(H_4)$, $u \in \mathcal{D}'(X)$ satisfies $Pu=f$ with $f \in \mathcal{D}'(X)$, $\rho_0 \in N \setminus WF(f)$ and $(0.1)_+$ holds.
As we already observed in remark 3, ${}^tP$ verifies the same assumptions of P on $-N=\{(x, \xi) \,|\, (x, -\xi) \in N\}$. Hence we can use the same arguments of the previous Sections to construct microlocal right parametrix $E_\pm$ for ${}^tP$, near the point $-\rho_0=(\bar{x}, -\bar{\xi})$. It is easy to verify that, in some conic neighborhood Γ

of ρ_0 we have:

$$WF(E_\pm)\cap(-N)\cap\Gamma\subset(-C'_\mp(\Gamma)\cup -C''_\mp(\Gamma)),$$

where $-C'_\mp$ (resp. $-C''_\mp$) is the relation obtained from $C'_\mp$ (resp. $C''_\mp$) by changing the sign of the fiber variable in both terms.
Passing to the transposed operator ${}^tE_\pm$, we get microlocal left parametrices for P with

$$WF'({}^tE_\pm) = -WF'(E_\mp).$$

Now, if ω is a conic neighborhood of ρ_0 in which $(0.1)_+$ holds, by using standard cut off procedures, we can suppose that $WF(u)\subset\omega$ and $WF({}^tE_-Pu-u)\cap\omega=\emptyset$. Arguing by contradiction, let us suppose that $\rho_0\in WF(u)\setminus WF(f)$ i.e. $\rho_0\in WF({}^tE_-f)\setminus WF(f)\cap\omega$.
Then, since simple bicharacteristics for P do not have limit points in N, it would exist $\rho'\in N\cap\omega\cap WF(f)$, $\rho'\neq\rho_0$, such that $(\rho_0, \rho')\in WF'({}^tE_-)$ i.e.

$$\rho'\in WF(f)\cap\omega\cap((C'_+(\rho_0)\cup C''_+(\rho_0))\setminus\{\rho_0\})\subset WF(u)\cap\omega\cap((C'_+(\rho_0)\cup \cup C''_+(\rho_0))\cup\setminus\{\rho_0\}) = \emptyset,$$

which is impossible.

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