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On the Non-Triviality of Some Kinds of Knots

By Hidetaka TERASAKA

In our earlier paper [6] we have introduced a union of knots as an extension of the product of knots and proved among other things that a union of non-trivial knots is always non-trivial. A new class of knots with Alexander polynomial unity which Kinoshita introduced in the same paper, was proved there to be non-trivial by using the theorem of Bankwitz [3], [7], [4] that an irreducible alternating knot is non-trivial.

Now these two theorems referred to have to all appearance geometric character and purely geometric proofs have been desirable. The main purpose of the present paper is the presentation of geometric proofs of the above theorem with some extensions. In § 1 is given a geometric proof of the non-triviality of torus knots, as well as that of parallel knots, in § 2 that of doubled knots of Whitehead, which we have presented here not for the sake of their novelty but for completeness, because they were the starting point of the subsequent arguments. In the same § is given a rather lengthy proof of the theorem above mentioned that a union of non-trivial knots is non-trivial in somewhat extended form, and in § 3 is proved a theorem giving sufficient conditions to determine from the projections of knots alone whether these are non-trivial. The rest of the paper is devoted to some applications of this theorem and among others a new kind of knots, called semi-alternating knots, is introduced as a generalization of irreducible alternating knots and proved that semi-alternating knots are always non-trivial (Theorem 6).

§ 1. Torus Knots and Parallel Knots.

The following considerations are based upon the semi-linear point of view.

1. Let κ_0 be a given knot, let $\dot{T}$ be a torus, i.e. a closed polyhedral surface of genus 1, with κ_0 as its core, and let T be the full torus bounded by $\dot{T}$. Span the knot κ_0 with a polyhedral surface F such that the intersection $F\frown T$ of F with the full torus T forms a ring bounded by κ_0 and a *closed line*, i.e. a simple closed polygonal line, $\xi_0=F\frown\dot{T}$ on the torus. ξ_0 is then a *basic longitude* of $\dot{T}$. Let η_0 be another closed line on $\dot{T}$ intersecting ξ_0 in a single point O, and bounding a disk H within T. η_0 is then a *basic meridian* of $\dot{T}$. Let H^* be a disk obtained from H by adjoining outside T a ring bounded by η_0 and a closed line $\eta_0{}^*$ surrounding T. The longitudes and the meridians on $\dot{T}$ may be obtained by mapping $\dot{T}$ homeomorphically onto an ordinary geometric torus by making correspond ξ_0 and η_0 to a longitude and a meridian of the latter and by re-mapping onto $\dot{T}$.

A closed line on $\dot{T}$ will be said to be *trivial*, if it bounds a simply connected domain on $\dot{T}$. It is an easy matter to show that if a closed line λ is not trivial, then λ can be isotopically deformed either to ξ_0 or to η_0 or to a closed line which intersects each longitude and meridian always in the same direction. A closed line of this latter type as well as a longitude and a meridian λ^* will be said to be *in normal position.* If l and m denote the numbers of times the line λ^* cuts η_0 (or H^*) and ξ_0 (or F) respectively, then

$$l = L(\lambda, \eta_0^*)\,, \qquad m = L(\lambda, \kappa_0)\,,$$

where $L(a, b)$ denotes in general the linking number of closed curves a and b. Then we can set symbolically

$$\lambda \approx l\xi_0 + m\eta_0\,.$$

$l = l(\lambda)$ will be called the *rotation-number* and $m = m(\lambda)$ the *twisting number*, of λ.

Deforming a closed line on $\dot{T}$ into its normal position, we see

Lemma 1. *If the rotation- or the twisting-number of a closed line $\lambda \subset \dot{T}$ is zero, then λ is either isotopic to ξ_0 or to η_0 or trivial on $\dot{T}$.*

If a closed line λ bounds a disk inside or outside T, then at least one of $L(\lambda, \eta_0)$ and $L(\lambda, \xi_0)$ must be 0, and by virtue of Lemma 1 we have

Lemma 2. *If a closed line $\lambda \subset \dot{T}$ bounds a disk lying wholly inside or wholly outside T except along λ, then λ is either isotopic to ξ_0 or to η_0 or trivial on $\dot{T}$.*

2. Let M be a *polyhedron*, i.e. a closed polyhedral surface, and let κ be a trivial knot bounding a disk F: $F = f(E)$, $\kappa = f(\dot{E})$, where E is a standard disk, $\dot{E}$ its boundary and f a homeomorphism. If F is taken in general position in E^3, then the intersection $F \frown M$ consists

(i) of at most a finite number of disjoint closed lines if F is disjoint from M,

(ii) of closed lines, if any, and *arcs*, i.e. a simple polygonal line, joining two intersection-points of $\kappa \frown M$, all disjoint from one another, if κ has a finite number of points in common with M,

(iii) of closed lines and arcs joining points of κ, disjoint from one another, if κ lies wholly on M.

We shall call such closed lines and arcs *intersection-polygons* and *intersection-arcs*.

The inverse images on E of a point set of $F \frown \dot{T}$ will be called briefly its *trace*, thus the inverse images of intersection-polygons and intersection-arcs are called *trace-polygons* and *trace-arcs*. A *minimal polygon* is a trace-polygon such that it contains no other traces in its interior. A *minimal arc* is a trace-arc such that it bounds together with an arc of the boundary $\dot{E}$ of E a domain, a *minimal domain*, containing no other traces.

3. Lemma 3. *If α and β are disjoint closed lines on a torus $\dot{T}$, then either one of them is trivial on $\dot{T}$ or α and β are isotopic to each other* : $\alpha \approx \beta$.

Proof. Let α and β be neither isotopic to ξ_0 nor to η_0 nor trivial. Put α in normal position α^*. Then by suitable modifications of β we can put it in normal position β^* without changing α^*. Then evidently $\alpha^* \approx \beta^*$ and so

$$\alpha \approx \beta .$$

4. Torus knots. A *torus knot* is a closed line lying on an unknotted torus. Then we have

Theorem 1.[1] *A torus knot is non-trivial, if and only if the rotation- and twisting-numbers are at least* 2 *in absolute value.*

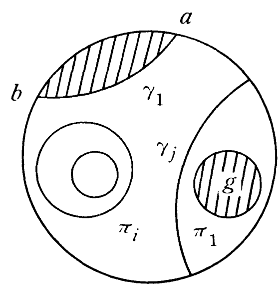
Proof. Let κ be a torus knot on $\dot{T}$, let $l(\kappa)=l$, $m(\kappa)=m$, $|l| \geqq 2$ and $|m| \geqq 2$. Supposing the contrary let κ bound a disk F : $F=f(E)$, $\kappa=f(\dot{E})$, where E is the standard disk, $\dot{E}$ the circumference of E and f a homeomorphism. In the course of our proof we denote by the same letter F and f disks and homeomorphisms resulting from modifications of the original F and f.

Supposing that F is in general position, the intersection $F \frown \dot{T}$ of F with $\dot{T}$ consists other than κ of disjoint closed lines Π_i and arcs Γ_j joining points of κ.

1) Let π_i be the trace, i.e. the inverse image of Π_i. Let π_1 be a minimal polygon, i.e. one of these π_i's such that there is no other π_i

1) First proved geometrically by L. Antoine [1].

in the interior of π_1. Then the image $f(\bar{g})$ of the closed domain $\bar{g}$ bounded by π_1 is a disk spanning Π_1 outside or inside T, and consequently the linking number of Π_1 either with ξ_0 or with η_0 is equal to zero and therefore Π_1 is isotopic to ξ_0 or to η_0 or bounds a simply connected domain on $\dot{T}$. But since Π_1 is disjoint from κ, whose rotation- and twisting-numbers are greater than 1 in absolute values, the former two cases cannot occur by Lemma 3, and $f(\pi_1)$ must bound a simply connected domain G. Then if we cut off the disk F by G and spanning again the intersection with a smaller disk or disks suitably and then detaching them from $\dot{T}$, we obtain a new disk F which has no more intersection with $\dot{T}$ in the neighbourhood of G. By successive modifications of F of this kind we can make disappear all intersection-polygons of $F \frown \dot{T}$ which are trivial on $\dot{T}$ (Cf. Fox [5], Schubert [8]).



Fig, 1.

2) Let γ_j be the trace of Γ_j, and let γ_1 be a minimal arc, i.e. one of these trace-arcs such that there is no other traces within the domain g bounded by γ_1 and by an arc $\widehat{ab}$ of the circumference $\dot{E}$. Then $\Gamma_1 + f(\widehat{ab}) = \Gamma$ bounds a disk lying wholly outside or inside T, and therefore by Lemma 3 the following three cases can occur:

(i) Γ is trivial. In this case we can get rid of such an intersection Γ_1 by a suitable modification of F.

(ii) Γ is isotopic to a longitude. Bring Γ to ξ_0 by a homeomorphism h of $\dot{T}$ onto itself. Then if Γ lies on the same side of κ in the neighbourhood of the endpoints $A = f(a)$ and $B = f(b)$, the intersection number of $h(\kappa)$ and ξ_0 is to be counted as 0, contradicting the assumption on κ. If Γ lies on the opposite side of κ in the neighbourhood of A and B, then the intersection number of $h(\kappa)$ with ξ_0 is to be counted as ± 1, contradicting again the assumption on κ. Thus this case cannot occur actually.

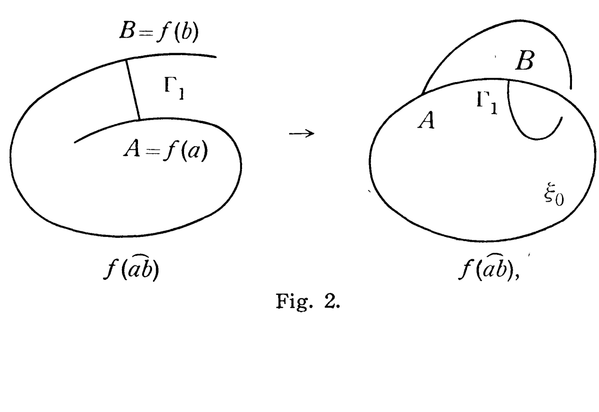


Fig. 2.

(iii) Γ is isotopic to a meridian. This case cannot occur by the same reason as (ii).

By the repetition of the procedure of (i) we can get rid of all intersections of $F \frown \dot{T}$ other than κ itself. Then κ comes to bound a disk F lying wholly inside or outside T, which contradicts again the assumption on κ that the rotation- and twisting-number of κ are $\geqq 2$ in absolute value. This completes the proof of the theorem.

5. Parallel knots. Let T be a full torus with the core κ_0. Any closed line κ on $\dot{T}$ with the rotation-number different from 0 is called a *parallel knot* of κ_0. If $l \geqq 1$ denotes the rotation-number of κ, then κ will be called an *l-fold* parallel knot of κ_0.

Theorem 2. *Every parallel knot of a non-trivial knot is non-trivial.*

Proof. Suppose again that a parallel knot κ of κ_0 is trivial and suppose that κ bounds a disk F. As in the proof of Theorem 1, the intersection $F \frown \dot{T}$ consists other than κ of disjoint intersection-polygons Π_i and intersection-arcs Γ_j joining points of κ. Then let the traces of Π_i and Γ_j on E be π_i and γ_j respectively.

1) Let π_1 be a minimal polygon. Then, denoting by g the domain bounded by π_1, since $\Pi_1 = f(\pi_1)$ bounds the disk $f(\bar{g})$ lying wholly outside or inside T, Π_1 is isomorphic either to ξ_0 or to η_0 or trivial. Then:

(i) Since ξ_0 is isotopic to the core κ_0, ξ_0 is a non-trivial knot by hypothesis. Then Π_1, being a trivial knot, cannot be isotopic to ξ_0.

(ii) If Π_1 is isotopic to η_0, then by Lemma 3 κ must be isotopic to η_0, which is impossible.

(iii) If Π_1 is trivial on $\dot{T}$, then this intersection can be made to disappear by a suitable modification of F.

By the repetition of the modifications of the kind (iii) we obtain a disk F such that $F \frown \dot{T}$ consists other than κ at most of intersection-arcs Γ_j joining points of κ.

The rest of the reasoning runs quite parallel to that of the second half of the proof of Theorem 1, and our proof is complete.

§ 2. Linked Sum of Knots

6. Let a knot κ meet a sphere S in 4 different points A, B, C, D in this order on κ, and let the arcs $\widehat{AB}$, $\widehat{CD}$ be inside S and $\widehat{AD}$, $\widehat{BC}$ outside. Let κ be trivial and let F be a disk bounded by κ: $F = f(E)$, where E is a standard disk.

Then, there are two cases to be considered.

(i) Intersection-arcs Γ_{AB} and Γ_{CD} of $F \cap S$ join A, B and C, D respectively. Let γ_{ab} and γ_{cd} be their traces. In this case, if $\overset{\frown}{ab}$ and $\overset{\frown}{cd}$ denote the disjoint arcs of $\dot{E}$, the images of the closed regions bounded by $\gamma_{ab} \cup \overset{\frown}{ab}$ and $\gamma_{cd} \cup \overset{\frown}{cd}$ are disjoint disks bounding $\Gamma_{ab} \cap \overset{\frown}{AB}$ and $\Gamma_{CD} \cup \overset{\frown}{CD}$. If there are no trace-polygons, these disks are totally inside S. If on the other hand there are trace-polygons, we can get rid by the known modifications of all those minimal polygons whose images do not separate the intersection-arcs Γ_{AB} and Γ_{CD}. Therefore let γ be a minimal polygon, whose image $f(\gamma)$ separates Γ_{AB} and Γ_{CD} on S. Then if $\bar{g}$ denotes the closed region bounded by γ, $f(\bar{g})$ divides S into two spheres S' and S'', of which S' contains $\overset{\frown}{AB}$ and S'' contains $\overset{\frown}{CD}$. The image of the closed region bounded by $\gamma_{ab} \cup \overset{\frown}{ab}$, which spans $\Gamma_{AB} \cup \overset{\frown}{AB}$, can again be modified (cf. e.g. [6], p. 136, proof of Lemma 1.) so that it has no intersection with S' other than Γ_{AB}.

Similarly for $\Gamma_{CD} \cup \overset{\frown}{CD}$; and so $\Gamma_{AB} \cup \overset{\frown}{AB}$ and $\Gamma_{CD} \cup \overset{\frown}{CD}$ bound disjoint disks inside S.

(ii) If intersection-arcs Γ_{AD} and Γ_{BC} join A, D and B, C respectively, their traces γ_{ad} and γ_{bc} on E bound with the arcs $\overset{\frown}{ab}$ and $\overset{\frown}{cd}$ of $\dot{E}$ a closed region $\bar{g}$, whose image $f(\bar{g})$ is a disk bounded by $\overset{\frown}{AB} \cup \Gamma_{BC} \cup \overset{\frown}{CD} \cup \Gamma_{AD}$.

We have thus proved the following

Lemma 4. *Let a knot κ meet a sphere S in 4 different points A, B, C, D in this order on κ, and let the arcs $\overset{\frown}{AB}$, $\overset{\frown}{CD}$ of κ be inside S and $\overset{\frown}{AD}$, $\overset{\frown}{BC}$ outside. In order that κ be trivial, it is necessary that either* (i) *there are disjoint arcs Γ_{AB} and Γ_{CD} joining A and B, C and D respectively on S such that $\Gamma_{AB} \cup \overset{\frown}{AB}$ and $\Gamma_{CD} \cup \overset{\frown}{CD}$ bound disjoint disks in the interior of S, or* (ii) *in general there are disjoint arcs Γ_{AD} and Γ_{BC} joining A and D, B and C respectively on S such that these arcs together with $\overset{\frown}{AB}$ and $\overset{\frown}{CD}$ form a trivial knot.*

7. Linked sum.

Now we are going to define a linked sum of knots.

Divide a full torus T by 4 disjoint disks bounded by meridians of the torus $\dot{T}$ in the interior of $\dot{T}$ into 4 cells, named the 1st *knot chamber* I, the 2nd *knot chamber* II, the *junction chamber* J, and the *linkage chamber* L, where I and II separate J and L. The disks will be called *walls* and denoted by W_J, W_J', W_L and W_L'.

The 1st knot κ consists of a segment AD in the interior of the wall

W_J, a segment BC in the interior of W_L, and the arcs $\widehat{AB}$ and $\widehat{CD}$ running in the interior of I except for their endpoints. The 2nd knot κ' consists likewise of segments $A'D'$ and $B'C'$ respectively in the interior of W'_J and W'_L and of the arcs $\widehat{A'B'}$ and $\widehat{C'D'}$ running inside II. Suppose further that the segments AA' and DD' lie wholly in the junction chamber J except for their endpoints and make together with AD and $A'D'$ the sides of a rectangle. Join next the endpoints B, C and B', C' of the segments BC and $B'C'$ by arcs $\widehat{BC}$ and $\widehat{B'C'}$ respectively in the interior of the linkage chamber L such that $\widehat{BC}\cup BC$ and $\widehat{B'C'}\cup B'C'$ link each other non-trivially, that is, that

Linking condition: $\widehat{BC}\cup BC$ *and* $\widehat{B'C'}\cup B'C'$ *do not bound any disjoint disks.*

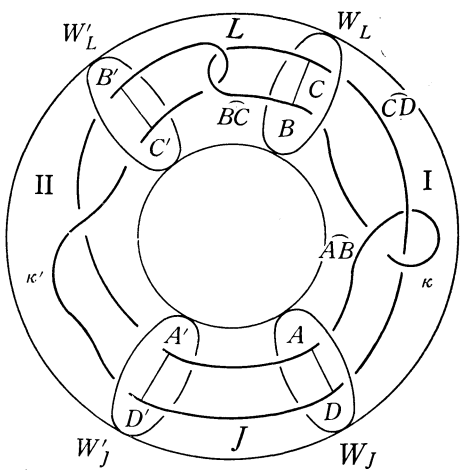


Fig. 3.

Then we call the knot

$$\kappa'' = \widehat{AB}\cup\widehat{BC}\cup\widehat{CD}\cup DD'\cup\widehat{D'C'}\cup\widehat{C'B'}\cup\widehat{B'A'}\cup A'A$$

a *linked sum* (*of* 1*st order*) of κ and κ'.

Our next purpose is to show that a linked sum κ'' is non-trivial under certain conditions. Therefore, if the linking of $\widehat{BC}$ and $\widehat{B'C'}$ in the linkage chamber L is taken from the beginning so to speak strongly enough to ensure that there can be no arcs joining B and B', C and C' respectively on the boundary of L forming together with $\widehat{BC}$ and $\widehat{B'C'}$ a trivial knot, then the resulting linked sum κ'' is as will be seen by Lemma 4 certainly non-trivial and nothing needs to be further proved. A true difficulty, if any, arises for the first time only when the linking is a weak one.

The simplest type of the linked sum was introduced in [6] under the name of a *union*: the linking $\widehat{BC}$ and $\widehat{B'C'}$ being a simple $2n$ times twisting. Theorem 4 below is an extension of Theorem 1 [6], p. 141, and a geometric proof will soon be given. Here some remarks concerning unions of knots may be added. A linked sum, even a union of knots, is not uniquely determined from the types of the original knots. The inverse of the union-making is not also unique and two different pairs of knots may give one and the same knots as their union: the knots of

11 crossings with Alexander polynominal unity introduced in [6], p. 151, given by S. Kinoshita, is represented there by a union of $(-)6_3$ and $(+)3_1$ with winding number 2, but it can also be represented by a union of $(-)6_2$ and $(+)3_1$. Nothing beyond this example is known up to present concerning the factorization by union.

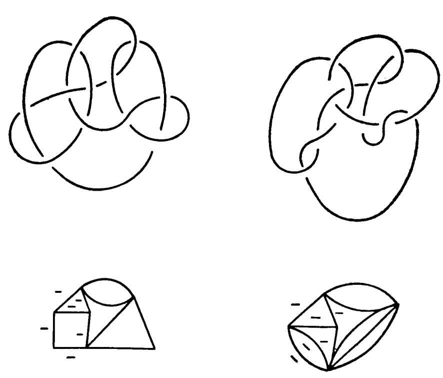


Fig. 4.

One more remark should be added here about a linked sum. The *torus of the linked sum*, that is the torus in which the linked sum is defined, may be trivial or not. But if the torus itself is non-trivial, that is, if the core of the torus is a non-trivial knot, then the situation is simpler and we have the following extension of a theorem of J. H. C. Whitehead on doubled knots [9]:

Theorem 3. *A linked sum of two knots, trivial or not, is always non-trivial, provided the torus of the linked sum itself is non-trivial.*

Proof. Supposing the contrary let F be a disk bounded by the linked sum κ'' of κ and κ': $F=f(E)$, where E is a standard disk.

1) We assert first that F intersects $\dot{T}$; for otherwise there would be intersection-arcs Γ and Γ' of F and the walls W_L and W'_L joining B and C, B' and C' respectively on themselves, whose traces γ and γ' on E bound with arcs $\widehat{bc}$ and $\widehat{b'c'}$ of $\dot{E}$ two disjoint domains. But then the images of the closure of these domains would be bounded by $\widehat{BC}\cup\Gamma$ and $\widehat{B'C'}\cup\Gamma'$ that link each other by hypothesis, which is a contradiction.

2) By the known modifications of F we can make disappear all intersection-polygons of $F\cap\dot{T}$ which are trivial.

3) $F\cap\dot{T}$ does not contain any intersection-polygon which is isotopic to a meridian; for otherwise all intersection-polygons Π_i of $F\cap\dot{T}$ would be by Lemma 3 isotopic to a meridian. Let π_i be their traces and let π_1 be a minimal polygon, g its interior. Then $f(\bar{g})$ would be a disk cutting T into a spherical domain which contains κ'' in its interior. Then κ'' would bound a disk inside T, contrary to what we have proved in 1).

4) Each intersection-polygon Π_i of $F\cap\dot{T}$ must have therefore a rotation-number different from 0, and Π_i must be a parallel knot of the core κ_0 of the torus T. But if π_1 is a minimal polygon on E with its

interior g, then Π_1 would bound a disk $f(\bar{g})$, which is impossible by Theorem 2.

The proof of Theorem 3 is thus complete.

8. We are now going to prove the following linked sum theorem:

Theorem 4. *Any linked sum of non-trivial knots is always non-trivial.*

Proof. Let κ'' be a linked sum of non-trivial knots κ and κ'. By virtue of the foregoing Theorem 3 we are only to prove the theorem when the torus of the linked sum T is an ordinary trivial one. Now supposing the contrary let κ'' be trivial and let κ'' bounds a disk F: $F=f(E)$, where E is again a standard disk. Then the following 1), 2) and 3) are already proved in the proof of Theorem 3.

1) The intersection $F\cap \dot{T}$ is not empty.

2) No trivial intersection-polygon may be supposed to appear.

3) $F\cap\dot{T}$ does not contain any intersection-polygon which is isotopic to a meridian.

4) Now let Λ_i $(i=1, 2, \cdots, N)$ be the system of intersection-polygons making up $F\cap\dot{T}$, and let $\lambda_i=f^{-1}(\Lambda_i)$ be their traces on E. Let λ_1 be a minimal polygon (cf. **2.**), g its interior. Then the disk $f(\bar{g})$ must lie wholly inside or wholly outside T except along Λ_1; but since the former occurrence is excluded by 2) and 3), $f(\bar{g})$ must be a disk bounded by Λ_1 outside T and consequently the linking number of Λ_1 with the core κ_0 of T is zero and Λ_1 turns out to be isotopic to a longitude of $\dot{T}$ (Lemma 1). Then, since all Λ_i $(i=1, 2, \cdots, N)$ are disjoint and non-trivial, they must all be isotopic to a longitude (Lemma 3).

Bring Λ_i $(i=1, 2, \cdots, N)$ by a modification of F to such positions that they are each equal to a longitude.

5) Consider next the intersection of F with the walls.

(i) First of all we notice that on the wall W_J of the junction chamber J the point A and D are not joined by any arc Γ of $F\cap W_J$; for otherwise the trace $f^{-1}(\Gamma)=\gamma$ of Γ would then join the points $f^{-1}(A)=a$ and $f^{-1}(D)=d$ within E and the image of the simply connected domain bounded by $\widehat{ab}\cup\widehat{bc}\cup\widehat{cd}\cup\gamma$, where b and c are the traces of B and C, would span the knot $\widehat{AB}\cup\widehat{BC}\cup\widehat{CD}\cup\Gamma$, contrary to the non-triviality of κ.

B and C are not joined also by any arc of $F\cup W_L$ on the wall W_L of the linkage chamber L; for if an arc Γ of $F\cup W_L$ joins B and C on W_L, then the map of the disk bounded by $f^{-1}(\Gamma\cup\widehat{BC})$ would span $\Gamma\cup\widehat{BC}$ and have no point in common with $\widehat{B'C'}\cup B'C'$, contrary to the linking condition (p. 119) of the linked sum.

Similarly for the rest of the walls.

(ii) If intersection-polygons appear in the intersections of F with the walls, make it disappear by suitable modifications of F.

(iii) Now let W be one of the walls. If Γ is an arc of $F \frown W$ dividing W into two parts such that one of these, say G, contains no other intersection-arc of $F \frown W$ in its interior, then by replacing a small portion P of F containing Γ by a disk Q, which together with P encloses G in the interior, we can obtain a new F which no more intersects W along Γ but intersects $\dot{T}$ along a new polygon which is trivial on $\dot{T}$. Make a further modification of F so that this last polygon disappear. Note that by this modification of F the number of intersection-polygons of $F \frown \dot{T}$ diminishes, the rest of the intersection-polygons of $F \frown \dot{T}$ still remaining in their positions as longitudes. If by this modification of F new polygons happen to appear as the intersections of F with the walls, make them disappear by further modifications of F.

(iv) The cyclical repetitions of these processes in this order come to an end when:

$F \frown W_J$ consists of a group of disjoint intersection-arcs with endpoints on the boundary $\dot{W}_J$ of W_J and of the pair of arcs $\widehat{AA_1}$ and $\widehat{DD_1}$ joining A and D in the interior of W_J to points A_1 and D_1 on $\dot{W}_J$ respectively, each of the former arcs separating the latter pair;

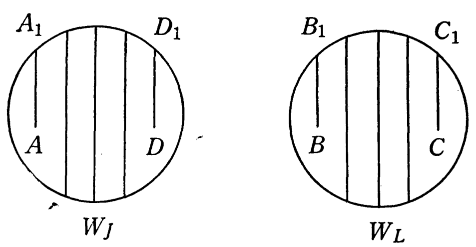


Fig. 5.

$F \frown W_L$ consists of a group of disjoint arcs with endpoints on the boundary $\dot{W}_L$ of W_L and of the pair of arc $\widehat{BB_1}$ and $\widehat{CC_1}$ joining B and C with endpoints B_1 and C_1 on $\dot{W}_L$ respectively, each of the former arcs separating the latter pair.

Similarly for $F \frown W_J'$ and $F \frown W_L'$.

(v) Finally let Λ_i be the system of longitudes making up $F \frown \dot{T}$, and λ_i their traces. Let $\{\lambda_{i'}\}$ be all of such a λ_i that is not contained in another λ_j. Span $\Lambda_{i'}$ outside T with disks $D_{i'}$ disjoint from one another, and substitute each of $f(\bar{g}_{i'})$, where $g_{i'}$ denotes the domain bounded by $\lambda_{i'}$, by $D_{i'}$. By this modification of F we obtain a new disk F bounded by κ'' which has all the properties of (iv) plus the property that the traces of $F \frown \dot{T}$ are polygons lying outside one another. We denote these trace-polygons again by λ_i $(i=1, 2, \cdots, N)$, without dashes. $\Lambda_i = f(\lambda_i)$ are then the longitudes constituting the whole intersections $F \frown \dot{T}$.

6) After we have thus put F in normal position we shall now pay attention to the intersection of F with the whole boundary of each chamber. It will be described schematically by the use of diagrams.

The intersection $F \cap \dot{J}$ looks like this:

(i) Let W_J be represented by the semicircular domain bounded by the diameter PQ and the semi-circumference PRQ, which should be identified to a point. N semicircles with centres A_1 and D_1 and segments

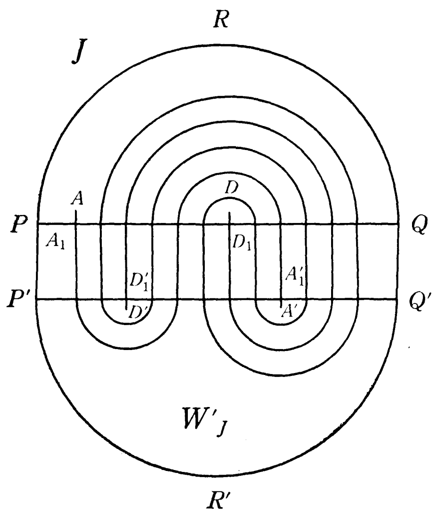


Fig. 6.

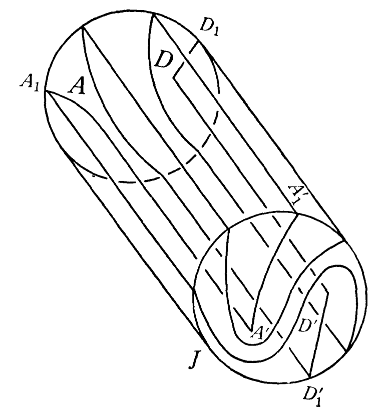


Fig. 7.

AA_1 and DD_1 represent then the intersection $F \cap W_J$. Likewise the semicircular domain bounded by the diameter $P'Q'$ and the semi-circumference $P'R'Q'$ represent the wall W'_J, where the semi-circumference $P'R'Q'$ is identified to a point, and N semicircles with centres A'_1 and D'_1 and segments $A'A'_1$, $D'D'_1$ represent the intersection $F \cap W'_J$. The rectangle $PQQ'P'$, the sides PP' and QQ' being identified, represents the cylindrical surface $\dot{J} - W_J - W'_J$, and N parallel segments joining endpoints of semicircles represent the intersection $F \cap \dot{J}$ which are subarcs of the longitudes Λ_i.

Certain connected successions of semicircles and parallel segments form a simple closed curve or an arc, which we call an *F-line*. Then there are three cases to be considered.

a) An F-line $F(A, A')$ joins A and A' and another one $F(D, D')$ joins D and D'.

b) An F-line joins A and D' and another one joins D and A'.

c) An F-line joins A and D and another one joins A' and D'.

Referring to the standard disk E we see that the cases b) and c) cannot occur; for let a, b, c, d and a', b', c', d' be the traces of A, B, C, D and A', B', C', D'. Then if b) occurs, a and d' as well as a' and d are to be joined by disjoint arcs in E, which is impossible. And if c) occurs, then the trace γ of an F-line would join a and d in E, showing that a disk would bound the closed line $\widehat{AB}\cup\widehat{BC}\cup\widehat{CD}\cup f(\gamma)$, which is impossible, since this is the product of knots one of which is the non-trivial knot κ, therefore non-trivial.

Thus only the case a) can occur.

Now let the F-line $F(A, A')$ joining A and A' contains n parallel segments. Then the F-line running closest to $F(A, A')$ must either be the F-line $F(D, D')$ joining D and D' containing n parallel segments or a closed F-line containing $2n$ parallel segments. In the latter case there can still be other F-lines, and in general the system of all F-lines consists of open F-lines $F(A, A')$ and $F(D, D')$, both containing n parallel segments, and of closed F-lines containing $2n$ parallel segments, where n is an *odd* number. (See Fig. 6)

(ii) Similarly for the intersection $F\cap\dot{I}$ of F with the boundary $\dot{I}$ of the 1st knot chamber I, and we see:

a) An open F-line $F(A, B)$ joins A and B and another one $F(C, D)$ joins C and D. They both contain m parallel segments, where m is an *odd* number. The remaining F-lines, if any, are closed and contain each $2m$ parallel segments.

b) The case that an F-line joins A and C and another one joins B and D cannot occur as seen by referring to the standard disk E.

c) If an F-line joins A and D and another one joins B and C, then, referring again to E we see that b, c would then be joined in E by an arc γ and so the image of the closed region bounded by the arc $\widehat{bc}$ of $\dot{E}$ and by γ would be a disk spanning $\widehat{BC}\cup f(\gamma)$, where $f(\gamma)$ is an arc of $\dot{I}$, thus $\widehat{BC}$ would be isotopically deformable to an arc of $\dot{I}$ without having any point in common with $\widehat{B'C'}$ during the deformation, contradicting the linking condition.

Thus the case a) can only occur.

(iii) The situation is the same for the intersection $F\cap\dot{II}$ as in the case of I and the following takes place:

a) An F-line $F(A', B')$ joins A' and B' and another one $F(C', D')$ joins C' and D'. Both of these F-lines contain m' parallel segments, where m' is an *odd* number. The remaining closed F-lines contain each $2m'$ parallel segments.

(iv) As regards the intersection $F \frown \dot{L}$ of F with the boundary $\dot{L}$ of the linkage chamber L, it is of no consequence in the following argument to determine whether B and C or B and B' are joined by an F-line. At any rate the two open F-lines contain both n' parallel segments and all the closed ones contain each $2n'$ parallel segments.

7) We have had until now several occasions of referring to the standard disk E and to traces of the intersections of F of several kinds to infer some or other consequences from them. Now let us have a full view of all the traces of the intersections of F with the boundaries of four chambers by naming it a *chart* of F.

In our chart there are first of all N polygons λ_i $(i=1, 2, \cdots, N)$ representing the traces of the longitudes Λ_i, that is, the intersection $F \frown \dot{T}$. Since a longitude meets exactly four walls W_J, W_L, W'_J, W'_L, there emerge from each λ_i four arcs, of which two consecutive arcs, called *J-arcs*, correspond to arcs on the walls W_J, W'_J of the junction chamber J and two other consecutive arcs, called *L-arcs*, correspond to arcs of the walls W_L, W'_L of the linkage chamber L. The traces a, b, c, d, and a', b', c', d' of the points A, B, C, D and A', B', C', D' of κ'' are joined correspondingly by arcs to λ's. Thus the standard disk E is divided by these traces into regions, *J-*, *L-*, *I-*, *II-regions*, and to *Λ-regions*.

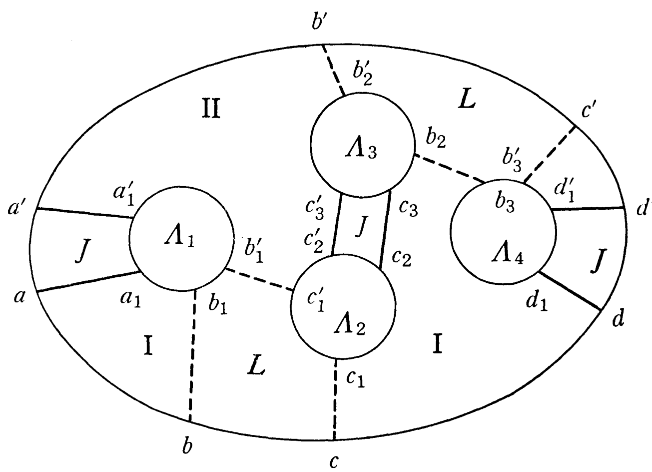


Fig. 8.

To explain by way of example, thickened lines aa_1, $a'a'_1$, etc. represent traces of the intersections $F \frown W_J$ and dotted lines those of $F \frown W_L$. Each region is marked with the corresponding letter.

Now draw this kind of chart on the northern hemisphere of a sphere S, taking the equator as $\dot{E}$, and draw the chart symmetric to it on the

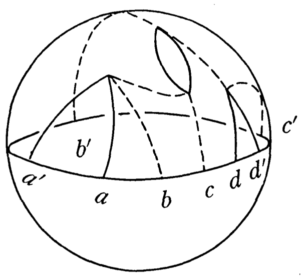


Fig. 9.

southern hemisphere. Letting shrink each λ_i to a point and deleting the equatorial line, we have a linear graph Φ on S: at each vertex meet four edges, and four polygons with $2n$, $2m$, $2n'$ and $2m'$ sides respectively meet there in this cyclic order corresponding to J-, I-, L- and II-regions, where n, m, m' are odd numbers. If Φ is not connected let Φ^* be an arbitrary component of it and let N_J, N_L, N_I, N_{II} be the numbers of J-, L-, I-, II-polygons of Φ^* corresponding to J-, L-, I-, and II-regions. If N_0, N_1 and N_2 denote the numbers of vertices, edges and polygons of the subgraph Φ^*, we have as easily be seen the following relations:

(1) $\quad 4N_0=2N_1$

(2) $\quad 2nN_J+2n'N_L=N_1$

(3) $\quad 2mN_I+2m'N_{II}=N_1$

(4) $\quad N_J+N_I+N_L+N_{II}=N_2$

(5) $\quad N_0-N_1+N_2=2$ (Euler relation)

From (1) and (5) we have

$$N_1 = 2N_0, \qquad N_2 = N_0+2$$

and substituting in (2), (3), (4) we have

(2′) $\quad nN_J+n'N_L=N_0$

(3′) $\quad mN_I+m'N_{II}=N_0$

(4′) $\quad (N_J+N_L)+(N_I+N_{II})=N_0+2$

8) We are going to determine by the help of the chart the values of m, m', n and n'.

(i) Let $m=m'=1$. (Fig. 10 illustrates this imaginary occurrence. Note that 2 disjoint cylindrical surfaces can be spanned inside I to 2 pairs of closed lines AA_1B_1B, $F_1F_2F_3F_4$ and CC_1D_1D, $E_1E_2E_3E_4$.) Then I- and II-regions are bounded by a J-arc, an L-arc and two longitudes. If one of them, say g, is a I-region and contains no further traces, then $f(\bar{g})$ spans a closed F-line inside the chamber I and divides it into two spheres, I' and I''. Let g' and g'' be the I-regions adjacent to the arcs $\widehat{ab}$ and $\widehat{cd}$ of $\dot{E}$; then $f(\bar{g}')$ is a disk spanning $\widehat{AB}\cup F(A,B)$, where $F(A,B)$

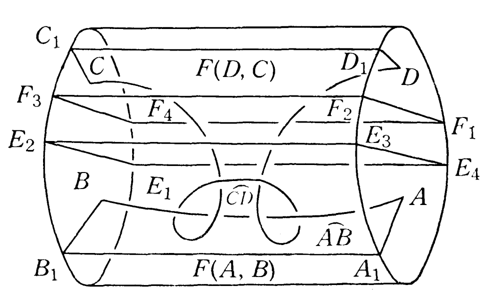


Fig. 10.

is the F-line joining A and B on the boundary $\dot{I}$. Let $\widehat{AB}\cup F(A, B)$ belong to I'. Then, since I' is a sphere, we can modify if necessary so that the disk becomes wholly inside I' except along $F(A, B)$. Now by the nature of an F-line the F-line $F(D, C)$ joining D and C on $\dot{I}$ cannot belong to I' but to I'', and $\widehat{CD}\cup F(D, C)$ can be spanned with a disk inside I'' by a modification of the disk $f(\bar{g}'')$. But then, since $F(A, B)\cup \widehat{BC}\cup F(D, C)\cup\widehat{DA}$, where $\widehat{DA}$ denotes an arc joining D and A on the wall W_J having no point in common with $\widehat{AA_1}$ and $\widehat{DD_1}$ of $F(A, B)$ and $F(D, C)$, is clearly unknotted, the knot $\widehat{AB}\cup\widehat{BC}\cup\widehat{CD}\cup\widehat{DA}$ turns out to be trivial, contrary to the hypothesis. Thus the supposition that a I-region, and likewise a II-region, does not contain any other traces leads to a contradiction.

But if every I-region and every II-region contains other traces, then it must contain another I-region or II-region, which contains in turn still another one, and so on ad infinitum, which is impossible. Thus $m=m'=1$ is impossible, and we can assume that $m\geqq 1$, $m'\geqq 3$, m (and m') being odd numbers.

(ii) Let $m=1$, $m'\geqq 3$. Then in the connected subgraph Φ^* each II-polygon has $2m'$, that is, at least 6 vertices. But since at each vertex of a graph appear cyclically J-, I-, L- and II-regions, there must be at least 6 I-polygons which are two-sided. Therefore there are at least 6 I-polygon in each Φ^*. Now, returning to the chart, every I-region g bounded by a J-arc, an L-arc and two longitudes must contain, as we have shown in (i), still another traces, and hence, according to the above consideration on Φ^*, g must contain still another I-region, and so on ad infinitum, which is impossible.

Thus we must have $m\geqq 3$, $m'\geqq 3$.

(iii) Let $m\geqq 3$, $m'\geqq 3$, $n=n'=1$. Consider the connected subgraph Φ^* of Φ. Then (2′) becomes in our case $N_J+N_L=N_0$, which, if substituted in (4′), yields $N_I+N_{II}=2$, whence $N_I=N_{II}=1$. Therefore Φ^* is a closed chain of $2m$ ($\geqq 6$) two-sided polygons. Thus, returning to the chart, a chain of L-, Λ- and J-regions appears separated from other traces. It may happen that only a half of this chain appears on the northern hemisphere of S, but there is at least one L-region g on the chart, since m is $\geqq 3$. Now if g does not contain any other traces, the disk $f(\bar{g})$ would span a closed F-line on the boundary of the linkage chamber L inside L and separate L in two parts. But then, since B and C is separated by the F-line on account of the nature of an F-line and since B and C is joined by the arc $\widehat{BC}$ of the linkage, the disk $f(\bar{g})$ would meet $\widehat{BC}$, which is impossible.

Therefore every L-region contains some other traces, and hence another chain of L-, Λ- and J-regions, which contains in turn still another L-region, and so on ad infinitum, which is again impossible.

Thus we must have $m \geqq 3$, $m' \geqq 3$, and $n \geqq 3$ or $n' \geqq 2$.

(iv) Let $m \geqq 3$, $m' \geqq 3$ and $n \geqq 3$. If $n' = 1$, then, substituting each two-sided L-polygon of Φ^* by a segment, we have a graph on a sphere, dividing this into polygons with sides $2m$, $2m'$, $2n$, that is, at least equal to 6. But this is as is well known impossible by the Euler relation. If $n' = 2$, then, joining in each 4-sided L-polygon a point in the interior to its 4 vertices by segments, and substituting 4 sides by these segments, we obtain a graph which divides the sphere into polygons with sides at least equal to 6, and this is again a contradiction. By the same reason n' cannot be greater than 2.

Hence $n \geqq 3$ is proved to be impossible.

(v) Since by the same reason as in (iv) the case $n = 1$, $n' = 3$ is excluded, there remains only the case $n = 1$, $n' = 2$ to be considered.

Shrinking each 4-sided L-polygon further to a point and substituting each two-sided J-polygon by a segment, we obtain from Φ^* a graph Φ^{**} dividing the sphere into m-sided J-polygons and m'-sided L-polygons meeting two by two opposite to each other at vertices. Shrinking each m'-sided L-polygon further to a point, we obtain finally a regular graph Φ^{***} on the sphere dividing this into m-sided polygons, at each vertex meeting m' of these polygons. Consequently, since m and m' are odd numbers $\geqq 3$, only the following three cases can occur:

a) $m = 3, \quad m' = 3,$

b) $m = 3, \quad m' = 5,$

c) $m = 5, \quad m' = 3.$

a) $m = m' = 3$. The graph Φ^{***} is given by the vertices and the edges of a regular tetrahedron. We obtain the graph Φ^{**} if we join in each face of the tetrahedron the middle points of edges. Φ^{**} is therefore given by the vertices and edges of a regular octahedron. If we bulge each vertex to a quadrangle, we get Φ^*. It may happen that there are further some "island" of graphs in some regions quite detached from the "mainland", but they have no bearing to the following reasoning. It should be remarked also that even this chart itself is unnecessary for the reasoning which immediately follows.

Under the hypothesis that $m = m' = 3$ let us examine the diagrams of $F \cap \dot{I}$, $F \cap \dot{II}$, $F \cap \dot{J}$ and $F \cap \dot{L}$ and deduce from it a contradiction.

To draw the diagram of $F \cap \dot{I}$, let A_1 be placed at the left end.

Then, since $m=3$, the F-lines $F(A, B)$, and $F(C, D)$ contain each 3 parallel segments:
$F(A, B)=AA_1X_1X_2X_3X_4B_1B$, $F(C, D)=CC_1Y_1Y_2Y_3Y_4D_1D$ where A_1X_1, X_2X_3, X_4B_1, C_1Y_1, Y_2Y_3, Y_4D_1 are parallel segments and X_1X_2, X_3X_4, Y_1Y_2, Y_3Y_4 are circular arcs.

Here two cases occur (See Fig. 11).

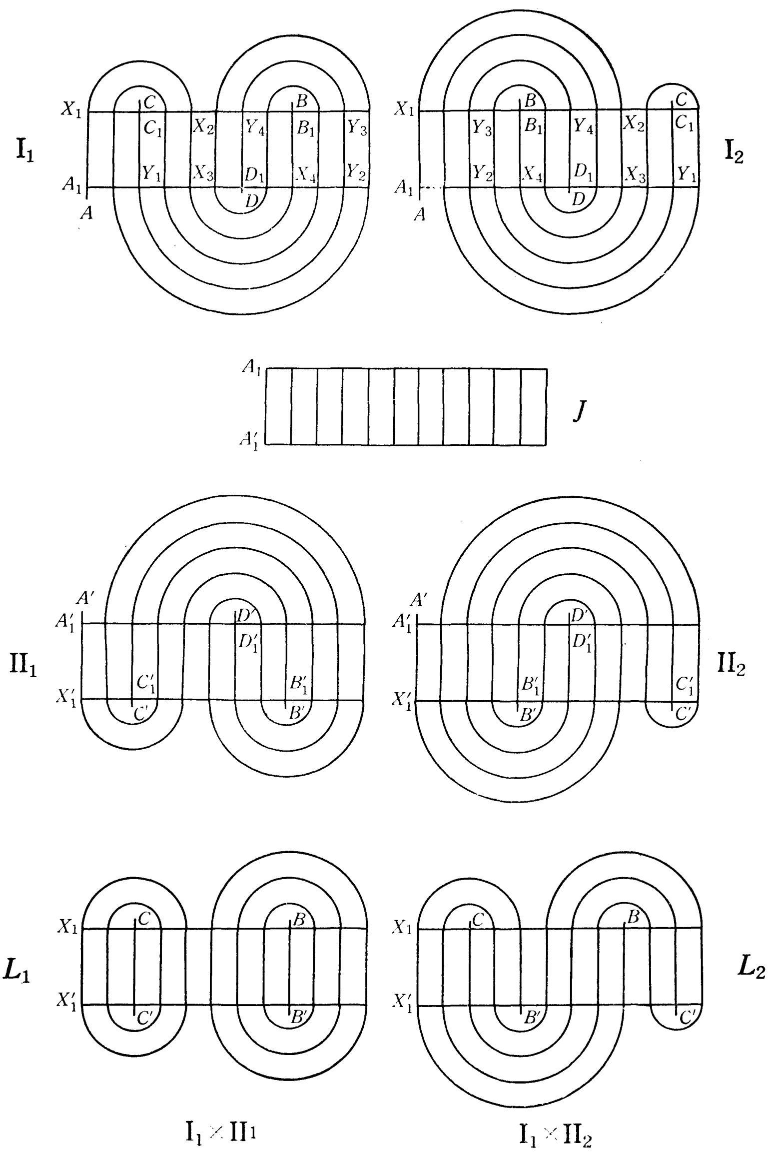


Fig. 11.

I_1. X_4 lies on the right of X_3. Then C_1 and D_1 must be the middle points of X_1X_2 and X_3X_4 respectively.

I_2. X_4 lies on the left of X_3. Then B_1 and D_1 are the middle points of X_1X_2 and X_3X_4 respectively.

If the parallel segments are placed at equal distances, then between each pair of consecutive segments A_1X_1 etc. already written there must be p segments more, where p is the number of closed F-lines, making together in all $6(p+1)=N$ parallel segments.

Similarly for the diagram $F\frown\dot{II}$: II_1 and II_2.

Combining $F\frown\dot{I}$ and $F\frown\dot{II}$, taking into consideration $n=1$ on $F\frown\dot{J}$, we must have as the diagram of $F\frown\dot{L}$ the forms written schematically by $\mathrm{I}_1\times\mathrm{II}_1$, $\mathrm{I}_1\times\mathrm{II}_2$ $\mathrm{I}_2\times\mathrm{II}_1$ or $\mathrm{I}_2\times\mathrm{II}_2$, of which there are only two different types as shown in Fig. 11. But they have either $n'=1$ or $n'=3$, contradicting our hypothesis that $n'=2$.

Thus a) does not occur actually.

b) If $m=3$ and $m'=5$ or $m=5$ and $m'=3$, Φ^{***} is given by a regular dodecahedron or by a regular icosahedron and Φ^{**} is given by a semi-regular polyhedron with 12 pentagons and 20 triangles meeting two by two at vertices, and can by no means be cut by any plane into two symmetric parts, the plane having interior points in common with only one pair of edges. But this would have been necessary in order that Φ^{**} yield a chart of F.

These contradictions complete the proof of our Theorem.

§ 3. Semi-Alternating Knots

The objects of this section is to obtain sufficient conditions to decide from the projection of a knot κ alone whether κ is non-trivial.

9. By a *closed surface* M *of genus* g *in normal position* we mean the sum of two polyhedral surfaces, an *over-surface* and an *under-surface*, bounded by $g+1$ polygons $C_0, C_1, \cdots, C_g$, on the ground plane E^2, lying outside one another except C_0 which encloses the other. The over- and the under-surface are supposed to be in one-to-one correspondence with the planar domain bounded by $C_0, C_1, \cdots, C_g$ by the orthogonal projection. We call $C_0, C_1, \cdots, C_g$ the *horizontal sections* of M and the splitting of the surface M into the over- and under-surface a *horizontal division.* Sometimes we consider also an unbounded surface M all whose horizontal sections $C_0, C_1, \cdots, C_g$ lie outside one another without exception. Naturally there is no essential difference between these two types of surfaces.

Throughout the rest of the paper M denotes always a closed surface in normal position.

Side by side with the horizontal division into over- and under-surface, we make use dually of the vertical division: Let $\widehat{ab}$ be an *arc*, i.e. a polygonal line, joining a point a of C_i and a point b of C_j of the horizontal sections on the ground plane E^2, otherwise disjoint from all horizontal sections. The polygon on M whose projection coincides with $\widehat{ab}$ will be called a *vertical section.* A *vertical division* of M is a splitting of M into a finite number of "perforated spheres" with suitable disjoint vertical sections $C_0', C_1', \cdots, C_{g'}'$.

In either division, each of the splitted sub-surfaces will be called a *component* of M.

The following definitions are equally applicable to both divisions.

A transversal arc, or briefly a *cross-cut*, is an arc joining points of two different sections and lying wholly on a component. A regressive arc, or an *end-cut*, is an arc lying on a component and joining two different points on the same section, but not isotopic to an arc of the section. Cross-cut and end-cut will be called *irreducible arcs.*

Aclosed line on M that is either contained in a component—then it is called a *loop*—or composed of irreducible arcs lying alternatingly or consecutively on different components of M will be called a *closed line in reduced form* or simply a *reduced closed line.* If a given closed line λ on M meets a section in two points a and b such that the arc $\widehat{ab}$ of λ bounds together with the arc γ of the section a simply connected domain on the component, then taking points a' and b' on the prolongation of $\widehat{ab}$ and substituting the arc $\widehat{a'b'}$ of λ by a suitable arc in the next component, we can obtain a new closed line isotopic to λ and having fewer points in common with the sections. By this *reduction* we can reduce any closed line on M isotopically to a closed line in reduced form.

Then we have the following

Lemma 5. *Every closed line λ on M which bounds a disk outside [inside] M is either isotopic to a loop contained in a component of the horizontal [vertical] division or to a reduced closed line containing at least two end-cuts of the horizontal [vertical] division.*

Proof. It will suffice to prove the lemma for the horizontal divison.

If a closed line on M bounds a disk outside M, then its reduced form will also bounds a disk outside M. We assume therefore λ to be in a reduced form, different from a loop. Let F be the disk spanning λ outside M: $F=f(E)$. If D_i $(i=0, 1, \cdots, g)$ denotes for $1 \leqq i \leqq g$ the planar domain bounded by C_i and for $i=0$ the domain outside C_0, then F can be modified if necessary so that all the intersections $F \cap D_i$ consist only of intersection-arcs joining points of C_i in the interior of D_i, and the traces of $F\cap(\bigcup_{i=1}^{g} D_i\cup M)$ consist only of trace-arcs on E other than $\dot{E}$. If γ denotes a minimal arc (cf. **2.**) on E bounding with an arc $\widehat{ab}$ of $\dot{E}$ a minimal domain (cf. **2.**), then $f(\gamma)$ is an intersection-arc of a D_i with endpoints $f(a)$ and $f(b)$ lying on C_i, and the image $f(\widehat{ab})$ turns out to be an end-cut. Since clearly there are at least two minimal domains on E, our Lemma is thus proved.

10. Let M^* be a component of the horizontal or a vertical division of M. A system Σ of arcs on M^* will be said to be *tight*, (1) if every non-trivial loop of M^* meets at least two members of Σ and (2) if every end-cut on M^* meets at least one member of Σ (see Fig. 13). *If Σ is tight, then it is composed of cross-cuts alone.* For if λ contained an end-cut η, then there could be found an end-cut η' of M^* running near η and meeting no member of Σ, which is a contradiction. Moreover, if M^* has more than two boundary sections, the condition (1) may be replaced by (1)$'$: every non-trivial loop meets at least one member of Σ. For if a non-trivial loop λ meets but a single cross-cut γ of Σ, let C_1 and C_2 be the sections of M^* joined by γ. If C_3 is one of the remaining sections, we may suppose that C_3 is separated from C_1 by λ. Modifying λ if necessary, we may suppose that λ meets γ just in a point. Then, replacing a small arc of λ containing this point by two arcs running near γ and joining points of C_2 and the endpoints of this small arc, we get an end-cut of M^* which meets no member of Σ, contradicting (2).

A closed line or a system of disjoint closed lines on M will be said to be *tight*, if it is tight on each component M^* of M. In this case each closed line is in a reduced form, as seen from what we have just proved.

The following theorem has a character of a lemma but is fundamental in the subsequent arguments. To avoid unnecessary restrictions as we have applications in view, let us take into consideration not only knots, but also links. A link will be said to *contain a trivial knot*, if one of its components bounds a disk having no points in common with the rest of the components.

Theorem 5. *Let κ be a closed line [or a system of disjoint closed lines] on M. Then if κ is tight, i.e. if every non-trivial loop of the horizontal division and of a suitable vertical division of M meets κ at least in two points and if every end-cut of both divisions meets κ at least in a point, then κ is a non-trivial knot [or contains no trivial knot] in E^3.*

Proof. Supposing the contrary let F be a disk spanning κ [its component]: $F=f(E)$, $\kappa=f(\dot{E})$, where E is a standard disk.

(i) Let γ be a minimal polygon (cf. **2.**), if any, on E, g its interior. Then $f(\gamma)=\Gamma$ is a closed line on M having no point in common with κ and bounds the disk $f(\bar{g})$ inside or outside M. If we deform Γ to its reduced form Γ' without meeting κ during the deformation, Γ' bounds also a disk inside or outside M, and is by Lemma 5 either a loop or has at least an end-cut. But the latter case is excluded by hypothesis, and even the first case is excluded if Γ' is non-trivial. Thus Γ' must be trivial and so the closed intersection-line $f(\gamma)=\Gamma$ is also trivial and can be made to disappear by the known modification. By the repetition of this process of modifications we can therefore make disappear all closed intersection-lines of $F\frown M$.

(ii) Let γ be a minimal arc bounding together with an arc $\widehat{ab}$ of $\dot{E}$ a minimal domain g. Then $f(\gamma\smile\widehat{ab})$ bounds a disk $f(\bar{g})$ inside or outside M. If we take points A and B very near $f(a)$ and $f(b)$ on $f(\gamma)$, we can join A and B with an arc running near $f(\widehat{ab})$ and meeting κ at most once. Then we have a closed line λ isotopic to $f(\gamma\smile\widehat{ab})$ and meeting κ at most once. Deform λ to its reduced form λ' without meeting κ during the deformation except for one eventual intersection. Then since λ, and hence λ', bounds a disk inside or outside M at the same time with $f(\gamma\smile\widehat{ab})$, λ' cannot meet κ by Lemma 5 and by the hypothesis of our theorem. Hence again as in (i) F can be modified so that the intersection-arc $f(\gamma)$ disappears. By the repetition of this process we can get rid also of all intersection-arcs of $F\frown M$ other than κ itself. But clearly κ [its component] can by no means bound any disk inside or outside M, and this completes the proof.

An immediate application of Theorem 5 is a proof of

Theorem 1 (bis) *Torus knot with $l, m\geqq 2$ is non-trivial.*

Second proof[1]. Let $\dot{T}$ be a torus represented by two concentric circles C_1 and C_2 as horizontal sections. Let Γ_i $(i=1, \cdots, m)$ be a system of cross-cuts on the over-surface lying cyclically in this order, where Γ_i joins a point a_i of C_1 to a point b_i of C_2 radiusly. Next let Γ'_j $(j=1, \cdots, m)$ be another system of cross-cuts on the under-surface joining b_i to a_{i+l} $(a_p=a_q$ if $p\equiv q \pmod{m})$, provided that the orthogonal projections of Γ'_j $(i=1, \cdots, m)$ divide each Γ_i into l equal parts. Then the cross-cuts

1) Cf. Theorem 1. This second proof of the non-triviality of torus knots do not make any use of the linking number as in the first proof.

form a closed line, i.e. a knot κ, if $(l, m)=1$. Take two arbitrary meridians as vertical sections. Then since l and m are supposed to be $\geqq 2$, every non-trivial loop on $\dot{T}$ has at least two points in common with κ, and since moreover there can be no end-cut on $\dot{T}$, κ is non-trivial by Theorem 5.

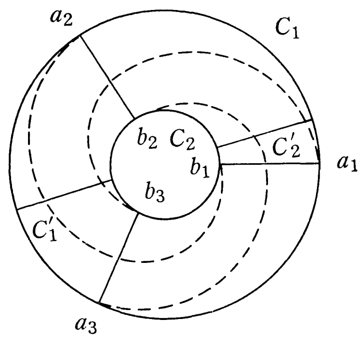


Fig. 12.

11. There is a convenient way of deciding whether a system of cross-cuts is tight or not.

Supposing the component M^* of the horizontal or a vertical division, i.e. a perforated sphere, is given by the domain of the plane E^2 bounded by the sections $C_1, \cdots, C_n$, shrink each C_i to a point c_i. Then we have corresponding to the system Σ a planar linear graph ψ with vertices $c_1, \cdots, c_n$, called the *graph* of Σ. That every non-trivial loop λ of M^* meets at least one member of Σ corresponds in the language of ψ that ψ is connected, and that every end-cut η meets at least one member of Σ corresponds in ψ that there is no polygon π meeting ψ in a single vertex and having points of ψ inside as well as outside π, or, which is the same thing, that ψ remains connected after the removal of a vertex. When a graph possesses these properties, we say it is *strongly connected.* A strongly connected graph contains naturally no end vertex nor looped edge. Then we have:

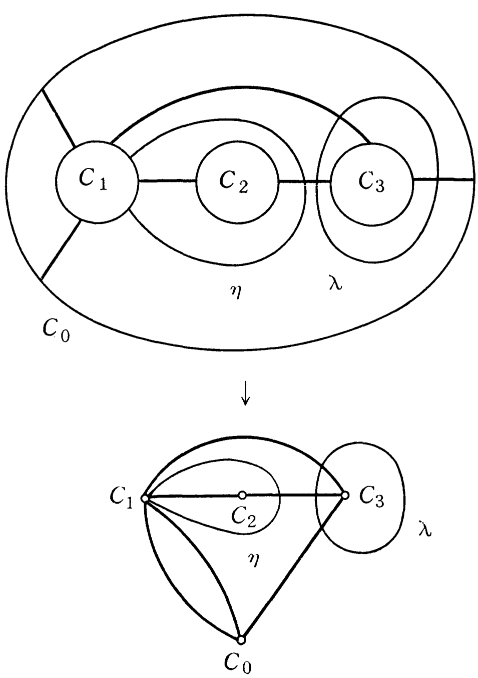


Fig. 13.

Lemma 6. *A system of cross-cuts on a perforated sphere is tight if and only if its graph is strongly connected.*

If κ is a closed line on M, then the *graph of κ on a component M^** of M means the graph of the system $\kappa \cap M^*$. Then Theorem 5 may be stated as follows:

Theorem 5 (bis). *Let κ be a closed line [or a system of disjoint* *closed lines] on M. If the graph of κ on every component of the horizontal division and of a suitable vertical division is strongly connected, then κ is a non-trivial knot [or contains no trivial knot] in E^3.*

12. The graph of a knot.

The plane is divided by the regular projection of a knot into white and black regions [7]. Take a point as centre in each black [white] region and for each crossing point p join the centres of the adjoining black [white] regions by an arc over p attached with the sign + or − according as the knot is positively or negatively twisted over p. The linear graph γ thus obtained is called the *graph with signs* or simply the *graph* of the knot. To a projection of a knot there are in general two different graphs dual to each other, one obtained from the black regions and the other from the white ones.

The graph (with signs) γ of a knot gives conversely the original knot uniquely . In general, any connected planar linear graph with signs gives the projection (the diagram) of a knot or a link in the following way:

In the interior of each polygonal region of the plane divided by the graph join the middle points of the edges consecutively with arcs to make a new polygon in the region. If the region is the interior of a looped edge or if there is an end vertex, the joining arc is self-returning and gives a new loop. All the arcs thus constructed make up then a curve on the plane which is the projection of a knot or a link if each over and under crossing is suitably chosen in accordance with the sign of the graph, the middle points of the edges of the graph corresponding to the crossing points of the knot or the link obtained.

If the projection κ of a knot has a loop, its graph has a looped edge or an end-vertex. If a graph contains neither looped edge nor end-vertex, it will be called *reduced*.

13. The carrier of a knot.

If a knot κ is given, we must first of all construct suitably a closed surface M in normal position on which κ comes to lie, in order to apply Theorem 5 for the determination of the non-triviality of κ. Such a closed surface is called the *carrier* of the knot.

Let κ be a knot given by its projection, denoted by the same letter κ, on the plane. Calling the arcs into which κ is divided by the crossing points *elementary arcs*, determine on each elementary arc a point, called the *alternation-point*, in the following way: α being an elementary arc with endpoints a and b, the alternation-point of α is by definition any arbitrarily chosen point on α, if one of the a, b is an over- and the other an under-crossing point; otherwise no alternation-point is defined on the elementary arc α.

Next construct in each black (or if one prefer, white) region a polygon, called the *inscribed polygon*, passing through all the alternation-points on the boundary arcs of the region, otherwise lying wholly within it and containing moreover the centre of the region in its interior; if there is no alternation-point on the boundary of the region, no polygon is naturally constructed in it.

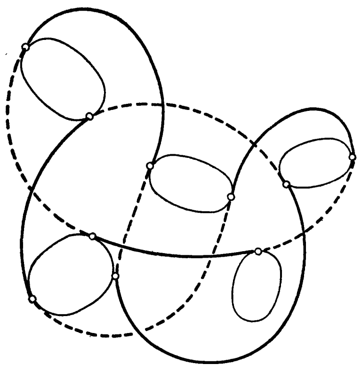


Fig. 14.

Now let M be a closed surface in normal position with the inscribed polygons above defined as its horizontal sections. Starting from an over-crossing point of κ as lying on the over-surface M_o of M, let a point move along κ in a definite direction as far as it comes at an alternation-point. Then it enter the under-surface M_u and remains there so far as it does not come at the next alternation-point, when it re-emerges into the over-surface M_o, and so on. Thus κ can be considered as a closed line lying on the surface M. M is then the *carrier* of κ. The arcs of κ that are defined to be lying on the over-surface M_o are then *over-arcs*, and the rest of arcs are *under-arcs*.

The number of alternation-points on the boundary of the black region play a part in the following. To compute this let B be a black region and let ab be an elementary arc on the boundary. There is an alternation-point defined on ab, if and only if one of the a, b is an over- and the other an under-crossing point, which, if interpolated in the language of the graph of κ, is expressed by saying, if and only if the edges of the graph over a and b have the same sign. Consequently, if c denotes the centre of B which by the definition is a vertex of the graph at the same time, and if i is the number of different pairs of consecutive edges with the same sign emerging from c, then i is precisely the number of alternation-points on the boundary of B. The number i will be called the *alternation-index* of the region or the *alternation-index* of the vertex of the graph. It is by the definition of inscribed polygon equal to the number of alternation-points on the inscribed polygon corresponding to the region.

If, for example, 4 edges with signs $+-+-$ emerge from c, there is no inscribed polygon in the region, the alternation-index being 0, and if

$++--$, the alternation-index is 2, and there are two alternation-points on the inscribed polygon.

The minimum of alternation-indices for all vertices of a graph will be called the *alternation-index* of the graph, or of the knot projection.

It will be now in order to prove the following lemma which is a part of the proof of the theorem that follows.

Lemma 7. *Let a projection of a knot κ have no loop, that is, let the graph of κ be reduced. Then, if the alternation-index of κ is different from* 0, *κ is tight on every component of a certain vertical division of the carrier M of κ.*

Proof. Join in each black region B the crossing points on the boundary of B with points of the inscribed polygon P, which exists by hypothesis, by disjoint arcs within $B-P$. Then for each pair of adjacent black region the inscribed polygons are seen to be joined by an arc crossing over the intervening crossing point. These joining arcs as projections construct vertical sections $C_1', C_2', \cdots, C_{N'}'$ of the carrier M of κ. Then we assert that κ is tight in each component of this vertical division. For let W be a white region and let $B_1, \cdots, B_n$ be a series of black regions around W in this order, P_i denoting the inscribed polygon of B_i $(P_{n+1}=P_1)$. Let γ_i be the joining arc of P_i and P_{i+1} just defined and let C_i' be the vertical section with the projection γ_i. Then if M is cut along these sections, we have a perforated sphere M^* corresponding to W. Since each pair of sections C_i' and C_{i+1}' are joined by an elementary arc of W, any non-trivial loop on M^* meet at least two of these arcs and any end-cut of M^* meet at least one of them. Since all such M^* exhaust the components of the vertical division, the proof of the lemma is complete.

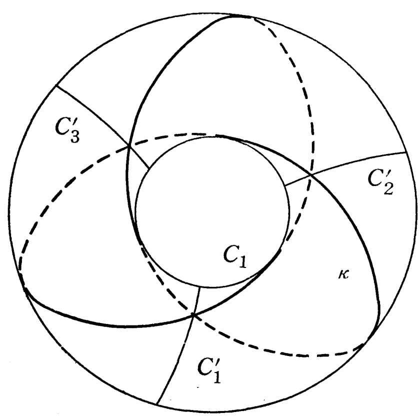


Fig. 15.

14. Over-graph and under-graph.

It now remains to see under what condition the knot is also tight with respect to the over- and under-surface M_o and M_u of M. But this is directly supplied by Lemma 6, according to which κ is tight on M_o and M_u, if and only if the graph of $\kappa\cap M_o$ or $\kappa\cap M_u$ is strongly connected respectively. Our next question is then, how to draw the graphs of $\kappa \cap M_o$ and $\kappa \cap M_u$ directly from that of κ.

We have explained in **12.** how to reconstruct the original knot from its graph γ. Now let c be a vertex of the graph and let $\alpha_1, \cdots, \alpha_p, \alpha_{p+1} = \alpha_1$ be the edges emerging from c in this order cyclically in the positive sense. If m_i denotes the middle points of α_i, then the arcs $m_i m_{i+1} (m_{p+1} = m_1)$ joining m_i and m_{i+1} give the elementary arcs of κ bounding together the black region B with centre c. On the arc $m_i m_{i+1}$ is defined the alternation-point b_i if and only if α_i and α_{i+1} have the same sign, and then the inscribed polygon P touches the boundary of B from within at b_i. Consider the following 4 cases.

(1) $\alpha_i(+)$, $\alpha_{i+1}(+)$, that is, α_i and α_{i+1} have both $+$ signs. In this case $m_i b_i$ is an over-arc, i. e. belongs to M_o, and $b_i m_{i+1}$ belongs to M_u.

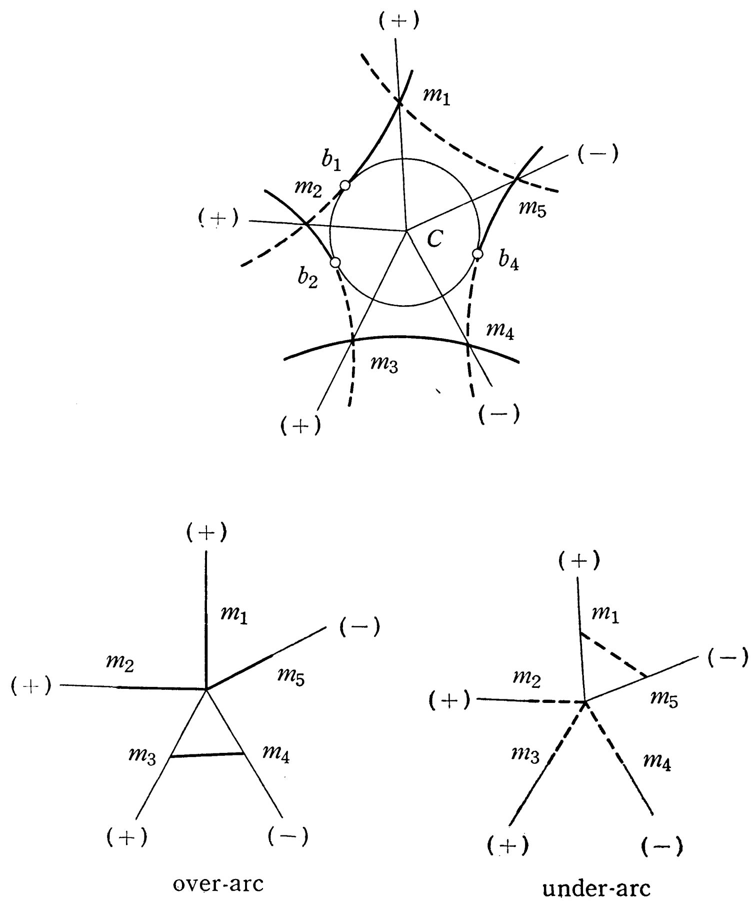


Fig. 16.

(2) $\alpha_i(+)$, $\alpha_{i+1}(-)$. Then the whole m_im_{i+1} belongs to M_o.

(3) $\alpha_i(-)$, $\alpha_{i+1}(+)$. Then m_im_{i+1} belongs to M_u, and there is no over-arc.

(4) $\alpha_i(-)$, $\alpha_{i+1}(-)$. Then m_ib_i belongs to M_u, and b_im_{i+1} to M_o.

The above rule applied for all vertices of the graph γ gives the over-arcs $\kappa\cap M_o$ and the under-arcs $\kappa\cap M_u$ of κ.

Now shrink the inscribed polygon P to the centre c, and with it bring m_ib_i and b_im_{i+1} to coincidence with m_ic and cm_{i+1} respectively. Leave m_im_{i+1} unaltered, if no alternation-point b is defined on it. Then we obtain the required graphs of over-arcs and under-arcs of κ, which we shall call hereafter the *over-graph* γ_o and *under-graph* γ_u of κ or of γ, by observing the following rule (see Fig. 16).

(1) $\alpha_i(+)$, $\alpha_{i+1}(+)$. Then let cm_i belong to M_o, and cm_{i+1} to M_u.

(2) $\alpha_i(+)$, $\alpha_{i+1}(-)$. Let m_im_{i+1} belong to M_o.

(3) $\alpha_i(-)$, $\alpha_{i+1}(+)$. Let m_im_{i+1} belong to M_u.

(4) $\alpha_i(-)$, $\alpha_{i+1}(-)$. Let cm_i belong to M_u, and cm_{i+1} to M_o.

We shall call m_im_{i+1} a *converting arc.*

To give more facilities to operate according to the above rule we note the following procedure (see Fig. 17).

Let α, α' be any pair of consecutive edges of γ emerging from a vertex c such that α can be brought to coincidence with α' by turning around c in the positive sense without meeting other edges of γ. Let m, m' be the middle points of α and α'.

To obtain the over-graph γ_o of κ, perform the following operation for all couples of edges (α, α') of the above type :

(1) $\alpha(+)$, $\alpha'(+)$, join c and m by a thickened line on α.

(2) If $\alpha(+)$, $\alpha'(-)$, join m and m' by a thickened line (converting arc).

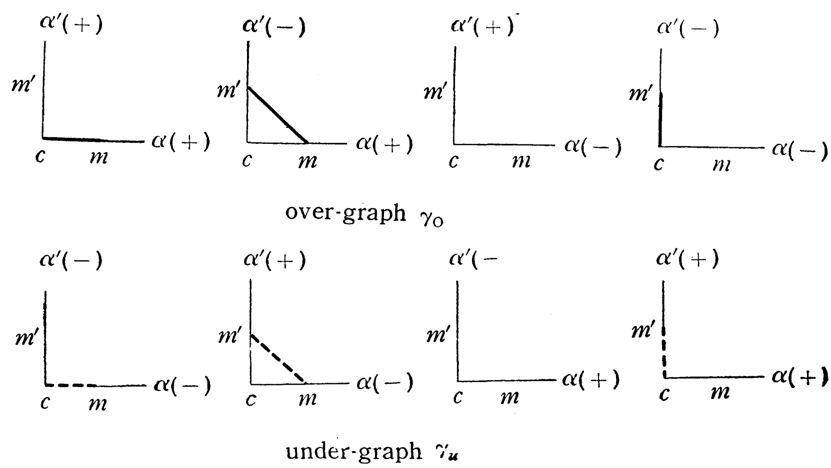
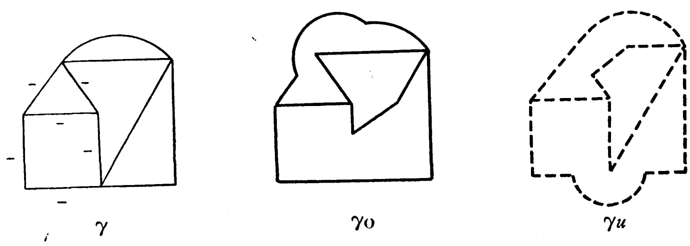


Fig. 17.

(3) If $\alpha(-)$, $\alpha'(+)$, draw no line whatever.
(4) If $\alpha(-)$, $\alpha'(-)$, join c and m' by a thickened line on α'.

Then the thickened lines give as a whole the required over-graph γ_o. *To obtain the under-graph* γ_u perform the following operations:

(1) If $\alpha(-)$, $\alpha'(-)$, join c and m by a dotted line on α.
(2) If $\alpha(-)$, $\alpha'(+)$, join m and m' by a dotted line.
(3) If $\alpha(+)$, $\alpha'(-)$, draw no dotted line.
(4) If $\alpha(+)$, $\alpha'(+)$, join c and m' by a dotted line on α'.



γ = graph of k_1

Fig. 18.

If the graph γ of a knot κ has the alternation-index at least 2 and if the over-graph γ_o and under-graph γ_u of γ are both strongly connected, then the knot will be called *semi-alternating.*

Then the theorem that we obtain finally is

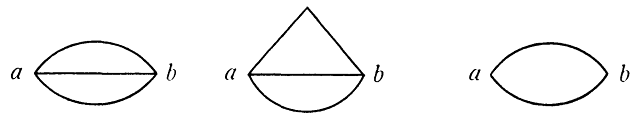
Theorem 6. *Any semi-alternating knot is non-trivial.*

Proof. From the second condition of the statement follows that γ has neither end-vertex nor looped edge. Since the alternation-index is supposed to be >0, κ is by Lemma 7 tight on every component of a certain vertical division of the carrier M of κ. From the second condition follows by Lemma 6 that κ is tight also on the over- and under-surface of M. The conclusion follows then from Theorem 5.

15. Some specifications of Theorem 6.

Under a *graph* we shall mean a planar (or rather spherical) connected linear graph. Under a *domain of the graph* γ we shall mean any one of $E^2-\gamma$. The closure of a domain will sometimes be called a *region.* Among the domains of γ we specify the unbounded one, the *outer domain,* and denote it by D_∞. The others are called inner *domains.* The *outer boundary* of γ is then the boundary of the domain D_∞. An *outer edge* is an edge on the outer boundary. A domain of γ is called *regular,* if its boundary is a polygon (i. e. a simple polygon).

A *linkage* λ joining points a and b is a graph with alternation-index at least 2 such that a and b lie on the outer boundary of γ and that every cut-vertex, if any, of its over-graph or under-graph cuts it just into two components, of which one contains a and the other b. A linkage



strong linkage weak linkage

Fig. 19.

is *strong*, if, whenever an edge of + or − sign is attached outward to each of the vertices a and b to make a graph λ^*, then its over- and under-graph λ_o^* and λ_u^* are both connected. If λ fails to possess this property, λ is said to be *weak*.

A *simple linkage* (m) *of winding* m, where m is a positive or a negative integer different from 0, is a graph consisting of $|m|$ edges joining two points a and b, the sign of each edge coinciding with that of m. The simple linkage (m) is *strong* if $|m| \geqq 3$, and *weak* otherwise.

Theorem 7. *Let γ be a reduced graph of alternation-index at least 2 with the over- and under-graph γ_o and γ_u and let a and b be two distinct vertices of its outer boundary such that (i) every cut-vertex, if any, of γ (or γ_o or γ_u) cuts it just into two components, one containing a and the other b, (ii) a and b do not belong to the same region of γ (or γ_o or γ_u) other than (possibly) D_∞. Then $\gamma \cup \lambda$ is the graph of a non-trivial knot or a link containing no trivial knot, provided that λ is a strong linkage.*

Proof. 1) Let aa_1 and aa_2 be consecutive edges of the outer boundary of γ counted in the positive sense. If $aa_1(+)$, $aa_2(-)$, then, denoting m_1, m_2 the middle points of aa_1 and aa_2, the half-edges am_1 and am_2 are to be replaced by an arc m_1m_2 joining m_1 and m_2 outside γ in the formation of γ_o. The alternation-index being $\geqq 2$, there are at least two edges of γ_o emerging from a. Then the domain of γ_o adjacent to m_1m_2 is cut by half-edges am_1 and am_2 into three domains, which are all regular and in particular those two having common points with domains of γ. In all other cases there are no such *bulging* domains as that bounded by am_1, am_2 and m_1m_2.

The same is true at the vertex b and also for the linkage λ. If there is a bulging domain at a or b for γ_o as well as for λ_o, cut it off by half-edges such as am_1 and am_2. Then the domains of γ_o or λ_o so modified will be called the *modified domains* of γ_o or λ_o. These are of course regular. γ_o or λ_o itself thus modified will be denoted by γ_o' and λ_o'.

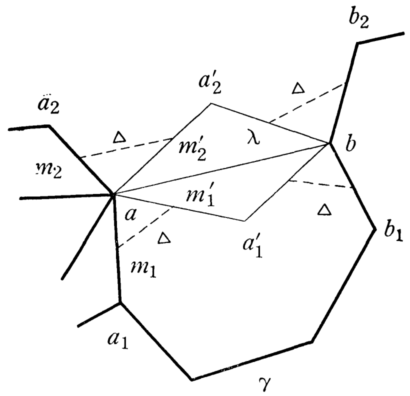


Fig. 20.

2) Let aa_2', aa_1' be the edges of the linkage λ counted in the positive sense, m_2', m_1' denoting their middle points. Then aa_1 and aa_1' are consecutive in the positive sense, and so are aa_2' and aa_2. Call the domain such as is bounded by the half-edges am_1 and am_1' and by an arc m_1m_1' a Δ-*domain.*

If $aa_1(+)$, $aa_1'(-)$, then the modified domain of γ_o adjacent to am_1 is connected to the modified domain of λ_o adjacent to am_1' through the Δ-domain bounded by am_1, am_1', m_1m_1' in the formation of $(\gamma\cup\lambda)_o$. Similarly if $aa_2'(+)$, $aa_2(-)$. These two cases may happen at the same time; but then the adjacent domains in question are all distinct. In all other cases no domains of γ_o' and λ_o' are joined together in the formation of $(\gamma\cup\lambda)_o$.

Now it is impossible that a modified domain of γ_o' is joined to one or two modified domains at a and at b at the same time; for otherwise there would exist edges aa_i and bb_j such that $(\lambda\cup aa_i\cup bb_j)_o$ is disconnected, contrary to the hypothesis of strongness of λ. Since by hypothesis a and b do not belong to the boundary of the same modified domain of γ_o, we see therefore that the domain of γ_o arising from the conjunction of modified domains of γ_o and λ_o at a or b is again regular.

3) A modified domain of γ_o or λ_o which is bounded and which is not augmented by a Δ-domain is unaffected in the formation of $(\gamma\cup\lambda)_o$ and becomes a domain of $(\gamma\cup\lambda)_o$. Consequently such a domain is regular.

4) Finally consider the modified outer domain D_∞'. By the conjunction of the modified linkage λ_o', D_∞' is divided into two domains, and by the hypothesis (i) these are easily seen to be regular. But a conjunction of a Δ-domain at a or b in the formation of $(\gamma\cup\lambda)_o$ in no way prevents the boundary of each of these domains from remaining a polygon, and hence the domains in question are proved to be regular.

A similar argument holds also in the formation of $(\gamma\cup\lambda)_u$ and the proof of the theorem is complete.

If the linkage is weak, further conditions must be imposed to obtain the same conclusion as Theorem 7, but we do not go into details further.

Let γ and γ' be the graphs of knots or links. Bring a vertex of γ with a vertex of γ' together to coincidence, otherwise disjoint, and join n pairs of disjoint vertices (a_i, a_i'), where $a_i\in\gamma$ and $a_i'\in\gamma'$, by disjoint linkages lying in the outer domain of $\gamma\cup\gamma'$. Then the resulting graph as well as knot or link will be called a *linked sum of n-th order* of γ and γ' along (a_i, a_i'). Then by repeated applications of Theorem 7 we have immediately

Theorem 8. *Let γ and γ' be the graphs of semi-alternating knots. Let a_i, a_i' $(i=1, 2, \cdots, n)$ be n pairs of vertices on the outer boundary of $\gamma \cup \gamma'$, $a_i \in \gamma$, $a_i' \in \gamma'$, such that a_i and a_i' do not belong to the same region of $(\gamma \cup \gamma')_o$ and $(\gamma \cup \gamma')_u$ other than possibly D_∞. Then a linked sum of γ and γ' with linkages λ_i $(i=1, \cdots, n)$ of end points (a_i, a_i') is non-trivial, provided λ_i are strongly connected.*

16. The model of a knot lying on a carrier surface M as described in this section may be extended to a knotted 2-sphere S^2 on a 3-dimensional carrier surface M^3 in a 4-space E^4 as follows.

By a *perforated 2-sphere P* we mean a homeomorph in E^4 of the closure of a 2-dimensional spherical domain G bounded by a finite number of disjoint circles. A *bounding circle* of the perforated 2-sphere P is then the image of a bounding circle of G. A system Σ of perforated 2-spheres P_i $(i=1, 2, \cdots, p)$ will make up an S^2, if (i) for $i \neq j$ P_i and P_j are either disjoint or have just one bounding circle C_{ij} in common, in which case P_i and P_j are said to be *consecutive*, (ii) for $i \neq j$ P_i and P_j can be joined uniquely by a sequence of consecutive P's of distinct members of Σ, and (iii) for each P_i and for each bounding circle C of P_i there is just one P_j of Σ having the bounding circle $C(=C_{ij})$ in common with P_i. The system Σ will then be called an *S^2-system.*

Next let $M_0, M_1, \cdots, M_m$ be disjoint 2-polyhedra in $E^3 \subset E^4$ bounding a 3-domain D in E^3, and let M_+^3 and M_-^3 be 3-surfaces in the upper and lower half-space E_+^4 and E_-^4 of E^4 divided by E^3 respectively bounded by $M_0, M_1, \cdots, M_m$, the points of M_+^3(and M_-^3) and D corresponding one-to-one in the direction orthogonal to E^3. Now if Σ is an S^2-system of perforated 2-spheres P_i $(i=1, 2, \cdots, p)$ such that (i) the bounding circles of P_i are circles on some M_k's (ii) each P_i is contained either in M_+^3 or in M_-^3, then the S^2-system Σ forms a knotted sphere S^2 in E^4. Examples of non-trivial knotted 2-sphere S^2 of this kind may be obtained by the use of Artin's method [2].

Problem: Find some sufficient conditions that S^2 be non-trivial.

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