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STRUCTURES OF FULL HAKEN MANIFOLDS

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1. Introduction

In this paper, we consider some relations between a Heegaard splitting and the torus decomposition of a Haken manifold. The first result of this paper is:

Theorem 1. *Let M be a Haken manifold without boundary or with incompressible toral boundary. Suppose that M admits a Heegaard splitting of genus $g(\geq 2)$. Then M is decomposed into at most $3g$-3 components by the torus decomposition. Moreover, if M is decomposed into $3g$-3 components, then each component is simple i.e. every incompressible torus in it is boundary parallel.*

For the definition of a Heegaard splitting, and the torus decomposition of a 3-manifold with boundary in this context, see section 2.

The classical Haken's theorem ([H], [J]) shows that a Heegaard genus g 3-manifold is decomposed into at most g components by the prime decomposition. Theorem 1 is an analogy to this fact.

REMARK. We note that the above estimation is best possible. In section 8, we will show that for each $g(\geq 2)$ there are infinitely many Haken manifolds with Heegaard splittings of genus g, each of which is decomposed into $3g$-3 components by the torus decomposition.

The key of the proof of Theorem 1 is Proposition 4.1, which is an analogy to the Haken's theorem.

Proposition 4.1. *Let M be a Haken manifold as in Theorem* 1, *and $\mathcal{T}$ be a union of tori which gives the torus decomposition of M. If the number of the components of $\mathcal{T}$ is greater than or equal to $3g$-4, then there is a component T of $\mathcal{T}$ such that T is ambient isotopic to T' which intersects the genus g Heegaard surface in a circle.*

Let M be a Haken manifold as in Theorem 1. We say that M is *full* if it

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is decomposed into $3g$-3 components by the torus decomposition. In section 7, we will investigate the structure of full Haken manifolds from the viewpoint of torus decomposition.

Theorem 2 (cf. [Ko 2, Theorem]). *Let M be a full Haken manifold, and $M=M_1\cup\cdots\cup M_{3g-3}$ be the torus decomposition of M. Then:*

(i) *If $\mathrm{Fr}_M\ M_i$ consists of a torus, then M_i admits a Seifert fibration,*

(ii) *g-1 components of $\{M_i\}$, say $M_1, \cdots, M_{g-1}$, are homeomorphic to the exteriors of two bridge links,*

(iii) *$M_g, \cdots, M_{3g-3}$ admit Seifert fibrations,*

(iv) *Suppose that $M_i\cap M_j\neq\phi$, where $i<g$. Then $j\geq g$, and M_j admits a Seifert fibration such that a regular fiber of M_j in $M_i\cap M_j$ is identified with a meridian loop of M_i.*

Theorems 1,2 together with the arguments in [C] implies:

Corollary 1. *Let L be a tunnel number n link in a closed 3-manifold. Suppose that the exterior of L is a Haken manifold with incompressible boundary. Then, the exterior is decomposed into at most $3n$ components by the torus decomposition. Moreover, if it is decomposed into $3n$ components, then the components satisfies the conclusions of Theorems 1,2.*

Bonahon-Siebenmann [B-S] showed that a classical link has a cannonical splitting by a system of tori and a system of 2-spheres each of which intersects the link in two or four points. The idea for the proof of this fact is to consider the prime and torus decomposition of the 2-fold covering space of the link. Theorem 1 together with this fact, the Haken's theorem, and a theorem of Birman-Hilden [B-H] implies:

Corollary 2. *Suppose that L is an $n(>2)$ bridge link. Then, L is decomposed into at most $3(n-2)$ pieces by the above splitting.*

The bulk of this work was done while I was a member of the Mathematical Science Research Institute, Berkeley. I would like to express my thanks for the generous hospitality of the institute. I thank to Andrew Casson for teaching me the results in [C-G], and several useful conversations. I also thank to Kanzi Morimoto for pointing out errors in the original paper of this work.

2. Preliminaries

Thorughout this paper, we will work in the piecewise linear category. For the definitions of *irreducible manifold, incompressible surface, parallel surface* we refer to [He]. For the definitions of *essential surface, ∂-incompressible surface, Haken manifold, Seifert fibered manifold, exceptional fiber,* and *orbit manifold* we refer to [J].

Let M be a Haken manifold as in Theorem 1. Then, by [J], there is a maximal, perfectly embedded Seifert fibered manifold Σ, which is called a characteristic Seifert pair for M. Then $\mathrm{Fr}_M\Sigma$ consists of tori in Int M. If some components of them are parallel in M, then we eliminate one of them from the system of tori. If a component of the system is parallel to a boundary component of M, then we eliminate it from the system. By performing these eliminations finitely many times, we get a system of tori $\mathcal{T}$ in M which are mutually non parallel, and each component of which is not parallel to a boundary component of M. In this paper, we call the decomposition of M by $\mathcal{T}$, the *torus decomposition* of M. Then, by corresponding each component of M-$\mathcal{T}$ to a vertex, and each component of $\mathcal{T}$ a edge, we get a graph G_M. We call G_M the *characteristic graph* for M.

Let S be a closed surface of genus g. A genus g *compression body* C is a 3-manifold obtained from $S\times[0, 1]$ by attaching 2-handles along mutually disjoint simple loops on $S\times\{1\}$, and then attaching some 3-handles to it (cf. [Bo]). Let $\partial_0 C$ be the boundary component of C which corresponds to $S\times\{0\}$. We note that a handlebody (:cube with handles) H is a compression body such that $\partial H=\partial_0 H$. Let M be a compact 3-manifold. $(C_1, C_2: F)$ is a genus g *generalized Heegaard splitting* (or simply a *Heegaard splitting*) of M if each C_i is a genus g compression body, $M=C_1\cup C_2$, and $C_1\cap C_2=\partial_0 C_1=\partial_0 C_2=F$ (cf. [C-G]). The minimal genus of all Heegaard splittings of M is called the *Heegaard genus* of M.

The next theorem follows from the fact that every 3-manifold admits a triangulation (cf. [He]).

Theorem 2.1. *Every compact* 3-*manifold admits a Heegaard splitting.*

Now, we will see some fundamental properties of compression bodies.

Lemma 2.2 ([Bo, corollary B.3]). *Let C be an irreducible compression body, and D be an essential disk properly embedded in C. Then, D cuts C into a (possibly, disconnected) compression body C' such that $\partial_0 C'-D\subset\partial_0 C$.*

Lemma 2.3 ([C-G]). *Let S be an incompressible, ∂-incompressible surface properly embedded in an irreducible compression body C. Then, S is either a disk, or an annulus A, where one component of ∂A is contained in $\partial_0 C$, and the other component is contained in a distinct component of ∂C.*

3. Incompressible surfaces and isotopies of type A

The problems concerning the relations between a Heegaard surface and an incompressible surface in a 3-manifolds were considered by several authors ([C-G, H, J, Ko 1, Ko 2, Mo, O]). In this section, we will show that the techniques used there, say hierarchy for a 2-manifold, isotopy of type A, $\cdots$, can be applied to generalized Heegaard splittings. We note that the first half of this section is a combination of the results by Casson-Gordon ([C-G]), and Ochiai ([O]), which are based on the argument by Jaco for the proof of the Haken's theorem ([J]). And the last half is a broad generalization of results in [Ko 1, Ko 2].

Let S be a (possibly disconnected) compact 2-manifold. A properly embedded arc a in S is *inessential* if there exists an arc $b(\subset\partial S)$ such that $a\cup b$ bounds a disk in S. a is *essential* if it is not inessential. A *partial hierarchy* (cf. [J, Chapter IV]) for S is a finite sequence $(S^{(0)}, a_0), \cdots, (S^{(m)}, a_m)$, where $S^{(0)}=S$, a_i is an essential arc in $S^{(i)}$, and $S^{(i+1)}$ is obtained from $S^{(i)}$ by cutting along a_i. A partial hierarchy for S, $(S^{(0)}, a_0), \cdots, (S^{(m)}, a_m)$ is a *hierarchy* if each component of $S^{(m+1)}$ is a disk. It is an *almost hierarchy* if each component of $S^{(m+1)}$ is a disk, or an annulus such that one boundary component is a component of ∂S. An essential arc a in S is of *type* 1 if a joins distinct components of ∂S, a is of *type* 2 if a joins one component of ∂S, and a separates the component of S containing a, and a is of *type* 3 if a joins one component of ∂S, and a does not separate the component of S containing a. Let $\mathcal{A}$ be a system of mutually disjoint, essential arcs in S. We say that an element a of $\mathcal{A}$ is a *d-arc related to* $\mathcal{A}$ if a is of tyep 1, and there is a component C of ∂S such that a is the only element of $\mathcal{A}$ which meets C.

Throughout this section, M denotes a compact 3-manifold, S denotes a closed or bounded, incompressible, ∂-incompressible surface properly embedded in M.

Let $(C_1, C_2: F)$ be a Heegaard splitting of M. Then, the proof of the next lemma is left to the reader.

Lemma 3.1. *There esists an incompressible, ∂-incompressible surface S' such that S' is homeomorphic to S, each component of $S'\cap C_i$ $(i=1, 2)$ is incompressible in C_i, and $\partial S'=\partial S$. Moreover, if M is irreducible, then S' is ambient isotopic to S* rel ∂.

We suppose that $S(\subset M)$ satisfies the conclusion of Lemma 3.1. Let $S_i= S\cap C_i (i=1, 2)$. Then, by Lemma 2.3, there is an almost hierarchy $(S_1^{(0)}, a_0), \cdots, (S_1^{(m)}, a_m)$ for S_1 and a sequence of isotopies of type A which realizes the almost hierarchy i.e. if $S^{(0)}=S$, and $S^{(i)}$ is the image of $S^{(i-1)}$ after the i-th isotopy of type A ([J, Chapter II]) at a_{i-1}, then $S^{(i)}\cap C_1=S_1^{(i)}$. We may suppose that $a_i\cap a_j=\phi (i\neq j)$. So, we can consider $a_1\cup\cdots\cup a_m$ are arcs properly embedded in S_1. Let $A_p (0\leq p\leq m)$ be the system of arcs $\{a_0, \cdots, a_p\}$ in S_1.

Lemma 3.2. *Let M, $(C_1, C_2: F)$, S, S_i, A_p be as above. Suppose that there are i, p $(i\leq p\leq m)$ such that a_i is a d-arc related to A_p, and there exits a disk component D of S_2 such that a_i is the only arc in A_p which meets ∂D. Then S is rel ∂ ambient isotopic to S' such that the number of the components of $S'\cap F$ is less* *than that of* $S \cap F$. *Moreover, if* S_1 *consists of disks, then* $S' \cap C_1$ *also consists of disks.*

Proof. See [O, Lemma 1]. The arguments there work in this situation.

Lemma 3.2 assures that we can prove the theorems in [J, Ko 1, Ko 2, Mo, O] for generalized Heegaard splittings without changing proofs. And, we can prove more theorems by using the same argument (cf. [C-G]).

In the rest of this section, we suppose that M is a Haken manifold without boundary, or with incompressible toral boundary, $\{T_1, \cdots, T_l\}$ be a system of mutually disjoint, non-parallel incompressible tori in M, and let $\mathcal{T} = T_1 \cup \cdots \cup T_l$.

By moving $\mathcal{T}$ by an ambient isotopy, we may suppose that each component of $\mathcal{T}_1 = \mathcal{T} \cap C_1$ is a disk, and each component of $\mathcal{T}_2 = \mathcal{T} \cap C_2$ is incompressible in C_2. Then, by Lemmas 2.3, 3.1, we have a hierarchy $(\mathcal{T}_2^{(0)}, a_0), \cdots, (\mathcal{T}_2^{(m)}, a_m)$ for $\mathcal{T}_2$ and a sequence of isotopies of type A which realizes the hierarchy. We note that if we perform an isotopy of type A at a_i then it produces a band b_i which connects component(s) of $\mathcal{T}^{(i)} \cap C_1$. We say that b_i is of type 1, 2, or 3 if a_i is of type 1, 2, or 3 respectively. Let A_p $(0 \leq p \leq m)$ be the system of essential arcs $\{a_1, \cdots, a_p\}$ on $\mathcal{T}_2$.

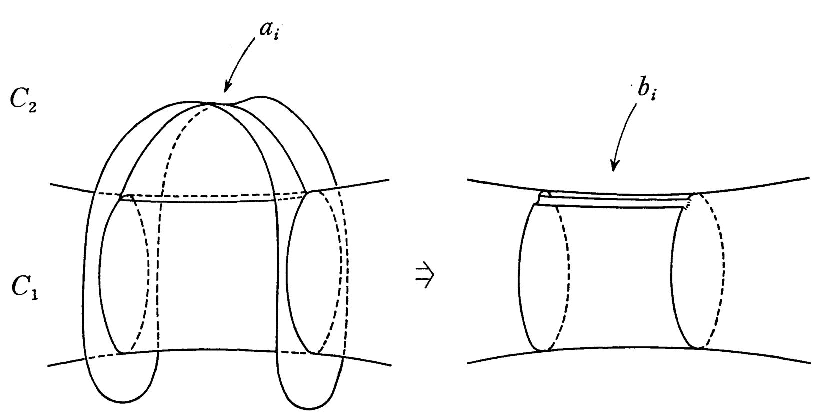


Figure 3.1

Lemma 3.3. *If some* a_i *is of type* 2, *then* $\mathcal{T}$ *is ambient isotopic to* $\mathcal{T}'$ *such that each component of* $\mathcal{T}' \cap C_1$ *is a disk, and the number of the components is less than that of* $\mathcal{T} \cap C_1$.

Proof. Let T be the component of $\mathcal{T}_2$ containing a_i. Then a_i separates T into a punctured torus and a planar surface P. Then, by the induction on the number of the components of ∂P, we can show that some a_j $(\subset P)$ is a d-arc related to A_m. Hence, by Lemma 3.2, we have the conclusion of Lemma 3.3.

Lemma 3.4. *Let C be a component of $\partial\mathcal{T}_2$, and a_i be the first arc which meets C i.e. $a_i \cap C \neq \phi$, $a_j \cap C = \phi (j<i)$. If a_i is not of type* 3, *then $\mathcal{T}$ is ambient isotopic to $\mathcal{T}'$ as in Lemma* 3.3.

Proof. If a_i is of type 2, then by Lemma 3.3, we have the conclusion. If a_i is of type 1, then a_i is a d-arc related to A_i. Hence, by Lemma 3.2, we have the conclusion.

Lemma 3.5. *Let T be a component of $\mathcal{T}$ and let $T_2 = T \cap C_2$. Suppose that $T \cap C_1$ consists of more than one disks, and that there are two arcs a_i, a_j which are of type* 3 *and meet a component C of ∂T_2. Then $\mathcal{T}$ is ambient isotopic to $\mathcal{T}'$ as in Lemma* 3.3.

Proof. Let D be the component of $T \cap C_1$ such that $\partial D = C$, and let $T' = \mathrm{cl}(T - D)$. Then, $a_i \cup a_j$ $(\subset T')$ cuts T' into a disk, or into a disk and an annulus. Let D' be the component of T' cut along $a_i \cup a_j$, which is a disk. Let P be the component of T_2 cut along $a_i \cup a_j$, which corresponds to D'. Then, we see that some a_k $(\subset P)$ is a d-arc related to A_m. Hence, by Lemma 3.2, we have the conclusion.

REMARK. Suppose that $T \cap C_1$ consists of a disk. Then T_2 contains just two arcs a_i, a_j, which are of type 3.

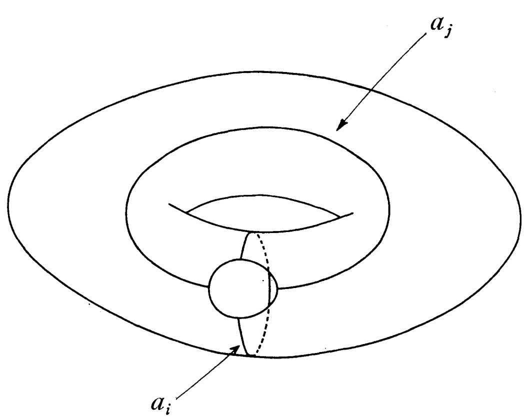


Figure 3.2

Lemma 3.6. *Suppose that there are three arcs a_i, a_j, a_k which are of type* 1 *such that each of them meets a component C of $\partial\mathcal{T}_2$. Then $\mathcal{T}$ is ambient isotopic to $\mathcal{T}'$ as in Lemma* 3.3.

Proof. Let T be the component of $\mathcal{T}$ containing $a_i \cup a_j \cup a_k$, and let $T_2 = T \cap C_2$. Since T_2 contains an arc of type 1, ∂T_2 consists of more than one component. By Lemmas 3.4, 3.5, we may suppose that, for each component of ∂T_2, there is only one arc of type 3 which meets the component. Let A be the union of all type 3 arcs on T_2. Then A cuts T_2 into annuli $B_1, \cdots, B_s$ $(s\geq 2)$. We may suppose that a_i, and a_j are contained in B_1, and $i<j$. Since a_j is an essential arc in $\mathcal{T}_2^{(j)}$, a_i is a d-arc related to A_i. Hence, by Lemma 3.2, we have the conclusion.

Before stating Lemma 3.7, we prepare some terminologies. A *link* L is a finite union of circles embedded in the 3-sphere S^3. If L consists of one component, then it is called a *knot*. The *exterior*, $Q(L)$, of the link L is the closure of the complement of a regular neighborhood of L. A *meridian loop* of L is a non-trivial loop in $\partial Q(L)$ which bounds a disk in the regular neighborhood of L. L is a *two bridge link* (or *knot*), if it can be represented as a union of two trivial tangles with two strings ([R]). Then, the next lemma follows from the definition easily.

Lemma 3.7. *Let V be an orientable genus two handlebody. Suppose that there are pairwise disjoint annuli A_1, A_2 in ∂V, and pairwise disjoint disks D_1, D_2 properly embedded in V such that $D_1\cup D_2$ cuts V into a 3-cell, $D_i\cap A_i$ is an essential arc in A_i, $A_i\cap D_j=\phi$ $(i, j=1, 2, i\neq j)$. Suppose that l is a simple loop in $\mathrm{cl}(\partial V-(A_1\cup A_2))$ which separates it into two disks with two holes, and that N is the 3-manifold obtained from V by attaching a 2-handle along l. Then N is homeomorphic to the exterior of a two bridge link, or a two bridge knot, where the core of A_i is a meridian loop.*

Proposition 3.8. *Suppose that $\mathcal{T}$ gives the torus decomposition of M, and the number of the components of $\mathcal{T}_1$ is minimal among all systems of tori which are ambient isotopic to $\mathcal{T}$, and each of which intersects C_1 in disks. If four disks D_1, D_2, D_3, D_4 of $\mathcal{T}_1$ are mutually parallel in C_1, then there is a component T of $\mathcal{T}$ such that $T\cap C_1=D_i (i=1, 2, 3, or\ 4)$.*

Proof. We suppose that D_1, D_2, D_3, D_4 are in C_1 in this order, and call the direction in which D_1 (D_4 resp.) is settled 'left' ('right' resp.). Let b_{j_n} be the n-th band which is attached to $D_1\cup D_2\cup D_3\cup D_4$. Assume that the conclusion of the proposition does not hold. By Lemma 3.4, b_{j_1} is of type 3. Hence, b_{j_1} is attached to the left side of D_1, or the right side of D_4 to produce an essential annulus A_1 in C_1. We may suppose that b_{j_1} is attached to D_1.

If b_{j_2} is attached to D_1 (A_1, correctly speaking), then, by Lemmas 3.2, 3.3, 3.5, b_{j_2} is of type 1, and is attached to the left side of D_1. Then, we can exchange the order of the isotopies of type A so that (j_1+1)-th isotopy is performed on a_{j_2}. We note that a_{j_2} is a d-arc related to $\{a_0, \cdots, a_{j_1-1}, a_{j_2}\}$. But, by Lemma 3.2, this contradicts the minimality assumption of $\mathcal{T}$. Hence, b_{j_2} is not attached to D_1.

Then, we divide the proof into two cases.

Case 1. b_{j_2} is attached to D_2.

In this case, b_{j_2} is of type 3, and produces an annulus A_2 which is parallel to A_1. Assume that b_{j_3} is attached to D_1 or D_2. Then, by the argument as above, we see that b_{j_3} is of type 1, and is attached between D_1 and D_2 ($:A_1$ and A_2). Then b_{j_3}, together with A_1 and A_2, produces a disk with two holes P properly embedded in C_1. A component of ∂P bounds a disk D in F. Let T' be the component of $\mathcal{T}^{(j_3+1)}$ containing P. Then ∂D bounds a disk D' on T'. Let $T''=(T-D')\cup D$. Since M is irreducible, T'' is ambient isotopic to T'. Let $\mathcal{T}'=(\mathcal{T}^{(j_3+1)}-T')\cup T''$. Then, $\mathcal{T}'$ is ambient isotopic to $\mathcal{T}''$ such that each component of $\mathcal{T}''\cap C_1$ is a disk, and the number of the components of $\mathcal{T}''\cap C_1$ is less than that of $\mathcal{T}\cap C_1$, a contradiction. Hence, b_{j_3} is not attached to D_1 or D_2.

Assume that b_{j_3} is attached to D_3. Then, b_{j_3} is of type 3, and produces an annulus A_3 which is parallel to A_2. Then, there are two annuli A', A'' in F such that $(\text{Int } A'\cup \text{Int } A'')\cap(A_1\cup A_2\cup A_3)=\phi$, a component of $\partial A'$ is a component of ∂A_1, the other component of $\partial A'$ is a component of ∂A_2 and is also a component of $\partial A''$, and the other component of $\partial A''$ is a component of ∂A_3. Let M' (M'' resp.) be the closure of the component of $M-\mathcal{T}^{(j_3+1)}$, which contains A' (A'' resp.). It is possible that $M'=M''$. By the minimality of $\mathcal{T}$ we see that A' (A'' resp.) is an essential annulus in M' (M'' resp.). Then, by [J], M' (M'' resp.) admits a Seifert fibration such that A' (A'' resp.) is a union of fibers. Hence, a Seifert fibration on M' can be extended to $M'\cup M''$ through a component of $\partial M'\cap\partial M''$. But, this contradicts the definition of the torus decomposition.

Hence, b_{j_3} is of type 3, and is attached to the right side of D_4 to produce an incompressible annulus A_4. By the argument as above, we see that b_{j_4} is of type 3, and is attached to the right side of D_3 to produce an incompressible annulus A_3 in C_1, which is parallel to A_4. By the argument as above, we see that b_{j_5} is attached between A_2 and A_3 to produce a disk with two holes P'. Let $\partial P'=l_1\cup l_2\cup l_3$. We suppose that l_2 (l_3 resp.) is a component of ∂A_2 (∂A_3 resp.). Since b_{j_2} and b_{j_4} are of type 3, there is a disk component D' of $\mathcal{T}^{(j_5+1)}\cap C_2$ such

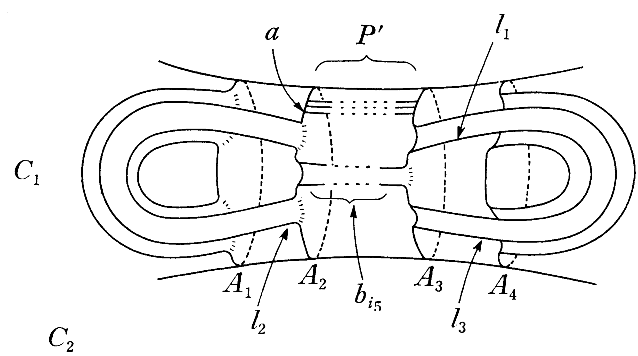


Figure 3.3

that $\partial D'=l_1$. $\partial P'$ bounds a disk with two holes P'' in F. Let a be an arc properly embedded in P'', which joins l_2 and l_3, and let b be a regular neighborhood of a in P''. We can consider that b is a band which is attached to P'. Let T'' be a twice punctured torus obtained from P' by attahcing b and isotoping it slightly so that T'' is properly embedded in C_1. We note that one boundary component of $\partial T''$ is l_1. Let l_4 be the other component of $\partial T''$. Since $l_1 \cup l_4$ bounds an annulus A^1 in F, l_4 bounds a disk D'' in C_2 such that Int $D'' \cap \mathcal{T}^{(j_5+1)}=\phi$. Let V' be the closure of the component of C_1-T'' which contains A^1. Let B be the product region between D' and D'' in C_2. Then, by Lemma 3.7, $N=V' \cup B$ is homeomorphic to the exterior of a two bridge knot, where the core of A_i $(i=2, 3)$ corresponds to a meridian loop. By [R], we see that N is simple. Let M' be the closure of the component of $M-\mathcal{T}^{(j_5+1)}$ which contains N, and $A^2=\mathrm{Fr}_{M'} N$. Then, A^2 is an annulus properly embedded in M'. Let T^* be the component of $\mathcal{T}^{(j_5+1)}$ which contains ∂A^2. Then ∂A^2 separates T^* into two annuli A_1^* $(=P' \cup D')$, and A_2^*, where $A_1^* \cup A^2 =\partial N$. Let $N'=\mathrm{cl}(M'-N)$.

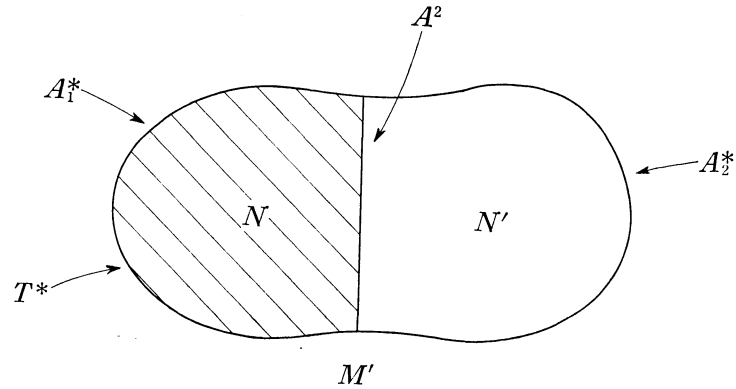


Figure 3.4

Then, we claim that (N', A_2) is homeomorphic to $(D^2 \times S^1, \alpha \times S^1)$ as a pair, where D^2 is a disk, and α is an arc in ∂D^2. Assume that ∂N is compressible in N. Then N is homeomorphic to the exterior of a trivial knot, a solid torus. Then, (N, A^2) is homeomorphic to $(D^2 \times S^1, \alpha \times S^1)$ as a pair. Then, let $\mathcal{T}^*= (\mathcal{T}^{(j_5+1)}-T^*) \cup (A^2 \cup A_2^*)$. $\mathcal{T}^{(j_5+1)}$ is ambient isotopic to $\mathcal{T}^*$, and $\mathcal{T}^*$ is ambient isotopic to $\mathcal{T}'$ such that each component of $\mathcal{T}' \cap C_1$ is a disk and the number of the components of $\mathcal{T}' \cap C_1$ is less than that of $\mathcal{T} \cap C_1$, a contradiction. Hence, ∂N is incompressible in N i.e. N is homeomorphic to the exterior of a non-trivial two bridge knot. Assume that (N', A^2) is not homeomorphic to $(D^2 \times S^1, \alpha \times S^1)$. Then A^2 is an essential annulus in M'. Hence, by [J], M' admits a Seifert fibration such that A^2 is a union of fibers. Then N admits a Seifert fibration such that a fiber in ∂N is a meridian loop. But, since two bridge knots whose exterior admit Seifert fibrations are $(2, 2n+1)$ torus knots, this is impossible. Hence, (N', A^2) is homeomorphic to $(D^2\times S^1, \alpha\times S^1)$.

Let $\bar{\mathcal{T}}=(\mathcal{T}^{(j_5+1)}-T^*)\cup(A^2\cup A_1^*)$. Then, by the above claim, $\bar{\mathcal{T}}$ is ambient isotopic to $\mathcal{T}^{(j_5+1)}$ and $\bar{\mathcal{T}}\cap C_1=((\mathcal{T}^{(j_5+1)}\cap C_1)-P')\cup T''$. Hence, we may suppose that $b_{j_6}=\mathrm{cl}(T''-P')$. Then, b_{j_7} is attached between D_1 and D_4, and, by using the same arguments as above, we see that the closure of the component M'' of $M-\mathcal{T}^{(j_5+1)}$ containing the region between D_1 and D_4 is homeomorphic to the exterior of a two bridge knot. Clearly, $M'\subset M''$. Since the exterior of a two bridge knot is simple, we see that $\partial M'$ and $\partial M''$ are parallel in M, a contradiction.

Hence, in Case 1, we have the conclusion of Proposition 3.8.

Case 2. b_{j_2} is attached to D_4.

In this case, b_{j_2} is attached to the right side of D_4. Then, by the arguments in Case 1, we see that b_{j_3}, and b_{j_4} are of type 3, b_{j_3} is attached to the left side of D_2 (or the right side of D_3), and b_{j_4} is attached to the right side of D_3 (or the left side of D_2). Hence, we have the conclusion of Proposition 3.8 by Case 1.

Since b_{j_2} is of type 3, it is not attached to D_3, and this completes the proof of Proposition 3.8.

Recall that $\mathcal{T}$ is a union of mutually disjoint, non-parallel incompressible tori in M such that each component of $\mathcal{T}_1=\mathcal{T}\cap C_1$ is a disk, and each component of $\mathcal{T}_2=\mathcal{T}\cap C_2$ is incompressible in C_2. Then, there is a hierarchy $(\mathcal{T}_2^{(0)}, a_0), \cdots, (\mathcal{T}_2^{(m)}, a_m)$ for $\mathcal{T}_2$ and a sequence of isotopies of type A which realizes the hierarchy i.e. if $\mathcal{T}^{(j)}$ is the image of $\mathcal{T}^{(j-1)}$ after the j-th isotopy, then $\mathcal{T}^{(j)}\cap C_2=\mathcal{T}_2^{(j)}$. Let $\Delta_0, \cdots, \Delta_m$ be a system of disks which defines the sequence of isotopies of type A with $\Delta_i\cap\Delta_j=\phi(i\neq j)$. Then, $\mathcal{T}_2^{(i)}\cap\Delta_i=a_i$, $\Delta_i\cap F=d_i$ an arc such that $\partial a_i=\partial d_i$, $a_i\cup d_i=\partial\Delta_i$. Let Δ_i' be a dual disk of Δ_i (see the fourth paragraph of [O, 462p.], or Figure 3.5), where $\Delta_i'\cap(\mathcal{T}^{(i+1)}\cap C_1)=a_i'$, $\Delta_i'\cap F=d_i'$ an arc such

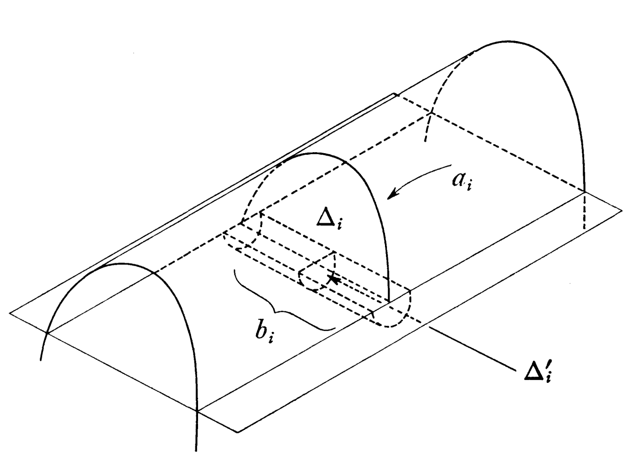


Figure 3.5

that $\partial a_i'=\partial d_i'$, $a_i'\cup d_i'=\partial\Delta_i'$, and $\Delta_i'\cap\Delta_j'=\phi(i\neq j)$. We may suppose that $d_0, \cdots, d_m$ and $d_0', \cdots, d_m'$ are in general position i.e. for each pair (i, j) $(0\leq i, j\leq m)$ d_i and d_j' intersects transversely in their interiors. We say that the band b_j goes through b_i, if $j>i$, and $d_i'\cap d_j\neq\phi$.

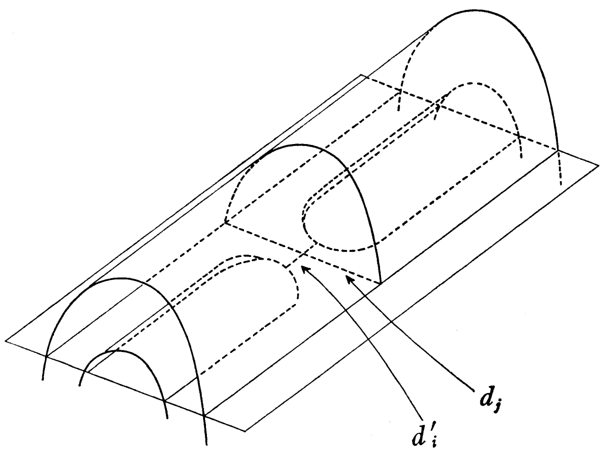


Figure 3.6

Then, we define the complexity of the system of disks $\Delta_0, \cdots, \Delta_m$ which realizes the hierarchy $(\mathcal{I}_2^{(0)}, a_0), \cdots, (\mathcal{I}_2^{(m)}, a_m)$ with $\Delta_i\cap\Delta_j=\phi(i\neq j)$ as follows:

$$c(\Delta_0, \cdots, \Delta_m)=\sum_{i=0}^{m-1}\#\{d_i'\cap(\bigcup_{j=i+1}^{m}d_j)\}.$$

$c(\Delta_0, \cdots, \Delta_m)$ denotes the number of times when the bands $b_0, \cdots, b_m$ go through themselves.

Then, we have:

Lemma 3.9. *Let D be a component of $\mathcal{I}_1$, and b_{j_n} be the n-th band which is attached to D. Suppose that b_{j_k} $(k\geq 1)$ is of type* 1, *and b_{j_k} does not go through b_{j_l} for each $l(<k)$. Then, $\mathcal{I}$ is ambient isotopic to $\mathcal{I}'$ as in Lemma* 3.3.

Proof. Since b_{j_k} does not go through b_{j_l}, we can change the order of the isotopies such that the (j_1+1)-th isotopy is performed at a_{j_k}. a_{j_k} is a d-arc related to $\{a_0, \cdots, a_{j_1-1}, a_{j_k}\}$. Hence, by Lemma 3.2, we have the conclusion of Lemma 3.9.

Lemma 3.10. *We consider the submanifold $F\cap\mathcal{I}^{(i)}=\partial\mathcal{I}_2^{(i)}$ in F. Suppose that there exists a rectangle R in F such that* Int $R\cap\mathcal{I}^{(i)}=\phi$, *two opposite edges of R are contained in $d_j'(j<i)$ with b_j is of type* 1, *one edge of R is contained in the boundary of a band $b_k(j<k<i)$, and the last edge of R is contained in a component C of $F\cap\mathcal{I}^{(i)}$ such that C bounds a disk component D of $\mathcal{I}_2^{(i)}$. Then, there exists a system of disks $\bar{\Delta}_0, \cdots, \bar{\Delta}_m$ in M such that $\bar{\Delta}_p\cap\bar{\Delta}_q=\phi$ $(p\neq q)$, $\bar{\Delta}_0, \cdots, \bar{\Delta}_m$ realizes the hierarchy $(\mathcal{I}_2^{(0)}, a_0), \cdots, (\mathcal{I}_2^{(m)}, a_m)$ and $c(\bar{\Delta}_0, \cdots, \bar{\Delta}_m)<c(\Delta_0, \cdots, \Delta_m)$.*

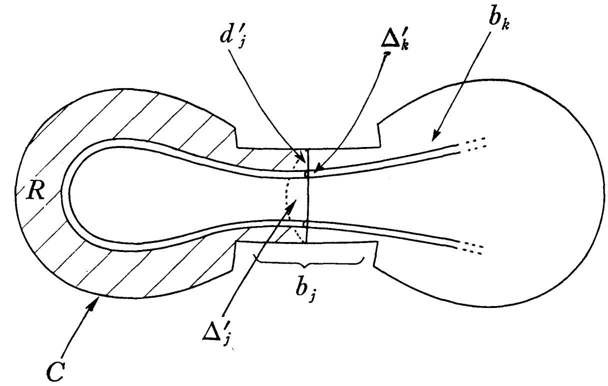


Figure 3.7

Proof. Let D^* be the frontier of a small regular neighborhood of D in C_2 which intersects R. Then, D^* is a disk properly embedded in C_2 such that $D^* \cap \partial \mathcal{T}_2^{(k)} = \phi$, and $D^* \cap R = \partial D^* \cap R$ consists of an arc. We get a disk Δ_k^* which defines an isotopy of type A at a_k, by joining Δ_k and D^* with a band which lie in a neighborhood of the arc d'_j. Let $d_k^* = \Delta_k^* \cap F$, and b_k^* be the band which is attached to $\mathcal{T}^{(k)} \cap C_1$ as the result of the isotopy of type A at a_k along Δ_k^*. Then, by isotoping Δ_k^* in a neighborhood of R as in Figure 3.8 we get $\bar{\Delta}_k$ which defines an isotopy of type A at a_k such that $\#(\bar{d}_k \cap d'_l) \leq \#(d_k \cap d'_l)$ $(l<k)$, and $\#(\bar{d}_k \cap d'_j) < \#(d_k \cap d_j)$, where $\bar{d}_k = \bar{\Delta}_k \cap F$. Then, we easily see that there is a system of disks $\bar{\Delta}_{k+1}, \cdots, \bar{\Delta}_m$, which define isotopies of type A at $a_{k+1}, \cdots, a_m$, and $c(\Delta_0, \cdots, \Delta_{k-1}, \bar{\Delta}_k, \bar{\Delta}_{k+1}, \cdots, \bar{\Delta}_m) < c(\Delta_0, \cdots, \Delta_m)$.

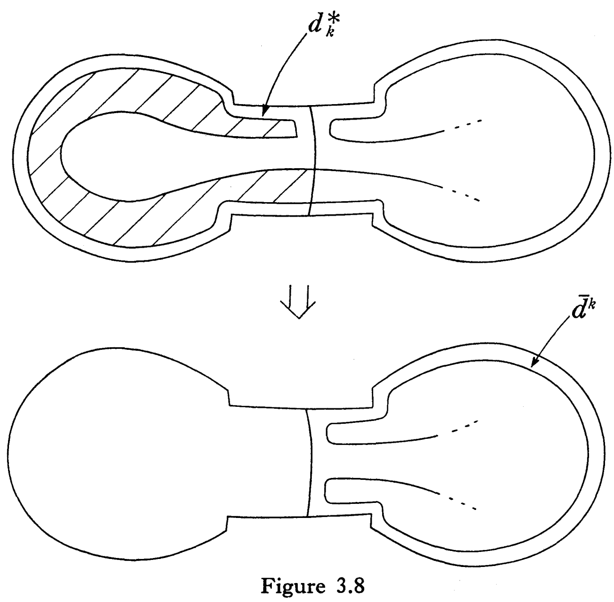


Figure 3.8

This completes the proof of Lemma 3.10.

4. Find an incompressible torus which intersects the Heegaard surface in a circle

Let M be a Haken manifold which is closed or with incompressible toral boundary, $(C_1, C_2; F)$ be a genus g Heegaard splitting of M, and $\mathcal{T}$ be a union of tori which gives the torus decomposition of M. The purpose of this section is to show:

Proposition 4.1. *If the number of the components of $\mathcal{T}$ is greater than or equal to $3g$-4, then there exists a component T of $\mathcal{T}$ such that T is ambient isotopic to a torus which intersects F in a circle.*

We note that if we omit the assumption on the number of the components of $\mathcal{T}$, then the conclusion of Proposition 4.1 does not hold in general. We will give such examples in Example 4.5.

We may suppose that each component of $\mathcal{T}_1=\mathcal{T}\cap C_1$ is a disk, and the number of the components of $\mathcal{T}\cap C_1$ is minimal among all systems of tori which are ambient isotopic to $\mathcal{T}$ and each of which intersects C_1 in disks. Then, by section 3, there is a hierarchy $(\mathcal{T}_2^{(0)}, a_0), \cdots, (\mathcal{T}_2^{(m)}, a_m)$ for $\mathcal{T}_2=\mathcal{T}\cap C_2$, and a sequence of isotopies of type A which realizes the hierarchy. Let $\mathcal{T}^{(i)}, \Delta_i (i=1, \cdots, m)$, Δ_i', d_i, d_i' be as in section 3. Let $\mathcal{T}'=\mathcal{T}^{(m+1)}$, $\mathcal{T}_i'=\mathcal{T}'\cap C_i (i=1, 2)$. Then $\Delta_m', \cdots, \Delta_0'$ defines a hierarchy for $\mathcal{T}_1'$, and a sequence of isotopies of type A which realizes the hierarchy. Then, we have:

Lemma 4.2. $c(\Delta_0, \cdots, \Delta_m)=c(\Delta_m', \cdots, \Delta_0')$. *Moreover, we can take dual disks $(\Delta_m'', \cdots, \Delta_0'')$ of $(\Delta_m', \cdots, \Delta_0')$ and a sequence of isotopies of type A so that $\Delta_i''=\Delta_i$.*

Proof. We will prove Lemma 4.2 in the case when $m=1$. The proof of the general case will follow easily by using the same argument. Suppose that

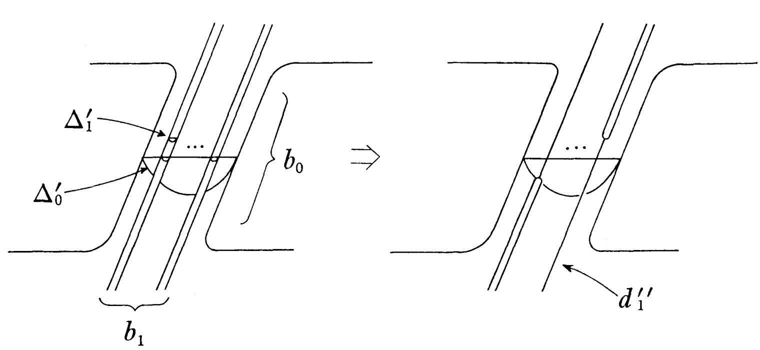


Figure 4.1

the band b_1 goes through b_0 n times i.e. $b_1 \cap \Delta_0'$ consists of n arcs. Then, we can take a dual disk of Δ_1' such that $d_1'' \cap d_0'$ consists of n points (Figure 4.1). Hence, $c(\Delta_0, \Delta_1)=c(\Delta_0', \Delta_1')$. It is clear tht that we can take an isotopy of type A at Δ_i' so that $\Delta_i''=\Delta_i$.

Lemma 4.3. *$\partial\mathcal{D}_1$ $(=\partial\mathcal{D}_2)$ contains at most $3g$-5 parallel classes in F.*

Proof. We prove Lemma 4.3 in the case when C_1 is a handlebody i.e. $\partial C_1=F$. The proof in the general case is essentially the same. Since F can contain $3g$-3 mutually non parallel simple closed curves, it is enough to show:

(*) Two components of C_1 cut along $\mathcal{D}_1$ is not simply connected, or there is a non-simply connected component V of C_1 cut along $\mathcal{D}_1$ such that $\chi(V \cap F) \leq -2$.

Assume that all components of $C_1-\mathcal{D}_1$ are simply connected. Then, if we perform the first isotopy of type A at a_0, then, by section 3, it produces an incompressible annulus A in C_1. But A can be pushed into a component of $C_1-\mathcal{D}_1$, a contradiction. Hence, at least one component of $C_1-\mathcal{D}_1$ is not simply connected. Suppose that just one component V' of $C_1-\mathcal{D}_1$ is not simply connected. Let $V=\text{cl}\, V'$. Assume that $\chi(V \cap F)>-2$. Then, since V is not simply connected, we see that V is a solid torus, and $V \cap F$ is a once punctured torus. Let D be the component of $\mathcal{D}_1$ such that $D \subset \partial V$. Then, the first band b_0 is attached to D to produce an incompressible annulus A in V. We note that ∂A bounds an annulus A^* in $V \cap F$. Let M' be the closure of the component of $M-\mathcal{D}^{(1)}$ which contains A^*. By the minimality of $\mathcal{D}$, we see that A^* is an essential annulus in M'. Hence, by [J], M' admits a Seifert fibration such that A^* is a union of fibers. Let D' be the component of $\mathcal{D}_1$ to which b_1 is attached. Assume that $D'=D$. Then, by the minimality of $\mathcal{D}$, and Lemmas 3.2, 3.3, we see that b_1 is of type 3, and it produces a once punctured torus T' properly embedded in C_1. Since V is a solid torus, we see that T' is compressible in

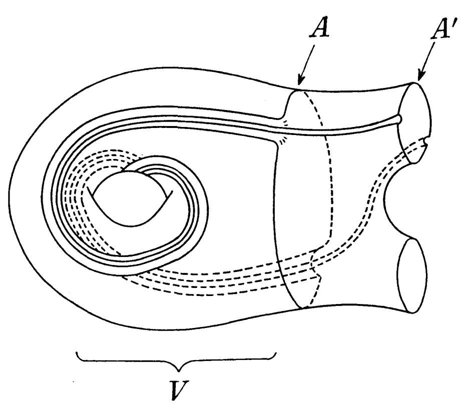


Figure 4.2

C_1, a contradiction. Hence, $D' \neq D$. Then, b_1 produces an incompressible annulus A' properly embedded in C_1. Then, we can span an annulus A^{**} between the core of A and the core of A' in C_1. Let M'' be the closure of the components of $M - \mathcal{S}^{(2)}$ which contains A^{**}. Then, by the minimality of $\mathcal{S}$, we see that A^{**} is an essential annulus in M''. By [J], M'' admits a Seifert fibration such that A^{**} is a union of regular fibers. Hence, a Seifert fibration on M' extends to M'' through the component of $\mathcal{S}^{(2)}$ which contains A, a contradiction.

This completes the proof of Lemma 4.3.

Lemma 4.4. *Suppose that there exist three components D_1, D_2, D_3 of $\mathcal{S}_1$ which are mutually parallel in C_1, and no component of $\mathcal{S}$ intersects C_1 in one disk. We may suppose that D_1, D_2, D_3 are in C_1 in this order, and define the direction 'left' and 'right' as in the proof of Proposition* 3.8. *Let b_{j_n} be the n-th band which is attached to $D_1 \cup D_2 \cup D_3$. Then, if needed by exchanging the suffix, we have:*

(i) *b_{j_i} $(i=1, 2)$ is attached to the left side of D_i to produce an essential annulus A_i, and A_1 and A_2 are parallel in C_1,*

(ii) *b_{j_3} is attached to the right side of D_3,*

(iii) *If M_1 is the closure of the component of $M - \mathcal{S}^{(j_3+1)}$ which contains the region between D_2 and D_3, then M_1 is homeomorphic to the exterior of a two bridge knot, where the core of A_i $(i=2, 3)$ is a meridian loop, and*

(iv) *If M_2 is the closure of the component of $M - \mathcal{S}^{(j_2+1)}$ which contains the product region between A_1 and A_2, then M_2 admits a Seifert fibration such that A_j $(j=2, 3)$ is a union of fibers.*

The proof of Lemma 4.4 is done by using the same case by case argument as in the proof of Proposition 3.8. So, we will omit it.

Proof of Proposition 4.1.

Assume that the conclusion does not hold. We suppose that each component of $\mathcal{S} \cap C_1$ is a disk, and $c_1(\mathcal{S})$ denotes the number of the components. Then we define a complexity for a pair $(\mathcal{S}, (\Delta_0, \cdots, \Delta_m))$ by $(c_1(\mathcal{S}), c(\Delta_0, \cdots, \Delta_m))$ with lexicographic order. Then, we suppose that $(\mathcal{S}, c(\Delta_0, \cdots, \Delta_m))$ is minimal with respect to this order.

Since each component of $\mathcal{S}$ intersects C_1 in more than one component, $\mathcal{S}_1$ consists of at least $6g$-8 components. If no three components of $\mathcal{S}_1$ are mutually parallel in C_1, then, by Lemma 4.3, we see that $\mathcal{S}_1$ consists of at most $6g$-10 components, a contradiction. On the other hand, if four components of $\mathcal{S}_1$ are mutually parallel in C_1, then, by Proposition 3.8, we see that the conclusion holds. Hence, we need to analyze mutually parallel three components for the proof of Proposition 4.1.

Suppose that two components D_1, D_2 of $\mathcal{S}_1$ are mutually parallel in C_1. Let b_{j_i} be the i-th band which is attached to $D_1 \cup D_2$. We call the direction to which D_1 (D_2 resp.) is settled 'left' ('right' resp.). By Lemmas 3.2, 3.3, we see that b_{j_1} is of type 3, and we may suppose that b_{j_1} is attached to the left side of D_1 to produce an essential annulus properly embedded in C_1. By the argument in the proof of Proposition 3.8, we see that b_{j_2} is attached to D_2 to produce an essential annulus. We say that the pair D_1, D_2 is of *type** in $\{(\mathcal{D}_2^{(i)}), \Delta_i\}$ if b_{j_2} is attached to the right side of D_2.

Let D_1, D_2 be of type*. Then, we claim that b_{j_3} and b_{j_4} are attached between the right side of D_1 and the left side of D_2. By Lemma 3.9, we see that b_{j_3} is attached between the right side of D_1 and the left side of D_2. By the proof of Proposition 3.8, we see that b_{j_4} is attached between D_1 and D_2. Assume that b_{j_4} is attached between the left side of D_1 and the right side of D_2. Then, it is clear that no band goes through b_{j_4}. By Lemma 3.9, we see that b_{j_4} goes through b_{j_3}. On the other hand, by the proof of Proposition 3.8, we have a band b'_{j_4} which is attached between the right side of D_1 and the left side of D_2. Since b_{j_4} goes through b_{j_3}, we see that b'_{j_4} goes through $b_1 \cup \cdots \cup b_{j_4-1}$ less times than b_{j_4}. And we can move $\mathcal{D}' = \mathcal{D}^{(m+1)}$ by an isotopy to $\mathcal{D}''$ such that $\mathcal{D}'' \cap C_1$ is obtained from $\mathcal{D} \cap C_1$ by attaching bands $b_0, b_1, \cdots, b_{j_4-1}, b'_{j_4}, b_{j_4+1}, \cdots, b_m$. But this contradicts the minimality of $c(\Delta_0, \cdots, \Delta_m)$, and we establish the claim.

Hence, b_{j_3} and b_{j_4} are attached between the right side of D_1 and the left side of D_2 to produce a twice punctured tori Q properly embedded in C_1. Then, ∂Q consists of pairwise parallel simple loops in F, and ∂Q bounds two disks D'_1, D'_2 which are the components of $\mathcal{D}'_2$. Since b_{j_3}, and b_{j_4} are attached between the right side of D_1 and the left side of D_2, we see:

(*) The pair D'_1, D'_2 is of type* in $\{(\mathcal{D}_1^{(m+1-i)}, a'_{m-i}), \Delta'_{m-i}\}$.

Let $\{E_1, \cdots, E_p\}$ ($\{E'_1, \cdots, E'_{p'}\}$ resp.) be the parallel classes of the disks in $\mathcal{D}_1$ ($\mathcal{D}'_2$ resp.) in C_1 (C_2 resp.). We may suppose that $\{E_1, \cdots, E_q\}$ ($\{E'_1, \cdots, E'_{q'}\}$ resp.) is the subset of $\{E_1, \cdots, E_p\}$ ($\{E'_1, \cdots, E'_{p'}\}$ resp.) each element of which contains a pair of disks which is of type*. Then, by (*), there is a correspondence ψ: $\{E_1, \cdots, E_q\} \to \{E'_1, \cdots, E'_{q'}\}$. Since no four components of $\mathcal{D}'_2$ are mutually parallel in C_2, ψ is 1–1. By Lemma 4.2, and (*), we see that ψ is onto. Hence, $q=q'$, and we may suppose that $\psi(E_i)=E'_i (i=1, \cdots, q)$. We note that each E_i(E'_i resp.) ($i=1, \cdots, q$) contains two, or three components of $\mathcal{D}_1$ ($\mathcal{D}'_2$ resp.), and, by Lemma 4.4, we see that each E_j ($j>q$) contains at most two components of $\mathcal{D}_1$ ($\mathcal{D}'_2$ resp.). Then, each E_i ($i=1, \cdots, q$) is one of the following four types.

Type a. Both E_i, E'_i contain exactly two components.

Type b. Both E_i, E'_i contain three components.

Type c. E_i contains exactly two components, and E'_i contains three components.

Type d. E_i contains three components, and E'_i contains exactly two components.

By Lemma 4.2, we may suppose that:

(**) $\#\{E_i \mid E_i \text{ is of type } c\} \geq \#\{E_i \mid E_i \text{ is of type } d\}$

In the following, for the proof of Proposition 4.1, we investigate type b, c parallel classes intimately.

Type b. Suppose that D_1, D_2, D_3 (D_1', D_2', D_3' resp.) belong to the parallel class $E_i(E_i'$ resp.), where the pair D_2, $D_3(D_2', D_3'$ resp.) is of type*. We call the direction in which D_2, $D_2'(D_3, D_3'$ resp.) is settled 'left' ('right' resp.). We may suppose that $D_1(D_1'$ resp.) is settled in the left side of $D_2(D_2'$ resp.). Let b_{j_i} be the i-th band which is attached to $D_1 \cup D_2 \cup D_3$. Then, by Lemma 4.4, we may suppose that $b_{j_i}(i=1, 2, 3)$ is attached to D_i to produce an essential annulus A_i, where A_1 and A_2 are parallel in C_1. Then $\partial A_1 \cup \partial A_2$ bounds pairwise disjoint annuli A^1, A^2 in F. Then, there are three annuli A_1', A_2', A_3' in $\mathcal{G}^{(j_3+1)} \cap C_2$ such that A_i' is obtained from D_i' by attaching a type 3 band, ∂A^1 is a union of a component of $\partial A_1'$ and a component of $\partial A_2'$, and one component of ∂A^2 is a component of $\partial A_3'$. Then, there is an annulus $A^3(\neq A^1)$ in F such that Int $A^3 \cap$

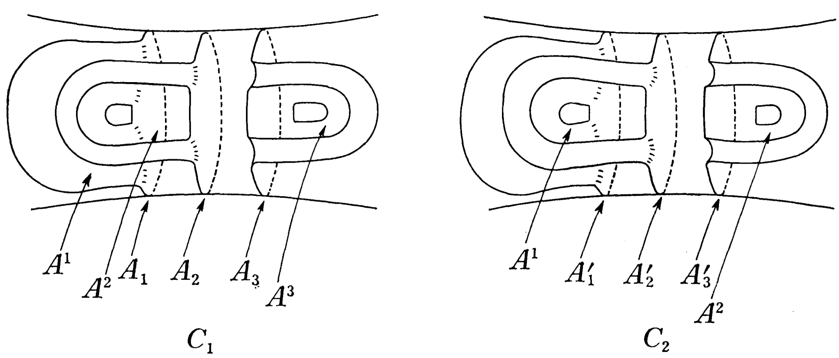


Figure 4.3

$(A_1' \cup A_2')=\phi$, and ∂A^3 is a union of a component of $\partial A_1'$ and a component of $\partial A_2'$. By the fourth paragraph in the proof of Proposition 4.1, we may suppose that b_{j_4} and b_{j_5} are attached between the right side of D_2 and the left side of D_3. Then, b_{j_6} is of type 1, and is attached to D_1.

Assertion 1. b_{j_6} is attached to the right side of D_1.

Proof. Assume that b_{j_6} is attached to the left side of D_1. If b_{j_6} does not go through b_{j_4} or b_{j_5}, then b_{j_6} does not go through b_{j_1}. Hence, by Lemma 3.9, we can decrease $c_1(\mathcal{G})$, a contradiction. If b_{j_6} goes through b_{j_4} or b_{j_5}, then we can find a rectangle which satisfies the assumption of Lemma 3.10. See Figure 4.4. Hence, we can decrease $c(\Delta_0, \cdots, \Delta_m)$ without changing $c_1(\mathcal{G})$, a contradiction.

Let $D_4(\neq D_1)$ be the component of $\mathcal{G}_1$ to which b_{j_6} is attached.

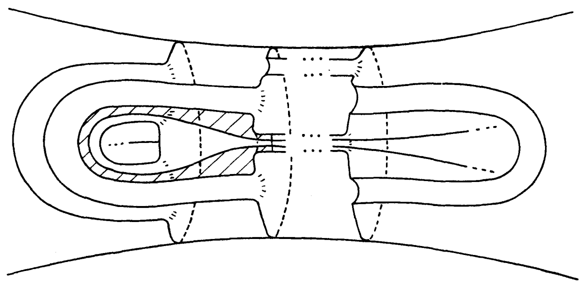


Figure 4.4

Assertion 2. There is no component of $\mathcal{S}_1$ which is parallel to D_4.

Proof. Assume that a component D_5 of $\mathcal{S}_1$ is parallel to D_4. Let b_{k_i} ($i=1, 2, \cdots$) be the i-th band which is attached to $D_4 \cup D_5$. By the argument in the proof of Proposition 3.8, we may suppose that b_{k_1}(b_{k_2} resp.) is attached to D_4 (D_5 resp.) to produce an essential annulus A_4(A_5 resp.). Since b_{j_6} is attached to D_4, we see that D_4, D_5 is not of type*. Hence A_4 and A_5 are parallel in C_1. We call the direction to which D_4(D_5 resp.) is settled 'right' ('left' resp.). Assume that b_{j_6} is attached to the right side of D_4. Let M_1(M_2 resp.) be the closure of the component of $M-\mathcal{S}^{(k_2+1)}$ corresponding to the product region between A_1 and A_2(A_4 and A_5 resp.). It is possible that $M_1=M_2$. By the minimality of $\mathcal{S}$, and [J], we see that M_1(M_2 resp.) admits a Seifert fibration such that A_1(A_4 resp.) is a union of fibers. Hence, a Seifert fibration on M_1 extends to M_2 through the component of ∂M_1 containing A_4, a contradiction.

Assume that b_{j_6} is attached to the left side of D_4. Then, there is a type 1 band $b_s(k_2<s<j_3)$ which is attached to the left side of D_5, and through which b_{j_6} goes. Then, we can find a rectangle which satisfies the assumption of Lemma 3.10. See Figure 4.5. Hence, we can decrease $c(\Delta_0, \cdots, \Delta_m)$ without changing $c_1(\mathcal{S})$, a contradiction.

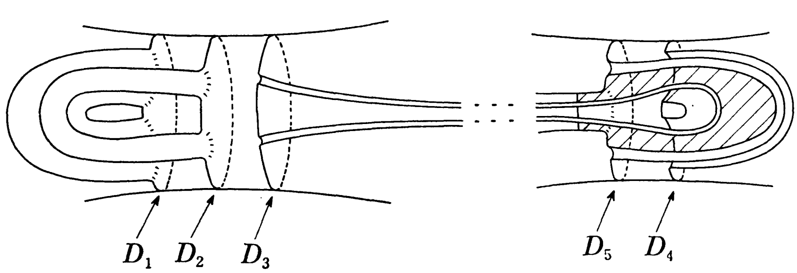


Figure 4.5

Type c. Suppose that D_1, D_2(D'_1, D'_2, D'_3 resp.) belong to the parallel class E_i(E'_i resp.), where the pair $\{D'_1, D'_2\}$ is of type*. We call the direction in which D_1, D'_1(D_2, D'_2 resp.) is settled 'left' ('right' resp.). We may suppose that D'_3 is settled in the right side of D'_2. Let b_{j_i} be the i-th band which is attached to $D_1 \cup D_2$. Then, we may suppose that $b_{j_i}(i=1, 2)$ is atached to D_i to produce an essential annulus A_i. Then, $\mathcal{S}^{(j_2+1)} \cap C_2$ contains three annuli A'_1, A'_2, A'_3, where $A'_i(i=1, 2, 3)$ is obtained from D'_i by attaching a type 3 band, and A'_2 and A'_3 are parallel in C_2. Then, $\partial A'_2 \cup \partial A'_3$ bounds pairwise disjoint annuli A^1, A^2 in F. Let $D_3, D_4(\neq D_1, D_2)$ be the components of $\mathcal{S}_1$, such that there are type 3 bands

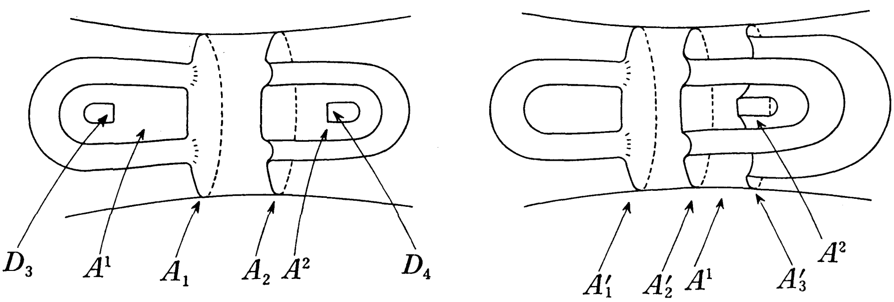


Figure 4.6

which are attached to D_3, and D_4 to produce annuli A_3, and A_4, where a component of $\partial A_3(\partial A_4$ resp.) is a component of $\partial A^1(\partial A^2$ resp.). Since no component of $\mathcal{S}$ intersects C_1 in a disk, we see that $D_3 \neq D_4$. Then, by using the arguments in the proof of Assertion 2, we can show:

Assertion 3. There is no component of $\mathcal{S}_1$ which is parallel to D_3 or D_4.

Then, we continue the proof of Proposition 4.1. Recall that $\{E_1, \cdots, E_q\}$ ($\{E'_1, \cdots, E'_q\}$ resp.) is the set of parallel classes of $\mathcal{S}_1(\mathcal{S}'_2$ resp.) each element of which contains a pair of type* disks, and $\psi(E_i)=E'_i$. We may suppose that $\{E_1, \cdots, E_r\}$ $(r \leq q)$ is the subset of $\{E_1, \cdots, E_q\}$, each element of which is of type c. Then, by Assertion 3, for each $E_i(i=1, \cdots, r)$, there are two elements $E_{l(i)}$, $E_{m(i)}$ $(l(i) \neq m(i))$, each of which contains exactly one component of $\mathcal{S}_1$, and, hence, $l(i), m(i)>q$. Let $\mathcal{C}=\bigcup_{i=1}^{r} \{E_{l(i)}, E_{m(i)}\}$. Since, for each element D of $\mathcal{S}_1$, there are two type 1 bands which are attached to D (Lemma 3.6), $\mathcal{C}$ contains at least r elements.

We may suppose that $\{E_j \mid r<j \leq r+s(s \geq 0)\}$ is the subset of $\{E_1, \cdots, E_q\}$, each element of which is of type b. By Assertion 2, for each $E_i(r<i \leq r+s)$, there is an element $E_{n(i)}$ which contains exactly one component of $\mathcal{S}_1$, and, hence, $n(i)>q$. Suppose that $E_{n(i)} \in \mathcal{C}$. Let D_1 be the component of $\mathcal{S}_1$, which belongs to E_i, and is not a component of the type* pair. Let T_1 be the component of $\mathcal{S}$ which contains D_1. Then, $T_1 \cap C_1$ consists of more than two components, and at least two components of $T_1 \cap C_1$ belong to $\mathcal{C}$. Hence, if we eliminate D_1 from $\mathcal{S}_1$, then we still have at least $6g$-8 components By applying this elimination from $\mathcal{T}_1$ for each $E_i(r<i\leq r+s)$ with $E_{n(i)}\in\mathcal{C}$, we get a subset $\mathcal{D}$ of $\mathcal{T}_1$ such that $\#\mathcal{D}\geq 6g\text{-}8$.

Then, suppose that $E_{n(i)}\notin\mathcal{C}\,(r<i\leq r+s)$. Let D_1 be as above. Suppose that there exists $j(r<j\leq r+s)$ such that $j\neq i$, and $E_{n(i)}=E_{n(j)}$. By Lemma 3.6, we can have at most one j as above for each i. Let T_1 be the component of $\mathcal{T}$ which contains D_1. Then, T_1 intersects C_1 in more than two components. Then, we eliminate D_1 from $\mathcal{D}$. By applying this elimination from $\mathcal{D}$ for each pair i, j as above, we get a subset $\mathcal{D}'$ of $\mathcal{D}$ such that $\#\mathcal{D}'\geq 6g\text{-}8$.

We may suppose that $\{E_j\,|\,r+s<j\leq r+s+t(t\geq 0)\}$ is the subset of $\{E_1, \cdots, E_q\}$ which consists of type d elements. By (**), we have $t\leq r$. Hence, if the number of elements of $\mathcal{C}'=\{E_1, \cdots, E_q\}\cup\mathcal{C}\cup\{E_{n(r+1)}, \cdots, E_{n(r+s)}\}$ is u, then the number of the elements of $\mathcal{D}'$ which belong to $\mathcal{C}'$ is at most $2u$. We note that if $i\geq r+s+t$, then E_i contains at most two components. By Lemma 4.3, we have $p\leq 3g\text{-}5$. Hence, we have $\#\mathcal{D}'\leq 6g\text{-}10$. But this contradicts the inequality in the last paragraph.

This completes the proof of Proposition 4.1.

Let T_1 be a component of $\mathcal{T}$ which intersects C_1 in a disk D. Let b_{j_1} be the first band which is attached to D. Let T be the image of T_1 in $\mathcal{T}^{(j_1+1)}$. Then $A_i=T\cap C_i(i=1, 2)$ is an essential annulus in C_i. We say that T_1 is *bad* if $A_i(i=1, 2)$ cuts C_i into a genus 1 compression body, and a genus g compression body. T_1 is *good* if it is not bad.

Proposition 4.1.′ *If the number of the components of $\mathcal{T}$ is greater than or equal to $3g\text{-}4$, and $g>2$, then there is a component T of $\mathcal{T}$ such that T is ambient isotopic to T' which intersects C_1 in a disk, and is good.*

Proof. Let $T_1, \cdots, T_k$ be the components of $\mathcal{T}$ such that each T_i intersects C_1 in a disk. Assume that all $T_1, \cdots, T_k$ are bad. Let $D_i=T_i\cap C_1$ $(i=1, \cdots, k)$.

Assertion 1. There is no component of $\mathcal{T}_1$ which is parallel to D_i.

Proof. Assume that there is a component D of $\mathcal{T}_1$ which is parallel to some D_i. Let b_j be the first band which is attached to D. D_i cuts C_1 into a genus 1 compression body V, and a genus g-1 (>1) compression body. Then, we have:

$$V\cap D=\phi\,.$$

Proof. Assume that this is not true. Then $\mathcal{T}^{(j+1)}\cap C_1$ contains pairwise parallel annuli A_1, A_2 such that A_1 is obtained from D_i by attaching a type 3 band, and A_2 is obtained from D by attaching b_j. Since V is a genus 1 compression body, there is an annulus A in F such that $A\cap(A_1\cup A_2)=A\cap A_1=\partial A=\partial A_1$. Hence, we have a contradiction as in the proof of Lemma 4.3.

By using the same argument we can show:

b_j is attached to D to the side opposite to the side in which D_i is settled.

Then, by Lemma 3.9, we see that there is a component T of $\mathcal{T}$ such that $T \cap C_1 = D$, and T is good, a contradiction.

Let $\mathcal{D}'$ be a subset of $\mathcal{T}_1$, which is obtained in the proof of Proposition 4.1.

Assertion 2. $\mathcal{D}' \supset \{D_1, \cdots, D_k\}$

Proof. Assume that $D_i \notin \mathcal{D}'$. Then, by definition, there is a component of $\mathcal{T}_1$ which is parallel to D_i, contradicting Assertion 1.

Let T' be a component of $\mathcal{T}$ such that $T' \neq T_i (i=1, \cdots, k)$. Then, by the proof of Proposition 4.1, we see that $\mathcal{D}'$ contains at least two components of $T' \cap C_1$. Hence, $\mathcal{D}'$ contains at least $6g$-8-k components. On the other hand, by the estimation in the last paragraph in the proof of Proposition 4.1, and Assertions 1, 2, we see that $\mathcal{D}'$ contains at most $6g$-10-k components, a contradiction.

This completes the proof of Proposition 4.1′.

EXAMPLE 4.5. We will show that there are infinitely many Haken manifolds with Heegaard splitting of genus two, each of which is decomposed into two pieces by the torus decomposition, and the torus which give the torus decomposition does not intersect any genus two Heegaard surface in a circle.

Let M_1 be the exterior of a hyperbolic two bridge knot (for example, figure eight knot [T2]), M_2 be a Seifert fibered manifold whose orbit manifold is a Möbius band with two exceptional fibers, and M be a closed 3-manifold obtained from M_1 and M_2 by identifying their boundaries by a homeomorphism such that a meridian loop on ∂M_1 is identified with a fiber in ∂M_2. Then, by [Ko 2, Theorem], we see that M admits a genus two Heegaard splitting. It is clear that $M_1 \cup M_2$ gives the torus decomposition of M. Let $T = \partial M_1 = \partial M_2$ $(\subset M)$.

Assume that T intersects a genus two Heegaard surface in a circle. Then, by the argument in [Ko 2, Case 2.2.1], we see that M_2 admits a Seifert fibration with orbit manifold a disk and two exceptional fibers. But this contradicts the uniqueness of the torus decomposition.

5. Closing boundary of a Haken manifold

Let $C_i (i=1, 2)$ be a compression body, $\{A_1^i, \cdots, A_p^i\}$ $(p \geq 1)$ be a system of mutually disjoint annuli in $\partial_0 C_i$, and $g\colon \mathrm{cl}(\partial_0 C_1 - \bigcup_{i=1}^{p} A_i^1) \to \mathrm{cl}(\partial_0 C_2 - \bigcup_{i=1}^{p} A_i^2)$ be a homeomorphism such that $g(\partial A_i^1) = \partial A_i^2 (i=1, \cdots, p)$. Then $N = C_1 \cup_g C_2$ is a compact 3-manifold with boundary. Suppose that N is a Haken manifold with incompressible toral boundary. In this section, we will investigate the generic structure of the manifold $N'=C_1\underset{g'}{\cup}C_2$, where g': $\mathrm{cl}(\partial_0C_1-\bigcup_{i=2}^{p}A_i^1)\to\mathrm{cl}(\partial_0C_2-\bigcup_{i=2}^{p}A_i^2)$ is an extension of g.

For the proof of the next lemma, see [J, Chapter VI].

Lemma 5.1. *Let S be a Seifert fibered manifold with boundary. If S is not homeomorphic to $D^2\times S^1$, $S^1\times S^1\times[0, 1]$, or the twisted $[0, 1]$ bundle over the Klein bottle, then Seifert fibrations on S are unique up to ambient isotopy of S. Moreover, if S is the twisted $[0, 1]$ bundle over the Klein bottle, then S can admit exactly two different Seifert fibrations up to ambient isotopy of S such that one is with orbit manifold a disk and two exceptional fibers of index 2, and the other is with orbit manifold a Möbius band and no exceptional fibers.*

Lemma 5.2. *Let C_i, A_i^j, g, N be as above. Suppose that N is decomposed into $q(>1)$ components by the torus decomposition. Let $\mathcal{T}$ be the system of tori which gives the torus decomposition, and Σ be the closure of the component of $N-\mathcal{T}$ which contains $A_1^1\cup A_1^2$. Then, there is a homeomorphism g' : $\mathrm{cl}(\partial_0C_1-\bigcup_{i=2}^{p}A_i^1)\to\mathrm{cl}(\partial_0C_2-\bigcup_{i=2}^{p}A_i^2)$ such that:*

(i) *g' is an extension of g,*

(ii) *$N'=C_1\underset{g'}{\cup}C_2$ is a Haken manifold which is closed, or with incompressible toral boundary. If Σ does not admit a Seifert fibration with orbit manifold an annulus and one exceptional fiber such that A_1^1, A_1^2 are unions of fibers, or with orbit manifold a disk with two holes and no exceptional fibers such that A_1^1, A_1^2 are unions of fibers, then the image of $\mathcal{T}$ in N' gives the torus decomposition of N'. Hence, N' is decomposed into q components by the torus decomposition,*

(iii) *If Σ admits a Seifert fibration with orbit manifold an annulus and one exceptional fiber such that A_1^1, A_1^2 are unions of fibers, then the image of $\mathcal{T}-T$ in N' gives the torus decomposition of N', where $T=\mathrm{Fr}_M\Sigma$. Hence, N' is decomposed into $q-1$ components, and*

(iv) *If Σ admits a Seifert fibration with orbit manifold a disk with two holes and no exceptional fibers such that A_1^1, A_1^2 are unions of fibers, then the image of $\mathcal{T}-T'$ in N' gives the torus decomposition of N', where T' is a component of $\mathrm{Fr}_M\Sigma$. Hence, N' is decomposed into $q-1$ components.*

Proof. First, we consider the rel ∂A_1^1 isotopy classes of homeomorphisms h: $A_1^1\to A_1^2$ with $h|_{\partial A_1^1}=g|_{\partial A_1^1}$. Let p_i: $[0, 1]\times R\to A_1^i(i=1, 2)$ be the universal cover of A_1^i, where the covering translations are generated by $(x, y)\to(x, y+1)$. Let $\bar{h}$ be a lift of h to the universal cover. We may suppose that $\bar{h}(0, 0)=(0, 0)$. Then, rel ∂A_1^1 isotopy class of h is determined by $\varepsilon\,(\in Z)$ with $\bar{h}\,(1, 0)=(1, \varepsilon)$. We fix a homeomorphism $h_\varepsilon(\varepsilon\in Z)$ such that $h_\varepsilon|_{\partial A_1^1}=g|_{\partial A_1^1}$, and $\bar{h}_\varepsilon(1, 0)=(1, \varepsilon)$.

Let $g_\varepsilon = g \cup h_\varepsilon$: $\mathrm{cl}\,(\partial_0 C_1 - \bigcup_{i=2}^{p} A_i^1) \to \mathrm{cl}\,(\partial_0 C_2 - \bigcup_{i=2}^{p} A_i^2)$.

Suppose that Σ admits a hyperbolic structure ([T1]), then by Thurston's hyperbolic Dehn surgery theory ([T2]), we see that if we take ε sufficiently large, then g_ε satisfies the conclusions (i), (ii).

Hence, suppose that Σ admits a Seifert fibration. Let $l\,(\subset \partial N)$ be a component of ∂A_1^1 with an orientation, $m(\subset \partial N)$ be a simple loop $p_1([0, 1] \times \{0\}) \cup p_2([0, 1] \times \{0\})$ with an orientation. Let $[l]$, $[m]$ be the homology class represented by l, m. Then $\{[l], [m]\}$ is a generator of the first homology group of the torus $A_1^1 \cup A_1^2 (\subset \partial N)$. Let $l_1(\subset A_1^1 \cup A_1^2)$ be a fiber of Σ with an orientation. Then $[l_1] = a\,[l] + b\,[m]$, where $a, b \in Z$, $(a, b) = 1$. Let $N_\varepsilon = C_1 \cup_{g_\varepsilon} C_2$, and Σ_ε be the image of Σ in N_ε. Then, N_ε is homeomorphic to the manifold which is obtained from N and $D^2 \times S^1$ by identifying $A_1^1 \cup A_1^2$ and $\partial(D^2 \times S^1)$ by a homeomorphism such that $\partial D^2 \times \{pt.\}$ is identified with a loop representing $\varepsilon\,[l] + [m]$.

Suppose that Σ does not admit a Seifert fibration such that A_1^1, A_1^2 are unions of fibers. Then $b \neq 0$, and the algebraic intersection number of $\varepsilon\,[l] + [m]$ and $a\,[l] + b[m]$ is $\det \begin{pmatrix} \varepsilon & 1 \\ a & b \end{pmatrix} = b\varepsilon - a$. If we take ε sufficiently large, then we can make the absolute value of the intersection number greater than two. Then Σ_ε admits a Seifert fibration such that one boundary component of Σ is exchanged to an exceptional fiber with index greater than two. By Lemma 5.1, it is easily checked that Seifert fibrations on Σ_ε are unique up to ambient isotopies of Σ_ε, and each component of $\partial \Sigma_\varepsilon$ is incompressible. Hence, g_ε satisfies the conclusions (i), (ii).

Suppose that Σ admits a Seifert fibration such that A_1^1, A_1^2 are unions of fibers. Then, Σ_ε admits a Seifert fibration such that one boundary component of Σ is exchanged to a regular fiber. If Σ_ε is not homeomorphic to $D^2 \times S^1$, $S^1 \times S^1 \times [0, 1]$, or the twisted $[0, 1]$ bundle over the Klein bottle, then, by Lemma 5.1, we see that N_ε is a Haken manifold and the image of $\mathcal{T}$ in N_ε gives the torus decomposition of N_ε, for each ε. Hence, g_ε satisfies the conclusions (i), (ii).

Let Σ' be the union of the closure of the components of $N - \mathcal{T}$, each component of which intersects Σ. Σ'_ε denotes the image of Σ' in N_ε.

Suppose that Σ_ε is the twisted $[0, 1]$ bundle over the Klein bottle. Then, Σ_ε is not a piece of the torus decomposition of N', if and only if a Seifert fibration on Σ_ε extends to a Seifet fibration on $\Sigma_\varepsilon \cup \Sigma'_\varepsilon$. But, it is easily seen that for almost all ε, any Seifert fibrations on Σ_ε do not extend to $\Sigma_\varepsilon \cup \Sigma'_\varepsilon$. Hence, we have the conclusions (i), (ii).

Suppose that $\Sigma_\varepsilon = D^2 \times S^1$. Then, Σ admits a Seifert fibration with orbit manifold an annulus and one exceptional fiber. Let $N' = \mathrm{cl}\,(N - \Sigma)$, $T_1 = N' \cap \Sigma$, $l_1(\subset T_1)$ be a fiber of Σ with an orientation. Let $m'(\subset T_1)$ be a non-trivial simple loop which bounds a disk in Σ_0, $l'(\subset T_1)$ be a non trivial simple loop which intersects m' transversely in a point. Then, $[l_1]=a_1[m']+b_1[l']\in H_1(T_1; Z)$ $(a_1, b_1 \in Z, |b_1|>1, (a_1, b_1)=1)$. Then N_ε is homeomorphic to the manifold obtained from N' and $D^2\times S^1$ by identifying T_1 and $\partial(D^2\times S^1)$ by a homeomorphism which takes a loop representing $[m']+\varepsilon b_1(a_1[m']+b_1[l'])=(1+\varepsilon a_1 b_1)[m']+\varepsilon b_1^2[l']$ to a loop $\partial D^2\times\{pt.\}$. If Σ' is hyperbolic, then, for sufficiently large ε, the image of $\mathcal{T}-T_1$ in N_ε gives the torus decomposition of N_ε. Suppose that Σ' admits a Seifert fibration. Let $l_2(\subset T_1)$ be an oriented fiber of a Seifert fibration on Σ'. Then, $[l_2]=a_2[m']+b_2[l']$ $(a_2, b_2\in Z, (a_2, b_2)=1)$, where $\det\begin{pmatrix}a_1 & b_1\\ a_2 & b_2\end{pmatrix}=a_1 b_2-a_2 b_1\neq 0$. Since, $\det\begin{pmatrix}(1+\varepsilon a_1 b_1) & \varepsilon b_1^2\\ a_2 & b_2\end{pmatrix}=b_2+\varepsilon b_1(a_1 b_2-a_2 b_1)$, we can extend the Seifert fibration on Σ'_ε to $\Sigma'_\varepsilon\cup\Sigma_\varepsilon$ with creating a new exceptional fiber, for sufficiently large ε. Then, by the argument as above, we see that N_ε is a Haken manifold and the image of $\mathcal{T}-T_1$ in N_ε gives the torus decomposition of N_ε. Hence, we have the conclusion (iii).

Suppose that $\Sigma_\varepsilon=S^1\times S^1\times[0, 1]$. Then Σ admits a Seifert fibration with orbit manifold a disk with two holes and no exceptional fibers i.e. Σ is homeomorphic to (disk with two holes)$\times S^1$. Suppose that Σ' does not admit a Seifert fibration i.e. one component of Σ' does not admit a Seifert fibration, then the image of $\mathcal{T}-T'$ gives a torus decomposition of N_ε for each ε, where T' is a component of $\mathrm{Fr}_M\Sigma$. Suppose that Σ' admits a Seifert fibration, then, by Lemma 5.1, it is easily seen that any Seifert fibrations on Σ'_ε do not extend to $\Sigma'_\varepsilon\cup\Sigma_\varepsilon$ for almost all ε. Hence, the image of $\mathcal{T}-T'$ gives the torus decomposition of N_ε, and we have the conclusion (iv).

This completes the proof of Lemma 5.2.

For the statement of Lemma 5.3, we define a terminology. Let C be a genus $g(>1)$ compression body, and $\mathcal{A}=\{A_1, \cdots, A_m\}$ $(m\geq 1)$ be a system of mutually disjoint annuli in $\partial_0 C$. We say that A_1 is *simple with respect to* $\mathcal{A}$ if there is a disk D properly embedded in C such that D cuts C into a solid torus V and a genus $g-1$ compression body with $\partial D\cap A_i=\phi$ $(i=1, \cdots, m)$, $A_1\subset\partial V$, $A_i\cap\partial V=\phi$ $(i=2, \cdots, m)$, and (V, A_1) is homeomorphic to $(A_1\times[0, 1], A_1\times\{0\})$ as a pair. Then, we have:

Lemma 5.3. *Let C_i, A^i_j $(i=1, 2, j=1, \cdots, p)$, g, N, Σ be as in Lemma 5.2. Suppose that A^1_1 is simple with respect to $\{A^1_1, \cdots, A^1_p\}$, and Σ admits a Seifert fibration such that A^i_1 $(i=1, 2)$ is a union of fibers. Then, there is an embedding g': $\mathrm{cl}\,(\partial_0 C_1-\bigcup_{j=1}^{p} A^1_j)\rightarrow\partial_0 C_2$ such that:*

(i) *$N'=C_1\cup_{g'} C_2$ is homeomorphic to N, and*

(ii) *If Σ' is the component of the torus decomposition of N' which contains A^1_1, then Σ' does not admit a Seifert fibration such that A^1_1 is a union of fibers.*

Proof. By definition, there is a disk D in C_1 such that D cuts C_1 into a solid torus V, and a compression body C_1', where $A_1^1 \subset \partial V$, and (V, A_1^1) is homeomorphic to $(A_1^1 \times [0, 1], A_1^1 \times \{0\})$ as a pair. Let $\tilde{g}$ be the restriction of g to $\partial_0 C_1 - (\partial V \cup \bigcup_{2 \leq j \leq p} A_j^1)$. We consider $\tilde{g}$ an embedding from a subsurface of $\partial_0 C_1'$ to $\partial_0 C_2$. Then $N'' = C_1' \cup_{\tilde{g}} C_2$ is homeomorphic to N. Let T be the component of $\partial N''$ which contains A_1^2, D' be the copy of D in T. By Lemma 5.1, there is a simple loop l on T such that l is not isotopic to a regular fiber of any Seifert fibration on Σ'. We may suppose that l intersects D' in an arc. Let $N(l)$ be a regular neighborhood of l in N'', $A' = N(l) \cap T$, and $C_1'' = C_1' \cup N(l)$. Then, there is a homeomorphism $h: (C_1, A_1^1) \to (C_1'', A')$ such that $h|_{A_j^1} = id_{A_j^1}$ $(1 < j \leq m)$. Let $C_2' = \mathrm{cl}(N'' - C_1'')$. Then, there is a homeomorphism h': $C_2' \to C_2$, and an embedding $\tilde{\tilde{g}}$: $\mathrm{cl}(\partial_0 C_1'' - (A' \cup \bigcup_{j=2}^{p} A_j^1)) \to \partial_0 C_2'$ such that $C_1'' \cup_{\tilde{\tilde{g}}} C_2'$ is homeomorphic to N''. Then, $g' = h' \circ \tilde{\tilde{g}} \circ h|_{\mathrm{Cl}(\partial_0 C_1 - \bigcup_{j=1}^{p} A_j^1)}$ satisfies the conclusion of Lemma 5.3.

This completes the proof of Lemma 5.3.

The next lemma will be needed for the proof of Theorem 1.

Lemma 5.4. *Let A be an essential annulus in a genus g compression body C such that $\partial A \subset \partial_0 C$. Then, A cuts C into two compression bodies C', C'' such that genus (C')+genus $(C'') = g+1$, or A cuts C into a genus g compression body $\bar{C}$. Moreover, if A', A'' denote the image of A in C', C'' (or $\bar{C}$), then one of A', A'', say A', is simple with respect to A' (or $\{A', A''\}$) in C' (or $\bar{C}$).*

This can be proved by using the same argument as in [Ko 2, Lemma 3.2] together with Lemmas 3.2, 3.3. So we will omit the proof.

6. Proof of Theorem 1

Let $\mathcal{T}$ be the union of tori which give the torus decomposition of M, and $(C_1, C_2; F)$ be a genus g Heegaard splitting of M. We may suppose that each component of $\mathcal{T} \cap C_1$ is a disk and the number of the component of $\mathcal{T} \cap C_1$ is minimal among all systems of tori which are ambient isotopic to $\mathcal{T}$, and intersect C_1 in disks. Let $c(M)$ be the first Betti number of the characteristic graph G_M. Then, we order $(g, c(M))$ lexicographically. The proof will be done by the induction on $(g, c(M))$. Let N be a Haken manifold as in Theorem 1, and $\mathcal{S}$ be the union of tori which gives the torus decompositoin of N. Then, $n(N)$ denotes the number of the components of $N - \mathcal{S}$.

As we see later (section 8), we can construct a Haken manifold with genus g Heegaard splitting and decomposed into $3g$-3 components by the torus decomposition. Hence, we may suppose that $\mathcal{T}$ contains at least $3g$-4 components.

As the first step of the induction, we will show:

Lemma 6.1. *Let M, g be as in Theorem* 1. *Suppose that $g=2$. Then, M is decomposed into at most* 3 *components by the torus decomposition. Moreover, if M is decomposed into* 3 *components, then G_M is $\overset{\bullet}{M_1}\!-\!\overset{\bullet}{M_2}\!-\!\overset{\bullet}{M_3}$, where M_i $(i=1, 3)$ is a simple Seifert fibered manifold, and M_2 is homeomorphic to the exterior of a two bridge link.*

REMARK. By [Ko 2, Lemma 4.3], we see that M_2 is simple.

Proof. This can be proved by using the arguments in [Ko 1, Ko 2]. We do not think it worth to repeat the argument here, and, hence, we will only state how the logic proceeds.

By using the argument in [Ko 1], we see that if M contains a non-separating incompressible torus, then M is decomposed into at most 2 components by the torus decomposition. Hence, we may suppose that each component of $\mathcal{T}$ separates M. By the argument in [Ko 2, section 6, Case 2], we can show that $\mathcal{T}$ consists of two tori T_1, T_2. Then, by [Ko 2, section 6, Case 3], we see that $T_1 \cup T_2$ can be isotoped so that $T_i \cap C_j$ $(i, j=1, 2)$ consists of an annulus which separates C_j. Then, by seeing the position of the annulus, we have the conclusion of Lemma 6.1.

In the rest of this section, we suppose that $g>2$. By Proposition 4.1′, there is a component T_1 of $\mathcal{T}$ such that $T_1 \cap C_1$ consists of a disk D_1, which is good. Then, as in section 3, let $\mathcal{T}_i=\mathcal{T} \cap C_i (i=1, 2)$, $(\mathcal{T}_2^{(0)}, a_0), \cdots, (\mathcal{T}_2^{(m)}, a_m)$ be a hierarchy for $\mathcal{T}_2$, which is realized by a sequence of isotopies of type A, $\mathcal{T}^{(0)}=\mathcal{T}$, and $\mathcal{T}^{(i)}$ $(i \geq 1)$ be the image of $\mathcal{T}^{(i-1)}$ after the isotopy of type A at a_{i-1}. Let k be the number such that $a_k \cap D_1 \neq \phi$, $a_l \cap D_1=\phi (0 \leq l<k)$. Let T_1' be the image of T_1 in $\mathcal{T}^{(k+1)}$, and $A_i=T_1' \cap C_i (i=1, 2)$. Then, A_i is an essential annulus in C_i.

First, suppose that T_1 separates M. Then, by Lemma 5.4, A_i cuts C_i into two compression bodies C_i^1, C_i^2, where $\text{genus}(C_i^1)+\text{genus}(C_i^2)=g+1$. A_i^j denotes the copy of A_i in $\partial_0 C_i^j$. We may suppose that $\text{cl}(\partial_0 C_1^j - A_1^j)$ and $\text{cl}(\partial_0 C_2^j - A_2^j)$ $(j=1, 2)$ are identified in M. Let $g_j=\text{genus}(C_1^j)=\text{genus}(C_2^j)$ $(j=1, 2)$. Let M_1, M_2 be the 3-manifold obtained from M by cutting along T_1'. Then, M_i $(i=1, 2)$ has a decomposition $M_i=C_1^i \cup_{h_i} C_2^i$, where h_i: $\text{cl}(\partial_0 C_1^i - A_1^i) \rightarrow \text{cl}(\partial_0 C_2^i - A_2^i)$ is a homeomorphism induced from the Heegaard splitting of M. By Lemma 5.4, we have essentially two cases.

Case 1. A_1^1 $(\subset \partial_0 C_1^1)$, and $A_2^2 (\subset \partial_0 C_2^2)$ are simple.

In this case, by Lemmas 5.2. (ii), 5.3, we see that there is a homeomorphism h_i': $\partial_0 C_1^i \rightarrow \partial_0 C_2^i (i=1, 2)$ such that $M_i'=C_1^i \cup_{h_i'} C_2^i$ is a Haken manifold and decomposed into the same number of the components as M_i by the torus decomposition. Then by the assumption of the induction, we have:

$$n(M) = n(M_1)+n(M_2) = n(M_1')+n(M_2') \leq (3g_1\text{-}3)+(3g_2\text{-}3) = 3g\text{-}3 .$$

Suppose that the equality holds. Let $\Sigma_i (i=1, 2)$ be the closure of the component of $M-\mathcal{T}^{(k+1)}$ such that $T_1' \subset \Sigma_i$, and $\Sigma_i \subset M_i$. Let Σ_i' be the image of Σ_i in M_i'. Then, by the assumption of the induction, we see that Σ_i' is simple. If Σ_i' is hyperbolic, then Σ_i is also hyperbolic, and, hence, simple ([T1]). If Σ_i' admits a Seifert fibration, then, by the proof of Lemma 5.2, we see that Σ_i' admits a Seifert fibration with at least one exceptional fiber. Then, by [J, 155p.], Σ_i' admits a Seifert fibration with orbit manifold a disk and two exceptional fibers, or with orbit manifold an annulus and one exceptional fiber. Hence, Σ_i admits a Seifert fibration with orbit manifold an annulus and one exceptional fiber, or with orbit manifold a disk with two holes and no exceptional fibers, and we see that Σ_i is simple. Then, by Lemma 5.2 (ii), we see that the closure of each component of $M-\mathcal{T}$ is simple.

Case 2. $A_1^1(\subset \partial_0 C_1^1)$, and $A_2^1(\subset \partial_0 C_2^1)$ are simple.

Let Σ_i be as in Case 1. Suppose that Σ_2 is hyperbolic, or does not admit a Seifert fibration such that $A_i^2 (i=1, 2)$ is a union of fibers. Then the arguments in the proof of Case 1 holds, and we see that M satisfies the conclusions of Theorem 1. Hence, suppose that Σ_2 admits a Seifert fibration such that A_i^2 is a union of fibers. Then, by the definition of the torus decomposition, we see that Σ_1 does not admit a Seifert fibration such that A_i^1 is a union of fibers. We can attach a solid torus to $C_i^1 (i=1, 2)$ along the annulus A_i^1 as in Figure 6.1 to produce a genus g_1 compression body $\bar{C}_i^1$. Let h': $\partial_0 \bar{C}_1^1 \rightarrow \partial_0 \bar{C}_2^1$ be a homeomor-

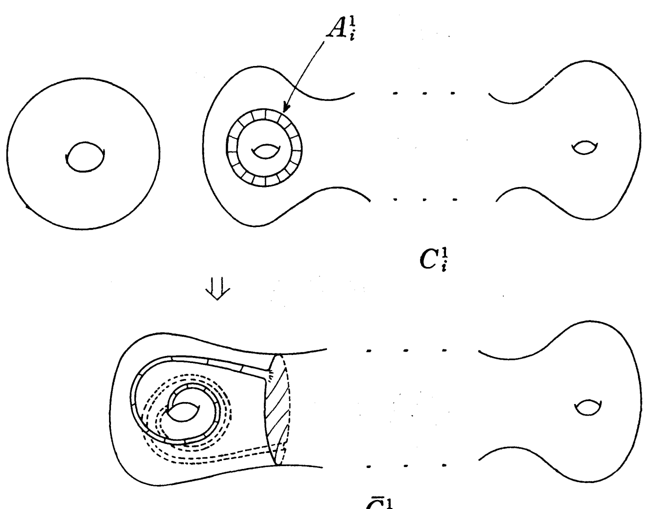


Figure 6.1

phism which is an extension of h_1, and let $M'=\bar{C}_1^1 \underset{h'}{\cup} \bar{C}_2^1$. Then, M' is homeomorphic to $M_1 \underset{T_1'}{\cup} S_{3,3}$, where $S_{3,3}$ is a Seifert fibered manifold with orbit manifold a disk and two exceptional fibers of index 3, and A_i^1 is a union of fibers. By Lemma 5.1, we easily see that any Seifert fibrations on $S_{3,3}$ do not extend to Σ_1. Hence, $n(M')=n(M)+1$.

By Lemma 5.2, there is a homeomorphism h_2': $\partial_0 C_1^2 \rightarrow \partial_0 C_2^2$ which is an extension of h_2, and $M_2'=C_1^2 \underset{h_2'}{\cup} C_2^2$ is a Haken manifold with $n(M_2')\geq n(M_2)-1$. Hence, by the assumption of the induction, we have:

$$n(M) = n(M_1)+n(M_2)\leq n(M_1')+n(M_2')\leq(3g_1\text{-}3)+(3g_2\text{-}3) = 3g\text{-}3\,.$$

Then $n(M_2')=n(M_2)-1$ and, by Lemma 5.2 (iii), (iv), we see that Σ_2 is simple. Hence, by the argument as in Case 1, we see that the closure of each component of $M-\mathcal{T}$ is simple.

Now, suppose that T_1 does not separate M. Then, $A_i(i=1, 2)$ cuts C_i into a compression body C_i'. Let A_i^1, A_i^2 be the copies of A_i in $\partial_0 C_i'$. Let M' be the 3-manifold obtained from M by cutting along T_1. Then, M' has a decomposition $M'=C_1' \underset{h}{\cup} C_2'$, where h: cl$(\partial_0 C_1'-(A_1^1\cup A_1^2))\rightarrow$cl$(\partial_0 C_2-(A_2^1\cup A_2^2))$ is a homeomorphism induced from the Heegaard splitting. We may suppose that $h(\partial A_1^i)=\partial A_2^i(i=1, 2)$. Then, we have essentially two cases.

Case 3. $A_1^1(\subset\partial_0 C_1')$, and $A_2^2(\subset\partial_0 C_2')$ are simple.

In this case, by applying Lemma 5.2 twice and Lemma 5.3 once, if needed, for h or h^{-1}, we see that there is a homoemorphism h': $\partial_0 C_1'\rightarrow\partial_0 C_2'$ such that $M'=C_1' \underset{h'}{\cup} C_2'$ is a Haken manifold with $n(M')=n(M)$. Clearly, $c(M')<c(M)$. Hence, by the assumption of the induction, we see that $n(M)\leq 3g\text{-}3$. And, if the equality holds, then, by the argument as in Case 1, the closure of each component of $M-\mathcal{T}$ is simple.

Case 4. $A_1^1(\subset\partial_0 C_1')$, and $A_2^1(\subset\partial_0 C_2')$ are simple.

Let $\Sigma^i(i=1, 2)$ be the component of M' cut along the image of $\mathcal{T}^{(k)}$ such that $A_1^i\cup A_2^i\subset\partial\Sigma^i$. If Σ^2 does not admit a Seifert fibration such that A_1^2, A_2^2 are unions of fibers, then, by the argument in Case 3, we see that Theorem 1 holds. Hence, suppose that Σ^2 admits a Seifert fibration such that A_1^2, A_2^2 are unions of fibers. Then, we attach a solid torus to $C_i'(i=1, 2)$ along the annulus A_i^1 to produce a genus g compression body $\bar{C}_i$ as in Figure 6.1. Then, there is a homeomorphism h': cl$(\partial_0\bar{C}_1-A_1^2)\rightarrow$cl$(\partial_0\bar{C}_2-A_2^2)$, which is an extension of h', and $M''=\bar{C}_1 \underset{h'}{\cup} \bar{C}_2$ is homeomorphic to $M' \underset{A_1^1\cup A_2^1}{\cup} S_{3,3}$, where $S_{3,3}$ is as in Case 2. By the assumption on Σ^2 and Lemma 5.1, we see that any Seifert fibrations on

$S_{3,3}$ do not extend to Σ^1. Hence, $n(M'')=n(M')+1=n(M)+1$. By Lemma 5.2, we see that there is a homeomorphism $h''\colon \partial_0\bar{C}_1\to\partial_0\bar{C}_2$ which is an extension of h' such that $M'''=\bar{C}_1\cup_{h''}\bar{C}_2$ is a Haken manifold, and $n(M''')=n(M'')$, or $n(M''')=n(M'')-1$. Clearly, $c(M''')=c(M'')=c(M')<c(M)$. Suppose that $n(M''')=n(M'')$. Then, by the assumption of the induction, we have $n(M''')=n(M')+1\leq 3g\text{-}3$. Hence, $n(M)\leq 3g\text{-}4$, a contradiction. Suppose that $n(M''')=n(M'')-1$. Then, $n(M)=n(M''')\leq 3g\text{-}3$. If the equality holds, then, by the argument as in Case 1, we see that the closure of each component of $M-\mathcal{T}$ is simple.

This completes the proof of Theorem 1.

7. Proof of Theorem 2

In this section, we will give a proof of Theorem 2. The proof is done by using the induction on a complexity which is different from the complexity in section 6. Let g, $c(M)$ be as in section 6. Then, we order $(c(M), g)$ lexicographically. Throughout this section, we will adopt this complexity. We note that Lemma 6.1 gives the first step of the induction.

Let $(C_1, C_2; F)$, $\mathcal{T}$, $\mathcal{T}_1^{(j)}$, $\mathcal{T}_i$, $\mathcal{T}^{(j)}$, T_1, T_1', and $A_i (i=1, 2)$ be as in section 6. Recall that $M=M_1\cup\cdots\cup M_{3g-3}$ is the torus decomposition of M. Let M' be M cut along T_1', $C_i' (i=1, 2)$ be C_i cut along A_i, A_i^1, $A_i^2 (\subset\partial_0 C_i')$ be the copies of A_i. Then M' admits a decomposition $M'=C_1'\cup C_2'$. We may suppose that ∂A_1^j $(j=1, 2)$ and ∂A_2^j are identified in M'. Let $\mathcal{T}'$ be the image of $\mathcal{T}^{(k)}-T_1'$ in M'. $\mathcal{T}'$ gives the torus decomposition of M', $M_1'\cup\cdots\cup M_{3g-3}'$, where each M_i' is the image of M_i. Suppose that $\partial M_s'$ contains $T_1^1=A_1^1\cup A_2^1$, and $\partial M_t'$ contains $T_1^2=A_1^2\cup A_2^2$. We note that possibly $M_s'=M_t'$.

Lemma 7.1. *If A_1^1, and A_2^1 are simple with respect to $\{A_1^1, A_1^2\}$, and $\{A_2^1, A_2^2\}$, then M_t' admits a Seifert fibration such that A_i^2 is a union of fibers.*

Proof. We give a proof in the case when T_1 is non separating. The arguments apply in the case when T_1 is separating. Assume that M_t' does not admit a Seifert fibration as above. Let $h'\colon \mathrm{cl}(\partial_0 C_1-(A_1^1\cup A_1^2))\to\mathrm{cl}(\partial_0 C_2-(A_2^1\cup A_2^2))$ be the homeomorphism induced from the Heegaard sewing homeomorphism h: $\partial_0 C_1\to\partial_0 C_2$. Then, by Lemma 5.2 (ii), h' can be extended to a homeomorphism $h''\colon \mathrm{cl}(\partial_0 C_1-A_1^1)\to\mathrm{cl}(\partial_0 C_2-A_2^1)$ such that the image of $\mathcal{T}'$ in $M''=C_1\cup_{h''}C_2$ gives the torus decomposition of M''. Then, we attach to a solid torus to C_i along A_i^1 as in Figure 7.1, to get a genus g compression body C_i''. Let $h'''\colon \partial_0 C_1''\to\partial_0 C_2''$ be a homeomorphism which is an extension of h'', and let $M'''=C_1''\cup_{h'''}C_2''$. Then, M''' has a decomposition $M''\cup S_{3,3}$, where $S_{3,3}$ is as in section 6.

Suppose that M_s' does not admit a Seifert fibration such that A_i^1 is a union of fibers. Then, M''' is decomposed into $3g\text{-}2$ components by the torus de-

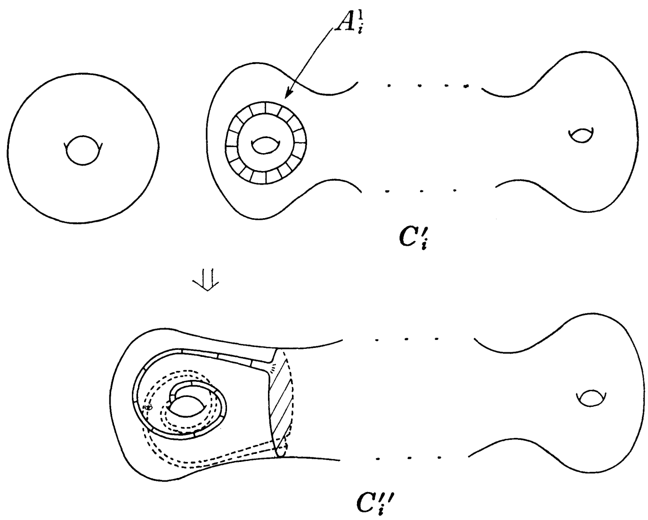


Figure 7.1

composition, contradicting Theorem 1.

Suppose that M_s' admits a Seifert fibration such that A_i^1 is a union of fibers. Then, $M''{}'$ is decomposed into $3g$-3 components by the torus decomposition, and $M_s' \cup S_{3,3}$ is a component of the decomposition. Clearly, $M_s' \cup S_{3,3}$ is not simple, contradicting Theorem 1.

This completes the proof of Lemma 7.1.

We will give a proof of the following two assertions of Theorem 2.

(i) If $\mathrm{Fr}_M M_i$ consists of a torus, then M_i admits a Seifert fibration.

(iii) $2g$-2 components of $\{M_i\}$ admit Seifert fibrations.

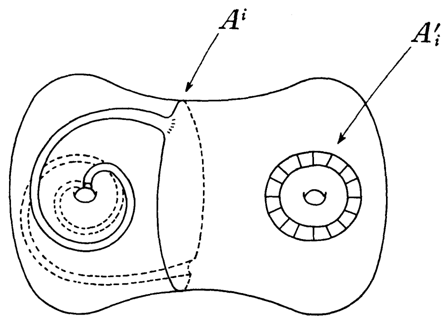
Proof of Theorem 2 (i), (iii).

By the proof of Theorem 1, we can construct a (possibly, disconnected) Haken manifold M^*, by closing boundary components of M', each component of which has a complexity less than that of M. It is easily seen that if $\mathrm{Fr}_M M_i$ consists of a torus, then the frontier of the image of M_i in M^* also consists of a torus, and if M_i admits a hyperbolic structure, then the image of M_i in M^* also admits a hyperbolic structure. Hence, by applying the assumption of the induction, we see that (i), and (iii) hold.

This completes the proof of Theorem 2 (i), (iii).

We will prepare an example for the proof of Theorem 2 (ii), (iv).

EXAMPLE. Let $V_i (i=1, 2)$ be a genus two handlebody, and $A_i' (\subset \partial V_i)$ be the annulus as in Figure 7.2. Then, there exists a homeomorphism h_1: cl $(\partial V_1 - A_1') \to$ cl $(\partial V_2 - A_2')$ such that $V_1 \underset{h_1}{\cup} V_2$ is a Haken manifold, which is decom-



V_i $(i=1, 2)$

Figure 7.2

posed into two components N_1, N_2 by the torus decomposition, where N_1 is homeomorphic to $S_{3,3}$ in section 6, and N_2 is homeomorphic to the exterior of Whitehead link ([T2]), hence, N_2 is hyperbolic.

Proof. Let $A^i(i=1, 2)$ be the annulus properly embedded in V_i as in Figure 7.1. Then, by [Ko2, Theorem], we see that there is a homeomorphism h_1: $\mathrm{cl}(\partial V_1 - A_1') \to \mathrm{cl}(\partial V_2 - A_2')$ such that $h_1(\partial A^1) = \partial A^2$ and the torus $A^1 \cup A^2$ gives the torus decomposition of $V_1 \underset{h_1}{\cup} V_2$ into N_1, and N_2 as above.

Proof of Theorem 2 (ii), (iv).

By Lemma 6.1, we see that the conclusions hold if $g=2$. Hence, we suppose that $g>2$. Then, by Proposition 4.1′ we can find a component T_1 as in section 6. Let T_1', $A_i^j(i, j=1, 2)$ be as in section 6. Then, we divide the proof into several cases.

Case 1. A_1^1, and A_2^2 are simple annuli.

Let $N_1(N_2$ resp.) be a regular neighborhood of $A_1^2(A_2^1$ resp.) in $C_1'(C_2'$ resp.). Then, $\mathrm{cl}((N_1 \cap \partial C_1') - A_1^2)$ ($\mathrm{cl}((N_2 \cap \partial C_2') - A_2^1)$ resp.) consists of two annuli. By attaching $N_1(N_2$ resp.) to $C_2'(C_1'$ resp.) along these annuli by the homeomorphism induced from the Heegaard sewing map $\partial_0 C_1 \to \partial_0 C_2$, we get a (possibly, disconnected) compression body $C_2''(C_1''$ resp.), and there is natural homeomorphism h'': $\partial_0 C_1'' \to \partial_0 C_2''$ such that $C_1'' \underset{h''}{\cup} C_2''$ is homeomorphic to M'. It is easily seen that each component of M' has a complexity less than that of M and the image of $\mathcal{T}^{(k)} - T_1$ gives the torus decomposion of M'. Hence, by the assumption of the induction, we have the conclusions of Theorem 2.

Case 2. A_1^1, and A_2^1 are simple annuli, and T_1 is non separating in M.

Let V_1, V_2, A_1', A_2' be as in Example 1. Then, we get a genus $g+1$ com-

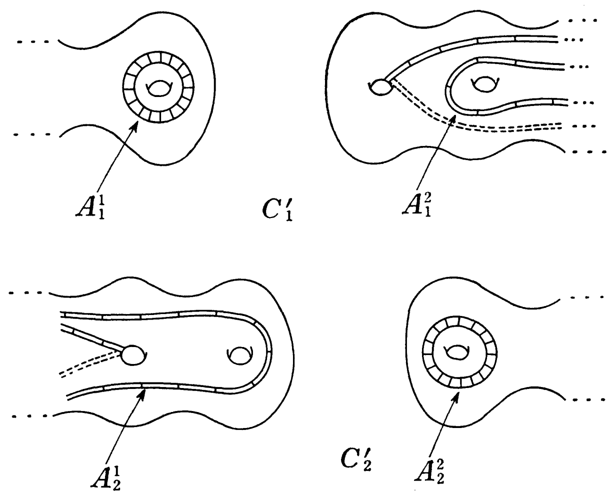


Figure 7.3

pression body C_i'' $(i=1, 2)$ from C_i' and V_i by identifying A_i^2 and A_i'. We denote the image of A_i^1 on $\partial_0 C_i''$ by A_i^1. Let h': $\mathrm{cl}\,(\partial_0 C_1' - (A_1^1 \cup A_1^2)) \to \mathrm{cl}\,(\partial_0 C_2' - (A_2^1 \cup A_2^2))$ be the homeomorphism induced from the splitting $M' = C_1' \cup C_2'$. Let h'': $\mathrm{cl}\,(\partial_0 C_1'' - A_1^1) \to \mathrm{cl}\,(\partial_0 C_2'' - A_2^1)$ be a homeomorphism which is a union of h' and h_1 in Example. Then $M'' = C_1'' \cup_{h''} C_2''$ is a Haken manifold and decomposed into $3g$-1 components by the torus decomposition.

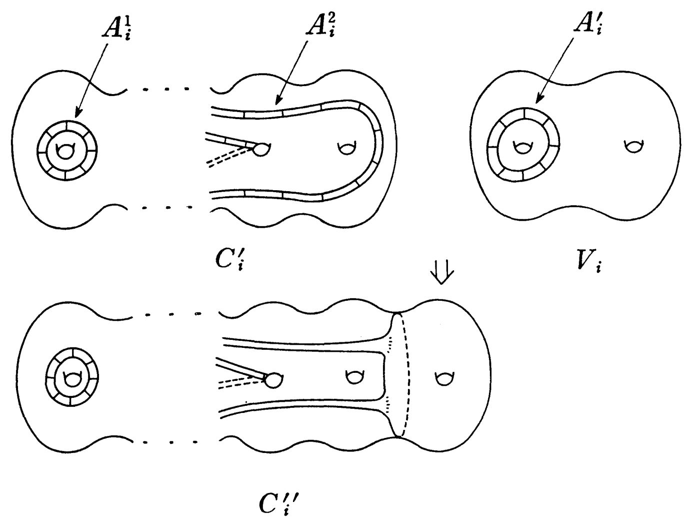


Figure 7.4

We attach a solid torus to C_i'' $(i=1, 2)$ along A_i' as in Figure 7.5 to get a genus $g+1$ compression body C_i'''. Let h''': $\partial_0 C_1''' \to \partial_0 C_2'''$ be a homeomorphism which is an extension of h''. Then, $M'''=C_1''' \cup_{h'''} C_2'''$ is a Haken manifold which is obtained from M'' by attaching $S_{3,3}$ along their boundary components. By Lemma 7.1, we see that M''' is decomposed into $3g$ components by the torus decomposition. Hene, M''' is full. Clearly, M''' has a complexity less than that of M. Then, by the assumption of the induction, we see that the conclusions of Theorem 2 hold.

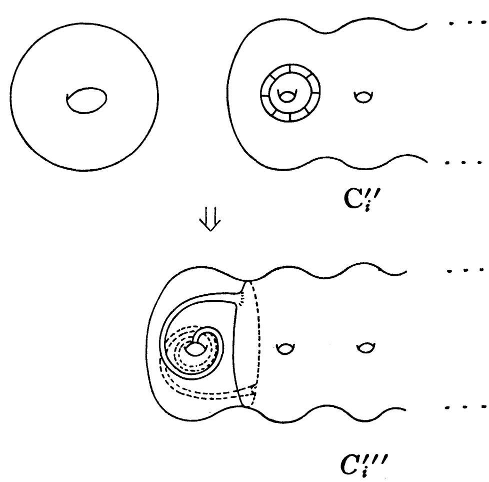


Figure 7.5

Case 3. A_1^1, and A_2^1 are simple annuli, and T_1 is separating in M.

In this case, $A_i (i=1, 2)$ separates C_i into two compression bodies C_i^1, C_i^2 such that $A_i^j \subset \partial_0 C_i^j$. Suppose that genus$(C_i^1)>2$. Then, by the construction in Case 2, we get two full Haken manifolds M_1', and M_2', each of which has a complexity less than that of M. Hence, by the assumption of the induction, we have the conclusions of Theorem 2. Since T_1 is good, we have genus $(C_i^1)>1$. Hence, the rest case that we should consider is:

(*) Case 3 with genus$(C_i^1)=2$.

Assume:

(**) every component of $\mathcal{T}$ which intersects C_1 in a disk, and which is good satisfies the above condition (*).

Then, we will proceed a long distance toward a contradiction, and that will complete the proof.

Let $T_1, \cdots, T_p$ be the components of $\mathcal{T}$ each of which intersects C_1 in a disk, and is good.

Assertion 1. No three components of $\mathcal{T}_1$ are mutually parallel in C_1.

Prcof. Assume that three components D_1, D_2, D_3 of $\mathcal{T}_1$ are mutually parallel in C_1. We may suppose that D_1, D_2, D_3 are settled in C_1 in this order, and we call the direction to which D_1 (D_3 resp.) settled left (right resp.). Let b_{j_i} be the i-th band which is attached to $D_1 \cup D_2 \cup D_3$. Then, by the argument in the proof of Proposition 3.8, we may suppose that $b_{j_i}(i=1, 2, 3)$ is attached to D_i to produce an essential annulus A_i, such that A_1 and A_2 are parallel in C_1.

Assume that b_{j_4} is attached between the right side of D_2 and the left side of D_3. Let $\bar{M}$ be the closure of the component of $M-\mathcal{T}^{(j_3+1)}$ which contains the product region between D_2 and D_3. Then, by the proof of Proposition 3.8, we see that $\bar{M}$ is homeomorphic to the exterior of a two bridge knot. Then, by exchanging the Heegaard sewing map $h: \partial_0 C_1 \to \partial_0 C_2$ in $\bar{M} \cap \partial_0 C_1$, we may suppose that $\bar{M}$ is homeomorphic to the exterior of figure eight knot. But, this contradicts Theorem 2 (i).

Assume that b_{j_4} is of type 3, and attached to D_1. By the proof of Proposition 3.8, we see that b_{j_4} is attached to the left side of D_1. By the minimality assumption on $\mathcal{T}$, we see that b_{j_4} is of type 3. Let $\bar{T}$ be the component of $\mathcal{T}$ such that $\bar{T} \cap C_1 = D_1$. Then, $\bar{T}$ satisfies the condition in Case 1, contradicting the assumption (**).

Assume that b_{j_4} is of type 3, and attached to D_2. Let $\bar{T}$ be the component of $\mathcal{T}$ such that $\bar{T} \cap C_1 = D_2$. Since there exist two disks, D_1, and D_2, we see that $\bar{T}$ is good. By the assumption (**), we see that D_2 separates C_1 into a genus one compression body and the other component V. Since $\mathcal{T}$ gives the torus decomposition of M, we see that $D_1 \subset V$. Then, by using the case by case argument as in the proof of Proposition 3.8, we see that there is a component $\bar{\bar{T}}$ of $\mathcal{T}$ such that $\bar{\bar{T}} \cap C_1 = D_1$, and $\bar{\bar{T}}$ is good. Since $g>2$, $\bar{\bar{T}}$ does not satisfy (*), a contradiction.

Assume that b_{j_4} is of type 3, and attached to D_3. By assumption (**), we see that D_2 separates C_1 into a genus one compression body V' and the other component. By (*), we see that $D_1, D_2 \subset V'$. But, since A_1 and A_2 are parallel, this contradicts the definition of the torus decomposition.

By the argument in the proof of Proposition 3.8, we see that no other possibility of the ways of attaching b_{j_4} can occur, and this completes the proof of Assertion 1.

Recall that $T_1, \cdots, T_p$ are the components of $\mathcal{T}$ each of which intersects C_1 in a disk, and is good. Let $D_i = T_i \cap C_1 (i=1, \cdots, p)$ and b_{k_i} be the first band which is attached to D_i. We may suppose that $k_1 < k_2 < \cdots < k_p$. By (**), we see that each D_i separates C_1 into a genus one compression body V_i and a genus $g-1$ compression body, and b_{k_i} is attached to the side of D_i in which the genus $g-1$ compression body is settled. Let $T_i^{(r)}$ be the image of T_i in $\mathcal{T}^{(r)}$.

Assertion 2. For each $i\,(1 \leq i \leq p)$, there is a component $D(\neq D_i)$ of $\mathcal{T}_1$

such that $D \subset V_i$, and D and D_i are parallel in C_1.

Proof. Assume that there is no component of $\mathcal{T}_1$ which is contained in $V_i - D_i$. Then, by applying the argument in the proof of Theorem 1 to T_i, we can construct a genus $g-1$ manifold which is decomposed into at least $3g$-5 components by the torus decomposition, a contradiction. Hence, we have a component D of $\mathcal{T}_1$ such that $D \subset V_i$. Assume that D and D_i are not parallel in C_1. Since V_i is a genus one compression body, D cuts V_i into a 3-cell. Let b_r be the first band which is attached to D. Since b_r is of type 3, we see that $r > k_i$. Let b_s be the second band which is attached to D_i.

Assume that $r > s$ $(> k_i)$. Then, by (**), we see that b_s is attached to D_i to the side in which V_i is settled. Then, $T_i^{(s+1)} \cap C_1$ is a once punctured torus, and is compressible in C_1, a contradiction. Hence, $s > r$.

Then, $\mathcal{T}^{(r+1)} \cap C_1$ contains two annuli A', and A'', where $A'(A''$ resp.) is obtained from $D(D_i$ resp.) by attaching $b_r(b_{k_i}$ resp.). Then, we can span an annulus A^* between the core of A' and the core of A'' in C_1. But, by Lemma 7.1, we see that this contradicts the definition of the torus decomposition.

This completes the proof of Assertion 2.

Assertion 3. $\mathcal{T}_1$ contains at most $3g$-p-5 parallel classes.

Proof. By Assertion 2, we see that $\mathcal{T}_1$ contains at most $(3g\text{-}3)\text{-}p = 3g\text{-}p\text{-}3$ parallel classes. If needed, by exchanging the order of the isotopies of type A we may suppose that b_0 is not attached to a disk contained in $\bigcup_{i=1}^{p} (V_i - D_i)$. Then, by the argument in the proof of Lemma 4.3 we see that $\mathcal{T}_1$ contains at most $3g$-p-4 parallel classes. Assume that $\mathcal{T}_1$ contains just $3g$-p-4 parallel classes. Then, $\mathcal{T}_1$ cuts C_1 into genus one compression bodies $V_1', \cdots, V_p'$ V, and some 3-cells, where $V_i' \subset V_i (i = 1, \cdots, p)$, $V \cap F$ is a once punctured torus and $b_0 \subset V$. If b_0 is attached to some D_i, then we see that C_1 is a genus two compression body, a contradiction. Hence, we may suppose that b_1 is not attached to a disk contained in $\bigcup_{i=1}^{p} (V_i - D_i)$. Then, by the argument in the proof of Lemma 4.3, we see that this contradicts the definition of the torus decomposition. Hence, $\mathcal{T}_1$ contains at most $3g$-p-5 parallel classes.

Let $\{T_i\}_{p < i \leq q}$ be the components of $\mathcal{T}$ which intersects C_1 in a disk and $T_i \cap C_1 \not\subset V_1 \cup \cdots \cup V_p$. By the definition of $\{T_i\}_{1 \leq i \leq p}$, we see that $T_i (p < i \leq q)$ is bad, and there is no component of $\mathcal{T}_1$ which is parallel to $T_i \cap C_1 (p < i \leq q)$. Hence, by Assertion 1, we see that $\mathcal{T}_1$ contains at least $2(3g-4-2p-q)+2p+q = 6g-8-2p-q$ components. On the other hand, by Assertions 1,3, we see that $\mathcal{T}_1$ contains at most $2(3g-p-5)-q = 6g-10-2p-q$ components, a contradiction.

This completes the proof of Theorem 2.

8. Examples

In this section, we will show that, for each $g(\geq 2)$, there exist infinitely many closed Haken manifolds with genus g Heegaard splittings, and each of which is decomposed into $3g$-3 components by the torus decomposition. We will give two constructions of such examples. It is easy to construct such examples with incompressible toral boundaries by using the arguments stated below.

CONSTRUCTION 1.

EXAMPLE 1. *Closed Haken manifold with a genus two Heegaard splitting, which is decomposed into three components by the torus decomposition* ([Ko2]).

Let $V_i(i=1, 2)$ be a genus two handlebody, A_i^1, A_i^2 be annuli properly embedded in V_i as in Figure 8.1. Let $g: \partial V_1 \to \partial V_2$ be a homeomorphism such that $g(\partial A_1^i)=\partial A_2^i(i=1, 2)$. Then, $T^i=A_1^i \cup A_2^i$ is a torus in the closed 3-manifold $N=V_1 \cup_g V_2$, and $T^1 \cup T^2$ cuts N into three components N_1, N_2, and N_3, where N_1, N_3 are homeomorphic to $S_{3,3}$ in section 6, and N_2 is homeomorphic to the exterior of a two bridge link ([Ko2, section 4]). Let g_n: $\partial V_1 \to \partial V_2(n=1, 2, \cdots)$ be a homeomorphism such that $g_n(\partial A_1^i)=\partial A_2^i(i=1, 2)$ and $T^1 \cup T^2$ cuts $N_n = V_1 \cup_{g_n} V_2$ into three components, two of which are homeomorphic to $S_{3,3}$, and the rest one is homeomorphic to the exterior of $(2, 2n)$ torus link, where the core of A_i^j is a meridian loop. Then, N_n is a Haken manifold and the above decomposition is the torus decomposition of N_n provided $|n| \geq 2$. By the uniqueness of the torus decomposition, we see that if $|m| \neq |n|$, then N_m is not homeomorphic to N_n.

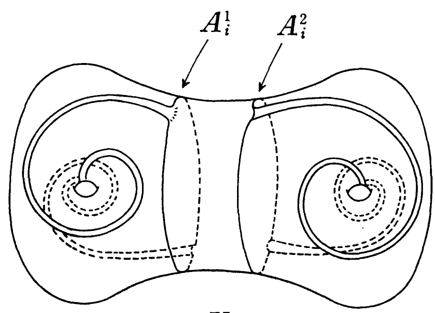


Figure 8.1

EXAMPLE 2. *Closed Haken manifold with Heegaard splitting of genus three, which is decomposed into six components by the torus decomposition.*

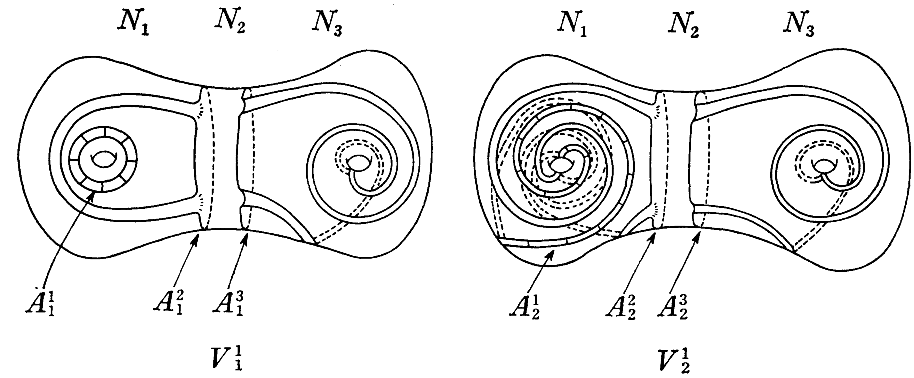


Figure 8.2

Let $V_i^1(i=1, 2)$ be a genus two handlebody, $A_i^1(\subset \partial V_i^1)$ be the annulus as in Figure 8.2, and A_i^2, A_i^3 be annuli properly embedded in V_i^1 as in Figure 8.2. Let g: $\mathrm{cl}\,(\partial V_1^1 - A_1^1) \to \mathrm{cl}\,(\partial V_2^1 - A_2^1)$ be a homeomorphism such that $g(\partial A_1^j)=\partial A_2^j$ $(j=2, 3)$. Let $T^j = A_1^j \cup A_2^j$. Then $T^2 \cup T^3$ cuts $N^1 = V_1^1 \cup_g V_2^1$ into three components N_1^1, N_2^1, N_3^1, where N_1^1 admits a Seifert fibration with orbit manifold an annulus and one exceptional fiber of index 2 where A_i^1 is a union of fibers, N_2^1 is homeomorphic to the exterior of a two bridge link ([Ko1, section 4]), where the core of $A_i^j(j=2, 3)$ is a meridian loop, and N_3^1 is homeomorphic to $S_{3,3}$ where A_i^3 is a union of fibers. Let g_n: $\mathrm{cl}\,(\partial V_1^1 - A_1^1) \to \mathrm{cl}\,(\partial V_2^1 - A_2^1)$ be a homeomorphism such that $g(\partial A_1^j) = \partial A_2^j (j=2, 3)$, and $T^2 \cup T^3$ cuts $N_{(n)}^1 = V_1^1 \cup_{g_n} V_2^1$ into three components, where two of them are homeomorphic to N_1^1, N_3^1 as above, and the rest one is homeomorphic to the exterior of $(2, 2n)$ torus link.

Let (V_1^2, A_1^2) be a copy of (V_1^1, A_1^1), and V_2^2 be a copy of V_1^2. Then, by Lemma 5.3, there is an embedding g_n': $\mathrm{cl}\,(\partial V_1^2 - A_1^2) \to \partial V_2^2$ such that $N_{(n)}^2 = V_1^2 \cup_{g_n'} V_2^2$ is homeomorphic to $N_{(n)}^1$, and if N_1^2 is the component of the torus decomposition of $N_{(n)}^2$ which intersects $\partial N_{(n)}^2$, then N_1^2 does not admit a Seifert fibration such that A_2^1 is a union of fibers. Let $A_2^2 = \mathrm{cl}\,(\partial V_2^2 - g_n'(\partial V_1^2 - A_1^2))$. Then, by attaching V_1^1 to V_2^2 (V_1^2 to V_2^1 resp.) along A_1^1 and A_2^2 (A_1^2 and A_2^1 resp.) we get a genus three handlebody V_1^3 (V_2^3 resp.) Let $g_n^{(3)}$: $\partial V_1^3 \to \partial V_2^3$ be a homeomorphism which is a union of g_n and $g_n'^{-1}$. Then $N_n^{(3)} = V_1^3 \cup_{g_n^{(3)}} V_2^3$ is a closed Haken manifold, and decomposed into six components by the torus decomposition.

CONSTRUCTION 2. We will give another construction of full Haken manifolds. First, we will prepare five ways of attaching handlebodies, each of which is a fundamental block of the full Haken manifolds.

1. Let V be a genus two handlebody, T, T' be a pair of once punctured tori embedded in ∂V as in Figure 8.3. It is directly seen that if we attach a

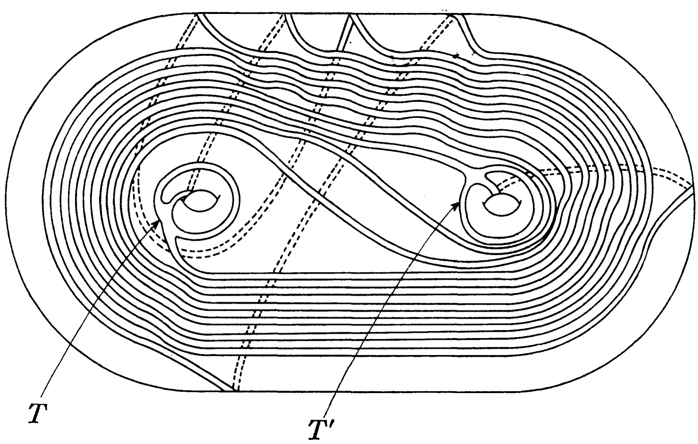


Figure 8.3

2-handle to V along the annulus $\mathrm{cl}(\partial V-(T\cup T'))$, then we get the exterior of Whitehead link.

2. Let V be a genus two handlebody, and T be a punctured torus embedded in ∂V_1 as in Figure 8.4. Let V' be a genus one handlebody, and D be a disk embedded in $\partial V'$. Let h: $\mathrm{cl}(\partial V-T)\rightarrow\mathrm{cl}(\partial V'-D)$ be a homeomorphism which takes the arc a to b. Then, by calculating the fundamental group, we see that $N=V\underset{h}{\cup}V'$ admits a Seifert fibration with orbit manifold a disk and two exceptional fibers of index three. Moreover, we may suppose that l is a fiber of the fibration.

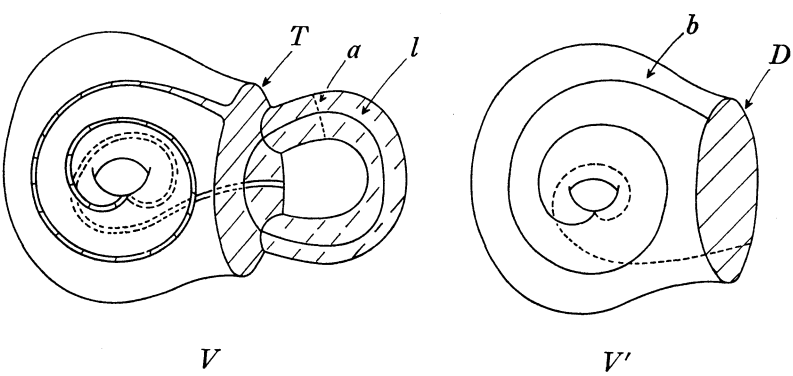


Figure 8.4

3. Let V be a genus three handlebody, and T, T' be a pair of punctured tori embedded in ∂V as in Figure 8.5. Let V' be a genus one handlebody, and D, D' be a pair of disks in $\partial V'$. Let h: $\mathrm{cl}(\partial V-(T\cup T'))\rightarrow\mathrm{cl}(\partial V'-(D\cup D'))$ be a homeomorphism which takes the arc a to b. Then, by calculat-

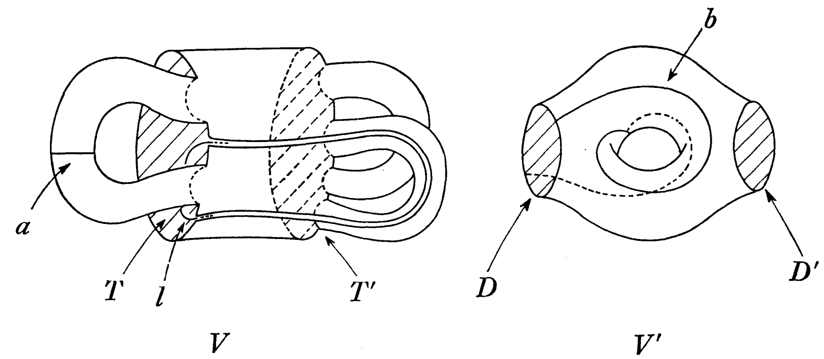


Figure 8.5

ing the fundamental group, we see that $N = V \cup_h V'$ admits a Seifert fibration with orbit manifold an annulus and one exceptional fiber of index two. Moreover, we may suppose that l is a fiber of the fibration.

4. Let V be a genus two handlebody, and T, T' be a pair of tori embedded in ∂V as in Figure 8.6. Let N be a 3-manifold obtained from V by attaching a 2-handle along the annulus $\mathrm{cl}(\partial V - (T \cup T'))$. Then, N admits a Seifert fibration with orbit manifold an annulus and one exceptional fiber of index two. Moreover, we may suppose that l is a fiber of the fibration.

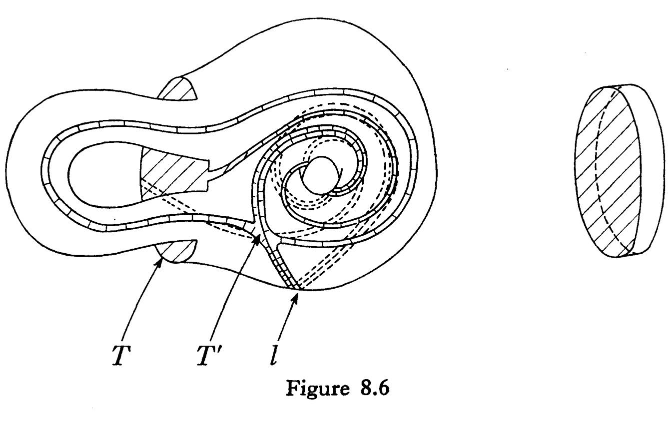


Figure 8.6

5. Let V be a genus three handlebody, T, T', T'' be a system of punctued tori embedded in ∂V as in Figure 8.7. Let V' be a 3-cell and D, D', D'' be a system of disks in $\partial V'$. Let h: $\mathrm{cl}(\partial V - (T \cup T' \cup T'')) \to \mathrm{cl}(\partial V' - (D \cup D' \cup D''))$ be a homeomorphism. Then, $N = V \cup_h V'$ admits a Seifert fibration with orbit manifold a disk with two holes and no exceptional fiber i.e. N is homeomorphic to (disk with two holes)$\times S^1$. Moreover, we may suppose that l is a fiber of the fibration.

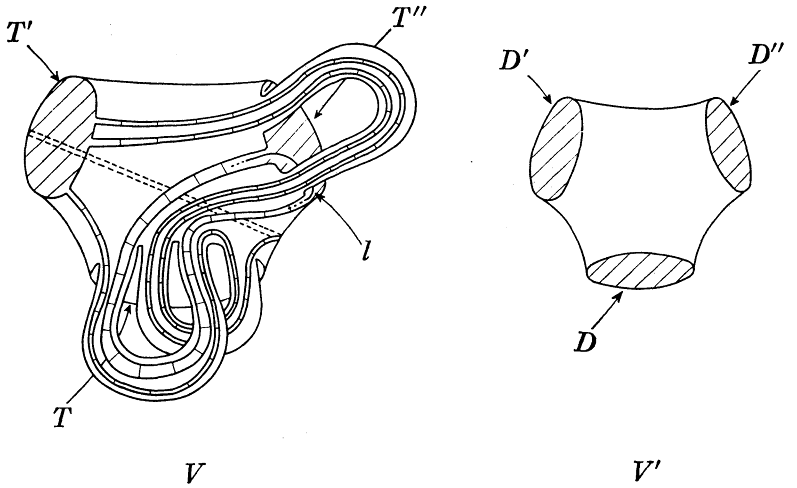


Figure 8.7

By using the above fundamental blocks, we will give another description of Example 2.

EXAMPLE 2′. Let $T_i (i=1, \cdots, 5)$ be a punctured torus properly embedded in a genus three handlebody V_1 such that $T_1 \cup \cdots \cup T_5$ cuts V_1 into six handlebo-

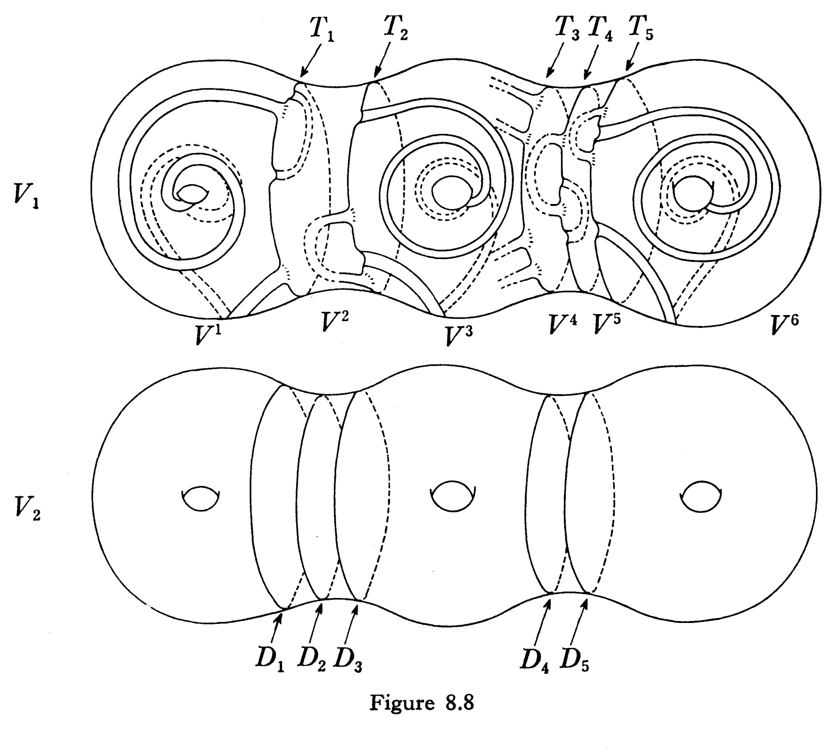
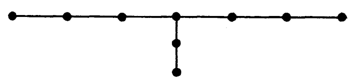


Figure 8.8

dies $V^1, \cdots, V^6$, where (V^1, T_1), (V^6, T_5) are homeomorphic to (V, T) in the above 2, (V^2, T_1, T_2), (V^5, T_4, T_5) are homeomorphic to (V, T, T') in the above 1, (V^3, T_2, T_3) is homeomorphic to (V, T, T') in the above 4, and (V^4, T_3, T_4) is homeomorphic to (V, T', T) in the above 3. By Figure 8.8, we see that such $T_1, \cdots, T_5$ actually exist. Let $D_1, \cdots, D_5$ be a system of disks properly embedded in a genus three handlebody V_2 as in Figure 8.8. Then, by the above constructions 1, $\cdots$, 5, we see that there is a homeomorphism f: $\partial V_1 \to \partial V_2$ such that $f(\partial T_i) = \partial D_i (i=1, \cdots, 5)$ and $M = V_1 \cup_f V_2$ is a full Haken manifold such that the system of tori $(T_1 \cup D_1) \cup \cdots \cup (T_5 \cup D_5)$ gives the torus decomposition.

EXAMPLE 3. genus four full Haken manifold whose characteristic graph is:



By Figure 8.9 and the arguments as above, we see that the above example actually exists.

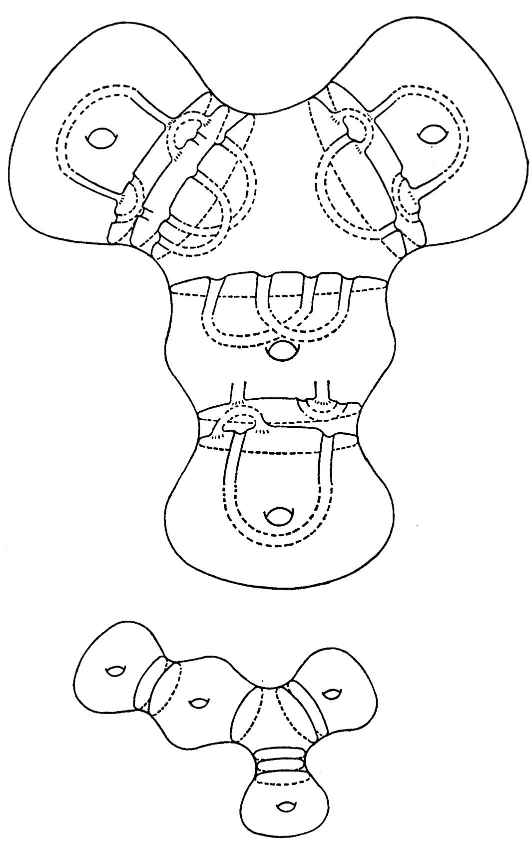
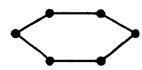


Figure 8.9

EXAMPLE 4. genus three full Haken manifold whose characteristic graph is:



See Figure 8.10.

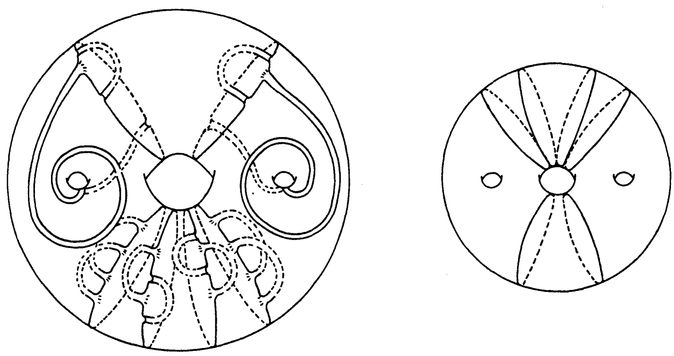
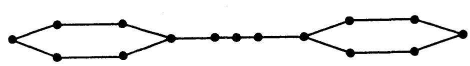


Figure 8.10

EXAMPLE 5. genus six full Haken manifold whose characteristic graph is:



See Figure 8.11.

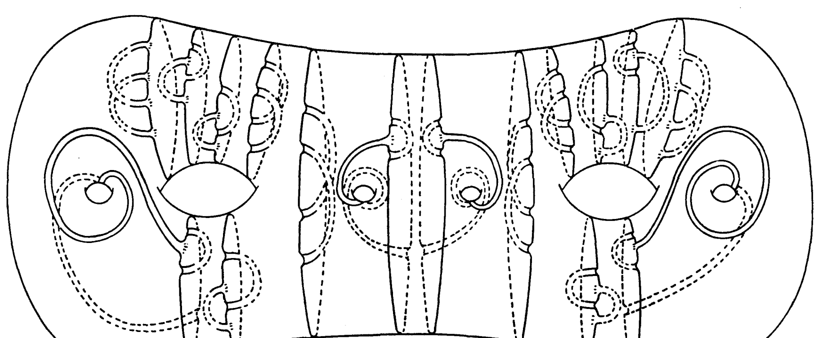


Figure 8.11

REMARK. By using the same arguments, we can construct full Haken manifolds such that the characteristic graphs have arbitrarily high first Betti numbers.

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