ENUMERATIVE AND ALGEBRAIC INVARIANTS OF LATTICE POLYTOPES

Submitted to
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Contents

Chapter 1

Introduction

Given a convex polytope as a subset of $\mathbb{R}^d$, how many integral points are contained in it and its integer dilates?

This question was investigated by Eugène Ehrhart in the 1960s and has since become a subject of intense research. Ehrhart found that the counting function which maps the magnitude of the dilation to the number of lattice points in the dilated polytope is a quasi-polynomial if one of the dilates has all integral vertices. If the first dilate has all integral vertices already (i.e., the polytope is a *lattice polytope*), the function is a polynomial. Thus, we call this function the *Ehrhart (quasi-)polynomial* of a polytope.

The Ehrhart polynomial of a lattice polytope encodes a variety of geometric information about it that goes beyond merely counting lattice points. For example, Alexander Barvinok showed in [Bar94] that the coefficients of the Ehrhart polynomial could be computed in polynomial time from the volumes of the faces of the polytope.

In the early 2000s, Ehrhart polynomials made a surprise appearance in number theory. While studying a version of the Riemann hypothesis, the authors of [Bum+00] noticed that one family of functions they investigated are the Ehrhart polynomials of the cross-polytope. This prompted [Rod02] to study polynomials whose roots also have real part $-\frac{1}{2}$ more closely. Both of these papers then led to [Bec+05], which initiated the study of Ehrhart polynomial roots. In particular, they considered geometric properties of polytopes whose Ehrhart polynomial roots have real part $-\frac{1}{2}$. From there, the study of Ehrhart polynomial roots can be split into two categories. Firstly, the study of the roots of classes of polynomials, which include Ehrhart polynomials, as showcased in publications such as [BD08; Bra08; Hig12; HHK19], and secondly, the study of specific families of polytopes with a focus on their Ehrhart polynomial roots, as showcased in publications such as [OS12; HKM17; HHY22]. In [HHK19], the term *CL-polytope* was coined for polytopes with all Ehrhart polynomial roots on the *canonical line*, i.e. the set $\left\{-\frac{1}{2} + \alpha i : \alpha \in \mathbb{R}\right\}$. Chapter 4 falls into the first category. It uses techniques similar to those in [BD08] and [Bra08] to prove a conjecture about Ehrhart polynomial roots from [BD08] in the case of polynomials that have all roots on the canonical line.

Symmetric edge polytopes (also sometimes referred to as *adjacency polytopes*) are a class of graph polytopes that have been the subject of intense study in recent years, for example [HKM17; CD22; DDM22; KT22] to name a few. They are a family of reflexive polytopes that are constructed from simple graphs. The dimension of a symmetric edge polytope is equal to the number of edges of the largest spanning forests of its graph, making this family of polytopes an ideal provider of examples of reflexive polytopes in high dimensions. Since every CL-polytope is reflexive (a fact that follows from Ehrhart reciprocity), symmetric edge polytopes also play a role in the study of CL-polytopes [HKM17].

In the study of CL-polytopes, one useful tool is the theory of *interlacing polynomials*. Given real-rooted polynomials p and q of degrees d and $d+1$ respectively, we say that they *interlace* if their roots alternate on $\mathbb{R}$. Interlacing polynomials have been making appearances in mathematics for a long time, for example in form of the classical result that orthogonal polynomials interlace, but only recently have all these results been collected and organised [Fis06]. In recent years, the theory of interlacing polynomials has been used to show that certain families of polytopes are CL-polytopes. In [HKM17], it was applied to symmetric edge polytopes of complete bipartite graphs of types $(1, n)$, $(2, n)$, and $(3, n)$. In Chapter 5, we extend these results to complete multipartite graphs of type $(1, 1, n)$, $(1, 2, n)$, and $(1, 1, 1, n)$.

Since its advent, Ehrhart theory has seen several generalisations. One of these is *equivariant Ehrhart theory*, introduced in [Sta11], which considers lattice polytopes that are fixed by a group action on the lattice $\mathbb{Z}^d$. Instead of simply counting the number of lattice points in the dilations of a polytope, we count for every element g in the acting group only the number of lattice points in every dilation, fixed by g. Ehrhart polynomial, Ehrhart series, and h^*-polynomial all have equivariant analogues where values from $\mathbb{Z}$ (or $\mathbb{Q}$ in the case of Ehrhart polynomial coefficients) have been replaced by $\mathbb{Z}$- (or $\mathbb{Q}$-)valued class functions. One peculiarity however is that the equivariant analogue of the h^*-polynomial need not be a polynomial, which is why we call it *equivariant H^*-series*. Whenever the equivariant H^*-series is not a polynomial, it also has a coefficient which does not correspond to an effective representation. Conversely, all currently available data suggests that if the equivariant H^*-series is a polynomial, all of its coefficients correspond to effective representations. This is known as the *effectiveness conjecture* and has been the main focus of investigation into equivariant Ehrhart theory. In Chapter 6, we investigate the effectiveness conjecture for symmetric edge polytopes coming from cycle graphs.

Another of the conjectures Stapledon has posed is concerned with the value of $H^*[1]$. It posits that if the equivariant H^*-series of a polytope is a polynomial, then $H^*[1]$ is a permutation representation. In Chapter 7 we investigate this question for hypersimplices when the symmetric group acts via coordinate permutation. The equivariant Ehrhart theory of hypersimplices has been studied before in [EKS24], but with a focus on the effectiveness conjecture. In particular, we propose the first closed formula for the coefficients of the equivariant H^*-series under this action.

Every lattice polytope P gives rise to an *Ehrhart ring* defined by taking the cone over $P \times \{1\}$ and forming the semigroup ring over the set of lattice points in that cone. This correspondence gives rise to a correspondence of notions. For example, a lattice polytope has the *integer decomposition property* if and only if its Ehrhart ring is normal. Famously, the *Gorenstein* property for rings indirectly corresponds to reflexivity in polytopes [Hib92] in the sense that a polytope is Gorenstein if one of its integer dilations is reflexive. The

Gorenstein property has numerous generalisations, and a popular subject of study is to attempt to understand them in terms of lattice polytopes. One of these properties is the *nearly Gorenstein* property [HHS19] defined in terms of the trace ideal of the canonical module. Both the canonical module and its trace ideal have combinatorial interpretations in terms of lattice points in the cone over a lattice polytope and are thus closely related to the integer decomposition property. We study the nearly Gorenstein property of lattice polytopes in Part IV.

Summary of the thesis

Part I contains background on the objects we study in this thesis. In Chapter 2 we recall some background about polytopes and Ehrhart theory and introduce the classes of polytopes that will play an important role later on. Alongside them, we will introduce the methods we use to study them. Specifically, we will introduce symmetric edge polytopes and Gröbner basis techniques, as well as CL-polytopes and the theory of interlacing polynomials and some notions from ring theory. In Chapter 3 we will introduce equivariant Ehrhart theory. In order to do that, we will first give some background in representation theory.

Part II contains two chapters that deal with the roots of Ehrhart polynomials. For this we will make heavy use of the theory of interlacing polynomials. In Chapter 4, we study SNN-polynomials, as introduced in [BD08], whose roots have real parts $-\frac{1}{2}$. We call this class of polynomials $\mathfrak{C} \cap \mathfrak{S}$: The set $\mathfrak{S}$ is the set of SNN-polynomials and $\mathfrak{C}$ is the set of real polynomials whose zeros all have real part $-\frac{1}{2}$. In particular, in Theorem 7 we find that in degree d the imaginary parts of the roots are bounded by those of the polynomial

$$p_0^d(z) = \binom{z}{d} + \binom{z+d}{d}.$$

This confirms a conjecture from [Bra08] in the case of $\mathfrak{C} \cap \mathfrak{S}$. Further, in Theorem 8 we show that within this bound, every root can be obtained by a degree d polynomial in $\mathfrak{C} \cap \mathfrak{S}$. The p_0^d are not themselves Ehrhart polynomials of any polytope. Up to dimension 9, we identify the standard reflexive simplex as being the polytope whose Ehrhart polynomials have the largest spread across the canonical line. For higher dimensions, we show how that might not be the case without providing a concrete polytope as a counterexample. Lastly, in Proposition 20 we provide a sufficient criterion for a polynomial in $\mathfrak{C}$ to be contained in $\mathfrak{S}$ in the form of inequalities on the roots.

In Chapter 5, we study symmetric edge polytopes from complete multipartite graphs. The goal is to find further evidence for a conjecture from [HKM17]. First, we use Gröbner basis techniques to derive a formula for the h^*-polynomials of complete tripartite graphs (Theorem 10, Theorem 11). Then, in Proposition 22 we use this formula, as well as results from [OT21] to compute the h^*-polynomials of the complete multipartite graphs of types $(1, m, n)$, $(1, 1, 1, n)$, and $(2, 2, n)$. Using techniques from [HKM17], in Theorem 9 we show interlacing relationships among the Ehrhart polynomials of some of these graphs. Finally, in Theorem 12 we develop a systematic approach to these techniques and show that their effectiveness depends on the γ-polynomial, which limits their usefulness in the further study of the conjecture from [HKM17].

Part III contains two chapters that deal with equivariant Ehrhart theory. In Chapter 6, we focus on testing the effectiveness conjecture. First, we consider the symmetric edge polytopes of cycle graphs where the actions are induced by the actions of the automorphism groups on the graphs. We confirm the conjecture in two cases (Theorem 13): firstly, for cycle graphs of prime order under the action of the dihedral group, and secondly for all cycle graphs under the action of the order 2 reflection subgroup of the dihedral group. In Theorem 17 we also study a family of modified cross-polytopes with rational coefficients under reflections across a hyperplane. All of these modified cross-polytopes have a polynomial equivariant H^*-series, but one of them has a non-effective coordinate. This shows that even if the effectiveness conjecture is true, it cannot be extended to the rational case.

In Chapter 7, we study the equivariant Ehrhart theory of hypersimplices under the action of the symmetric group. From [EKS24] it is already known that the effectiveness conjecture holds, so we focus on a different conjecture, namely whether in the effective case, $H^*[1]$, the sum of all the coefficients of the equivariant H^*-series, is a permutation representation. We show in Theorem 20 that this is indeed the case and detail an interpretation via *decorated ordered set partitions* (DOSPs for short), which is known to exist both in the non-equivariant case [Kim20] and in the case of a cyclic group action. In Theorem 18 we also give an explicit description of the individual coordinates of the equivariant H^*-series and show that they are not necessarily permutation actions.

Part IV contains one chapter. In it, the goal is to find a characterisation of the nearly Gorenstein property for the Ehrhart rings of lattice polytopes. We start by defining the *floor polytope* $\lfloor P \rfloor$ and the *remainder polytope* $\{P\}$ of a given lattice polytope. Then in Theorem 24 we show that every nearly Gorenstein lattice point with negated a-invariant a can be written as the Minkowski sum $\lfloor aP \rfloor + \{P\}$. We show that the converse does not necessarily hold, but if P is representable as such a sum, then at least most of its integer dilates are nearly Gorenstein. In Theorem 26 we show that for every nearly Gorenstein polytope its facet data is encoded by some reflexive polytope. Lastly, in Theorem 27 we give a full classification of nearly Gorenstein $(0, 1)$-polytopes in the case when they have the integer decomposition property. Using that, we prove in Corollary 7 that all IDP $(0, 1)$-polytopes are level. Furthermore, we characterise nearly Gorenstein edge polytopes in the IDP (Corollary 8) case and nearly Gorenstein matroid base polytopes from graphic matroids (Corollary 9).

Part I

Preliminaries

Chapter 2

Convex polytopes and Ehrhart theory

In this chapter we will introduce our main object of study: convex polytopes. We will start by introducing the most basic notions and give an overview of Ehrhart theory, the methodology for the study of convex polytopes used throughout most of this thesis. We will also introduce the classes of polytopes that will play a special role during our investigations. For any undefined terms and notations throughout this chapter, we may refer to standard texts like [BR15] and [Zie12].

2.1 Basic notions

Polytopes. A *lattice* of dimension d is an abelian group $M \cong \mathbb{Z}^d$. For every M we get the real vector space $M \otimes_{\mathbb{Z}} \mathbb{R}$. In the following we will always identify M with $\mathbb{Z}^d$ and $M \otimes_{\mathbb{Z}} \mathbb{R}$ with $\mathbb{R}^d$ unless stated otherwise. Ehrhart theory is the study of convex polytopes via counting lattice points it contains. We will start by defining convex polytopes.

Definition 1 (Convex polytope, lattice polytope, rational polytope)**.** Let d be a positive integer. Then a *convex polytope* P is the convex hull of a finite subset of $\mathbb{R}^d$. If a convex polytope can be written as the convex hull of elements of $\mathbb{Z}^d$ (resp. $\mathbb{Q}^d$) exclusively, we refer to it as a *lattice polytope* (resp. *rational polytope*).

Throughout this thesis, we will refer to convex polytopes simply as *polytopes*. While rational non-lattice polytopes play a vital role in parts of this thesis, the focus in general lies on lattice polytopes. We provide a number of useful examples.

Example 1 (Some useful families of lattice polytopes)**.** Fix a positive integer d. Henceforth, we will refer to the i-th unit vector in $\mathbb{R}^d$ as e_i.

(a) The *standard* $(d-1)$*-simplex* is given by

$$\Delta^{d-1} \coloneqq \mathrm{conv}\{e_1, e_2, \ldots, e_d\}.$$

(b) The *standard reflexive d-simplex* is given by

$$\Delta^d_{sr} := \operatorname{conv}\left\{e_1, e_2, \ldots, e_d, -\sum_{i=1}^{d} e_i\right\}.$$

(c) The *d-th cross-polytope* is given by

$$\diamondsuit^d := \operatorname{conv}\{\pm e_1, \pm e_2, \ldots, \pm e_d\}.$$

(d) The *d-th hypercube* is given by

$$\square^d := \operatorname{conv}\left\{\sum_{i=1}^{d} \varepsilon_i e_i \colon (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_d) \in \{-1, 1\}^d\right\}.$$

(e) The *d-th unit hypercube* is given by

$$U^d := \operatorname{conv}\left\{\sum_{i=1}^{d} \varepsilon_i e_i \colon (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_d) \in \{0, 1\}^d\right\}.$$

(f) Let $0 < k < n$ be integers. The *hypersimplex of type* (k, n) is given by

$$\Delta^n_k := \operatorname{conv}\{e_{i_1} + e_{i_2} + \cdots + e_{i_k} \colon \{i_1, i_2, \ldots, i_k\} \subseteq [n]\}.$$

In particular, the hypersimplex of type $(1, n)$ is identical with the standard $(n-1)$-simplex

There exists a natural notion of isomorphism for polytopes called *unimodular equivalence*. Let M and N be two lattices and $P \subset M \otimes_{\mathbb{Z}} \mathbb{R}$ and $Q \subset N \otimes_{\mathbb{Z}} \mathbb{R}$ polytopes. Then P and Q are called unimodularly equivalent if there exists a map $f \colon M \to N$ such that $f(M \cap P) = N \cap Q$, f is invertible of $M \cap P$, and for a fixed $n \in N$ and every M-basis $B \subset M$, $f(N) + n$ is an N-basis (see Figure 2.1).

Let $(\mathbb{R}^d)^*$ denote the dual space of $\mathbb{R}^d$. For $n \in (\mathbb{R}^d)^*$ and $x \in \mathbb{R}^d$, we denote by $n(x)$ their natural pairing. Given $n \in (\mathbb{R}^d)^*$ and $h \in \mathbb{R}$, a *hyperplane in* $\mathbb{R}^d$ is a subset $\mathcal{H}_{n,h}$ of the form

$$\mathcal{H}_{n,h} = \left\{x \in \mathbb{R}^d \colon n(x) = -h\right\}.$$

Every hyperplane $\mathcal{H}_{n,h}$ defines a *closed half-space* $\mathcal{H}^+_{n,h}$ by

$$\mathcal{H}^+_{n,h} = \left\{x \in \mathbb{R}^d \colon n(x) \geq -h\right\}.$$

A hyperplane $\mathcal{H}_{n,h}$ is called *supporting hyperplane* of a polytope P if

(i) $P \subset \mathcal{H}^+_{n,h}$ or $P \subset \mathcal{H}^+_{-n,-h}$,

(ii) $P \cap \mathcal{H}_{n,h}$ is not empty.

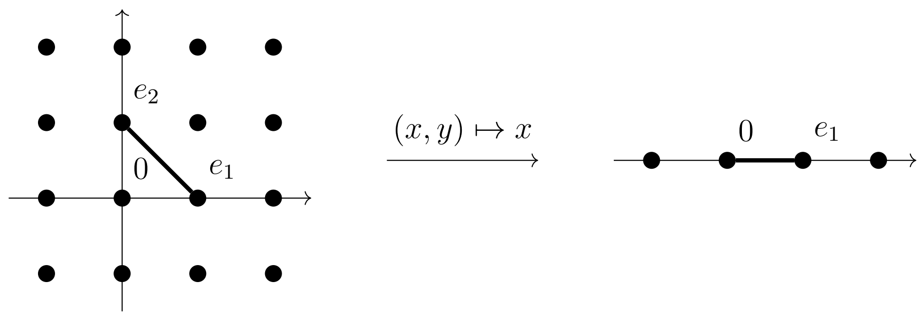

Figure 2.1: The polytopes Δ^1 and U^1 are unimodularly equivalent but lie in different lattices.

Definition 2 (Face, dimension)**.** A *face* of a polytope P is a subset of the form $P \cap \mathcal{H}$ where $\mathcal{H}$ is a supporting hyperplane of P. Notice that P is a face of itself given by $P \cap \mathcal{H}_{0,0}$.

The *dimension* of a face F is the length d of the longest chain

$$F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_{d-1} \subsetneq F_d = F$$

where the $F_0, F_1, \ldots, F_{d-1}$ are faces. The dimension of the polytope P is its dimension as a face.

Some authors regard the empty set as a face of dimension -1, but here, this case is excluded. Faces of dimension 0 are called *vertices*, those of dimension 1 are called *edges*, and those of codimension 1 *facets*. We denote the set of vertices of P by $\mathrm{vert}(P)$ and the set of facets by $\mathcal{F}(P)$. In particular, $P = \mathrm{conv}\{\mathrm{vert}(P)\}$. We can also describe P in terms of its facets. Assume P is is rational polytope. Then we can write

$$P = \{x \in \mathbb{R}^d \colon n_F(x) \geq -h_F \text{ for all } F \in \mathcal{F}(P)\} \tag{2.1}$$

where h_F is a rational number and n_F is a *primitive* integer vector, i.e., the greatest common divisor of its coordinates is 1. We call h_F the *height* and n_F the *facet normal* of F. In particular, if P is a lattice polytope, h_F is an integer. We call the set in Equation 2.1 call the *hyperplane description* or *facet presentation* of P. The set $\partial P \coloneqq \bigcup_{F \in \mathcal{F}(P)} F$ is called the *boundary* of P and the set $\mathrm{int}(P) \coloneqq P \setminus \partial P$ is called the *strict interior* of P.

Constructions. Given one or several polytopes, it is always possible to construct new polytopes. As we will see on many occasions, understanding polytopes in relationship with each other often tremendously boosts our understanding of them.

Definition 3 (Polar dual, reflexivity)**.** Given a polytope $P \subset \mathbb{R}^d$ of dimension d with $0 \in \mathrm{int}(P)$, its *polar dual* P^* is given by

$$P^* = \big\{n \in (\mathbb{R}^d)^* \colon n(x) \geq -1 \text{ for all } x \in P\big\}.$$

If P is a lattice polytope, it is called *reflexive* if P^* is also a lattice polytope.

One can verify that $(P^*)^* = P$. The condition of having full dimension and including the origin in the strict interior is a technicality that must be satisfied to guarantee that P^* is a polytope. Whenever we encounter a polytope P that contains a lattice point in its strict interior but does not satisfy these conditions, we consider an appropriate unimodularly equivalent polytope.

Given two polytopes $P, Q \subset \mathbb{R}^d$, we can define their *Minkowski sum*

$$P + Q = \{x + y \colon x \in P,\ y \in Q\}.$$

One can check that $P + Q$ is also a polytope. In particular, it is a lattice polytope if P and Q are. For a non-negative integer k, the *k-th dilation* kP of P is given by

$$kP = \{kx \colon x \in P\}.$$

The 0-th dilation is just the origin and for every $k > 1$, kP coincides with the k-fold Minkowski sum of P with itself.

The *(direct) product* of two polytopes $P \subset \mathbb{R}^d$ and $Q \subset \mathbb{R}^e$ is defined as the Cartesian product of P and Q and denoted by $P \times Q \subset \mathbb{R}^{d+e}$. Note that we can regard $P \times Q$ as the Minkowski sum of polytopes, as follows. Let

$$P' = \{(p, \underbrace{0, \ldots, 0}_{e}) \in \mathbb{R}^{d+e} \colon p \in P\} \text{ and } Q' = \{(\underbrace{0, \ldots, 0}_{d}, q) \in \mathbb{R}^{d+e} \colon q \in Q\}.$$

Then, we can see that $P \times Q = P' + Q'$. Conversely, suppose two polytopes $P', Q' \subset \mathbb{R}^d$ satisfy the following condition: for all $i \in [d] := \{1, \ldots, d\}$, we have that $\pi_i(P') = \{0\}$ or $\pi_i(Q') = \{0\}$, where $\pi_i \colon \mathbb{R}^d \to \mathbb{R}$ is the projection onto the i-th coordinate. Then we can regard $P' + Q'$ as the product of two polytopes.

The *direct sum* or *free sum* of two polytopes $P \subset \mathbb{R}^d$ and $Q \subset \mathbb{R}^e$ is defined as

$$P \oplus Q = \operatorname{conv}\left(\{(p, \underbrace{0, \ldots, 0}_{e}) \colon p \in P\} \cup \{(\underbrace{0, \ldots, 0}_{d}, q) \colon q \in Q\} \right) \subset \mathbb{R}^{d+e}.$$

Ehrhart theory. We define the *lattice point enumerator* of a set $P \subset \mathbb{R}^d$ as the function $E_P \colon \mathbb{N} \to \mathbb{N}$ via

$$E_P(k) = |kP \cap \mathbb{Z}^d|.$$

If P is a lattice polytope, E_P is a polynomial which we call the *Ehrhart polynomial* of P. If P is a rational polytope, E_P is a *quasi-polynomial*, i.e. a function

$$E_P(k) = c_0\,(k) + c_1(k)\,k + c_2(k)\,k^2 + \cdots + c_d(k)\,k^d$$

such that there exists an integer p with $c_i(k) = c_i(k + p)$ for all i and k. We call p the *period* of P. If $p = 1$, then E_P is a polynomial. A rational, non-lattice polytope whose Ehrhart quasi-polynomial has period $p = 1$ is called a *pseudo-integral polytope* (PIP). The degree of the Ehrhart (quasi-)polynomial is equal to the dimension of its polytope. In the lattice polytope case, the leading coefficient of E_P is equal to the volume of P and the coefficient with the second-highest degree is equal to half the boundary volume of P. In both cases the volumes are suitably normalised (i.e. the volumes of the polytope and each of its facets are defined with respect to the volume of the unit hypercubes in the sublattices they lie in).

The Ehrhart (quasi-)polynomial also contains information about the number of lattice points in the strict interior of P.

Proposition 1 (Ehrhart-Macdonald reciprocity [Mac71])**.** *Let P be a rational polytope of dimension d with Ehrhart quasi-polynomial E_P. Then the following equality holds.*

$$E_P(-k) = (-1)^d E_{\mathrm{int}(P)}(k)$$

The generating function of the Ehrhart (quasi)-polynomial is called its *Ehrhart series* and can be written as

$$\mathrm{ehr}_P(t) = \sum_{k \geq 0} E_P(k)\, t^k = \frac{h_P^*(t)}{(1-t^p)^{d+1}},$$

where $h_P^*(t)$ is a polynomial with non-negative integer coefficients of degree dp or less, and p is the period of the Ehrhart (quasi-)polynomial. We call this polynomial the *h^*-polynomial* of P.

If $p = 1$, E_P can be easily inferred from h_P^* via the equation

$$E_P(k) = \sum_{j=0}^{d} h_j^* \binom{k+d-j}{d}. \tag{2.2}$$

where h_k^* is the k-th coefficient of h_P^*. When P is clear from context, we will usually just omit the index.

Example 2 (Ehrhart polynomial of the d-simplex)**.** Let d and k be positive integers. The set of lattice points in $k\Delta^d$ is given by

$$\{\alpha_1 e_1 + \alpha_2 e_2 + \cdots + \alpha_{d+1} e_{d+1} \colon \text{ for all } i,\ 0 \leq \alpha_i \leq k \text{ and } \alpha_1 + \alpha_2 + \cdots + \alpha_{d+1} = k\}$$

To get $E_{\Delta^d}(k)$, we need to count the number of ways to partition k elements into $d+1$ many parts. By stars and bars, we obtain

$$E_{\Delta^k}(k) = \binom{k+d}{d}.$$

Equation 2.2 implies that $h_{\Delta^d}^*(t) = 1$.

A popular subject of study are the properties of the coefficients of the h^*-polynomial. For example, it is known that $h_0^* = 1$, $h_d^* = |\mathrm{int}(P) \cap \mathbb{Z}^d|$, $h_1^* = |P \cap \mathbb{Z}^d| - d - 1$, and $h_P^*(1)$ is equal to the *normalised volume* of P, i.e., the volume of P expressed with respect to the volume of Δ^d. Another relationship is given by the following classical result due to Hibi.

Theorem 1 (Hibi's Lower Bound Theorem [Hib94])**.** *Let P be a lattice polytope of dimension d with h^*-polynomial $h^*(t) = \sum_{k=0}^{d} h_k^* t^k$. Further, suppose that $h_d^* \neq 0$. Then the inequalities $h_1^* \leq h_k^*$ hold for every $1 \leq k < d$.*

A remarkable result by Hibi even shows us that the h^*-polynomial has a connection with reflexivity.

Theorem 2 (Corollary 2.2 in [Hib92])**.** *Let f be a degree d polynomial and set $h^*(t) = (1-t)^{d+1} \sum_{k \geq 0} f(k)\, t^k$. Then f satisfies the functional equation*

$$f(z-1) = (-1)^d f(-z) \tag{2.3}$$

if and only if h^* has palindromic coefficients, i.e., $h^*(t) = t^d\, h^*\left(\frac{1}{t}\right)$.

Ehrhart polynomials and h^*-polynomials behave nicely under certain polytope constructions. For example, for polytopes $P \subset \mathbb{R}^d$ and $Q \subset \mathbb{R}^e$, $E_{P \times Q} = E_P \cdot E_Q$ and $h^*_{P \oplus Q} = h^*_P \cdot h^*_Q$. For most other constructions however (like Minkowski-summation and dilation), E_P and h^* are difficult to deduce in general.

Unimodular triangulations. The most basic way of computing Ehrhart polynomials is straightforward: if your polytope has dimension d, count the number of lattice points in the first d dilations and infer the coefficients of Ehrhart polynomial from it. In many cases however, there is a better way. We need two key definitions.

Definition 4 (Unimodular simplex, unimodular triangulation)**.** A *unimodular simplex* of dimension d is a polytope which is unimodularly equivalent to Δ^d. A *unimodular triangulation* of a d-dimensional polytope P is a decomposition

$$P = S_1 \cup S_2 \cup \cdots \cup S_r$$

where every S_i is a unimodular simplex of dimension d such that for any pair $1 \leq i < j \leq r$, $S_i \cap S_j$ is either empty or a face of S_i and S_j.

A polytope does not necessarily have a unimodular triangulation. For example, almost by definition, only lattice polytopes can have one. However, in the case when a polytope does admit a unimodular triangulation, the h^*-polynomial is fully encoded by it.

Definition 5 (Visible facets, half-open simplex)**.** Fix integers $d > 0$ and $0 \leq m \leq d$. Let the facets of Δ^d be denoted by $F_0, F_1, \ldots, F_d$ and assume Δ^d lies in $\mathbb{R}^d$. Let $q \in \mathbb{R}^d$ be a point in general position, i.e., assume it does not lie on any of the supporting hyperplanes of the F_i. We call a facet *visible from* q if for every point $f \in F_i$, the half-open line segment $[q, f)$ does not intersect P. The set of visible facets from q shall be denote by V_q. The *half-open simplex* of dimension d viewed from q is defined as

$$H_q \Delta^d := \Delta^d \setminus \bigcup_{F \in V_q} F.$$

With an argument similar to that from Example 2, one can show that the lattice point enumerator of this set is given by

$$E_{H_d\Delta^d}(k) = \binom{k+d-|V_q|}{d}.$$

Equation 2.2 implies that $h^*_{H^q\Delta^d}(t) = t^{|V_q|}$.

Suppose P has a unimodular triangulation $S_1, S_2, \dots, S_r$. There exists a $q \in P$ in general position with respect to every S_i. We can then define the *half-open triangulation viewed from q* by

$$H_q S_1 \sqcup H_q S_2 \sqcup \cdots \sqcup H_q S_r.$$

All the elements of the half-open triangulation are disjoint [KV08] and their union is exactly P. This means that the Ehrhart polynomial of P is equal to the sum of the lattice point enumerators of the half-open simplices. The same goes for the Ehrhart series and, in particular, the h^*-polynomial.

2.2 Symmetric edge polytopes

We will now introduce a family of polytopes, first defined in [Mat+11], which we will study in Chapters 5 and 7. Let $G = (V, E)$ be a simple graph with vertex set V and edge set E. We define the *symmetric edge polytope* of G as

$$P_G = \left\{ \pm(e_v - e_w) \in \mathbb{R}^{|V|} \colon \{v, w\} \in E \right\}.$$

The dimension of P_G is equal to $|V| - c(G)$ where $c(G)$ is the number of connected components of G. In practice it is never necessary to consider disconnected graphs because joining connected components in a common vertex yields the same symmetric edge polytope. In particular, if $G_1, G_2, \dots, G_n$ are the 2-connected components of G, we get

$$P_G = P_{G_1} \oplus P_{G_2} \oplus \cdots \oplus P_{G_n}.$$

This leads us to our first family of examples.

Example 3. Let G be a tree with d edges. Then

$$P_G = \Diamond^d$$

after an appropriate unimodular transformation.

The geometric and combinatorial properties of a symmetric edge polytope can usually be expressed in terms of its underlying graph. One example is its facet structure.

Proposition 2 (Theorem 3.1 in [HJM19])**.** *Let $G = (V, E)$ be a finite simple connected graph. Then $f \colon V \to Z$ is facet defining if and only if* *(i) for any edge* $e = \{u, v\}$ *we have* $|f(u) - f(v)| \leq 1$*, and*

(ii) the subset of edges $E_f = \{e = \{u, v\} \in E \colon |f(u) - f(v)| = 1\}$ *forms a spanning subgraph of* G*.*

In Chapter 5 we will be studying symmetric edge polytopes of *complete multipartite graphs*, i.e., graphs of the form

$$K_{a_1,a_2,\ldots,a_n} = (A_1 \sqcup A_2 \sqcup \cdots \sqcup A_n, \{\{u, v\} \colon \text{for } u \in A_i \text{ and } v \in A_j \text{ if } i \neq j\})$$

where the a_i are positive integers and the A_i are finite sets with $|A_i| = a_i$ for all i. The facet description of symmetric edge polytopes from multipartite graphs is as follows.

Proposition 3 (Proposition 3.5 in [HJM19])**.** *Let* $k \geq 3$ *and* $G = K_{a_1,\ldots,a_k}$ *be a complete* k*-partite graph with vertex set* $V = \bigsqcup_{i=1}^{k} A_i$*. Then* $\lambda \colon V \to \mathbb{Z}$ *is facet defining if and only if* λ*, up to a constant, satisfies one of the following conditions.*

(i) $\lambda(A_i) = \{-1, 1\}$ *for some* $1 \leq i \leq k$ *and* $\lambda|_{A_j} = 0$ *for all* $i \neq j$*, or*

(ii) $\lambda(V) = \{0, 1\}$ *and*

  (a) $\lambda|_{A_i}$ *is constant for every* A_i*, or*

  (b) there exists an i *such that* $\lambda(A_i) = \{0, 1\} = \lambda\left(\bigcup_{j=1}^{k} A_j \setminus A_i\right)$*.*

In particular, the symmetric edge polytope of G *has* $2^{\sum_{i=1}^{k} a_i} - \sum_{i=1}^{k}(2^{a_i} - 2) - 2$ *facets.*

Gröbner bases and unimodular triangulations. Symmetric edge polytopes have a unimodular triangulation, as stated in [HJM19]. It was obtained using an algebraic technique which we will briefly introduce now. More detailed information can be found in [Stu96].

Let $\mathbf{k}$ be a field and let $\mathbf{k}[t_1^{\pm}, t_2^{\pm}, \ldots, t_d^{\pm}, s]$ be the Laurent monomial ring in $d + 1$ variables. We define the *toric ring* of a d-dimensional lattice polytope P as the subring

$$\mathbf{k}[P] = \mathbf{k}[t^p s \colon p \in P \cap \mathbb{Z}^d]$$

where $t^p = t_1^{p_1} \cdots t_d^{p_d}$ and $p = (p_1, \ldots, p_d) \in P \cap \mathbb{Z}^d$. Define now the ring $\mathbf{k}[x_p \colon p \in P \cap \mathbb{Z}^d]$ which associates a formal variable to every lattice point in P. We define the *toric ideal* I_P of $\mathbf{k}[P]$ as the kernel of the map

$$\pi \colon \mathbf{k}[x_p \colon p \in P \cap \mathbb{Z}^d] \to \mathbf{k}[P]$$

where $\pi(x_p) = t^p s$. It is well known that toric ideals arising from polytopes in this way are homogeneous binomial prime ideals.

A *monomial ordering* is a total ordering of monomials in a polynomial ring $\mathbf{k}[x_1, \ldots, x_n]$ such that three given monomials a, b, c, $a < b$ and $1 < c$ imply $ac < bc$ for every monomial

c. There exist several standard examples, such as *lex* (lexicographic), *deglex* (degree-lexicographic) and *degrevlex* (degree-reverse lexicographic), but we will focus only on the last one. First we impose a total ordering on the variables of $\mathbf{k}[x_1, x_2, \ldots, x_n]$. Without loss of generality, assume $x_i < x_j$ if and only if $i < j$. Then, given two monomials $x^a := x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$ and $x^b := x_1^{b_1} x_2^{b_2} \cdots x_n^{b_n}$, we say that $x^a < x^b$ if either $\sum_{i=1}^n a_i < \sum_{i=1}^n b_i$ or $\sum_{i=1}^n a_i = \sum_{i=1}^n b_i$ and $a_j > b_j$ where $j = \min\{i \colon a_i \neq b_i\}$.

Given a polynomial $p \in \mathbf{k}[x_1, x_2, \ldots, x_n]$, the *leading term* of p, denoted $\operatorname{lt}(p)$, is the largest term of p with respect to the chosen monomial ordering. For a subset $S \subseteq \mathbf{k}[x_1, x_2, \ldots, x_n]$, we define

$$\operatorname{lt} S := \{\operatorname{lt}(p) \colon p \in S\}.$$

This gives us the necessary tools for the following definition.

Definition 6 (Gröbner basis)**.** Let I be an ideal in a polynomial ring $\mathbf{k}[x_1, x_2, \ldots, x_n]$ equipped with a monomial ordering. A *Gröbner basis* B_I of I is a finite subset of I such that B_I generates I and $\operatorname{lt} B_I$ generates $\operatorname{lt} I$.

A Gröbner basis B_I is called *square-free* if every element in $\operatorname{lt} B_I$ is square free, i.e., for every element of $\operatorname{lt} B_I$, every variable x_i has at most degree 1. A Gröbner basis is called *quadratic* (resp. cubic) if every element in it is at most quadratic (resp. cubic).

Let I_P be the toric ideal of a d-dimensional polytope P. Denote its Gröbner basis by B_P and assume B_P is square-free. Then the elements of $\operatorname{lt} B_P$ encode a unimodular triangulation in the following way. In the ring $\mathbf{k}[x_p \colon p \in P \cap \mathbb{Z}^d]$, define the set

$$U := \left\{ x_S = \prod_{p \in S} x_p \colon |S| = d+1,\ x_S \notin \operatorname{lt} I_P \right\} \subset \mathbf{k}[x_p \colon p \in P \cap \mathbb{Z}^d]$$

of square-free degree $d+1$ monomials not in $\operatorname{lt} I_P$. Then the set of polytopes

$$\mathcal{T} := \{\operatorname{conv}\{S\} \colon x_S \in U\}$$

defines a unimodular triangulation of P. And since the condition $x_S \notin \operatorname{lt} I_P$ is equivalent to the condition that no $m \in \operatorname{lt} B_P$ divides x_S, we can think of $\mathcal{T}$ as being encoded by B_P.

For symmetric edge polytopes, there exists a known square-free Gröbner basis.

Proposition 4 (Proposition 3.8 in [HJM19])**.** *Let $z < x_{e_1} < y_{e_1} < \cdots < x_{e_k} < y_{e_k}$ be an ordering on the edges. Then the following collection of three types of binomials forms a Gröbner basis of the toric ideal of the symmetric edge polytope of G with respect to the degrevlex ordering:*

(1) For every $2k$-cycle C, with fixed orientation, and any k-element subset I of edges of C not containing the smallest edge

$$\prod_{e \in I} p_e - \prod_{e \in C \setminus I} q_e.$$

(2) For every $(2k+1)$*-cycle* C*, with fixed orientation, and any* $(k+1)$*-element subset* I *of edges of* C

$$\prod_{e \in I} p_e - z \prod_{e \in C \setminus I} q_e.$$

(3) For any edge e

$$x_e y_e - z^2.$$

The leading monomial is always chosen to have positive sign.

The h^*-polynomials of symmetric edge polytopes. We almost have all necessary parts for the machine that lets us compute h^*-polynomials from symmetric edge polytopes. The only thing missing is a way to make the triangulation half-open. The authors of [HJM19] found a solution in terms of graphs.

First, note that type (3) of the Gröbner basis elements implies that every unimodular simplex comes from a monomial of the form $z \prod_{e \in E} v_e$ where v_e is either x_e or y_e. The variables x_e and y_e can be regarded as directed versions of e which go in opposite directions. Consequently, every simplex in $\mathcal{T}$ can be identified with a directed spanning subgraph of G. Now we fix a vertex r of G. Given a directed spanning tree $T \in \mathcal{T}$, we call an edge e of S ingoing if the unique path starting at the foot of e and ending in r, the path includes e. Otherwise we call it outgoing. We denote the number of ingoing edges of S by $\text{in}(S)$. With all this information, one can compute the h^*-polynomial of P_G.

Proposition 5 (Proposition 4.6 in [HJM19])**.** *Let* $h^*_G(t) = \sum_{i=0}^{d} h^*_i t^i$*. Then*

$$h^*_i = |\{T \in \mathcal{T} \colon \text{ in}(T) = i\}|.$$

Notice that there is a symmetry in these coefficients. Whenever there is a $T \in \mathcal{T}$ with $\text{in}(T) = i$, reversing all the edge gives a $T' \in \mathcal{T}$ with $\text{in}(T') = d - i$. Hence, the h^*-polynomials of symmetric edge polytopes are palindromic, which means that they are reflexive.

The authors use Proposition 5 to compute the h^*-polynomial in the case of complete bipartite graphs.

Proposition 6 (Theorem 4.1 in [HJM19])**.** *For all* $a, b \geq 0$ *let* $h^*_{a,b}(t)$ *denote the* h^**-polynomial of the symmetric edge polytope of* $K_{a+1,b+1}$*. Then*

$$h^*_{a,b}(t) = \sum_{i=0}^{\min\{a,b\}} \binom{2i}{i} \binom{a}{i} \binom{b}{i} t^i (1+t)^{a+b+1-2i}.$$

In Chapter 5 we will make use of two other results that both come from [OT21]. The first one out of the two needs some preparation. A *hypergraph* $\mathcal{H}$ is a set V and a set E of nonempty subsets of V called *hyperedges*. We can associate a bipartite graph $\text{Bip}(\mathcal{H})$ to $\mathcal{H}$ whose bipartite classes are given by elements of V and E respectively with an edge between a $v \in V$ and an $e \in E$ if $v \in e$. A *hypertree* is a function $f\colon E \to \{0, 1, \ldots\}$ such that there exists a spanning tree Γ of $\mathrm{Bip}(\mathcal{H})$ whose vertices $e \in E$ have degree $f(e) + 1$. In this case, we say that Γ *induces* f. The set of hypertrees of $\mathcal{H}$ shall be denoted by $\mathrm{ht}(\mathcal{H})$. Let us now fix a total ordering of E. A hyperedge e is called *internally active* with respect to f is there exists no $e' < e$ such that increasing $f(e')$ by 1 and decreasing $f(e)$ results in a different hypertree. A hyperedge that is not internally active with respect to f is called *internally inactive*. We denote the number of internally inactive edges of f by $\iota(f)$. The *interior polynomial* $I_{\mathcal{H}}$ of $\mathcal{H}$ is then defined as

$$I_{\mathcal{H}}(t) = \sum_{f \in \mathrm{ht}(\mathcal{H})} t^{\iota(f)}.$$

Given a graph $G = \mathrm{Bip}(\mathcal{H})$ for some hypergraph $\mathcal{H}$, we define $I_G = I_{\mathcal{H}}$.

Next, recall that a *cut* of a graph $G = (V, E)$ is a subgraph $G_C = (V, E_C)$ of G where C is a subset of V and $E_C \subset E$ is the set of edges of G with one end in C and one end not in C. We denote the set of cuts of G by $\mathrm{Cuts}(G)$.

Lastly, we need two special graph constructions. Given a graph G with vertex set $[d]$, let $\widehat{G}$ describe the the *suspension* of G, i.e., the graph on the set $[d+1]$ with the same edge set as G but with the vertex $d+1$ connected to all the others. If G is bipartite with bipartite classes V and W, its *joint bipartite suspension* $\widetilde{G}$ is the graph on $[d+2]$ such that $d+1$ connects to all the edges in V, $d+2$ connects to all the edges in W and $d+1$ and $d+2$ connect to each other. Like this, we can cite the following theorem.

Proposition 7 (Theorem 4.3 in [OT21])**.** *Let G be a finite graph on the vertex set $[d]$. Then the symmetric edge polytope of $\widehat{G}$ is unimodularly equivalent to a reflexive polytope whose h^*-polynomial is*

$$h^*_{\widehat{G}}(t) = (1+t)^d f_G\left(\frac{4t}{(1+t)^2}\right),$$

where $f_G(t) = \frac{1}{2^{d-1}} \sum_{H \in \mathrm{Cuts}(G)} I_{\widetilde{H}}(t)$.

The other useful theorem is the following.

Proposition 8 (Proposition 4.4 in [OT21])**.** *Let G be a bipartite graph on $[d]$ and let e be an edge of G. Then we have*

$$h^*_G(t) = (1+t)h^*_{G/e}(t)$$

where G/e denotes the graph obtained from G by contracting the edge e.

2.3 CL-polytopes

We define the *canonical line* (CL for short) as the set

$$\mathrm{CL} = \left\{-\frac{1}{2} + \alpha i \colon \alpha \in \mathbb{R}\right\} \subset \mathbb{C}.$$

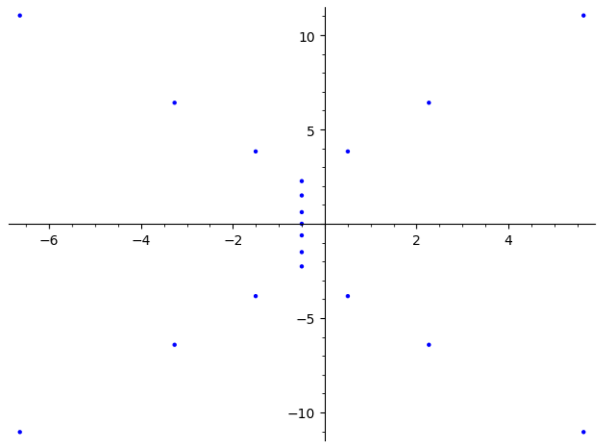

Figure 2.2: The root distribution of the Ehrhart polynomial of a 20-dimensional polytope studied in [OS12]. Notice the reflection symmetry across both $\mathbb{R}$ and CL, which tells us that the underlying polytope is reflexive. It is, however, not a CL-polytope.

Theorem 2 implies the fact that a polytope is reflexive if and only if its Ehrhart polynomial roots are distributed symmetrically across the canonical line (see Figure 2.2). A special case of that are reflexive polytopes whose Ehrhart polynomial roots all lie on CL. We call these polytopes *CL-polytopes*.

The prototypical family of CL-polytopes are the cross-polytopes, which also happen to be symmetric edge polytopes (Example 3). There are several ways to prove that cross-polytopes are in fact CL, but a historically relevant one follows from the main theorem in [Rod02].

Theorem 3. *Let f be a degree d and h a degree $d-1$ polynomial. Also, assume*

$$\sum_{k=0}^{\infty} f(k)\, t^k = \frac{h(t)}{(1-t)^{d+1}}$$

holds. If the roots of h lie on the unit circle, then the roots of f all lie on CL.

The original statement also considers roots on lines parallel to CL, but for our purposes, it is enough to present it like this. The h^*-polynomial of $\diamondsuit^1$ is $1+t$. As a consequence, using the fact that cross-polytopes are direct sums of copies of $\diamondsuit^1$, the h^*-polynomial of $\diamondsuit^n$ is $(1+t)^n$. By Theorem 3 and the behaviour of h^*-polynomials with respect to direct sums, it follows immediately that $\diamondsuit^n$ is CL-polytope.

Interlacing polynomials. The converse of Theorem 3 does not hold in general. Thus, other methods of studying CL-polytopes have been employed. One that has become popular in recent years is the technique of interlacing polynomials. We will define the term and give some results.

Definition 7. Let f and g be polynomials of degrees d and $d+1$ respectively. Further, let L be a totally ordered subset of $\mathbb{C}$. We say that f *L-interlaces g* or *f and g interlace on L* if all the roots $a_1, \ldots, a_d$ of f and $b_1, \ldots, b_{d+1}$ of g lie on L and satisfy

$$b_1 \leq a_1 \leq b_2 \leq a_2 \leq \cdots \leq a_d \leq b_{d+1}$$

with respect to the ordering on L.

The first result is an extension of Theorem 3.

Proposition 9 (Theorem 2.1.10, [Rod10])**.** *Let f and g be CL-polynomials with degrees d and $d+1$ respectively. Let h_f^* and h_g^* be the polynomials $(1+t)^{d+1}\sum_{k\geq 0} f(k)\, t^k$ and $(1+t)^{d+2}\sum_{k\geq 0} g(k)\, t^k$ respectively. Assume h_f^* and h_g^* also have degrees d and $d+1$ and their roots interlace on the unit circle. Then f CL-interlaces g.*

The next two theorems come from Steven Fisk's vast work on interlacing polynomials [Fis06]. They have been selected because we will make use of them in Chapter 4.

Proposition 10 (Lemma 1.26, [Fis06], "Leibnitz Rule")**.** *Suppose that f, f_1, g, g_1 are polynomials with positive leading coefficients, and with all real roots. Assume that f and g have no common roots. If f_1 $\mathbb{R}$-interlaces f and g_1 $\mathbb{R}$-interlaces g, then $f_1 g_1$ $\mathbb{R}$-interlaces $fg_1 + f_1 g$ which in turn $\mathbb{R}$-interlaces fg, fg_1, and $f_1 g$. In particular, $fg_1 + f_1 g$ has all real roots.*

Proposition 11 (Corollary 1.41, [Fis06])**.** *Suppose that $f_1, f_2, \ldots$ and $g_1, g_2, \ldots$ are sequences of polynomials with all real roots that converge to polynomials f and g respectively. If f_n and g_n $\mathbb{R}$-interlace for all positive integers n, then f and g $\mathbb{R}$-interlace.*

Both these statements refer to interlacing on the real line, but can be transported to any line of the form $c_1\,\mathbb{R} + c_2$ for complex numbers c_1, c_2 by performing an appropriate affine transformation. Further, since roots are invariant under scaling, positive leading coefficients can always be obtained.

Next, we will recall some results from [HKM17]. This paper studies the Ehrhart polynomials of symmetric edge polytopes of complete bipartite graphs. This family can be seen as a natural extension of cross-polytopes, considering that $P_{K_{1,d}}$ is the cross-polytope of dimension d. We start by citing the following useful result.

Proposition 12 (Lemmas 2.3, 2.4, 2.5 in [HKM17])**.** *Let $f, g, h_1, \ldots, h_n$ be real polynomials such that $\deg f = \deg g + 1 = \deg h_i + 2$ for all $1 \leq i \leq n$ which all satisfy Equation 2.3. Assume the identity*

$$f(x) = (2x+1)\,\alpha\, g(x) + \sum_{i=1}^{n} \alpha_i h_i(x)$$

where $\alpha, \alpha_i > 0$ for all i. Then $\sum_{i=1}^{n} \alpha_i h_i$ CL-interlaces g if for every i, h_i CL-interlaces g. Also, the following are equivalent.

(a) $\sum_{i=1}^{n} \alpha_i h_i$ CL-interlaces g,

(b) g CL-interlaces f.

If this is the case, $(2x+1)\sum_{i=1}^{n}\alpha_i h_i$ CL-interlaces f.

Among other things, it gives an alternative proof for the CL-ness of cross-polytopes. We saw the h^*-polynomials of cross-polytopes before. Notice that that implies that the Ehrhart polynomials $E_{\Diamond^d} := \mathcal{C}_d$ are of the form

$$\mathcal{C}_d(k) = \sum_{i=0}^{d} \binom{d}{i}\binom{d+k-i}{d}.$$

We call the polynomials $\mathcal{C}_d(k)$ *cross-polynomials*. They fulfil a recursive relation.

Proposition 13 (Example 3.3 in [HKM17]). *For any $n \geq 2$, cross-polynomials satisfy the recursive relation*

$$\mathcal{C}_n(x) = \frac{1}{n}(2x+1)\mathcal{C}_{n-1}(x) + \frac{n-1}{n}\mathcal{C}_{n-2}(x).$$

With Proposition 12, it follows that cross-polytopes are CL-polytopes.

The next results concern the Ehrhart polynomials of $P_{K_{a,b}}$, denoted by $E_{a,b}$. In the course of this thesis, we will continue using this notation and, by analogy, use $E_{a_1,a_2,\ldots,a_k}$ to denote the Ehrhart polynomial of the complete k-partite graph $K_{a_1,a_2,\ldots,a_k}$.

Proposition 14 (Proposition 4.5 in [HKM17]). *The following relations hold:*

$$\begin{aligned}
E_{2,n}(x) &= \frac{1}{2}(2x+1)\,E_{1,n}(x) + \frac{1}{2}E_{1,n-1}(x),\\
E_{2,n}(x) &= \frac{1}{n}(2x+1)\,E_{2,n-1}(x) + \frac{1}{2n}\left(n\,E_{1,n-1}(x) + (n-2)(2x+1)E_{1,n-2}(x)\right),\\
E_{3,n+1}(x) &= \frac{(2x+1)(3n^2+13n+16)}{8(n^2+5n+6)}E_{2,n+1}(x)\\
&\quad + \frac{n^3 13n^2+18n}{8(n-1)(n^2+5n+6)}E_{2,n}(x) + \frac{4n^3+9n^2-13n-32}{8(n-1)(n^2+5n+6)}E_{1,n+1}(x).
\end{aligned}$$

With Proposition 12, the authors derived the following result.

Proposition 15 (Lemmas 4.6, 4.7, 4.8, Theorem 4.9 in [HKM17]). *The following statements hold.*

(a) For every $n \geq 1$, $E_{1,n}$ CL-interlaces $E_{1,n+1}$.

(b) For every $n \geq 1$, the Ehrhart polynomials of $K_{1,n}$ and $(2k+1)K_{1,n-1}$ CL-interlace $E_{2,n}$.

(c) For every $n \geq 1$, $E_{2,n}$ CL-interlaces $E_{2,n+1}$.

(d) For every $n \geq 1$, $E_{2,n}$ CL-interlaces $E_{3,n}$.

In particular, for every $n \geq 1$ the Ehrhart polynomial of $K_{m,n}$ is a CL-polynomial if $m \leq 2$.

2.4 Algebraic aspects of polytopes

In this section, we will give some background from commutative algebra necessary to understand the content of Part IV. We will focus more on the algebraic rather than the combinatorial side, so the reader is advised to consult standard references such as [CLS24], [Mat89], or [Eis13] for any undefined terms and notations.

Ehrhart rings. We begin by defining the *codegree* a_P of a lattice polytope P by

$$a_P \coloneqq \min\{k \in \mathbb{N} \colon \operatorname{int}(kP) \cap \mathbb{Z}^d \neq \varnothing\},$$

i.e. the minimum positive integer you have to dilate P by until its interior contains lattice points [Bat06]. Next, let C_P be the *cone over* P, that is,

$$C_P = \mathbb{R}_{\geq 0}(P \times \{1\}) = \{(x,k) \in \mathbb{R}^d \times \mathbb{R} \colon n_F(x) \geq -kh_F \text{ for all } F \in \mathcal{F}(P)\}.$$

In our discussion of Gröbner bases, we defined toric rings. A similar object is the *Ehrhart ring* of P, defined as

$$A(P) = \mathbf{k}[C_P \cap \mathbb{Z}^{d+1}] = \mathbf{k}[t^p s^k \colon k \in \mathbb{N} \text{ and } p \in kP \cap \mathbb{Z}^d],$$

where $t^p = t_1^{p_1} \cdots t_d^{p_d}$ and $x = (x_1, \ldots, x_d) \in kP \cap \mathbb{Z}^d$. Note that the Ehrhart ring of P is a normal affine semigroup ring, and hence it is Cohen-Macaulay [BG09, Prop. 6.10]. Moreover, we can regard $A(P)$ as an $\mathbb{N}$-graded $\mathbf{k}$-algebra by setting $\deg(t^p s^k) = k$ for each $p \in kP \cap \mathbb{Z}^d$. The toric ring of P is a standard $\mathbb{N}$-graded $\mathbf{k}$-algebra.

We say that P has the *integer decomposition property* (i.e. P is *IDP*) if for all positive integers k and all $p \in kP \cap \mathbb{Z}^d$, there exist $q_1, \ldots, q_k \in P \cap \mathbb{Z}^d$ such that $p = q_1 + \cdots + q_k$. It is known that $\mathbf{k}[P] = A(P)$ if and only if P has the integer decomposition property.

Gorensteinness and its generalisations. Let R be a finitely generated $\mathbb{N}$-graded $\mathbf{k}$-algebra with unique graded maximal ideal $\mathbf{m}$. We will assume that R is Cohen-Macaulay and admits a canonical module ω_R and, consequently, an anticanonical module ω_R^{-1}. In particular, every Ehrhart ring admits a canonical module. We call $a(R)$ the a-invariant of R, i.e.

$$a(R) = -\min\{i \in \mathbb{N} \colon (\omega_R)_i \neq 0\},$$

where $(\omega_R)_i$ is the i-th graded piece of ω_R.

For Ehrhart rings, the canonical and anti-canonical modules as well as the a-invariant can be interpreted in combinatorial terms. For a cone σ, we denote its strict interior by $\operatorname{int}(\sigma)$. Note that

$$\operatorname{int}(C_P) = \{(x,k) \in \mathbb{R}^{d+1} \colon n_F(x) > -kh_F \text{ for all } F \in \mathcal{F}(P)\}.$$

Moreover, we define

$$\operatorname{ant}(C_P) \coloneqq \{(x,k) \in \mathbb{R}^{d+1} \colon n_F(x) \geq -kh_F - 1 \text{ for all } F \in \mathcal{F}(P)\}.$$

Then the following is true.

Proposition 16 (see [HMP19, Proposition 4.1 and Corollary 4.2])**.** *The canonical module of $A(P)$ and the anti-canonical module of $A(P)$ are given by the following, respectively:*

$$\omega_{A(P)} = \langle t^x s^k \colon (x,k) \in \operatorname{int}(C_P) \cap \mathbb{Z}^{d+1} \rangle \text{ and } \omega_{A(P)}^{-1} = \langle t^x s^k \colon (x,k) \in \operatorname{ant}(C_P) \cap \mathbb{Z}^{d+1} \rangle.$$

Further, the negated a-invariant of $A(P)$ coincides with the codegree of P, i.e.

$$a(A(P)) = -\min\{k \in \mathbb{Z}_{\geq 1} \colon \operatorname{int}(kP) \cap \mathbb{Z}^d \neq \varnothing\}.$$

The canonical module is very closely related to the Gorenstein property. In the following we will use them to study two of its numerous generalisations: *nearly Gorensteinness* and *levelness*. For this, we will use trace ideals.

Definition 8 (Trace ideal)**.** For a graded R-module M, let the *trace ideal* of M, $\operatorname{tr}_R(M)$, be the sum of the ideals $\phi(M)$ over all $\phi \in \operatorname{Hom}_R(M,R)$, i.e.

$$\operatorname{tr}_R(M) = \sum_{\phi \in \operatorname{Hom}_R(M,R)} \phi(M).$$

When there is no risk of confusion about the ring, we simply write $\operatorname{tr}(M)$.

With this, we can define *Gorensteinness* and *nearly Gorensteinness*.

Definition 9 ([HHS19, Definition 2.2])**.** We say that R is *nearly Gorenstein* if $\operatorname{tr}(\omega_R) \supseteq \mathbf{m}$. In particular, R is Gorenstein if and only if $\operatorname{tr}(\omega_R) = R$.

With the next proposition, the anti-canonical module enters the picture.

Proposition 17 ([HHS19, Lemma 1.1])**.** *Let R be a ring and I an ideal of R containing a non-zero divisor of R. Let $Q(R)$ be the total quotient ring of fractions of R and $I^{-1} := \{x \in Q(R) \colon xI \subseteq R\}$. Then*

$$\operatorname{tr}(I) = I \cdot I^{-1}.$$

Lastly, we shall introduce *levelness*.

Definition 10 ([Sta07, Chapter III, Proposition 3.2])**.** We say that R is *level* if all the degrees of the minimal generators of ω_R are the same.

Segre products. Let $R = \bigoplus_{n\geq 0} R_n$ and $S = \bigoplus_{n \geq 0} S_n$ be standard graded $\mathbf{k}$-algebras and define their *Segre product* $R\#S$ as the graded algebra

$$R\#S = (R_0 \otimes_{\mathbf{k}} S_0) \oplus (R_1 \otimes_{\mathbf{k}} S_1) \oplus \cdots \subseteq R \otimes_{\mathbf{k}} S.$$

We denote a homogeneous element $x \otimes_{\mathbf{k}} y \in R_i \otimes_{\mathbf{k}} S_i$ by $x\#y$.

If P and Q are lattice polytopes, it is known that $\mathbf{k}[P \times Q]$ is isomorphic to the Segre product $\mathbf{k}[P]\#\mathbf{k}[Q]$.

Proposition 18 ([HMP19, Proposition 2.2 and Theorem 2.4])**.** *Let $R_1, \cdots, R_s$ be standard graded Cohen-Macaulay toric $\mathbf{k}$-algebras with Krull dimension at least 2. Let $R = R_1\#R_2\#\cdots\#R_s$ be the Segre product. Then the following is true.*

$$\omega_R = \omega_{R_1}\#\omega_{R_2}\#\cdots\#\omega_{R_s} \quad \text{and} \quad \omega_R^{-1} = \omega_{R_1}^{-1}\#\omega_{R_2}^{-1}\#\cdots\#\omega_{R_s}^{-1}.$$

Chapter 3

Equivariant Ehrhart theory

Equivariant Ehrhart theory concerns the study of polytopes and their lattice points under a given group action. In this chapter we introduce the necessary preliminaries and fix the main setup following [Sta11]. We begin with some background on the representation theory of finite groups [Isa94; CR66].

3.1 Representations of groups

Let G be a finite group and $\mathbf{k}$ a field. A *finite dimensional* $\mathbf{k}$*-representation of* G is a homomorphism $\rho\colon G \to \mathrm{GL}(V)$ from G to the group of invertible linear maps of an n-dimensional $\mathbf{k}$-vector space V. Fixing a basis for V identifies $\rho(g)$ with an $n \times n$ matrix, for each $g \in G$. Equivalently, a representation is a module V for the group ring $\mathbf{k}G$ where $g \in G \subseteq \mathbf{k}G$ acts via the linear map $\rho(g)$. The *character of* ρ is the function $\chi\colon G \to \mathbf{k}$ defined by the trace $\chi(g) = \mathrm{tr}(\rho(g))$. We say that a representation is *irreducible* if it contains no proper G-invariant subspaces, *indecomposable* if it cannot be written as a non-trivial direct sum of representations, and *semisimple* if it is a direct sum of irreducible representations.

The *representation ring* $R(G)$ is the set of formal differences with respect to direct sums of isomorphism classes of representations of G. The term "formal difference" here means that for two isomorphism classes of representations $[V]$ and $[W]$ we define the element $[V]-[W]$ which satisfy $([V]+[W])-[V] = [W]$. The addition and multiplication structure of $R(G)$ are given by direct sums and tensor products respectively. So given $[V]$ and $[W]$ in $R(G)$ we have $[V]+[W] = [V \oplus W]$ and $[V]\cdot[W] = [V \otimes_K W]$. Throughout this thesis, we work with representations defined over $\mathbb{C}$. In this case Maschke's Theorem holds, so all representations are semisimple. In particular, all indecomposable representations are irreducible and any representation is a direct sum of irreducible representations. Therefore, $R(G)$ is a free Abelian group generated by the irreducible representations of G. Since the isomorphism class of a representation is determined uniquely by its character, we identify elements of $R(G)$ with $\mathbb{Z}$-linear combinations of characters.

Suppose G acts on a finite set S. Then the action induces a so-called *permutation representation* constructed as follows. Let V be the vector space over some field $\mathbf{k}$ with basis $\{e_s \colon s \in S\}$. We define the permutation representation $\rho\colon G \to \mathrm{GL}(V)$ by its action on the basis $\rho(g)(e_s) = e_{g(s)}$. Each matrix $\rho(g)$ is a permutation matrix, hence the character of the representation is given by $\chi(g) = |\{s \in S \colon g(s) = s\}|$. We say that a $\mathbf{k}G$-module V is a permutation representation if it is isomorphic to a permutation representation.

3.2 Group actions on lattices

Let $M \cong \mathbb{Z}^{n+1}$ be a lattice with a distinguished basis and G a finite group. We say that G *acts on* M if there is a homomorphism $\rho\colon G \to \mathrm{GL}_{n+1}(\mathbb{Z})$ from G to the group of invertible $(n+1) \times (n+1)$ matrices with entries in $\mathbb{Z}$. Note, this action extends naturally to the vector space $M_\mathbb{R} = M \otimes_\mathbb{Z} \mathbb{R}$. Assume that G fixes a lattice point $m \in M \setminus \{0\}$. We proceed to describe how M decomposes into a disjoint union of G-invariant affine lattices.

By assumption M has a basis, so we denote by $\langle \cdot, \cdot \rangle \colon M \times M \to \mathbb{Z}$ the standard inner-product. We construct a new inner-product by averaging over the group:

$$\langle u, v \rangle_G := \frac{1}{|G|} \sum_{g \in G} \langle \rho(g)u, \rho(g)v \rangle \in \mathbb{Q}\,.$$

Using the above inner-product, we observe two important properties about the orthogonal space $m^\perp \subseteq M_\mathbb{R}$. Firstly, we have that $m^\perp$ is G-invariant, which follows from the fact that $\langle \rho(g)u, \rho(g)v \rangle_G = \langle u, v \rangle_G$ for all $u, v \in M_\mathbb{R}$ and $g \in G$. Secondly, we may choose a basis for $m^\perp$ that lies in M, since $\langle u, v \rangle_G \in \mathbb{Q}$ for all $u, v \in M$. It follows that the lattice N generated by $m^\perp \cap M$ and m has rank $n+1$. Therefore, N is a finite index subgroup of M and we write $[M : N]$ for the index. We define the *affine space* $(M_i)_\mathbb{R}$ and the *affine lattice* M_i at height $i \in \mathbb{Z}$ as follows:

$$(M_i)_\mathbb{R} = \frac{i}{[M:N]} m + m^\perp \quad \text{and} \quad M_i = (M_i)_\mathbb{R} \cap M.$$

Since $m^\perp$ and M are G-invariant, we have that M_i is G-invariant for each $i \in \mathbb{Z}$. Note that $M = \bigcup_{i \in \mathbb{Z}} M_i$ is a disjoint union and for each $v \in M_i$ we have $v + M_j = M_{i+j}$.

Example 4. Let $G = \{1, \sigma\} \leq S_4$ be a subgroup of the symmetric group on four letters with $\sigma = (1,2)(3,4)$. The permutation representation ρ maps σ to the permutation matrix

$$\rho(\sigma) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \in \mathrm{GL}_4(\mathbb{R}).$$

In particular, this matrix lies in $\mathrm{GL}_4(\mathbb{Z})$, hence G preserves the lattice $M = \mathbb{Z}[e_1, e_2, e_3, e_4]$. Notice that $m = e_1 + e_2 + e_3 + e_4$ is fixed by the action of G. We compute a basis F that decomposes $\rho(\sigma)$ as a block diagonal matrix:

$$F = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} \right\} \quad \text{and} \quad \rho(\sigma)_F = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & -1 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

The orthogonal lattice M_0 is the 3-dimensional lattice generated by $F \setminus \{m\}$. Observe that the sublattice $N = \mathbb{Z}[F]$ has index 4 inside M. Therefore, the affine lattice $M_1 = (\frac{1}{4}m + (M_0)_{\mathbb{R}}) \cap M$ is equal to the lattice affinely generated by $\{e_1, e_2, e_3, e_4\}$.

3.3 Main setup

Let $M \cong \mathbb{Z}^{n+1}$ be a lattice with a distinguished basis and G a finite group that acts on M by $\rho\colon G \to \mathrm{GL}_{n+1}(\mathbb{Z})$. Assume that there is a lattice point $m \in M \setminus \{0\}$ fixed by G. Let $P \subseteq (M_1)_{\mathbb{R}}$ be a rational G-invariant polytope. For each non-negative integer $k \in \mathbb{Z}_{\geq 0}$, we obtain a permutation representation of the lattice points $kP \cap M \subseteq M_k$ and denote by χ_{kP} its character. The *equivariant Ehrhart series* is an element of the ring of formal power series $R(G)[[t]]$ given by:

$$\mathrm{ehr}_\rho(P, t) = \sum_{k \geq 0} \chi_{kP} t^k = \frac{H^*[t]}{\det[I - t \cdot \rho]} = \frac{H^*[t]}{(1-t)\det[I - t \cdot \rho|_{M_0}]}$$

where $H^*[t] \in R(G)[[t]]$ is the *equivariant H^*-series*. The denominator $\det[I - t \cdot \rho]$ denotes the formal alternating sum $\sum_{i=0}^{n+1} [\Lambda^i M_{\mathbb{R}}](-t)^i \in R(G)[t]$, where $\Lambda^i M_{\mathbb{R}}$ is the i-th alternating power of the representation $M_{\mathbb{R}}$. If the character of the above alternating sum is evaluated at an element $g \in G$, then the resulting polynomial is equal to $\det[I - t \cdot \rho(g)]$ where I is the identity matrix, see [Sta11, Lemma 3.1].

By assumption, $M_{\mathbb{R}} = \langle m \rangle_{\mathbb{R}} \oplus (M_0)_{\mathbb{R}}$ is a G-invariant decomposition of $M_{\mathbb{R}}$. So, for each $g \in G$, we may write $\rho(g) = [1] \oplus \rho(g)|_{M_0}$ as a block diagonal matrix, hence $\det[I - t \cdot \rho(g)] = (1-t)\det[I - t \cdot \rho(g)|_{M_0}]$.

Remark 1. The equivariant Ehrhart series and H^*-series are a generalisation of the usual Ehrhart series and h^*-polynomial. If the equivariant Ehrhart series is evaluated at the identity element, then each character $\chi_{kP}(1_G)$ is equal to the number of lattice points of kP. Since $\det[I - t \cdot \rho(1_G)] = (1-t)^{n+1}$, it follows that the equivariant Ehrhart series evaluated at 1_G is equal to the classical Ehrhart series $\mathrm{ehr}(P, t)$.

The equivariant Ehrhart series contains all the data about the Ehrhart series for fixed sub-polytopes of P. Let $M_{\mathbb{R}}^g = \{x \in M_{\mathbb{R}} \colon g(x) = x\}$ be the subspace of $M_{\mathbb{R}}$ fixed by $g \in G$. For each $k \geq 0$ and $g \in G$, the value $\chi_{kP}(g)$ is the number of lattice points of kP fixed by g. Equivalently, $\chi_{kP}(g)$ is the number of lattice points in the k-th dilate of the *fixed polytope* $P^g = P \cap M_{\mathbb{R}}^g$. Therefore, the evaluation of the equivariant Ehrhart series at $g \in G$ is the Ehrhart series $\mathrm{ehr}(P^g, t)$.

Remark 2. The setup may be equivalently defined by fixing: a group action $\rho|_{M_0}$ of G on a lattice $M_0 \cong \mathbb{Z}^n$; a rational polytope $P \subseteq (M_1)_{\mathbb{R}}$, where $M_1 \cong \mathbb{Z}^n$ is a lattice of the same rank; and a lattice-preserving isomorphism between $(M_1)_{\mathbb{R}}$ and $(M_0)_{\mathbb{R}}$, which induces an action of G on P. We require that, for each $g \in G$, the polytope $g(P) = (-v_g) + P$ differs from P only by a translation $v_g \in M_0$. So, for all $g, h \in G$ we have that

$$(gh)(P) + v_{gh} = P = g(P) + v_g = g(h(P) + v_h) + v_g = (gh)(P) + g(v_h) + v_g,$$

hence $v_{gh} = g(v_h) + v_g$.

We recover the original setup by taking $m \in |G| \cdot P \subseteq (M_{|G|})_{\mathbb{R}}$ to be any G-invariant lattice point of the $|G|$-th dilate of P. Explicitly, for all $g \in G$, we require $g(m) + |G| \cdot v_g = m$. For example, such a point can always be constructed from any lattice point $p \in P$ by summing over the group: $m = \sum_{g \in G} (g(p) + v_g)$. We define M to be the lattice generated by M_0 and M_1 where M_0 is a lattice that contains the origin and M_1 is the affine lattice at height 1 such that the orthogonal projection of $(M_1)_{\mathbb{R}}$ onto $(M_0)_{\mathbb{R}}$ sends $\frac{1}{|G|} m \in (M_1)_{\mathbb{R}}$ to $0 \in M_0$ and differs from the lattice-preserving isomorphism by a translation. Concretely, we may take $M = \mathbb{Z} \times M_0 \cong \mathbb{Z}^{n+1}$ and define the action of G on M by the matrix $\rho(g) = \begin{bmatrix} 1 & 0 \\ v_g & \rho|_{M_0}(g) \end{bmatrix}$. Note that ρ is indeed a group homomorphism. That is, for all g and h in G we have

$$\rho(g)\rho(h) = \begin{bmatrix} 1 & 0 \\ v_g & \rho|_{M_0}(g) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ v_h & \rho|_{M_0}(h) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ g(v_h) + v_g & \rho|_{M_0}(gh) \end{bmatrix} = \rho(gh)$$

since $g(v_h) + v_g = v_{gh}$.

Let $\lambda \in \mathbb{Z}_{>0}$ be the smallest positive integer such that $\frac{\lambda}{|G|} e$ is a lattice point. The value of λ coincides with the index of the sublattice N in M from the original setup.

Example 5 (Continuation of Example 4)**.** Recall $G = \{1, \sigma\} \leq S_4$, with $\sigma = (1,2)(3,4)$, acting by a permutation representation on $M = \mathbb{Z}^4$. Let $P = \text{conv}\{e_1, e_2, e_3, e_4\} \subseteq (M_1)_{\mathbb{R}}$ be a G-invariant 3-dimensional simplex. The permutation character χ_{kP} counts the number of lattice points of $kP \subseteq M_k$ fixed by each $g \in G$. Explicitly, we have

$$\chi_{kP}(1) = \binom{k+3}{3} \quad \text{and} \quad \chi_{kP}(\sigma) = \begin{cases} \frac{k}{2} + 1 & \text{if } 2 \mid k, \\ 0 & \text{otherwise.} \end{cases}$$

Computing the equivariant Ehrhart series, we have

$$\sum_{k \geq 0} \chi_{kP}(1) t^k = \frac{1}{(1-t)^4} \quad \text{and} \quad \sum_{k \geq 0} \chi_{kP}(\sigma) t^k = \frac{1}{(1-t^2)^2}.$$

For each $g \in G$, we observe that the equivariant Ehrhart series is given by $\frac{1}{\det[I - t \cdot \rho(g)]}$. Therefore, the equivariant H^*-series is a polynomial given by $H^*[t] = 1$.

Example 6. Following the alternative setup in Remark 2, let $G = \{1, \sigma\}$ be the group with two elements that acts on a rank 3 lattice $M_0 = \mathbb{Z}[e_1, e_2, e_3]$ by the map

$$\sigma \mapsto \begin{bmatrix} -1 & -1 & -1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

Let $P = \text{conv}\{0, e_1, e_2, e_3\}$ and notice that $\sigma(P) = (-e_1) + P$, hence the above map defines a valid setup. This setup is equivalent to the setup in Example 5, which can be seen as follows. By averaging the vertex $0 \in P$ over G, we obtain the G-invariant point $m = \frac{1}{2} e_1$, verified by the fact that $m = \sigma(e) + e_1$. We define the lattice $M = \mathbb{Z}[e_0, e_1, e_2, e_3]$ and identify the affine sublattice of M containing P with the affine span of $\{e_0+e_1, e_0+e_2, e_0+e_3\}$. In particular, the polytope P is identified in $M_{\mathbb{R}}$ as $\operatorname{conv}\{e_0, e_0+e_1, e_0+e_2, e_0+e_3\}$. The action of G on P extends to an action of G on M given by

$$\sigma \mapsto \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

The point m in $M_{\mathbb{R}}$ is identified with $e_0 + \frac{1}{2}e_1$ which spans a 1-dimensional G-invariant subspace. Observe that the vertices of $P \subseteq M_{\mathbb{R}}$ are a basis for the lattice M. Rewriting the action of G in terms of this basis identifies it with Example 5.

Effectiveness of the equivariant H^*-series. We say that the equivariant H^*-series $H^*[t] = \sum_{i \geq 0} H_i^* t^i \in R(G)[[t]]$ is *effective* if each $H_i^* \in R(G)$ is the isomorphism class of a representation of G. In other words, H_i^* is a non-negative sum of irreducible representations of G. One of the main problems in equivariant Ehrhart theory is to understand when $H^*[t]$ is effective.

Conjecture 1 ([Sta11, Conjecture 12.1])**.** *Let G be a finite group that acts on a lattice and P a G-invariant lattice polytope. Let Y be the toric variety with ample line bundle L associated to P. Then the following are equivalent:*

(1) *L admits a G-invariant section that defines a non-degenerate hypersurface of Y,*

(2) *$H^*[t]$ is effective,*

(3) *$H^*[t]$ is a polynomial.*

It is known that (1) $\Rightarrow$ (2) $\Rightarrow$ (3), see [Sta11], and a counterexample has been constructed by Santos and Stapledon [EKS24, Theorem 1.2] showing that (2) $\not\Rightarrow$ (1) and (3) $\not\Rightarrow$ (1). It is currently open whether (3) $\Rightarrow$ (2).

Another conjecture of interest asks about the character obtained by summing up the coefficients of the H^*-series.

Conjecture 2 ([Sta11, Conjecture 12.2])**.** *Let P be a lattice polytope. If the equivariant H^*-series $H^*[t]$ is effective, then $H^*[1]$ is a permutation representation.*

Part II

On CL-polytopes

Chapter 4

The roots of the Ehrhart polynomials of CL-polytopes

In this chapter we study the distribution of the roots of Ehrhart polynomials of CL-polytopes. The work in this chapter is motivated by the way roots of Ehrhart polynomials are studied in [Bra08] and [BD08] and the first main result confirms a conjecture in [BD08] in the case of CL-polynomials. The chapter also includes brief discussions about attainability of roots, CL-polytopes whose Ehrhart polynomial roots have large magnitude, and a more refined set of inequalities on the magnitude of the Ehrhart polynomial, which is satisfied by many – possibly all – CL-polytopes. The content of this chapter is fully contained in the author's paper [Köl25].

4.1 Background on Ehrhart polynomial roots and the main results

The study of the bounds of Ehrhart polynomial roots goes back to [Bec+05] and starts with the following theorem.

Theorem 4 (Theorem 1.2 in [Bec+05]). *(a) The roots of Ehrhart polynomials of lattice polytopes of dimension d are bounded in norm by $1 + (d + 1)!$.*

(b) All real roots of Ehrhart polynomials of d-dimensional lattice polytopes lie in the half-open interval $\left[-d, \left\lfloor \frac{d}{2} \right\rfloor\right)$.

The authors noticed that this bound was far from being optimal and conjectured, based on experimental data, the following.

Conjecture 3 (Conjecture 1.4 in [Bec+05]). *All roots α of Ehrhart polynomials of lattice polytopes of dimension d satisfy $-d \leq \mathrm{Re}(\alpha) \leq d - 1$.*

This conjecture holds true for the real roots of Ehrhart polynomials of degree 5 or less, but has been disproven in general by counterexamples in [Hig12] and [OS12]. Meanwhile, Braun gave an improvement of the bound in Theorem 4.

Theorem 5 (Theorem 1 in [Bra08]). *If P is a lattice polytope of dimension d, then all the roots of E_P lie inside the disc with centre $-\frac{1}{2}$ of radius $d\left(d-\frac{1}{2}\right)$.*

Braun obtained this result by studying a larger class of polynomials, called *Stanley non-negative polynomials* (*SNN-polynomials*). They are defined as the class of non-zero polynomials f such that $h^*(t) = (1-t)^{\deg f+1}\sum_{k\geq 0} f(k)\, t^k$ has only non-negative coefficients. Notice that for every (not necessarily reflexive) lattice polytope P, E_P lies in $\mathfrak{S}$. SNN-polynomials were also used in [BD08] to give a bound for the imaginary part of Ehrhart polynomial roots.

Theorem 6 (Theorem 2.3 in [BD08]). *For the polynomial $M_d(t) = \binom{t+d}{d} + \binom{t}{d}$, which is not an Ehrhart polynomial, if β_d is the root of $M_d(t)$ of maximal norm, then*

$$\left|\beta_d + \frac{1}{2}\right| = \frac{d^2}{\pi} + O(1)$$

as d goes to infinity.

The authors also conjecture the following.

Conjecture 4 (Conjecture 2.4 in [BD08]). *The root of the polynomial $M_d(t)$ with largest norm has the maximal imaginary part among all roots of degree d polynomials in $\mathfrak{S}$.*

In this study, we will use a similar idea to study the roots of CL-polytopes and define the class $\mathfrak{C} \subset \mathbb{R}[z]$ of *CL-polynomials*. Its elements are the polynomials of the form

$$f(z) = b(z)(z^2+z+c_0)(z^2+z+c_1)\cdots(z^2+z+c_m), \tag{4.1}$$

where the c_k are real numbers $\geq \frac{1}{4}$ and

$$b(z) = \begin{cases} a & \text{if } \deg f \text{ is even,} \\ a(2z+1) & \text{otherwise} \end{cases}$$

for a non-zero real number a. Notice that if P is a CL-polytope, E_P does indeed fall into this class: if $-\frac{1}{2}+\alpha i$ is a root of E_P with $\alpha > 0$, then so is $-\frac{1}{2}-\alpha i$ and E_P is divisible by $z^2+z+\frac{1}{4}+\alpha^2$. If E_P has odd degree, then $-\frac{1}{2}$ is necessarily a root, thus E_P is divisible by $2z+1$. Furthermore, notice that every $f \in \mathfrak{C}$ satisfies Equation (2.3) and thus

$$h^*(t) = (1-t)^{\deg f+1}\sum_{k\geq 0} f(k)\, t^k$$

is a palindromic polynomial.

However, not every CL-polynomial is an SNN-polynomial. For example, for $f(z) = \frac{2}{15}\left(z^2+z+\frac{13}{4}\right)\left(z^2+z+\frac{1}{4}\right)$, we get $h^*(t) = 1+\frac{2}{3}t-\frac{2}{15}t^2+\frac{2}{3}t^3+t^4$. Hence, we will focus on the class $\mathfrak{C}\cap\mathfrak{S}$.

Our first result proves Conjecture 4 in the case of CL-polynomials.

Theorem (Theorem 7)**.** The root of the polynomial $M_d(t)$ with largest norm has the maximal imaginary part among all roots of degree d polynomials in $\mathfrak{C} \cap \mathfrak{S}$.

In Section 3 we present a sufficient condition for a given $f \in \mathfrak{C}$ to lie in $\mathfrak{S}$.

Theorem (Proposition 20)**.** Let f be a CL-polynomial of degree d. Assume that the c_k are ordered by size. Then $f \in \mathfrak{S}$ if the c_k satisfy

$$\frac{1}{4} \leq c_k \leq \begin{cases} 2k+2, & d \text{ is odd} \\ 2k+1, & d \text{ is even.} \end{cases}$$

While this condition is only sufficient, we find a number of examples of CL-polytopes whose Ehrhart polynomials satisfy it.

4.2 Possible roots of polynomials in $\mathfrak{C} \cap \mathfrak{S}$

Let Ω_d denote the set of points $z \in \mathrm{CL}$ such that there exists a polynomial $f \in \mathfrak{C} \cap \mathfrak{S}$ of degree d with $f(z) = 0$. In the course of this section, we will characterise the sets Ω_d for every non-negative integer d, using techniques from [Bra08]. We start with some helpful definitions.

Let a bracketed term with a lower integer index refer to the *Pochhammer symbol* $(x)_j = x(x-1)(x-2)\cdots(x-j+1)$ where $(x)_0 := 1$. For positive integers d and j, we define the functions

$$p_j^d(z) = \begin{cases} (z+d-j)_d + (z+j)_d & \text{if } 2j \neq d, \\ (z+j)_d & \text{if } 2j = d. \end{cases}$$

If a degree d polynomial f is in $\mathfrak{C}$, with the help of Equation (2.2), it can be expressed in terms of the p_j^d;

$$d!\; f(z) = \sum_{k=0}^{\lfloor \frac{d}{2} \rfloor} h_k^* p_k^d(z),$$

where h_k^* refers to the k-th coefficient of the polynomial

$$h^*(t) = (1-t)^{d+1} \sum_{k=0}^{d} h_k^* \, t^k.$$

Notice however, that for $j > 0$, the p_j^d themselves are not in $\mathfrak{C}$.

Lastly, let f be a polynomial with root set $A = \{\alpha_1, \alpha_2, \ldots, \alpha_d\}$. Let the *CL-span* of f denote the set $\operatorname{clspan} f = \operatorname{conv}\{A \cap \mathrm{CL}\}$. If $\operatorname{clspan} f$ is non-empty, it is an interval of CL.

4.2.1 An upper bound for the roots of CL-polynomials

The main result in this subsection is a proof of Conjecture 4 in the case of CL-polytopes. Notice that the polynomials M_d mentioned in this conjecture are equal to the polynomials p_0^d above.

We start with a useful lemma telling us that CL-polynomials take on either exclusively real values or imaginary values on CL, depending on their degree.

Lemma 1. *Let $f \in \mathfrak{C}$ be a degree d polynomial. Then for every $z_0 \in \mathrm{CL}$, we find that $f(z_0) \in \mathbb{R}\, i^d$.*

Proof. We can use once again the functional equation in Lemma 2,

$$f(z-1) = (-1)^d f(-z).$$

Further, we can use that for any $z_0 \in \mathrm{CL}$, $-z_0 - 1 = \overline{z_0}$ holds. Since f has real coefficients, the equality

$$\overline{f(z_0)} = f(\overline{z_0}) = f(-z_0-1) = (-1)^d f(z_0)$$

holds, which implies the statement. □

Remark 3. Notice that this result holds more generally for polynomials with palindromic h^*-polynomials, which includes polynomials not in $\mathfrak{C}$.

Lemma 1 enables us to find roots of $p_d^d\big|_{\mathrm{CL}}$ using a variant of the intermediate value theorem. We use this to study the limit behaviour and the extremal roots these functions. In the following, we will use the convention that t is a real number. Its purpose will be to parametrise CL via $it - \frac{1}{2}$.

Lemma 2. *Let d and j be non-negative integers with $2j \leq d$. Then*

(a) $\lim_{t\to\infty} p_j^d\left(it - \frac{1}{2}\right) i^{-d} = \infty$,

(b) For $2j \neq d$, $p_j^d\left(it - \frac{1}{2}\right) = 0$ if and only if $\left(it - \frac{1}{2} + d - j\right)_{d-2j} \in \mathbb{R}\, i^{d-2j+1}$.

(c) $\operatorname{clspan} p_j^d \subset \operatorname{clspan} p_{j-1}^d$ *for every j with $0 < 2j \leq d$.*

Proof. We begin with (a).

$$p_j^d\left(it - \frac{1}{2}\right) = \begin{cases} \left(it - \frac{1}{2} + d - j\right)_d + \left(it - \frac{1}{2} + j\right)_d & \text{if } 2j \neq d, \\ \left(it - \frac{1}{2} + j\right)_d & \text{if } 2j = d. \end{cases}$$

Observe that this results in a degree d polynomial with leading coefficient $2i^d t$ if $2j < d$ and $i^d t$ if $2j = d$. Multiplying by i^{-d} makes the leading coefficient positive, which proves the statement.

For (b), we start by noticing the identity

$$(z+m-n)_m = (-1)^m \overline{(z+n)_m} \tag{4.2}$$

where m and n are non-negative integers. Next, we rewrite p_j^d as follows.

$$p_j^d(z) = (z+j)_{2j}\left((z-j)_{d-2j} + (z+d-j)_{d-2j}\right) = (z+j)_{2j}\, p_{d-j}^{d-2j}(z). \tag{4.3}$$

Since $\left(it - \frac{1}{2} + j\right)_{2j} \neq 0$ for all t, p_j^d and p_{d-j}^{d-2j} have the same CL-span. It follows that $p_j^d\left(it - \frac{1}{2}\right) = 0$ if and only if

$$\left(it - \frac{1}{2} + d - j\right)_{d-2j} = -\left(it - \frac{1}{2} + -j\right)_{d-2j} = (-1)^{d-2j+1}\overline{\left(it - \frac{1}{2} + d - j\right)_{d-2j}}.$$

The second equality follows from Equation (4.2). From these equalities, we can see that $\left(it - \frac{1}{2} + d - j\right)_{d-2j}$ is an element of $\mathbb{R}\, i^{d-2j+1}$.

For (c), we first notice that if $d = 2j$, p_j^d has an empty CL-span. Without loss of generality, we can assume that d is odd. Thanks to (b), we have

(i) $p_{j-1}^d\left(it - \frac{1}{2}\right) = 0$ if and only if $\left(it - \frac{1}{2} - j + 1\right)_{d-2j+2} \in \mathbb{R}$,

(ii) $p_j^d\left(it - \frac{1}{2}\right) = 0$ if and only if $\left(it - \frac{1}{2} - j\right)_{d-2j} \in \mathbb{R}$.

We make another observation: Statement (b) is equivalent to the following statement.

(b') For $2j \neq d$, $p_j^d\left(it - \frac{1}{2}\right) = 0$ if and only if $\sum_{k=0}^{d-2j-1} \arg\left(it - \frac{1}{2} + d - j - k\right) \in \{0, \pi\}$, where $\arg(z)$ denotes the complex argument of z.

We reverse the order of the sum.

$$\sum_{k=0}^{d-2j-1} \arg\left(it - \frac{1}{2} + d - j - k\right) = \sum_{k=0}^{d-2j-1} \arg\left(it + j + \frac{1}{2} + k\right).$$

Hence, we see that for positive t, we get $0 < \arg\left(it + j + \frac{1}{2} + k\right) < \frac{\pi}{2}$. Also for each k, $\arg\left(it + j + \frac{1}{2} + k\right)$ is monotonic and tends to $\frac{\pi}{2}$ as t tends to ∞. Thus we can rewrite the arguments with error terms $\varepsilon_k(t)$

$$\arg\left(it + j + \frac{1}{2} + k\right) = \frac{\pi}{2} - \varepsilon_k(t).$$

Summarising all this, we can restate (i) and (ii) for positive t.

(i') $p_{j-1}^d\left(it - \frac{1}{2}\right) = 0$ if and only if $\frac{(d-2j)\pi}{2} - \sum_{k=0}^{d-2j-1} \varepsilon_k(t) \in \left\{0, \frac{\pi}{2}\right\}$,

(ii') $p_j^d\left(it - \frac{1}{2}\right) = 0$ if and only if $\frac{(d-2j-2)\pi}{2} - \sum_{k=1}^{d-2j-2} \varepsilon_k(t) \in \left\{0, \frac{\pi}{2}\right\}$.

Since d is odd, $\left\{\frac{(d-2j)\pi}{2}, \frac{(d-2j-2)\pi}{2}\right\} = \left\{\frac{\pi}{2}, \frac{3\pi}{2}\right\}$. As a consequence, $t > 0$ is a root of p_{j-1}^d if and only if $\sum_{k=0}^{d-2j-1} \varepsilon_k(t) \in \left\{\frac{\pi}{2}, \frac{3\pi}{2}\right\}$.

Since the $\varepsilon_k(t)$ are monotonic functions, there exists an $a > 0$ such that

$$\begin{cases} \sum_{k=0}^{d-2j-1} \varepsilon_k(t) = \frac{\pi}{2} & \text{if } t = a, \\ \sum_{k=0}^{d-2j-1} \varepsilon_k(t) < \frac{\pi}{2} & \text{if } t < a. \end{cases}$$

We can conclude that the CL-span of p_{j-1}^d is bounded by the values $\pm ia - \frac{1}{2}$. Finally, we see that

$$\begin{cases} \sum_{k=1}^{d-2j-2} \varepsilon_k(t) < \frac{\pi}{2} & \text{if } t = a, \\ \sum_{k=1}^{d-2j-2} \varepsilon_k(t) < \frac{\pi}{2} & \text{if } t < a, \end{cases}$$

which implies that the values $\pm ia - \frac{1}{2}$ lie outside the CL-span of p_j^d.

□

Finally, we may discuss the bound of the roots.

Theorem 7. *For every degree d polynomial $f \in \mathfrak{C} \cap \mathfrak{S}$,* $\operatorname{clspan} f \subseteq \operatorname{clspan} p_0^d$.

Proof. Let $b > 0$ be a real number such that $ib - \frac{1}{2} \notin \operatorname{clspan} p_0^d$. By Lemma 2(c), we also get $ib - \frac{1}{2} \notin \operatorname{clspan} p_j^d$ for every integer j with $0 < 2j \leq d$. Write

$$i^{-d}\, d!\, f\left(ib - \frac{1}{2}\right) = \sum_{k=0}^{\lfloor \frac{d}{2} \rfloor} h_k^*\, i^{-d}\, p_k^d \left(ib - \frac{1}{2}\right)$$

where the h_k^* are non-negative real numbers. Lemma 2(a) indicates that $i^{-d}\, d!\, f\left(ib - \frac{1}{2}\right)$ is greater than 0 and thus not a root.

□

4.2.2 The standard reflexive simplex

Theorem 1 shows that the polynomials p_0^d are not themselves Ehrhart polynomials of any polytope. Hence it is natural to ask which CL-polytopes have Ehrhart polynomials with large extremal roots. In dimension at most 9, this question can be answered by the *standard reflexive simplex* (see Example 1(b)).

We can write Δ_{sr}^d as a union of simplices

$$\operatorname{conv}\left(\{0\} \cup \left\{e_1, e_2, \ldots, e_d, -\sum_{k=1}^{d} e_k\right\} \setminus \{e\}\right)$$

where e is an element of $\left\{e_1, e_2, \ldots, e_d, -\sum_{k=1}^{d} e_k\right\}$. This is a unimodular triangulation into $d+1$ elements and implies that Δ_{sr}^d has lattice volume $d+1$. Thus $h_{\Delta_{sr}^d}^*(1) = d+1$ (see Introduction) and using Hibi's Lower Bound Theorem, we can see that $h_k^* = 1$ for every coefficient of Δ_{sr}^d.

Proposition 19. *For every reflexive polytope P of dimension $d \leq 9$,* $\operatorname{clspan} E_P \subseteq \operatorname{clspan} E_{\Delta_{sr}^d}$.

Proof. There are two cases: $d \leq 5$ and $5 < d \leq 9$. In the case $d \leq 5$, we verify with a computer that

$$\operatorname{clspan} p_1^d \subset \operatorname{clspan} E_{\Delta_{sr}^d} \subset \operatorname{clspan} p_0^d.$$

Let $ia - \frac{1}{2}$ be the boundary point of $\operatorname{clspan} E_{\Delta_{sr}^d}$ in the upper half plane. Lemma 2(a) implies that for $j > 0$ and $b \geq a$,

$$i^{-d} p_j^d \left(ib - \frac{1}{2} \right) > 0.$$

Assume the Ehrhart polynomial of P is given by

$$E_P(z) = \sum_{k=0}^{\lfloor \frac{d}{2} \rfloor} h_k^* \, p_k^d(z).$$

Since $h_0^* = 1$,

$$i^{-d} E_P \left(ib - \frac{1}{2} \right) > i^{-d} E_{\Delta_{sr}^d} \left(ib - \frac{1}{2} \right) \geq 0.$$

In the case $5 < d \leq 9$, we can verify with a computer that

$$\operatorname{clspan} p_2^d \subset \operatorname{clspan} E_{\Delta_{sr}^d} \subset \operatorname{clspan} p_1^d.$$

Let $ia - \frac{1}{2}$ be the boundary point of $\operatorname{clspan} E_{\Delta_{sr}^d}$ in the upper half plane. Lemma 2(a) implies that for $j > 1$ and $b \geq a$,

$$i^{-d} p_j^d \left(ib - \frac{1}{2} \right) > 0.$$

Assume the Ehrhart polynomial of P is given by

$$E_P(z) = \sum_{k=0}^{\lfloor \frac{d}{2} \rfloor h_k^* \, p_k^d(z)} .$$

Since $h_0^* = 1$ and, by Hibi's Lower Bound Theorem, $h_k^* \geq h_1^*$ for $k > 1$, we get

$$i^{-d} E_P \left(ib - \frac{1}{2} \right) \geq h_1^* \sum_{k=1}^{d} i^{-d} p_k^d \left(ib - \frac{1}{2} \right) + i^{-d} p_0^d \left(ib - \frac{1}{2} \right) > i^{-d} E_{\Delta_{sr}^d} \left(ib - \frac{1}{2} \right) \geq 0.$$

□

For higher degrees, it is no longer true that $\operatorname{clspan} p_2^d \subset \operatorname{clspan} E_{\Delta_{sr}^d} \subset \operatorname{clspan} p_1^d$ and thus Hibi's Lower Bound Theorem can no longer guarantee that the h_k^* for $k \geq 3$ are large enough to balance out h_3^*. In particular, in degree 10, for $2 \leq m \leq 14$ the polynomial

$$f(z) = p_0^5(z) + p_1^5(z) + m p_2^5(z) + p_3^5(z) + p_4^5(z) + p_5^5(z)$$

is a CL-polynomial whose extremal roots have a larger absolute imaginary part than those of the Ehrhart polynomial of Δ_{sr}. We still conjecture the following.

Conjecture 5. *Let P be a reflexive polytope of dimension d whose h^*-polynomial is unimodal, i.e., satisfies the inequalities*

$$h_0^* \leq h_1^* \leq \cdots \leq h^*_{\lfloor \frac{d}{2} \rfloor} \geq \cdots \geq h^*_{d-1} \geq h_d^*$$

where h_k^* is the k-th coefficient of h_P^*. Then $\operatorname{clspan} E_P \subseteq \operatorname{clspan} E_{\Delta^d_{sr}}$.

Remark 4. The following table compares the maximal roots $i\beta_d - \frac{1}{2}$ of $E_{\Delta^d_{sr}}$ to the maximal roots $i\alpha_d - \frac{1}{2}$ of p_0^d to the bounds from Theorems 6 and 5. The values were obtained using SAGEMATH [The22].

d	α_d	β_d	$\frac{d^2}{\pi}$	$d(d-\frac{1}{2})$
2	0.866	0.645	1.273	3
3	2.398	1.658	2.865	7.5
4	4.603	3.040	5.093	14
5	7.457	4.761	7.958	22.5
6	10.952	6.811	11.459	33
7	15.085	9.186	15.597	45.5
8	19.857	11.882	20.372	60
9	25.267	14.899	25.783	76.5
10	31.313	18.236	31.831	95
20	126.802	69.147	127.324	390
30	285.956	151.904	286.479	885
100	3182.575	1622.493	3183.099	9950
150	7161.449	3627.845	7161.972	22425

4.2.3 Connectedness of the set of possible roots

We return to the characterisation of the sets Ω_d we defined in the very beginning of this section. After establishing a sharp bound, it is natural to ask, which values on CL within that bound can be assumed by the roots of an appropriate degree d polynomial in $\mathfrak{C}$.

Lemma 3. *For any positive integer d, p_0^d CL-interlaces p_0^{d+1}.*

Proof. Equation (2.2) tells us that

$$h^*_{p_0^d}(t) = (1+t)^{d+1} \sum_{k\geq 0} p_0^d(k)\, t^k = d!\left(1+t^d\right).$$

An analogous results holds for p_0^{d+1}. The roots of $h^*_{p_0^d}$ and $h^*_{p_0^{d+1}}$ are $\exp\left(\frac{(1+2n)\pi}{d} i\right)$ and $\exp\left(\frac{(1+2n)\pi}{d+1} i\right)$ respectively, where n ranges from 0 to $d-1$ (resp. d). These roots interlace on the unit circle and hence, by Proposition 9, they interlace on CL. □

Lemma 4. *For any positive integer d and every positive real number w, p_0^d CL-interlaces by $p_0^{d+1} + w(2z+1)p_0^d$.*

Proof. Since w is positive, we can without loss of generality assume that $w = 1$. We start with the case when d is odd. From Lemma 3 we know that p_0^d CL-interlaces p_0^{d+1}. Further, $2z + 1$ trivially CL-interlaces $(2z + 1)^2$. Since

$$p_0^{d+1}\left(-\frac{1}{2}\right) = \left(-\frac{1}{2} + d + 1\right)_{d+1} + \left(-\frac{1}{2}\right)_{d+1}$$

is not a root, p_0^{d+1} does not share a root with $(2z+1)^2$. Hence, by Proposition 10, $(2z+1)p_0^d$ interlaces $(2z + 1)(p_0^{d+1} + (2z + 1)p_0^d)$. Dividing $2z + 1$ from both expressions yields the statement.

In the case where d is even, p_0^{d+1} has a root at $-\frac{1}{2}$ due to symmetry. The root has multiplicity 1, because if it had a higher multiplicity, p_0^d would need to have two roots at $-\frac{1}{2}$ as well due to interlacing, but we already saw that this is not the case. Hence we define polynomials

$$g_k(z) = z^2 + z + \frac{1}{4} + \varepsilon_k^2$$

where $\varepsilon_1 > \varepsilon_2 > \cdots$ is a sequence of positive reals that goes to 0. The roots of g_k are $-\frac{1}{2} \pm \epsilon_k i$. Hence, they are CL-interlaced by $2z+1$ and with appropriately chosen ϵ_k, none of them have a common root with p_0^{d+1}. Hence, by Proposition 10, $(2z + 1)p_0^d$ interlaces $(2z + 1)p_0^{d+1}(z) + p_0^d(z)\, g_k(z)$. Using Proposition 11, we get that $(2z + 1)(p_0^{d+1}(z) + (2z + 1)p_0^d(z))$ interlaces $(2z + 1)p_0^{d+1}(z)$ and dividing by $2z + 1$ again yields the statement. ☐

Lemma 5. *Let f be a degree d SNN-polynomial. Then $(2z + 1)f(z)$ is also an SNN-polynomial.*

Proof. Since by Equation (2.2) f is a non-negative linear combination of polynomials $\binom{z+d-k}{d}$, we may restrict ourselves to these. Using $z = (z + d - k + 1) - (d - k + 1)$ and then

$$\binom{z+d-k}{d} = \binom{z+d-k+1}{d+1} - \binom{z+d-k}{d+1},$$

we get

$$z\binom{z+d-k}{d} = k\binom{z+d-k+1}{d+1} + (d-k+1)\binom{z+d-k}{d+1}$$

and thus

$$(2z+1)\binom{z+d-k}{d} = (2k+1)\binom{z+d-k+1}{d+1} + (2(d-k)+1)\binom{z+d-k}{d+1}.$$

This is a positive linear combination of polynomials $\binom{z+(d+1)+k}{d+1}$. Hence, $(2z+1)f(z)$ is an SNN-polynomial. ☐

With these three lemmas, we can prove the main statement of this subsection. For simplicity, we will use the convention

$$h_f^*(t) = (1+t)^d \sum_{k\geq 0} f(k)\, t^k$$

for any degree d polynomial f.

Theorem 8. *For every positive integer d, Ω_d is connected.*

Proof. In the case $d = 1$, $\Omega_1 = \{-\frac{1}{2}\}$ is a singleton and hence connected.

Consider the case $d = 2$. Let c be a positive real number. Then $h^*_{f_c}(t) = 1 + ct + t^2$ corresponds to an SNN-polynomial f_c whose roots are $-\frac{1}{2} \pm \frac{\sqrt{c^2-4c-12}}{2c+4}$. For $0 \leq c \leq 6$, the roots of f_c lie on CL and f_0 is exactly p_0^2, which marks the boundary of Ω_2. The roots of f_6 are both $-\frac{1}{2}$. Since the root depend continuously on c, Ω_2 is connected.

The proof for higher degrees $d+1$ can be built inductively. First, take an element $z_0 \in \Omega_d$ and a degree d polynomial $f \in \mathfrak{C} \cap \mathfrak{S}$ with $f(z_0) = 0$. The polynomial $g(z) = (2z+1)f(z)$ is a degree $d+1$ polynomial with $g(z_0) = 0$ and it is in $\mathfrak{C}$. By Lemma 5, g lies also in $\mathfrak{S}$ and thus, $z_0 \in \Omega_{d+1}$.

Now, pick $z_0 = ci - \frac{1}{2} \in \operatorname{clspan} p_0^{d+1} \setminus \Omega_d$ in the upper half plane. Denote the roots of p_0^d by $b_k i - \frac{1}{2}$ where $b_m < b_n$ if $m < n$. Analogously, we denote the roots of p_0^{d+1} by $a_k i - \frac{1}{2}$. From Lemma 3, it follows that $a_d < b_d < c < a_{d+1}$. Define the function $g(z) = (2z+1)p_0^d(z)$. Lemmas 1 and 2(a) imply

$$i^{-d-1}p_0^{d+1}(z_0) < 0 \quad \text{and} \quad i^{-d-1}(2z_0+1)p_0^d(z_0) > 0$$

Thus, for an appropriate number $w > 0$, the linear combination

$$\lambda(z) = p_0^{d+1}(z) + w(2z+1)p_0^d(z)$$

satisfies $\lambda(z_0) = 0$. In particular, $\lambda \in \mathfrak{S}$. Since by Lemma 3 λ is interlaced by p_0^d, it follows that $\lambda \in \mathfrak{C}$. Thus $z_0 \in \Omega_{d+1}$. □

4.3 Inequalities for $\mathfrak{C} \cap \mathfrak{S}$

In Equation (4.1), we define CL-polynomials in terms of parameters $c_k \geq \frac{1}{4}$. Every c_k corresponds to a pair of roots $-\frac{1}{2} \pm \sqrt{c_k - \frac{1}{4}}i$, which is a fact we used several times throughout the previous section. Thus, Theorem 7 can be interpreted as an inequality that gives a necessary condition for the c_k to correspond to an SNN-polynomial.

Theorem (Restatement of Theorem 7)**.** Let f be a CL-polynomial of degree d with parameters c_k. If $f \in \mathfrak{S}$, the inequality

$$c_k \leq m_0^d$$

is satisfied for every k, where m_0^d is the maximal parameter of p_0^d.

However, this is very far from being sufficient. For example, the polynomial $f(z) = \frac{1}{400}(z^2+z+20)^2$ has its roots at around $-\frac{1}{2} \pm 4.44i$, which by the table in Remark 4 lies within Ω_4, but

$$h^*_f(t) = 1 - \frac{379}{100}t + \frac{564}{100}t^2 - \frac{379}{100}t^3 + t^4.$$

In the following, we give a sufficient condition. We base it on a computational lemma.

Lemma 6. *Let d be a positive integer and $j \leq d$ be a non-negative integer. Further, let $c \geq \frac{1}{4}$ be a real number. Then*

$$(z^2 + z + c)\binom{z+d-j}{d} = \alpha\binom{z+d-j+2}{d+2} + \beta\binom{z+d-j+1}{d+2} + \gamma\binom{z+d-j}{d+2}$$

where

$$\begin{aligned}\alpha &= j^2 + j + c,\\ \beta &= 2(dj - j^2 + d + 1 - c),\\ \gamma &= d^2 - 2dj - j + j^2 + d + c.\end{aligned}$$

Proof. By adapting the technique used in Lemma 5,

$$(z+c)\binom{z+d-j}{d} = (j+c)\binom{z+d-j+1}{d+1} + (d-j+1-c)\binom{z+d-j}{d+1}.$$

Equation (2.2) implies that

$$\sum_{k\geq 0}\binom{k+d-j}{d} t^k = \frac{t^j}{(1-t)^{d+1}},$$

which means that we can write

$$\sum_{k\geq 0}(k+c)\binom{k+d-j}{d} t^k = \frac{(j+c)t^j + (d-j+1-c)t^{j+1}}{(1-t)^{d+2}}.$$

We can use the same identity to compute $\sum_{k\geq 0} k^2\binom{k+d-j}{d} t^k$ by applying it twice with $c = 0$ both times.

$$\sum_{k\geq 0} k^2\binom{k+d-j}{d} t^k = \frac{j^2t^j + \Big(j(d+2-j) + (d+1-j)(j+1)\Big)t^{j+1} + (d+1-j)^2t^{j+2}}{(1-t)^{d+3}}$$

Summing up gives the values for α, β, and γ as stated. □

Proposition 20. *Let f be a CL-polynomial of degree d. Assume that the c_k are ordered by size. Then $f \in \mathfrak{S}$ if the c_k satisfy*

$$\frac{1}{4} \leq c_k \leq \begin{cases} 2k+2, & d \text{ is odd} \\ 2k+1, & d \text{ is even.}\end{cases} \tag{4.4}$$

Proof. The proof proceeds inductively. The idea is to take a degree d element f of $\mathfrak{C} \cap \mathfrak{S}$ and multiply it with $z^2 + z + c$ where c is chosen so that it preserves non-negativity of the coefficients of the h_f^*. That is in particular the case when the three factors from Lemma 6, $\alpha = j^2 + j + c$, $\beta = 2(dj - j^2 + d + 1 - c)$, and $\gamma = d^2 - 2dj - j + j^2 + d + c$, are non-negative.

Since c is positive, α and γ are always non-negative. For β, the largest possible choice for c is $d+1$ since j ranges from 0 up to d.

To complete the induction, we only have to look at the cases of $d = 1$ and $d = 2$. We start with the former. If f has degree 1, it is of the form $z + \frac{1}{2}$ and $h_f^*(t) = 1 + t$. Thus $c_0 \leq 2$, $c_1 \leq 4$, $c_2 \leq 6$ etc. If f has degree 2, it is of the form $z^2 + z + c_0$ and has $h_f^*(t) = c_0 + 2(1 - c_0)t + c_0 t^2$. Thus, $c_0 \leq 1$, $c_1 \leq 3$, $c_2 \leq 5$ etc. □

The class of CL-polynomials that satisfy this proposition trivially includes the Ehrhart polynomials $E_{[-1,1]^d}$ of reflexive hypercubes since they satisfy

$$c_1 = c_2 = \cdots = c_d = \frac{1}{4}.$$

It is possible to construct further examples.

Example 7. Let P be a CL-polytope of dimension d. Then there exists a non-negative integer n, such that the Ehrhart polynomial of $P \times [-1,1]^n$ satisfies Inequalities (4.4).

If E_P is defined by the parameters $c_1, c_2, \ldots, c_d$, then $E_{P \times [-1,1]^n}$ is defined by the parameters $\frac{1}{4}, \frac{1}{4}, \ldots, \frac{1}{4}, c_1, c_2, \ldots, c_d$ where $\frac{1}{4}$ appears n times. For the c_k, this changes the equations to

$$\frac{1}{4} \leq c_k \leq \begin{cases} 2(k+n)+2, & d \text{ is odd} \\ 2(k+n)+1, & d \text{ is even.} \end{cases}$$

which is always satisfied for a sufficiently large n.

Using the same idea, we also get another example.

Example 8. Let P be a CL-polytope of dimension d and let Q be a CL-polytope of dimension $2m+1$. Then there exists a non-negative integer n, such that the Ehrhart polynomial of $P \times Q^n$ satisfies Inequalities (4.4).

However, there exist counter-examples as well. The Ehrhart polynomial of standard reflexive 4-simplex Δ_{sr}^4 can be written as

$$E_{\Delta_{sr}^4}(k) = \frac{5}{24}(x^2 + x + 0.505558989151154)(x^2 + x + 9.49444101084885),$$

which does not satisfy the inequalities.

Chapter 5

The CL-property of symmetric edge polytopes from complete multipartite graphs

In this chapter we investigate the Ehrhart polynomial roots of symmetric edge polytopes from complete multipartite graphs. We compute the h^*-polynomials of a number of classes of symmetric edge polytopes of complete multipartite graphs and confirm a conjecture from [HKM17] for some of them. Also we systematise the interlacing methods from [HKM17] and show their limitations. The content of this chapter is entirely contained in the author's paper [Köl24].

5.1 A conjecture about interlacing polynomials and the main result

In [HKM17], the authors studied the roots of the Ehrhart polynomials of symmetric edge polytopes of the complete bipartite graphs $K_{2,n}$ and $K_{3,n}$ and were able to prove that both these classes are CL-polytopes. This extends the case of cross-polytopes, which are the symmetric edge polytopes of $K_{1,n}$. They accomplished that by using the technique of *interlacing polynomials*, i.e., polynomials whose roots alternate on a given totally ordered set. For an in-depth treatment of the theory of interlacing polynomials, see [Fis06]. The authors gave the following conjecture.

Conjecture 6 (Conjecture 4.10 in [HKM17]). *(i) For any complete multipartite graph $K_{a_1,\dots,a_k}$ the Ehrhart polynomial $E_{a_1,\dots,a_k}$ has its roots on CL.*

(ii) Suppose $a_1 \leq \cdots \leq a_k$. Any two Ehrhart polynomials $E_{a_1,\dots,a_k}$ and $E_{a_1,a_2,\dots,a_k-1}$ interlace on CL.

After finding a general formula for the h^*-polynomial of symmetric edge polytopes of complete tripartite graphs in the Section 5.2, we confirm Conjecture 6 partially in our main result.

Theorem 9. *The following statements hold for every positive integer n.*

(a) $E_{1,n}$ *CL-interlaces* $E_{1,1,n}$.

(b) $E_{1,1,n}$ *CL-interlaces* $E_{1,1,n+1}$.

(c) $E_{1,1,n}$ *CL-interlaces* $E_{1,2,n}$.

(d) $E_{1,1,n}$ *CL-interlaces* $E_{1,1,1,n}$.

(e) $E_{3,n}$ *CL-interlaces* $E_{4,n}$ *if* $E_{1,n+1}$ *CL-interlaces* $E_{3,n}$.

(f) $E_{1,2,n}$ *CL-interlaces* $E_{1,3,n}$ *if* $E_{1,n+1}$ *CL-interlaces* $E_{1,2,n}$.

(g) $E_{1,2,n}$ *CL-interlaces* $E_{2,2,n}$ *if* $E_{1,n+1}$ *CL-interlaces* $E_{1,2,n}$.

In particular, for every positive integer n, $E_{x,y,z,n}$ is a CL-polynomial for $x + y + z \leq 3$ and $x, y, z \geq 0$.

Finally, in Section 5.4, we investigate a connection between the γ-vector of the h^*-polynomial of an Ehrhart polynomial and the existence of recursive relations that can be used to prove interlacing. In particular, in Theorem 12 and Corollary 1, we show that the type of recursive relations in [HKM17] and Proposition 23 can be found for arbitrary complete bipartite graphs.

5.2 A Reduced Gröbner Basis

We start by describing an edge ordering. First we denote the multipartite classes of vertices of $K_{a_1,\dots,a_k}$ by $A_1, A_2, \dots, A_k$ and then we pick an ordering of the vertices which satisfies the following condition. If $v \in A_i$ and $w \in A_j$, then $v < w$ if and only if $i < j$. Let $e = \{v, w\}$ and $e' = \{v', w'\}$ be edges in $K_{a_1,\dots,a_k}$. Without loss of generality, we may assume $v < w$ and $v' < w'$. Then $e < e'$ if and only if $v < v'$ or $v = v'$ and $w < w'$.

Let a, b be vertices with an edge between them. We will denote by $x_{a,b}$ the directed edge from a to b and by $x_{b,a}$ the edge going the other way. The variable which corresponds to the unique interior lattice point of $P_{K_{a_1,\dots,a_k}}$ will be denoted by z.

Theorem 10. *With the described edge ordering, the Gröbner basis from Proposition 4 is at most cubic for every complete multipartite graph $K_{a_1,a_2,\dots,a_k}$. The elements of the reduced Gröbner basis are of the following form.*

(1) Let $a \in A_i$ and $b \in A_j$ with $i \neq j$. Then the following polynomial is a Gröbner basis element.

$$x_{a,b}x_{b,a} - z^2$$

(2) Let $a \in A_i, b \in A_j, c \in A_\ell$ with i, j, ℓ all different. Then the following polynomial is a Gröbner basis element.

$$x_{a,b}x_{b,c} - zx_{a,c}$$

(3) Let a, b, c, d be vertices such that the edges $\{a, b\}, \{b, c\}, \{c, d\}, \{a, d\}$ all exist and a is the smallest vertex. Then the following polynomial is a Gröbner basis element if and only if b and d lie in the same A_i.

$$x_{b,c}x_{c,d} - x_{b,a}x_{a,d}$$

We call these polynomials Gröbner basis elements of type (3a). *Furthermore, the following polynomial is a Gröbner basis element if and only if $b < d$.*

$$x_{b,c}x_{d,a} - x_{b,a}x_{d,c}$$

We call these polynomials Gröbner basis elements of type (3b). *In particular, a, b, c, d lie across either 2, 3, or 4 multipartite classes.*

(4) Let a, b, c, d, e be vertices such that the edges $\{a, b\}$, $\{b, c\}$, $\{c, d\}$, $\{d, e\}$, $\{a, e\}$ all exist. Then the following polynomial is a Gröbner basis element if and only if $a, b, c \in A_1 \cup A_2$, a, c lie in the same A_i, and b is the smallest vertex in A_1 or A_2.

$$x_{a,b}x_{b,c}x_{d,e} - z\, x_{d,c}x_{a,e}$$

In particular, a, b, c, d, e lie across either 3 or 4 multipartite classes.

(5) Let a, b, c, d, e, f be vertices such that the edges $\{a, b\}$, $\{b, c\}$, $\{c, d\}$, $\{d, e\}$, $\{e, f\}$, $\{a, f\}$ all exist. Then the following polynomial is a Gröbner basis element if and only if

(i) c and f lie in the same A_i,

(ii) $b > d$, or b and e lie in the same A_i, or $c < e$,

(iii) a and d lie in the same A_i, or $f < d$.

$$x_{a,b}x_{b,c}x_{d,e} - z\, x_{d,c}x_{a,e}$$

In particular, a, b, c, d, e lie across either 3, 4, or 5 multipartite classes.

More generally, for every complete multipartite graph $K_{a_1,\ldots,a_k}$ which contains $K_{2,2,2}$ as a subgraph, the Gröbner basis in Proposition 4 has an element of degree 3 regardless of the edge ordering.

Proof. One can check that all the listed elements indeed come from directed cycles in the way described in Proposition 4. To check the reducedness of a Gröbner basis element p, it is enough to find another element q of lower degree such that $\mathrm{lt}(q) | \mathrm{lt}(p)$, where $\mathrm{lt}(p)$ and $\mathrm{lt}(q)$ are the leading terms of p and q respectively. Since all elements are of degree at least 2, we can see that (1), (2), and (3) are indeed not redundant. For (4) and (5) we may notice that the given restrictions correspond to indivisibility of the polynomials by the leading terms of Gröbner basis elements of type (2) or (3).

Now we can go on to show that no further elements are contained in the Gröbner basis. Firstly, let C be a directed cycle of length 7 or greater. Assume the set I which defines the leading term of the polynomial $p_{C,I}$ contains two adjacent directed edges (a, b) and (b, c). One can check that there exists an element of type (2) or (3) whose leading term contains these edges unless a and c both lie in A_i and b is the smallest vertex in A_j with $\{i, j\} = \{1, 2\}$. In this case, C cannot be even because I contains the smallest edge. That means that there exists a vertex d not in A_1 or A_2. Further, we assume that (a, b) and (b, c) are the only pair of adjacent edges in C. Due to the size constraint of I, every vertex other than b is part of one directed edge in I. Thus, there exists a directed edge (d, e) or (e, d) whose associated variable forms the leading term of a Gröbner basis element of type (4) together with $x_{a,b}$ and $x_{b,c}$.

Next, let C be a directed cycle of length 8 or greater. We assume that the set I which defines the leading term of the polynomial $p_{C,I}$ contains no adjacent directed edge. Thus, C is necessarily an even cycle. We denote the vertices of the cycle in order by $a_0, b_0, a_1, b_1, \ldots, a_n, b_n$ such that a_0 is the smallest vertex and we can assume up to orientation that $(b_i, a_{i+1}) \in I$ for $0 \leq i \leq n-1$ and $(b_n, a_0) \in I$. If a_1 and b_i for $i > 1$ lie in the same multipartite class, we get a smaller cycle C' containing $a_0, b_0, a_1, b_i, a_{i+1}, \ldots, b_n$ with $I' \subset I$ such that $\operatorname{lt}(p_{C',I'}) | \operatorname{lt}(p_{C,I})$. Thus, we may assume that a_1 and b_i for $i > 1$ all lie in the same multipartite class. This puts a_2 in a different class from b_n. As a consequence, the directed cycle C' on the vertices $a_0, b_0, a_1, b_1, a_2, b_n$ with $I' = \{(b_0, a_1), (b_1, a_2), (b_n, a_0)\}$ yields a polynomial $p_{C',I'}$ whose leading term divides that of $p_{C,I}$.

Lastly, we prove the second part of the theorem. Let $K_{2,2,2}$ be a subgraph of $K_{a_1,\ldots,a_k}$. Then there exists a directed 6-cycle with vertices a, b, c, d, e, f with edges

$$(a, b), (b, c), (c, d), (d, e), (e, f), (f, a)$$

such that (a, b) is the smallest edge of $K_{2,2,2}$. This gives rise to the polynomial $c = x_{b,c}x_{d,e}x_{f,a} - x_{b,a}x_{d,c}x_{f,e}$ which is an element of the Gröbner basis. We can verify that there does not exist another Gröbner basis element whose leading monomial divides that of c. □

If we want to use this Gröbner basis to find a unimodular triangulation, we may notice that not all elements need to be considered. We know that for every unimodular simplex in the triangulation, its vertices that lie in the boundary of P_G all lie within the same facet. Further we know by Proposition 3 that these facets are given by labelings of the vertices of $K_{a_1,\ldots,a_k}$ which satisfy specific conditions. Indeed, edge configurations induced by Gröbner basis elements of types (1), (2), and (4) do not occur in any facet-inducing spanning subgraph. Configurations induced by elements of type (5) only appear in spanning trees of type (ii). Among the configurations induced by elements of type (3), both varieties appear in facet-inducing spanning subgraphs of type (i), whereas in type (ii) only type (3b) elements appear.

5.2.1 Simplices in type (i) facets

To start, we will establish some terminology and notation. Let $G = (V, E)$ be a graph and P_G its symmetric edge polytope. As seen in Proposition 3, a facet of P_G is induced by an integer valued function on V. We will henceforth call such a function a *labeling* on V. We will denote it with a lowercase Greek letter such as λ. Following this, we call a vertex v *ℓ-labled* if $\lambda(v) = \ell$. The facet of P_G induced by λ shall be denoted by $\mathcal{F}_\lambda$. The spanning subgraph of G induced by λ shall be denoted by $G|_\lambda$. The simplices in the unimodular triangulation of P_G will be denoted by the symbol Δ and the associated directed spanning tree by T_Δ. For the unimodular triangulation itself, we will write $\mathcal{T}$. Given a labeling λ, the set $\mathcal{T}_\lambda \subset \lambda$ is the set of simplices which lie in $\mathcal{F}_\lambda$. Lastly, we define the set $\mathcal{T}_{(i)}$ as the union of all the $\mathcal{T}_\lambda$ where λ is a type (i) facet, and the set $\mathcal{T}_{(ii)}$ analogously.

The following definition should be viewed with an eye toward Gröbner basis elements of type (3b): Let λ be a facet-inducing labeling and let A and B be sets of vertices such that no $a \in A$ lies in the same multipartite class as a $b \in B$. Further, assume that $\lambda(a) = \lambda(b) - 1$ for every $a \in A$ and $b \in B$, and that $\lambda|_A$ and $\lambda|_B$ are constant. The spanning subgraph corresponding to this situation would contain a directed edge from every element of A to every element of B. The following definition tells us which subsets of edges from A to B can be included in "legal" spanning trees with respect to the Gröbner basis from Theorem 10.

Definition 11. Let A and B be disjoint finite totally ordered sets. A *planar spanning tree* between A and B is a subset E of $A \times B$ such that

(i) $|E| = |A \cup B| - 1$,

(ii) every element of A and B is contained in at least one element of E,

(iii) if (a, b) and (a', b') are elements of E, then $a < a'$ implies $b < b'$.

The number of planar spanning trees is $\binom{a+b-2}{b-1}$ where a and b are the cardinalities of A and B respectively.

Proposition 21. *Let $K_{a_1,a_2,\ldots,a_k}$ be a complete multipartite graph with multipartite classes of vertices $A_1, A_2, \ldots, A_k$ and let $P_{K_{a_1,a_2,\ldots,a_k}}$ be its associated symmetric edge polytope.*

Then the polynomial $h^{(i)}_{a_1,a_2,\ldots,a_k} = \sum_{\Delta \in \mathcal{T}_{(i)}} t^{\mathrm{in}(T_\Delta)}$ is given by

$$h^{(i)}_{a_1,a_2,\ldots,a_k}(t) = \sum_{i=0}^{a-a_i-1} \sum_{j=1}^{a_1-1} p(a, a_1, i, j) \binom{a_1 + j - i - 2}{j - 1} \left(t^{i+j+1} + t^{a-i-j-2}\right)$$
$$+ \sum_{m=2}^{k} \sum_{i=0}^{a-a_m-1} \sum_{j=1}^{a_m-1} p(a, a_m, i, j) \binom{a - a_m + j - i - 2}{a - a_m - i - 1} \left(t^{i+j} + t^{a-i-j-1}\right)$$

where $\mathrm{in}(T)$ *is the number of ingoing edges of T, $a = a_1 + a_2 + \cdots + a_k$ and*

$$p(x, y, i, j) = \binom{x - y - 1}{i} \binom{y - 1}{j} \binom{y + i - j - 1}{i}.$$

Proof. We fix a labeling $\lambda\colon \bigsqcup_{j=1}^{k} A_j \to \{-1,0,1\}$ corresponding to a facet of type (i). This means that for one A_m we get $\lambda|_{A_m} = \{-1,1\}$ and all the remaining vertices are mapped to 0. For the graph $K_{a_1,a_2,\dots,a_k}|_\lambda$ this means that every vertex $v \notin A_m$ has an edge (v,w) if w is 1-labeled and an edge (w,v) if w is -1-labeled. On the other hand, all the vertices in A_m only have edges leading into them or out of them, depending on their labeling. Consider now a spanning tree T_Δ with $\Delta \in \mathcal{T}_\lambda$. Within T_Δ, the Gröbner basis elements of type (3a) block every vertex 0-labeled vertex v (with one exception) from having edges of the form (v,w) and (w,v) at the same time. The exception is the smallest 0-labeled vertex in the graph, which we will denote by v_0. Thus, we obtain two subsets of $\bigsqcup A_j \setminus A_m$: the subset P of vertices v whose edges are of the form (v,w), and the subset N of vertices v whose edges are of the form (w,v). In particular, $P \cap N = \{v_0\}$. With these conditions A_m naturally splits into two disjoint subsets $A_m^+ = \{v \in A_m \colon \lambda(v) = 1\}$ and $A_m^- = \{v \in A_m \colon \lambda(v) = -1\}$. Taking the Gröbner basis elements of type (3b) into account, T_Δ is the disjoint union of a planar spanning tree between P and A_m^+ and a planar spanning tree between N and A_m^-.

Next, we want to count the number of ingoing edges. Let r denote the smallest vertex in $\bigsqcup A_j$ and let v be some element in P different from r. The edge e containing v in the unique path from r to v is ingoing. In a similar way, if v is any element in A_m^- different from r, the edge containing v in the unique path from r to v is also ingoing. Every other edge is outgoing. We get a total of four cases:

(a) $m = 1$ and $\lambda(\min A_m) = 1$,

(b) $m = 1$ and $\lambda(\min A_m) = -1$,

(c) $m > 1$ and $\lambda(\min A_m) = 1$,

(d) $m > 1$ and $\lambda(\min A_m) = -1$.

Notice that if Case (a) applies, the direction of all edges can be reversed and it results in another spanning tree $T_{\Delta'}$ with $\Delta' \in \mathcal{T}_{(i)}$ for which Case (b) applies and vice versa. The same holds for Cases (c) and (d). Thus, we get $\operatorname{in}(T_{\Delta'}) = a - 1 - \operatorname{in} T_\Delta$, which means that when we count the elements in $\mathcal{T}_{(i)}$ with the number of their respective ingoing edges, we can fix without loss of generality the value for $\lambda(\min A_m)$.

$\mathbf{m = 1}$: We choose $\lambda(\min A_1) = \lambda(r) = 1$. Let i denote the number of vertices in $P \setminus N$ and let j denote the number of vertices in A_1^-. The number of ingoing edges in this situation is $i + 1 + j$. This gives rise to the first line of the formula $h^{(i)}$: we sum over all possible choices of i and j and multiply the number of ways to pick $P \setminus N$, the number of ways to pick A_1^-, the number of planar spanning trees between P and A_1^+, the number of planar spanning trees between N and A_1^-, and polynomial $t^{i+j+1} + t^{a-i-j-2}$, which counts the number of ingoing edges in Cases (a) and (b).

$\mathbf{m > 1}$: We choose $\lambda(\min A_m) = 1$. Again, we let i denote the number of vertices in $P \setminus N$ and let j denote the number of vertices in A_1^-. In this case the number of ingoing edges is $i + j$ because r itself is an element of P now and thus cannot be counted in. By an analogous statement to the on in the previous case and by summing over all the A_m with $m > 1$, we get the second line of the formula which concludes the proof. □

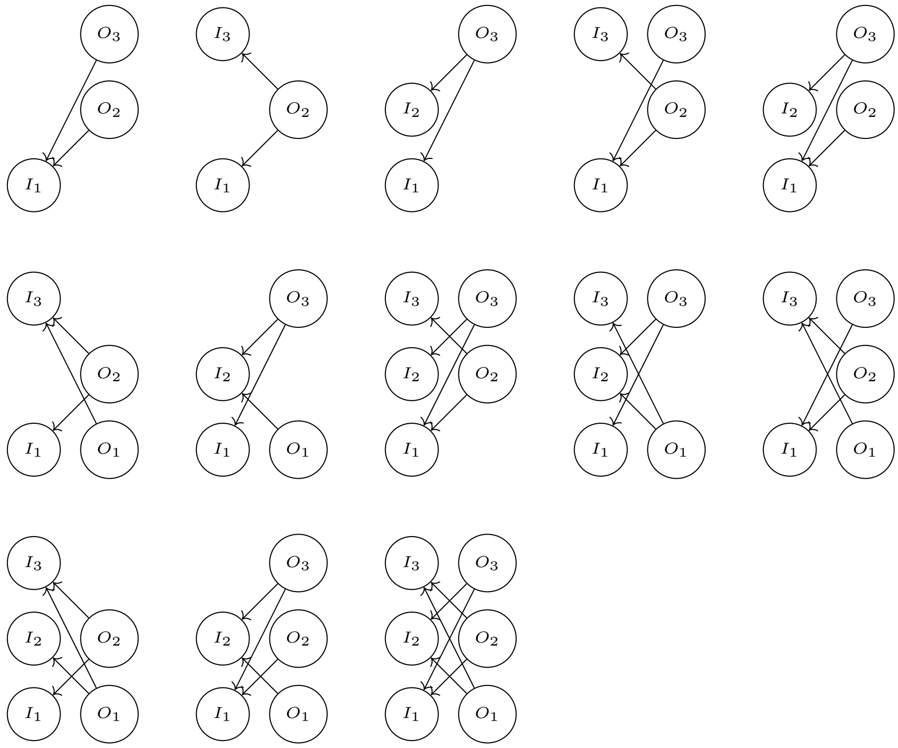

Figure 5.1: The 13 types of facet graphs of $K_{a,b,c}$.

5.2.2 Simplices in type (ii) facets

The situation for type (ii) facets is more complicated. To make things easier, we first define a *labeling in normal form* to be any facet-inducing labeling ν such that for every multipartite class A_i of $K_{a_1,a_2,\dots,a_k}$ and every $v, w \in A_i$ that $\nu(v) = 1$ and $\nu(w) = 0$ implies $v < w$. Further, define the *opposite labeling* of a facet-inducing λ to be the labeling $\bar{\lambda}$ such that the edge set of $K_{a_1,a_2,\dots,a_k}|_{\bar{\lambda}}$ consists of all the reversed edges of $K_{a_1,a_2,\dots,a_k}|_{\lambda}$. In the case of type (ii) facets, that means that $\bar{\lambda}(v) = 1 - \lambda(v)$ vor every v. Lastly, for any facet-inducing labeling λ, the *associated labeling in nomral form* is the labeling in normal form ν_λ such that for every multipartite class of vertices A_i, $|\lambda|_{A_i}{}^{-1}(1)| = |\nu_\lambda|_{A_i}{}^{-1}(1)|$.

We collect some facts about these objects.

Lemma 7. *Let λ be a facet-inducing labeling of a type (ii) facet and let $\Delta, \Delta' \in \mathcal{T}_\lambda$ be simplices. Further, let r denote the smallest vertex in $K_{a_1,a_2,\dots,a_k}$. The following statements hold.*

(a) $\operatorname{in}(T_\Delta) = \operatorname{in}(T_{\Delta'})$. *Because of this, we write* $\operatorname{in}(\lambda)$ *to refer to the number of ingoing edges of the spanning trees of the simplices in* $\mathcal{T}_\lambda$.

(b) $\operatorname{in}(\bar{\lambda}) = a - \operatorname{in}(\lambda - 1)$ *where* $a = a_1 + a_2 + \cdots + a_n$.

(c) $\operatorname{in}(\nu) = |\nu^{-1}(0)|$ *where ν denotes a labeling in normal form.*

(d) *The number of simplices in $\mathcal{F}_\lambda$ is equal to the number of simplices in $\mathcal{F}_{\nu_\lambda}$ and $\mathcal{F}_{\bar{\nu}_\lambda}$.*

(e) *Then* $\mathrm{in}(\lambda) = \mathrm{in}(\nu_\lambda)$ *if r is* 1*-labeled and* $\mathrm{in}\,\lambda = \mathrm{in}(\bar{\nu}_{\bar{\lambda}})$ *if r is* 0*-labeled.*

Proof. The crucial insight is the fact that for any given simplex $\Delta \in \mathcal{T}_\lambda$, the number of ingoing edges does not depend on Δ: $\mathrm{in}(T_\Delta) = |\lambda^{-1}(0)) \setminus \{r\}|$. This can be easily observed by considering that every edge of a 0-labeled vertex points away from it – and if it is the one connecting it to the part of the tree which contains r, that is an ingoing while the others are outgoing. (a) and (c) are immediate corollaries of this. (b) is true because by reversing every edge, the outgoing edges become ingoing and vice versa. For (d) we may notice that a permutation π of the vertices of $K_{a_1,a_2,\ldots,a_k}$ within the multipartite classes A_i induces a map from facets to facets. If, in addition, we make sure that for all ℓ-labeled vertices with $\ell \in \{0,1\}$, $w < v$ implies that $\pi(w) < \pi(v)$, then π induces a mapping of the simplices in $\mathcal{T}_{(ii)}$. For (e), we get two cases: $\lambda(r) = 1$ and $\lambda(r) = 0$. By default, if A_1 contains a single 1-labeled vertex, r will be 1-labeled under ν_λ. Thus, if $\lambda(r) = 1$, the first half of the statement follows from (c). If $\lambda(r) = 0$, $\bar{\lambda}(r) = 1$ and $\mathrm{in}(\bar{\lambda}) = \mathrm{in}(\nu_{\bar{\lambda}})$. Thus, $\bar{\nu}_{\bar{\lambda}}(r) = 0$ and the statement follows. □

Now we define $q(\nu) = |\{\lambda \colon \nu = \nu_\lambda,\ \lambda(r) = 1\}|$ and $r(\nu) = |\mathcal{T}_\nu|$. Since ν is uniquely defined by the number of 0-labeled vertices in every multipartite class, we can identify it with the tuple $(\nu_1, \nu_2, \ldots, \nu_k)$ which readily gives us a formula for q:

$$q(\nu_1, \nu_2, \ldots, \nu_k) = \prod_{i=1}^{k} \binom{a_i - \delta_{1,i}}{\nu_i}$$

where $\delta_{1,i}$ denotes the Kronecker delta, whose function here is to exclude r for the choice. With the previous lemma, we get

$$h^{(ii)}_{a_1,a_2,\ldots,a_k} = \sum_{\Delta \in \mathcal{T}_{(ii)}} t^{\mathrm{in}(T_\Delta)} = \sum_{\substack{\text{labelings in} \\ \text{normal form } \nu}} q(\nu) r(\nu) (t^{\nu_1+\nu_2+\cdots+\nu_k} + t^{a-1-\nu_1-\nu_2-\cdots-\nu_k})$$

where $a = a_1 + \cdots + a_k$ again.

To understand r, some more work is necessary. In particular, we will restrict ourselves to the tripartite case. Figure 5.1 shows the 13 different types of facet graphs that are possible. We call these graphs *class graphs*. Its vertices and edges are called *class vertices* and *class edges*. Every class vertex named O_j (resp. I_j) represents the set of 0-marked (resp. 1-marked) vertices in the corresponding layer. Every class edge represents the edges of the directed complete bipartite graph between the two corresponding classes of vertices.

Next, we investigate what class graphs tell us about spanning trees corresponding to unimodular simplices. Firstly, we notice that not every class edge can contain edges in such a spanning tree. The Gröbner basis elements of type (5) give configurations with involve 6 class vertices and 3 class edges. More precisely, the class edges $\{O_1, I_2\}$, $\{O_2, I_3\}$, and $\{O_3, I_1\}$ cannot all be non-empty at the same time, which turns the last class graph in Figure 5.1 into three reduced class graphs, each of them missing one of these class edges.

Let us now assume a reduced class graph. Let A be a class vertex connected to class vertices B and C. Without loss of generality, assume that for every $b \in B$ and every $c \in C$, $b < c$. The Gröbner basis elements of type (3b) imply that for two distinct vertices $a, a' \in A$ where a is connected to a vertex in B and a' is connected to a vertex in C, $a < a'$. In particular there can be only one vertex in $\hat{a} \in A$ which connects to both classes. We denote the set of vertices in A which connect to B (resp. C) by A_B (resp. A_C). Analogously, we define the subsets B_A and C_A of vertices which connect to A. Thus we end up with planar spanning trees between the sets B_A and A_B as well as C_A and A_C respectively.

Notice that all reduced class graphs are paths of length 3, 4, or 5. Thus, consider a class path with vertices $C_1, C_2, \dots, C_n$ of sizes $c_1, c_2, \dots, c_n$. We define the following function

$$\begin{aligned} &c\left(c_1, c_2, \dots, c_n\right) \\ &= \sum_{j_2=0}^{c_2-1} \sum_{j_3=0}^{c_3-1} \cdots \sum_{j_{n-1}=0}^{c_{n-1}} \binom{c_1 + j_2 - 1}{j_2} \binom{c_{n-1} - j_{n-1} + c_n - 2}{c_n - 1} \prod_{i=2}^{n-2} \binom{c_i - j_i + j_{i+1} - 1}{j_{i+1}}. \end{aligned}$$

Although this formula looks complicated, its function is fairly straightforward: Every binomial coefficient $\binom{c_i - j_i + j_{i+1} - 1}{j_{i+1}}$ counts the number of planar spanning trees between the sets $C_{i C_{i+1}}$ and $C_{i+1 C_i}$ where the former has cardinality $j_i + 1$ and the latter has cardinality $c_{i+1} - j_{i+1}$. With this, we get a formula for r in the case of complete tripartite graphs $K_{a,b,c}$.

$$r(\nu_1, \nu_2, \nu_3) = \begin{cases} c(b, a, c) & \nu_1 = 0, \nu_2 = b, \nu_3 = c \\ c(a, b, c) & \nu_1 = 0, \nu_2 = b, \nu_3 = 0 \\ c(a, c, b) & \nu_1 = 0, \nu_2 = 0, \nu_3 = c \\ c(\nu_3, a, b, c - \nu_3) & \nu_1 = 0, \nu_2 = b, \nu_3 \notin \{0, c\} \\ c(\nu_2, a, c, b - \nu_2) & \nu_1 = 0, \nu_2 \notin \{0, b\}, \nu_3 = c \\ c(\nu_1, c, b, a - \nu_1) & \nu_1 \notin \{0, a\}, \nu_2 = b, \nu_3 = 0 \\ c(\nu_1, b, c, a - \nu_1) & \nu_1 \notin \{0, a\}, \nu_2 = 0, \nu_3 = c \\ c(b - \nu_2, \nu_3, a, \nu_2, c - \nu_3) & \nu_1 = 0, \nu_2 \notin \{0, b\}, \nu_3 \notin \{0, c\} \\ c(a - \nu_1, \nu_3, b, \nu_1, c - \nu_3) & \nu_1 \notin \{0, a\}, \nu_2 = 0, \nu_3 \notin \{0, c\} \\ c(\nu_1, c - \nu_3, b, a - \nu_1, \nu_3) & \nu_1 \notin \{0, a\}, \nu_2 = b, \nu_3 \notin \{0, c\} \\ c(a - \nu_1, \nu_2, c, \nu_1, b - \nu_2) & \nu_1 \notin \{0, a\}, \nu_2 \notin \{0, b\}, \nu_3 = 0 \\ c(\nu_1, b - \nu_2, c, a - \nu_1, \nu_2) & \nu_1 \notin \{0, a\}, \nu_2 \notin \{0, b\}, \nu_3 = c \\ c(a - \nu_1, \nu_2, c - \nu_3, \nu_1, b - \nu_2, \nu_3) & \\ +c(\nu_1, c - \nu_3, \nu_2, a - \nu_1, \nu_3, b - \nu_2) & \\ +c(b - \nu_2, \nu_1, \nu_3, \nu_2, a - \nu_1, c - \nu_3) & \nu_1 \notin \{0, a\}, \nu_2 \notin \{0, b\}, \nu_3 \notin \{0, c\} \\ 0 & \text{otherwise} \end{cases}$$

We can finally assemble the h^*-polynomial of $P_{K_{a,b,c}}$.

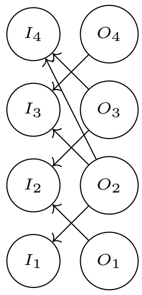

Figure 5.2: The O_2-labeled class vertex has degree 3.

Theorem 11. *The h^*-polynomial of $P_{K_{a,b,c}}$ is given by*

$$h^*_{a,b,c}(t) = h^{(i)}_{a,b,c} + h^{(ii)}_{a,b,c}(t).$$

Here, $h^{(i)}_{a,b,c}$ is given by

$$\begin{aligned} h^{(i)}_{a,b,c}(t) = &\sum_{i=0}^{b+c-1}\sum_{j=1}^{a-1} p(a+b+c,a,i,j)\binom{a+j-i-2}{j-1}\left(t^{i+j+1}+t^{a+b+c-i-j-2}\right) \\ &+ \sum_{i=0}^{a+c-1}\sum_{j=1}^{b-1} p(a+b+c,b,i,j)\binom{a+c+i-j-1}{a+c-i-1}\left(t^{i+j}+t^{a+b+c-i-j-1}\right) \\ &+ \sum_{i=0}^{a+b-1}\sum_{j=1}^{c-1} p(a+b+c,c,i,j)\binom{a+b+i-j-1}{a+b-i-1}\left(t^{i+j}+t^{a+b+c-i-j-1}\right) \end{aligned}$$

with

$$p(x,y,i,j) = \binom{x-y-1}{i}\binom{y-1}{j}\binom{y+i-j-1}{i},$$

and $h^{(ii)}_{a,b,c}$ is given by

$$h^{(ii)}_{a,b,c}(t) = \sum_{\nu_1=0}^{a-1}\sum_{\nu_2=0}^{b}\sum_{\nu_3=0}^{c} q(\nu_1,\nu_2,\nu_3)r(\nu_1,\nu_2,\nu_3)\left(t^{\nu_1+\nu_2+\nu_3}+t^{a+b+c-1-\nu_1-\nu_2-\nu_3}\right)$$

with

$$q(\nu_1,\nu_2,\nu_3) = \binom{a-1}{\nu_1}\binom{b}{\nu_2}\binom{c}{\nu_3}$$

and $r(\nu_1,\nu_2,\nu_3)$ as stated above.

For general complete multipartite graphs, the number of possible class graphs grows rapidly as k grows. Furthermore, in the complete tetrapartite case, class graphs which are not paths start to appear, see e.g. Figure 5.2.

5.3 New Recursive Relations

In this section, we gather new evidence for Conjecture 6. First, we state the relevant h^*-polynomials.

Proposition 22. *The h^*-polynomials of the symmetric edge polytopes of graphs $K_{1,m,n}$, $K_{1,1,1,n}$, and $K_{2,2,n}$, are given as follows.*

(a) $h^*_{1,m,n}(t) = \sum_{i=0}^{\min(m,n)} \binom{2i}{i}\binom{m}{i}\binom{n}{i} t^i (1+t)^{m+n-2i}$

(b) $h^*_{1,1,1,n}(t) = 3(n-1)n(1+t)^{n-2}t^2 + 2(2n+1)(1+t)^n t + (1+t)^{n+2}$

(c) $h^*_{2,2,n}(t) = 20\binom{n}{3}(1+t)^{n-3}t^3 + 2\binom{3n}{2}(1+t)^{n-1}t^2 + 2\binom{3n+1}{1}(1+t)^{n+1}t + (1+t)^{n+3}$

Proof. Notice that (a) is a direct consequence of Propositions 6 and 8. The formula in (b) can be obtained by applying Proposition 7. The graph $G = K_{1,1,n}$ gives rise via suspension to the graph $\widehat{G} = K_{1,1,1,n}$. We denote the the vertices in the first two tripartite classes of G by a and b respectively. The remaining vertices shall be denoted by the integers $1,\dots,n$. First, we need to understand the cuts G_C of G. There are two primary types: one type where without loss of generality $a, b \notin C$, and one type where $a \in C$, $b \notin C$. Thus, for every subset $S \subseteq [n]$ we get a cut set $C_1 = S$ and a cut set $C_2 = S \cup \{a\}$. Assume $|S| = m$. Now we need to understand the hypertrees associated to the joint bipartite suspensions $\widetilde{G}_{C_i}$ for $i = 1,2$, which we regard as functions $f\colon \{n+3\} \cup C_i \to \{0,1,\dots\}$. In the order of the hyperedges, it is convenient to regard $n+3$ as the smallest edge. This way, it can never be an internally inactive edge and we can focus on the elements of C_i instead. One can check that a hyperedge $e \in C_i$ is internally inactive if and only if $f(e) > 0$. Without loss of generality, we can assume that every inducing spanning tree of a hypertree contains the edge $\{n+3, n+4\}$ and for every $c \in C_i$ the edge $\{n+4, c\}$. From here, one can check that the interior polynomial of $\widetilde{G}_{C_1}$ is $\binom{m}{2}t^2 + 2mt + 1$ and that of $\widetilde{G}_{C_2}$ is $m(n-m)t^2 + (n+1)t + 1$. Summing up, we get

$$\begin{aligned} f_G(t) &= \frac{1}{2^{n+1}} \sum_{i=0}^{n} \binom{n}{m} \left(\left(\binom{m}{2} + m(n-m) \right) t^2 + (2m+n+1)t + 2 \right) \\ &= \frac{3(n-1)n}{2^4} t^2 + \frac{2n+1}{2} t + 1. \end{aligned}$$

Thus, we obtain the h^*-polynomial of the symmetric edge polytope of $K_{1,1,1,n}$:

$$h^*_{1,1,1,n}(t) = 3(n-1)n(1+t)^{n-2}t^2 + 2(2n+1)(1+t)^n t + (1+t)^{n+2}$$

For (c), one can use the description of $h^*_{a,b,c}$ of Theorem 11 to derive the coefficients of $h^*_{2,2,n}$. Then these coefficients can be checked against those of (c). Since this is a very tedious process, the reader may consult the corresponding file on

https://github.com/maxkoelbl/seps_multipartite_graphs/.

It was programmed with SAGEMATH [The22]. □

In the following we will denote the Ehrhart polynomial of $P_{K_{a_1,\ldots,a_k}}$ by $E_{a_1,\ldots,a_k}$.

Proposition 23. *For every $n \geq 2$ there exist non-negative rational numbers $\alpha_1, \ldots, \alpha_{35}$ such that the following statements hold.*

$$
\begin{aligned}
E_{1,1,n}(x) &= \alpha_1\,(2x+1)\,E_{1,n}(x) + \alpha_2\,E_{1,n-1}(x),\\
E_{1,1,n+1}(x) &= \alpha_3\,(2x+1)\,E_{1,1,n}(x) + \alpha_4\,E_{1,1,n-1}(x) + \alpha_5\,E_{1,n}(x),\\
E_{1,2,n}(x) &= \alpha_6\,(2x+1)\,E_{1,1,n}(x) + \alpha_7\,E_{1,1,n-1}(x) + \alpha_8\,E_{1,n}(x),\\
E_{1,2,n+1}(x) &= \alpha_9\,(2x+1)\,E_{1,2,n}(x) + \alpha_{10}\,E_{1,2,n-1}(x) + \alpha_{11}\,E_{1,1,n}(x) + \alpha_{12}\,E_{1,n+1}(x)\\
E_{1,1,1,n}(x) &= \alpha_{13}\,(2x+1)\,E_{1,1,n}(x) + \alpha_{14}\,E_{1,1,n-1}(x) + \alpha_{15}\,E_{1,n}(x)\\
E_{4,n}(x) &= \alpha_{16}\,(2x+1)\,E_{3,n}(x) + \alpha_{17}\,E_{3,n-1}(x) + \alpha_{18}\,E_{2,n}(x) + \alpha_{19}\,E_{1,n+1}(x),\\
E_{3,n+1}(x) &= \alpha_{20}\,(2x+1)\,E_{3,n}(x) + \alpha_{21}\,E_{3,n-1}(x) + \alpha_{22}\,E_{2,n}(x) + \alpha_{23}\,E_{1,n+1}(x),\\
E_{2,2,n}(x) &= \alpha_{24}\,(2x+1)\,E_{1,2,n}(x) + \alpha_{25}\,E_{1,2,n-1}(x) + \alpha_{26}\,E_{1,1,n}(x) + \alpha_{27}\,E_{1,n+1}(x),\\
E_{1,3,n}(x) &= \alpha_{28}\,(2x+1)\,E_{1,2,n}(x) + \alpha_{29}\,E_{1,2,n-1}(x) + \alpha_{30}\,E_{1,1,n}(x) + \alpha_{31}\,E_{1,n+1}(x),\\
E_{1,1,1,n+1}(x) &= \alpha_{32}\,(2x+1)\,E_{1,1,1,n}(x) + \alpha_{33}\,E_{1,1,1,n-1}(x) + \alpha_{34}\,E_{1,1,n}(x) + \alpha_{35}\,E_{1,n+1}(x).
\end{aligned}
$$

Proof. With the formulas in Propositions 6 and 22, these relations can be obtained algorithmically[1]. We explain the method of proof using the first relation. The proof follows that of Proposition 4.5 in [HKM17]. Since taking the generating function of a polynomial is a linear operation, addition and scalar multiplication translate immediately to Ehrhart series. For (a), we need the Ehrhart series of $E_{1,1,n}$, $(2x+1)E_{1,n}$, and $E_{1,n-1}$. Notice that multiplying an Ehrhart polynomial by x corresponds to differentiating its Ehrhart series and then multiplying t to it. Since the Ehrhart series of $E_{1,n}$ can be written as $\frac{(1+t)^n}{(1-t)^{n+1}}$, we get

$$\frac{(2\,nt+t+1)(t+1)^n}{(t^3-t^2-t+1)(1-t)^n}$$

for the Ehrhart series of $(2x+1)E_{1,n}$. Next, we form the equation

$$1 = \frac{\alpha\sum_{k\geq 0}(2k+1)E_{1,n}(k)t^k + \alpha_0\sum_{k\geq 0}E_{1,n-1}(k)t^k}{\sum_{k\geq 0}E_{1,1,n}(k)t^k}.$$

Note that there need not be any solutions for α and α_0. The right-hand side is a rational function of two polynomials where the numerator polynomial involves α and α_1. Since the right-hand-side is assumed to be equal to one, obtaining a solution is equivalent to a finding asolution of the system of equations

$$n_i(\alpha, \alpha_0) = d_i$$

[1]The code for computing explicitly all the coefficients is also available on

https://github.com/maxkoelbl/seps_multipartite_graphs/.

It was also written using SAGEMATH.

where n_i is the i-th degree coefficient of the numerator polynomial and d_i is the i-th degree coefficient of the denominator polynomial. Since n_i and d_i both depend on n, α and α_0 do as well. We get $\alpha = \frac{n+2}{2(n+1)}$ and $\alpha_0 = \frac{n}{2(n+1)}$. □

With this, we can prove our first main result, Theorem 9.

Proof. The six labeled statements in this theorem rest entirely on the recursive relations from Proposition 23, the relation from Proposition 13, and Proposition 12. The concluding statement follows from the labeled statements and Proposition 15. □

5.4 Recursive Relations and the γ-vector

Looking at the recursive relations in Propositions 14 and 23, we may notice that as the parameters $a_1, \ldots, a_{k-1}$ of the multipartite graphs increase, then so does the complexity of the formulas surrounding them. The results of this section show that this is not a coincidence. We will show how the existence of a recursion as well as, to some extent, the number of terms it has, are related with the γ-*vectors* of the h^*-polynomials of all the Ehrhart polynomials involved.

Definition 12. Let h be a palindromic polynomial of degree d. We define the γ-*vector* as the polynomial $\sum_{i=0}^{\lfloor \frac{d}{2} \rfloor} \gamma_i t^i$ such that $h(t) = \sum_{i=0}^{\lfloor \frac{d}{2} \rfloor} \gamma_i (1+t)^{d-2i} t^i$. We call the degree of the γ-vector the γ-*degree* of h.

Lemma 8. *For every integer $d \geq 1$ and every integer $n \geq 0$, the following equation holds.*

$$\sum_{k \geq 0} \left(\sum_{i=0}^{n} (-1)^i \binom{n}{i} \mathcal{C}_{d+2(n-i)}(k) \right) t^k = \frac{(1+t)^d (4t)^n}{(1-t)^{d+2n+1}}$$

Proof. There are two key insights. The first is the well-known fact that the generating function of $\mathcal{C}_d$ is $\frac{(1+t)^d}{(1-t)^{d+1}}$. The second is that the generating function of $\mathcal{C}_2(x) - \mathcal{C}_0(x)$ can be written as $\frac{4t}{(1-t)^3} = \frac{(1+t)^2}{(1-t)^3} - \frac{1}{1-t}$, which can be checked easily.

The first insight tells us that for real numbers $c_0, c_1, \ldots, c_n$, the generating function of $\sum_{i=0}^{n} c_i \mathcal{C}_i$ can be written as

$$\frac{1}{1-t} \sum_{i=0}^{n} c_i \frac{(1+t)^i}{(1-t)^i}.$$

Using the second insight tells us that

$$\frac{(1+t)^d (4t)^n}{(1-t)^{d+2n+1}} = \frac{1}{1-t} \frac{(1+t)^d}{(1-t)^d} \left(\frac{(1+t)^2}{(1-t)^2} - 1 \right)^n.$$

Finally, with the binomial theorem, we get

$$\frac{1}{1-t} \sum_{i=0}^{n} (-1)^i \binom{n}{i} \frac{(1+t)^{d+2(n-i)}}{(1-t)^{d+2(n-i)}},$$

which concludes the proof. □

Proposition 24. *Let p be a polynomial of degree d and let h be a polynomial defined by*

$$h(t) = (1-t)^{d+1} \sum_{k \geq 0} p(k) t^k.$$

If h is a palindromic polynomial with γ-vector γ, we get

$$p(x) = \sum_{i=0}^{\deg \gamma} (-1)^i c_i \mathcal{C}_{d-2i}(x).$$

where $c_i = \sum_{j=i}^{\deg \gamma} \frac{1}{4^j} \binom{j}{i} \gamma_j$. We call the polynomial $\sum_{i=0}^{\deg \gamma} (-1)^i c_i x^i$ the cross-polynomial coefficients *of p.*

Proof. We rewrite the generating function of p.

$$\frac{h(t)}{(1+t)^{d+1}} = \frac{\sum_{i=0}^{d} \gamma_i \frac{1}{4^i} (1+t)^{d-2i} (4t)^i}{(1+t)^{d+1}}$$

Splitting up the sum and applying Lemma 8, we get

$$\begin{aligned} &\sum_{k \geq 0} \gamma_0 \mathcal{C}_d(k) t^k \\ &+ \sum_{k \geq 0} \left(\frac{\gamma_0}{4} \mathcal{C}_d(k) - \frac{\gamma_1}{4^2} \mathcal{C}_{d-2}(k) \right) t^k + \dots \\ &+ \sum_{k \geq 0} \left(\sum_{i=0}^{n} (-1)^i \frac{\gamma_n}{4^n} \binom{n}{i} \mathcal{C}_{d+2(n-i)}(k) \right) t^k \end{aligned}$$

Rearranging to sort the sum by the $\mathcal{C}_i$ yields the claim. □

In the setting of Proposition 24, we call the γ-degree of h the *cross-degree* of p.

Theorem 12. *Let f be a degree $d+1$ polynomial with cross-degree $m+1$, let g be a degree d polynomial with cross-degree m, and let h_i be degree $d-1$ polynomials with cross degree i for $1 \leq i \leq m$. Then there exist real numbers $\alpha, \alpha_1, \alpha_2, \dots, \alpha_m$ which satisfy*

$$f(x) = (2x+1)\alpha g(x) + \sum_{i=1}^{m} \alpha_i h_i(x).$$

Proof. Using Proposition 13, we can see that the degree $d+1$ polynomial $(2x+1)g(x)$ has cross-degree $m+1$. Thus, the right-hand side of the equation can be written as

$$\begin{aligned} &\alpha c_{g,0} \mathcal{C}_{d+1} \\ &+(-\alpha c_{(2x+1)g,1} + \alpha_1 c_{h_1,0} + \alpha_2 c_{h_2,0} + \dots + \alpha_m c_{h_m,0}) \mathcal{C}_{d-1} \\ &-(-\alpha c_{(2x+1)g,2} + \alpha_2 c_{h_2,1} + \dots + \alpha_m c_{h_m,1}) \mathcal{C}_{d-3} \\ &\vdots \\ &+(-1)^m (-\alpha c_{(2x+1)g,m+1} + \alpha_m c_{h_m,m}) \mathcal{C}_{d-2m+1} \end{aligned}$$

where $c_{(2x+1)g,i}$ is the i-th cross-polynomial coefficient of $(2x+1)g(x)$ and the $c_{h_j,i}$ are the cross-polynomial coefficients of the h_j. This means that in order to get the left-hand side, all we need to do is choose $\alpha, \alpha_m, \alpha_{m-1}, \dots, \alpha_1$ in order. □

For complete bipartite graphs, Proposition 6 shows that the γ-degree of the h^*-polynomial of $K_{m,n}$ is $\min\{m,n\} - 1$. Thus, we get the following immediate corollary.

Corollary 1. *Let n be a positive integer. For $1 \le m \le n$ there exist unique α, α_0, $\alpha_1, \dots$ alpha$_{m-1}$ and β, β_0, $\beta_1, \dots, \beta_{m-1}$ in $\mathbb{R}$ such that the following equations are satisfied.*

$$E_{m+1,n+1}(x) = (2x+1)\,\alpha\,E_{m,n+1}(x) + \sum_{i=0}^{m-1} \alpha_i\,E_{m-i,n+i}(x)$$

$$E_{m,n+1}(x) = (2x+1)\,\beta\,E_{m,n}(x) + \sum_{i=0}^{m-1} \beta_i\,E_{m-i,n+i-1}(x)$$

Remark 5. This corollary alone are not enough to prove Conjecture 6 for all $K_{m,n}$ for two crucial reasons. Firstly, as m increases, the number of interlacings having to be satisfied increases too, and they are between polynomials whose cross-degrees puts them outside the scope of Theorem 12. This is noticeable in the last four statements of Theorem 9 where the interlacing of cross-degree 3 polynomials by cross-degree 2-polynomials depend on the interlacing of a cross-degree 2-polynomial by a cross-degree 0 polynomial.

Secondly, there is no guarantee that the coefficients $\alpha, \alpha_1, \dots, \alpha_m$ are non-negative, although explicit computations for low m in the context of Corollary 1 always yield positive coefficients. In fact, for $m \ge 4$, explicit computations reveal that $\alpha_2, \dots, \alpha_{m-2}$ are always negative. In the case $m = 4$, we get $\alpha_2 = \frac{n-n^3}{8(5n^3+39n^2+100n+96)}$. To see the parameters for every $1 \le m \le 10$, we refer once again to the corresponding SAGEMATH code on

https://github.com/maxkoelbl/seps_multipartite_graphs/.

We close the chapter by stating a conjecture.

Conjecture 7. *Let $a_1 \le a_2 \le \cdots \le a_k \le n$ be positive integers and let m denote the cross-degree of the Ehrhart polynomial of the symmetric edge polytope of $K_{a_1,a_2,\dots,a_k}$. Then we conjecture the inequalities*

$$\left\lfloor \frac{\sum_{i=1}^{k} a_i}{2} \right\rfloor \le m + 1 \le \sum_{i=1}^{k} a_i.$$

hold. Furthermore, the Ehrhart polynomial of the symmetric edge polytope of the graph $K_{1^k,n}$ interlaces that of $K_{1^{k+1},n}$, where 1^k represents a list of ones.

Part III

On equivariant Ehrhart theory

Chapter 6

The equivariant Ehrhart Theory of order-two symmetries

In this chapter we study the equivariant Ehrhart theory of two families of polytopes: the symmetric edge polytopes of the cycle graph under the induced action of the automorphism group of the graph, and rational cross-polytopes under the action of coordinate reflections. We compute the equivariant Ehrhart series in each case to verify the effectiveness conjecture. In particular, in Example 9 we see that pseudo-integral polytopes need not satisfy the effectiveness conjecture if the assumption that P is a lattice polytope is dropped. The content of this chapter is fully contained in the author's paper [CHK23] with Oliver Clarke and Akihiro Higashitani.

6.1 The main results

We fix the setup of the first main result. Let $\Gamma = (V, E)$ be an undirected graph and $\mathbb{Z}^{|V|}$ a lattice whose basis elements e_v are associated to the vertices $v \in V$. Throughout this chapter, we will consider the symmetric edge polytope $P_\Gamma \subset \mathbb{R}^{|V|}$ and its automorphism group of Γ, denoted $\mathrm{Aut}(\Gamma)$. One can see that $\mathrm{Aut}(\Gamma)$ naturally induces a permutation representation ρ_Γ on $\mathbb{R}^{|V|}$, which leaves P_Γ invariant. We focus on the case when Γ is the cycle graph C_d for some integer $d \geq 3$. In this case, $\mathrm{Aut}(C_d) \cong D_{2d} = \langle r, s \mid s^2 = r^d = (sr)^2 = 1 \rangle$ is the dihedral group of order $2d$.

We identify D_{2d} with the automorphism group of C_d. We fix the generator $s \in D_{2d}$, in the presentation of the group, to be a reflection that fixes the fewest number of vertices of C_d. Let $\rho_d \coloneqq \rho_{C_d} \colon D_{2d} \to \mathrm{GL}(\mathbb{R}^d)$ denote the associated permutation representation. From now on, we label the vertices of C_d with $\{v_0, \dots, v_{\lceil (d-2)/2 \rceil}, w_0, \dots, w_{\lceil (d-2)/2 \rceil}\}$, where $w_0 = v_0$ if d is odd, so that: $(v_0, v_1, \dots, v_{\lceil (d-2)/2 \rceil})$ and $(w_0, w_1, \dots, w_{\lceil (d-2)/2 \rceil})$ are distinct paths in C_d; for each $0 \leq i \leq \lceil (d-2)/2 \rceil$ the s-orbits are $\{v_i, w_i\}$; if d is odd, then $v_0 = w_0$ is the unique fixed vertex of s; if d is even, then v_0 and w_0 are neighbours; and r is the rotation that maps w_0 to w_1 (see Figure 6.1).

We state the first main result.

Theorem 13 (Theorems 15 and 16)**.** *Let $d \geq 3$ be an integer. Then*

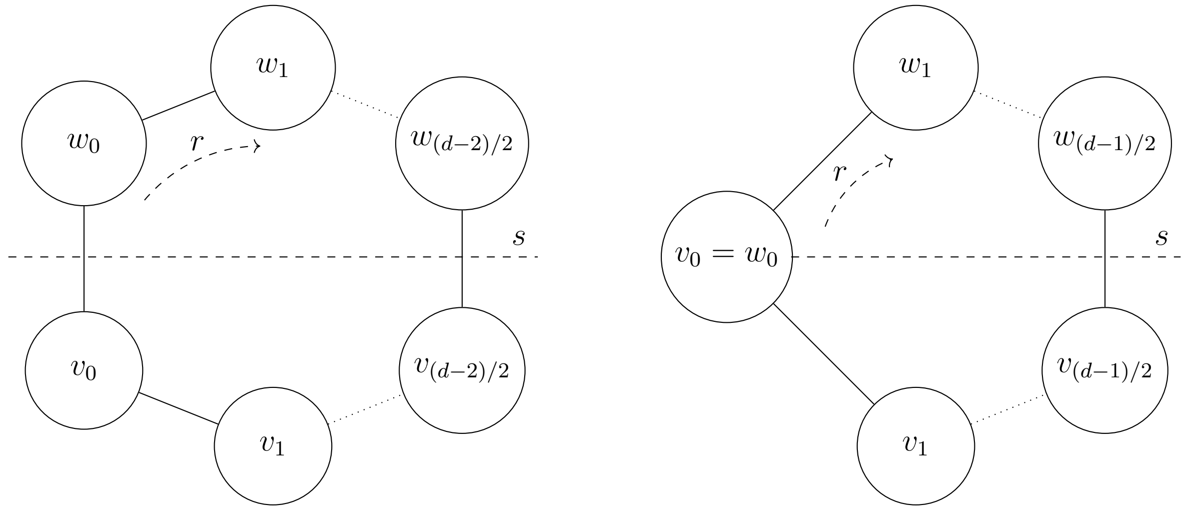

Figure 6.1: The vertex labelings for even (left) and odd (right) cycle graphs and the action of the generators of the dihedral group.

(i) the H^*-series of $H^*_{(d)}$ of P_d with respect to the action of the dihedral group D_{2d} is effective if d is prime,

(ii) the H^*-series of $H^*_{(d)}$ of P_d with respect to the action of the group $\{1, s\}$ is effective.

Our second main theorem explores the limits of the effectiveness conjecture. Let $k, d \in \mathbb{Z}$ be positive integers with k odd and $d \geq 2$. Throughout this section we consider the polytope

$$P(k,d) = \operatorname{conv}\left\{\pm e_1, \ldots, \pm e_{d-1}, \pm \frac{k}{2} e_d\right\} \subseteq M_{\mathbb{R}} \cong \mathbb{R}^d .$$

Theorem 14 (Theorem 17 and Example 9)**.** *With the setup above, we have $H^*[t] = \sum_{j=0}^{d} (a_j \chi_1 + b_j \chi_2) t^j$ where*

$$a_j = \binom{d-2}{j} + \frac{1}{2}(k+1)\binom{d-1}{j-1} \text{ and } b_j = \frac{1}{2}(k-1)\binom{d-1}{j-1} - \binom{d-2}{j-1}$$

and $\binom{n}{k}$ is defined to be zero if $k < 0$ or $k > n$.

In particular, the equivariant Ehrhart H^*-series of $P(1,2)$ is $H^*[t] = \chi_1 + (\chi_1 - \chi_2)t + \chi_1 t^2$ and hence not effective.

6.2 Symmetric edge polytopes of cycle graphs

Studying the equivariant Ehrhart theory of $P_d := P_{C_d}$ under the action of D_{2d} involves understanding the Ehrhart series of the individual sub-polytopes P_d^g fixed by the individual elements $g \in D_{2d}$. Let us begin with the trivial element $1 \in D_{2d}$.

Proposition 25 ([OS12, Theorem 3.3])**.** *The Ehrhart series of P_d is given by*

$$\mathrm{ehr}(P_d, t) = \frac{h_0^{(d)} + h_1^{(d)} t + \cdots + h_{d-1}^{(d)} t^{d-1}}{(1-t)^d}$$

where: $h_0^{(d)} = 1$; for $1 \leq j \leq \lfloor \frac{d}{2} \rfloor$, we have

$$h_j^{(d)} = (-1)^j \sum_{i=0}^{j} (-2)^i \binom{d}{i} \binom{d-1-i}{j-i} = \begin{cases} 2^{d-1} & \text{if } d \text{ is odd and } j = \frac{d-1}{2}, \\ h_{j-1}^{(d-1)} + h_j^{(d-1)} & \text{otherwise;} \end{cases}$$

and for each $\frac{d}{2} < j < d$, the coefficients are $h_j^{(d)} = h_{d-1-j}^{(d)}$.

For odd cycle graphs $C_{2\ell+1}$, all reflections in $D_{4\ell+2}$ are conjugate and so the corresponding fixed polytopes are unimodularly equivalent. Hence, it suffices to compute the fixed polytope for a single reflection, say $s \in D_{4\ell+2}$.

Proposition 26. *Let $\ell \geq 1$ be an integer. The fixed sub-polytopes $P_{2\ell+1}^s$ and $P_{2\ell+2}^s$ are unimodularly equivalent to the cross-polytope of dimension ℓ dilated by the factor $\frac{1}{2}$ and their Ehrhart series are given by*

$$\mathrm{ehr}(P_{2\ell+1}^s, t) = \mathrm{ehr}(P_{2\ell+2}^s, t) = \frac{(1+t^2)^\ell}{(1-t)(1-t^2)^\ell}.$$

Proof. We start by giving a full description of the vertices of $P_{2\ell+1}^s$ and $P_{2\ell+2}^s$. Each s-orbit is given by $\{v_i, w_i\}$ for each $0 \leq i \leq \ell$. Note, the s-orbit that is an edge of $C_{2\ell+1}$ is $\{v_\ell, w_\ell\} \in E$, while those of $C_{2\ell+2}$ are $\{v_0, w_0\}$ and $\{v_\ell, w_\ell\}$. The s-orbits of the vertices of $P_{2\ell+1}$ and $P_{2\ell+2}$ are hence given by $\{\pm(e_{w_i} - e_{w_{i+1}}), \pm(e_{v_i} - e_{v_{i+1}})\}$ as well as $\{e_{w_\ell} - e_{v_\ell}, e_{v_\ell} - e_{w_\ell}\}$. In the case of $P_{2\ell+2}$, we have the additional vertex $\{e_{w_0} - e_{v_0}, e_{v_0} - e_{w_0}\}$. By Lemma 5.4 in [Sta11], $P_{2\ell+1}^s$ (resp. $P_{2\ell+2}$) is given by the convex hull of points of the form $\frac{\sum_{p \in I} p}{|I|}$ where I is an s-orbit of the vertices of $P_{2\ell+1}^s$ (resp. $P_{2\ell+2}$). The orbits $\{e_{w_0} - e_{v_0}, e_{v_0} - e_{w_0}\}$ and $\{e_{w_\ell} - e_{v_\ell}, e_{v_\ell} - e_{w_\ell}\}$ correspond to the origin and do not contribute to the description of $P_{2\ell+1}^s$ (resp. $P_{2\ell+2}$). The remaining orbits yield

$$P_{2\ell+1}^s = P_{2\ell+2}^s = \mathrm{conv}\left\{ \pm\frac{1}{2}(e_{v_i} + e_{w_i} - e_{v_{i+1}} - e_{w_{i+1}}) \colon 0 \leq i \leq \ell - 1 \right\}.$$

One can see that the points $\{e_{v_i} + e_{w_i} - e_{v_{i+1}} - e_{w_{i+1}}\}$ form a lattice basis for the fixed subspace (note that it does not matter whether v_0 and w_0 are identical or not), and with respect to that basis, $P_{2\ell+1}^s$ is unimodularly equivalent to the cross-polytope of dimension ℓ dilated by the factor $\frac{1}{2}$. By [BJM13, Theorem 1.4] and the fact that the Ehrhart series of the interval $[-\frac{1}{2}, \frac{1}{2}]$ is $\frac{1+t^2}{(1-t)(1-t^2)}$, the result follows by induction on ℓ. □

Remark 6. For even cycle graphs $C_{2\ell+2}$, there is another type of reflection: one that fixes two antipodal vertices. For such a reflection $sr \in D_{4\ell+4}$, the fixed sub-polytope $P_{2\ell+2}^{sr}$ cannot be studied using the same method as in Proposition 26. The one-element sr-orbits are $\{v_0\}$ and $\{w_\ell\}$ and the other orbits are $\{v_i, w_{i-1}\}$. By a similar argument as above, the vertices of the sub-polytope $P^{sr}_{2\ell+2}$ are of the form

$$\pm\frac{1}{2}(e_{v_i} + e_{w_{i-1}} - e_{v_{i+1}} - e_{w_i}) \text{ for } i = 1, \ldots, \ell - 1,$$
$$\pm\frac{1}{2}(e_{v_1} + e_{w_0} - 2e_{v_0}) \text{ and } \pm\frac{1}{2}(e_{v_\ell} + e_{w_{\ell-1}} - 2e_{w_\ell}).$$

For $\ell = 1$, this is unimodularly equivalent to a dilated square containing the origin in its interior. For $\ell > 1$, one can cut through the points $\pm(e_{v_1} + e_{w_0} - 2e_{v_0})$ and $\pm(e_{v_1} + e_{w_0})$, which yields a subpolytope of $2P^{sr}_{2\ell+2}$ containing the origin and four of its vertices. Again, this is unimodularly equivalent to a square containing the origin. Hence, $P^{sr}_{2\ell+2}$ is not unimodularly equivalent to a dilated cross-polytope.

We have computed the invariant polytopes of the symmetric edge polytope fixed by reflections of D_{2d}. The remaining conjugacy classes are the rotations. For odd d, the irreducible characters of D_{2d} are determined by the following table:

	1	r^k	sr^k
ψ_1	1	1	1
ψ_2	1	1	-1
χ_j	2	$2\cos\frac{2jk\pi}{d}$	0

.

where j ranges from 1 to $\frac{d-1}{2}$ and k ranges from 1 to d.

In general, the fixed polytope $P^{r^k}_d$ with respect to a rotation r^k is very difficult to compute directly. Not only does the description of the vertices of $P^{r^k}_d$ depend on the cycle decomposition of the permutation action of r^k on the basis vectors of $\mathbb{R}^{|V|}$, but also on the adjacency of these vertices in the cycle graph.

However, the rotation $r^k \in D_{2d}$, where k and d are coprime, does not fix any vertex of C_d. Therefore, the induced action on P_d fixes only the origin, whose Ehrhart series is simply a geometric series $\operatorname{ehr}(\{0\}, t) = 1 + t + t^2 + \cdots = \frac{1}{1-t}$. This yields the following result when d is prime.

Theorem 15. *Let $p \geq 3$ be a prime number. The H^*-series $H^*_{(p)}$ of P_p with respect to the action of the dihedral group D_{2p} is a polynomial of degree $p-1$ and its coefficients $H^*_{(p),j}$ are given by*

$$H^*_{(p),j} = \frac{1}{2p}\begin{cases} (h^{(p)}_j - 1 + p(g^{(p)}_j + 1))\psi_1 + (h^{(p)}_j - 1 - p(g^{(p)}_j - 1))\psi_2 + (2h^{(p)}_j - 2)\chi & \text{if } 2 \mid j, \\ (p + h^{(p)}_j - 1)\psi_1 + (p + h^{(p)}_j - 1)\psi_2 + (2h^{(p)}_j - 2)\chi & \text{if } 2 \nmid j. \end{cases}$$

with where $h^{(p)}_j$ follows the notation from Proposition 25, $g^{(p)}_j := \binom{(p-1)/2}{j/2}$, and $\chi = \sum_j \chi_j$. In particular, $H^*_{(p)}$ is effective.

To prove Theorem 15, we require the following technical lemma.

Lemma 9. *Let $d \geq 3$ be an odd integer and let $0 \leq j \leq \frac{d-1}{2}$ be even. Define*

$$g_j^{(d)} = \binom{(d-1)/2}{j/2} \quad \text{and} \quad h_j^{(d)} = (-1)^j \sum_{i=0}^{j} (-2)^i \binom{d}{i} \binom{d-1-i}{j-i}.$$

Then the inequality $h_j^{(d)} \geq d \cdot (g_j^{(d)} - 1) + 1$ holds.

Proof. In the case of $j = 0$, the statement follows because $h_0^{(d)} = g_0^{(d)} = 1$. Hence, we let $0 < j \leq \frac{d-1}{2}$. In particular, we have $d \geq 5$.

We start by observing the recurrence relations

$$g_j^{(d)} = g_{j-2}^{(d-2)} + g_j^{(d-2)} \quad \text{and} \quad h_j^{(d)} \geq h_{j-2}^{(d-2)} + 2h_{j-1}^{(d-2)} + h_j^{(d-2)}$$

for $0 < j \leq \frac{d-1}{2}$ and $g_0^{(d)} = h_0^{(d)} = 1$. The inequality for $h_j^{(d)}$ is an equality if $j < (d-1)/2$. If $j = \frac{d-1}{2}$ then we get

$$h_{\frac{d-1}{2}}^{(d)} = 4h_{\frac{d-3}{2}}^{(d-2)} > 2h_{\frac{d-3}{2}}^{(d-2)} + 2h_{\frac{d-5}{2}}^{(d-2)} = h_{\frac{d-5}{2}}^{(d-2)} + 2h_{\frac{d-3}{2}}^{(d-2)} + h_{\frac{d-1}{2}}^{(d-2)}.$$

For $j > 0$, we prove the statement by induction on odd d. Assume $h_j^{(d)} > d \cdot (g_j^{(d)} - 1) + 1$. Then, by the recurrences, we have:

$$\begin{aligned} h_j^{(d+2)} &\geq h_{j-2}^{(d)} + 2h_{j-1}^{(d)} + h_j^{(d)} > d(g_{j-2}^{(d)} - 1) + 1 + 2h_{j-1}^{(d)} + d(g_j^{(d)} - 1) + 1 \\ &= d(g_j^{(d+2)} - 2) + 2 + 2h_{j-1}^{(d)}. \end{aligned}$$

At the same time, we can write:

$$(d+2)(g_j^{(d+2)} - 1) + 1 = d(g_j^{(d+2)} - 2) + 2 + 2(g_{j-2}^{(d)} + g_j^{(d)}) + d - 3.$$

Hence it remains to prove that $h_{j-1}^{(d)} \geq g_{j-2}^{(d)} + g_j^{(d)} + \frac{d-3}{2}$.

Here, by our assumption, we let $j \coloneqq 2k$ and $d \coloneqq 2n + 1$, where $k \geq 1$, $n \geq 2$ and $2k \leq n$. Since $h_\ell^{(d)} \geq \binom{d-1}{\ell}$ holds for any ℓ, we get the following inequalities:

$$\begin{aligned} h_{j-1}^{(d)} = h_{2k-1}^{(2n+1)} &\geq \binom{2n}{2k-1} \geq \binom{2n}{k} \geq \binom{n+1}{k} + n - 1 \\ &= \binom{n}{k} + \binom{n}{k-1} + n - 1 = g_{j-2}^{(d)} + g_j^{(d)} + \frac{d-3}{2}. \end{aligned}$$

This concludes the proof. □

Proof of Theorem 15. For the reflection s, we obtain

$$\det(I - t \cdot \rho_p(s)) = \det \begin{bmatrix} 1-t & 0 & 0 & \cdots \\ 0 & \begin{matrix} 1 & -t \\ -t & 1 \end{matrix} & 0 & \cdots \\ 0 & 0 & \begin{matrix} 1 & -t \\ -t & 1 \end{matrix} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} = (1-t)(1-t^2)^{\frac{p-1}{2}}.$$

For the rotation r, note that p is odd, so we get $\det(I - t\cdot\rho_p(r)) = 1+(-t)^p = 1-t^p$. Since p is a prime number, recall that the rotation r, and any power r^k with $1 \le k \le p-1$, fixes only the origin. That is $P_p^{r^k} = \{0\}$, and so $\operatorname{ehr}(P_p^r, t) = \frac{1}{1-t}$. Using this and the description of the Ehrhart series in Propositions 25 and 26, we obtain:

$$\begin{aligned} H^*_{(p)}[t](1) &= h_0^{(p)} + h_1^{(p)}t + \cdots + h_{p-1}^{(p)}t^{p-1}, \\ H^*_{(p)}[t](s) &= (1+t^2)^{\frac{p-1}{2}}, \\ H^*_{(p)}[t](r) &= \frac{1-t^p}{1-t} = 1 + t + \cdots + t^{p-1}, \end{aligned}$$

where $h_j^{(p)}$ are the values specified in Proposition 25.

Consider now the character of the regular module $\mathbb{R}\, D_{2p}$, which is given by $\psi_1 + \psi_2 + 2\sum_j \chi_j$. It is well known that this character evaluates to zero at every element of D_{2p} except at 1 where it evaluates to $2p$. Hence, we deduce that the composite character $\chi = \sum_j \chi_j$, obtained by adding together all irreducible two-dimensional characters of D_{2p}, is given by:

	1	r^k	sr^k
χ	$p-1$	-1	0

The coefficients $H^*_{(p),j}$ of the H^*-series are given by

$$H^*_{(p),j} = \frac{1}{2p}\begin{cases} (h_j^{(p)} - 1 + p(g_j^{(p)}+1))\psi_1 + (h_j^{(p)} - 1 - p(g_j^{(p)}-1))\psi_2 + (2h_j^{(p)} - 2)\chi & \text{if } 2 \mid j, \\ (p + h_j^{(p)} - 1)\psi_1 + (p + h_j^{(p)} - 1)\psi_2 + (2h_j^{(p)} - 2)\chi & \text{if } 2 \nmid j. \end{cases}$$

It remains to show that these quantities are non-negative integers. Non-negativity follows from Lemma 9 and integrality follows immediately from the fact that $H^*[t]$ is an element of $R(D_{2p})[[t]]$. □

In the last part of this section, we study the equivariant Ehrhart theory of the order 2 subgroups associated to the reflections described in Proposition 26. Fix the subgroup $S_2 = \{1, s\}$ of D_{2d}. We write χ_1 and χ_2 for the trivial and non-trivial characters of S_2 respectively.

Theorem 16. *Let $d \ge 3$ be an integer and let $\ell = \lfloor d/2 \rfloor$ and $b \in \{0,1\}$ be integers such that $d = 2\ell + 1 + b$. The equivariant H^*-series of P_d under the action of S_2, denoted $H^*_{(d)}[t]$, is a polynomial of degree $d-1$ and its coefficients $H^*_{(d),j}$ are given by*

$$H^*_{(d),j} = \frac{1}{2}\left[(h_j^{(d)} + g_j^{(d)})\chi_1 + (h_j^{(d)} - g_j^{(d)})\chi_2\right].$$

where $h_j^{(d)}$ follows the notation from Proposition 25 and $g_j^{(d)}$ are the coefficients of the polynomial $(1+t)^b(1+t^2)^\ell := g_0^{(d)} + g_1^{(d)}t + \cdots + g_{d-1}^{(d)}t^{d-1}$. In particular, $H^*_{(d)}[t]$ is effective.

Proof. By a similar argument to the proof of Theorem 15, we obtain

$$\det(I - t\cdot\rho_d(s)) = (1-t)^{1-b}(1-t^2)^{\ell+b}.$$

By the description of $\mathrm{ehr}(P_d, t)$ in Proposition 26, we have:

$$H^*_{(d)}[t](1) = h_0^{(d)} + h_1^{(d)}t + \cdots + h_{d-1}^{(d)}t^{d-1},$$
$$H^*_{(d)}[t](s) = (1+t)^b(1+t^2)^\ell = g_0^{(d)} + g_1^{(d)}t + \cdots + g_{d-1}^{(d)}t^{d-1}.$$

For the coefficients $H^*_{(d),j}$ of the H^*-series, we obtain

$$H^*_{(d),j} = \frac{1}{2}\left[(h_j^{(d)} + g_j^{(d)})\chi_1 + (h_j^{(d)} - g_j^{(d)})\chi_2\right].$$

It remains to show that $H^*_{(d)}$ is effective, for which it suffices to show that $h_j^{(d)} \geq g_j^{(d)}$. If d is odd, this follows directly from Lemma 9. If d is even, we start with the case where j is also even. We can use that in this case, $g_j^{(d)} = g_j^{(d-1)}$, giving us

$$h_j^{(d)} \geq h_{j-1}^{(d-1)} + h_j^{(d-1)} \geq h_j^{(d-1)} \geq g_j^{(d-1)} = g_j^{(d)}.$$

For the case where j is odd, we may assume without loss of generality that $j \leq \ell - 1$. In this case, we use $g_j^{(d)} = g_{j-1}^{(d)}$ and the fact that $H^*_{(d)}(1)$ is unimodal, to conclude

$$h_j^{(d)} \geq h_{j-1}^{(d)} \geq g_{j-1}^{(d)} = g_j^{(d)}.$$

So we have shown that $H^*_{(d)}[t]$ is effective, completing the proof. □

6.3 Rational cross-polytopes

In this section we prove Theorem 17 which gives a complete description of the equivariant H^*-series of $P(k,d)$ under the action of a reflection group. We observe in Example 9 that a rational analogue of Conjecture 1 does not hold for rational polytopes with period one.

The Ehrhart series $\mathrm{ehr}(P(k,d),t)$ has the following explicit description.

Proposition 27 (An application of [BJM13, Theorem 1.4])**.** *For each k odd and $d \geq 2$ we have*

$$\mathrm{ehr}(P(k,d),t) = (1-t)\,\mathrm{ehr}([k/2,-k/2],t)\frac{(1+t)^{d-1}}{(1-t)^d} = \frac{(1+(k-1)t+kt^2)(1+t)^{d-2}}{(1-t)^{d+1}}.$$

In the following, we will refer to $(1+(k-1)t+kt^2)(1+t)^{d-2}$ by $\tilde{h}_{P(k,d)}$. We denote by $G = \{1,\sigma\}$ the group of order two. We fix its two irreducible characters: the trivial character χ_1 and non-trivial character χ_2. Fix some index $i \in [n]$. We let G act on the lattice $\mathbb{Z}[e_1,\ldots,e_d]$ by a coordinate reflection $\sigma(e_i) = -e_i$ and $\sigma(e_j) = e_j$ for all $j \in [n]\backslash\{i\}$.

Proposition 28. *If $i \in \{1,2,\ldots,d-1\}$, then $H^*[t] = \chi_1 \cdot \tilde{h}_{P(k,d)}(t)$.*

Proof. The reflection σ acts on $P(k,d)$ by the diagonal matrix $A = \mathrm{Diag}(1,\ldots,1,-1,1,\ldots,1)$ where -1 appears in position i. Therefore, we may compute $\det(I - tA) = (1-t)^{d-1}(1+t)$.

We proceed by taking cases on d; either $d = 2$ or $d > 2$. Fix $d = 2$. In this case, the fixed polytope $P(k,2)^\sigma$ is a line segment $[k/2, -k/2]$ and so its Ehrhart series is

$$\mathrm{ehr}(P(k,2)^\sigma, t) = \frac{1 + (k-1)t + kt^2}{(1-t)(1-t^2)} = \frac{\tilde{h}_{P(k,2)}(t)}{(1-t)\det(I - tA)}.$$

On the other hand, the identity element $e \in G$ acts by the identity matrix I and so $\det(I - tI) = (1-t)^3$. Clearly, this fixes the entire polytope $P(k,d)$, so its Ehrhart series is given by

$$\mathrm{ehr}(P(k,2), t) = \frac{1 + (k-1)t + kt^2}{(1-t)^3} = \frac{\tilde{h}_{P(k,2)}(t)}{(1-t)\det(I - tI)}.$$

And so we have that $H^*[t] = \chi_1 \cdot \tilde{h}_{P(k,2)}(t)$ and we are done for the case $d = 2$.

Next, let $d > 2$. The fixed polytope $P(k,d)^\sigma$ is equal to $P(k, d-1)$ in a one-dimension-higher ambient space, and so, by Proposition 27, its Ehrhart series is given by

$$\mathrm{ehr}(P(k,d)^\sigma, t) = \frac{(1 + (k-1)t + kt^2)(1+t)^{d-3}}{(1-t)^d}\frac{(1+t)}{(1+t)} = \frac{\tilde{h}_{P(k,d)}(t)}{(1-t)\det(I - tA)}.$$

On the other hand the identity element $e \in G$ fixes the entire polytope $P(k,d)$ and so its Ehrhart series is

$$\mathrm{ehr}(P(k,d), t) = \frac{\tilde{h}_{P(k,d)}(t)}{(1-t)^{d+1}} = \frac{\tilde{h}_{P(k,d)}(t)}{(1-t)\det(I - tI)}.$$

And so it follows that $H^*[t] = \chi_1 \cdot \tilde{h}_{P(k,d)}(t)$ and we are done for the case $d > 2$. □

Proposition 29. *If $i = d$, then $H^*[t] = \sum_{j=0}^{d}(a_j\chi_1 + b_j\chi_2)t^j$ where*

$$a_j = \binom{d-2}{j} + \frac{1}{2}(k+1)\binom{d-1}{j-1} \text{ and } b_j = \frac{1}{2}(k-1)\binom{d-1}{j-1} - \binom{d-2}{j-1}$$

and $\binom{n}{k}$ is defined to be zero if $k < 0$ or $k > n$.

Proof. The identity $e \in G$ acts by the identity matrix I, hence $\det(I - tI) = (1-t)^d$. So, by Proposition 27, we have

$$\mathrm{ehr}(P(k,d), t) = \frac{(1 + (k-1)t + kt^2)(1+t)^{d-2}}{(1-t)^{d+1}} = \frac{(1 + (k-1)t + kt^2)(1+t)^{d-2}}{(1-t)\det(I - tI)}.$$

On the other hand, the reflection acts by the diagonal matrix $A = \mathrm{Diag}(1,\ldots,1,-1)$ hence $\det(I - tA) = (1-t)^{d-1}(1+t)$. Observe that the fixed polytope $P(k,d)^\sigma$ is a $(d-1)$-dimensional cross-polytope, therefore we have

$$\mathrm{ehr}(P(k,d)^\sigma, t) = \frac{(1+t)^{d-1}}{(1-t)^d} = \frac{(1+t)^d}{(1-t)\det(I - tA)}.$$

Write $H^*[t] = \sum_{j=0}^{d}(a_j\chi_1 + b_j\chi_2)t^j$ for some a_j and b_j. By evaluating $H^*[t]$ at each group element $g \in G$, we have $H^*[t](g) = \mathrm{ehr}(P(k,d)^g, t)(1-t)\det(I - t\rho(g))$. It follows that

$$\begin{cases} a_j + b_j = \binom{d-2}{j} + (k-1)\binom{d-2}{j-1} + k\binom{d-2}{j-2}, \\ a_j - b_j = \binom{d}{j} = \binom{d-2}{j} + 2\binom{d-2}{j-1} + \binom{d-2}{j-2} \end{cases}$$

for $j \in \{0, 1, \ldots, d\}$ where $\binom{n}{k}$ is defined to be zero if $k < n$ or $k > n$. By solving this, we obtain the desired conclusion. □

Example 9. Consider the case $d = 2$ and $k = 1$. The polytope $P(1,2)$ is given by

$$P(1,2) = \mathrm{conv}\{(1,0), (-1,0), (0,1/2), (0,-1/2)\}.$$

The group $G = \{1, \sigma\}$ acts by a coordinate reflection: $\sigma(e_2) = -e_2$ and $\sigma(e_1) = e_1$. The equivariant Ehrhart H^*-series is $H^*[t] = \chi_1 + (\chi_1 - \chi_2)t + \chi_1 t^2$. In particular, $H^*[t]$ is polynomial but not effective since $\chi_1 - \chi_2$ is not the character of a representation of G.

Remark 7. Consider the dilate of the polytope $2P(1,2)$ with the same group action as in Example 9. In this case the equivariant H^*-series is given by $H^*[t] = \chi_1 \cdot (1 + 4t + 3t^2) = \chi_1 \cdot \tilde{h}_{2P(k,d)}(t)$. The example $P(1,2)$ does not extend to an example of a lattice polytope since all lattice points of $P(1,2)$ are fixed by G. However, if G is a non-trivial group acting non-trivially on a full dimensional lattice polytope, then at least one lattice point of P is not fixed by G. Concretely, we can say the following about two dimensional polytopes.

Suppose that G is the group of order 2 and irreducible characters χ_1 and χ_2. Assume G acts on a 2-dimensional lattice M and let P be a G-invariant lattice polytope with a polynomial equivariant H^*-series given by $H^*[t] = \chi_1 + (a\chi_1 + b\chi_2)t + c\chi_1 t^2$ for some $a, b, c \in \mathbb{Z}$. By Corollary 6.7 in [Sta11], $H^*[t]$ is effective. Moreover, since χ_1 corresponds to a trivial permutation representation and $\chi_1 + \chi_2$ corresponds to the regular representation, which is a permutation representation as well, the linear coefficient of $H^*[t]$ is itself a permutation representation if $a \geq b \geq 0$. To see that this is satisfied, one first should notice that $2P^\sigma$ is a lattice polytope by Corollary 5.4 in [Sta11] and so it is either a line segment or a point whose vertices have coordinates lying in $\frac{1}{2}\mathbb{Z}$. If P^σ is a non-lattice point, then the result follows from a simple computation. So, by Lemma 7.3 in [Sta11] and our assumption that $H^*[t]$ is a polynomial, we only need to consider the case where P^σ contain a lattice point. So, it follows that P^σ is unimodularly equivalent to a line segment $[v, w] \subseteq \mathbb{R}$ with v, $w \in \frac{1}{2}\mathbb{Z}$. By taking cases on whether v or w lie in $\mathbb{Z}$ we can show that the Ehrhart series has the form

$$\mathrm{ehr}(P^\sigma, t) = \frac{1 + rt + st^2}{(1-t)(1-t^2)}$$

with $r, s \geq 0$. Evaluating $H^*[t]$ at σ and comparing coefficients gives us $a - b = r \geq 0$

Let $G = (\mathbb{Z}/2\mathbb{Z})^d = \langle \sigma_1, \sigma_2, \ldots, \sigma_d \rangle$ be the group of coordinate reflections of $\mathbb{R}^d$. Explicitly, for each $i, j \in \{1, 2, \ldots, d\}$ we have $\sigma_i(e_i) = -e_i$ and $\sigma_i(e_j) = e_j$ if $i \neq j$. Let

χ_1 denote the trivial character of G and χ_2 denote the character satisfying $\chi_2(\sigma_d) = -1$ and $\chi_2(\sigma_i) = 1$ for all $i \in \{1, 2, \ldots, d-1\}$. The polytope $P(k, d)$ is invariant under G. By Propositions 28 and 29 it follows that the equivariant H^*-series $H^*[t]$ of P is a polynomial whose coefficients are integer multiples of χ_1 and χ_2. Moreover, we obtain the following result.

Theorem 17. *With the setup above, we have* $H^*[t] = \sum_{j=0}^{d}(a_j\chi_1 + b_j\chi_2)t^j$ *where*

$$a_j = \binom{d-2}{j} + \frac{1}{2}(k+1)\binom{d-1}{j-1} \text{ and } b_j = \frac{1}{2}(k-1)\binom{d-1}{j-1} - \binom{d-2}{j-1}$$

and $\binom{n}{k}$ *is defined to be zero if* $k < 0$ *or* $k > n$.

Example 10. Let $d \geq 3$ and $k = 1$. The polytope $P(k, d)$ has Ehrhart series

$$\operatorname{ehr}(P(1, d), t) = \frac{(1 + t + t^2 + t^3)(1 + t)^{d-3}}{(1 - t)^{d+1}}.$$

We note that this coincides with the Ehrhart series of the lattice polytope $Q_d \subseteq \mathbb{R}^3 \times \mathbb{R}^{d-3}$ given by $Q_d = \operatorname{conv}\{e_1, e_2, e_3, -e_1-e_2-e_3\} \oplus [-1, 1]^{\oplus(d-3)}$. By a result of Stapledon [Sta11, Proposition 6.1], the equivariant H^*-series of the simplex $S = \operatorname{conv}\{e_1, e_2, e_3, -e_1-e_2-e_3\}$ is always effective. If a group $G = \{1, \sigma\}$ acts on Q_d with an action that factors $\sigma(x, y) = (\sigma|_{\mathbb{R}^3}(x), \sigma|_{\mathbb{R}^{d-3}}(y))$ such that $\sigma|_{\mathbb{R}^{d-3}}$ acts by a coordinate reflection, then the equivariant H^*-series of Q_d is $(1 + t)^{d-3}$ times the H^*-series of S, meaning that it is effective.

On the other hand, if we take the polytope $P(1, d)$ with respect to the action of $G = \{1, \sigma\}$ given by $\sigma(e_d) = -e_d$ and $\sigma(e_i) = e_i$ for all $i \in \{1, \ldots, d-1\}$ then the equivariant H^*-series is not effective.

Chapter 7

The equivariant Ehrhart Theory of the hypersimplex

In this chapter we study the equivariant Ehrhart theory of hypersimplices under the action of the symmetric group. The authors of [EKS24] already showed that the equivariant H^*-series is effective in that case, but we find a new, more direct proof. Further, we show that $H^*[1]$ is a permutation action, confirming Conjecture 2 in this case. The content of this chapter is fully contained in the author's paper [CK24] with Oliver Clarke.

7.1 The main results

Let $0 < k < n$ be integers and let Δ_k^n be the hypersimplex of type (k, n). In the following, we will consider the action of the symmetric group S_n on $\mathbb{R}^n$ via coordinate permutation. Note that Δ_k^n is fixed by this action.

Fix $\sigma \in S_n$ a permutation with cycle type $(s_1, \dots, s_r)$. For each $0 < k < n$ and $m \geq 0$, we define the set of functions

$$\Phi_k(\sigma, m) = \left\{ f \colon [r] \to \{0, 1, \dots, k-1\} \colon \sum_{i=1}^{r} f(i) s_i = m \right\}.$$

By convention, we define $\Phi_\ell(\sigma, m) = \emptyset$ for all $m < 0$. For each $h \geq 0$, we define the set

$$\mathcal{I}_h = \left\{ I = (I_1, I_2, \dots, I_{k-1}) \in \mathbb{Z}_{\geq 0}^{k-1} \colon \sum_{i=1}^{k-1} i \cdot I_i = h \right\}.$$

For each $I \in \mathcal{I}_h$, we write $|I| = I_1 + I_2 + \cdots + I_{k-1}$. The number of functions $|\Phi_k(\sigma, m)|$ is the evaluation of the permutation character of S_n.

Proposition 30. *Fix $0 < k < n$ and $m \geq 0$. Let χ be the permutation character of S_n acting on the set of function functions $\{f : [n] \to \{0, 1, \dots, k-1\} : \sum_{i=1}^{n} f(i) = m\}$ by $(\sigma \cdot f)(i) = f(\sigma^{-1}(i))$. Then $\chi(\sigma) = |\Phi_k(\sigma, m)|$.*

Proof. Follows immediately from the definition. □

With this, we can state our first main result.

Theorem 18. *Fix $0 < k < n$ and $\sigma \in S_n$ with cycle type $(s_1, s_2, \dots, s_r)$. For each $m \geq 0$, the m-th coefficient of the equivariant H^*-series of $H^*_{\Delta^n_k}$ is*

$$H^*_m(\sigma) = \sum_{h=0}^{k-1} \left(\sum_{I \in \mathcal{I}_h} (-1)^{|I|} \binom{\lambda_1}{I_1} \cdots \binom{\lambda_{k-1}}{I_{k-1}} \right) |\Phi_{k-h}(\sigma, m(k-h) - h)|.$$

In particular, H^* has degree $\left\lfloor \frac{(k-1)n}{k} \right\rfloor$.

The Ehrhart theory of hypersimplices is closely related to a class of combinatorial objects called *decorated ordered set partitions* (DOSPs for short) which we shall introduce now.

Definition 13. A (k, n)-DOSP is an ordered partition $(L_1, \dots, L_r)$ of $\{1, 2, \dots, n\}$ together with a sequence of positive integers $(\ell_1, \dots, \ell_r)$ such that $\ell_1 + \ell_2 + \cdots + \ell_r = k$. We write a DOSP as a sequence of pairs $D = ((L_1, \ell_1), \dots, (L_r, \ell_r))$. A DOSP is defined up to cyclic permutation. So, for example, we have $D = ((L_2, \ell_2), \dots, (L_r, \ell_r), (L_1, \ell_1))$. We say D is *hypersimplicial* if $|L_i| > \ell_i$ for each $i \in \{1, \dots, r\}$.

For every DOSP D, Early [Ear17] defines the *winding number* $w(D) \in \{0, 1, \dots, n-1\}$, see Definition 14, and conjectured that the h^*-polynomial is given by

$$h^*_{\Delta^n_k}(t) = \sum_D t^{w(D)}$$

where the sum is taken over all hypersimplicial (k, n)-DOSPs. This conjecture was proved by Kim [Kim20].

In [EKS24], this result is brought to the equivariant realm.

Theorem 19 ([EKS24, Theorem 3.33])**.** *Let $0 < k < n$ and let the cyclic group C_n act on $\mathbb{R}^n$ via cyclic permutation of the coordinates. Further, let H^*_m denote the m-th coordinate of the equivariant H^*-series of the hypersimplex of type (k, n) under the action of C_n. Then H^*_m is the permutation representation of the (k, n)-DOSPs with winding number m where C_n acts by cyclically permuting the set $[n]$.*

In particular, this satisfies Stapledon's Conjecture 2. Under the full action of the symmetric group, an analogous result does not hold because in general, the coefficients of the equivariant Ehrhart series of any given hypersimplex are not permutation representations. Our second main result shows that something similar is still true.

Theorem 20 (Theorem 22)**.** *The character $H^*[1]$ of the equivariant H^*-polynomial of Δ^n_k under the action of S_n is equal to the permutation character of S_n acting on the set of hypersimplicial (k, n)-DOSPs.*

7.2 Coefficients of the equivariant H^*-polynomial

The goal of this section is to prove Theorem 18.

7.2.1 Katzman's method

In this section, we apply the method used by Katzman [Kat05] to obtain a formula for the coefficients of the equivariant H^*-series. First, we introduce two pieces of useful notation. Given $\sigma \in S_n$ with cycle type $(s_1, \dots, s_r)$, we define the formal power series

$$u^\sigma = \sum_{i \geq 0} u_i^\sigma t^i = \prod_{i=1}^{k} 1 + t^{s_i} + t^{2s_i} + \cdots = \prod_{i=1}^{k} \frac{1}{1 - t^{s_i}} \in \mathbb{Z}[[t]].$$

If σ is clear from context, then we write u for u^σ and u_i for u_i^σ. For each subset $S \subseteq [r]$, we write $\Sigma^\sigma S = \sum_{i \in S} s_i$. If the permutation σ is clear from context then we write ΣS for $\Sigma^\sigma S$.

Lemma 10. *Fix $0 < k < n$ and $\sigma \in S_n$ with cycle type $(s_1, \dots, s_r)$. For each $i \in [n]$, let λ_i be the number of length i cycles of σ. For each $m \geq 0$ we have*

$$H_m^*(\sigma) = \sum_{S \subseteq [r]} (-1)^{|S|} \sum_{h=0}^{k-1} \left(\sum_{I \in \mathcal{I}_h} (-1)^{|I|} \binom{\lambda_1}{I_1} \cdots \binom{\lambda_{k-1}}{I_{k-1}} \right) u_{(m - \Sigma S)(k-h)-h}.$$

Proof. By conjugating σ, we may assume without loss of generality that

$$\sigma = (1\ 2\ \dots\ s_1)(s_1+1\ s_1+2\ \dots\ s_1+s_2)\dots(n-s_r+1\ n-s_r+2\ \dots\ n).$$

Fix $d \geq 0$. We have that $(d\Delta_k^n)_\sigma \cap \mathbb{Z}^n$ is equal to

$$\left\{ (\underbrace{x_1, \dots, x_1}_{s_1}, \underbrace{x_2, \dots, x_2}_{s_2}, \dots, \underbrace{x_r, \dots, x_r}_{s_r}) \in \mathbb{Z}^n : \sum_{i=1}^{r} x_i s_i = kd,\ 0 \leq x_i \leq d \text{ for all } i \in [r] \right\}.$$

So there is a bijection between the set solutions $(x_1, x_2, \dots, x_r) \in \{0, 1, \dots, d\}^r$ to $\sum_{i=1}^r x_i s_i = kd$ and $(d\Delta_k^n)_\sigma \cap \mathbb{Z}^n$. Consider the polynomial

$$f_d(t) = \prod_{i=1}^{r} (1 + t^{s_i} + t^{2s_i} + \cdots + t^{ds_i}) = \prod_{i=1}^{r} \frac{1 - t^{(d+1)s_i}}{1 - t^{s_i}}.$$

For each solution $(x_1, \dots, x_r)$ to the above equation, we have a term $t^{kd} = t^{x_1 s_1} t^{x_2 s_2} \dots t^{x_r s_r}$. Moreover, each term t^{kd} in the expansion of $f_d(t)$ arises from such a solution. So we have that $|(d\Delta_k^n)_\sigma \cap \mathbb{Z}^n|$ is equal to the coefficient of t^{kd} in $f_d(t)$, which we denote by $[f_d]_{kd}$.

For each $s_i \geq k$, we have that $t^{(d+1)s_i}$ does not divide t^{kd}. It follows that $[f_d]_{kd}$ is equal to the coefficient of t^{kd} in the formal power series:

$$\begin{aligned}
[f_d]_{kd} &= \left[\prod_{j=1}^{k-1}(1-t^{(d+1)j})^{\lambda_j}\prod_{i=1}^{r}\frac{1}{1-t^{s_i}}\right]_{kd} \\
&= \left[\prod_{j=1}^{k-1}\sum_{h=0}^{\lambda_j}(-1)^h\binom{\lambda_j}{h}t^{(d+1)jh}\prod_{i=1}^{r}\frac{1}{1-t^{s_i}}\right]_{kd} \\
&= \left[\sum_{h=0}^{k-1}\left(\sum_{I\in\mathcal{I}_h}(-1)^{|I|}\binom{\lambda_1}{I_1}\binom{\lambda_2}{I_2}\cdots\binom{\lambda_{\ell-1}}{I_{k-1}}\right)t^{(d+1)h}\prod_{i=1}^{r}\frac{1}{1-t^{s_i}}\right]_{kd} \\
&= \sum_{h=0}^{k-1}\left(\sum_{I\in\mathcal{I}_h}(-1)^{|I|}\binom{\lambda_1}{I_1}\binom{\lambda_2}{I_2}\cdots\binom{\lambda_{\ell-1}}{I_{k-1}}\right)u_{kd-(d+1)h}.
\end{aligned}$$

So the Ehrhart series of $(\Delta_k^n)_\sigma$ is given by

$$\sum_{d\geq 0}[f_d]_{kd}t^d = \sum_{d\geq 0}\left(\sum_{h=0}^{k-1}\left(\sum_{I\in\mathcal{I}_h}(-1)^{|I|}\binom{\lambda_1}{I_1}\cdots\binom{\lambda_{k-1}}{I_{k-1}}\right)u_{kd-h(d+1)}\right)t^d = \frac{H^*[t](\sigma)}{\prod_{i=1}^{r}1-t^{s_i}},$$

where the right-most equality follows from definition of the equivariant H^*-series. So, by clearing the denominator, we obtain a formula for the coefficients of equivariant H^*-series. For each $m \geq 0$, we have

$$\begin{aligned}
H_m^*(\sigma) &= \left[\prod_{i=1}^{r}(1-t^{s_i})\sum_{d\geq 0}\left(\sum_{h=0}^{k-1}\left(\sum_{I\in\mathcal{I}_h}(-1)^{|I|}\binom{\lambda_1}{I_1}\cdots\binom{\lambda_{k-1}}{I_{k-1}}\right)u_{kd-h(d+1)}\right)t^d\right]_m \\
&= \sum_{S\subseteq[r]}(-1)^{|S|}\sum_{h=0}^{k-1}\left(\sum_{I\in\mathcal{I}_h}(-1)^{|I|}\binom{\lambda_1}{I_1}\cdots\binom{\lambda_{k-1}}{i_{\ell-1}}\right)u_{(m-\Sigma S)(k-h)-h}.
\end{aligned}$$

This concludes the proof of the result. □

We illustrate the steps in the above proof with the following example.

Example 11. Let $k = 3$. In this case, we consider the sets $\mathcal{I}_0$, $\mathcal{I}_1$, and $\mathcal{I}_2$, which are given by

$$\mathcal{I}_0 = \{(0,0)\}, \quad \mathcal{I}_1 = \{(1,0)\}, \quad \mathcal{I}_2 = \{(2,0),(0,1)\}.$$

So, we have

$$[f_d]_{3d} = \left[\left(1-\lambda_1 t^{d+1}+\left(\binom{\lambda_1}{2}-\lambda_2\right)t^{2(d+1)}\right)u^\sigma\right]_{3d} = u_{3d}-\lambda_1 u_{2d-1}+\left(\binom{\lambda_1}{2}-\lambda_2\right)u_{d-2}.$$

The Ehrhart series of $(\Delta_3^n)_\sigma$ is given by

$$\frac{H^*[t](\sigma)}{\prod_{i=1}^r (1-t^{s_i})} = \sum_{d\geq 0} \left(u_{3d} - \lambda_1 u_{2d-1} + \left(\binom{\lambda_1}{2} - \lambda_2 \right) u_{d-2} \right) t^d.$$

The coefficient of t^m in the H^* series is given by

$$H_m^* = \sum_{S\subseteq[k]} (-1)^{|S|} \left(u_{3(m-\Sigma S)} - \lambda_1 u_{2(m-\Sigma S)-1} + \left(\binom{\lambda_1}{2} - \lambda_2 \right) u_{m-\Sigma S-2} \right).$$

7.2.2 Permutation representation interpretation

In this section we prove Theorem 18. To do this, we use the sets of functions $\Phi_k(\sigma, m)$, see Section 7.1, to give an interpretation of terms appearing in formula for H_m^* in Lemma 10.

Lemma 11. *Fix $0 < k < n$ and $\sigma \in S_n$ with cycle type $(s_1, \ldots, s_r)$. With our usual notation, we have*

$$\sum_{S\subseteq[r]} (-1)^{|S|} u_{m-k\Sigma S} = |\Phi_k(\sigma, m)|.$$

Proof. We prove the result by induction on r. For the base case, assume $r = 1$, i.e., σ is an n-cycle. We have $\Sigma\emptyset = 0$ and $\Sigma\{1\} = n$, and $u = 1 + t^n + t^{2n} + \cdots = 1/(1-t^n)$. Therefore the left-hand sum is given by

$$\sum_{S\subseteq[r]} (-1)^{|S|} u_{m-k\Sigma S} = u_m - u_{m-kn} = \begin{cases} 1 & \text{if } m \in \{0, n, 2n, \ldots, (k-1)n\}, \\ 0 & \text{otherwise.} \end{cases}$$

On the other hand, there are exactly k functions $f : [r] \to \{0, 1, \ldots, k-1\}$, and, any such function f satisfies $\sum_i f(i)s_i = f(1)n$. Therefore

$$|\Phi_k(\sigma, m)| = \begin{cases} 1 & \text{if } m \in \{0, n, 2n, \ldots, (k-1)n\} \\ 0 & \text{otherwise} \end{cases} = \sum_{S\subseteq[r]} (-1)^{|S|} u_{m-k\Sigma S},$$

and we are done with the base case.

For the induction step, let σ be a permutation with cycle type $(s_1, s_2, \ldots, s_{r+1})$ and assume that the result holds for any permutation with r disjoint cycles. Let τ be a permutation with cycle type $(s_1, \ldots, s_r)$. For ease of notation, we define $s := s_{r+1}$. We note that $u^\sigma = u^\tau(1 + t^s + t^{2s} + \ldots)$, so it follows that $u_i^\sigma = \sum_{j\geq 0} u_{i-sj}^\tau$. Then have the following chain of equalities:

$$\begin{aligned}\sum_{S\subseteq[r+1]}(-1)^{|S|}u^{\sigma}_{m-k\Sigma^{\sigma}S} &= \sum_{S\subseteq[r]}(-1)^{|S|}\left(u^{\sigma}_{m-k\Sigma^{\sigma}S}-u^{\sigma}_{m-k\Sigma^{\sigma}S-ks}\right)\\ &= \sum_{S\subseteq[r]}(-1)^{|S|}\left(u^{\sigma}_{m-k\Sigma^{\tau}S}-u^{\sigma}_{m-k\Sigma^{\tau}S-ks}\right)\\ &= \sum_{S\subseteq[r]}(-1)^{|S|}\sum_{j\geq 0}\left(u^{\tau}_{m-k\Sigma^{\tau}S-sj}-u^{\tau}_{m-k\Sigma^{\tau}S-s(j+k)}\right)\\ &= \sum_{S\subseteq[k]}(-1)^{|S|}\sum_{j=0}^{k-1}u^{\tau}_{m-k\Sigma^{\tau}S-sj}\\ &= \sum_{j=0}^{k-1}|\Phi_k(\tau,m-sj)|.\end{aligned}$$

To conclude the proof, we note that there is a natural bijection between the sets

$$\Phi_k(\sigma,m)\leftrightarrow\bigsqcup_{j=0}^{k-1}\Phi_k(\tau,m-sj)$$

$$f\mapsto f|_{[r]}\in\Phi_k(\tau,m-sf(k))$$

$$\left(i\mapsto\begin{cases}f(i) & \text{if } i\in[r]\\ j & \text{if } i=r+1\end{cases}\right)\leftarrow f\in\Phi_k(\tau,m-sj)\text{ for some } j\in\{0,1,\dots,k-1\}.$$

It follows that $|\Phi_k(\sigma,m)|=\sum_{j=0}^{k-1}|\Phi_k(\tau,m-sj)|=\sum_{S\subseteq[r+1]}(-1)^{|S|}u^{\sigma}_{m-k\Sigma^{\sigma}S}$. This concludes the proof of the result. □

With this result, we can give a proof of Theorem 18.

Proof of Theorem 18. By Lemma 11, we have

$$\sum_{S\subseteq[r]}(-1)^{|S|}u_{m(k-h)-h-(k-h)\Sigma S}=|\Phi_{k-h}(\sigma,m(k-h)-h)|.$$

By Lemma 10, we have

$$\begin{aligned}H^*_m(\sigma) &= \sum_{S\subseteq[r]}(-1)^{|S|}\sum_{h=0}^{k-1}\left(\sum_{I\in\mathcal{I}_h}(-1)^{|I|}\binom{\lambda_1}{I_1}\cdots\binom{\lambda_{k-1}}{I_{k-1}}\right)u_{(m-\Sigma S)(k-h)-h}\\ &= \sum_{h=0}^{k-1}\left(\sum_{I\in\mathcal{I}_h}(-1)^{|I|}\binom{\lambda_1}{I_1}\cdots\binom{\lambda_{k-1}}{I_{k-1}}\right)\sum_{S\subseteq[r]}(-1)^{|S|}u_{m(k-h)-h-(k-h)\Sigma S}\\ &= \sum_{h=0}^{k-1}\left(\sum_{I\in\mathcal{I}_h}(-1)^{|I|}\binom{\lambda_1}{I_1}\cdots\binom{\lambda_{k-1}}{I_{k-1}}\right)|\Phi_{k-h}(\sigma,m(k-h)-h)|.\end{aligned}$$

We will now prove that H^* is indeed a polynomial of degree $\lfloor (k-1)n/k \rfloor$. Let $\sigma \in S_n$ be a permutation with cycle type $(s_1, \ldots, s_r)$ and fix $h \geq 0$. For any function $f : [r] \to \{0, 1, \ldots, k-h-1\}$, we have

$$\sum_{i=1}^{r} f(i) s_i \leq (k-h-1)n.$$

If m satisfies $(k-1)n < km$ then we have

$$\sum_{i=1}^{r} f(i) s_i \leq (k-h-1)n < \frac{(k-h-1)km}{k-1} = km - \frac{khm}{k-1} = m(k-h) - \frac{hm}{k-1} < m(k-h) - h,$$

hence the set $\Phi_k(\sigma, m(k-h)-h)$ is empty. Thus the coefficient $H^*_m(\sigma) = 0$ for all $m > (k-1)n/k$, so the H^* series is a polynomial.

On the other hand, let $e \in S_n$ be the identity. The set $\Phi_k(e, mk)$ is non-empty if and only if m satisfies $(k-1)n \geq km$. Therefore, the degree of the H^*-polynomial is at least $\lfloor (k-1)n/k \rfloor$, which concludes the proof.

□

7.3 Decorated ordered set partitions

In this section we show that $H^*(\Delta_k^n; S_n)[1]$ is a permutation character of S_n acting naturally on the set of hypersimplicial (k,n)-DOSPs. From Section 7.1 we recall the definition of the set $\mathcal{I}_h$. Our main result gives a formula for the number of hypersimplicial σ-fixed (k,n)-DOSPs.

Theorem 21 (Theorem 22)**.** *Let $2 \leq k < n$ and H^* be the equivariant H^*-polynomial of Δ_k^n under the action of S_n. Then $H^*[1]$ is equal to the permutation character of the action of S_n on hypersimplicial (k,n)-DOSPs. Let $\sigma \in S_n$ be a permutation with r disjoint cycles, and write λ_i for the number cycles of length i. Then the number of σ-fixed hypersimplicial (k,n)-DOSPs is*

$$H^*[1](\sigma) = g \sum_{h=0}^{k-1} \left(\sum_{I \in \mathcal{I}_h} (-1)^{|I|} \binom{\lambda_1}{I_1} \cdots \binom{\lambda_{k-1}}{I_{k-1}} \right) (k-h)^{r-1}$$

where $g = \gcd(\{k\} \cup \{i \in [n] : \lambda_i \geq 1\})$.

In the following sections we state and prove Theorem 21 in two steps. First, we use Theorem 18 to show that the above formula for $H^*[1]$ holds. We count the total number of σ-fixed (k,n)-DOSPs, including non-hypersimplicial DOSPs. We observe that this total corresponds to the $h = 0$ term in the above sum, which is equal to gk^{r-1}. Second, we give an explicit formula for the number of σ-fixed non-hypersimplicial DOSPs using the inclusion-exclusion principle. We simplify the formula using the falling factorial identity for Stirling numbers to prove the theorem. In the remainder of this section, we give an alternative but equivalent definition for DOSPs under the action of S_n and define a notion of *directed distance* within a DOSP.

Alternative DOSP definition. Fix $k < n$. Let $\Psi = \{f : [n] \to \mathbb{Z}/k\mathbb{Z}\}$ be the set of functions modulo the equivalence relation $f \sim g$ if and only if $f - g$ is constant. Then there is an action of S_n on Ψ given by $(\sigma \cdot f)(i) = f(\sigma^{-1}(i))$ for each $\sigma \in S_n$ and $f \in \Psi$. We now describe the natural S_n-set isomorphism between Ψ and the set of (k,n)-DOSPs. Given a DOSP $D = ((L_1, \ell_1), (L_2, \ell_2), \ldots, (L_t, \ell_t))$, its corresponding function is $f_D(i) = 0$ if $i \in L_1$ and $f_D(i) = \ell_1 + \ell_2 + \cdots + \ell_{j-1}$ if $i \in L_j$ with $j \geq 2$. It is straightforward to check that the map $D \mapsto f_D$ is an isomorphism of S_n-sets.

Definition 14 (Distance in DOSPs, winding and turning number)**.** Let $i, j \in [n]$ and let $D = ((L_1, \ell_1), \ldots, (L_t, \ell_t))$ be a (k,n)-DOSP. We define the *directed distance* $d_D(i,j)$ from i to j in D as follows. Without loss of generality, we may assume $i \in L_1$. Suppose that $j \in L_u$ for some $u \in [t]$. Then $d_D(i,j) := \ell_1 + \ell_2 + \cdots + \ell_{u-1} \in \{0, 1, \ldots, k-1\}$. The *winding number* of D is $w(D) = (d_D(1,2) + d_D(2,3) + \cdots + d_D(n-1,n) + d_D(n,1))/k$. If we think of the DOSP as a function $f : [n] \to \mathbb{Z}/k\mathbb{Z}$, then, for each $i, j \in [n]$, the directed distance $d_f(i,j) = f(j) - f(i)$ by taking a representative in $\{0, 1, \ldots, k-1\} \subseteq \mathbb{Z}$.

Fix a permutation $\sigma \in S_n$. Given a σ-fixed DOSP $f : [n] \to \mathbb{Z}/k\mathbb{Z}$, we define the *turning number* τ of f to be the $\tau \in \mathbb{Z}/k\mathbb{Z}$ such that $\tau + f(i) = (\sigma \cdot f)(i)$ for any $i \in [n]$. This notion is well-defined since σ fixes f. See Figure 7.1 for an example with turning number 3.

Recall that the cycle sets of σ are denoted $C_1, C_2, \ldots, C_r$ and partition $[n]$ into sets of size $s_1, \ldots, s_r$ respectively. For every $i \in [r]$, we fix a distinguished element $q_i \in C_i$.

Lemma 12. *Fix $2 \leq k < n$ and let $\sigma \in S_n$ be a permutation with cycle type $(s_1, s_2, \ldots, s_r)$. Let $g = \gcd(s_1, \ldots, s_r, k)$. If D is a σ-fixed (k,n)-DOSP with turning number τ, then $g\tau = 0$.*

Proof. Suppose that $f : [n] \to \mathbb{Z}/k\mathbb{Z}$ is a σ-fixed DOSP with non-zero turning number τ. Notice that $(\sigma^{s_i} \cdot f)(q_i) = f(q_i)$. So we have $s_i\tau = 0$ for every $1 \leq i \leq r$. Since $\tau \in \mathbb{Z}/k\mathbb{Z}$ we have $k\tau = 0$, and it follows that $g\tau = 0$. □

7.3.1 Interpreting terms with DOSPs

Throughout, we fix $k < n$ and write $H^*[t] = H^*(\Delta_k^n; S_n)[t]$ for the equivariant H^*-polynomial. By Theorem 18, let $d = \lfloor (k-1)n/k \rfloor$ be the degree of H^*. We have that

$$H^*[1](\sigma) = \sum_{m=0}^{d} \sum_{h=0}^{k-1} \left(\sum_{I \in \mathcal{I}_h} (-1)^{|I|} \binom{\lambda_1}{I_1} \cdots \binom{\lambda_{k-1}}{I_{k-1}} \right) |\Phi_{k-h}(\sigma, m(k-h) - h)|.$$

The $h = 0$ term in the above sum is $\sum_{m=0}^{d} |\Phi_\ell(\sigma, m\ell)|$, which we will show corresponds to the number of σ-fixed (k,n)-DOSPs. We subsequent sections, we show that the remaining terms count the number of non-hypersimplicial (k,n)-DOSPs. Hence, the overall value $H^*[1]$ is the number of σ-fixed hypersimplicial (k,n) DOSPs.

For ease of notation, let us write functions as tuples. The function $f : [n] \to \{0,\dots,k-1\}$ is written as $(f(1), f(2), \dots, f(n)) \in \{0,\dots,k-1\}^n$.

Example 12. Consider the case $n = 6$, $k = 3$, and take the permutation $\sigma = (1\,2\,3\,4)(5\,6)$. In this case we have the functions

$$\bigcup_{m=0}^{4} \Phi_3(\sigma, 3m) = \{(0,0,0,0,0,0), (1,1,1,1,1,1), (2,2,2,2,2,2)\}.$$

Indeed there are three DOSPs that are fixed by σ, which are given by

$$D_1 = ((123456, 3)), \quad D_2 = ((1234, 2), (56, 1)), \quad D_3 = ((1234, 1), (56, 2)).$$

Lemma 13. *Fix $0 < k < n$. Let $\sigma \in S_n$ be a permutation with cycle type $(s_1, \dots, s_r)$ and define $g = \gcd(k, s_1, s_2, \dots, s_r)$. Then the number of σ-fixed DOSPs is gk^{r-1}. In particular, there is a bijection between the set of σ-fixed DOSPs and the set*

$$\{(\alpha_1, \alpha_2, \dots, \alpha_r) : 0 \le \alpha_1 \le g-1,\ 0 \le \alpha_i \le k-1,\ 2 \le i \le r\}.$$

Proof. Throughout, we consider DOSPs as functions $f : [n] \to \mathbb{Z}/k\mathbb{Z}$ up to equivalence. If σ is the identity permutation, then every DOSP is fixed by σ. The number of σ-fixed DOSPs is k^{n-1}, because we may take $f(1) = 0$ and freely choose the values $f(i) \in \mathbb{Z}/k\mathbb{Z}$ for each $i \in \{2, 3, \dots, n\}$. Note that each such choice gives a distinct DOSP.

Now suppose that σ is not the identity. Let $C_1, C_2, \dots, C_r$ be the cycle sets of σ. Without loss of generality, we may assume that $s_1 > 1$ and $1 \in C_1$. We define $q_1 = \sigma(1)$ and, for each $i \in \{2, \dots, r\}$, let us fix a distinguished element $q_i \in C_i$. Let $f : [n] \to \mathbb{Z}/k\mathbb{Z}$ be a σ-fixed (k,n)-DOSP. Without loss of generality we assume that $f(1) = 0$. We will show that the sequence of integers

$$\alpha = (\alpha_1, \alpha_2, \dots, \alpha_k) = (d_f(1, q_1),\ d_f(1, q_2),\ \dots,\ d_f(1, q_r))$$

uniquely determines the DOSP. By assumption $f(1) = 0$. By definition, we have $f(q_1) = d_f(1, q_1)$. Since D is invariant under σ, it follows that

$$d_D(1, \sigma(1)) = d_D(\sigma(1), \sigma^2(1)) = \cdots = d_D(\sigma^{s_1-1}(1), \sigma^{s_1}(1)) = d_D(1, q_1).$$

So the value $f(\sigma^i(1))$ for each element of $C_1 = \{1, \sigma(1), \sigma^2(1), \dots, \sigma^{s_1-1}(1)\}$ is determined by $d_f(1, q_1)$. Explicitly, we have $f(\sigma^i(1)) = i \cdot d_f(1, q_1) \mod k$. By a similar argument, the value $f(\sigma^i(q_2))$ for each element of $C_2 = \{q_2, \sigma(q_2), \sigma^2(q_2), \dots, \sigma^{s_2-1}(q_2)\}$ is determined by $d_f(1, q_2)$. To see this, observe that $f(q_2) = d_f(1, q_2)$ and, since f is invariant under σ, it follows that $d_f(q_2, \sigma(q_2)) = d_f(1, \sigma(1))$. We deduce that the DOSP f is uniquely determined by α.

We now consider the possible vectors α. By definition, we have that $d_f(1, q_1) = d_f(1, \sigma(1))$ is the turning number of f. By Lemma 12 we have that $g \cdot d_f(1, q_1) \equiv 0$

mod k. The possible values for $d_f(1,q_1)\cdot g$ are $\beta\cdot k$ for each $\beta\in\{0,1,\ldots,g-1\}$. Hence, the possible values for $d_f(1,q_1)$ are $\beta k/g$ for each $0\le\beta<g$. For each such value of $\alpha_1=d_f(1,q_1)$, we may freely choose the values $\alpha_2,\ldots,\alpha_k$ in $\{0,1,\ldots,k-1\}$. Each choice gives a distinct DOSP and every σ-fixed (k,n)-DOSP arises in this way. So the total number of DOSPs is gk^{r-1}. □

Proposition 31. *Let $\sigma\in S_n$ be a permutation with cycle type $s_1,s_2,\ldots,s_r$, and fix $k\in[n]$. Define*

$$\Phi=\left\{(f_1,f_2,\ldots,f_r)\in\{0,1,\ldots,k-1\}^r:\sum_{i=1}^r f_is_i\equiv 0\mod k\right\},$$

and let $g:=\gcd(s_1,s_2,\ldots,s_r,k)$. Then $|\Phi|=gk^{r-1}$.

Proof. Consider the homomorphism of abelian groups

$$\varphi:(\mathbb{Z}/k\mathbb{Z})^r\to\mathbb{Z}/k\mathbb{Z},\quad (f_1,f_2,\ldots,f_r)\mapsto\sum_{i=1}^r f_is_i.$$

The image of φ is the subgroup of $\mathbb{Z}/k\mathbb{Z}$ generated by $s_1,s_2,\ldots,s_r$. By Bezout's identity

$$\langle s_1,s_2,\ldots,s_k\rangle=\langle\gcd(s_1,s_2,\ldots,s_r,k)\rangle=\langle g\rangle\subseteq\mathbb{Z}/k\mathbb{Z}.$$

So we have $|\operatorname{Im}(\varphi)|=k/g$. Therefore

$$|\Phi|=|\ker(\varphi)|=\frac{k^r}{k/g}=gk^{r-1}.$$

□

The two results above, give us the following.

Corollary 2. *Fix $0<k<n$. Let $\sigma\in S_n$ be a permutation with cycle type $s_1,\ldots,s_r$ and define $g=\gcd(k,s_1,s_2,\ldots,s_r)$. There is a bijection between the set of all σ-fixed DOSPs and the set*

$$\Phi:=\bigcup_{m=0}^{\lfloor(k-1)n/k\rfloor}\Phi_k(\sigma,mk).\qquad \textit{Therefore}\quad\sum_{m\ge 0}|\Phi_k(\sigma,mk)|=gk^{r-1}.$$

Proof. Suppose that σ has cycle type $(s_1,s_2,\ldots,s_r)$. By Lemma 13, the number of σ-fixed (k,n)-DOSPs is gk^{r-1}. By Proposition 31, we have that $|\Phi|=gk^{r-1}$, and we are done. □

Lemma 14. *Let $\sigma\in S_n$ be a permutation with cycle type $(s_1,s_2,\ldots,s_r)$. Fix $h\in\{0,1,\ldots,k-1\}$. Define $g'=\gcd(s_1,s_2,\ldots,s_r,k-h)$. Then we have*

$$\sum_{m\ge 0}|\Phi_{k-h}(\sigma,m(k-h)-h)|=\begin{cases}g'(k-h)^{r-1} & \text{if } g' \text{ divides } h,\\ 0 & \text{otherwise.}\end{cases}$$

Proof. Consider the homomorphism of abelian groups

$$\varphi : (\mathbb{Z}/(k-h)\mathbb{Z})^r \to \mathbb{Z}/(k-h)\mathbb{Z}, \quad (f_1, f_2, \ldots, f_r) \mapsto \sum_{i=1}^{r} f_i s_i.$$

Observe that there is a natural bijection between $\bigcup_{m\geq 0} \Phi_{k-h}(\sigma, m(k-h)-h)$ and the set $\Phi := \{f \in (\mathbb{Z}/(k-h)\mathbb{Z})^r : \varphi(f) = -h \mod (k-h)\}$. By Bezout's identity, it follows that the image of φ is principally generated by g'. It follows that

$$|\Phi| = \begin{cases} |\ker(\Phi)| = g'(k-h)^{r-1} & \text{if } -h \in \langle g' \rangle \subseteq \mathbb{Z}/(k-h)\mathbb{Z}, \\ 0 & \text{otherwise.} \end{cases}$$

In the first case, we have that $-h \in \langle g' \rangle$ if and only if g' divides h. □

Proposition 32. *Fix $0 \leq h < k$ and some positive integers $s_1, \ldots, s_r$. For each $0 \leq i < k$ define $g_i = \gcd(k-i, s_1, \ldots, s_r)$. Then $g_h|h$ if and only if $g_0|h$.*

Proof. Define $\tilde{g} = \gcd(s_1, \ldots, s_r)$, so $g_h = \gcd(k-h, \tilde{g})$ and $g_0 = \gcd(k, \tilde{g})$. We have $g_h|h$ if and only if $(k-h)|h$ and $\tilde{g}|h$. On the other hand $g_0|h$ if and only if $k|h$ and $\tilde{g}|h$. So it suffices to show that $(k-h)|h$ if and only if $k|h$, which easily follows from the assumption that $k > h \geq 0$. □

Proposition 33. *Let $\sigma \in S_n$ be a permutation with cycle type $(s_1, \ldots, s_r)$. For each $h \geq 0$, define $g := \gcd(k, s_1, \ldots, s_r)$ and $g_h := \gcd(k-h, s_1, \ldots, s_r)$. We have*

$$H^*[1](\sigma) = \sum_{h=0}^{k-1} \left(\sum_{I \in \mathcal{I}_h} (-1)^{|I|} \binom{\lambda_1}{I_1} \cdots \binom{\lambda_{k-1}}{I_{k-1}} \right) g_h (k-h)^{r-1} d(g,h),$$

where $d(g,h) = 1$ if g divides h and $d(g,h) = 0$ otherwise.

Proof. Follows immediately from Theorem 18, Lemma 14, and Proposition 32. □

Corollary 3. *Let $\sigma \in S_n$ be a permutation with cycle type $(s_1, \ldots, s_r)$. For each $h \geq 0$, define $g := \gcd(k, s_1, \ldots, s_r)$. We have*

$$H^*[1](\sigma) = g \sum_{h=0}^{k-1} \left(\sum_{I \in \mathcal{I}_h} (-1)^{|I|} \binom{\lambda_1}{I_1} \cdots \binom{\lambda_{k-1}}{I_{k-1}} \right) (k-h)^{r-1}.$$

Proof. For each $h \geq 0$, define $g_h := \gcd(k-h, s_1, \ldots, s_r)$. Consider the formula in Proposition 33. Let $h \geq 0$ and $I \in \mathcal{I}_h$. Assume that the product of binomials $\binom{\lambda_1}{I_1} \cdots \binom{\lambda_{k-1}}{I_{k-1}}$ is nonzero. For each $s \in [k-1]$ such that $\lambda_s \geq 1$, we have that $g|s$. Therefore each nonzero term of $1 \cdot I_1 + \cdots + (k-1) \cdot I_{k-1}$ is divisible by g, hence g divides h and so $d(g,h) = 1$. Since g divides h, we have that $g_h = \gcd(k-h, s_1, \ldots, s_r) = \gcd(k, s_1, \ldots, s_r) = g$. The result immediately follows. □

7.3.2 Counting non-hypersimplicial DOSPs

We start this subsection by recalling some facts about *Stirling numbers of the second kind*, which, for brevity, we will refer to simply as *Stirling numbers*. Given non-negative integers n and k, the Stirling number $\left\{ n \atop k \right\}$ is the number of partitions of the set $[n]$ into k non-empty unlabelled parts. For example the set $[3]$ is partitioned into 2 non-empty parts in three ways: $(1|23)$, $(2|13)$, $(3|12)$. Therefore $\left\{ 3 \atop 2 \right\} = 3$. The Stirling numbers satisfy the following defining relation, which is similar to binomial coefficients.

Proposition 34. *For all $n \geq 0$, we have $\left\{ n \atop n \right\} = 1$. For each $n \geq 1$, we have $\left\{ n \atop 0 \right\} = \left\{ 0 \atop n \right\} = 0$. And for all $0 < k < n$, we have the following recurrence relation:*

$$\left\{ n+1 \atop k \right\} = k \left\{ n \atop k \right\} + \left\{ n \atop k-1 \right\}.$$

Proof. Consider a partition of $[n+1]$ into k non-empty parts and remove the element $n+1$. If this results in a partition of $[n]$ into k parts, but one of them may be empty. If that is the case, we may consider it a partition into $k-1$ non-empty parts. Otherwise, we get a partition into k non-empty parts. There are k choices to recover the original partition. □

We also require the following identity, which involves *falling factorials*.

Proposition 35. *The Stirling numbers satisfy the relationship*

$$\sum_{k=1}^{n} \left\{ n \atop k \right\} (x)_k = x^n$$

where $(x)_k = \prod_{i=0}^{k-1}(x-i)$ denotes the kth falling factorial.

Proof. The proposition can be proven via induction on k using the identity from Proposition 34 as well as the identity $x(x)_k = (x)_{k+1} + k(x)_k$. □

In this section we count the number of non-hypersimplicial σ-fixed (k,n)-DOSPs. Throughout this section, we define the following collection of Laurent polynomials. Let $j \geq 1$ be an integer. Define the Laurent polynomial $F_j(y) \in \mathbb{Q}[y, y^{-1}]$ as follows:

$$F_j(y) := \left(\frac{1}{y}\right)^{j-1} \sum_{h=1}^{j} (-1)^{h+1} \left\{ j \atop h \right\} (y+1)(y+2)\cdots(y+h-1).$$

These Laurent polynomials are, in fact, constants.

Lemma 15. *We have $F_j(y) = (-1)^{j+1}$.*

Proof. For each $h \geq 0$, notice that

$$(-1)^{h+1}(y+1)(y+2)\cdots(y+h-1) = \frac{(-y)_h}{y}.$$

By Proposition 35, with $x = -y$, we get $F_j(y) = \frac{(-y)^{j-1}}{y^{j-1}}$, which concludes the proof. □

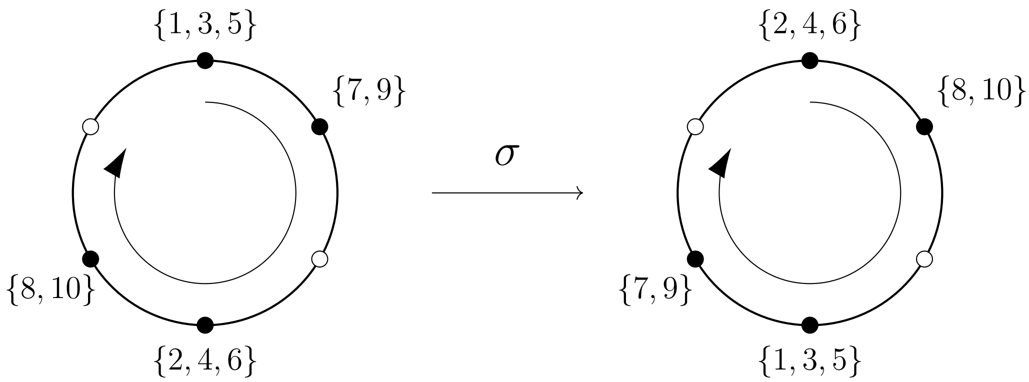

Figure 7.1: A σ-fixed DOSP of type $(6, 10)$ where σ acts with cycles of length 6 and 4 on the subsets $[6]$ and $[10] \setminus [6]$. The turning number is 3. Note that this DOSP is not hypersimplicial because of the placement of the 2-element sets.

Throughout this section, we will frequently make use of the alternative definition of DOSPs in terms of functions $[n] \to \mathbb{Z}/k\mathbb{Z}$. See the beginning of Section 7.3 and Definition 14.

Setup for counting non-hypersimplicial DOSPs. Fix $k < n$ and $\sigma \in S_n$. We denote by $\mathcal{D}$ the set of σ-fixed non-hypersimplicial DOSPs and by $\mathcal{D}^\tau \subseteq \mathcal{D}$ the subset of DOSPs with turning number $\tau \in \mathbb{Z}/k\mathbb{Z}$. We define the set Λ of non-empty unions of cycles in σ:

$$\Lambda = \{C_{i_1} \cup C_{i_2} \cup \cdots \cup C_{i_s} : \{i_1, i_2, \ldots, i_s\} \subseteq [r], s > 0\}.$$

For each $u = C_{i_1} \cup C_{i_2} \cup \cdots \cup C_{i_s} \in \Lambda$, we will denote the corresponding set $\{i_1, i_2, \ldots, i_s\} \subseteq [r]$ by $\operatorname{ind}(u)$. Furthermore, we define the subset $\mathcal{D}^\tau_u \subseteq \mathcal{D}^\tau$ of DOSPs containing a tuple (L, ℓ) such that:

- $|L| \leq \ell$ (we call such a set a *bad set*),
- L completely lies in u, and
- for every $C_i \subseteq u$, $L \cap C_i$ is non-empty.

In other words, $\mathcal{D}^\tau_u$ is the set of all σ-fixed DOSPs D such that u is a disjoint union of bad sets of D, and those bad sets form a single σ orbit. Note, for any $D \in \mathcal{D}^\tau$ there exists $u \in \Lambda$ such that $D \in \mathcal{D}^\tau_u$. Explicitly, D is non-hypersimplicial so it has a bad set, say (L, ℓ), then $D \in \mathcal{D}^\tau_u$ where $u = L \cup \sigma(L) \cup \sigma^2(L) \cup \cdots \cup \sigma^{o(\sigma)-1}(L)$ is the σ-orbit of L.

Lemma 16. *Fix $2 \leq k < n$ and let $\sigma \in S_n$ be a permutation with cycle type $(s_1, s_2, \ldots, s_r)$. Define $g = \gcd(s_1, \ldots, s_r, k)$ and let $\tau \in \mathbb{Z}/k\mathbb{Z}$ such that $g\tau = 0$. Fix $h \in [k-1]$ and let $J \in \binom{\Lambda}{h}$ be a non-empty subset of Λ such that the elements of J are pairwise disjoint. Then*

$$\left| \bigcap_{u \in J} \mathcal{D}^\tau_u \right| = \frac{((k-i)/o(\tau) + h - 1)!}{((k-i)/o(\tau))!} o(\tau)^{j-1} (k-i)^{r-j}$$

where $o(\tau)$ denotes the order of τ, j is the number of elements in $\bigcup_{u\in J} \operatorname{ind}(u)$, and i is the number of elements in $\bigcup_{u\in J} u$.

Before giving the proof, we will outline the concepts in the proof with an example.

Example 13. Fix $k = 12$, $n = 24$, and $\sigma \in S_n$ with cycle type $(3, 3, 6, 3, 9)$. This means that $r = 5$ and $g = 3$. For simplicity, we assume that the cycle sets are

$$C_1 = \{1, 2, 3\},\ C_2 = \{4, 5, 6\},\ C_3 = \{7, 8, \dots, 12\},\ C_4 = \{13, 14, 15\},\ C_5 = \{16, 17, \dots, 24\}.$$

For each cycle C_i, we fix a distinguished element $q_i \in C_i$. Explicitly, we choose q_i to be the smallest element: $q_1 = 1$, $q_2 = 4$, $q_3 = 7$ etc. Let us fix a subset $J = \{u_1, u_2\}$ where $u_1 = C_1 \cup C_2$ and $u_2 = C_4$. This gives us

$$\bigcup_{u\in J} u = \{1, 2, 3, 4, 5, 6, 13, 14, 15\} \text{ and } \bigcup_{u\in J} \operatorname{ind}(u) = \{1, 2, 4\},$$

hence $i = 9$ and $j = 3$. Lastly we fix $\tau = 8 \in \mathbb{Z}/12\mathbb{Z}$ meaning that $o(\tau) = 3$. We will give an overview of the proof of Lemma 16, which counts the number of non-hypersimplicial DOSPs in $\mathcal{D}^\tau_{u_1} \cap \mathcal{D}^\tau_{u_2}$. To do this, we construct DOSPs in this set. We imagine starting with an *empty DOSP* $(L_1 = \{\}, \dots, L_{12} = \{\})$ of $k = 12$ empty sets. We then consider the possible ways to place the cycles into the DOSP. Note that the turning number $\tau = 8$ is fixed, so each σ-orbit consists of $o(\tau) = 3$ sets of the DOSP. We will place the cycles into the DOSP with three steps.

Our first step is to distribute u_1 across a single σ-orbit of $o(\tau) = 3$ sets of 2 elements each and to adorn each of these three sets with a decoration $\ell_i \geq 2$ so that each set is a bad set of the resulting DOSP. Our second step is to distribute u_2 across 3 singletons, which we note always result in bad sets in the final DOSP. The third step is to put the rest of the elements into the remaining spaces. See Figure 7.2 for a specific instance.

Step 1. A σ-fixed DOSP D with turning number τ is completely determined by the values the function f_D takes on the distinguished elements q_i. In Figure 7.2, the q_i are the underlined elements. As a starting point, we will assume that $f_D(q_1) = 0$, or in other words $1 \in L_1$. This choice fixes the positions of the elements in $C_1 = \{1, 2, 3\}$. Since $\tau = 8$, it follows that $f_D(u_1) = \{0, 4, 8\}$, meaning that we have $o(\tau) = 3$ choices for the position of $4 = q_2 \in u_1$. Once we have placed $u = C_1 \cup C_2$, we mark the positions $0, 1, 4, 5, 8, 9$ as *filled*, this guarantees that each set in the resulting DOSP containing elements of u_1 are bad sets. In Figure 7.2, these filled sets include the white circles.

Step 2. The placement of the element $13 = q_4 \in u_2$ is restricted to the locations $\{2, 6, 10\}$ and $\{3, 7, 11\}$ because $\{1, 5, 9\}$ (the white spaces in Figure 7.2) need to remain clear. This gives us 6 choices for q_4. Notice that we count possible locations for a q_i in sets of 3. This corresponds to the factor $o(\tau)^{j-1}$ in the formula in the lemma.

Step 3. Finally, we must choose placements for the remaining cycles C_3 and C_5. After having placed u_1 and u_2, there are only 3 spaced left in the DOSP. So, we have 3 choices for $q_3 = 7$ and $q_5 = 16$, which corresponds to 9 choices to finish off the DOSP. This part corresponds to the right-most factor $(k - i)^{r-j}$ in the formula from the lemma.

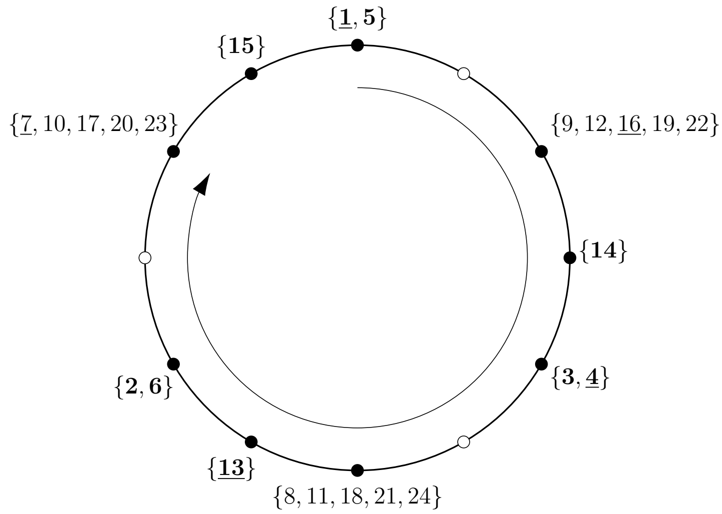

Figure 7.2: A DOSP with the setup of Example 13, the choices for the q_i are $f_D(1) = 0$, $f_D(4) = 4$, $f_D(7) = 10$ $f_D(13) = 7$, $f_D(16) = 3$, are the underlined elements. The bold sets are the bad sets corresponding to u_1 and u_2

Proof of Lemma 16. For the purpose of this proof, we will endow the set $\mathbb{Z}/k\mathbb{Z}$ with a total ordering induced by identifying it with the set $\{0, 1, \ldots, k-1\}$. We write the elements of J as $u_1, u_2, \ldots, u_h$. Let

$$D = ((L_1, \ell_1), (L_2, \ell_2), \ldots, (L_t, \ell_t)) \in \bigcap_{u \in J} \mathcal{D}_u^\tau$$

be a DOSP. Without loss of generality, we may assume that $L_1 \subseteq u_1$. Since the turning number of D is τ, we may assume that $f_D(L_1) = 0$ and $f_D(\sigma(L_1)) = \tau$. For each u_a with $a \in \{2, \ldots h\}$, there exists a set L of D such that $0 < f_D(L) < \tau$ with $L \subseteq u_a$ and we write $p_a = f_D(L)$ for the value of this function on L. We also fix $p_1 = 0$. Let us count the number of DOSPs D, as above, such that $p_2 < \cdots < p_h$.

Suppose that we are given the values of function $f_D(L)$ for each set $L \subseteq u_a$ over all u_a. Let us count the number of ways to distribute the elements of the u_a into the DOSP if u_a is already placed. For each $b \in \mathrm{ind}(u_a)$, there are $k/o(\tau)$ different possible values for $f_D(q_b)$. So, in total, there are $o(\tau)^{j-1}$ possible choices for the positions of the q's, where the first q is put in the first position and the other q's, of which there are $j-1$, are placed relatively to the first.

Now let us count the number of ways to position the sets $u_1, \ldots, u_h$ in the DOSP. The position of u_1 and its corresponding sets of the DOSP is fixed. Then we use a stars and bars argument to place the remaining spaces between the u_a's:

$$(L(u_1)) \,\square\, (L(u_2)) \,\square\, (L(u_3)) \,\square\, \ldots \,\square\, (L(u_h)) \,\square,$$

where $L(u_a)$ is the set of the DOSP with $f_D(L(u_a)) = p_a$ and the boxes represent some number of spaces. Since each set $L(u_a)$ is a bad set, it takes up at least $|L(u_a)|$ spaces of the DOSP. Since u_a is a σ-orbit of bad sets, we have that each set that partitions u_a is bad and together they take up $|u_a|$ spaces of the DOSP. So the bad sets whose union is $u_1 \cup \cdots \cup u_h$ take up i spaces of the DOSP. Hence, in total there are $k-i$ spaces to place between the u_a's. However, whenever we place one space, its σ-orbit has length $o(\tau)$. So we are free to choose the positions of $(k-i)/o(\tau)$ spaces and the rest are determined. So by stars and bars there are $\binom{(k-i)/o(\tau)+h-1}{(k-i)/o(\tau)} = \binom{(k-i)/o(\tau)+h-1}{h-1}$ many ways place the spaces.

So far, we have fixed the position of all cycles of σ in $u_1, \ldots, u_h$, i.e., we have placed j cycles into the DOSP. There are $r-j$ remaining cycles. Placing a cycle C_c into the DOSP is equivalent to choosing the position of q_c. Each q_c can be placed into the $k-i$ spaces. Hence, there are $(k-i)^{r-j}$ DOSPs with the given u_a positions.

Finally there are $(h-1)!$ different total orderings of $p_2, \ldots, p_h$ and each gives the same number of DOSPs. So, the total number of σ-fixed non-hypersimplicial DOSPs with turning number τ is:

$$\binom{(k-i)/o(\tau)+h-1}{h-1}(h-1)!o(\tau)^{j-1}(k-i)^{r-j} = \frac{((k-i)/o(\tau)+h-1)!}{((k-i)/o(\tau)!}o(\tau)^{j-1}(k-i)^{r-j}.$$

□

We proceed to counting the total number of σ-fixed non-hypersimplicial DOSPs.

Lemma 17. *Fix $2 \leq k < n$ and let $\sigma \in S_n$ be a permutation with cycle type $(s_1, s_2, \ldots, s_r)$. For each $i \in [n]$, let λ_i be the number of length i-cycles of σ. Let $g = \gcd(s_1, \ldots, s_r, k)$ and define the set $T = \{\tau \in \mathbb{Z}/k\mathbb{Z} \colon g\tau = 0\}$. The number of σ-fixed non-hypersimplicial DOSPs is*

$$\sum_{\tau\in T}\sum_{h=1}^{k-1}(-1)^{h+1}\sum_{i=h}^{k-1}\sum_{j=1}^{i}\kappa_h^i(\tau)o(\tau)^{j-1}(k-i)^{r-j}\left(\sum_{\substack{I\in\mathcal{I}_i\\|I|=j}}\binom{\lambda_1}{I_1}\cdots\binom{\lambda_{k-1}}{I_{k-1}}\right)\left\{{j \atop h}\right\}. \tag{7.1}$$

where $\kappa_h^i(\tau) := \frac{((k-i)/o(\tau)+h-1)!}{((k-i)/o(\tau))!}$.

Proof. Let $\tau \in \mathbb{Z}/k\mathbb{Z}$ satisfy $g\tau = 0$ and let $\mathcal{D}^\tau$ be the set of non-hypersimplicial σ-fixed DOSPs with turning number τ. The number of σ-fixed non-hypersimplicial DOSPs is

$$\sum_{\tau\in T}|\mathcal{D}^\tau|.$$

By the inclusion-exclusion principle, we have

$$|\mathcal{D}^\tau| = \left|\bigcup_{u\in\Lambda}\mathcal{D}_u^\tau\right| = \sum_{h\geq 1}(-1)^{h+1}\sum_{J\in\binom{\Lambda}{h}}\left|\bigcap_{u\in J}\mathcal{D}_u^\tau\right|.$$

Given a non-empty subset $J \subseteq \Lambda$, suppose we have $u_1, u_2 \in J$. For a DOSP D to lie both in $\mathcal{D}_{u_1}$ and $\mathcal{D}_{u_2}$, it means there exist (not necessarily distinct) sets L_1 and L_2 whose

σ-orbits are u_1 and u_2 respectively. In particular, if $u_1 \cap u_2 \neq \emptyset$, the σ-orbits must be the same and $u_1 = u_2$. Hence, we may always assume that the sets u_i contained in J are disjoint. It also follows that h is bounded by the number of sets L_i, which is k. In the case $h = k$, every L_i satisfies $|L_i| = \ell_i = 1$, which means that $n = k$, a contradiction. Thus $1 \leq h \leq k-1$. We introduce one more notation:

$$\Lambda(h,i,j) = \left\{ J \in \binom{\Lambda}{h} : \left| \bigcup_{u \in J} u \right| = i \text{ and } \left| \bigcup_{u \in J} \operatorname{ind}(u) \right| = j \right\}$$

This is the set of h-element subsets of Λ involving exactly j distinct cycles and contain a total of i elements across all cycles. The cardinality of $\Lambda(h,i,j)$ is exactly

$$\left(\sum_{\substack{I \in \mathcal{I}_i \\ |I|=j}} \binom{\lambda_1}{I_1} \cdots \binom{\lambda_{k-1}}{I_{k-1}} \right) \left\{ \begin{matrix} j \\ h \end{matrix} \right\},$$

which follows from the argument that there are $\left\{ \begin{smallmatrix} j \\ h \end{smallmatrix} \right\}$ ways to partition j distinct cycles into h sets and that we choose I_1 many fixed points of σ, I_2 many 2-cycles of σ, and so on, such that $|I| = j$ and $1 \cdot I_1 + 2 \cdot I_2 + \cdots + (k-1) \cdot I_{k-1} = i$. With this, we apply Lemma 16 and rearrange the previous formula:

$$\begin{aligned}
|\mathcal{D}^\tau| &= \sum_{h=1}^{k-1} (-1)^{h+1} \sum_{i=h}^{k-1} \sum_{j=1}^{r} \sum_{J \in \Lambda(h,i,j)} \left| \bigcap_{u \in J} \mathcal{D}_u^\tau \right| \\
&= \sum_{h=1}^{k-1} (-1)^{h+1} \sum_{i=h}^{k-1} \sum_{j=1}^{r} \sum_{J \in \Lambda(h,i,j)} \frac{((k-i)/o(\tau)+h-1)!}{((k-i)/o(\tau))!} o(\tau)^{j-1} (k-i)^{r-j} \\
&= \sum_{h=1}^{k-1} (-1)^{h+1} \sum_{i=h}^{k-1} \sum_{j=1}^{r} \sum_{J \in \Lambda(h,i,j)} \kappa_h^i(\tau) o(\tau)^{j-1} (k-i)^{r-j} \\
&= \sum_{h=1}^{k-1} (-1)^{h+1} \sum_{i=h}^{k-1} \sum_{j=1}^{r} \kappa_h^i(\tau) o(\tau)^{j-1} (k-i)^{r-j} \sum_{J \in \Lambda(h,i,j)} 1 \\
&= \sum_{h=1}^{k-1} (-1)^{h+1} \sum_{i=h}^{k-1} \sum_{j=1}^{r} \kappa_h^i(\tau) o(\tau)^{j-1} (k-i)^{r-j} \left(\sum_{\substack{I \in \mathcal{I}_i \\ |I|=j}} \binom{\lambda_1}{I_1} \cdots \binom{\lambda_{k-1}}{I_{k-1}} \right) \left\{ \begin{matrix} j \\ h \end{matrix} \right\}.
\end{aligned}$$

□

7.3.3 Proof of the main result

With the results from the previous section, we are now ready to give a proof of Theorem 21, which we restate in the following way.

Theorem 22. *Fix $2 \leq k < n$ and $\sigma \in S_n$ be a permutation with cycle type $(s_1, s_2, \ldots, s_r)$. For each $i \in [n]$, let λ_i be the number of length i cycles of σ. Let $g = \gcd(s_1, \ldots, s_r, k)$. The number of σ-fixed non-hypersimplicial DOSPs is*

$$gk^{r-1} - H^*[1](\sigma) = -g \sum_{h=1}^{k-1} \left(\sum_{I \in \mathcal{I}_i} (-1)^{|I|} \binom{\lambda_1}{I_1} \cdots \binom{\lambda_{k-1}}{I_{k-1}} \right) (k-i)^{r-1}.$$

In particular, we have that $H^*[1](\sigma)$ is equal to the number of σ-fixed hypersimplicial (k,n)-DOSPs.

Proof. Like before, let $\kappa_h^i(\tau)$ refer to $\frac{((k-i)/o(\tau)+h-1)!}{((k-i)/o(\tau))!}$. Define the set $T = \{\tau \in \mathbb{Z}/k\,\mathbb{Z} \colon g\tau = 0\}$ and note that $|T| = g$. By Lemma 17, we have that the number of σ-fixed non-hypersimplicial DOSPs is

$$\sum_{\tau \in T} \sum_{h=1}^{k-1} (-1)^{h+1} \sum_{i=h}^{k-1} \sum_{j=1}^{i} \kappa_h^i(\tau) o(\tau)^{j-1} (k-i)^{r-j} \left(\sum_{\substack{I \in \mathcal{I}_i \\ |I|=j}} \binom{\lambda_1}{I_1} \cdots \binom{\lambda_{k-1}}{I_{k-1}} \right) \left\{ {j \atop h} \right\}.$$

We reorder the sums in this expression to obtain

$$\begin{aligned}
&\sum_{\tau \in T} \sum_{h=1}^{k-1} (-1)^{h+1} \sum_{i=h}^{k-1} \sum_{j=1}^{i} \kappa_h^i(\tau) o(\tau)^{j-1} (k-i)^{r-j} \left(\sum_{\substack{I \in \mathcal{I}_i \\ |I|=j}} \binom{\lambda_1}{I_1} \cdots \binom{\lambda_{k-1}}{I_{k-1}} \right) \left\{ {j \atop h} \right\} \\
&= \sum_{\tau \in T} \sum_{i=1}^{k-1} \sum_{j=1}^{i} \sum_{h=1}^{i} (-1)^{h+1} \kappa_h^i(\tau) o(\tau)^{j-1} (k-i)^{r-j} \left(\sum_{\substack{I \in \mathcal{I}_i \\ |I|=j}} \binom{\lambda_1}{I_1} \cdots \binom{\lambda_{k-1}}{I_{k-1}} \right) \left\{ {j \atop h} \right\} \\
&= \sum_{\tau \in T} \sum_{i=1}^{k-1} \sum_{j=1}^{i} \left(\sum_{\substack{I \in \mathcal{I}_i \\ |I|=j}} \binom{\lambda_1}{I_1} \cdots \binom{\lambda_{k-1}}{I_{k-1}} \right) (k-i)^{r-j} o(\tau)^{j-1} \sum_{h=1}^{i} (-1)^{h+1} \kappa_h^i(\tau) \left\{ {j \atop h} \right\}.
\end{aligned}$$

Next, we apply Lemma 15 to the above expression by setting $y = (k-i)/o(\tau)$. Note that

$o(\tau)^{j-1} = (k-i)^{j-1} \cdot (1/y)^{j-1}$. So, the above expression is equal to the following

$$\sum_{\tau \in T} \sum_{i=1}^{k-1} \sum_{j=1}^{i} \left(\sum_{\substack{I \in \mathcal{I}_i \\ |I|=j}} \binom{\lambda_1}{I_1} \cdots \binom{\lambda_{k-1}}{I_{k-1}} \right) (k-i)^{r-j} o(\tau)^{j-1} \sum_{h=1}^{i} (-1)^{h+1} \kappa_h^i(\tau) \left\{ {j \atop h} \right\}$$

$$= \sum_{\tau \in T} \sum_{i=1}^{k-1} \sum_{j=1}^{i} \left(\sum_{\substack{I \in \mathcal{I}_i \\ |I|=j}} \binom{\lambda_1}{I_1} \cdots \binom{\lambda_{k-1}}{I_{k-1}} \right) (k-i)^{r-j} (k-i)^{j-1} F_j((k-i)/o(\tau))$$

$$= \sum_{\tau \in T} \sum_{i=1}^{k-1} \sum_{j=1}^{i} \left(\sum_{\substack{I \in \mathcal{I}_i \\ |I|=j}} \binom{\lambda_1}{I_1} \cdots \binom{\lambda_{k-1}}{I_{k-1}} \right) (k-i)^{r-1} (-1)^{j+1}$$

$$= -g \sum_{i=1}^{k-1} \left(\sum_{I \in \mathcal{I}_i} (-1)^{|I|} \binom{\lambda_1}{I_1} \cdots \binom{\lambda_{k-1}}{I_{k-1}} \right) (k-i)^{r-1}.$$

Finally, we recall that the number of σ-fixed (k,n)-DOSPs is equal to gk^{r-1}, so we have $H^*[1](\sigma)$ is the number of σ-fixed hypersimplicial (k,n)-DOSPs. This completes the proof. □

7.3.4 Recurrence relation

In this section, we show that $H^*(\Delta_k^n; S_n)[1](\sigma)$ satisfies a recurrence relation similar to that for Eulerian numbers. Given $k \in \mathbb{Z}$, a tuple $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n) \in \mathbb{Z}_{\geq 0}^n$, and $r \geq 1$, we define

$$B(k, \lambda, r) = g(k, \lambda) \sum_{h=0}^{k-1} \left(\sum_{I \in \mathcal{I}_h} (-1)^{|I|} \binom{\lambda_1}{I_1} \cdots \binom{\lambda_{k-1}}{I_{k-1}} \right) (k-h)^{r-1}$$

where $g(k, \lambda) = \gcd(\{k\} \cup \{i \in [n] : \lambda_i \geq 1\})$.

Suppose that $\sigma \in S_n$ has cycle type $(s_1, \dots, s_r)$ and for each $i \in [n]$ we denote by λ_i the number of cycles of σ of length i. Then, by Theorem 21, we have $H^*(\Delta_k^n; S_n)[1](\sigma) = B(k, \lambda, r)$.

Proposition 36. *We have $B(k, \lambda, r) = 0$ if $k < 1$, $B(1, \lambda, r) = gk^{r-1}$, and $B(k, \lambda, r) = gk^{r-1}$ if $\lambda_1 = \cdots = \lambda_{k-1} = 0$. Suppose that there exists $a \in [k-1]$ such that $\lambda_a \geq 1$. Define $\lambda' = (\lambda_1, \dots, \lambda_{a-1}, \lambda_a - 1, \lambda_{a+1}, \dots, \lambda_n)$. Then, we have*

$$B(k, \lambda, r) = \frac{g(k, \lambda)}{g(k, \lambda')} B(k, \lambda', r) - \frac{g(k, \lambda)}{g(k-a, \lambda')} B(k-a, \lambda', r).$$

Proof. The first part of the result follows easily from the definition of $B(k, \lambda, r)$. For the recurrence relation, fix $a \in [k-1]$ such that $\lambda_a \geq 1$. To simplify notation, we write

$g = g(k, \lambda)$, $g' = g(k, \lambda')$, and $g'' = g(k-a, \lambda')$. First, we apply Pascal's identity

$$\begin{aligned}
B(k,\lambda,r) &= g\sum_{h=0}^{k-1}\left(\sum_{I\in\mathcal{I}_h}(-1)^{|I|}\binom{\lambda_1}{I_1}\cdots\binom{\lambda_{k-1}}{I_{k-1}}\right)(k-h)^{r-1} \\
&= g\sum_{h=0}^{k-1}\left(\sum_{I\in\mathcal{I}_h}(-1)^{|I|}\binom{\lambda_1}{I_1}\cdots\binom{\lambda_a-1}{I_a}\cdots\binom{\lambda_{k-1}}{I_{k-1}}\right)(k-h)^{r-1} \\
&\quad + g\sum_{h=0}^{k-1}\left(\sum_{I\in\mathcal{I}_h}(-1)^{|I|}\binom{\lambda_1}{I_1}\cdots\binom{\lambda_a-1}{I_a-1}\cdots\binom{\lambda_{k-1}}{I_{k-1}}\right)(k-h)^{r-1}.
\end{aligned}$$

The first sum coincides with $(g/g')B(k, \lambda', r)$. For the second sum, we re-index as follows

$$\begin{aligned}
B(k,\lambda,r)-\frac{g}{g'}B(k,\lambda',r) &= g\sum_{h=0}^{k-1}\left(\sum_{I\in\mathcal{I}_h}(-1)^{|I|}\binom{\lambda_1}{I_1}\cdots\binom{\lambda_a-1}{I_a-1}\cdots\binom{\lambda_{k-1}}{I_{k-1}}\right)(k-h)^{r-1} \\
&= g\sum_{h=0}^{k-1-a}\left(\sum_{I\in\mathcal{I}_{h+a}}(-1)^{|I|}\binom{\lambda'_1}{I_1}\cdots\binom{\lambda'_a}{I_a-1}\cdots\binom{\lambda'_{k-1}}{I_{k-1}}\right)(k-h-a)^{r-1} \\
&= g\sum_{h=0}^{(k-a)-1}\left(\sum_{I\in\mathcal{I}_h}(-1)^{|I|+1}\binom{\lambda'_1}{I_1}\cdots\binom{\lambda'_{k-1}}{I_{k-1}}\right)((k-a)-h)^{r-1} \\
&= -\frac{g}{g''}B(k-a,\lambda',r).
\end{aligned}$$

So we have shown that B satisfies the recursive relation and concludes the proof. □

Remark 8. The recurrence relation in Proposition 36 differs a little from the typical one for Eulerian numbers given by $A(n,k) = (k+1)A(n-1,k) + (n-k)A(n-1,k-1)$. The evaluation of $H^*(\Delta_k^n; S_n)[1]$ at the identity is equal to $A(n-1,k-1)$, which is equal to $B(k,(n,0,\dots,0),n)$.

7.4 The second hypersimplex

In this section we give a complete description of the coefficients of the H^*-polynomial for the second hypersimplex Δ_2^n. We interpret these coefficients in terms of DOSPs, see Definition 13, as well as actions of S_n on subsets and partitions of $[n]$.

Notation. For each $m \in [n]$, we denote by ρ_m the character of the permutation representation of S_n acting on $\binom{[n]}{m}$. So, we have $\rho_m(\sigma) = |\{S \subseteq [n] : |S| = m,\ \sigma(S) = S\}|$. By taking complements, we have $\rho_{n-m} = \rho_m$ for each m. And so $\rho_{n-1} = \rho_1 = \chi_{\text{nat}} = \lambda_1$ is the character of the natural representation and $\rho_n = \chi_0$ is the trivial character. We define τ_m to be the character of the permutation representation of S_n acting on the set of partitions of $[n]$ into two parts: one of size m and the other of size $n-m$. Note that, unless n is even and $m = n/2$, we have that $\rho_m = \tau_m$.

With this notation, we have the following characterisation of the coefficients of the equivariant H^* polynomial of Δ_2^n.

Theorem 23. *Let $n > 2$. The coefficients of the equivariant H^*-polynomial of the hypersimplex Δ_2^n:*

- $H_0^* = \chi_0$ *is the trivial character,*
- $H_1^* = \rho_2 - \rho_1$,
- $H_m^* = \rho_{2m}$ *for each* $2 \leq m \leq \lfloor n/2 \rfloor$.

The evaluation at one is given by $H^*[1] = \chi_0 + \tau_2 + \tau_3 + \cdots + \tau_{\lfloor n/2 \rfloor}$, which is a permutation character. If n is odd, then the leading coefficient is $H^*_{(n-1)/2} = \rho_1$. Otherwise, if n is even, then the leading coefficient is $H^*_{n/2} = \chi_0$.

Before we prove the theorem, we note that the formula of coefficients H_m^* in Theorem 18 has a simple description in the special case when $h = k - 1$.

Proposition 37. *Let $\sigma \in S_n$ and $m \geq 0$. We have*

$$|\Phi_1(\sigma, m)| = \begin{cases} 1 & \text{if } m = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. The set $\Phi_1(\sigma, m)$ consists of functions $f : [r] \to \{0\}$ such that $\sum_{i=1}^r f(i) s_i = m$. There is only one such function, which belongs to the set $\Phi_1(\sigma, 0)$. □

To prove Theorem 23, we also require the following result about S_n representations.

Lemma 18. *Let n be even. The following equation of S_n-representations holds*

$$\sum_{m=0}^{n/2} \rho_{2m} = \sum_{m=0}^{n/2} \tau_m.$$

Proof. Fix a permutation σ with cycle type $s_1, s_2, \dots, s_r$. It suffices to show that the number of subsets of $[n]$ with even size that are fixed by σ is equal to the number of two-part partitions of $[n]$ that are fixed by σ. Suppose that $A \sqcup B = [n]$ is a two-part partition, then we write $AB := \{A, B\}$ for the partition of $[n]$ into A and B. Given a partition AB of $[n]$, we write $\sigma(AB)$ for the partition of $[n]$ with parts $\sigma(A)$ and $\sigma(B)$. We define the sets

$$L^\sigma = \{A \subseteq [n] : |A| \text{ is even}, \sigma(A) = A\} \text{ and } R^\sigma = \{AB : A \sqcup B = [n], \sigma(AB) = AB\}.$$

We will show that $|L^\sigma| = |R^\sigma|$.

We first consider the special case where each cycle of σ has odd length. Each element $A \in L^\sigma$ is the union of the supports of cycles of σ. Since $|A|$ is even and each cycle has odd length, it follow that A is the union of an even number of cycle supports. The indices of the cycles whose supports form A uniquely determine A, and any even subset of cycles forms a unique set A. So we have

$$\begin{aligned} |L^\sigma| &= |\{S \subseteq [r] : \Sigma S \text{ is even}\}| \\ &= |\{S \subseteq [r] : |S| \text{ is even}\}| \\ &= 2^{r-1}. \end{aligned}$$

On the other hand, for each subset $S \subset [r]$, we obtain a partition TU where T is the union of supports of the cycles in σ indexed by S and $U = [n] \setminus T$ is the complement of T. Observe that every σ-invariant partition arises in this way because each cycle has odd length. Moreover, each partition arises from a subset $S \subseteq [k]$ or its complement $[k] \setminus S$. So we conclude that $|R^\sigma| = 2^{k-1} = |L^\sigma|$. This concludes the special case.

Suppose that σ contains a cycle of even length. Without loss of generality we assume that $s := s_r$ is even. We prove $|L^\sigma| = |R^\sigma|$ by induction on r. For the base case with $r = 1$, we have that $\sigma = (\sigma_1\ \sigma_2\ \dots\ \sigma_n)$ is an n-cycle where n is even. It is easy to see that

$$L^\sigma = \{\emptyset, [n]\} \quad \text{and} \quad R^\sigma = \{\{\emptyset, [n]\}, \{\sigma_1\sigma_3 \dots \sigma_{n-1}, \sigma_2\sigma_4 \dots \sigma_n\}\}.$$

So we have $|L^\sigma| = 2 = |R^\sigma|$.

For the induction step, assume that $r > 1$ and consider a permutation τ that has cycle type $s_1, s_2, \dots, s_{r-1}$. Without loss of generality, let us assume that τ is equal to the permutation σ restricted to $[n - s_r]$. Define the set $S = [n] \setminus [n - s_r]$. It is easy to see that

$$L^\sigma = L^\tau \sqcup \{A \cup S : A \in L^\tau\}$$

and so we have $|L^\sigma| = 2|L^\tau|$. On the other hand, let us consider a partition $AB \in R^\tau$. If $\sigma(A) = A$, then we have that the partitions $(A \cup S)B$ and $A(B \cup S)$ lie in R^σ. On the other hand, if $\sigma(A) = B$, then write $(c_1, c_2, \dots, c_{s_r})$ for the cycle of σ supported on S. Then we have

$$(A \cup \{c_1, c_3, \dots, c_{s_k-1}\})(B \cup \{c_2, c_4, \dots, c_{s_k}\}) \text{ and } (A \cup \{c_2, c_4, \dots, c_{s_k}\})(B \cup \{c_1, c_3, \dots, c_{s_k-1}\})$$

are elements of R^σ. Every element of R^σ arises uniquely in one of the ways described above. So it follows that $|R^\sigma| = 2|R^\tau|$. By induction, we have $|L^\tau| = |R^\tau|$ and so we deduce that $|L^\sigma| = |R^\sigma|$ and we are done. □

Proof of Theorem 23. Fix $\sigma \in S_n$ with cycle type $(s_1, \dots, s_r)$ and denote by $C_1, \dots, C_r$ the cycle sets of σ such that $|C_i| = s_i$ for each $i \in [r]$. Let us consider the coefficients given by Theorem 18 and Proposition 37 for the second hypersimplex Δ_2^n. We have

$$H_m^*(\sigma) = |\Phi_2(\sigma, 2m)| - \lambda_1 |\Phi_1(\sigma, m-1)| = \begin{cases} |\Phi_2(\sigma, 2m)| & \text{if } m \neq 1, \\ |\Phi_2(\sigma, 2m)| - \lambda_1 & \text{if } m = 1. \end{cases}$$

The value λ_1 is equal to the number of fixed points of σ. So $\lambda_1(\sigma) = \rho_1(\sigma)$ is the character of the natural representation of S_n. On the other hand, the set $\Phi_2(\sigma, 2m)$ contains all functions $f : [r] \to \{0, 1\}$ such that $\sum_{i=1}^{r} f(i)s_i = 2m$. There is a natural correspondence between $f \in \Phi_2(\sigma, 2m)$ and subsets $F \subseteq [n]$ with $|F| = 2m$ and $\sigma(F) = F$, which is given by

$$f \mapsto F = \bigcup_{i:f(i)=1} S_i.$$

So $|\Phi_2(\sigma, 2m)| = \rho_{2m}$ is equal to the permutation character of S_n acting on $\binom{[n]}{2m}$. This proves that $H_0^* = \chi_0$, $H_1^* = \rho_2 - \rho_1$, and $H_m^* = \rho_{2m}$ for each $2 \le m \le \lfloor n/2 \rfloor$. By Theorem 18, if n is odd then the leading coefficient is $H^*_{(n-1)/2} = \rho_{n-1} = \rho_1$, otherwise if n is even then the leading coefficient is $H^*_{n/2} = \rho_n = \chi_0$.

It remains to show that $H^*[1] = \chi_0 + \tau_2 + \cdots + \tau_{\lfloor n/2 \rfloor}$. If n is odd the result follows from the above and the fact that $\rho_m = \tau_m$ for each $m \in [n]$. On the other hand if n is even, then result follows from Lemma 18. □

Corollary 4. *Fix n and let $0 \le m \le \lfloor n/2 \rfloor$. Then the coefficient H_m^* of the equivariant H^*-polynomial of Δ_2^n is a permutation character if and only if $m \ne 1$. Moreover, the trivial character does not appear in H_1^*.*

Proof. Suppose $m \ne 1$. By Theorem 23, we have that H_m^* is the permutation character ρ_{2m} if $m > 0$ and χ_0 if $m = 0$. Otherwise, let $m = 1$ and assume by contradiction that H_1^* is a permutation character. For any permutation character ρ, a consequence of the Orbit-Stabiliser Theorem is that $\langle \chi_0, \rho \rangle = \frac{1}{n!} \sum_{\sigma \in S_n} \rho(\sigma)$ is equal to the number of orbits of the action. By Theorem 23, we have $H_1^* = \rho_2 - \rho_1$. Since S_n acts transitively on $[n]$ and the 2-subsets of $[n]$, the action associated to H_1^* has $\langle \chi_0, \rho_2 - \rho_1 \rangle = \langle \chi_0, \rho_2 \rangle - \langle \chi_0, \rho_1 \rangle = 0$ orbits, a contradiction. This completes the proof. □

Theorem 23 also allows us to give a complete combinatorial proof of the effectiveness of the H^*-polynomial.

Corollary 5. *Fix n. Each coefficient of $H^*(\Delta_2^n, S_n)$ is an effective representation.*

Proof. Let $m \in \{0, 1, 2, \ldots, \lfloor n/2 \rfloor\}$ and consider the t^m coefficient H_m^* of $H^*(\Delta_2^n, S_n)$. If $m \ne 1$, then H_m^* is a permutation character, hence it is effective. Otherwise if $m = 1$ then let V and W be $\mathbb{C}\, S_n$ modules with characters ρ_1 and ρ_2 respectively. Explicitly, we assume V has basis e_i with $i \in [n]$ and action $\sigma(e_i) = e_{\sigma(i)}$; and W has basis f_I with $I \in \binom{[n]}{2}$ and action $\sigma(f_I) = f_{\sigma(I)}$. Define the map $\varphi : V \to W$ given by $\varphi(e_i) = \sum_{j \ne i} f_{ij}$. It is straightforward to show that φ is an injective $\mathbb{C}\, S_n$-module homomorphism, hence $H_1^* = \rho_2 - \rho_1$ is effective.

□

Remark 9. The coefficients of the H^*-polynomial may be interpreted as permutation characters of DOSPs. The set of $(2, n)$-DOSPs consists of: the *trivial DOSP* $(([n], 2))$; and the DOSPs with two parts $((A, 1), (B, 1))$, where $\{A, B\}$ is a partition of $[n]$. The trivial DOSP is fixed by every element of S_n, hence it is naturally associated with H_0^*.

For each $2 \leq m \leq \lfloor (n+1)/2 \rfloor - 1$, by Theorem 23, we have $H^*_m = \rho_{2m} = \tau_{2m}$ is the permutation character of S_n acting on partition $\{A, B\}$, where $|A| = 2m$. Hence $H^*_m(\sigma)$ counts the number of σ-fixed DOSPs $((A,1),(B,1))$ such that $|A| = 2m$. Suppose that n is odd. Then $H^*_1 + H^*_{(n-1)/2} = \tau_2$ counts the number of σ-fixed DOSPs with $|A| = 2$. Hence the character $H^*[1]$ counts the number of σ-fixed DOSPs that do not have a set of size one. Since a $(2,n)$-DOSP is hypersimplicial if and only if it has no set of size one. On the other hand, if n is even, then fix an odd value m. The permutation character of S_n acting on the set of DOSPs with $|A| = m$ does not immediately arise from a coefficient of the H^*-polynomial.

Remark 10. Each coefficient of the h^*-polynomial of the hypersimplex has a combinatorial interpretation in terms of DOSPs. Explicitly h^*_m is the number of hypersimplicial (k,n)-DOSPs with *winding number m*. We note that the winding number is not invariant under the action of S_n so the same interpretation does not apply in the most general setting. However, the winding number is invariant under the cyclic group $C_n \leq S_n$. It is shown in [EKS24] that the coefficient $H^*(\Delta^n_k; C_n)_m$ is the number of σ-fixed hypersimplicial (k,n)-DOSPs with winding number m. In the case $k = 2$, this result can be deduced from Theorem 23 as follows. If $D = ((A,1),(B,1))$ is a DOSP, then we define the set $J(D)$ of *jumping points* to be the set of $i \in [n]$ such that i, $i+1$ belong to different sets of D. Since $k = 2$, the winding number of D is equal to half the number of jumping poitns. For $m = 0$ and $m \geq 2$, the restriction $\mathrm{Res}^{S_n}_{C_n} \rho_{2m}(\sigma)$ counts the number of σ-fixed partitions $\{A, B\}$ of $[n]$ with $|A| = 2m$. For each such partition there is a unique DOSP with jumping points A. This DOSP is σ-fixed and has winding number m. Every such DOSP arises in this way and so H^*_m is the number of σ-fixed DOSPs with winding number m. In the case $m = 1$, we have that $\mathrm{Res}^{S_n}_{C_n}(\rho_2 - \rho_1)$ is isomorphic to the permutation representation of C_n acting on the set of 2-subsets $ij \in \binom{[n]}{2}$ such that $|i - j| > 1$. For each such 2-subset, we obtain a σ-fixed hypersimplicial DOSP, which concludes the proof.

Part IV

On nearly Gorenstein polytopes

Chapter 8

Nearly Gorenstein polytopes

The Ehrhart ring of a lattice polytope P is Gorenstein if and only if there exists a positive integer k such that kP is reflexive, cf. [BN08]. In this chapter, we work towards a characterisation of lattice polytopes by the nearly Gorensteinness of their Ehrhart rings. We find both necessary and sufficient conditions as well as a full classification in the case of IDP $(0, 1)$-polytopes. The content of this chapter is fully contained in the author's paper [Hal+23] with Thomas Hall, Koji Matsushita, and Sora Miyashita.

8.1 The main results

Let $P \subset \mathbb{R}^d$ be a lattice polytope with codegree a. We define its *floor polytope* and *remainder polytopes* as

$$\lfloor P \rfloor := \operatorname{conv}(\operatorname{int}(P) \cap \mathbb{Z}^d) \qquad \text{and} \qquad \{P\} := \operatorname{conv}(\operatorname{ant}(C_P)_{1-a} \cap \mathbb{Z}^d),$$

respectively. Note that $\lfloor P \rfloor$ coincides with $\operatorname{conv}(\operatorname{int}(C_P)_1 \cap \mathbb{Z}^d)$. Our first result gives a necessary condition and a sufficient condition for a lattice polytope to be nearly Gorenstein.

Theorem 24 (Proposition 39 and Theorem 28)**.** *Let $P \subset \mathbb{R}^d$ be a lattice polytope with codegree a.*

1. *If P is nearly Gorenstein, then it has the Minkowski decomposition $P = \lfloor aP \rfloor + \{P\}$.*
2. *Conversely, if $P = \lfloor aP \rfloor + \{P\}$, then there exists some K such that, for all integers $k \geq K$, the polytope kP is nearly Gorenstein.*

The next main result gives facet presentations for the floor and remainder polytopes appearing in the Minkowski decomposition of a nearly Gorenstein polytope.

Theorem 25 (Theorem 31)**.** *Let $P \subset \mathbb{R}^d$ be a lattice polytope with codegree a. Suppose that $P = \lfloor aP \rfloor + \{P\}$. Then*

$$\lfloor aP \rfloor = \{x \in \mathbb{R}^d : n_F(x) \geq 1 - ah_F \text{ for all } F \in \mathcal{F}(P)\} \text{ and}$$
$$\{P\} = \{x \in \mathbb{R}^d : n_F(x) \geq (a-1)h_F - 1 \text{ for all } F \in \mathcal{F}(P)\}.$$

Furthermore, if $\lfloor P \rfloor \neq \varnothing$, then $\{P\}$ is reflexive.

These results allow us to prove the next main result. It reveals that the primitive inner normal vectors of a nearly Gorenstein polytope come from boundary points of reflexive polytopes.

Theorem 26. *Let $P \subset \mathbb{R}^d$ be a nearly Gorenstein polytope. Then there exists a reflexive polytope $Q \subset \mathbb{R}^d$ such that*

$$P = \{x \in \mathbb{R}^d \colon n(x) \geq -h_n \text{ for all } n \in \partial Q^* \cap (\mathbb{Z}^d)^*\},$$

where h_n are integers. Moreover, the inequalities defined by $n \in \operatorname{vert}(Q^*)$ are irredundant. Furthermore, the number of facets of a nearly Gorenstein polytope is bounded by a constant depending on the dimension d.

The final main result is a classification of IDP $(0,1)$-polytopes, which generalises prior work on nearly Gorenstein Hibi rings [HHS19] and Ehrhart rings of stable set polytopes arising from perfect graphs [HS21; Miy22].

Theorem 27 (Theorem 34)**.** *Let P be a $(0,1)$-polytope which has the integer decomposition property. Then, P is nearly Gorenstein if and only if $P = P_1 \times \cdots \times P_s$, for some Gorenstein $(0,1)$-polytopes $P_1, \ldots, P_s$ which satisfy $|a_{P_i} - a_{P_j}| \leq 1$, where a_{P_i} and a_{P_j} are the respective codegrees of P_i and P_j, for $1 \leq i < j \leq s$.*

8.2 Nearly Gorensteinness of lattice polytopes

Throughout this section, the lattice polytope P has the facet presentation (2.1).

Definition 15. We say that P is *Gorenstein* (resp. *nearly Gorenstein*) if the Ehrhart ring $A(P)$ is Gorenstein (resp. nearly Gorenstein).

There are well-known equivalent conditions of Gorensteinness in terms of the lattice polytope P itself. For instance, P is Gorenstein if and only if there exists a positive integer a such that a *lattice translation* of aP is *reflexive*, i.e. aP has a unique interior lattice point which has lattice distance 1 to all facets of aP.

In this section, we will determine a necessary condition for P to be nearly Gorenstein, in terms of the polytope P itself. This condition demands that P has a particular Minkowski decomposition. By taking a dual perspective, we see exactly the connection to reflexive polytopes. Next, we will show that if P satisfies the aforementioned necessary condition and is in some sense "big enough", then P will be nearly Gorenstein. We end the section by investigating the nearly Gorensteinness of Minkowski indecomposable lattice polytopes.

8.2.1 Necessary conditions

The main aim of this subsection is to show the first half of Theorem 24. Before we proceed, let us first introduce some helpful notation. For a subset X of $\mathbb{R}^{d+1}$ and $k \in \mathbb{Z}$, let $X_k = \{x \in \mathbb{R}^d : (x,k) \in X\}$ be the k-th *piece* of X. Note the subtlety in our notation: while X is a subset of $\mathbb{R}^{d+1}$, its k-th piece X_k is a subset of $\mathbb{R}^d$. Moreover, for a lattice polytope P, we denote its *codegree* by a_P – see below Proposition 16 for the definition. When it is clear from context, we simply write a instead of a_P.

Proposition 38. *Let $P \subset \mathbb{R}^d$ be a lattice polytope with codegree a. Then P is nearly Gorenstein if and only if*

$$(C_P \cap \mathbb{Z}^{d+1}) \setminus \{0\} \subseteq \operatorname{int}(C_P) \cap \mathbb{Z}^{d+1} + \operatorname{ant}(C_P) \cap \mathbb{Z}^{d+1}. \tag{8.1}$$

In particular, if P is nearly Gorenstein, then

$$P \cap \mathbb{Z}^d = \operatorname{int}(C_P)_a \cap \mathbb{Z}^d + \operatorname{ant}(C_P)_{1-a} \cap \mathbb{Z}^d. \tag{8.2}$$

The converse also holds if P is IDP.

Proof. By definition, P is nearly Gorenstein if and only if the trace $\operatorname{tr}(\omega)$ of the canonical ideal ω of $A(P)$ contains the maximal ideal $\mathbf{m}$ of $A(P)$. By Proposition 17, this trace is exactly the product $\omega_{A(P)} \cdot \omega^{-1}_{A(P)}$. Then, Proposition 16 tells us the monomial generators of ω and ω^{-1} in terms of the lattice points of $\operatorname{int}(C_P)$ and $\operatorname{ant}(C_P)$. We finally note that the maximal ideal $\mathbf{m}$ can be generated by the monomials $t^x s^k$, where (x,k) are lattice points in $C_P \setminus \{0\}$. From this, it is clear to see that P is nearly Gorenstein if and only if (8.1) holds.

We next prove that (8.2) follows from nearly Gorensteinness of P. First, note that the right hand side of (8.1) is contained in $C_P \cap \mathbb{Z}^{d+1}$ by definition. Therefore, when we take the 1-st piece of all three sets, we obtain the equality

$$P \cap \mathbb{Z}^d = (\operatorname{int}(C_P) \cap \mathbb{Z}^{d+1} + \operatorname{ant}(C_P) \cap \mathbb{Z}^{d+1})_1.$$

Note that when P is Gorenstein, $\operatorname{int}(C_P)_a \cap \mathbb{Z}^d$ and $\operatorname{ant}(C_P)_{-a} \cap \mathbb{Z}^d$ are singleton sets; therefore, the result easily follows. Otherwise, we claim that $\operatorname{ant}(C_P)_{1-b} \cap \mathbb{Z}^d$ is empty for all $b \geq a+1$. Since $\operatorname{int}(C_P)_b$ is empty for $b < a$, we obtain the desired result.

Finally, we show that the converse holds when P is IDP. Let $(x,k) \in C_P \cap \mathbb{Z}^d \setminus \{0\}$. Since P is IDP, there are $x_1, \ldots, x_k \in P \cap \mathbb{Z}^d$ such that $(x,k) = (x_1, 1) + \cdots + (x_k, 1)$. Further, each $x_i \in P \cap \mathbb{Z}^d$ can be written as the sum of lattice points in $\operatorname{int}(C_P)$ and $\operatorname{ant}(C_P)$. Therefore, (8.1) holds and so P is nearly Gorenstein. □

We now collate a couple of easy facts about the floor and remainder polytopes and reformulate part of Proposition 38 into the following statement.

Lemma 19. *Let $P \subset \mathbb{R}^d$ be a lattice polytope with codegree a. Then:*

1. $\lfloor aP \rfloor \subseteq \{x \in \mathbb{R}^d : n_F(x) \geq 1 - ah_F$ *for all* $F \in \mathcal{F}(P)\}$;

2. *$\{P\} \subseteq \{x \in \mathbb{R}^d \colon n_F(x) \geq (a-1)h_F - 1$ for all $F \in \mathcal{F}(P)\}$;*

3. *If P is nearly Gorenstein, then $P \cap \mathbb{Z}^d = \lfloor aP \rfloor \cap \mathbb{Z}^d + \{P\} \cap \mathbb{Z}^d$;*

4. *If P is IDP and $P \cap \mathbb{Z}^d = \lfloor aP \rfloor \cap \mathbb{Z}^d + \{P\} \cap \mathbb{Z}^d$, then P is nearly Gorenstein.*

Proof. Statements (1) and (2) follow immediately from the definition of the floor and remainder polytope. To prove statements (3) and (4), notice that the lattice points of $\mathrm{int}(C_P)_a$ coincide with those of $\lfloor aP \rfloor$ and the lattice points of $\mathrm{ant}(C_P)_{1-a}$ coincide with those of $\{P\}$. Then simply substitute this into Proposition 38. □

The following proposition is the first half of Theorem 24:

Proposition 39. *If P is nearly Gorenstein, then $P = \lfloor aP \rfloor + \{P\}$, where a is the codegree of P.*

Proof. Let $x \in \lfloor aP \rfloor$ and $y \in \{P\}$. By statements (1) and (2) of Lemma 19, we have that, for all facets F of P, $n_F(x+y) \geq 1 - ah_F + (a-1)h_F - 1 = -h_F$. So, $x + y \in P$. Therefore, we obtain that $\lfloor aP \rfloor + \{P\} \subseteq P$.

On the other hand, let v be a vertex of P. Since P is a lattice polytope, $v \in P \cap \mathbb{Z}^d$. Thus, by statement (3) of Lemma 19, can write v as the sum of an element of $\lfloor aP \rfloor \cap \mathbb{Z}^d$ and an element of $\{P\} \cap \mathbb{Z}^d$. This implies $P \subseteq \lfloor aP \rfloor + \{P\}$. □

Example 14. Consider the stop sign polytope, given by

$$P = \mathrm{conv}\{(1,0),(2,0),(3,1),(3,2),(2,3),(1,3),(0,2),(0,1)\}.$$

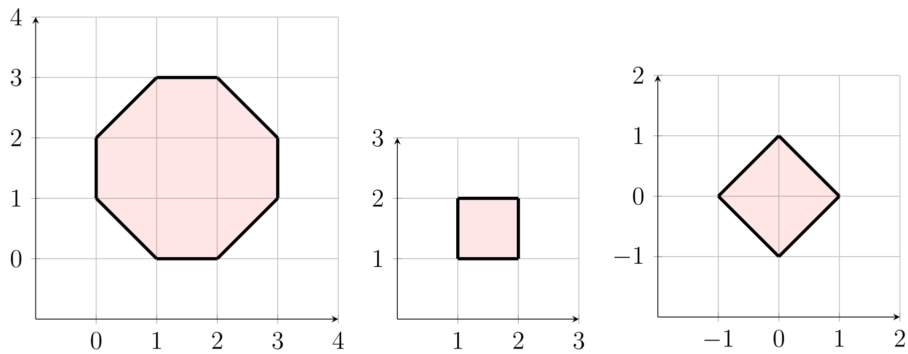

Figure 8.1: The stop sign polytope P (left) with its floor polytope $\lfloor P \rfloor$ (middle) and remainder polytope $\{P\}$ (right).

First, we note that $a_P = 1$. Next, we may compute the floor and remainder polytopes:

$$\lfloor P \rfloor = \mathrm{conv}\{(1,1),(2,1),(1,2),(2,2)\} \quad \text{and} \quad \{P\} = \mathrm{conv}\{(1,0),(0,1),(-1,0),(0,-1)\}.$$

By taking the Minkowski sum of these polytopes, we see that P satisfies the necessary condition to be Gorenstein given by Proposition 39, i.e. $P = \lfloor P \rfloor + \{P\}$. On the other hand, it is straightforward to verify that every lattice point of P can be written as the sum of a lattice point of $\lfloor P \rfloor$ and a lattice point of $\{P\}$. Since P is IDP (as is true for all *polygons*), statement (4) of Lemma 19 informs us that P is nearly Gorenstein.

Finally, we remark that the remainder polytope $\{P\}$ is reflexive. This is not coincidence, as we will prove in Proposition 31.

8.2.2 A sufficient condition

In this subsection, we will explore sufficient conditions for a lattice polytope to be nearly Gorenstein; in particular, we will prove the second half of Theorem 24.

We first note that the converse of Proposition 39 does not hold in general.

Example 15 (compare [MP05, Example 1.1])**.** Let $f = \frac{1}{3}(e_1 + \cdots + e_6) \in \mathbb{R}^6$, where $e_1, \ldots, e_6$ is a basis of the lattice $\mathbb{Z}^6$. Define a new lattice $L \coloneqq \mathbb{Z}^6 + f \cdot \mathbb{Z}$, and consider the lattice polytope

$$Q \coloneqq \operatorname{conv}\{e_1, \ldots, e_6, e_1 - f, \ldots, e_6 - f\}$$

with respect to the lattice L. Set $P \coloneqq 2Q$. Since $\lfloor P \rfloor = \{P\} = Q$, it's easy to see that $P = \lfloor P \rfloor + \{P\}$, meeting the necessary condition of Proposition 39 for nearly Gorensteinness.

On the other hand, Q is not IDP. In particular, $2Q \cap L \neq (Q \cap L) + (Q \cap L)$. Thus, $P = 2Q$ fails the necessary condition of statement (3) in Lemma 19, and so P is not nearly Gorenstein.

So, we need to make more assumptions about P in order to be guaranteed nearly Gorensteinness. This brings us to the following result, which is the second half of Theorem 24:

Theorem 28. *Let $P \subset \mathbb{R}^d$ be a lattice polytope satisfying $P = \lfloor aP \rfloor + \{P\}$, where a is the codegree of P. Then there exists some integer $K \geq 1$ (depending on P) such that for all $k \geq K$, the polytope kP is nearly Gorenstein.*

In order to prove the above, we rely on a few key ingredients. The first ingredient is an extension of known results from the reflexive case, which appear in [Hib92].

Lemma 20. *Let $P \subset \mathbb{R}^d$ be a lattice polytope satisfying $P = \lfloor aP \rfloor + \{P\}$, where a is the codegree of P. Then the following statements hold:*

1. *$kP = \lfloor (k + a - 1)P \rfloor + \{P\}$, for all $k \geq 1$;*
2. *$\lfloor k'P \rfloor = \lfloor aP \rfloor + (k' - a)P$, for all $k' \geq a$.*

Before we give the proof, we will restrict these statements to the reflexive case for the sake of comparison. First, we have $a = 1$. Next, since $\lfloor P \rfloor$ is the origin, $P = \{P\}$. So, for reflexive polytopes, the statement (1) is equivalent to $kP = \lfloor kP \rfloor + P$. After cancellation by P, we obtain the reflexive version of statement (2): $\lfloor kP \rfloor = (k - 1)P$.

Proof of Lemma 20. Let $k \geq 1$ be an integer. Throughout this proof, we repeatedly use the two inequalities appearing in statements (1) and (2) of Lemma 19. We also use the inequalities appearing in the facet presentations for P and its dilates.

We first prove the "$\supseteq$" part of statement (1), i.e. that

$$kP \supseteq \lfloor (k+a-1)P \rfloor + \{P\}, \text{ for all } k \geq 1. \tag{8.3}$$

Let $x \in \lfloor (k+a-1)P \rfloor$ and $y \in \{P\}$. Then $n_F(x+y) \geq (1-(k+a-1)h_F)+((a-1)h_F-1) = -kh_F$, for all facets F of P. Thus, $x+y \in kP$.

Next, we note that $kP = (k-1)P + \lfloor aP \rfloor + \{P\}$. We substitute this into (8.3), then cancel $\{P\}$ from both sides to obtain $\lfloor (k+a-1)P \rfloor \subseteq (k-1)P + \lfloor aP \rfloor$.

We now prove the reverse inclusion of the above. Let $x \in (k-1)P$ and $y \in \lfloor aP \rfloor$. Then, $n_F(x+y) \geq -(k-1)h_F + (1-ah_F) = 1-(k+a-1)h_F$. Therefore, $x+y \in \lfloor (k+a-1)P \rfloor$. Thus, we obtain the equality $\lfloor (k+a-1)P \rfloor = (k-1)P + \lfloor aP \rfloor$. Setting $k' := k+a-1$ then gives us statement (2). Adding $\{P\}$ to both sides gives us statement (1). □

The main ingredient in proving Theorem 28 is a result of Haase and Hofmann, which allows us to guarantee that the second condition of statement (4) of Lemma 19 holds.

Theorem 29 ([HH17, Theorem 4.2])**.** *Let $P, Q \subset \mathbb{R}^d$ be rational polytopes such that the normal fan $\mathcal{N}(P)$ of P is a refinement of the normal fan $\mathcal{N}(Q)$ of Q. Suppose also that for each edge E of P, the corresponding face E' of Q has lattice length $\ell_{E'}$ satisfying $\ell_E \geq d\ell_{E'}$. Then $(P+Q) \cap \mathbb{Z}^d = (P \cap \mathbb{Z}^d) + (Q \cap \mathbb{Z}^d)$.*

In order to guarantee the first condition of statement (4) of Lemma 19, we need this next result:

Theorem 30 ([VGB97, Theorem 1.3.3])**.** *Let $P \subset \mathbb{R}^d$ be a lattice polytope. Then $(d-1)P$ is IDP.*

We are now ready to give the proof.

Proof of Theorem 28. We first wish to find a suitable K which satisfies

$$kP \cap \mathbb{Z}^d = \lfloor kP \rfloor \cap \mathbb{Z}^d + \{kP\} \cap \mathbb{Z}^d, \text{ for all } k \geq K.$$

Let a be the codegree of P. Looking at statement (2) of Lemma 20, we see that $(k-a)P$ is a Minkowski summand of $\lfloor kP \rfloor$; thus, we get a crude lower bound on the length of the edges of $\lfloor kP \rfloor$: for $k \geq a$, every edge E of $\lfloor kP \rfloor$ has lattice length $\ell_E \geq k-a$. Denote by L the maximum edge length of $\{aP\}$ and set $K := dL + a$. Note that for $k \geq a$, the polytopes $\{kP\}$ and $\{aP\}$ coincide. So, for all $k \geq K$, every edge E of $\lfloor kP \rfloor$ will have lattice length $\ell_E \geq k-a \geq dL$.

Further, statement (2) of Lemma 20 implies that, for $k \geq a+1$, the normal fan $\mathcal{N}(\lfloor kP \rfloor)$ coincides with $\mathcal{N}(P)$. Hence, $\mathcal{N}(\lfloor kP \rfloor)$ is a refinement of the normal fan of $\{kP\}$. Thus, we may apply Theorem 29, obtaining that $kP \cap \mathbb{Z}^d = \lfloor kP \rfloor \cap \mathbb{Z}^d + \{kP\} \cap \mathbb{Z}^d$.

Finally, since $a, L \geq 1$, we see that $K \geq d-1$. Thus, by Theorem 30, we have that kP is IDP. Therefore, by statement (4) of Lemma 19, we can conclude that kP is nearly Gorenstein for all $k \geq K$. □

Remark 11. We say that a graded ring R is *Gorenstein on the punctured spectrum* [HHS19] if $\mathrm{tr}(\omega_R)$ contains $\mathbf{m}^k$ for some integer $k \geq 0$. If $k = 0$, this is just the Gorenstein condition; if $k = 1$, it is the nearly Gorenstein condition. Now, for a lattice polytope $P \subset \mathbb{R}^d$, it can be shown that its Ehrhart ring $A(P)$ is Gorenstein on the punctured spectrum if there exists a positive integer K such that $kP \cap \mathbb{Z}^d$ coincides with $(\mathrm{int}(C_P) \cap \mathbb{Z}^{d+1} + \mathrm{ant}(C_P) \cap \mathbb{Z}^{d+1})_k$, for all $k \geq K$. Therefore, using Theorem 28, it's straightforward to show that all lattice polytopes P satisfying $P = \lfloor aP \rfloor + \{P\}$ are Gorenstein on the punctured spectrum.

8.2.3 Decompositions of nearly Gorenstein polytopes

In this subsection, we first prove Theorem 25. This naturally leads to an investigation of whether nearly Gorenstein polytopes decompose into the Minkowski sum of Gorenstein polytopes (Questions 1 and 2). We prove Theorem 26, which leads to a way to systematically construct examples of nearly Gorenstein polytopes. This is then used to find a counterexample to Questions 1 and 2. Finally, we conclude the section with a result about indecomposable nearly Gorenstein polytopes.

Theorem 31 (Theorem 25)**.** *Let $P \subset \mathbb{R}^d$ be a lattice polytope which satisfies $P = \lfloor aP \rfloor + \{P\}$, where a is the codegree of P. Then we have*

$$\lfloor aP \rfloor = \{x \in \mathbb{R}^d \colon n_F(x) \geq 1 - ah_F \text{ for all } F \in \mathcal{F}(P)\} \text{ and}$$

$$\{P\} = \{x \in \mathbb{R}^d \colon n_F(x) \geq (a-1)h_F - 1 \text{ for all } F \in \mathcal{F}(P)\}.$$

In particular, the right hand sides of the equalities are lattice polytopes. Furthermore, if $a = 1$, then $\{P\}$ is a reflexive polytope.

Proof. Label the two polytopes on the right-hand sides as Q_1 and Q_2, respectively. It's straightforward to see that $\lfloor aP \rfloor = \mathrm{conv}(Q_1 \cap \mathbb{Z}^d)$ and $\{P\} = \mathrm{conv}(Q_2 \cap \mathbb{Z}^d)$. Thus, $\lfloor aP \rfloor \subseteq Q_1$ and $\{P\} \subseteq Q_2$. Ultimately, we want to prove the reverse inclusions but first, we must show an intermediate equality: $P = Q_1 + Q_2$. Let $x \in Q_1$ and $y \in Q_2$. Then, for all facets F of P, we have $n_F(x + y) \geq 1 - ah_F + (a-1)h_F - 1 = -h_F$. Thus, $x + y \in P$ and so, $Q_1 + Q_2 \subseteq P$. Conversely, if we combine this with our assumption that $P = \lfloor aP \rfloor + \{P\}$, we obtain that, in fact, $P = Q_1 + Q_2$.

We now use the above equality to obtain that $\lfloor aP \rfloor = Q_1$ and $\{P\} = Q_2$, as follows. Assume towards a contradiction that $Q_1 \not\subseteq \lfloor aP \rfloor$, i.e. there exists a vertex v of Q_1 which doesn't belong to $\lfloor aP \rfloor$. Choose a normal vector $n \in (\mathbb{R}^d)^*$ which achieves its minimal value h_1 over Q_1 *only* at v (i.e. n lies in the interior of the cone σ_v in the (inner) normal fan $\mathcal{N}(Q_1)$ which corresponds to v). Denote by h_2 the minimal evaluation of n over Q_2 Then, the minimal evaluation of n over P is $h_1 + h_2$. However, for all $x \in \lfloor aP \rfloor$ and $y \in \{P\}$, we have that $n(x+y) > h_1 + h_2$. This contradicts the fact that $P = \lfloor aP \rfloor + \{P\}$. Therefore, the vertices of Q_1 coincide with the vertices of $\lfloor aP \rfloor$; in particular, $\lfloor aP \rfloor = Q_1$. We similarly obtain that $\{P\} = Q_2$.

Next, since $\lfloor aP \rfloor$ and $\{P\}$ are lattice polytopes by definition, we note that Q_1 and Q_2 are lattice polytopes in this situation.

Finally, suppose we are in the case when P has an interior lattice point, i.e. $a = 1$. By substituting this into the second equality, we see that the remainder polytope $\{P\}$ is indeed reflexive as all its facets lie at height 1.

□

In contrast, when P has no interior points, the remainder polytope $\{P\}$ is not necessarily even Gorenstein.

Example 16. Consider the polytope

$$P = \mathrm{conv}\{(0,0,0),(2,0,0),(1,1,0),(0,1,0),(0,0,1),(2,0,1),(1,1,1),(0,1,1)\}.$$

We can verify that P is nearly Gorenstein and IDP, but the remainder polytope $\{P\}$ is not Gorenstein. However, $\{P\}$ can be written as the Minkowski sum of

$$\mathrm{conv}\{(0,0,0),(1,0,0),(0,1,0)\} \quad \text{and} \quad \mathrm{conv}\{(-1,-1,-1),(-1,-1,0)\},$$

which are both Gorenstein.

We see similar behavior when studying the nearly Gorensteinness for certain restricted classes of polytopes. This motivated us to pose the following question.

Question 1. If P is nearly Gorenstein, then can we write $P = P_1 + \cdots + P_s$ for some Gorenstein lattice polytopes $P_1, \ldots, P_s$?

We recall that P is (*Minkowski*) *indecomposable* if P is not a singleton and if there exist lattice polytopes P_1 and P_2 with $P = P_1 + P_2$, then either P_1 or P_2 is a singleton. Note that if P is not a singleton, then we can write $P = P_1 + \cdots + P_s$ for some indecomposable lattice polytopes $P_1, \ldots, P_s$.

Then, Question 1 can be rephrased as:

Question 2. If P has an indecomposable non-Gorenstein lattice polytope as a Min-kowski summand, then is P not nearly Gorenstein?

This question has a positive answer for IDP $(0,1)$-polytopes, which is shown in Section 8.3. For the remainder of this section, we will build up some machinery which allows for the efficient construction of nearly Gorenstein polytopes. We then use this in Example 17 to give an answer to Questions 1 and 2.

Theorem 32 (Theorem 26)**.** *Let $P \subset \mathbb{R}^d$ be a nearly Gorenstein polytope. Then there exists a reflexive polytope $Q \subset \mathbb{R}^d$ such that*

$$P = \{x \in \mathbb{R}^d \colon n(x) \geq -h_n \text{ for all } n \in \partial Q^* \cap (\mathbb{Z}^d)^*\},$$

where h_n are integers. Moreover, the inequalities defined by $n \in \mathrm{vert}(Q^*)$ are irredundant. Furthermore, the number of facets of a nearly Gorenstein polytope is bounded by a constant depending on the dimension d.

Before we dive into the proof, it will be useful to have the following lemma.

Lemma 21. *Let P be a lattice polytope satisfying $P = \lfloor aP \rfloor + \{P\}$, where a is the codegree of P. Then $aP = \lfloor aP \rfloor + \{aP\}$. Moreover, $\{aP\} = (a-1)P + \{P\}$.*

Proof. We first wish to show that $(a-1)P + \{P\} \subseteq \{aP\}$. Let $x \in (a-1)P$ and $y \in \{P\}$. Then, by Lemma 19 (2), $n_F(x+y) \geq -(a-1)h_F + (a-1)h_F - 1 = -1$, for all facets F of P. So, $x + y \in \{aP\}$. Thus, $(a-1)P + \{P\} \subseteq \{aP\}$.

We can add $\lfloor aP \rfloor$ to both sides of the inclusion to get $aP \subseteq \lfloor aP \rfloor + \{aP\}$.

We next wish to show the reverse inclusion of the above. Let $z \in \lfloor aP \rfloor$ and $w \in \{aP\}$. Then $n_F(z+w) \geq (1 - ah_F) - 1 = -ah_F$, for all facets F of P. So, $z + w \in aP$. Therefore, $\lfloor aP \rfloor + \{aP\} \subseteq aP$. Combining the two inclusions gives the desired equality: $aP = \lfloor aP \rfloor + \{aP\}$.

Moreover, we obtain that $\lfloor aP \rfloor + \{P\} + (a-1)P = \lfloor aP \rfloor + \{aP\}$. Since Minkowski addition of convex sets satisfies the cancellation law, we may cancel both sides by $\lfloor aP \rfloor$ to obtain the equality $\{aP\} = (a-1)P + \{P\}$. □

Proof of Theorem 32. We wish to study the (inner) normal fan $\mathcal{N}(P)$ of P, as it's enough to show that its primitive ray generators all lie in $\partial Q^* \cap (\mathbb{Z}^d)^*$, for some reflexive polytope $Q \subset \mathbb{R}^d$. Let a be the codegree of P. Since dilation has no effect on the normal fan, we may pass to the normal fan of aP. Now, by Lemma 21, aP has a Minkowski decomposition into $\lfloor aP \rfloor$ and $\{aP\}$. Thus, $\mathcal{N}(aP)$ is the common refinement of $\mathcal{N}(\lfloor aP \rfloor)$ and $\mathcal{N}(\{aP\})$. By Proposition 31, we obtain that $Q := \{aP\}$ is a reflexive polytope. Hence, the primitive ray generators of $\mathcal{N}(Q)$ are vertices of the reflexive polytope $Q^* \subset (\mathbb{R}^d)^*$; in particular, they are lattice points lying in the boundary of Q^*.

We next look at the contribution to $\mathcal{N}(aP)$ coming from $\lfloor aP \rfloor$. Let $n \in (\mathbb{Z}^d)^*$ be a primitive ray generator of $\mathcal{N}(\lfloor aP \rfloor)$. Then, by definition of the remainder polytope, $n(x) \geq -1$, for all $x \in Q$. But now, this means that n lies in Q^*. So, since $n \neq 0$ and Q is reflexive, we obtain that $n \in \partial Q^* \cap (\mathbb{Z}^d)^*$. Therefore, we have now shown that the primitive ray generators of $\mathcal{N}(P) = \mathcal{N}(aP)$ contain the vertices of Q^*, and that they all lie in $\partial Q^* \cap (\mathbb{Z}^d)^*$.

Finally, we note that the number of facets of a nearly Gorenstein polytope $P \subset \mathbb{R}^d$ is bounded by $c_d := \sup_Q |\partial Q^* \cap (\mathbb{Z}^d)^*|$, where Q runs over all d-dimensional reflexive polytopes. Since there are only finitely reflexive polytopes in each dimension d, and all polytopes only have a finite number of boundary points, we see that c_d is a finite number. □

We will now detail how to construct nearly Gorenstein polytopes. First, choose a reflexive polytope $Q \subset \mathbb{R}^d$. Then, choose a (possibly empty) subset S' of the boundary lattice points of Q^* which are not vertices of Q^*. Now, for each $n \in S := S' \cup \operatorname{vert}(Q^*)$, choose the height $h_n \in \mathbb{Z}$. Construct a polytope P' defined by $n(x) \geq -h_n$ for all $n \in S$, and assert that none of these inequalities are redundant. Next, we can dilate P' to rP' so that it's a lattice polytope which contains an interior lattice point. By construction, its remainder polytope $\{rP'\}$ coincides with the reflexive polytope Q. In practice, rP' has a Minkowski decomposition into $\lfloor rP' \rfloor$ and $\{rP'\}$, but we don't yet have a proof that this always holds. Finally, we can use Theorem 28 to dilate rP' even further to $P := krP'$ so that $P = \lfloor P \rfloor + \{P\}$ is nearly Gorenstein.

Example 17. Consider the polytope

$$P = \operatorname{conv}\{(-4,-3,-4),(-3,-1,-3),(-2,-2,-3),(0,1,4),(0,4,1),(3,1,1)\}.$$

Note that P has many interior lattice points, it has codegree 1. Its floor polytope is

$$\lfloor P \rfloor = \operatorname{conv}\{(-3,-2,-3),(0,3,1),(0,1,3),(2,1,1)\}.$$

This is an indecomposable simplex, which is not Gorenstein. Its remainder polytope is

$$\{P\} = \operatorname{conv}\{(-1,-1,-1),(1,0,0),(0,1,0),(0,0,1)\},$$

which is clearly reflexive. We have $P = \lfloor P \rfloor + \{P\}$. We use Magma [BCP97] to verify that $P \cap \mathbb{Z}^3 = (\lfloor P \rfloor \cap \mathbb{Z}^3) + (\{P\} \cap \mathbb{Z}^3)$ and that P is IDP. Thus, we may conclude by Lemma 19 that P is a nearly Gorenstein polytope.

It can be shown that $P = \lfloor P \rfloor + \{P\}$ is the only non-trivial Minkowski decomposition of P. Thus, we may conclude that the nearly Gorenstein polytope P cannot be decomposed into Gorenstein polytopes. Therefore, we may answer Questions 1 and 2 in the negative.

We end this section by giving the following theorem about nearly Gorensteinness of indecomposable polytopes, which plays an important role in the characterisation of nearly Gorenstein $(0,1)$-polytopes in Section 8.3.

Theorem 33. *Let P be an indecomposable lattice polytope. Then, P is nearly Gorenstein if and only if P is Gorenstein.*

Proof. It is already clear that Gorensteinness implies nearly Gorensteinness, so we just have to treat the converse implication. Suppose that P is nearly Gorenstein. By Proposition 39, we have that $P = \lfloor aP \rfloor + \{P\}$, where a is the codegree of P. Since P is indecomposable, either (i) $\lfloor aP \rfloor$ is a singleton or (ii) $\{P\}$ is a singleton.

We first deal with case (i). Consider aP. By Lemma 21, $aP = \lfloor aP \rfloor + \{aP\}$. Thus, aP is a translation of $\{aP\}$. By Proposition 31, $\{aP\}$ is reflexive. Thus, P is Gorenstein.

The argument for case (ii) is similar. We consider $\{aP\}$. By Lemma 21, $\{aP\} = (a-1)P + \{P\}$. Proposition 31 tells us that $\{aP\}$ is reflexive; therefore, $(a-1)P$ is a translation of a reflexive polytope. But this is an absurdity as it implies that $(a-1)P$ has an interior lattice point, contradicting that the codegree of P is a. Thus, this case cannot occur.

□

8.3 Nearly Gorenstein $(0,1)$-polytopes

In this section, we consider the case of $(0,1)$-polytopes. We provide the characterisation of nearly Gorenstein $(0,1)$-polytopes which are IDP. Moreover, we also characterise nearly

Gorenstein edge polytopes of graphs satisfying the odd cycle condition and characterise nearly Gorenstein graphic matroid polytopes.

Beforehand, we need a technical lemma.

Lemma 22. *Let $R_1, \dots, R_s$ be homogeneous normal affine semigroup rings over infinite field* $\mathbf{k}$ *which have Krull dimension at least 2. Let $R = R_1 \# \cdots \# R_s$ be the Segre products. Then the following are true:*

(1) If R is nearly Gorenstein, then R_i is nearly Gorenstein for all i.

(2) If R_i is level for all i, then R is level.

Proof. It suffices to prove the case $s = 2$. Let $x_1, \dots, x_n$ be $\mathbf{k}$-basis of $(R_1)_1$ and $y_1, \dots, y_m$ be a $\mathbf{k}$-basis of$(R_2)_1$.

(1): In this case, by using Proposition 18, we get $\omega_R \cong \omega_{R_1} \# \omega_{R_2}$ and $\omega_R^{-1} \cong \omega_{R_1}^{-1} \# \omega_{R_2}^{-1}$. Then we may identify ω_R and $\omega_R{}^{-1}$ with $\omega_{R_1} \# \omega_{R_2}$ and $\omega_{R_1}^{-1} \# \omega_{R_2}^{-1}$, respectively.

It is enough to show that $x_i \in \operatorname{tr}(\omega_{R_1})$ for any $1 \leq i \leq n$. Since R is nearly Gorenstein, there exist homogeneous elements $v_1 \# v_2 \in \omega_{R_1} \# \omega_{R_2}$ and $u_1 \# u_2 \in \omega_{R_1}^{-1} \# \omega_{R_2}^{-1}$ such that $x_i \# y_1 = (v_1 \# v_2)(u_1 \# u_2) = (v_1 u_1 \# v_2 u_2)$, by [Miy24, Proposition 4.2]. Thus, we get $x_i = v_1 u_1 \in \operatorname{tr}(\omega_{R_1})$, so R_1 is nearly Gorenstein. In the same way as above, we can show that R_2 is also nearly Gorenstein.

(2): First, $\omega_R \cong \omega_{R_1} \# \omega_{R_2}$ by Proposition 18. Let a_1 and a_2 be the a-invariants of R_1 and R_2, respectively, and assume that $a_1 \leq a_2$. Since R_1 and R_2 are level, $\omega_{R_1} \cong \langle f_1, \cdots, f_r \rangle R_1$ and $\omega_{R_2} \cong \langle g_1, \cdots, g_l \rangle R_2$ where $\deg f_i = -a_1$ and $\deg g_j = -a_2$ for all $1 \leq i \leq r$, $1 \leq j \leq l$. Thus, since $\omega_R \cong \omega_{R_1} \# \omega_{R_2}$, we may identify ω_R with $\langle f_1, \cdots, f_r \rangle R_1 \# \langle g_1, \cdots, g_l \rangle R_2$. We set

$$V \coloneqq \{ y^b g_j \colon 1 \leq j \leq l,\ a \in \mathbb{N}^m,\ \sum_{i=1}^{m} b_i = a_2 - a_1 \},$$

where $y^a \coloneqq y_1^{a_1} \cdots y_m^{a_m}$. Then $\omega_R = \langle f_i \# v :\ 1 \leq i \leq r, v \in V \rangle R$. Therefore, R is level. □

8.3.1 The characterisation of nearly Gorenstein $(0,1)$-polytopes

Lemma 23. *Let $P \subset \mathbb{R}^d$ be a $(0,1)$-polytope. Then, after a change of coordinates, we can write $P = P_1 \times \cdots \times P_s$ for some indecomposable $(0,1)$-polytopes $P_1, \dots, P_s$.*

Proof. As mentioned in Section 8.2, we can write $P = P_1' + \cdots + P_s'$ for some indecomposable lattice polytopes $P_1', \dots, P_s'$.

First, we show that we can choose $P_1', \dots, P_s'$ so that these are $(0,1)$-polytopes. Suppose that we can write $P = P_1' + P_2'$ for some lattice polytopes P_1' and P_2'. Then, for any $v \in P_1' \cap \mathbb{Z}^d$ and for any $u \in P_2' \cap \mathbb{Z}^d$, $v + u$ is a $(0,1)$-vector. Therefore, for any $i \in [d]$, $\pi_i(P_1' \cap \mathbb{Z}^d)$ can take one of the following forms: (i) $\{w_i\}$ or (ii) $\{w_i, w_i + 1\}$ for some $w_i \in \mathbb{Z}$. In case (i), $\pi_i(P_2' \cap \mathbb{Z}^d)$ is equal to $\{-w_i\}$, $\{-w_i + 1\}$ or $\{-w_i, -w_i + 1\}$. In case (ii), $\pi_i(P_2' \cap \mathbb{Z}^d)$ is equal to $\{-w_i\}$. Thus, in all cases, $P_1' - w$ and $P_2' + w$ are $(0,1)$-polytopes and we have $P = (P_1' - w) + (P_2' + w)$, where $w = (w_1, \dots, w_d)$.

Moreover, if we can write $P = P'_1 + P'_2$ for some $(0,1)$-polytopes P'_1 and P'_2, then we can see that either $\pi_i(P'_1)$ or $\pi_i(P'_2)$ is equal to $\{0\}$ for any $i \in [d]$. Therefore, after a change of coordinates, we can write $P = P_1 \times P_2$ for some $(0,1)$-polytopes P_1 and P_2. □

Now, we provide the main theorem of this section.

Theorem 34. *Let P be an IDP $(0,1)$-polytope. Then, P is nearly Gorenstein if and only if you can write $P = P_1 \times \cdots \times P_s$ for some Gorenstein $(0,1)$-polytopes $P_1, \ldots, P_s$ with $|a_{P_i} - a_{P_j}| \leq 1$, where a_{P_i} and a_{P_j} are the respective codegrees of P_i and P_j, for $1 \leq i < j \leq s$.*

Proof. It follows from Lemma 23 that we can write $P = P_1 \times \cdots \times P_s$ for some indecomposable $(0,1)$-polytopes $P_1, \ldots, P_s$. Thus, we have $\mathbf{k}[P] \cong \mathbf{k}[P_1]\# \cdots \#\mathbf{k}[P_s]$. Note that if P is IDP, then so is P_i for each $i \in [s]$, and $A(P)$ (resp. $A(P_i)$) coincides with $\mathbf{k}[P]$ (resp. $\mathbf{k}[P_i]$). Therefore, since P is nearly Gorenstein, $\mathbf{k}[P]$ is nearly Gorenstein, and hence $\mathbf{k}[P_i]$ is also nearly Gorenstein from Lemma 22 (1). Furthermore, P_i is nearly Gorenstein. Since P_i is indecomposable, P_i is Gorenstein by Theorem 33. Moreover, it follows from [HMP19, Corollary 2.8] that $|a_{P_i} - a_{P_j}| \leq 1$ for $1 \leq i < j \leq s$.

The converse also holds from [HMP19, Corollary 2.8]. □

From this theorem, we immediately obtain the following corollaries:

Corollary 6. *Question 1 is true for IDP $(0,1)$-polytopes.*

Corollary 7. *Let P be an IDP $(0,1)$-polytope. If $\mathbf{k}[P]$ is nearly Gorenstein, then $\mathbf{k}[P]$ is level.*

Proof. It follows immediately from Lemma 22 (2) and Theorem 34. □

The result of Theorem 34 can be applied to many classes of $(0,1)$-polytopes such as order polytopes and stable set polytopes.

Order polytopes, which were introduced by Stanley [Sta86], arise from posets. Let Π be a poset equipped with a partial order $\preceq$. The Ehrhart ring of the order polytope of a poset Π is called the Hibi ring of Π, denoted by $\mathbf{k}[\Pi]$. It is known that Hibi rings are standard graded ([Hib87]). For a subset $I \subset P$, we say that I is a *poset ideal* of P if $p \in I$ and $q \preceq p$ then $q \in I$. According to [Sta86], the characteristic vectors of poset ideals in $\mathbb{R}^\Pi$ are precisely the vertices of the order polytope of Π (hence order polytopes are $(0,1)$-polytopes). By this fact, we can see that the order polytope of a poset Π is indecomposable if and only if Π is connected. Nearly Gorensteinness of Hibi rings have been studied in [HHS19]. It is shown that $\mathbf{k}[\Pi]$ is nearly Gorenstein if and only if Π is the disjoint union of pure connected posets $\Pi_1, \ldots, \Pi_q$ such that their ranks of any two also can only differ by at most 1. Moreover, in this case, $\mathbf{k}[\Pi_i]$ is Gorenstein and $\mathbf{k}[\Pi] \cong \mathbf{k}[\Pi_1]\# \cdots \#\mathbf{k}[\Pi_s]$. Therefore, its characterisation can be derived from Theorem 34.

Stable set polytopes, which were introduced by Chvátal [Chv75], arise from graphs. For a finite simple graph G on the vertex set $V(G)$ with the edge set $E(G)$, the stable set polytope of G, denoted by Stab_G, is defined as the convex hull of the characteristic vectors of stable sets of G in $\mathbb{R}^{V(G)}$, hence Stab_G is a $(0,1)$-polytope. Here, we say that a subset S of $V(G)$ is a *stable set* if $\{v,u\} \notin E(G)$ for any $v,u \in S$. This implies that Stab_G is indecomposable if and only if G is connected. Stable set polytopes behave well for perfect graphs. For example, Stab_G is IDP if G is perfect (cf.[OH01]). Moreover, the characterisation of nearly Gorenstein stable set polytopes of perfect graphs has been given in [HS21; Miy22]. Let G be a perfect graph with connected components $G_1,\dots,G_s$ and let δ_i denote the maximal cardinality of cliques of G_i. Then, it is known that Stab_G is nearly Gorenstein if and only if the maximal cliques of each G_i have the same cardinality and $|\delta_i - \delta_j| \le 1$ for $1 \le i < j \le s$. In this case, as in the case of order polytopes, $\mathbf{k}[\mathrm{Stab}_{G_i}]$ is Gorenstein and $\mathbf{k}[\mathrm{Stab}_G] \cong \mathbf{k}[\mathrm{Stab}_{G_1}]\#\cdots\#\mathbf{k}[\mathrm{Stab}_{G_s}]$. Therefore, its characterisation can also follow from Theorem 34.

Furthermore, by using this theorem, we can study the nearly Gorensteinness of other classes of $(0,1)$-polytopes.

8.3.2 Nearly Gorenstein edge polytopes

First, we define the edge polytope and edge ring of a graph. We refer the reader to [HHO18, Section 5] and [Vil01, Chapters 10 and 11] for an introduction to edge rings.

Let G be a finite simple graph on the vertex set $V(G) = \{1,\dots,d\}$ with the edge set $E(G)$. Given an edge $e = \{i,j\} \in E(G)$, let $\rho(e) \coloneqq e_i + e_j$, where e_i denotes the i-th unit vector of $\mathbb{R}^d$ for $i \in [d]$. We define the *edge polytope* P_G of G as follows:

$$P_G = \mathrm{conv}\{\rho(e) \colon e \in E(G)\}.$$

The toric ring of P_G is called the *edge ring* of G, denoted by $\mathbf{k}[G]$ instead of $\mathbf{k}[P_G]$.

Let $G_1,\dots,G_s$ be the connected components of G. From the definition of edge polytope, we can see that $\mathbf{k}[G] \cong \mathbf{k}[G_1] \otimes \cdots \otimes \mathbf{k}[G_s]$. Therefore, in considering the characterisation of nearly Gorenstein edge polytopes, we may assume that G is connected.

Moreover, for a connected graph G, P_G is IDP if and only if G satisfies the *odd cycle condition*, in other words, for each pair of odd cycles C and C' with no common vertex, there is an edge $\{v,v'\}$ with $v \in V(C)$ and $v' \in V(C')$ (see [OH98; SVV94]).

Gorenstein edge polytopes have been investigated in [OH06]. We now state the characterisation of nearly Gorenstein edge polytopes.

Corollary 8. *Let G be a connected simple graph satisfying the odd cycle condition. Then, the edge polytope P_G of G is nearly Gorenstein if and only if P_G is Gorenstein or G is the complete bipartite graph $K_{n,n+1}$ for some $n \ge 2$.*

Proof. If P_G is nearly Gorenstein, then Theorem 34 allows us to write $P_G = P_1 \times \cdots \times P_s$ for some indecomposable Gorenstein $(0,1)$-polytopes $P_1,\dots,P_s$. Then, we have $s \le 2$ since $P_G \subset \{(x_1,\dots,x_d) \in \mathbb{R}^d \colon x_1 + \cdots + x_d = 2\}$, where $d = |V(G)|$. In the case $s = 1$, P_G is Gorenstein. If $s = 2$, we can see that $P_1 = \mathrm{conv}\{e_1,\dots,e_n\} \subset \mathbb{R}^n$ and $P_2 = \mathrm{conv}\{e_1,\dots,e_{d-n}\} \subset \mathbb{R}^{d-n}$ for some $1 < n < d-1$. Therefore, we have $G = K_{n,d-n}$, and it is shown by [HS21, Proposition 1.5] that for any $1 < n < d-1$, $P_{K_{n,d-n}}$ is nearly Gorenstein if and only if $d - n \in \{n, n+1\}$. Since $P_{K_{n,n}}$ is Gorenstein, we get the desired result. □

8.3.3 Nearly Gorenstein graphic matroid polytopes

We start by giving one of several equivalent definitions of a matroid.

Definition 16. Let E be a finite set and let $\mathcal{B}$ be a subset of the power set of E satisfying the following properties:

1. $\mathcal{B} \neq \varnothing$.
2. If $A, B \in \mathcal{B}$ with $A \neq B$ and $a \in A \setminus B$, then there exists some $b \in B \setminus A$ such that $(A \setminus \{a\}) \cup \{b\} \in \mathcal{B}$.

Then the tuple $M = (E, \mathcal{B})$ is called a *matroid* with *ground set* E and *set of bases* $\mathcal{B}$.

Let now $G = (V, E)$ be a multigraph. The *graphic matroid* associated to G is the matroid M_G whose ground set is the set of edges E and whose bases are precisely the subsets of E which induce a spanning tree of G. Given two matroids $M_E = (E, \mathcal{B}_E)$ and $M_F = (F, \mathcal{B}_F)$, their *direct sum* $M_E \oplus M_F$ is the matroid with ground set $E \sqcup F$ such that for each basis B of $M_E \oplus M_F$, there exist bases $B_E \in \mathcal{B}_E$ and $B_F \in \mathcal{B}_F$ with $B = B_E \sqcup B_F$. If such a decomposition is not possible for a matroid M, we call it *irreducible*.

A graphic matroid with underlying multigraph G is irreducible if and only if its underlying graph is 2-connected. If it is not irreducible, its irreducible components correspond precisely to the 2-connected components of G.

For any matroid $M = (E, \mathcal{B})$, we can define its *matroid base polytope* (or simply *base polytope*) by

$$B_M = \operatorname{conv}\left\{\sum_{b\in B} e_b \colon B \in \mathcal{B}\right\} \subset \mathbb{R}^{|E|}$$

where e_b is the incidence vector in $\mathbb{R}^{|E|}$ corresponding to the basis b. If B_M comes from a graphic matroid M_G, we will call it B_G.

An alternative definition of matroid base polytopes is as follows.

Definition 17 ([Gel+87, Section 4])**.** A $(0,1)$-polytope $P \subset \mathbb{R}^d$ is called *(matroid) base polytope* if there is a positive integer h such that every vertex $v = (v_1, \dots, v_n)$ satisfies $\sum_{i=1}^{d} v_i = h$ and every edge (i.e. dimension 1 face) of P is a translation of a vector $e_i - e_j$ with $i \neq j$.

It is shown in [Gel+87, Theorem 4.1] that this definition is indeed equivalent to that of a base polytope as given above and that the underlying matroid is uniquely determined. This gives us the following two lemmas.

Lemma 24. *Let G be a multigraph and let $G_1, \dots G_n$ be its 2-connected components. Then B_G can be written as a direct product of the base polytopes $B_{G_1}, \dots, B_{G_n}$. Conversely, if B_G can be written as a direct product of polytopes $P_1, \dots, P_n$, where no P_i is itself a direct product, then these polytopes correspond to the base polytopes of the 2-connected components $G_1, \dots, G_n$ of G.*

Proof. The first statement is trivially satisfied.

The converse follows from two key insights. Firstly, the fact that if a base polytope B_M associated to a (not necessarily graphic) matroid M can be written as a direct product $P_1 \times P_2$, then P_1 and P_2 are again base polytopes. Secondly, if a graphic matroid M_G can be written as a direct sum $M_1 \oplus M_2$, then M_1 and M_2 are again graphic matroids corresponding to subgraphs of G which have at most one vertex in common.

The first insight follows from the alternative definition of a base polytope: Every edge of B_M is given by an edge in P_1 and a vertex of P_2, or vice versa. Hence, P_1 and P_2 must satisfy the definition as well, making them base polytopes with unique underlying matroids M_1 and M_2. The second insight is a classical result and can be found, among other places, in [Tru92, Lemma 8.2.2]. □

The following proposition is the polytopal version of a classical result due to White.

Lemma 25 ([Whi77, Theorem 1])**.** *Matroid base polytopes are IDP.*

We can now define Gorensteinness, nearly Gorensteinness, and levelness of a matroid by identifying it with its base polytope. In [Hib+21] and [Köl20], a constructive, graph-theoretic criterion of Gorensteinness for graphic matroids was found. Since the direct product of two Gorenstein polytopes that have the same codegree is again Gorenstein, the characterisation is presented in terms of 2-connected graphs.

Proposition 40 ([Köl20, Theorems 2.22 and 2.25])**.** *Let G be a 2-connected multigraph. Then the following are equivalent.*

1. *B_G is Gorenstein with codegree $a = 2$*
2. *Either G is the 2-cycle or G can be obtained from copies of the clique K_4 and Construction 2.15 in [Köl20].*

The following are also equivalent.

1. *B_G is Gorenstein with codegree $a > 2$*
2. *G can be obtained from copies of the cycle C_a and Constructions 2.15, 2.17, 2.18 in [Köl20] with $\delta = a$.*

The full characterisation of nearly Gorenstein graphic matroids is thus an immediate corollary of Theorem 34 and Proposition 40.

Corollary 9. *Let G be a multigraph with 2-connected components $G_1, \dots, G_n$, then the following are equivalent.*

1. *B_G is nearly Gorenstein*
2. *$B_{G_1}, \dots, B_{G_n}$ are Gorenstein with codegrees $a_1, \dots, a_n$, where $|a_i - a_j| \leq 1$ for $1 \leq i < j \leq s$.*

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