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NON-STANDARD REPRESENTATIONS OF DISTRIBUTIONS I

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1. Introduction

We fix an infinite natural number H in $^*\boldsymbol{N}$-$\boldsymbol{N}$. We assume that H is even. Let $\varepsilon=1/H$. Then, ε is a positive infinitesimal element of $^*\boldsymbol{Q}$, hence, of $^*\boldsymbol{R}$. We denote by $\boldsymbol{L}$ the set $^*\boldsymbol{Z}\cdot\varepsilon$ of all integer multiples of ε. Then, $^*\boldsymbol{Z}\subsetneqq\boldsymbol{L}\subsetneqq{}^*\boldsymbol{R}$. $\boldsymbol{L}$ is a lattice with infinitesimal mesh ε. Put

$$X=\left\{x\in\boldsymbol{L}\,|\,-\frac{H}{2}\leq x<\frac{H}{2}\right\}.$$

Then, X is a $*$-finite subset of $\boldsymbol{L}$ of cardinality H^2.

Now, consider the set

$$R(X)=\{\varphi\,|\,\varphi\colon X\to{}^*\boldsymbol{C},\ \text{internal}\}.$$

By the above, $R(X)$ is an internal H^2-dimensional vector space over $^*\boldsymbol{C}$. From now on, we will assume that every element φ of $R(X)$ is extended to a function defined on $\boldsymbol{L}$ with period H.

Let Ω be an open set in $\boldsymbol{R}$. Every function $\Omega\to\boldsymbol{C}$ we consider is assumed to be extended to a function $\boldsymbol{R}\to\boldsymbol{C}$ which takes zero outside Ω.

Let's consider $f\in\mathcal{D}(\Omega)$, where $\mathcal{D}(\Omega)$ is the space of indefinitely differentiable functions with compact support on Ω. We have $^*f\colon{}^*\boldsymbol{R}\to{}^*\boldsymbol{C}$, and denoting $K=\mathrm{supp}(f)$, the following statement holds:

$$x\in X,\quad x\notin{}^*K\cap X \text{ implies } {}^*f(x)=0.$$

In the following, we shall define several mappings each from an external subspace of $R(X)$ to some space consisting of distributions on Ω.
Let

$$A(\Omega)=\{\varphi\in R(X)\,|\,\sum_{x\in X}\varepsilon\varphi(x)^*f(x)\text{ is finite for every } f\in\mathcal{D}(\Omega)\}.$$

We remark that the sum $\sum_{x\in X}\varepsilon\varphi(x)^*f(x)$ in this definition always exists in $^*\boldsymbol{C}$ as a

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$*$-finite sum, and it can be represented as $\sum_{x \in {}^*K \cap X} \mathcal{E}\varphi(x)^*f(x)$ with $K=\text{supp}(f)$. We also write these sums $\sum_X \mathcal{E}\varphi^*f$ and $\sum_{{}^*K\cap X} \mathcal{E}\varphi^*f$ respectively.

When $\alpha \in {}^*\boldsymbol{C}$ is finite, ${}^\circ\alpha$ denotes the standard part of α. For each $\varphi \in A(\Omega)$ and $f \in \mathcal{D}(\Omega)$, put

$$P_\varphi(f) = {}^\circ(\textstyle\sum_X \mathcal{E}\varphi^*f)\,.$$

P_φ: $\mathcal{D}(\Omega) \ni f \mapsto P_\varphi(f) \in \boldsymbol{C}$ is a linear form on $\mathcal{D}(\Omega)$. $\mathcal{D}(\Omega)^*$ denotes the algebraic dual of $\mathcal{D}(\Omega)$ which is the set of all linear forms on $\mathcal{D}(\Omega)$. Using these, we define a map

$$P\colon\ A(\Omega) \ni \varphi \mapsto P_\varphi \in \mathcal{D}(\Omega)^*\,.$$

Lemma 1. *Let X be a set, k a field, and $f_1, \cdots, f_n$ maps from X to k. If $f_1, \cdots, f_n$ are linearly independent over k, then there are $x_1, \cdots, x_n$ in X such that* $\det\|f_i(x_j)\| \neq 0$.

Theorem 1. *P is an onto map.*

Proof. Choose T from $\mathcal{D}(\Omega)^*$. For each f in $\mathcal{D}(\Omega)$, let

$$A(f) = \{\varphi \in R(X) \mid \sum_X \mathcal{E}\varphi^*f = T(f)\}\,.$$

For any $f_1, \cdots, f_n$ in $\mathcal{D}(\Omega)$ we will show that $\cap_{i=1}^n A(f_i) \neq \emptyset$. Then, we can see that

$$\bigcap_{f \in \mathcal{D}(\Omega)} A(f) \neq \emptyset$$

by saturation principle. Picking φ from $\cap_{f \in \mathcal{D}(\Omega)} A(f)$, we have

$$\sum_X \mathcal{E}\varphi^*f = T(f)$$

for every f in $\mathcal{D}(\Omega)$. In other words, $P_\varphi = T$.

In the following, we show that $\cap_{n=1}^i A(f_i) \neq \emptyset$ for any $f_1, \cdots, f_n$ in $\mathcal{D}(\Omega)$ by induction on n.

First of all, we remark that, for any t in $\boldsymbol{R}$, there is a unique x in X which satisfies $x \leq t < x + \mathcal{E}$, that is the maximum element $x \in X$ satisfying $x \leq t$. We denote this x by ${}^\vartriangle t$. Also, if $t \in \Omega$, then the above $x(=t^\vartriangle)$ is an element of ${}^*\Omega \cap X$, because $x \simeq t$ and Ω is an open set.

For $n=1$, let $f \in \mathcal{D}(\Omega)$. If $f=0$, then $0 \in A(f)$. If $f \neq 0$, then $f(t_1) \neq 0$ for some $t_1 \in \Omega$. Choose ${}^\vartriangle t_1 \in {}^*\Omega \cap X$ as we have just remarked above and put $x_1 = {}^\vartriangle t_1$. Then, ${}^*f(x_1) \neq 0$. Otherwise, the continuity of f and $x_1 \simeq t_1$ would imply $0 = {}^*f(x_1) = f(t_1)$. We define $\varphi\colon X \to {}^*\boldsymbol{C}$ by

$$\varphi(x) = \begin{cases} \dfrac{T(f)}{\varepsilon^* f(x_1)} & (x = x_1) \\ 0 & (x \neq x_1) . \end{cases}$$

Then, we have $\sum_X \varepsilon\varphi^* f = T(f)$, i.e. $\varphi \in A(f)$.

Assuming $n > 1$ and that the assertion holds for $n-1$, we prove it for n. Let $f_1, \cdots, f_n \in \mathcal{D}(\Omega)$. If they are linearly dependent over $\boldsymbol{C}$, by changing the suffixes if necessary, we can assume that $f_n = \sum_{i=1}^{n-1} c_i f_i$ with c_i's in $\boldsymbol{C}$. Choosing φ from $\bigcap_{i=1}^{n-1} A(f_i)$, we have $T(f_i) = \sum_X \varepsilon\varphi^* f_i$ for $1 \leq i \leq n-1$. Using this equation, we have

$$T(f_n) = \sum_{i=1}^{n-1} c_i T(f_i) = \sum_{i=1}^{n-1} c_i \sum_X \varepsilon\varphi^* f_i = \sum_X \varepsilon\varphi \sum_{i=1}^{n-1} c_i{}^* f_i = \sum_X \varepsilon\varphi^* f_n .$$

Hence, $\varphi \in \bigcap_{i=1}^{n} A(f_i)$.

If $f_1, \cdots, f_n$ are linearly independent over $\boldsymbol{C}$, then we can find $t_1, \cdots, t_n$ in Ω such that $\det \|f_i(t_j)\| \neq 0$ by Lemma 1. Choose ${}^\triangle t_1, \cdots, {}^\triangle t_n \in {}^*\Omega \cap X$, and put $x_j = {}^\triangle t_j$ $(1 \leq j \leq n)$. Since

$$\det \|{}^* f_i(x_j)\| \simeq \det \|f_i(t_j)\| ,$$

we have $\det \|{}^* f_i/(x_j)\| \neq 0$. There are $\alpha_1, \cdots, \alpha_n \in {}^*\boldsymbol{C}$ which satisfy

$$\sum_{j=2}^{n} {}^* f_i(x_j)\alpha_j = T(f_i) \qquad (1 \leq i \leq n) .$$

With this statement and reminding that $x_j \neq x_k$ if $j \neq k$, we can define a map φ: $X \to {}^*\boldsymbol{C}$ as follows:

$$\varphi(x) = \begin{cases} \dfrac{\alpha_j}{\varepsilon} & (x = x_j) \\ 0 & (x \neq x_1, \cdots, x_n) . \end{cases}$$

Then, $$\sum_X \varepsilon\varphi^* f_i = \sum_{j=1}^{n} \varepsilon \frac{\alpha_j}{\varepsilon} {}^* f_i(x_j) = T(f_i) \qquad (1 \leq i \leq n) .$$

Hence, $\varphi \in \bigcap_{i=1}^{n} A(f_i)$. □

Theorem 1 above is a modification of a theorem of Robinson ([8], §5.3) by reducing ${}^*\boldsymbol{R}$ to X. The proof is almost same as that of the Robinson's theorem given by M. Saito [9]. Another proof which makes use of the lattice structure of X can be found, for example, in *H.J.* Keisler [4].

Following this theorem, we will show that by defining several external subspaces $D(\Omega)$, $D_F(\Omega)$, $M(\Omega)$, $M_1(\Omega)$, $E(\Omega)$, and $S(\Omega)$ of $A(\Omega)$ appropriately, the images of these sets by P: $A(\Omega) \to \mathcal{D}(\Omega)^*$ are $\mathcal{D}'(\Omega)$, $\mathcal{D}'_F(\Omega)$, $\mathcal{D}'^{(0)}(\Omega) = \mathcal{M}(\Omega)$, $\mathcal{M}_1(\Omega)$, $\mathcal{L}_{1,\mathrm{loc}}(\Omega)$ and $C(\Omega)$ respecivetly. Here, $\mathcal{D}'_F(\Omega)$ is the set of distributions of finite oder on Ω, $\mathcal{D}'^{(0)}(\Omega) = \mathcal{M}(\Omega)$ the set of distributions of order 0 on

Ω which is the set of all (complex) measures on Ω, and $\mathcal{M}_1(\Omega)$ the set of all bounded (complex) measures on Ω. $\mathcal{D}'(\Omega)$, $\mathcal{L}_{1,\mathrm{loc}}(\Omega)$ and $\mathcal{C}(\Omega)$ are frequently used symbols. For the knowledge of distributions, refer to L. Schwartz [10].

2. Complex Measures

Definition. $M(\Omega)=\{\varphi\in R(X)\,|\,\sum_{{}^*K\cap X}\varepsilon|\varphi|$ is finite for every compact subset K of $\Omega\}$.

Theorem 2. (*a*) $M(\Omega)\subseteq A(\Omega)$.
(*b*) *The image of* $M(\Omega)$ *by* P *is* $\mathcal{D}'^{(0)}(\Omega)=\mathcal{M}(\Omega)$.

Proof. (*a*) Let $\varphi\in M(\Omega), f\in\mathcal{D}(\Omega)$, and $K=\mathrm{supp}(f)$. Then,

$$|\sum_X \varepsilon\varphi^*f|=|\sum_{{}^*K\cap X}\varepsilon\varphi^*f|\leq(\sum_{{}^*K\cap X}\varepsilon|\varphi|)\cdot\sup|f|\,.$$

So, $\sum_X\varepsilon\varphi^*f$ is finite and $\varphi\in A(\Omega)$.

The first half of (*b*): Let $\varphi\in M(\Omega)$, and K a compact subset of Ω. Put $C_K={}^*\sum_{{}^*K\cap X}\varepsilon|\varphi|$. By the formulas in the proof of (*a*), we get $|P_\varphi(f)|\leq C_K\cdot\sup|f|$ for each $f\in\mathcal{D}(\Omega)$ such that $\mathrm{supp}(f)\subseteq K$ and thus, $P_\varphi\in\mathcal{D}'^{(0)}(\Omega)=\mathcal{M}(\Omega)$.

The second half of (*b*): For each $T\in\mathcal{M}(\Omega)$, we show that $T=P_\varphi$ for some $\varphi\in M(\Omega)$. Let $\mathcal{K}(\Omega)=\{f:\boldsymbol{R}\to\boldsymbol{C}$, continuous$\,|\,\mathrm{supp}(f)$ is compact$\subset\Omega\}$ and for every map T: $\mathcal{D}(\Omega)\to\boldsymbol{C}$, we will extend it to a map from $\mathcal{K}(\Omega)$ to $\boldsymbol{C}$ and denote it by the same letter T. we also put $\mathcal{K}_+(\Omega)=\{f\in\mathcal{K}(\Omega)\,|\,f\geq 0\}$.

Suppose first that T: $\mathcal{K}(\Omega)\to\boldsymbol{C}$ is a positive linear form. For $f\in\mathcal{K}_+(\Omega)$ and $e>0$, let $A(f,e)=\{\varphi\in R(X)\,|\,\varphi\geq 0, |T(f)-\sum_X\varepsilon\varphi^*f|\leq e\}$. For any $f_1,\cdots,f_n\in\mathcal{K}_+(\Omega)$ and $e_1,\cdots,e_r>0$, we will show that $\bigcap_{1\leq i\leq n,1\leq j\leq r}A(f_i,e_j)\neq\emptyset$. This will yield the following by saturation principle:

$$\bigcap_{f\in\mathcal{K}_+(\Omega),\,e>0}A(f,e)\neq\emptyset\,.$$

Choosing φ from $\bigcap_{f\in\mathcal{K}_+(\Omega),\,e>0}A(f,e)$, we have $\varphi\geq 0$ and

$$|\sum_X\varepsilon\varphi^*f-T(f)|\leq e$$

for every $f\in\mathcal{K}_+(\Omega)$ and $e>0$. Hence, $\sum_X\varepsilon\varphi^*f\simeq T(f)$.

Now, it is enough to show that $\bigcap_{i=1}^n A(f_i,e)\neq\emptyset$ for each $f_1,\cdots,f_n$ and $e=\mathrm{Min}\{e_1,\cdots,e_r\}$.

Let $f_0\in\mathcal{K}_+(\Omega)$ be such that $f_0\geq f_1,\cdots,f_0\geq f_n$ (e.g. $f_0=f_1+\cdots+f_n$). Let $S_0=\{t\in\Omega\,|\,f_0(t)\neq 0\}$.

If $T(f_0)=0$, then we get $T(f_i)=0$ for $1\leq i\leq n$, since $f_0\geq f_i$ implies $0=T(f_0)\geq T(f_i)\geq 0$. So, we have $0\in\bigcap_{i=1}^n A(f_i,e)$.

Now, assume that $T(f_0)>0$. Then, $S_0\neq\emptyset$. We can see that the the point $Q=(T(f_1)/T(f_0), \cdots, T(f_n)/T(f_0))$ in $\boldsymbol{R}^n$ is contained in the closed convex closure C of the subset $\{(f_1(t)/f_0(t), \cdots, f_n(t)/(f_0(t)) \mid t\in S_0\}$ of $\boldsymbol{R}^n$ as follows: Assuming $Q\notin C$, the point Q and the set C are strictly separated by some hyperplane in $\boldsymbol{R}^n$. Hence, for some $b_0, \cdots, b_n\in\boldsymbol{R}$,

$$\sum_{i=1}^{n} b_i(T(f_i)/T(f_0))>b_0> \sup_{t\in S_0} \sum_{i=1}^{n} b_i(f_i(t)/f_0(t)) .$$

Put $g=\sum_{i=1}^{n} b_i f_i\in\mathcal{K}(\Omega)$. Then,

$$\text{(1)} \qquad \frac{T(g)}{T(f_0)}>b_0>\sup_{t\in S_0}\frac{g(t)}{f_0(t)} .$$

For $t\in S_0$, by the right half of (1), $g(t)<b_0f_0(t)$ and $t\in\Omega-S_0$ imply $f_0(t)=0$. Hence, $f_i(t)=0$ for $1\leq i\leq n$. Thus, $g(t)=0$ and we haev $g(t)=b_0f_0(t)$. So, $g\leq b_0f_0$. Since T is positive, we get $T(g)\leq b_0T(f_0)$ and $(T(g)/T(f_0))\leq b_0$, which contradicts the first inequality of (1).

Now, since $Q\in C$, for each $e>0$, there are $t_1, \cdots, t_r\in S_0$ and $a_1, \cdots, a_r\in\boldsymbol{R}$ such that $a_j>0$, $\sum_{j=1}^{r} a_j=1$ and $|T(f_i)/T(f_0)-\sum_{j=1}^{r} a_j(f_i(t_j)/f_0(t_j))|\leq e/2T(f_0)$ $(1\leq i\leq n)$. So, we have

$$\text{(2)} \qquad |T(f_i)-\sum_{j=1}^{r}\frac{a_jT(f_0)}{f_0(t_j)}f_i(t_j)|\leq\frac{e}{2} \qquad (1\leq i\leq n) .$$

Here, we can assume that $t_1, \cdots, t_r$ are pairwise distinct. (If $t_j=t_k$ for $j\neq k$, then we can write $a_j\dfrac{T(f_0)}{f_0(t_j)}f_i(t_j)+a_k\dfrac{T(f_0)}{f_0(t_k)}f_i(t_k)=(a_j+a_k)\dfrac{T(f_0)}{f_0(t_j)}f_i(t_j)$.) Now, let $x_j={}^{\Delta}t_j\in{}^*\Omega\cap X$ for each j such that $1\leq j\leq n$. Since $x_1, \cdots, x_r$ are pairwise distinct, by defining $\varphi: X\to{}^*\boldsymbol{C}$, $\varphi\geq 0$ by

$$\varphi(x)=\begin{cases}\dfrac{a_jT(f_0)}{\varepsilon f_0(t_j)} & (x=x_j)\\[2mm] 0 & (x\neq x_1, \cdots, x_r)\end{cases}$$

we have

$$\begin{aligned}\sum_X \varepsilon\varphi^*f_1 &= \sum_{j=1}^{r}\varepsilon\varphi(x_j)^*f_i(x_j)\\ &= \sum_{j=1}^{r}\frac{a_jT(f_0)}{f_0(t_j)}{}^*f_i(x_j)\simeq\sum_{j=1}^{r}\frac{a_jT(f_0)}{f_0(t_j)}f_i(t_j)\end{aligned}$$

for each i satisfying $1\leq i\leq n$, and by combining with (2), we get

$$\begin{aligned}|T(f_i)-\sum_X\varepsilon\varphi^*f_i| &\leq |T(f_i)-\sum_{j=1}^{r}\frac{a_jT(f_0)}{f_0(t_j)}f_i(t_j)|\\ &+|\sum_{j=1}^{r}\frac{a_jT(f_0)}{f_0(t_j)}f_i(t_j)-\sum_X\varepsilon\varphi^*f_i|\leq\frac{e}{2}+\frac{e}{2}=e \qquad (1\leq i\leq n) .\end{aligned}$$

Hnece, $\varphi \in \bigcap_{i=1}^{n} A(f_i, e)$.

Now, we have verified as we said above that, for each positive linear form $T: \mathcal{K}(\Omega) \to \boldsymbol{C}$, there is a map $\varphi \in R(X)$ such that $\varphi \geq 0$ and, for every map $f \in \mathcal{K}_+(\Omega)$, the following holds:

$$\sum_X \varepsilon \varphi^* f \simeq T(f) . \tag{3}$$

It is easily seen that (3) holds for every function $f \in \mathcal{K}(\Omega)$, and in particular, for every function $f \in \mathcal{D}(\Omega)$.

We show that $\varphi \in M(\Omega)$ as follows. For every compact subset K of Ω and every non-negative function $f \in \mathcal{D}(\Omega)$/such that $0 \leq f \leq 1$ which equals 1 on K, we have

$$\sum_{*K \subset X} \varepsilon |\varphi| = \sum_{*K \cap X} \varepsilon \varphi = \sum_{*K \cap X} \varepsilon \varphi^* f \simeq T(f) .$$

Therefore, $\varphi \in M(\Omega)$.

Returning to general case, every measure T can be written in the form $T = T_1 - T_2 + i(T_3 - T_4)$ where T_i $(1 \leq i \leq 4)$ are positive linear forms on $\mathcal{K}(\Omega)$. For each i satisfying $1 \leq i \leq 4$, we can find $\varphi_i \in M(\Omega)$ such that $\varphi_i \geq 0$ and, for every function $f \in \mathcal{D}(\Omega)$, $\sum_X \varepsilon \varphi_i^* f \simeq T_i(f)$ holds. Putting $\varphi = \varphi_1 - \varphi_2 + i(\varphi_3 - \varphi_4)$, we have $\varphi \in M(\Omega)$ and $P_\varphi = T$. □

The above proof is almost same as that in M. Saito [9], §2.2.

DEFINITION. $M_1(\Omega) = \{\varphi \in R(X) \mid \sum_{*\Omega \cap X} \varepsilon |\varphi| \text{ is finite}\}$.

We can immediately see from the definition that $M_1(\Omega) \subseteq M(\Omega)$.

Theorem 3. *The image of* $M_1(\Omega)$ *by* $P: A(\Omega) \to \mathcal{D}(\Omega)^*$ *coincides with* $\mathcal{M}_1(\Omega)$. *Moreover, if* T *is real, we can find a function* $\varphi: X \to {}^*\boldsymbol{R}$ *such that* $\varphi \in \mathcal{M}_1(\Omega)$, $P_\varphi = T$ *and* $\sum_{*\Omega \cap X} \varepsilon |\varphi| \simeq \|T\|$. *If* T *is not real, we only have the inequality* $\sum_{*\Omega \cap X} \varepsilon |\varphi| \geq \|T\|$. *Here,* $\|T\|$ *is the norm of* T.

Proof. If $\varphi \in M_1(\Omega)$ and $f \in \mathcal{D}(\Omega)$, then

$$\left|\sum_X \varepsilon \varphi^* f\right| = \left|\sum_{*\Omega \cap X} \varepsilon \varphi^* f\right| \leq \left(\sum_{*\Omega \cap X} \varepsilon |\varphi|\right) \cdot \sup |f| .$$

Hence, P_φ is a bounded measure in Ω.

Now, let $\mathcal{C}_B(\Omega)$ be the set of all complex-valued bounded continuous functions on Ω (assumed to take value 0 on $\boldsymbol{R}$–Ω). We write $\mathcal{C}_{B,+}(\Omega) = \{f \in \mathcal{B}_B(\Omega) \mid f \geq 0\}$. 1_Q denotes the characteristic function of Ω. And, for $T \in \mathcal{M}_1(\Omega)$, we extend it to a linear form $\mathcal{C}_B(\Omega) \to \boldsymbol{C}$ by integration and denote it by T again.

So, let $T: \mathcal{C}_B(\Omega)\to \boldsymbol{C}$ be a positive linear form. The proof of the second half of Theorem 2 (*b*) is also valid if we rsplace $\mathcal{K}(\Omega)$ and $\mathcal{K}_+(\Omega)$ by $\mathcal{C}_B(\Omega)$ and $\mathcal{C}_{B,+}(\Omega)$ respectively. This means, for each $T\in\mathcal{M}_1(\Omega)$ with T positive, there is $\varphi\in R(X)$ such that $\varphi\geq 0$ and $T(f)\simeq\sum_X \varepsilon\varphi{}^*f$ for each $f\in\mathcal{C}_B(\Omega)$. Putting $f=1_\Omega$, we get $\|T\|\simeq\sum_{^*\Omega\cap X}\varepsilon\varphi=\sum_{^*\Omega\cap X}\varepsilon|\varphi|$. Hence, we have $\varphi\in M_1(\Omega)$. The proof for the last part of the theorem is similar to that for Theorem 2. □

3. Complex-valued functions

We need the theory of Loeb measures on Ω and on Jordan measurable subsets of Ω ([1[, [6], and [7]). To avoid duplications, we use the following notations:

Z: *A* countable union of Jordan measurable compact subsets of Ω.

$\mathcal{A}_Z$: The set of all internal subsets of $^*Z\cap X$. We sometimes write $\mathcal{A}$ if there is no danger of confusion.

We define $\nu: \mathcal{A}\to{}^*\boldsymbol{R}$ by $\nu(A)=(\#(A))\cdot\varepsilon$ for each $A\in\mathcal{A}$. Then, $(^*Z\cap X, \mathcal{A}, \nu)$ is an internal finitely additive measure space. Let $(^*Z\cap X, L(\mathcal{A}), \nu_L)$ be the Loeb space associated with it.

An internal function φ: $^*Z\cap X\to{}^*\boldsymbol{R}$ is said to be S-integrable if the following three conditions are satisfied:

(1) $N\in{}^*\boldsymbol{N}-\boldsymbol{N}$ implies $\sum_{(|\varphi|\geq N)}\varepsilon|\varphi|\simeq 0$,

(2) $N\in{}^*\boldsymbol{N}-\boldsymbol{N}$ implies $\sum_{(|\varphi|\leq(1/N))}\varepsilon|\varphi|\simeq 0$, and

(3) $\sum_{^*Z\cap X}\varepsilon|\varphi|$ is finite.

If $\nu(^*Z\cap X)$ is finite, (1) implies (2) and (3).

The following theorems are due to Loeb:

Let $\overline{\boldsymbol{R}}$ be the set of extended real numbers.

(1) If φ: $^*Z\cap X\to{}^*\boldsymbol{R}$ is S-integrable, then $^\circ\varphi$: $^*Z\cap X\to\overline{\boldsymbol{R}}$ is Loeb integrable and

$$\sum_{^*Z\cap X}\varepsilon\varphi\simeq\int_{^*Z\cap X}{}^\circ\varphi d\nu_L\,.$$

(2) If g: $^*Z\cap X\to\overline{\boldsymbol{R}}$ is Loeb integrable, then there is an S-integrable function φ: $^*Z\cap X\to{}^*\boldsymbol{R}$ such htat $^\circ\varphi=g$ (ν_L-almost everywhere).

Moreover, let $\mathrm{Ns}(^*Z)=\{x\in{}^*Z\,|\,x\simeq t \text{ for some } t\in Z\}$. We have $\mathrm{Ns}(^*Z)\subseteq\mathrm{Ns}(^*\boldsymbol{R})\cap{}^*Z$, but the equality does not necessarily hold. Here, $\mathrm{Ns}(^*\boldsymbol{R})=\{\alpha\in{}^*\boldsymbol{R}\,|\,\alpha \text{ is finite}\}$. Define st_Z: $\mathrm{Ns}(^*Z)\cap X\to Z$ by $\mathrm{st}_Z(x)={}^\circ x$ when $x\in\mathrm{Ns}(^*Z)\cap X$. We sometimes omit Z in st_Z.

Let $(Z, \mathcal{L}, \mu)$ be a Lebesgue messure space over Z. The followings are known.

(3) For a subset E of Z, the condition $E\in\mathcal{L}$ is equivalent to the condition $\mathrm{st}^{-1}(E)\in L(\mathcal{A})$ and if this condition is satisfied, then we have $\mu(E)=\nu_L(\mathrm{st}^{-1}(E))$.

(4) Let $E\subseteq Z$ be $\mathcal{L}$-measurable and let h be a non-negative $\mathcal{L}$-measurable function: $E\to\overline{\boldsymbol{R}}$. Then, $h\circ\mathrm{st}\colon \mathrm{st}^{-1}(E)\to\overline{\boldsymbol{R}}$ is $L(\mathcal{A})$-measurable and

$$\int_E hd\mu=\int_{\mathrm{st}^{-1}(E)}(h\circ\mathrm{st})d\nu_L\,.$$

Now, if $K\subset\Omega$, then the compactness of K and the inclusion

$$\mathrm{st}^{-1}(K)\supseteq{}^*K\cap X$$

are equivalent. By this fact and by (3) above, if K is a compact subset of Ω, then

$$\nu_L({}^*K\cap X)\leq\nu_L(\mathrm{st}^{-1}(K))=\mu(K)\,.$$

Moreover, if K is a Jordan measurable compact set, we can prove that

$$\nu_L({}^*K\cap X)=\nu_L(\mathrm{st}^{-1}(K))\,.$$

We define the local S-integrability below.

Recall that $R(X)$ is the set of internal functions from X to ${}^*\boldsymbol{C}$. If $\varphi\in R(X)$, K is a compact subset of Ω, and $n\in{}^*\boldsymbol{N}$, then we write

$$A(\varphi,K,n)=\{x\in{}^*K\cap X\,|\,|\varphi(x)|\geq n\}\,.$$

DEFINTION. (1) A function $\varphi\in R(X)$ is said to be *locally S-integrable* over Ω if the following holds for every compact subset K of Ω and every infinite natural number $N\in{}^*\boldsymbol{N}-\boldsymbol{N}$:

$$\sum_{A(\varphi,K,N)}\varepsilon|\varphi|\simeq 0\,.$$

(2) $E(\Omega)=\{\varphi\in R(X)\,|\,\varphi$ is locally S-integrable over $\Omega\}$.

Proposition 1. *The following two conditions are equivalent. In particular we have* $E(\Omega)\subseteq M(\Omega)$.

(a) $\varphi\in E(\Omega)$.

(b) $\varphi\in M(\Omega)$,*and for any compact subset K of Ω and for any set $A\in\mathcal{A}$ such that $A\subseteq{}^*K\cap X$,*

$$\nu(A)\simeq 0\quad\textit{implies}\quad\sum_A\varepsilon|\varphi|\simeq 0\,.$$

Proof. *(a)→(b)*. Assume $\varphi\in E(\Omega)$. Let K be a compact subset of Ω and $e>0$. Since φ is locally S-integrable over Ω, we have ${}^*\boldsymbol{N}-\boldsymbol{N}\subseteq\{n\in{}^*\boldsymbol{N}|$ $\sum_{A(\sigma,K,n)}\varepsilon|\varphi|\leq e\}$. Hence, there is a natural number $n\in\boldsymbol{N}$ such that $\sum_{A(\varphi,K,n)}\varepsilon|\varphi|\leq e$. Thus, we have

$$\sum_{{}^*K\cap X}\varepsilon|\varphi|=\sum_{\{x\in{}^*K\subset X|\,|\varphi(x)|<n\}}\varepsilon|\varphi|+\sum_{A(\varphi,K,x)}\varepsilon|\varphi|\leq n\sum_{{}^*K\subset X}\varepsilon+e\,.$$

Since ${}^\circ\sum_{{}^*K\cap X}\varepsilon=\nu_L({}^*K\cap X)\leq\nu_L(\mathrm{st}^{-1}(K))=\mu(K)$, we can see that $\sum_{{}^*K\cap X}\varepsilon|\varphi|$

is finite and thus, we get $\varphi \in M(\Omega)$.

Now, let $A \in \mathcal{A}_\Omega$, K be a compact subet of Ω, $A \subseteq {}^*K \cap X$, and $\nu(A) \simeq 0$. Since $\boldsymbol{N} \subseteq \{n \in {}^*\boldsymbol{N} \mid n^2 \cdot \nu(A) \leq 1\}$, we have $N^2 \cdot \nu(A) \leq 1$ for some $N \in {}^*\boldsymbol{N} - \boldsymbol{N}$. Hence we get $N \cdot \nu(A) \simeq 0$. Here, we have

$$\sum_A \varepsilon |\varphi| \leq \sum_{\{x \in A \mid |\varphi(x)| \geq N\}} \varepsilon |\varphi| + \sum_{\{x \in A \mid |\varphi(x)| < N\}} \varepsilon |\varphi| \leq \sum_{A(\varphi, K, n)} \varepsilon |\varphi| + N \cdot \nu(A).$$

The first term $\simeq 0$ by the hypothesis, and the second term $\simeq 0$ by what we have just shown above. Hence, we get $\sum_A \varepsilon |\varphi| \simeq 0$.

(b)→(a). Let K be a compact subset of Ω, and $N \in {}^*\boldsymbol{N} - \boldsymbol{N}$. Put $A = A(\varphi, K, N) = \{x \in {}^*K \cap X \mid |\varphi(x)| \geq N\}$. We have $A \in \mathcal{A}_\Omega$, $A \subseteq {}^*K \cap X$ and the inequality

$$N \cdot \nu(A) \leq \sum_A \varepsilon |\varphi| \leq \sum_{{}^*K \cap X} \varepsilon |\varphi|$$

holds. But the right hand side is finite for $\varphi \in M(\Omega)$. So we have $\nu(A) \simeq 0$. This and the latter half of (*b*) yields that $\sum_A \varepsilon |\varphi| \simeq 0$. □

The lemma below will also be used later.

Now, we write $\mathrm{Ns}({}^*\boldsymbol{C}) = \{\alpha \in {}^*\boldsymbol{C} \mid \alpha \text{ is finite}\}$. This set is a commutative ring.

Lemma 2. *Let $Y(\Omega)$ be an* $\mathrm{Ns}({}^*\boldsymbol{C})$*-submodule of $R(X)$, $T \in \mathcal{D}'(\Omega)$, and $(f_i)_{i \in N}$ a partition of unity on Ω. Suppose that, for each $i \in \boldsymbol{N}$, there corresponds a function $\psi_i \in Y(\Omega)$ such that $P_{\psi_i} = f_i T$ and that ψ_i is 0 on ${}^*K \cap X$ provided K is a compact subset of Ω and $K \cap \operatorname{supp}(f_i) = \emptyset$. Put $\varphi_n = \sum_{i=1}^{n} \psi_i$ for $n \in \boldsymbol{N}$. The map from $\boldsymbol{N}$ to $Y(\Omega)$: $n \mapsto \varphi_n$ extends to an internal map from ${}^*\boldsymbol{N}$ to $R(X)$: $n \mapsto \varphi_n$.*

In the situation above, there exists an integer $N \in {}^*\boldsymbol{N}$ such that the following conditions hold:

(a) *In case $N \in \boldsymbol{N}$, then $\varphi_N \in Y(\Omega)$ and $P_{\psi_N} = T$;*

(b) *in case $N \in {}^*\boldsymbol{N} - \boldsymbol{N}$, then $\varphi_N \in A(\Omega)$, $P_{\psi_N} = T$*

and, for every compact subset $K \subseteq \Omega$, there exists an $n \in \boldsymbol{N}$ such that $\varphi_N = \varphi_n$ on ${}^*K \cap X$.

Proof. For each compact subset K of Ω, choose $n(K) \in \boldsymbol{N}$ so that $i > n(K)$ implies $\operatorname{supp}(f_i) \cap K = \emptyset$, which yields that ψ_i takes 0 on ${}^*K \cap X$. Then, $n \geq n(K)$ implies that $\sum_{i=0}^{n} f_i$ takes 1 on K.

For each $f \in \mathcal{D}(\Omega)$, put $n(f) = n(\operatorname{supp}(f))$ and consider the following internal set:

$$I(f) = \{n \in {}^*\boldsymbol{N} \mid n \geq n(f) \wedge \forall l \in {}^*\boldsymbol{N}(n(f) \leq l \leq n \rightarrow$$
$$|\sum_X \varepsilon \varphi_l {}^*f - T(f)| \leq \frac{1}{l+1})\}.$$

If $n \in \boldsymbol{N}$ and $n \geq n(f)$, then a natural number $l \in {}^*\boldsymbol{N}$ satisfying $n(f) \leq l \leq n$ turns out to be an element of $\boldsymbol{N}$, and since $l \geq n(f)$, reminding that $\sum_{i=0}^{n} f_i$ takes 1 on $\mathrm{supp}(f)$, we get

$$\sum_X \varepsilon\varphi_l{}^*f = \sum_{i=0}^{l} \sum_X \varepsilon\psi_i{}^*f \simeq \sum_{i=0}^{l} (f_iT)(f) = T((\sum_{i=0}^{l} f_i)f) = T(f)\,.$$

Now, we have $\{n \in \boldsymbol{N} \mid n \geq n(f)\} \subseteq I(f)$. Hence, by Permanence Principle, there is $N(f) \in {}^*\boldsymbol{N} - \boldsymbol{N}$ such that $\{n \in {}^*\boldsymbol{N} \mid n(f) \leq n \leq N(f)\} \subseteq I(f)$. Thus, the family of internal sets $\{I(f) \mid f \in \mathcal{D}(\Omega)\}$ has the finite intersection property and we have $\cap \{I(f) \mid f \in \mathcal{D}(\Omega)\} \neq \emptyset$ by saturation principle. Take $N \in \cap \{I(f) \mid f \in \mathcal{D}(\Omega)\}$. Then N belongs to ${}^*\boldsymbol{N}$ and the following holds:

For any $f \in \mathcal{D}(\Omega)$, $N \geq n(f)$ and for every $l \in {}^*\boldsymbol{N}$,

$$n(f) \leq l \leq N \quad \text{implies} \quad |\sum_X \varepsilon\varphi_l{}^*f - T(f)| \leq \frac{1}{l+1}\,.$$

If $N \in \boldsymbol{N}$, then $\varphi_N \in Y(\Omega)$ and since $N \geq n(f)$ for every $f \in \mathcal{D}(\Omega)$, we have

$$P_{\varphi_N}(f) = \sum_{i=0}^{N} P_{\psi_i}(f) = \sum_{i=0}^{N} (f_iT)(f) = T((\sum_{i=0}^{N} f_i)f) = T(f)\,.$$

If $N \in {}^*\boldsymbol{N} - \boldsymbol{N}$, then for every $L \in {}^*\boldsymbol{N} - \boldsymbol{N}$ satisfying $L \leq N$, we have

$$|\sum_X \varepsilon\varphi_L{}^*f - T(f)| \leq \frac{1}{L+1} \simeq 0$$

by what we have shown above and by the fact that $n(f) \leq L$ for any $f \in \mathcal{D}(\Omega)$. Thus we get $\varphi_L \in A(\Omega)$ and $P_{\varphi_L} = T$. We fix this N for a while.

Now, for a compact subset K of Ω, we put

$$J(K) = \{n \in {}^*\boldsymbol{N} \mid n \geq 1\,, \quad \varphi_{n-1} = \varphi_n \quad \text{on} \quad {}^*K \cap X\}\,.$$

If $n \in \boldsymbol{N}$ and $n \geq n(K)$, then $\mathrm{supp}(f_n) \cap K = \emptyset$ and thus ψ_n takes 0 on ${}^*K \cap X$ and so, we have $\varphi_n = \varphi_{n-1} + \psi_n = \varphi_{n-1}$ on ${}^*K \cap X$. Hence,

$$\{n \in \boldsymbol{N} \mid n > n(K)\} \subseteq J(K)\,,$$

With this, for each compact subset K of Ω, there exists $N(K) \in {}^*\boldsymbol{N} - \boldsymbol{N}$ such that

$$\{n \in {}^*\boldsymbol{N} \mid {}^*n(K) < n \leq N(K)\} \subseteq J(K)\,.$$

Moreover, we can show that, there is $M \in {}^*\boldsymbol{N} - \boldsymbol{N}$ such that for any compact subset K of Ω and for any $n \in {}^*\boldsymbol{N}$, $n(K) < n < M$ implies $\varphi_n = \varphi_{n-1}$ on ${}^*K \cap X$. To show this, choose a fundamental sequence of compact sets $(K_j)_{j \in \boldsymbol{N}}$ for Ω and choose $M \in {}^*\boldsymbol{N} - \boldsymbol{N}$ so that $M \leq N(K_j)$ for every $j \in \boldsymbol{N}$. Now, using N we have fixed above, consider the number $\mathrm{Min}(M, N)$ and rename it N. Then we have $\varphi_N \in A(\Omega)$, $P_{\varphi_N}=T$, and $\varphi_N=\varphi_{n(K)}$ on $^*K\cap X$ for every compact subset K of Ω. □

Theorem 4. (a) *For each* $\varphi\in E(\Omega)$, *there is* $h\in\mathcal{L}_{1,\mathrm{loc}}(\Omega)$ *such that* $P_\varphi=T_h$, *where* T_h *denotes the distribution determined by h.*

(b) *For each* $h\in\mathcal{L}_{1,\mathrm{loc}}(\Omega)$, *there is* $\varphi\in E(\Omega)$ *such that* $P_\varphi=T_h$.

Proof. (a) Let $\varphi\in E(\Omega)$ and $\varphi\geq 0$. We shall show that for any $g\in\mathcal{K}_+(\Omega)$ and $e>0$, there exists $d>0$ such that we have, for every $f\in\mathcal{K}_+(\Omega)$, $P_\varphi(f)\leq e$ provided $f\leq g$ and $\int_\Omega f d\mu\leq d$, where $d\mu$ is the Lebesgue measure on Ω. Then, P_φ will turn to be a measure on Ω with base μ; that is, there is $h\in\mathcal{L}_{1,\mathrm{loc}}(\Omega)$ and such that we have $P_\varphi(f)=\int_\Omega hf d\mu$ for every $f\in\mathcal{K}(\Omega)$ ([2], Chap. 5, §5, n° 5, Cor. 5).

So, let $K=\mathrm{supp}(g)$ and choose $c\geq 0$ so that $c\cdot\sup g\leq e/2$. Since $^*\boldsymbol{N}-\boldsymbol{N}\subseteq\{n\in{}^*\boldsymbol{N}\mid\sum_{A(\varphi,K,n)}\varepsilon\varphi\leq c\}$, there is $n\in\boldsymbol{N}$ such that $\sum_{A(\varphi,K,n)}\varepsilon\varphi\leq c$. With this n, choose $d>0$ so that $nd\leq e/3$. For g and d above, take $f\in\mathcal{K}_+(\Omega)$ such that $f\leq g$ and $\int_\Omega f d\mu\leq d$. We show that $P_\varphi(f)\leq e$. We get

$$\sum_X \varepsilon\varphi^*f=\sum_{^*K\cap X}\varepsilon\varphi^*f=\sum_{\{x\in{}^*K\cap X\mid\varphi(x)<n\}}\varepsilon\varphi^*f+\sum_{A(\varphi,K,n)}\varepsilon\varphi^*f$$
$$\leq n\sum_{^*K\cap X}\varepsilon^*f+\sup f\cdot\sum_{A(\varphi,K,n)}\varepsilon^*f,$$

but in the right hand side of the inequality,

$$\text{the first term}\simeq n\int_K f d\mu\leq nd\leq\frac{e}{3},$$

thus,

$$\text{the first term}\leq\frac{e}{2},$$

and

$$\text{the second term}\leq(\sup f)\cdot c\leq(\sup g)\cdot c\leq\frac{e}{2}.$$

Hence, $\sum_X\varepsilon\varphi^*f\leq\frac{e}{2}+\frac{e}{2}=e$ and immediately we get $P_\varphi(f)\leq e$.

(b) Let $h\in\mathcal{L}_{1,\mathrm{loc}}(\Omega)$. Let $(f_i)_{i\in N}$ be a partition of unity such that each $\mathrm{supp}(f_i)$ (we name it K_i) is a Jordan measurable compact set. Since $f_ih\colon K_i\to\boldsymbol{C}$ is μ-integrable, $(f_ih)\circ\mathrm{st}\colon{}^*K_i\cap X\to\boldsymbol{C}$ is Loeb integrable. Hence, for each i, there is an S-integrable function $\psi_i\colon{}^*K_i\cap X\to\boldsymbol{C}$ such that $^\circ\psi_i=(f_ih)\circ\mathrm{st}$ (ν_L-almost everywhere) on $^*K\cap X$. Extend ψ_i so that it takes 0 on $X-{}^*K_i\cap X$ and also denote it by ψ_i. We have $\psi_i\in R(X)$ and $\psi_i\in E(\Omega)$. We shall show that $P_{\psi_i}=f_iT_h$: For each $g\in\mathcal{D}(\Omega)$, we have

$$\sum_X \varepsilon\psi_i{}^*g = \sum_{^*K_i\cap X} \varepsilon\psi_i{}^*g \simeq \int_{^*K_i\cap X} {}^\circ\psi_i{}^\circ{}^*g d\nu_L$$
$$= \int_{^*K_i\cap X} ((f_i h)\circ\mathrm{st})(g\circ\mathrm{st})d\nu_L$$
$$= \int_{\mathrm{st}^{-1}(K_i)} (f_i hg)\circ\mathrm{st}\, d\nu_L$$
$$= \int_{K_i} f_i hg d\mu = (f_i T_h)(g)\,.$$

Here, we used the fact that Jordan measurable compact set K_i satisfies the equation $\nu_L(\mathrm{st}^{-1}(K_i)-{}^*K_i\cap X)=0$. Moreover, for a compact subset K of Ω such that $K\cap K_i=\emptyset$, ψ_i takes 0 on ${}^*K\cap X$ by our definition of ψ_i. Now, by applying Lemma 2 to the case $Y(\Omega)=E(\Omega)$, we get an internal function ${}^*\boldsymbol{N}\ni n\mapsto\varphi_n\in R(X)$ and a natural number $N\in{}^*\boldsymbol{N}$ such that $\varphi_n=\sum_{i=1}^n\psi_i$ for each $n\in\boldsymbol{N}$ and satisfy the following conditions:

(1) $N\in\boldsymbol{N}$ implies $\varphi_N\in E(\Omega)$ and $P_{\varphi_N}=T_h$.

(2) $N\in{}^*\boldsymbol{N}-\boldsymbol{N}$ implies that $\varphi_N\in A(\Omega)$, $P_{\varphi_N}=T_h$ and that, for each compact subset K of Ω, there is a suitable $n\in\boldsymbol{N}$ such that $\varphi_N=\varphi_n$ on ${}^*K\cap X$.

In the case (1), the proof is done. In the case (2), for each compact subset K of Ω and for each $M\in{}^*\boldsymbol{N}-\boldsymbol{N}$, we have $A(K,\varphi_N,M)=A(K,\varphi_n,M)$ and we get $\varphi_N\in E(\Omega)$ by the following:

$$\sum_{A(K,\varphi_N,M)} \varepsilon|\varphi_N| = \sum_{A(K,\varphi_n,M)} \varepsilon|\varphi_n| \simeq 0\,. \qquad \square$$

Proposition 2. *Let $\varphi\in E(\Omega)$, $\varphi\geq 0$, and $h\in\mathcal{L}_{1,\mathrm{loc}}(\Omega)$, $h\geq 0$. Then, the following two conditions are mutually equivalent*

(a) $P_\varphi=T_h$.

(b) ${}^\circ\varphi=h\circ\mathrm{st}$ a.e. *on* $\mathrm{Ns}({}^*\Omega)\cap X$.

Proof. (a)→(b). Let $f\in\mathcal{K}(\Omega)$ and C be a compact and Jordan measurable subset of Ω with $\mathrm{supp}(f)\subseteq C$. Then, $\mathrm{st}^{-1}(C)\supseteq{}^*C\cap X$ because C is compact, and

$$\nu_L(\mathrm{st}^{-1}(C)) = \nu_L({}^*C\cap X)$$

because C is Jordan measurable. We have then

$$\int_{\mathrm{st}^{-1}(C)} {}^\circ\varphi(f\circ\mathrm{st})d\nu_L = \int_{^*C\cap X} {}^\circ(\varphi{}^*f)d\nu_L = \int_X {}^\circ(\varphi{}^*f)d\nu_L$$
$$= {}^\circ\textstyle\sum_X \varepsilon\varphi{}^*f = \displaystyle\int_\Omega hfd\mu \text{ (by assumption)} = \int_C hfd\mu$$
$$= \int_{\mathrm{st}^{-1}(C)} (h\circ\mathrm{st})(f\circ\mathrm{st})d\nu_L\,.$$

For every compact subset K of Ω, there exists a sequence of functions $f_n \in \mathcal{K}(\Omega)$ $(n \in \boldsymbol{N})$ such that

(a) $f_n \downarrow 1_K$,

(2) $\mathrm{supp}(f_n) \subseteq C$ for some fixed compact and Jordan measurable subset C of Ω.

By the remark above, we have

$$\int_{\mathrm{st}^{-1}(C)} {}^{\circ}\varphi(f_n \circ \mathrm{st}) d\nu_L = \int_{\mathrm{st}^{-1}(C)} (h \circ \mathrm{st})(f_n \circ \mathrm{st}) d\nu_L$$

and hence,

$$\int_{\mathrm{st}^{-1}(K)} {}^{\circ}\varphi d\nu_L = \int_{\mathrm{st}^{-1}(K)} (h \circ \mathrm{st}) d\nu_L .$$

The positivity of the integrand implies

$${}^{\circ}\varphi = h \circ \mathrm{st} \quad \text{a.e.} \quad \text{on} \quad \mathrm{st}^{-1}(K) .$$

Take a sequence of compact sets K_n such that $\Omega = \bigcup_{n \in N} K_n$. Then we have $\mathrm{Ns}({}^*\Omega) \cap X = \bigcup_{n \in N} \mathrm{st}^{-1}(K_n)$ and therefore

$${}^{\circ}\varphi = h \circ \mathrm{st} \quad \text{a.e.} \quad \text{on} \quad \mathrm{Ns}({}^*\Omega) \cap X .$$

(b)→(a). Let $f \in \mathcal{K}(\Omega)$ and C be a comapct and Jordan measurable subset of Ω with $\mathrm{supp}(f) \subseteq C$. Then we have

$$\begin{aligned} {}^{\circ}\textstyle\sum_X \varepsilon\varphi {}^*f &= {}^{\circ}\textstyle\sum_{{}^*C \cap X} \varepsilon\varphi {}^*f = \int_{{}^*C \cap X} {}^{\circ}\varphi {}^{\circ *}f d\nu_L \\ &= \int_{\mathrm{st}^{-1}(C)} {}^{\circ}\varphi(f \circ \mathrm{st}) d\nu_L = \int_{\mathrm{st}^{-1}(C)} (h \circ \mathrm{st})(f \circ \mathrm{st}) d\nu_L \\ &= \int_C hf d\mu = \int_\Omega hf d\mu . \end{aligned}$$

□

DEFINITION. Recall that $M_1(\Omega)$ is the set of internal functions φ on X such that $\sum_{{}^*\Omega \cap X} \varepsilon |\varphi|$ is finite,and that $E(\Omega)$ is the set of internal functions on X which are locally S-integrable on Ω. Put $E_1(\Omega) = E(\Omega) \cap M_1(\Omega)$, and, for every $p \geq 1$ in $\boldsymbol{R}$, put

$$E_p(\Omega) = \{\varphi \in R(X) \mid |\varphi|^p \in E_1(\Omega)\} .$$

In case $p=1$, two definitions of $E_1(\Omega)$ coincide.

Lemma 3. *Let $\varphi \in E(\Omega)$ and $\varphi \geq 0$. Then we have $\varphi^{1/p} \in E(\Omega)$ for every $p \geq 1$ in $\boldsymbol{R}$.*

Proof. Recall that an internal function φ is said to be locally S-integrable on Ω if we have $\sum_{A(\varphi,K,N)} \varepsilon |\varphi| \simeq 0$ for every compact subset K of Ω and for every infinite natural number N, where $A(\varphi, K, N)$ is the internal set of all

$x \in {}^*K \cap X$ such that $|\varphi(x)| \geq N$.

Now let $\varphi \in E(\Omega)$, $\varphi \geq 0$ and $K \subset \Omega$ compact and N infinite. Then, $A(\varphi^{1/p}, K, N) = A(\varphi, K, N^p)$ for every $p \geq 1$. Since $|\varphi(x)|^{1/p} \leq |\varphi(x)|$ for $x \in A(\varphi, K, N^p)$, we have

$$\sum_{A(\varphi^{1/p},K,N)} \varepsilon\varphi^{1/p} = \sum_{A(\varphi,K,N^p)} \varepsilon\varphi^{1/p} \leq \sum_{A(\varphi,K,N^p)} \varepsilon\varphi \simeq 0 . \qquad \square$$

Proposition 3. *For every $p \geq 1$ in $\boldsymbol{R}$, we have $E_p(\Omega) \subseteq E(\Omega)$.*

Proof. Let $\varphi \in E_p(\Omega)$. The definition of $E_p(\Omega)$ gives $|\varphi|^p \in E_1(\Omega) \subseteq E(\Omega)$, hence $|\varphi| \in E(\Omega)$ by the above Lemma, so we have $\varphi \in E(\Omega)$. $\square$

Theorem 5. *Let $p \geq 1$ in $\boldsymbol{R}$. Recall that $\mathcal{L}_p(\Omega)$ is the set of measurable functions φ on Ω such that $|\varphi|^p$ is integrable on Ω. Then we have*

(a) *For every $\varphi \in E_p(\Omega)$, there exists an $h \in \mathcal{L}_p(\Omega)$ such that $P_\varphi = T_h$.*

(b) *For every $h \in \mathcal{L}_p(\Omega)$, there exists a $\varphi \in E_p(\Omega)$ such that $P_\varphi = T_h$.*

Proof. (a) Suppose first $p=1$. For every $\varphi \in E_1(\Omega) \subseteq E(\Omega)$, there exists an $h \in \mathcal{L}_{1,\mathrm{loc}}(\Omega)$ such that $P_\varphi = T_h$. Since $\varphi \in E_1(\Omega) \subseteq M_1(\Omega)$ we have $T_h = P_\varphi \in \mathcal{M}_1(\Omega)$, that is, $T_h = P_\varphi$ is a bounded measure. On the other hand, Bourbaki's "Integration" [2] Chap B §5. 5, n° 4, Theorem 1, Corollary says that, for every $h \in \mathcal{L}_{1,\mathrm{loc}}(\Omega)$, T_h is a bounded measure if and only if $h \in \mathcal{L}_1(\Omega)$. Applying this to our case, there exists an $h \in \mathcal{L}_1(\Omega)$ such that $P_\varphi = T_h$.

Suppose next $p>1$ and $\varphi \in E_p(\Omega)$, $\varphi \geq 0$. Then $\varphi^p \in E_1(\Omega)$. By the result for $p=1$, there exists a $g \in \mathcal{L}_1(\Omega)$, $g \geq 0$ such that $P_{\varphi^p} = T_g$. Proposition 2 implies

$${}^\circ\varphi^p = g \circ \mathrm{st} \quad \text{a.e.} \quad \text{on} \quad \mathrm{Ns}(\Omega^*) \cap X .$$

Putting $h = g^{1/p}$, we have $h \in \mathcal{L}_p(\Omega)$ and ${}^\circ\varphi^p = h^p \circ \mathrm{st}$ a.e. on $\mathrm{Ns}({}^*\Omega) \cap X$. Hence ${}^\circ\varphi = h \circ \mathrm{st}$ a.e. on $\mathrm{Ns}({}^*\Omega) \cap X$ and we have $P_\varphi = T_h$ by Proposition 2.

If φ is not positive, the result follows from the decomposition $\varphi = \varphi_1 - \varphi_2 + i\varphi_3 - i\varphi_4$, φ_1, φ_2, φ_3, φ_4 being positive.

(b). Let $h \in \mathcal{L}_p(\Omega)$, $h \geq 0$. Then $h^p \in \mathcal{L}_1(\Omega)$. We extend the function $h^p \circ \mathrm{st}$ on $\mathrm{Ns}(X)$ to whole X by giving the value 0 outside $\mathrm{Ns}(X)$, which we write $h^p \circ$ st. Then this function is positive and ν_L-measurable, and we have

$$\int_X h^p \circ \mathrm{st}\, d\nu_L = \int_{\mathrm{Ns}({}^*\Omega) \cap X} h^p \circ \mathrm{st}\, d\nu_L = \int_\Omega h^p d\mu < \infty .$$

The theory of Loeb integration assures us the existence of an S-integrable function $\psi \geq 0$ in $R(X)$ such that

$${}^\circ\psi = h^p \circ \mathrm{st} \quad \text{a.e.} \quad \text{on} \quad X .$$

We then have $\psi \in E_1(\Omega)$, because

$$\sum\nolimits_{*\Omega\cap X}\varepsilon\psi\leq\sum\nolimits_{X}\varepsilon\psi\simeq\int_{X}{}^{\circ}\psi d\nu_{L}=\int_{X}h^{p}\circ\mathrm{st}\,d\nu_{L}=\int_{\Omega}h^{p}d\mu\leq\infty\,.$$

Putting $\varphi=\psi^{1/p}$, Lemma 3 implies $\varphi\in E(\Omega)$. Moreover we have $\varphi^{p}=\psi\in E_{1}(\Omega)$, hence $\varphi\in E_{p}(\Omega)$. On the other hand, we have

$${}^{\circ}\varphi^{p}=h^{p}\circ\mathrm{st}\quad\text{a.e.}\quad\text{on}\quad \mathrm{Ns}({}^{*}\Omega)\cap X$$

and hence the equality

$${}^{\circ}\varphi=h\circ\mathrm{st}\quad\text{a.e.}\quad\text{on}\quad \mathrm{Ns}({}^{*}\Omega)\cap X\,,$$

which implies $P_{\varphi}=T_{h}$ by Proposition 2. □

DEFINITION. (1) After A. Robinson [8], we call a function $\varphi\in R(X)$ S-continuous on Ω if $\varphi(x)\simeq\varphi(y)$ whenever $x, y\in\mathrm{Ns}({}^{*}\Omega)\cap X$ and $x\simeq y$.

(2) Let $S(\Omega)$ be the set of functions $\varphi\in R(X)$ which are finite-valued and S-continuous on Ω.

(3) For each $\varphi\in S(\Omega)$, define the function ${}^{\vee}\varphi\colon\Omega\to\boldsymbol{C}$ by ${}^{\vee}\varphi(t)={}^{\circ}\varphi({}^{\vartriangle}t)$ for $t\in\Omega$ (recall that ${}^{\vartriangle}t\in\mathrm{Ns}({}^{*}\Omega)\cap X$, ${}^{\vartriangle}t\leq t<{}^{\vartriangle}t+\varepsilon$).

The property (a) in the following theorem is due to P. Loeb ([15]), and other parts can be easily deduced from theories and definitions by Loeb.

Theorem 6. (a) *$\varphi\in S(\Omega)$ implies ${}^{\vee}\varphi\in C(\Omega)$, that is, ${}^{\vee}\varphi$ is a continuous function on Ω.*

(b) *If $h\in C(\Omega)$ (by the convention that h is extended so that it takes value 0 outside Ω, we have ${}^{*}h\colon{}^{*}\boldsymbol{R}\to{}^{*}\boldsymbol{C}$ and ${}^{*}h(x)=0$ for $x\in{}^{*}\boldsymbol{R}-{}^{*}\Omega$), then ${}^{*}h|X\in S(\Omega)$ and ${}^{\vee}({}^{*}h|X)=h$.*

(c) *$S(\Omega)\subseteq E(\Omega)$ and $\varphi\in S(\Omega)$ implies $P_{\varphi}=T_{{}^{\vee}\varphi}$.*

4. Distributions

Proposition 4. *For each $\varphi\in R(X)$, the following two conditions are equivalent:*

(a) *For any compact subset K of Ω, there is $m\in\boldsymbol{N}$ such that $\sum_{{}^{*}K\cap X}\varepsilon^{m+1}|\varphi|$ is finite.*

(b) *For any compact subset K of Ω, there is $k\in\boldsymbol{N}$ such that $\sum_{{}^{*}K\cap X}\varepsilon^{k+1}|\varphi|^{2}$ is finite.*

Proof. (a)→(b). Let K be a compact subset of Ω, $m\in\boldsymbol{N}$, and $\sum_{{}^{*}K\cap X}\varepsilon^{m+1}|\varphi|$ finite. We have

$$\sum_{{}^{*}K\cap X}\varepsilon^{(2m+1)+1}|\varphi|^{2}\leq\Big(\sum_{{}^{*}K\cap X}\varepsilon^{m+1}|\varphi|\Big)^{2}$$

and the right hand side of the inequality is finite. Hence we get (b) with

$k=2m+1$.

(b)→(a). Let K be a compact subset of Ω, $k\in\boldsymbol{N}$, and $\sum_{*K\cap X}\varepsilon^{k+1}|\varphi|^2$ finite. Choose $m\in\boldsymbol{N}$ such that $k\leq 2m$. We have

$$\Big(\sum_{*K\cap K}\varepsilon^{m+1}|\varphi|\Big)^2\leq\nu_L({}^*K\cap X)\cdot H\cdot\sum_{*K\cap X}\varepsilon^{2m+2}|\varphi|^2$$

and

$$H\cdot\sum_{*K\cap X}\varepsilon^{2m+2}|\varphi|^2=\sum_{*K\cap X}\varepsilon^{2m+1}|\varphi|^2\leq\sum_{*K\cap X}\varepsilon^{k+1}|\varphi|^2 .$$

As the right hand side of the second inequality is finite, we get (a). □

DEFINITION. $Z(\Omega)$ denotes the set of all $\varphi\in R(X)$ which satisfies the condition (a) in Proposition 4.

Immediately, we have $M(\Omega)\subseteq Z(\Omega)$.

DEFINITION. For each $\varphi\in R(X)$, we define $D_+\varphi$ and $D_-\varphi$ as follows:

$$D_+\varphi(x)=\frac{\varphi(x+\varepsilon)-\varphi(x)}{\varepsilon}\quad\text{and}\quad D_-\varphi(x)=\frac{\varphi(x)-\varphi(x-\varepsilon)}{\varepsilon}.$$

(Note that we extend $\varphi\colon X\to{}^*\boldsymbol{C}$ to $\varphi\colon \boldsymbol{L}\to{}^*\boldsymbol{C}$ to have the period H.)

Proposition 5. (a) *$Z(\Omega)$ is stable under D_+ and D_-.*
(b) *If φ, $\psi\in Z(\Omega)$, then $\varphi\psi\in Z(\Omega)$.*

Proof. (a) Let $\varphi\in Z(\Omega)$ and K be a compact subset of Ω. Choose a compact subset K_1 of Ω so that $K\subseteq K_1\subseteq\Omega$ and:

$$\text{If}\quad x\in X,\quad\text{then}\quad x\in{}^*K\quad\text{implies}\quad x\pm\varepsilon\in{}^*K_1 .$$

By choosing $m\in\boldsymbol{N}$ so that $\sum_{*K_1\cap X}\varepsilon^{m+1}|\varphi|$ is finite, we have

$$\sum_{*K\cap X}\varepsilon^{m+2}|D_\pm\varphi|\leq\sum_{*K\cap X}\varepsilon^{m+1}|\varphi(x\pm\varepsilon)|+\sum_{*K\cap X}\varepsilon^{m+1}|\varphi| .$$

Both terms in the right hand side turn out to be less than or equal to $\sum_{*K\cap X}\varepsilon^{m+1}|\varphi|$ and thus, $\sum_{*K\cap X}\varepsilon^{(m+1)+1}|D_\pm\varphi|$ is finite.

(b) Let K be a compact subset of Ω and choose k, $l\in\boldsymbol{N}$ so that both $\sum_{*K\cap X}\varepsilon^{k+1}|\varphi|^2$ and $\sum_{*K\cap X}\varepsilon^{l+1}|\psi|^2$ are finite (Proposition 4). By choosing m, $n\in\boldsymbol{N}$ such that $k\leq 2m+1$ and $l\leq 2n+1$, we get

$$\Big(\sum_{*K\cap X}\varepsilon^{m+n+2}|\varphi\psi|\Big)^2\leq\sum_{*K\cap X}\varepsilon^{2m+2}|\varphi|^2\cdot\sum_{*K\cap X}{}^{2m+2}|\varphi|^2$$
$$\leq\sum_{*K\cap X}\varepsilon^{n+2}|\psi|^2\cdot\sum_{*K\cap X}\varepsilon^{l+1}|\varphi|^2 .$$ □

Proposition 6. *Let $\varphi\in A(\Omega)\cap Z(\Omega)$ and $h\in\mathcal{E}(\Omega)$. Then we have $D_\pm\varphi$, ${}^*h\varphi\in A(\Omega)\cap Z(\Omega)$ and if $f\in\mathcal{D}(\Omega)$, then $P_{D_\pm\varphi}(f)=-P_\varphi(f')$ and $P_{*h\varphi}(f)=P_\varphi(hf)$.*

Here, $\mathcal{E}(\Omega)$ denotes the set of $\boldsymbol{C}$-valued, indefinitely differentiable functions on Ω, and its elements are assumed to be extended to whole $\boldsymbol{R}$ so that they take 0 outside Ω. We have written simply $^*h$ for $^*h|X$. f' denotes the derived function of f.

Proof. (i) We have $D_{\pm}\varphi \in Z(\Omega)$ by Proposition 5. Now we show that $D_{\pm}\varphi \in A(\Omega)$ and $P_{D_{\pm}\varphi} = -P_{\varphi}(f')$ as follows. Suppose $f \in \mathcal{D}(\Omega)$ and $K = \mathrm{supp}(f)$. Choose a compact set K_1 satisfying $K \subseteq K_1 \subseteq \Omega$ so that, for each $x \in X$, $x \in {}^*K$ implies $x \pm \varepsilon \in {}^*K_1$, and then choose $m \in \boldsymbol{N}$ so that $\sum_{^*K_1 \cap X} \varepsilon^{m+1}|\varphi|$ is finite. With signs in the respective order, we have

$$\begin{aligned}
\sum_X \varepsilon (D_{\pm}\varphi)^* f &= \pm \sum_{x \in X} \varphi(x \pm \varepsilon)^* f(x) \mp \sum_X \varphi^* f \\
&= \pm \sum_{x \in X} \varphi(x)^* f(x \mp \varepsilon) \mp \sum_{x \in X} \varphi(x)^* f(x) \\
&= -\sum_{x \in X} \varepsilon \varphi(x) = \frac{^*f(x \mp \varepsilon) - {}^*f(x)}{\mp \varepsilon} \\
&= -\left\{ \sum_{k=1}^{m+1} \frac{(\mp 1)^{k-1}}{k!} \varepsilon^{k-1} \sum_X \varepsilon \varphi^* f^{(k)} \right. \\
&\quad \left. + \frac{(\mp 1)^{m+1}\varepsilon}{(m+2)!} \sum_{x \in X} \varepsilon^{m+1} \varphi(x) ({}^*\mathrm{Re} f^{(m+2)}(x \mp \sigma\varepsilon) + i\, {}^*\mathrm{Im} f^{(m+2)}(x \mp \tau\varepsilon)) \right\}
\end{aligned}$$

$(\sigma, \tau \in {}^*\boldsymbol{R},\ 0 < \sigma,\ \tau < 1)$.

As for the sums in the scope of negative sign, we have

$$\text{the first sum} \simeq \sum_{k=1}^{m+1} \frac{(\mp 1)^{k-1}}{k!} \varepsilon^{k-1} P_{\varphi}(f^{(k)}) \simeq P_{\varphi}(f')\,,$$

and

$$\text{the second sum} \leq \frac{\varepsilon}{(m+2)!} \sum_{^*K_1 \cap X} \varepsilon^{m+1} |\varphi| \cdot 2 \sup |f^{(m+2)}| \simeq 0\,.$$

Hence, $D_{\pm}\varphi \in A(\Omega)$ and $P_{D_{\pm}\varphi}(f) = -P_{\varphi}(f')$ for each $f \in \mathcal{D}(\Omega)$.
(ii) If $h \in \mathcal{E}(\Omega)$, then $^*h \in S(\Omega) \subseteq E(\Omega)$ by Theorem 6, and we have $E(\Omega) \subseteq M(\Omega) \subseteq Z(\Omega)$ by Proposition 1 and definitions. Hence, $^*h\varphi \in Z(\Omega)$ for $\varphi \in Z(\Omega)$ by Proposition 5. Now , since $hf \in \mathcal{D}(\Omega)$ for $f \in \mathcal{D}(\Omega)$, we have

$$\sum_X \varepsilon^* h \varphi^* f = \sum_X \varepsilon \varphi^*(hf) \simeq P_{\varphi}(hf)\,.$$

Thus we get $^*h\varphi \in A(\Omega)$ and $P_{^*h\varphi}(f) = P_{\varphi}(hf)$. □

DEFINITION. $D_F(\Omega)$ denotes the smallest subset of $R(X)$ which includes $M(\Omega)$ and closed under applications of D_+ and D_-, multiplication of $^*h$ for each $h \in \mathcal{E}(\Omega)$, and addition.

Theorem 7. (a) $D_F(\Omega) \subseteq A(\Omega) \cap Z(\Omega)$. *If $\varphi \in D_F(\Omega)$ and $h \in \mathcal{E}(\Omega)$, then $D_{\pm}\varphi$, $^*h\varphi \in D_F(\Omega)$ and $P_{\varphi} \in \mathcal{D}'_F(\Omega)$, $P_{D_{\pm}} = (P_{\varphi})'$ and $P_{^*h\varphi} = hP_{\varphi}$.*

(b) *Every* $T\in\mathcal{D}'_F(\Omega)$ *can be represented in the form* $T=P_\varphi$ *for some* $\varphi\in D_F(\Omega)$.

Proof. (a) Since $M(\Omega)\subseteq A(\Omega)\cap Z(\Omega)$ and $A(\Omega)\cap Z(\Omega)$ is stable under D_+, D_-, and $*h$, we have $D_F(\Omega)\subseteq A(\Omega)\cap Z(\Omega)$.

Now, if $\varphi\in M(\Omega)$, then we have $P_\varphi\in\mathcal{D}'^{(0)}(\Omega)\subseteq\mathcal{D}'_F(\Omega)$, and $\mathcal{D}'_F(\Omega)$ is stable under derivation and multiplication of h. On the other hand, by Proposition 6 we have

$$P_{D_\pm\varphi}(f)=-P_\varphi(f')\,,\quad\text{and}\quad P_{*h\varphi}(f)=P_\varphi(hf)\,,$$

and thus we can prove that, for each $\varphi\in D_F(\Omega)$, $P_\varphi\in\mathcal{D}'_F(\Omega)$ and

$$P_{D_\pm\varphi}=(P_\varphi)'\quad\text{and}\quad P_{*h\varphi}=hP_\varphi$$

by induction on the number of operations of D_+, D_- and $*h$ to an element of $M(\Omega)$.

(b) For each $T\in\mathcal{D}'_F(\Omega)$, we can represent it in the form $T=S^{(k)}$ for some $S\in\mathcal{D}'^{(0)}(\Omega)$ and $k\in\boldsymbol{N}$. Representing S in the form $S=P_\psi$ with $\psi\in M(\Omega)$, we have $D_+^k\psi\in D_F(\Omega)$ and $P_{D_+^k\psi}=(P_\psi)^{(k)}=T$. □

DEFINITION. Let $D(\Omega)$ denote the set of elements $\varphi\in R(X)$ such that, for each compact subset K of Ω, there is some $\psi\in D_F(\Omega)$ which satisfies $\varphi=\psi$ on $*K\cap X$.

REMARK. $D(\Omega)\subseteq A(\Omega)\cap Z(\Omega)$.

Theorem 8. (a) *If* $\varphi\in D(\Omega)$, *then* $P_\varphi\in\mathcal{D}'(\Omega)$.

(b) *If* $\varphi\in D(\Omega)$, *then* $D_\pm\varphi\in D(\Omega)$ *and* $P_{D_\pm\psi}=(P_\psi)'$.

(c) *If* $\varphi\in D(\Omega)$ *and* $h\in\mathcal{E}(\Omega)$, then $*h\varphi\in D(\Omega)$ *and* $P_{*h\varphi}=hP_\varphi$.

(d) *If* $T\in\mathcal{D}'(\Omega)$, *then there is some* $\varphi\in D(\Omega)$ *such that* $P_\varphi=T$.

Proof. (a) Suppose $\varphi\in D(\Omega)$. Let $(f_j)_{j\in N}$ be a sequence in $\mathcal{D}(\Omega, K)$ such that $f_j\to 0$ in $\mathcal{D}(\Omega, K)$. Choose $\psi\in D_F(\Omega)$ corresponding to K such that $\varphi=\psi$ on $*K\cap X$. Then by Theorem 7, we have $P_\psi\in\mathcal{D}'_F(\Omega)$ and thus $P_\phi(f_j)\to 0$. On the other hand, we have $P_\varphi(f_j)=P_\psi(f_j)$ for every $j\in\boldsymbol{N}$, so $P_\varphi(f_j)\to 0$. Hence $P_\varphi\in\mathcal{D}'(\Omega)$.

(b) Suppose $\varphi\in D(\Omega)$. We know that $D_\pm\varphi\in A(\Omega)\cap Z(\Omega)$ by Proposition 6. For a compact subset K of Ω, choose a compact set K_1 so that $K\subseteq K_1\subseteq\Omega$ and $x\pm\varepsilon\in *K_1$ for each $x\in *K\cap X$. Choose $\psi\in D_F(\Omega)$ so that $\varphi=\psi$ on $*K_1\cap X$. By Theorem 7, we hae have $D_\pm\psi\in D_F(\Omega)$ and $D_\pm\varphi=D_\pm\psi$ on $*K\cap X$. Thus $D_\pm\varphi\in D(\Omega)$.

Now, since $D(\Omega)\subseteq A(\Omega)\cap Z(\Omega)$, we have

$$P_{D_\pm\varphi}(f)=-P_\varphi(f')=(P_\varphi)'(f)$$

for each $f\in\mathcal{D}(\Omega)$ by Proposition 6.

(c) Suppose that $\varphi\in D(\Omega)$ and $h\in\mathcal{E}(\Omega)$. We have $*h\varphi\in A(\Omega)\cap Z(\Omega)$ by Proposition 6. For each compact subset K of Ω, choose $\psi\in D_F(\Omega)$ so that $\varphi=\psi$ on $*K\cap X$. Then $*h\psi\in D_F(\Omega)$ by Theorem 6, and obviously, $*h\varphi^*=*h\psi$ on $*K\subset X$. Hence, $*h\varphi\in D(\Omega)$.

Also, since $D(\Omega)\subseteq A(\Omega)\cap Z(\Omega)$, we have

$$P_{*h\varphi}(f)=P_\varphi(hf)=(hP_\varphi)(f)$$

for each $f\in\mathcal{D}(\Omega)$ by Proposition 6.

(d) Suppose $T\in\mathcal{D}'(\Omega)$. Let $(f_i)_{i\in N}$ be a partition of unity on Ω such that each $K_i=\mathrm{supp}(f_i)$ is a convex compact set which has an interior point. Each f_iT is a distribution on Ω with support contained in K_i. Thus, we can represent each f_iT as a finite sum of derivatives of elements belonging to $\mathcal{C}(\Omega)$ with each support contained in K_i ([3], Chap. 1, corollary to Theorem 1.5). Now we shall show that there is a function $\psi_i\in D_F(\Omega)$ for each i such that $P_{\psi_i}=f_iT$ and that ψ_i is 0 on $*K\cap X$ for every compact subset K of Ω with the property $K\cap K_i=\emptyset$. For it, we can assume that $f_iT=(T_h)^{(n)}$ with $h\in\mathcal{C}(\Omega)$, $\mathrm{supp}(h)\subseteq K_i$ and $n\geq 0$. By Theorem 6, we have

$$*h\,|\,X\in S(\Omega)\subseteq M(\Omega) \quad\text{and}\quad P_{*h|X}=T_h\,.$$

Clearly, $*h\,|\,X$ takes 0 outside $*K_1\cap X$. Then, $D_+^n(*h\,|\,X)$ belongs to $D_F(\Omega)$ and if you choose a compact set K such that $K\cap K_i=\emptyset$, then it takes 0 on $*K\cap X$, and moreover,

$$P_{D_+^n(*h|X)}=(P_{*h|X})^{(n)}=(T_h)^{(n)}=f_iT\,.$$

Thus we get the claim above.

Now, applying Lemma 2 for $Y(\Omega)=D_F(\Omega)$, we get an internal map $*\boldsymbol{N}\ni n\mapsto\varphi_n\in R(X)$ and $N\in*\boldsymbol{N}$ such that $n\in\boldsymbol{N}$ implies $\varphi_n=\sum_{i=0}^n\psi_i\in D_F(\Omega)$ and satisfy following conditions:

(1) $N\in\boldsymbol{N}$ implies $\varphi_N\in D_F(\Omega)$ and $P_{\varphi_N}=T$;

(2) $N\in*\boldsymbol{N}-\boldsymbol{N}$implies $\varphi_N\in A(\Omega)$, $P_{\varphi_N}=T$ and for each compact subset K of Ω, with an appropriate $n\in\boldsymbol{N}$, we have $\varphi_N=\varphi_n$ on $*K\cap X$.

So, let $\varphi=\varphi_N$. For the case (1), there is nothing more to prove. For the case (2), as each φ_n belongs to $D_F(\Omega)$, we have $\varphi\in D(\Omega)$ and $P_\varphi=T$, and we finish the proof. □

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