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NON-STANDARD REPRESENTATIONS OF DISTRIBUTIONS I

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1. Introduction

We fix an infinite natural number H in ${}^*\mathbf{N} \setminus \mathbf{N}$. We assume that H is even. Let $\varepsilon = 1/H$. Then, ε is a positive infinitesimal element of ${}^*\mathbf{Q}$, hence, of ${}^*\mathbf{R}$. We denote by \mathbf{L} the set ${}^*\mathbf{Z} \cdot \varepsilon$ of all integer multiples of ε . Then, ${}^*\mathbf{Z} \subsetneq \mathbf{L} \subsetneq {}^*\mathbf{R}$. \mathbf{L} is a lattice with infinitesimal mesh ε . Put

$$X = \left\{ x \in \mathbf{L} \mid -\frac{H}{2} \leq x < \frac{H}{2} \right\}.$$

Then, X is a $*$ -finite subset of \mathbf{L} of cardinality H^2 .

Now, consider the set

$$R(X) = \{ \varphi \mid \varphi: X \rightarrow {}^*\mathbf{C}, \text{ internal} \}.$$

By the above, $R(X)$ is an internal H^2 -dimensional vector space over ${}^*\mathbf{C}$. From now on, we will assume that every element φ of $R(X)$ is extended to a function defined on \mathbf{L} with period H .

Let Ω be an open set in \mathbf{R} . Every function $\Omega \rightarrow \mathbf{C}$ we consider is assumed to be extended to a function $\mathbf{R} \rightarrow \mathbf{C}$ which takes zero outside Ω .

Let's consider $f \in \mathcal{D}(\Omega)$, where $\mathcal{D}(\Omega)$ is the space of indefinitely differentiable functions with compact support on Ω . We have $*f: {}^*\mathbf{R} \rightarrow {}^*\mathbf{C}$, and denoting $K = \text{supp}(f)$, the following statement holds:

$$x \in X, \quad x \notin {}^*K \cap X \text{ implies } *f(x) = 0.$$

In the following, we shall define several mappings each from an external subspace of $R(X)$ to some space consisting of distributions on Ω .

Let

$$A(\Omega) = \{ \varphi \in R(X) \mid \sum_{x \in X} \varepsilon \varphi(x) * f(x) \text{ is finite for every } f \in \mathcal{D}(\Omega) \}.$$

We remark that the sum $\sum_{x \in X} \varepsilon \varphi(x) * f(x)$ in this definition always exists in ${}^*\mathbf{C}$ as a

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*-finite sum, and it can be represented as $\sum_{x \in {}^*K \cap X} \varepsilon \varphi(x) {}^*f(x)$ with $K = \text{supp}(f)$.

We also write these sums $\sum_x \varepsilon \varphi {}^*f$ and $\sum_{{}^*K \cap X} \varepsilon \varphi {}^*f$ respectively.

When $\alpha \in {}^*\mathcal{C}$ is finite, ${}^\circ\alpha$ denotes the standard part of α .
For each $\varphi \in A(\Omega)$ and $f \in \mathcal{D}(\Omega)$, put

$$P_\varphi(f) = {}^\circ(\sum_x \varepsilon \varphi {}^*f).$$

$P_\varphi: \mathcal{D}(\Omega) \ni f \mapsto P_\varphi(f) \in \mathcal{C}$ is a linear form on $\mathcal{D}(\Omega)$. $\mathcal{D}(\Omega)^*$ denotes the algebraic dual of $\mathcal{D}(\Omega)$ which is the set of all linear forms on $\mathcal{D}(\Omega)$. Using these, we define a map

$$P: A(\Omega) \ni \varphi \mapsto P_\varphi \in \mathcal{D}(\Omega)^*.$$

Lemma 1. *Let X be a set, k a field, and f_1, \dots, f_n maps from X to k . If f_1, \dots, f_n are linearly independent over k , then there are x_1, \dots, x_n in X such that $\det \|f_i(x_j)\| \neq 0$.*

Theorem 1. *P is an onto map.*

Proof. Choose T from $\mathcal{D}(\Omega)^*$. For each f in $\mathcal{D}(\Omega)$, let

$$A(f) = \{\varphi \in R(X) \mid \sum_x \varepsilon \varphi {}^*f = T(f)\}.$$

For any f_1, \dots, f_n in $\mathcal{D}(\Omega)$ we will show that $\bigcap_{i=1}^n A(f_i) \neq \emptyset$. Then, we can see that

$$\bigcap_{f \in \mathcal{D}(\Omega)} A(f) \neq \emptyset$$

by saturation principle. Picking φ from $\bigcap_{f \in \mathcal{D}(\Omega)} A(f)$, we have

$$\sum_x \varepsilon \varphi {}^*f = T(f)$$

for every f in $\mathcal{D}(\Omega)$. In other words, $P_\varphi = T$.

In the following, we show that $\bigcap_{i=1}^n A(f_i) \neq \emptyset$ for any f_1, \dots, f_n in $\mathcal{D}(\Omega)$ by induction on n .

First of all, we remark that, for any t in \mathbf{R} , there is a unique x in X which satisfies $x \leq t < x + \varepsilon$, that is the maximum element $x \in X$ satisfying $x \leq t$. We denote this x by ${}^{\wedge}t$. Also, if $t \in \Omega$, then the above $x (= {}^{\wedge}t)$ is an element of ${}^*\Omega \cap X$, because $x \simeq t$ and Ω is an open set.

For $n=1$, let $f \in \mathcal{D}(\Omega)$. If $f=0$, then $0 \in A(f)$. If $f \neq 0$, then $f(t_1) \neq 0$ for some $t_1 \in \Omega$. Choose ${}^{\wedge}t_1 \in {}^*\Omega \cap X$ as we have just remarked above and put $x_1 = {}^{\wedge}t_1$. Then, ${}^*f(x_1) \neq 0$. Otherwise, the continuity of f and $x_1 \simeq t_1$ would imply $0 = {}^*f(x_1) = f(t_1)$. We define $\varphi: X \rightarrow {}^*\mathcal{C}$ by

$$\varphi(x) = \begin{cases} \frac{T(f)}{\varepsilon^* f(x_1)} & (x=x_1) \\ 0 & (x \neq x_1) \end{cases}.$$

Then, we have $\sum_x \varepsilon \varphi^* f = T(f)$, i.e. $\varphi \in A(f)$.

Assuming $n > 1$ and that the assertion holds for $n-1$, we prove it for n . Let $f_1, \dots, f_n \in \mathcal{D}(\Omega)$. If they are linearly dependent over \mathcal{C} , by changing the suffixes if necessary, we can assume that $f_n = \sum_{i=1}^{n-1} c_i f_i$ with c_i 's in \mathcal{C} . Choosing φ from $\cap_{i=1}^{n-1} A(f_i)$, we have $T(f_i) = \sum_x \varepsilon \varphi^* f_i$ for $1 \leq i \leq n-1$. Using this equation, we have

$$T(f_n) = \sum_{i=1}^{n-1} c_i T(f_i) = \sum_{i=1}^{n-1} c_i \sum_x \varepsilon \varphi^* f_i = \sum_x \varepsilon \varphi \sum_{i=1}^{n-1} c_i^* f_i = \sum_x \varepsilon \varphi^* f_n.$$

Hence, $\varphi \in \cap_{i=1}^n A(f_i)$.

If f_1, \dots, f_n are linearly independent over \mathcal{C} , then we can find t_1, \dots, t_n in Ω such that $\det \|f_i(t_j)\| \neq 0$ by Lemma 1. Choose ${}^a t_1, \dots, {}^a t_n \in {}^a \Omega \cap X$, and put $x_j = {}^a t_j$ ($1 \leq j \leq n$). Since

$$\det \|{}^* f_i(x_j)\| \simeq \det \|f_i(t_j)\|,$$

we have $\det \|{}^* f_i(x_j)\| \neq 0$. There are $\alpha_1, \dots, \alpha_n \in {}^* \mathcal{C}$ which satisfy

$$\sum_{j=2}^n {}^* f_i(x_j) \alpha_j = T(f_i) \quad (1 \leq i \leq n).$$

With this statement and reminding that $x_j \neq x_k$ if $j \neq k$, we can define a map $\varphi: X \rightarrow {}^* \mathcal{C}$ as follows:

$$\varphi(x) = \begin{cases} \frac{\alpha_j}{\varepsilon} & (x=x_j) \\ 0 & (x \neq x_1, \dots, x_n) \end{cases}.$$

Then, $\sum_x \varepsilon \varphi^* f_i = \sum_{j=1}^n \varepsilon \frac{\alpha_j}{\varepsilon} {}^* f_i(x_j) = T(f_i) \quad (1 \leq i \leq n).$

Hence, $\varphi \in \cap_{i=1}^n A(f_i)$. □

Theorem 1 above is a modification of a theorem of Robinson ([8], §5.3) by reducing ${}^* \mathcal{R}$ to X . The proof is almost same as that of the Robinson's theorem given by M. Saito [9]. Another proof which makes use of the lattice structure of X can be found, for example, in H.J. Keisler [4].

Following this theorem, we will show that by defining several external subspaces $D(\Omega)$, $D_F(\Omega)$, $M(\Omega)$, $M_1(\Omega)$, $E(\Omega)$, and $S(\Omega)$ of $A(\Omega)$ appropriately, the images of these sets by $P: A(\Omega) \rightarrow \mathcal{D}(\Omega)^*$ are $\mathcal{D}'(\Omega)$, $\mathcal{D}'_F(\Omega)$, $\mathcal{D}'^{(0)}(\Omega) = \mathcal{M}(\Omega)$, $\mathcal{M}_1(\Omega)$, $\mathcal{L}_{1, \text{loc}}(\Omega)$ and $\mathcal{C}(\Omega)$ respectively. Here, $\mathcal{D}'_F(\Omega)$ is the set of distributions of finite order on Ω , $\mathcal{D}'^{(0)}(\Omega) = \mathcal{M}(\Omega)$ the set of distributions of order 0 on

Ω which is the set of all (complex) measures on Ω , and $\mathcal{M}_1(\Omega)$ the set of all bounded (complex) measures on Ω . $\mathcal{D}'(\Omega)$, $\mathcal{L}_{1,\text{loc}}(\Omega)$ and $\mathcal{C}(\Omega)$ are frequently used symbols. For the knowledge of distributions, refer to $L.$ Schwartz [10].

2. Complex Measures

DEFINITION. $M(\Omega) = \{\varphi \in R(X) \mid \sum_{*K \cap X} \varepsilon |\varphi| \text{ is finite for every compact subset } K \text{ of } \Omega\}$.

Theorem 2. (a) $M(\Omega) \subseteq A(\Omega)$.

(b) The image of $M(\Omega)$ by P is $\mathcal{D}'^{(0)}(\Omega) = \mathcal{M}(\Omega)$.

Proof. (a) Let $\varphi \in M(\Omega)$, $f \in \mathcal{D}(\Omega)$, and $K = \text{supp}(f)$. Then,

$$|\sum_X \varepsilon \varphi^* f| = |\sum_{*K \cap X} \varepsilon \varphi^* f| \leq (\sum_{*K \cap X} \varepsilon |\varphi|) \cdot \sup |f|.$$

So, $\sum_X \varepsilon \varphi^* f$ is finite and $\varphi \in A(\Omega)$.

The first half of (b): Let $\varphi \in M(\Omega)$, and K a compact subset of Ω . Put $C_K = \sum_{*K \cap X} \varepsilon |\varphi|$. By the formulas in the proof of (a), we get $|P_\varphi(f)| \leq C_K \cdot \sup |f|$ for each $f \in \mathcal{D}(\Omega)$ such that $\text{supp}(f) \subseteq K$ and thus, $P_\varphi \in \mathcal{D}'^{(0)}(\Omega) = \mathcal{M}(\Omega)$.

The second half of (b): For each $T \in \mathcal{M}(\Omega)$, we show that $T = P_\varphi$ for some $\varphi \in M(\Omega)$. Let $\mathcal{K}(\Omega) = \{f: \mathbf{R} \rightarrow \mathbf{C}, \text{continuous} \mid \text{supp}(f) \text{ is compact} \subset \Omega\}$ and for every map $T: \mathcal{D}(\Omega) \rightarrow \mathbf{C}$, we will extend it to a map from $\mathcal{K}(\Omega)$ to \mathbf{C} and denote it by the same letter T . we also put $\mathcal{K}_+(\Omega) = \{f \in \mathcal{K}(\Omega) \mid f \geq 0\}$.

Suppose first that $T: \mathcal{K}(\Omega) \rightarrow \mathbf{C}$ is a positive linear form. For $f \in \mathcal{K}_+(\Omega)$ and $e > 0$, let $A(f, e) = \{\varphi \in R(X) \mid \varphi \geq 0, |T(f) - \sum_X \varepsilon \varphi^* f| \leq e\}$. For any $f_1, \dots, f_n \in \mathcal{K}_+(\Omega)$ and $e_1, \dots, e_r > 0$, we will show that $\bigcap_{1 \leq i \leq n, 1 \leq j \leq r} A(f_i, e_j) \neq \emptyset$. This will yield the following by saturation principle:

$$\bigcap_{f \in \mathcal{K}_+(\Omega), e > 0} A(f, e) \neq \emptyset.$$

Choosing φ from $\bigcap_{f \in \mathcal{K}_+(\Omega), e > 0} A(f, e)$, we have $\varphi \geq 0$ and

$$|\sum_X \varepsilon \varphi^* f - T(f)| \leq e$$

for every $f \in \mathcal{K}_+(\Omega)$ and $e > 0$. Hence, $\sum_X \varepsilon \varphi^* f = T(f)$.

Now, it is enough to show that $\bigcap_{i=1}^n A(f_i, e) \neq \emptyset$ for each f_1, \dots, f_n and $e = \text{Min}\{e_1, \dots, e_r\}$.

Let $f_0 \in \mathcal{K}_+(\Omega)$ be such that $f_0 \geq f_1, \dots, f_0 \geq f_n$ (e.g. $f_0 = f_1 + \dots + f_n$). Let $S_0 = \{t \in \Omega \mid f_0(t) \neq 0\}$.

If $T(f_0) = 0$, then we get $T(f_i) = 0$ for $1 \leq i \leq n$, since $f_0 \geq f_i$ implies $0 = T(f_0) \geq T(f_i) \geq 0$. So, we have $0 \in \bigcap_{i=1}^n A(f_i, e)$.

Now, assume that $T(f_0) > 0$. Then, $S_0 \neq \emptyset$. We can see that the point $Q = (T(f_1)/T(f_0), \dots, T(f_n)/T(f_0))$ in \mathbf{R}^n is contained in the closed convex closure C of the subset $\{(f_1(t)/f_0(t), \dots, f_n(t)/f_0(t)) \mid t \in S_0\}$ of \mathbf{R}^n as follows: Assuming $Q \notin C$, the point Q and the set C are strictly separated by some hyperplane in \mathbf{R}^n . Hence, for some $b_0, \dots, b_n \in \mathbf{R}$,

$$\sum_{i=1}^n b_i (T(f_i)/T(f_0)) > b_0 > \sup_{t \in S_0} \sum_{i=1}^n b_i (f_i(t)/f_0(t)).$$

Put $g = \sum_{i=1}^n b_i f_i \in \mathcal{K}(\Omega)$. Then,

$$(1) \quad \frac{T(g)}{T(f_0)} > b_0 > \sup_{t \in S_0} \frac{g(t)}{f_0(t)}.$$

For $t \in S_0$, by the right half of (1), $g(t) < b_0 f_0(t)$ and $t \in \Omega - S_0$ imply $f_0(t) = 0$. Hence, $f_i(t) = 0$ for $1 \leq i \leq n$. Thus, $g(t) = 0$ and we have $g(t) = b_0 f_0(t)$. So, $g \leq b_0 f_0$. Since T is positive, we get $T(g) \leq b_0 T(f_0)$ and $(T(g)/T(f_0)) \leq b_0$, which contradicts the first inequality of (1).

Now, since $Q \in C$, for each $\epsilon > 0$, there are $t_1, \dots, t_r \in S_0$ and $a_1, \dots, a_r \in \mathbf{R}$ such that $a_j > 0$, $\sum_{j=1}^r a_j = 1$ and $|T(f_i)/T(f_0) - \sum_{j=1}^r a_j (f_i(t_j)/f_0(t_j))| \leq \epsilon/2T(f_0)$ ($1 \leq i \leq n$). So, we have

$$(2) \quad |T(f_i) - \sum_{j=1}^r \frac{a_j T(f_0)}{f_0(t_j)} f_i(t_j)| \leq \frac{\epsilon}{2} \quad (1 \leq i \leq n).$$

Here, we can assume that t_1, \dots, t_r are pairwise distinct. (If $t_j = t_k$ for $j \neq k$, then we can write $a_j \frac{T(f_0)}{f_0(t_j)} f_i(t_j) + a_k \frac{T(f_0)}{f_0(t_k)} f_i(t_k) = (a_j + a_k) \frac{T(f_0)}{f_0(t_j)} f_i(t_j)$.) Now, let $x_j = t_j \in {}^*\Omega \cap X$ for each j such that $1 \leq j \leq r$. Since x_1, \dots, x_r are pairwise distinct, by defining $\varphi: X \rightarrow {}^*\mathbf{C}$, $\varphi \geq 0$ by

$$\varphi(x) = \begin{cases} \frac{a_j T(f_0)}{\epsilon f_0(t_j)} & (x = x_j) \\ 0 & (x \neq x_1, \dots, x_r) \end{cases}$$

we have

$$\begin{aligned} \sum_X \epsilon \varphi * f_1 &= \sum_{j=1}^r \epsilon \varphi(x_j) * f_i(x_j) \\ &= \sum_{j=1}^r \frac{a_j T(f_0)}{f_0(t_j)} * f_i(x_j) = \sum_{j=1}^r \frac{a_j T(f_0)}{f_0(t_j)} f_i(t_j) \end{aligned}$$

for each i satisfying $1 \leq i \leq n$, and by combining with (2), we get

$$\begin{aligned} |T(f_i) - \sum_X \epsilon \varphi * f_i| &\leq |T(f_i) - \sum_{j=1}^r \frac{a_j T(f_0)}{f_0(t_j)} f_i(t_j)| \\ &\quad + |\sum_{j=1}^r \frac{a_j T(f_0)}{f_0(t_j)} f_i(t_j) - \sum_X \epsilon \varphi * f_i| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \quad (1 \leq i \leq n). \end{aligned}$$

Hence, $\varphi \in \bigcap_{i=1}^n A(f_i, e)$.

Now, we have verified as we said above that, for each positive linear form $T: \mathcal{K}(\Omega) \rightarrow \mathbf{C}$, there is a map $\varphi \in R(X)$ such that $\varphi \geq 0$ and, for every map $f \in \mathcal{K}_+(\Omega)$, the following holds:

$$(3) \quad \sum_X \varepsilon \varphi^* f \simeq T(f).$$

It is easily seen that (3) holds for every function $f \in \mathcal{K}(\Omega)$, and in particular, for every function $f \in \mathcal{D}(\Omega)$.

We show that $\varphi \in M(\Omega)$ as follows. For every compact subset K of Ω and every non-negative function $f \in \mathcal{D}(\Omega)$ such that $0 \leq f \leq 1$ which equals 1 on K , we have

$$\sum_{*K \subset X} \varepsilon |\varphi| = \sum_{*K \cap X} \varepsilon \varphi = \sum_{*K \cap X} \varepsilon \varphi^* f \simeq T(f).$$

Therefore, $\varphi \in M(\Omega)$.

Returning to general case, every measure T can be written in the form $T = T_1 - T_2 + i(T_3 - T_4)$ where T_i ($1 \leq i \leq 4$) are positive linear forms on $\mathcal{K}(\Omega)$. For each i satisfying $1 \leq i \leq 4$, we can find $\varphi_i \in M(\Omega)$ such that $\varphi_i \geq 0$ and, for every function $f \in \mathcal{D}(\Omega)$, $\sum_X \varepsilon \varphi_i^* f \simeq T_i(f)$ holds. Putting $\varphi = \varphi_1 - \varphi_2 + i(\varphi_3 - \varphi_4)$, we have $\varphi \in M(\Omega)$ and $P_\varphi = T$. \square

The above proof is almost same as that in M. Saito [9], §2.2.

DEFINITION. $M_1(\Omega) = \{\varphi \in R(X) \mid \sum_{*\Omega \cap X} \varepsilon |\varphi| \text{ is finite}\}.$

We can immediately see from the definition that $M_1(\Omega) \subseteq M(\Omega)$.

Theorem 3. *The image of $M_1(\Omega)$ by $P: A(\Omega) \rightarrow \mathcal{D}(\Omega)^*$ coincides with $\mathcal{M}_1(\Omega)$. Moreover, if T is real, we can find a function $\varphi: X \rightarrow {}^*\mathbf{R}$ such that $\varphi \in \mathcal{M}_1(\Omega)$, $P_\varphi = T$ and $\sum_{*\Omega \cap X} \varepsilon |\varphi| \simeq \|T\|$. If T is not real, we only have the inequality $\sum_{*\Omega \cap X} \varepsilon |\varphi| \geq \|T\|$. Here, $\|T\|$ is the norm of T .*

Proof. If $\varphi \in M_1(\Omega)$ and $f \in \mathcal{D}(\Omega)$, then

$$\left| \sum_X \varepsilon \varphi^* f \right| = \left| \sum_{*\Omega \cap X} \varepsilon \varphi^* f \right| \leq \left(\sum_{*\Omega \cap X} \varepsilon |\varphi| \right) \cdot \sup |f|.$$

Hence, P_φ is a bounded measure in Ω .

Now, let $\mathcal{C}_B(\Omega)$ be the set of all complex-valued bounded continuous functions on Ω (assumed to take value 0 on $\mathbf{R}-\Omega$). We write $\mathcal{C}_{B,+}(\Omega) = \{f \in \mathcal{C}_B(\Omega) \mid f \geq 0\}$. 1_Ω denotes the characteristic function of Ω . And, for $T \in \mathcal{M}_1(\Omega)$, we extend it to a linear form $\mathcal{C}_B(\Omega) \rightarrow \mathbf{C}$ by integration and denote it by T again.

So, let $T: \mathcal{C}_B(\Omega) \rightarrow \mathcal{C}$ be a positive linear form. The proof of the second half of Theorem 2 (b) is also valid if we replace $\mathcal{K}(\Omega)$ and $\mathcal{K}_+(\Omega)$ by $\mathcal{C}_B(\Omega)$ and $\mathcal{C}_{B,+}(\Omega)$ respectively. This means, for each $T \in \mathcal{M}_1(\Omega)$ with T positive, there is $\varphi \in R(X)$ such that $\varphi \geq 0$ and $T(f) \simeq \sum_x \varepsilon \varphi^* f$ for each $f \in \mathcal{C}_B(\Omega)$. Putting $f = 1_\Omega$, we get $\|T\| \simeq \sum_{\Omega \cap X} \varepsilon \varphi = \sum_{\Omega \cap X} \varepsilon |\varphi|$. Hence, we have $\varphi \in M_1(\Omega)$. The proof for the last part of the theorem is similar to that for Theorem 2. \square

3. Complex-valued functions

We need the theory of Loeb measures on Ω and on Jordan measurable subsets of Ω ([1], [6], and [7]). To avoid duplications, we use the following notations:

Z : A countable union of Jordan measurable compact subsets of Ω .

\mathcal{A}_Z : The set of all internal subsets of ${}^*Z \cap X$. We sometimes write \mathcal{A} if there is no danger of confusion.

We define $\nu: \mathcal{A} \rightarrow {}^*\mathbf{R}$ by $\nu(A) = (\#(A)) \cdot \varepsilon$ for each $A \in \mathcal{A}$. Then, $({}^*Z \cap X, \mathcal{A}, \nu)$ is an internal finitely additive measure space. Let $({}^*Z \cap X, L(\mathcal{A}), \nu_L)$ be the Loeb space associated with it.

An internal function $\varphi: {}^*Z \cap X \rightarrow {}^*\mathbf{R}$ is said to be S-integrable if the following three conditions are satisfied:

- (1) $N \in {}^*\mathbf{N} - \mathbf{N}$ implies $\sum_{(|\varphi| \geq N)} \varepsilon |\varphi| \simeq 0$,
- (2) $N \in {}^*\mathbf{N} - \mathbf{N}$ implies $\sum_{(|\varphi| \leq (1/N))} \varepsilon |\varphi| \simeq 0$, and
- (3) $\sum_{{}^*Z \cap X} \varepsilon |\varphi|$ is finite.

If $\nu({}^*Z \cap X)$ is finite, (1) implies (2) and (3).

The following theorems are due to Loeb:

Let $\bar{\mathbf{R}}$ be the set of extended real numbers.

(1) If $\varphi: {}^*Z \cap X \rightarrow {}^*\mathbf{R}$ is S-integrable, then ${}^\circ\varphi: {}^*Z \cap X \rightarrow \bar{\mathbf{R}}$ is Loeb integrable and

$$\sum_{{}^*Z \cap X} \varepsilon \varphi \simeq \int_{{}^*Z \cap X} {}^\circ\varphi d\nu_L.$$

(2) If $g: {}^*Z \cap X \rightarrow \bar{\mathbf{R}}$ is Loeb integrable, then there is an S-integrable function $\varphi: {}^*Z \cap X \rightarrow {}^*\mathbf{R}$ such that ${}^\circ\varphi = g$ (ν_L -almost everywhere).

Moreover, let $\text{Ns}({}^*Z) = \{x \in {}^*Z \mid x \simeq t \text{ for some } t \in Z\}$. We have $\text{Ns}({}^*Z) \subseteq \text{Ns}({}^*\mathbf{R}) \cap {}^*Z$, but the equality does not necessarily hold. Here, $\text{Ns}({}^*\mathbf{R}) = \{\alpha \in {}^*\mathbf{R} \mid \alpha \text{ is finite}\}$. Define $\text{st}_Z: \text{Ns}({}^*Z) \cap X \rightarrow Z$ by $\text{st}_Z(x) = {}^\circ x$ when $x \in \text{Ns}({}^*Z) \cap X$. We sometimes omit Z in st_Z .

Let (Z, \mathcal{L}, μ) be a Lebesgue measure space over Z . The followings are known.

(3) For a subset E of Z , the condition $E \in \mathcal{L}$ is equivalent to the condition $\text{st}^{-1}(E) \in L(\mathcal{A})$ and if this condition is satisfied, then we have $\mu(E) = \nu_L(\text{st}^{-1}(E))$.

(4) Let $E \subseteq Z$ be \mathcal{L} -measurable and let h be a non-negative \mathcal{L} -measurable function: $E \rightarrow \bar{\mathbf{R}}$. Then, $h \circ \text{st}: \text{st}^{-1}(E) \rightarrow \bar{\mathbf{R}}$ is $L(\mathcal{A})$ -measurable and

$$\int_E h d\mu = \int_{\text{st}^{-1}(E)} (h \circ \text{st}) d\nu_L.$$

Now, if $K \subset \Omega$, then the compactness of K and the inclusion

$$\text{st}^{-1}(K) \supseteq {}^*K \cap X$$

are equivalent. By this fact and by (3) above, if K is a compact subset of Ω , then

$$\nu_L({}^*K \cap X) \leq \nu_L(\text{st}^{-1}(K)) = \mu(K).$$

Moreover, if K is a Jordan measurable compact set, we can prove that

$$\nu_L({}^*K \cap X) = \nu_L(\text{st}^{-1}(K)).$$

We define the local S -integrability below.

Recall that $R(X)$ is the set of internal functions from X to *C . If $\varphi \in R(X)$, K is a compact subset of Ω , and $n \in {}^*\mathbf{N}$, then we write

$$A(\varphi, K, n) = \{x \in {}^*K \cap X \mid |\varphi(x)| \geq n\}.$$

DEFINITION. (1) A function $\varphi \in R(X)$ is said to be *locally S -integrable* over Ω if the following holds for every compact subset K of Ω and every infinite natural number $N \in {}^*\mathbf{N} - \mathbf{N}$:

$$\sum_{A(\varphi, K, N)} \varepsilon |\varphi| \simeq 0.$$

(2) $E(\Omega) = \{\varphi \in R(X) \mid \varphi \text{ is locally } S\text{-integrable over } \Omega\}.$

Proposition 1. *The following two conditions are equivalent. In particular we have $E(\Omega) \subseteq M(\Omega)$.*

(a) $\varphi \in E(\Omega)$.

(b) $\varphi \in M(\Omega)$, and for any compact subset K of Ω and for any set $A \in \mathcal{A}$ such that $A \subseteq {}^*K \cap X$,

$$\nu(A) \simeq 0 \text{ implies } \sum_A \varepsilon |\varphi| \simeq 0.$$

Proof. (a) \rightarrow (b). Assume $\varphi \in E(\Omega)$. Let K be a compact subset of Ω and $e > 0$. Since φ is locally S -integrable over Ω , we have ${}^*\mathbf{N} - \mathbf{N} \subseteq \{n \in {}^*\mathbf{N} \mid \sum_{A(\varphi, K, n)} \varepsilon |\varphi| \leq e\}$. Hence, there is a natural number $n \in \mathbf{N}$ such that $\sum_{A(\varphi, K, n)} \varepsilon |\varphi| \leq e$. Thus, we have

$$\sum_{{}^*K \cap X} \varepsilon |\varphi| = \sum_{\{x \in {}^*K \cap X \mid |\varphi(x)| < n\}} \varepsilon |\varphi| + \sum_{A(\varphi, K, n)} \varepsilon |\varphi| \leq n \sum_{{}^*K \cap X} \varepsilon + e.$$

Since $\sum_{{}^*K \cap X} \varepsilon = \nu_L({}^*K \cap X) \leq \nu_L(\text{st}^{-1}(K)) = \mu(K)$, we can see that $\sum_{{}^*K \cap X} \varepsilon |\varphi|$

is finite and thus, we get $\varphi \in M(\Omega)$.

Now, let $A \in \mathcal{A}_\Omega$, K be a compact subset of Ω , $A \subseteq {}^*K \cap X$, and $\nu(A) \simeq 0$. Since $N \subseteq \{n \in {}^*\mathbf{N} \mid n^2 \cdot \nu(A) \leq 1\}$, we have $N^2 \cdot \nu(A) \leq 1$ for some $N \in {}^*\mathbf{N} - \mathbf{N}$. Hence we get $N \cdot \nu(A) \simeq 0$. Here, we have

$$\sum_A \varepsilon |\varphi| \leq \sum_{\{x \in A \mid |\varphi(x)| \geq N\}} \varepsilon |\varphi| + \sum_{\{x \in A \mid |\varphi(x)| < N\}} \varepsilon |\varphi| \leq \sum_{A(\varphi, K, N)} \varepsilon |\varphi| + N \cdot \nu(A).$$

The first term $\simeq 0$ by the hypothesis, and the second term $\simeq 0$ by what we have just shown above. Hence, we get $\sum_A \varepsilon |\varphi| \simeq 0$.

(b) \rightarrow (a). Let K be a compact subset of Ω , and $N \in {}^*\mathbf{N} - \mathbf{N}$. Put $A = A(\varphi, K, N) = \{x \in {}^*K \cap X \mid |\varphi(x)| \geq N\}$. We have $A \in \mathcal{A}_\Omega$, $A \subseteq {}^*K \cap X$ and the inequality

$$N \cdot \nu(A) \leq \sum_A \varepsilon |\varphi| \leq \sum_{{}^*K \cap X} \varepsilon |\varphi|$$

holds. But the right hand side is finite for $\varphi \in M(\Omega)$. So we have $\nu(A) \simeq 0$. This and the latter half of (b) yields that $\sum_A \varepsilon |\varphi| \simeq 0$. \square

The lemma below will also be used later.

Now, we write $\text{Ns}({}^*\mathbf{C}) = \{\alpha \in {}^*\mathbf{C} \mid \alpha \text{ is finite}\}$. This set is a commutative ring.

Lemma 2. Let $Y(\Omega)$ be an $\text{Ns}({}^*\mathbf{C})$ -submodule of $R(X)$, $T \in \mathcal{D}'(\Omega)$, and $(f_i)_{i \in \mathbf{N}}$ a partition of unity on Ω . Suppose that, for each $i \in \mathbf{N}$, there corresponds a function $\psi_i \in Y(\Omega)$ such that $P_{\psi_i} = f_i T$ and that ψ_i is 0 on ${}^*K \cap X$ provided K is a compact subset of Ω and $K \cap \text{supp}(f_i) = \emptyset$. Put $\varphi_n = \sum_{i=1}^n \psi_i$ for $n \in \mathbf{N}$. The map from \mathbf{N} to $Y(\Omega)$: $n \mapsto \varphi_n$ extends to an internal map from ${}^*\mathbf{N}$ to $R(X)$: $n \mapsto \varphi_n$.

In the situation above, there exists an integer $N \in {}^*\mathbf{N}$ such that the following conditions hold:

(a) In case $N \in \mathbf{N}$, then $\varphi_N \in Y(\Omega)$ and $P_{\varphi_N} = T$;

(b) in case $N \in {}^*\mathbf{N} - \mathbf{N}$, then $\varphi_N \in A(\Omega)$, $P_{\varphi_N} = T$

and, for every compact subset $K \subseteq \Omega$, there exists an $n \in \mathbf{N}$ such that $\varphi_N = \varphi_n$ on ${}^*K \cap X$.

Proof. For each compact subset K of Ω , choose $n(K) \in \mathbf{N}$ so that $i > n(K)$ implies $\text{supp}(f_i) \cap K = \emptyset$, which yields that ψ_i takes 0 on ${}^*K \cap X$. Then, $n \geq n(K)$ implies that $\sum_{i=1}^n f_i$ takes 1 on K .

For each $f \in \mathcal{D}(\Omega)$, put $n(f) = n(\text{supp}(f))$ and consider the following internal set:

$$I(f) = \{n \in {}^*\mathbf{N} \mid n \geq n(f) \wedge \forall l \in {}^*\mathbf{N} (n(f) \leq l \leq n \rightarrow |\sum_x \varepsilon \varphi_l * f - T(f)| \leq \frac{1}{l+1})\}.$$

If $n \in N$ and $n \geq n(f)$, then a natural number $l \in {}^*N$ satisfying $n(f) \leq l \leq n$ turns out to be an element of N , and since $l \geq n(f)$, reminding that $\sum_{i=0}^n f_i$ takes 1 on $\text{supp}(f)$, we get

$$\sum_X \varepsilon \varphi_l * f = \sum_{i=0}^l \sum_X \varepsilon \psi_i * f \simeq \sum_{i=0}^l (f_i T)(f) = T((\sum_{i=0}^l f_i)f) = T(f).$$

Now, we have $\{n \in N \mid n \geq n(f)\} \subseteq I(f)$. Hence, by Permanence Principle, there is $N(f) \in {}^*N - N$ such that $\{n \in {}^*N \mid n(f) \leq n \leq N(f)\} \subseteq I(f)$. Thus, the family of internal sets $\{I(f) \mid f \in \mathcal{D}(\Omega)\}$ has the finite intersection property and we have $\cap \{I(f) \mid f \in \mathcal{D}(\Omega)\} \neq \emptyset$ by saturation principle. Take $N \in \cap \{I(f) \mid f \in \mathcal{D}(\Omega)\}$. Then N belongs to *N and the following holds:

For any $f \in \mathcal{D}(\Omega)$, $N \geq n(f)$ and for every $l \in {}^*N$,

$$n(f) \leq l \leq N \quad \text{implies} \quad \left| \sum_X \varepsilon \varphi_l * f - T(f) \right| \leq \frac{1}{l+1}.$$

If $N \in N$, then $\varphi_N \in Y(\Omega)$ and since $N \geq n(f)$ for every $f \in \mathcal{D}(\Omega)$, we have

$$P_{\varphi_N}(f) = \sum_{i=0}^N P_{\psi_i}(f) = \sum_{i=0}^N (f_i T)(f) = T((\sum_{i=0}^N f_i)f) = T(f).$$

If $N \in {}^*N - N$, then for every $L \in {}^*N - N$ satisfying $L \leq N$, we have

$$\left| \sum_X \varepsilon \varphi_L * f - T(f) \right| \leq \frac{1}{L+1} \simeq 0$$

by what we have shown above and by the fact that $n(f) \leq L$ for any $f \in \mathcal{D}(\Omega)$. Thus we get $\varphi_L \in A(\Omega)$ and $P_{\varphi_L} = T$. We fix this N for a while.

Now, for a compact subset K of Ω , we put

$$J(K) = \{n \in {}^*N \mid n \geq 1, \quad \varphi_{n-1} = \varphi_n \quad \text{on} \quad {}^*K \cap X\}.$$

If $n \in N$ and $n \geq n(K)$, then $\text{supp}(f_n) \cap K = \emptyset$ and thus ψ_n takes 0 on ${}^*K \cap X$ and so, we have $\varphi_n = \varphi_{n-1} + \psi_n = \varphi_{n-1}$ on ${}^*K \cap X$. Hence,

$$\{n \in N \mid n > n(K)\} \subseteq J(K),$$

With this, for each compact subset K of Ω , there exists $N(K) \in {}^*N - N$ such that

$$\{n \in {}^*N \mid {}^*n(K) < n \leq N(K)\} \subseteq J(K).$$

Moreover, we can show that, there is $M \in {}^*N - N$ such that for any compact subset K of Ω and for any $n \in {}^*N$, $n(K) < n < M$ implies $\varphi_n = \varphi_{n-1}$ on ${}^*K \cap X$. To show this, choose a fundamental sequence of compact sets $(K_j)_{j \in N}$ for Ω and choose $M \in {}^*N - N$ so that $M \leq N(K_j)$ for every $j \in N$. Now, using N we have fixed above, consider the number $\text{Min}(M, N)$ and rename it N . Then we

have $\varphi_N \in A(\Omega)$, $P_{\varphi_N} = T$, and $\varphi_N = \varphi_{n(K)}$ on $*K \cap X$ for every compact subset K of Ω . \square

Theorem 4. (a) For each $\varphi \in E(\Omega)$, there is $h \in \mathcal{L}_{1,\text{loc}}(\Omega)$ such that $P_\varphi = T_h$, where T_h denotes the distribution determined by h .

(b) For each $h \in \mathcal{L}_{1,\text{loc}}(\Omega)$, there is $\varphi \in E(\Omega)$ such that $P_\varphi = T_h$.

Proof. (a) Let $\varphi \in E(\Omega)$ and $\varphi \geq 0$. We shall show that for any $g \in \mathcal{K}_+(\Omega)$ and $e > 0$, there exists $d > 0$ such that we have, for every $f \in \mathcal{K}_+(\Omega)$, $P_\varphi(f) \leq e$ provided $f \leq g$ and $\int_\Omega f d\mu \leq d$, where $d\mu$ is the Lebesgue measure on Ω . Then, P_φ will turn to be a measure on Ω with base μ ; that is, there is $h \in \mathcal{L}_{1,\text{loc}}(\Omega)$ and such that we have $P_\varphi(f) = \int_\Omega h f d\mu$ for every $f \in \mathcal{K}(\Omega)$ ([2], Chap. 5, §5, n° 5, Cor. 5).

So, let $K = \text{supp}(g)$ and choose $c \geq 0$ so that $c \cdot \sup g \leq e/2$. Since $*N - N \subseteq \{n \in *N \mid \sum_{A(\varphi, K, n)} \varepsilon \varphi \leq c\}$, there is $\iota \in N$ such that $\sum_{A(\varphi, K, n)} \varepsilon \varphi \leq c$. With this n , choose $d > 0$ so that $nd \leq e/3$. For g and d above, take $f \in \mathcal{K}_+(\Omega)$ such that $f \leq g$ and $\int_\Omega f d\mu \leq d$. We show that $P_\varphi(f) \leq e$. We get

$$\begin{aligned} \sum_x \varepsilon \varphi^* f &= \sum_{*K \cap X} \varepsilon \varphi^* f = \sum_{\{x \in *K \cap X \mid \varphi(x) < n\}} \varepsilon \varphi^* f + \sum_{A(\varphi, K, n)} \varepsilon \varphi^* f \\ &\leq n \sum_{*K \cap X} \varepsilon^* f + \sup f \cdot \sum_{A(\varphi, K, n)} \varepsilon^* f, \end{aligned}$$

but in the right hand side of the inequality,

$$\text{the first term} \simeq n \int_K f d\mu \leq nd \leq \frac{e}{3},$$

thus,

$$\text{the first term} \leq \frac{e}{2},$$

and

$$\text{the second term} \leq (\sup f) \cdot c \leq (\sup g) \cdot c \leq \frac{e}{2}.$$

Hence, $\sum_x \varepsilon \varphi^* f \leq \frac{e}{2} + \frac{e}{2} = e$ and immediately we get $P_\varphi(f) \leq e$.

(b) Let $h \in \mathcal{L}_{1,\text{loc}}(\Omega)$. Let $(f_i)_{i \in N}$ be a partition of unity such that each $\text{supp}(f_i)$ (we name it K_i) is a Jordan measurable compact set. Since $f_i h: K_i \rightarrow \mathbb{C}$ is μ -integrable, $(f_i h) \circ \text{st}: *K_i \cap X \rightarrow \mathbb{C}$ is Loeb integrable. Hence, for each i , there is an S -integrable function $\psi_i: *K_i \cap X \rightarrow \mathbb{C}$ such that ${}^\circ \psi_i = (f_i h) \circ \text{st}$ (ν_L -almost everywhere) on $*K \cap X$. Extend ψ_i so that it takes 0 on $X - *K_i \cap X$ and also denote it by ψ_i . We have $\psi_i \in R(X)$ and $\psi_i \in E(\Omega)$. We shall show that $P_{\psi_i} = f_i T_h$: For each $g \in \mathcal{D}(\Omega)$, we have

$$\begin{aligned}
\sum_X \varepsilon \psi_i * g &= \sum_{*K_i \cap X} \varepsilon \psi_i * g = \int_{*K_i \cap X} {}^\circ \psi_i \circ * g d\nu_L \\
&= \int_{*K_i \cap X} ((f_i h) \circ \text{st})(g \circ \text{st}) d\nu_L \\
&= \int_{\text{st}^{-1}(K_i)} (f_i h g) \circ \text{st} d\nu_L \\
&= \int_{K_i} f_i h g d\mu = (f_i T_h)(g).
\end{aligned}$$

Here, we used the fact that Jordan measurable compact set K_i satisfies the equation $\nu_L(\text{st}^{-1}(K_i) - *K_i \cap X) = 0$. Moreover, for a compact subset K of Ω such that $K \cap K_i = \emptyset$, ψ_i takes 0 on $*K \cap X$ by our definition of ψ_i . Now, by applying Lemma 2 to the case $Y(\Omega) = E(\Omega)$, we get an internal function $*N \ni n \mapsto \varphi_n \in R(X)$ and a natural number $N \in *N$ such that $\varphi_n = \sum_{i=1}^n \psi_i$ for each $n \in N$ and satisfy the following conditions:

- (1) $N \in N$ implies $\varphi_N \in E(\Omega)$ and $P_{\varphi_N} = T_h$.
- (2) $N \in *N - N$ implies that $\varphi_N \in A(\Omega)$, $P_{\varphi_N} = T_h$ and that, for each compact subset K of Ω , there is a suitable $n \in N$ such that $\varphi_N = \varphi_n$ on $*K \cap X$.

In the case (1), the proof is done. In the case (2), for each compact subset K of Ω and for each $M \in *N - N$, we have $A(K, \varphi_N, M) = A(K, \varphi_n, M)$ and we get $\varphi_N \in E(\Omega)$ by the following:

$$\sum_{A(K, \varphi_N, M)} \varepsilon |\varphi_N| = \sum_{A(K, \varphi_n, M)} \varepsilon |\varphi_n| = 0. \quad \square$$

Proposition 2. *Let $\varphi \in E(\Omega)$, $\varphi \geq 0$, and $h \in \mathcal{L}_{1, \text{loc}}(\Omega)$, $h \geq 0$. Then, the following two conditions are mutually equivalent*

- (a) $P_\varphi = T_h$.
- (b) ${}^\circ \varphi = h \circ \text{st}$ a.e. on $Ns(*\Omega) \cap X$.

Proof. (a) \rightarrow (b). Let $f \in \mathcal{K}(\Omega)$ and C be a compact and Jordan measurable subset of Ω with $\text{supp}(f) \subseteq C$. Then, $\text{st}^{-1}(C) \supseteq *C \cap X$ because C is compact, and

$$\nu_L(\text{st}^{-1}(C)) = \nu_L(*C \cap X)$$

because C is Jordan measurable. We have then

$$\begin{aligned}
\int_{\text{st}^{-1}(C)} {}^\circ \varphi (f \circ \text{st}) d\nu_L &= \int_{*C \cap X} {}^\circ (\varphi^* f) d\nu_L = \int_X {}^\circ (\varphi^* f) d\nu_L \\
&= \int_X \varepsilon \varphi^* f = \int_\Omega h f d\mu \quad (\text{by assumption}) = \int_C h f d\mu \\
&= \int_{\text{st}^{-1}(C)} (h \circ \text{st})(f \circ \text{st}) d\nu_L.
\end{aligned}$$

For every compact subset K of Ω , there exists a sequence of functions $f_n \in \mathcal{K}(\Omega)$ ($n \in \mathbb{N}$) such that

- (a) $f_n \downarrow 1_K$,
- (2) $\text{supp}(f_n) \subseteq C$ for some fixed compact and Jordan measurable subset C of Ω .

By the remark above, we have

$$\int_{\text{st}^{-1}(C)} {}^\circ \varphi(f_n \circ \text{st}) d\nu_L = \int_{\text{st}^{-1}(C)} (h \circ \text{st})(f_n \circ \text{st}) d\nu_L$$

and hence,

$$\int_{\text{st}^{-1}(K)} {}^\circ \varphi d\nu_L = \int_{\text{st}^{-1}(K)} (h \circ \text{st}) d\nu_L.$$

The positivity of the integrand implies

$${}^\circ \varphi = h \circ \text{st} \quad \text{a.e. on } \text{st}^{-1}(K).$$

Take a sequence of compact sets K_n such that $\Omega = \bigcup_{n \in \mathbb{N}} K_n$. Then we have $\text{Ns}(*\Omega) \cap X = \bigcup_{n \in \mathbb{N}} \text{st}^{-1}(K_n)$ and therefore

$${}^\circ \varphi = h \circ \text{st} \quad \text{a.e. on } \text{Ns}(*\Omega) \cap X.$$

(b) \rightarrow (a). Let $f \in \mathcal{K}(\Omega)$ and C be a compact and Jordan measurable subset of Ω with $\text{supp}(f) \subseteq C$. Then we have

$$\begin{aligned} {}^\circ \sum_X \varepsilon \varphi * f &= {}^\circ \sum_{*C \cap X} \varepsilon \varphi * f = \int_{*C \cap X} {}^\circ \varphi \circ * f d\nu_L \\ &= \int_{\text{st}^{-1}(C)} {}^\circ \varphi(f \circ \text{st}) d\nu_L = \int_{\text{st}^{-1}(C)} (h \circ \text{st})(f \circ \text{st}) d\nu_L \\ &= \int_C h f d\mu = \int_\Omega h f d\mu. \end{aligned} \quad \square$$

DEFINITION. Recall that $M_1(\Omega)$ is the set of internal functions φ on X such that $\sum_{* \Omega \cap X} \varepsilon |\varphi|$ is finite, and that $E(\Omega)$ is the set of internal functions on X which are locally S -integrable on Ω . Put $E_1(\Omega) = E(\Omega) \cap M_1(\Omega)$, and, for every $p \geq 1$ in \mathbf{R} , put

$$E_p(\Omega) = \{\varphi \in R(X) \mid |\varphi|^p \in E_1(\Omega)\}.$$

In case $p=1$, two definitions of $E_1(\Omega)$ coincide.

Lemma 3. Let $\varphi \in E(\Omega)$ and $\varphi \geq 0$. Then we have $\varphi^{1/p} \in E(\Omega)$ for every $p \geq 1$ in \mathbf{R} .

Proof. Recall that an internal function φ is said to be locally S -integrable on Ω if we have $\sum_{A(\varphi, K, N)} \varepsilon |\varphi| \simeq 0$ for every compact subset K of Ω and for every infinite natural number N , where $A(\varphi, K, N)$ is the internal set of all

$x \in {}^*K \cap X$ such that $|\varphi(x)| \geq N$.

Now let $\varphi \in E(\Omega)$, $\varphi \geq 0$ and $K \subset \Omega$ compact and N infinite. Then, $A(\varphi^{1/p}, K, N) = A(\varphi, K, N^p)$ for every $p \geq 1$. Since $|\varphi(x)|^{1/p} \leq |\varphi(x)|$ for $x \in A(\varphi, K, N^p)$, we have

$$\sum_{A(\varphi^{1/p}, K, N)} \varepsilon \varphi^{1/p} = \sum_{A(\varphi, K, N^p)} \varepsilon \varphi^{1/p} \leq \sum_{A(\varphi, K, N^p)} \varepsilon \varphi = 0. \quad \square$$

Proposition 3. For every $p \geq 1$ in \mathbf{R} , we have $E_p(\Omega) \subseteq E(\Omega)$.

Proof. Let $\varphi \in E_p(\Omega)$. The definition of $E_p(\Omega)$ gives $|\varphi|^p \in E_1(\Omega) \subseteq E(\Omega)$, hence $|\varphi| \in E(\Omega)$ by the above Lemma, so we have $\varphi \in E(\Omega)$. \square

Theorem 5. Let $p \geq 1$ in \mathbf{R} . Recall that $\mathcal{L}_p(\Omega)$ is the set of measurable functions φ on Ω such that $|\varphi|^p$ is integrable on Ω . Then we have

- (a) For every $\varphi \in E_p(\Omega)$, there exists an $h \in \mathcal{L}_p(\Omega)$ such that $P_\varphi = T_h$.
- (b) For every $h \in \mathcal{L}_p(\Omega)$, there exists a $\varphi \in E_p(\Omega)$ such that $P_\varphi = T_h$.

Proof. (a) Suppose first $p=1$. For every $\varphi \in E_1(\Omega) \subseteq E(\Omega)$, there exists an $h \in \mathcal{L}_{1, \text{loc}}(\Omega)$ such that $P_\varphi = T_h$. Since $\varphi \in E_1(\Omega) \subseteq M_1(\Omega)$ we have $T_h = P_\varphi \in \mathcal{M}_1(\Omega)$, that is, $T_h = P_\varphi$ is a bounded measure. On the other hand, Bourbaki's "Integration" [2] Chap B §5. 5, $n^\circ 4$, Theorem 1, Corollary says that, for every $h \in \mathcal{L}_{1, \text{loc}}(\Omega)$, T_h is a bounded measure if and only if $h \in \mathcal{L}_1(\Omega)$. Applying this to our case, there exists an $h \in \mathcal{L}_1(\Omega)$ such that $P_\varphi = T_h$.

Suppose next $p > 1$ and $\varphi \in E_p(\Omega)$, $\varphi \geq 0$. Then $\varphi^p \in E_1(\Omega)$. By the result for $p=1$, there exists a $g \in \mathcal{L}_1(\Omega)$, $g \geq 0$ such that $P_{\varphi^p} = T_g$. Proposition 2 implies

$${}^\circ \varphi^p = g \circ \text{st} \quad \text{a.e. on } \text{Ns}(\Omega^*) \cap X.$$

Putting $h = g^{1/p}$, we have $h \in \mathcal{L}_p(\Omega)$ and ${}^\circ \varphi^p = h^p \circ \text{st}$ a.e. on $\text{Ns}(\Omega^*) \cap X$. Hence ${}^\circ \varphi = h \circ \text{st}$ a.e. on $\text{Ns}(\Omega^*) \cap X$ and we have $P_\varphi = T_h$ by Proposition 2.

If φ is not positive, the result follows from the decomposition $\varphi = \varphi_1 - \varphi_2 + i\varphi_3 - i\varphi_4$, $\varphi_1, \varphi_2, \varphi_3, \varphi_4$ being positive.

(b). Let $h \in \mathcal{L}_p(\Omega)$, $h \geq 0$. Then $h^p \in \mathcal{L}_1(\Omega)$. We extend the function $h^p \circ \text{st}$ on $\text{Ns}(X)$ to whole X by giving the value 0 outside $\text{Ns}(X)$, which we write $h^p \circ \text{st}$. Then this function is positive and ν_L -measurable, and we have

$$\int_X h^p \circ \text{st} d\nu_L = \int_{\text{Ns}(\Omega^*) \cap X} h^p \circ \text{st} d\nu_L = \int_\Omega h^p d\mu < \infty.$$

The theory of Loeb integration assures us the existence of an S -integrable function $\psi \geq 0$ in $R(X)$ such that

$${}^\circ \psi = h^p \circ \text{st} \quad \text{a.e. on } X.$$

We then have $\psi \in E_1(\Omega)$, because

$$\sum_{*\Omega \cap X} \varepsilon \psi \leq \sum_X \varepsilon \psi \simeq \int_X {}^\circ \psi d\nu_L = \int_X h^b \circ \text{st} d\nu_L = \int_\Omega h^b d\mu \leq \infty.$$

Putting $\varphi = \psi^{1/p}$, Lemma 3 implies $\varphi \in E(\Omega)$. Moreover we have $\varphi^b = \psi \in E_1(\Omega)$, hence $\varphi \in E_p(\Omega)$. On the other hand, we have

$${}^\circ \varphi^b = h^b \circ \text{st} \quad \text{a.e. on } \text{Ns}(*\Omega) \cap X$$

and hence the equality

$${}^\circ \varphi = h \circ \text{st} \quad \text{a.e. on } \text{Ns}(*\Omega) \cap X,$$

which implies $P_\varphi = T_h$ by Proposition 2. \square

DEFINITION. (1) After A. Robinson [8], we call a function $\varphi \in R(X)$ *S-continuous* on Ω if $\varphi(x) \simeq \varphi(y)$ whenever $x, y \in \text{Ns}(*\Omega) \cap X$ and $x \simeq y$.

(2) Let $S(\Omega)$ be the set of functions $\varphi \in R(X)$ which are finite-valued and *S-continuous* on Ω .

(3) For each $\varphi \in S(\Omega)$, define the function ${}^\vee \varphi: \Omega \rightarrow \mathcal{C}$ by ${}^\vee \varphi(t) = {}^\circ \varphi({}^\wedge t)$ for $t \in \Omega$ (recall that ${}^\wedge t \in \text{Ns}(*\Omega) \cap X$, ${}^\wedge t \leq t < {}^\wedge t + \varepsilon$).

The property (a) in the following theorem is due to P. Loeb ([15]), and other parts can be easily deduced from theories and definitions by Loeb.

Theorem 6. (a) $\varphi \in S(\Omega)$ implies ${}^\vee \varphi \in \mathcal{C}(\Omega)$, that is, ${}^\vee \varphi$ is a continuous function on Ω .

(b) If $h \in \mathcal{C}(\Omega)$ (by the convention that h is extended so that it takes value 0 outside Ω , we have $*h: *R \rightarrow *C$ and $*h(x) = 0$ for $x \in *R - *\Omega$), then $*h|X \in S(\Omega)$ and ${}^\vee(*h|X) = h$.

(c) $S(\Omega) \subseteq E(\Omega)$ and $\varphi \in S(\Omega)$ implies $P_\varphi = T_{{}^\vee \varphi}$.

4. Distributions

Proposition 4. For each $\varphi \in R(X)$, the following two conditions are equivalent:

(a) For any compact subset K of Ω , there is $m \in \mathbb{N}$ such that $\sum_{*K \cap X} \varepsilon^{m+1} |\varphi|$ is finite.

(b) For any compact subset K of Ω , there is $k \in \mathbb{N}$ such that $\sum_{*K \cap X} \varepsilon^{k+1} |\varphi|^2$ is finite.

Proof. (a) \rightarrow (b). Let K be a compact subset of Ω , $m \in \mathbb{N}$, and $\sum_{*K \cap X} \varepsilon^{m+1} |\varphi|$ finite. We have

$$\sum_{*K \cap X} \varepsilon^{(2m+1)+1} |\varphi|^2 \leq \left(\sum_{*K \cap X} \varepsilon^{m+1} |\varphi| \right)^2$$

and the right hand side of the inequality is finite. Hence we get (b) with

$k=2m+1$.

(b)→(a). Let K be a compact subset of Ω , $k \in \mathbb{N}$, and $\sum_{*K \cap X} \varepsilon^{k+1} |\varphi|^2$ finite. Choose $m \in \mathbb{N}$ such that $k \leq 2m$. We have

$$(\sum_{*K \cap X} \varepsilon^{m+1} |\varphi|)^2 \leq \nu_L(*K \cap X) \cdot H \cdot \sum_{*K \cap X} \varepsilon^{2m+2} |\varphi|^2$$

and

$$H \cdot \sum_{*K \cap X} \varepsilon^{2m+2} |\varphi|^2 = \sum_{*K \cap X} \varepsilon^{2m+1} |\varphi|^2 \leq \sum_{*K \cap X} \varepsilon^{k+1} |\varphi|^2.$$

As the right hand side of the second inequality is finite, we get (a). \square

DEFINITION. $Z(\Omega)$ denotes the set of all $\varphi \in R(X)$ which satisfies the condition (a) in Proposition 4.

Immediately, we have $M(\Omega) \subseteq Z(\Omega)$.

DEFINITION. For each $\varphi \in R(X)$, we define $D_+\varphi$ and $D_-\varphi$ as follows :

$$D_+\varphi(x) = \frac{\varphi(x+\varepsilon) - \varphi(x)}{\varepsilon} \quad \text{and} \quad D_-\varphi(x) = \frac{\varphi(x) - \varphi(x-\varepsilon)}{\varepsilon}.$$

(Note that we extend $\varphi: X \rightarrow *C$ to $\varphi: L \rightarrow *C$ to have the period H .)

Proposition 5. (a) $Z(\Omega)$ is stable under D_+ and D_- .

(b) If $\varphi, \psi \in Z(\Omega)$, then $\varphi\psi \in Z(\Omega)$.

Proof. (a) Let $\varphi \in Z(\Omega)$ and K be a compact subset of Ω . Choose a compact subset K_1 of Ω so that $K \subseteq K_1 \subseteq \Omega$ and:

If $x \in X$, then $x \in *K$ implies $x \pm \varepsilon \in *K_1$.

By choosing $m \in \mathbb{N}$ so that $\sum_{*K_1 \cap X} \varepsilon^{m+1} |\varphi|$ is finite, we have

$$\sum_{*K \cap X} \varepsilon^{m+2} |D_{\pm}\varphi| \leq \sum_{*K \cap X} \varepsilon^{m+1} |\varphi(x \pm \varepsilon)| + \sum_{*K \cap X} \varepsilon^{m+1} |\varphi|.$$

Both terms in the right hand side turn out to be less than or equal to $\sum_{*K \cap X} \varepsilon^{m+1} |\varphi|$ and thus, $\sum_{*K \cap X} \varepsilon^{(m+1)+1} |D_{\pm}\varphi|$ is finite.

(b) Let K be a compact subset of Ω and choose $k, l \in \mathbb{N}$ so that both $\sum_{*K \cap X} \varepsilon^{k+1} |\varphi|^2$ and $\sum_{*K \cap X} \varepsilon^{l+1} |\psi|^2$ are finite (Proposition 4). By choosing $m, n \in \mathbb{N}$ such that $k \leq 2m+1$ and $l \leq 2n+1$, we get

$$\begin{aligned} (\sum_{*K \cap X} \varepsilon^{m+n+2} |\varphi\psi|)^2 &\leq \sum_{*K \cap X} \varepsilon^{2m+2} |\varphi|^2 \cdot \sum_{*K \cap X} \varepsilon^{2n+2} |\psi|^2 \\ &\leq \sum_{*K \cap X} \varepsilon^{k+2} |\varphi|^2 \cdot \sum_{*K \cap X} \varepsilon^{l+2} |\psi|^2. \end{aligned} \quad \square$$

Proposition 6. Let $\varphi \in A(\Omega) \cap Z(\Omega)$ and $h \in \mathcal{E}(\Omega)$. Then we have $D_{\pm}\varphi$, $*h\varphi \in A(\Omega) \cap Z(\Omega)$ and if $f \in \mathcal{D}(\Omega)$, then $P_{D_{\pm}\varphi}(f) = -P_{\varphi}(f')$ and $P_{*h\varphi}(f) = P_{\varphi}(hf)$.

Here, $\mathcal{E}(\Omega)$ denotes the set of \mathbf{C} -valued, indefinitely differentiable functions on Ω , and its elements are assumed to be extended to whole \mathbf{R} so that they take 0 outside Ω . We have written simply $*h$ for $*h|X$. f' denotes the derived function of f .

Proof. (i) We have $D_{\pm}\varphi \in Z(\Omega)$ by Proposition 5. Now we show that $D_{\pm}\varphi \in A(\Omega)$ and $P_{D_{\pm}\varphi} = -P_{\varphi}(f')$ as follows. Suppose $f \in \mathcal{D}(\Omega)$ and $K = \text{supp}(f)$. Choose a compact set K_1 satisfying $K \subseteq K_1 \subseteq \Omega$ so that, for each $x \in X$, $x \in *K$ implies $x \pm \varepsilon \in *K_1$, and then choose $m \in \mathbf{N}$ so that $\sum_{*K_1 \cap X} \varepsilon^{m+1} |\varphi|$ is finite. With signs in the respective order, we have

$$\begin{aligned} \sum_X \varepsilon (D_{\pm}\varphi) * f &= \pm \sum_{x \in X} \varphi(x \pm \varepsilon) * f(x) \mp \sum_X \varphi * f \\ &= \pm \sum_{x \in X} \varphi(x) * f(x \mp \varepsilon) \mp \sum_{x \in X} \varphi(x) * f(x) \\ &= - \sum_{x \in X} \varepsilon \varphi(x) = \frac{*f(x \mp \varepsilon) - *f(x)}{\mp \varepsilon} \\ &= - \left\{ \sum_{k=1}^{m+1} \frac{(\mp 1)^{k-1}}{k!} \varepsilon^{k-1} \sum_X \varepsilon \varphi * f^{(k)} \right. \\ &\quad \left. + \frac{(\mp 1)^{m+1} \varepsilon}{(m+2)!} \sum_{x \in X} \varepsilon^{m+1} \varphi(x) (*\text{Re } f^{(m+2)}(x \mp \sigma \varepsilon) + i * \text{Im } f^{(m+2)}(x \mp \tau \varepsilon)) \right\} \\ (\sigma, \tau \in *R, 0 < \sigma, \tau < 1). \end{aligned}$$

As for the sums in the scope of negative sign, we have

$$\text{the first sum} \simeq \sum_{k=1}^{m+1} \frac{(\mp 1)^{k-1}}{k!} \varepsilon^{k-1} P_{\varphi}(f^{(k)}) \simeq P_{\varphi}(f'),$$

and

$$\text{the second sum} \leq \frac{\varepsilon}{(m+2)!} \sum_{*K_1 \cap X} \varepsilon^{m+1} |\varphi| \cdot 2 \sup |f^{(m+2)}| \simeq 0.$$

Hence, $D_{\pm}\varphi \in A(\Omega)$ and $P_{D_{\pm}\varphi}(f) = -P_{\varphi}(f')$ for each $f \in \mathcal{D}(\Omega)$.

(ii) If $h \in \mathcal{E}(\Omega)$, then $*h \in S(\Omega) \subseteq E(\Omega)$ by Theorem 6, and we have $E(\Omega) \subseteq M(\Omega) \subseteq Z(\Omega)$ by Proposition 1 and definitions. Hence, $*h\varphi \in Z(\Omega)$ for $\varphi \in Z(\Omega)$ by Proposition 5. Now, since $hf \in \mathcal{D}(\Omega)$ for $f \in \mathcal{D}(\Omega)$, we have

$$\sum_X \varepsilon *h\varphi * f = \sum_X \varepsilon \varphi * (hf) \simeq P_{\varphi}(hf).$$

Thus we get $*h\varphi \in A(\Omega)$ and $P_{*h\varphi}(f) = P_{\varphi}(hf)$. □

DEFINITION. $D_F(\Omega)$ denotes the smallest subset of $R(X)$ which includes $M(\Omega)$ and closed under applications of D_+ and D_- , multiplication of $*h$ for each $h \in \mathcal{E}(\Omega)$, and addition.

Theorem 7. (a) $D_F(\Omega) \subseteq A(\Omega) \cap Z(\Omega)$. If $\varphi \in D_F(\Omega)$ and $h \in \mathcal{E}(\Omega)$, then $D_{\pm}\varphi, *h\varphi \in D_F(\Omega)$ and $P_{\varphi} \in \mathcal{D}'_F(\Omega)$, $P_{D_{\pm}} = (P_{\varphi})'$ and $P_{*h\varphi} = hP_{\varphi}$.

(b) Every $T \in \mathcal{D}'_F(\Omega)$ can be represented in the form $T = P_\varphi$ for some $\varphi \in D_F(\Omega)$.

Proof. (a) Since $M(\Omega) \subseteq A(\Omega) \cap Z(\Omega)$ and $A(\Omega) \cap Z(\Omega)$ is stable under D_+ , D_- , and $*h$, we have $D_F(\Omega) \subseteq A(\Omega) \cap Z(\Omega)$.

Now, if $\varphi \in M(\Omega)$, then we have $P_\varphi \in \mathcal{D}'^{(0)}(\Omega) \subseteq \mathcal{D}'_F(\Omega)$, and $\mathcal{D}'_F(\Omega)$ is stable under derivation and multiplication of h . On the other hand, by Proposition 6 we have

$$P_{D_\pm \varphi}(f) = -P_\varphi(f'), \quad \text{and} \quad P_{*h\varphi}(f) = P_\varphi(hf),$$

and thus we can prove that, for each $\varphi \in D_F(\Omega)$, $P_\varphi \in \mathcal{D}'_F(\Omega)$ and

$$P_{D_\pm \varphi} = (P_\varphi)' \quad \text{and} \quad P_{*h\varphi} = hP_\varphi$$

by induction on the number of operations of D_+ , D_- and $*h$ to an element of $M(\Omega)$.

(b) For each $T \in \mathcal{D}'_F(\Omega)$, we can represent it in the form $T = S^{(k)}$ for some $S \in \mathcal{D}'^{(0)}(\Omega)$ and $k \in \mathbb{N}$. Representing S in the form $S = P_\psi$ with $\psi \in M(\Omega)$, we have $D_+^k \psi \in D_F(\Omega)$ and $P_{D_+^k \psi} = (P_\psi)^{(k)} = T$. \square

DEFINITION. Let $D(\Omega)$ denote the set of elements $\varphi \in R(X)$ such that, for each compact subset K of Ω , there is some $\psi \in D_F(\Omega)$ which satisfies $\varphi = \psi$ on $*K \cap X$.

REMARK. $D(\Omega) \subseteq A(\Omega) \cap Z(\Omega)$.

Theorem 8. (a) If $\varphi \in D(\Omega)$, then $P_\varphi \in \mathcal{D}'(\Omega)$.

(b) If $\varphi \in D(\Omega)$, then $D_\pm \varphi \in D(\Omega)$ and $P_{D_\pm \varphi} = (P_\varphi)'$.

(c) If $\varphi \in D(\Omega)$ and $h \in \mathcal{E}(\Omega)$, then $*h\varphi \in D(\Omega)$ and $P_{*h\varphi} = hP_\varphi$.

(d) If $T \in \mathcal{D}'(\Omega)$, then there is some $\varphi \in D(\Omega)$ such that $P_\varphi = T$.

Proof. (a) Suppose $\varphi \in D(\Omega)$. Let $(f_j)_{j \in \mathbb{N}}$ be a sequence in $\mathcal{D}(\Omega, K)$ such that $f_j \rightarrow 0$ in $\mathcal{D}(\Omega, K)$. Choose $\psi \in D_F(\Omega)$ corresponding to K such that $\varphi = \psi$ on $*K \cap X$. Then by Theorem 7, we have $P_\psi \in \mathcal{D}'_F(\Omega)$ and thus $P_\psi(f_j) \rightarrow 0$. On the other hand, we have $P_\varphi(f_j) = P_\psi(f_j)$ for every $j \in \mathbb{N}$, so $P_\varphi(f_j) \rightarrow 0$. Hence $P_\varphi \in \mathcal{D}'(\Omega)$.

(b) Suppose $\varphi \in D(\Omega)$. We know that $D_\pm \varphi \in A(\Omega) \cap Z(\Omega)$ by Proposition 6. For a compact subset K of Ω , choose a compact set K_1 so that $K \subseteq K_1 \subseteq \Omega$ and $x \pm \varepsilon \in *K_1$ for each $x \in *K \cap X$. Choose $\psi \in D_F(\Omega)$ so that $\varphi = \psi$ on $*K_1 \cap X$. By Theorem 7, we have $D_\pm \psi \in D_F(\Omega)$ and $D_\pm \varphi = D_\pm \psi$ on $*K \cap X$. Thus $D_\pm \varphi \in D(\Omega)$.

Now, since $D(\Omega) \subseteq A(\Omega) \cap Z(\Omega)$, we have

$$P_{D_\pm \varphi}(f) = -P_\varphi(f') = (P_\varphi)'(f)$$

for each $f \in \mathcal{D}(\Omega)$ by Proposition 6.

(c) Suppose that $\varphi \in D(\Omega)$ and $h \in \mathcal{E}(\Omega)$. We have $*h\varphi \in A(\Omega) \cap Z(\Omega)$ by Proposition 6. For each compact subset K of Ω , choose $\psi \in D_F(\Omega)$ so that $\varphi = \psi$ on $*K \cap X$. Then $*h\psi \in D_F(\Omega)$ by Theorem 6, and obviously, $*h\varphi = *h\psi$ on $*K \cap X$. Hence, $*h\varphi \in D(\Omega)$.

Also, since $D(\Omega) \subseteq A(\Omega) \cap Z(\Omega)$, we have

$$P_{*h\varphi}(f) = P_\varphi(hf) = (hP_\varphi)(f)$$

for each $f \in \mathcal{D}(\Omega)$ by Proposition 6.

(d) Suppose $T \in \mathcal{D}'(\Omega)$. Let $(f_i)_{i \in \mathbb{N}}$ be a partition of unity on Ω such that each $K_i = \text{supp}(f_i)$ is a convex compact set which has an interior point. Each $f_i T$ is a distribution on Ω with support contained in K_i . Thus, we can represent each $f_i T$ as a finite sum of derivatives of elements belonging to $\mathcal{C}(\Omega)$ with each support contained in K_i ([3], Chap. 1, corollary to Theorem 1.5). Now we shall show that there is a function $\psi_i \in D_F(\Omega)$ for each i such that $P_{\psi_i} = f_i T$ and that ψ_i is 0 on $*K \cap X$ for every compact subset K of Ω with the property $K \cap K_i = \emptyset$. For it, we can assume that $f_i T = (T_h)^{(n)}$ with $h \in \mathcal{C}(\Omega)$, $\text{supp}(h) \subseteq K_i$ and $n \geq 0$. By Theorem 6, we have

$$*h|X \in S(\Omega) \subseteq M(\Omega) \quad \text{and} \quad P_{*h|X} = T_h.$$

Clearly, $*h|X$ takes 0 outside $*K_i \cap X$. Then, $D_+^n(*h|X)$ belongs to $D_F(\Omega)$ and if you choose a compact set K such that $K \cap K_i = \emptyset$, then it takes 0 on $*K \cap X$, and moreover,

$$P_{D_+^n(*h|X)} = (P_{*h|X})^{(n)} = (T_h)^{(n)} = f_i T.$$

Thus we get the claim above.

Now, applying Lemma 2 for $Y(\Omega) = D_F(\Omega)$, we get an internal map $*N \ni n \mapsto \varphi_n \in R(X)$ and $N \in *N$ such that $n \in N$ implies $\varphi_n = \sum_{i=0}^n \psi_i \in D_F(\Omega)$ and satisfy following conditions:

- (1) $N \in N$ implies $\varphi_N \in D_F(\Omega)$ and $P_{\varphi_N} = T$;
- (2) $N \in *N - N$ implies $\varphi_N \in A(\Omega)$, $P_{\varphi_N} = T$ and for each compact subset K of Ω , with an appropriate $n \in N$, we have $\varphi_N = \varphi_n$ on $*K \cap X$.

So, let $\varphi = \varphi_N$. For the case (1), there is nothing more to prove. For the case (2), as each φ_n belongs to $D_F(\Omega)$, we have $\varphi \in D(\Omega)$ and $P_\varphi = T$, and we finish the proof. \square

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