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Geometric analysis on weighted Riemannian manifolds of Ricci curvature bounded from below

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Contents

Abstract

Geometric analysis on Riemannian manifolds under lower bounds of Ricci curvature has been generalized to weighted Riemannian manifolds. In this thesis, we review some recent developments on weighted Riemannian manifolds under lower bounds of N-weighted Ricci curvature with ε-range. Especially, we present analyses of harmonic functions, eigenfunctions, and porous medium equations. In particular, for harmonic functions, we address an L^p-Liouville type theorem (Theorem 3.2.4), a Cheng type Liouville theorem (Theorem 3.3.4), and a gradient estimate (Theorem 3.4.1). As for eigenfunctions, we give a Cheng type upper bound of the bottom spectrum (Theorem 3.5.1). Regarding the porous medium equations, we provide an Aronson-Bénilan type gradient estimate (Theorem 4.1.1). This thesis is based on [**36–38**] by the author.

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CHAPTER 1

Introduction

1.1. Lower bounds of weighted Ricci curvature

1.1.1. **Riemannian geometry and its synthetic notion.** In this thesis, we focus on the lower bounds of Ricci curvature. Under lower bounds of Ricci curvature, there are numerous researches in view of geometry and analysis. We first review some classical results (see also Peterson [**105**]). Before the 1970's, Myers [**89**] gave an upper bound of the diameter and Bochner [**8**] estimated the first Betti number. In 1971, Cheeger-Gromoll [**21**] obtained a splitting theorem. Their proof used an upper bound of Laplacian of the distance function, which is now referred to as the *Laplacian comparison* theorem. In the 1980's, Gromov [**45**] (see also Gallot [**40**]) obtained a Betti number estimate using a volume comparison property, which is now called the *Bishop-Gromov volume comparison* (see also Bishop-Crittenden [**7**]). In the beginning of 2000's, Perelman solved the Poincaré conjecture by using the Ricci flow and metric geometry (see e.g., [**102**–**104**]).

Synthetic notions of sectional curvature bounded from above and below on non-smooth spaces appeared in the theory of metric geometry, and had been deeply studied. As for the synthetic notion of Ricci curvature, one of the pioneering works was done by Cheeger-Colding (see e.g., [**17**–**20**]). They studied limit spaces of Riemannian manifolds with Ricci curvature bounded from below. Especially, Cheeger-Colding [**17**] obtained the Cheeger-Gromoll splitting theorem for this limit space. As applications, they showed the Gromov conjecture [**44**] for the limit spaces (see also Fukaya-Yamaguchi [**39**]). Recently, the Cheeger-Colding theory was essentially used to show the existence of Kähler-Einstein metrics on compact complex manifolds (see e.g., Cheng-Donaldson-Sun [**22**–**24**] and Tian [**123**, **124**]). Although Cheeger-Colding considered the limit spaces of convergent sequences of Riemannian manifolds with Ricci curvature bounded from below, we can now directly analyze such limit spaces, which are metric spaces equipped with measures, under some synthetic notion of Ricci curvature bounded from below. The optimal transport theory took an important role in formulating this synthetic notion of lower bounds of Ricci curvature on metric measure spaces.

1.1.2. **Optimal transport and weighted Ricci curvature.** To explain the synthetic notion of lower bounds of Ricci curvature on metric measure spaces, we introduce a generalization of Ricci curvature to a weighted Riemannian manifold, which is a Riemannian manifold (M, g) equipped with a weighted measure $\mathrm{e}^{-f}\,\mathrm{d}v_g$. Here, v_g is the Riemannian volume measure and $f \in C^\infty(M)$. For an n-dimensional weighted Riemannian manifold and a parameter $N \in (-\infty, 1] \cup [n, \infty]$, the *N-weighted Ricci curvature* is defined as follows:

$$\mathrm{Ric}_f^N := \mathrm{Ric}_g + \mathrm{Hess}\, f - \frac{\mathrm{d}f \otimes \mathrm{d}f}{N - n}.$$

In the unweighted case $f \equiv 0$, Ric_f^N coincides with the Ricci curvature Ric_g for (M, g). As a pioneering work, Lichnerowicz [**75**] generalized the Cheeger-Gromoll splitting theorem in the weighted case with $N = \infty$. After that, the case $N = \infty$ appeared in the analysis of linear diffusion operators by Bakry-Émery [**4**]. Hence, the weighted Ricci curvature is also called the

Bakry-Émery-Ricci curvature. The case $N = \infty$ is also meaningful in several other fields, such as Ricci flow and convex geometry. In the context of Ricci flow, if there is a potential f such that $\mathrm{Ric}_f^\infty = Kg$, the Riemannian manifold is called a gradient Ricci soliton, which is related to self-similar solutions of the Ricci flow. In addition, if we consider Euclidean space equipped with a measure $(\mathbb{R}^n, |\cdot|, \mathrm{e}^{-f}v_g)$, the condition $\mathrm{Ric}_f^\infty \geq 0$ implies that the weighted measure $\mathrm{e}^{-f}v_g$ is a log-concave measure. This class of measures appears in the field of convex geometry. Especially, the characteristic function of a convex set is log-concave. The case $N \in [n, \infty)$ was introduced by Bakry [**6**] and Qian [**108**]. In particular, a Bishop-Gromov type volume comparison theorem under $\mathrm{Ric}_f^N \geq Kg$ was obtained in [**108**]. On a weighted Riemannian manifold, if we assume

$$\mathrm{Ric}_f^N \geq Kg, \tag{1}$$

comparison geometric results similar to those of Riemannian manifolds with $\mathrm{Ric}_g \geq Kg$ and $\dim(M) \leq N$ hold true. The parameter N is called the *effective dimension*.

It turned out that lower bounds of the weighted Ricci curvature have characterizations in view of the optimal transport theory. In the 18th century, optimal transport theory was first introduced by Monge. This theory defines a distance on the set of probability measures, which is called the *Wasserstein distance*. It has many applications in various fields not only mathematics. The relation between Ricci curvature and the Wasserstein space, which is a space of probability measures equipped with the Wasserstein distance, was investigated by Otto [**101**]. Later, Otto-Villani [**100**] showed some functional inequalities, and pointed out that lower bounds of Ric_f^∞ imply the convexity of the relative entropy with respect to the measure $\mathrm{e}^{-f}v_g$ by a heuristic argument. Cordero-Erausquin-McCann-Schmuckenschläger [**29**, **30**] rigorously investigated this connection and proved that lower bounds of Ric_f^∞ imply the convexity of entropy along Wasserstein geodesics. This convexity is called the *displacement convexity*. In the weighted case $N = \infty$, the inverse implication was proved by von Renesse-Sturm [**126**]. Since the convexity of entropies can be formulated without the differentiable structure of spaces, this convexity was employed to formulate a synthetic notion of lower bounds of Ricci curvature on metric measure spaces. Indeed, Lott-Villani [**77**] and Sturm [**119**, **120**] formulated the synthetic notion of $\mathrm{Ric}_f^\infty \geq Kg$ by the convexity of the relative entropy along Wasserstein geodesics, and $\mathrm{Ric}_f^N \geq Kg$ with $N \in [n, \infty)$ by the convexity of the Rényi entropies. These conditions are called the *curvature dimension condition* $CD(K, N)$. We emphasize that this $CD(K, N)$ condition is defined even on non-smooth spaces. In particular, the stability of $CD(K, N)$ under some notions of convergences, such as the measured Gromov-Hausdorff convergence and the $\mathbb{D}$-convergence, were investigated in [**77**, **119**, **120**].

Another important application of $CD(K, N)$ is geometric analysis on Finsler manifolds (see e.g., Ohta [**92**] and Ohta-Sturm [**96**, **97**]). Later, the *Riemannian curvature dimension condition* $RCD(K, N)$ was formulated by adding some assumptions to the $CD(K, N)$ condition (see e.g., Ambrosio-Gigli-Savaré [**2**]). In general, a Finsler manifold satisfying the $CD(K, N)$ condition does not satisfy the $RCD(K, N)$ condition. While the Cheeger-Gromoll type isometric splitting theorem is not obtained for Finsler manifolds under $CD(K, N)$, it was generalized to RCD spaces by Gigli [**42**]. Subsequently, generalizations of the Cheeger-Colding theory on RCD spaces have been intensively developed. Actually, Cheeger-Colding [**18**] suggested studying metric spaces possibly with measures under some synthetic notion of Ricci curvature bounded from below, and RCD space gave an answer to this research direction. Nowadays, the theory of RCD space is regarded as a generalization of the Cheeger-Colding theory.

Developments of geometric analysis under lower bounds of Ric_f^N, including those mentioned above, are also inspired by the theory of convex geometry. Especially, the curvature dimension condition implies a Brunn-Minkowski type inequality. In convex geometry, it was known that the classical Brunn-Minkowski inequality implies the isoperimetric inequality on the Euclidean space. Since the isoperimetric inequality is an important inequality, which originates from ancient Greece, researches on the isoperimetric inequality is also very active in the weighted case. Bakry-Ledoux [**5**] showed a Lévy-Gromov type isoperimetric inequality in the case $N = \infty$. We review some rigidity results of the isoperimetric inequality. In the case $N = \infty$ on smooth spaces, Morgan [**85**] showed that the rigidity case is when the space isometrically splits to a product $\mathbb{R} \times \Sigma$, and $\mathbb{R}$ is equipped with the Gaussian measure. In a non-smooth framework, Cavalletti-Mondino [**15**] obtained the rigidity under $RCD(K, N)$ with $N \in [2, \infty)$. An important step to obtain the Bakry-Ledoux type isoperimetric inequality [**5**] was to obtain a Poincaré-Lichnerowicz type inequality, which is an inequality for lower bounds of the first eigenvalue of the Laplacian. We briefly review researches on the rigidity of the Poincaré-Lichnerowicz type inequality. In the unweighted case $f \equiv 0$, Obata [**90**] showed that only the sphere attains the equality. Ketterer [**53**] investigated the weighted case $N \in [n, \infty)$ in a non-smooth framework, and showed that the equality is attained by the spherical suspensions. In the weighted case $N = \infty$, Cheng-Zhou [**27**] showed that the equality is also attained by a product $\mathbb{R} \times \Sigma$, and $\mathbb{R}$ is equipped with the Gaussian measure, which is the same phenomenon as the case of the isoperimetric inequality. Although Cheng-Zhou [**27**] considered only smooth manifolds, Gigli-Ketterer-Kuwada-Ohta [**43**] generalized their result to the $RCD(K, \infty)$ spaces.

1.2. The case effective dimension $N \in (-\infty, 1]$

1.2.1. **Constant curvature bounds.** For the case $N \in (-\infty, 0)$, Ohta [**93**] showed the equivalence between $CD(K, N)$ and $\mathrm{Ric}_f^N \geq Kg$. Also for the case $N = 0$, Ohta [**94**] introduced $CD(K, 0)$, and showed its equivalence with $\mathrm{Ric}_f^0 \geq Kg$. At the same time, Kolesnikov-Milman [**56**] also considered the case $N \in (-\infty, 0]$.

We note that the case $N \in (-\infty, 1]$ is weaker than the case $N \in [n, \infty]$. Indeed, for $N \in (-\infty, 1)$ and $N' \in (n, \infty)$, we have

$$\mathrm{Ric}_f^{N'} \leq \mathrm{Ric}_f^{\infty} \leq \mathrm{Ric}_f^{N} \leq \mathrm{Ric}_f^{1}.$$

Hence, we see that $\mathrm{Ric}_f^{N'} \geq Kg$ implies $\mathrm{Ric}_f^{N} \geq Kg$. Very recently, De Luca-De Ponti-Mondino-Tomasiello [**33**] revealed a new relation between $N \in (-\infty, 0)$ and physics. Non-smooth framework for the case $N \in (-\infty, 0)$ has been also investigated. Especially, Magnabosco-Rigoni-Sosa [**81**] and Oshima [**99**] studied the stability under some notions of convergences. Later, Magnabosco-Rigoni [**80**] studied the local-to-global property.

Furthermore, when we consider some inequalities such as the Poincaré-Lichnerowicz inequality and the isoperimetric inequality in the case $N \in (-\infty, 0)$, some particular spaces appear in the rigidity situation. As for the Poincaré-Lichnerowicz type inequality, Ohta [**93**] and Kolesnikov-Milman [**55**] independently obtained the weighted case with $N \in (-\infty, 0)$, and its rigidity case was obtained by Mai [**82**]. For the isoperimetric inequality in the weighted case $N \in (-\infty, 0)$, Milman [**85**] showed the inequality and Mai [**83**] obtained the rigidity.

1.2.2. **Variable curvature bounds.** Wylie [**135**] obtained a splitting theorem under $\mathrm{Ric}_f^1 \geq 0$, which asserts that a manifold splits to a warped product, unlike a Riemannian product in the case $N \in (-\infty, 1) \cup [n, \infty]$. After that, Wylie-Yeroshkin [**136**] further pushed forward researches of the case $N = 1$, and introduced the following variable curvature bound:

$$\mathrm{Ric}_f^1 \geq K e^{\frac{-4f}{n-1}} g. \tag{2}$$

Under this curvature bound, they obtained the Laplacian comparison theorem, the Bishop-Gromov type volume comparison theorem, and several applications such as a Myers type diameter estimate. In addition, they introduced an affine connection, which we call the *Wylie-Yeroshkin type connection*. It turned out that the Ricci curvature for the Wylie-Yeroshkin type connection coincides with Ric_f^1. The relation between the displacement convexity of entropies and the curvature bounds (2) was obtained by Sakurai [**113**] in the smooth setting.

For the case $N \in (-\infty, 1]$, Kuwae-Li [**58**] introduced the curvature bound:

$$\mathrm{Ric}_f^N \geq K \mathrm{e}^{\frac{-4f}{n-N}} g. \tag{3}$$

This coincides with (2) when $N = 1$. Further generalization of them to the case $N \in (-\infty, 1] \cup [n, \infty]$ was given by Lu-Minguzzi-Ohta [**79**], where they introduced the curvature bound:

$$\mathrm{Ric}_f^N \geq K \mathrm{e}^{\frac{4(\varepsilon-1)f}{n-1}} g. \tag{4}$$

Here, $\varepsilon \in \mathbb{R}$ is a parameter in some interval, which is called the *ε-range*. This curvature bound with ε-range is a generalization of the curvature bound (3) in the case $N \in (-\infty, 1)$. Moreover, it is also a generalization of the constant curvature bound (1) in the case $N \in [n, \infty)$. Under (4), Kuwae-Sakurai [**59**] studied the rigidity cases of several comparison geometric results such as the Laplacian comparison property and the Bishop-Gromov type volume comparison property. Displacement convexity of entropies under (4) was investigated by Kuwae-Sakurai [**61**].

We note that researches of the Cheeger-Gromoll type isometric splitting theorem for manifolds with boundary also evolved. In particular, unlike the case without boundary, in the case with boundaries, we have a splitting to a warped product with a warping function depending on the parameter $\varepsilon \in \mathbb{R}$ in the ε-range under (4). We briefly review related researches on manifolds with boundary. First, in the unweighted case $f \equiv 0$ with boundary, Kasue [**52**] obtained the splitting theorem by generalizing the proof of Cheeger-Gromoll [**21**], and also Croke-Kleiner [**31**] showed it by following the line of a simpler proof of the Cheeger-Gromoll splitting theorem by Eschenburg-Heintze [**35**]. After that, Sakurai [**109–112**] gave a further generalization of them including the weighted case with $N \in (-\infty, 1]$. Finally, Kuwae-Sakurai [**60**] obtained the case with ε-range.

Recently, another motivation for the case $N = 1$ arose from general relativity theory. Wang-Wang-Zhang [**130**] showed, if a Lorentz manifold $(\mathbb{R} \times M, -V^2 \mathrm{d}t^2 + g)$ with $V \in C^\infty(M)$ satisfies the null energy condition, (M, g) satisfies the *substatic condition* $V \,\mathrm{Ric}_g + \mathrm{Hess}\, V + (\Delta V) g \geq 0$. In particular, the deSitter-Schwarzschild manifolds and the Reissner-Nordström manifolds satisfy the substatic condition. Regarding the relation between the subtstatic condition and the case $N = 1$, Li-Xia [**62**] introduced a family of affine connections that interpolate the Wylie-Yeroshkin type connection and the affine connection whose Ricci curvature is the static Ricci tensor. Notably, the non-negativity of the static Ricci tensor coincides with the substatic condition. Furthermore, Borghini-Fogagnolo [**9**] pointed out that the relation between Ric_f^1 and the substatic condition, and also showed a volume comparison theorem and a splitting theorem on manifolds under the substatic condition. We note that their Riccati inequality also appeared in a prior work by Brendle [**10**]. Recently, Ketterer [**53**], McCann [**84**] and Cavalletti-Manini-Mondino [**14**] conducted researches on the characterization of the null energy condition in view of the optimal transport theory.

1.3. Harmonic functions

1.3.1. **On Riemannian manifolds.** A classical Liouville theorem asserts that bounded harmonic functions on $\mathbb{R}^n$ must be constant. On Riemannian manifolds, a breakthrough under lower bounds of Ricci curvature was given by Yau [**137**] as follows:

Theorem 1.3.1 ([**137**]). *Let (M, g) be a complete Riemannian manifold. We assume*

$$\mathrm{Ric}_g \geq 0.$$

Then any positive harmonic function must be a constant function.

There are several ways to obtain this Liouville property. We review some of them. One may take an approach of gradient estimates (see e.g., [**67**]). Saloff-Coste [**114**] obtained the Liouville property under a volume doubling property and a local Poincaré inequality. Note that lower bounds of Ricci curvature imply those conditions. Also, the Alexandrov-Backlmann-Pucci type estimate (ABP estimate, for short) gives an alternative proof of the Liouville theorem. In particular, Cabré [**13**] conducted an ABP estimate on Riemannian manifolds under lower bounds of sectional curvature, and showed a Krylov-Safonov type Harnack inequality, which yields the Liouville property. This was generalized by Kim [**54**] to more general manifolds including Riemannian manifolds under lower bounds of Ricci curvature.

Cheng [**26**] replaced the boundedness condition with the sublinear growth condition of harmonic functions, and obtained the following Liouville property:

Theorem 1.3.2 ([**26**]). *Let (M, g) be a complete Riemannian manifold. We assume*

$$\mathrm{Ric}_g \geq 0.$$

Then any sublinear growth harmonic function must be a constant function.

We can prove this by the method of gradient estimate (see [**67**]). A probabilistic proof of Cheng's result was given by Stafford [**118**]. If we denote the space of harmonic functions with polynomial growth at most d by $\mathcal{H}^d(M)$, we see that Theorem 1.3.2 implies

$$\dim\left(\mathcal{H}^d(M)\right) = 1$$

for any $d < 1$. According to Li [**66**], this leads to Yau's conjecture (see e.g., [**139**, Problem 48]):

Conjecture 1.3.3 (Yau). Let (M, g) be a Riemannian manfiold. We assume

$$\mathrm{Ric}_g \geq 0.$$

Then $\mathcal{H}^d(M)$ is finite dimensional for any $d \geq 1$.

This conjecture motivated the analysis of harmonic functions with polynomial growth. Among them, Li-Tam [**69**] gave a sharp estimate for the case $d = 1$, and its rigidity was proved by Cheeger-Colding-Minicozzi [**16**]. Finally, this Yau's conjecture was proved by Colding-Minicozzi [**28**].

While the Yau type Liouville property implies the L^∞-Liouville property, researches on the L^p-Liouville property are also active. Actually, Yau [**138**] showed the L^p-Liouville theorem for the case $p \in (1, \infty)$. For the case $p = 1$, Garnett [**41**] showed it if manifolds have bounded geometry. Later, Li-Schoen [**68**] obtained the L^1-Liouville theorem under lower bounds of Ricci curvature, and Li [**64**] improved them using the theory of heat equation. The case $p \in (0, 1)$ was also given by Li-Schoen [**68**] under lower bounds of Ricci curvature as follows:

Theorem 1.3.4 ([**68**]). *Let (M, g) be an n-dimensional complete non-compact weighted Riemannian manifold. Then there exists a constant $\delta > 0$ depending only on n such that the following assertion holds*:

We assume that there exists $q \in M$ such that

$$\mathrm{Ric}_g \geq \delta d_q^{-2} g$$

when $d_q := d(q, \cdot)$ is sufficiently large. Let u be a non-negative L^p-function with $p \in (0, \infty)$ with $\Delta u \geq 0$. Then u is identically zero.

1.3.2. **On weighted Riemannian manifolds.** Li [**74**] studied the Liouville theorem for symmetric diffusion operator $\Delta_f := \Delta - \langle \nabla f, \nabla \cdot \rangle$, which is also called the *weighted Laplacian*. His motivation came from the relation between the symmetric diffusion operator and the Schrödinger operator, and its relation with probability theory and potential theory. He considered an *f-harmonic function*, i.e., a function u such that $\Delta_f u = 0$. We remark that an f-harmonic function is also called *weighted harmonic function*. In particular, Li [**74**] obtained the Yau type Liouville theorem and the L^1-Liouville theorem for the case $N \in [n, \infty)$. The L^p-Liouville theorem for the case $p \in (1, \infty)$ was proved by Pigola-Rigoli-Setti [**106**] (see also [**107**]). For the case $N = \infty$, Wu [**132**] showed the Yau type Liouville theorem under an additional assumption that $|\nabla f|$ is bounded, and also Wu [**133**] generalized L^p-Liouville theorem with $p \in (0, 1]$ under an additional assumption that f is bounded. Although there are several ways to prove the Yau type Liouville theorem, Li [**74**] and Wu [**132**] employed the method of gradient estimate.

Later, Brighton [**11**] gave further progress in this context for the case $N = \infty$. He applied the method of Yau's gradient estimate to a specific function and showed that bounded f-harmonic functions must be constant under $\mathrm{Ric}_f^\infty \geq 0$ without any assumptions on f. After that, Munteanu-Wang [**87**] employed an argument similar to [**11**], and obtained a gradient estimate of f-harmonic functions under $\mathrm{Ric}_f^\infty \geq 0$ and an additional assumption concerning the linear growth rate of f. It should be noted that, not only using a Brighton type gradient estimate, Munteanu-Wang [**87**] combined them with the De Giorgi-Nash-Moser theory to obtain the gradient estimate. This gradient estimate implies the Yau type Liouville theorem for positive f-harmonic functions under $\mathrm{Ric}_f^\infty \geq 0$ if f is of sublinear growth. In addition, they combined the De Giorgi-Nash-Moser theory with the weighted Bochner formula, and showed a Cheng type Liouville theorem for f-harmonic functions with sublinear growth under an additional assumption that f is bounded. We note that, in the proof of this Cheng type Liouville theorem, they did not use the Brighton type gradient estimate. Also, for the space of f-harmonic functions, they gave an estimate of its dimension and proved especially its finiteness. After [**87**], Munteanu-Wang [**88**] further pursued the study on gradient estimates, where they replaced the assumption $\mathrm{Ric}_f^\infty \geq 0$ with $\mathrm{Ric}_f^\infty \geq Kg$.

For the case $N \in (-\infty, 1]$, much less is known. It seems that the method of gradient estimates does not work straightforwardly in the case $N \in (-\infty, 1]$. To overcome this difficulty, the author [**38**] took an approach to utilize the arguments in Munteanu-Wang [**87**], and showed a Cheng type Liouville theorem for sublinear growth f-harmonic functions for the case $N \in (-\infty, 0)$ under an additional assumption that f is bounded. The case $N \in [0, 1]$ was excluded in [**38**] since the author did not know a suitable Bochner formula in this case. As for the L^p-Liouville theorem, the author [**38**] also showed them in the case $p \in (0, 1)$ and $N \in (-\infty, 1]$ under a strong assumption on f. There is another progress on harmonic maps in the case $N \in (-\infty, 0]$. Actually, Cheng [**26**] also showed a Liouville type theorem for harmonic maps, and an alternative proof using the probabilistic theory was given also by Stafford [**118**], and this was generalized to the case $N \in (-\infty, 0]$ by Kuwae-Li-Li-Sakurai [**57**].

1.3.3. **Related topics: Cheng type upper bound of the first eigenvalue.** It is worth mentioning that Munteanu-Wang [**87**] used the Brighton type Liouville theorem to show the rigidity of the Cheng type inequality, which estimates the upper bound of the bottom spectrum. We briefly review the history of this Cheng type inequality. Cheng [**25**] first obtained an upper bound of the first eigenvalue of the Laplacian as follows:

Theorem 1.3.5 ([**25**]). *Let (M, g) be an n-dimensional complete non-compact Riemannian manifold. For $K \geq 0$, we assume*

$$\mathrm{Ric}_g \geq -Kg.$$

Then we have

$$\lambda_1(M) \leq \frac{(n-1)K}{4}.$$

Cheng [**25**] obtained this by calculating the upper bounds of space forms. Its rigidity was obtained by Li-Wang [**70, 71**]. Although the calculation on space forms is necessary, a part of Cheng's proof can also be simplified using the heat kernel comparison (see e.g., [**117**]). We also note that this Cheng's inequality is also obtained as an application of a gradient estimate (see e.g.,[**67**]). This method of gradient estimate was generalized by Wang [**127**] and Wu [**131**] to the case $N \in [n, \infty]$. We remind that Wu [**132**] assumed the boundedness of $|\nabla f|$ in the case $N = \infty$. On the other hand, by using the Bishop-Gromov type volume comparison theorem, Munteanu-Wang [**87, 88**] and Su-Zhang [**121**] obtained Cheng type inequalities for the case $N = \infty$ with some additional assumption on f, and they also obtained the rigidity. Especially, in Munteanu-Wang [**87**], the rigidity of Cheng's inequality was applied to study the topology of steady gradient Ricci solitons. As for the L^p-spectrum, Wang [**128**] obtained an upper bound by using the volume comparison theorem. In the weighted case $N \in (-\infty, 1] \cup [n, \infty]$ with ε-range, the author [**37**] obtained an upper bound, while its sharpness and rigidity are left for future work. Recently, Cheng type inequalities for the case $N = 1$ gained some attention in view of the substatic condition (see also [**9**]).

1.4. Porous medium equation

1.4.1. **Heat flow as a gradient flow.** The solution of the heat equation $\partial_t u = \Delta u$ can be regarded as a gradient flow of the relative entropy with respect to the Wasserstein distance, which we call the *Wasserstein gradient flow.* On Euclidean space, this was shown by Jordan-Kinderlehrer-Otto [**51**]. Later, so called the *Otto calculus* proposed by Otto [**101**] enabled us to see those relations more intuitively. Actually, it turned out that the Wasserstein gradient flow of the relative entropy for the weighted measure $\mathrm{e}^{-f} v_g$ is the solution of the weighted heat equation $\partial_t u = \Delta_f u$, which is also known as the Fokker-Planck equation. Although the Wasserstein space does not have any differential structure, the Otto calculus introduced a differential structure heuristically. We note that the rigorous treatment is very active, and the general theory of gradient flow on metric spaces took an important role (see e.g., [**1**]).

On Riemannian manifolds, lower bounds of Ricci curvature and the behavior of heat flows are closely related. Indeed, von Renesse-Sturm [**126**] showed that the lower bound of Ricci curvature and the contraction of the heat flow are equivalent. This contraction is measured in view of the Wasserstein distance, and also referred to as the *Wassrstein expansion bound.* Erber-Kuwada-Sturm [**34**] further pursued this study. In particular, they formulated the (K, N)-convexity of the relative entropy as the *entropic curvature dimension condition* $CD^e(K, N)$. In a non-smooth framework, they showed $CD^e(K, N)$ implies the Wasserstein expansion bound. In addition, they also showed that $CD^e(K, N)$ yields the Bakry-Ledoux gradient estimate. Later, Ohta [**93**] investigated the (K, N)-convexity for $N \in (-\infty, 0)$. It turned out that the argument in [**34**] showing that $CD^e(K, N)$ implies the Bakry-Ledoux type gradient estimate does not hold in the case $N \in (-\infty, 0)$, and the technical difficulty comes from the lack of expansion bounds of the gradient flows of general (K, N)-convex functions. Not only the Bakry-Ledoux type gradient estimate, the Li-Yau type gradient estimate is not yet obtained for the case

$N \in (-\infty, 0)$. Gradient estimates for the case $N \in (-\infty, 0)$ are listed as one of the open questions in [**93**] (see also [**95**]).

1.4.2. **Porous medium equation as a gradient flow.** Porous medium equation $\partial_t u = \Delta u^m$ with $m > 1$, which is a generalization of the heat equation, appears in many fields, not only mathematics. Especially, the porous medium equation on Euclidean space is well investigated (see e.g., [**125**]). Recently, there is a growing interest in view of the Wasserstein distance. It is pointed out by Otto [**101**] that the porous medium equation can be interpreted as the Wasserstein gradient flow of the Rényi entropy.

One of the particular features of the heat flow is the stability of the Gaussian measures along the heat flow. As a generalization of this property, Ohara-Wada [**91**] showed the stability of the q-Gaussian measures along the porous medium equation. After that, Takatsu [**122**] obtained some functional inequalities as a generalization of inequalities by Otto-Villani [**100**], and showed that the equality is attained by the q-Gaussian measures. In Ohta-Takatsu [**98**], they investigated the relation between the displacement convexity for a more general class of entropies and lower bounds of Ric_f^N. In particular, Ric_f^N with $N \in (-\infty, 0)$ was used to characterize this convexity. They also applied the general theory to this displacement convexity, and obtained a contraction property of porous medium equation on compact Riemannian manifolds.

1.4.3. **Aronson-Bénilan estimate.** In Aronson-Bénilan [**3**], they obtained a gradient estimate for the porous medium equation $\partial_t u = \Delta u^m$ on Euclidean space $\mathbb{R}^n$ as follows:

$$\nabla \cdot \left(m u^{m-2} \nabla u \right) \geq -\frac{1}{2} \frac{n}{n(m-1)+2}. \tag{5}$$

Later, Li-Yau [**72**] obtained a gradient estimate for the heat equation. If we let $m \searrow 1$ in the Aronson-Bénilan type estimate, it coincides with the Li-Yau type estimate. On compact Riemannian manifolds, the Aronson-Bénilan type estimate was obtained in Vázquez [**125**]. After that, Lu-Ni-Vázquez-Villani [**78**] generalized the Aronson-Bénilan type gradient estimate to non-compact Riemannian manifolds. We note that, similarly to the Li-Yau gradient estimate, they obtained a local gradient estimate. We may consider their Aronson-Bénilan type local gradient estimate as a counterpart of the Li-Yau gradient estimate for the heat equation. The results in [**78**] were improved by Huang-Huang-Li [**49**]. Their improvements of the Aronson-Bénilan gradient estimate are compatible with the recent progress of gradient estimates for the heat equation obtained by Davies [**32**], Hamilton [**48**] and Li-Xu [**63**]. The weighted case with $N \in [n, \infty)$ was obtained by Huang-Li [**50**]. Furthermore, the author [**36**] obtained the case $N \in \left(-\infty, -\frac{2}{m-1}\right) \cup [n, \infty]$ with ε-range. This can be regarded as a gradient estimate for the case $N \in (-\infty, 0)$. As the porous medium equation becomes the heat equation by letting $m \searrow 1$, we see $\frac{-2}{m-1} \to -\infty$, and the range $(-\infty, \frac{-2}{m-1})$ degenerates. An approach different from the Li-Yau gradient estimate is needed to obtain a gradient estimate for the heat equation in the case $N \in (-\infty, 0)$, which is left for future work.

1.5. Organization

The aim of this thesis is a comprehensive understanding of the following assertions:

(i) L^p-Liouville theorem (Theorem 3.2.4);
(ii) Liouville theorem for sublinear growth f-harmonic functions (Theorem 3.3.4);
(iii) Gradient estimate for f-harmonic functions (Theorem 3.4.1);
(iv) Cheng type eigenvalue estimate (Theorem 3.5.1);
(v) Aronson-Bénilan gradient estimate (Theorem 4.1.1);

under lower bounds of N-weighted Ricci curvature with ε-range. In the unweighted case $f \equiv 0$, they are all obtained via the method of gradient estimate. Although some of the classical properties written in this thesis are not generalized to our settings, we consider that they also help us to grasp the current situation on this topic.

Section 2.1 is devoted to introducing the weighted Ricci curvature with ε-range, and the rest of Chapter 2 is devoted to listing some results related to harmonic functions, eigenvalue estimates, and the porous medium equation in the case $N \in [n, \infty]$. In section 2.1, we introduce weighted Ricci curvature and list several examples, and basic theorems such as the Laplacian comparison property (Proposition 2.1.15) and the Bishop-Gromov type volume comparison property (Proposition 2.1.19). We also address the Bochner inequality for the case $N \in (-\infty, 0) \cup \{\infty\}$ (Propositions 2.1.6 and 2.1.7). Section 2.2 is divided into several subsections. In subsection 2.2.1, we study functional inequalities which are useful in the analysis of harmonic functions. We provide the Li-Schoen type Poincaré inequaliy (Theorem 2.2.1), the Neumann-Sobolev inequality (Theorems 2.2.11 and 2.2.15), and the Saloff-Coste type Sobolev inequality (Theorem 2.2.7). We note that, in the case $N = \infty$, some different assumptions on the weight functions are employed. After that, we address the consequences of the De Giorgi-Nash-Moser theory. In particular, the mean value inequality (Propositions 2.2.16 and 2.2.17) and the Harnack inequality (Theorem 2.2.18) are spelled out. In subsection 2.2.2, we present several Liouville type theorems and gradient estimates, especially the various types in the case $N = \infty$. In the next section 2.3, we list Cheng type eigenvalue estimates. Several types are presented in the case $N = \infty$ (Theorems 2.3.1 to 2.3.6). In section 2.4, we study gradient estimates of Li-Yau type (Theorem 2.4.1) and Aronson-Bénilan type (Theorem 2.4.3).

Chapter 3 is devoted to the analysis of harmonic functions and related topics under lower bounds of Ric_f^N with ε-range for $N \in (-\infty, 1] \cup [n, \infty]$. In subsection 3.1.1, to obtain the Neumann-Sobolev type inequality (Theorem 3.1.7), we provide a Neumann-Poincaré type inequality (Proposition 3.1.1), a Saloff-Coste type Sobolev inequality (Proposition 3.1.2), and a Li-Schoen type Poincaré inequality (Proposition 3.1.6). In subsection 3.1.2, using Neumann-Poincaré inequality and Neumann-Sobolev inequality, we address the mean value inequality (Theorem 3.1.8). This is obtained by the De Giorgi-Nash-Moser theory. As applications of the mean value inequality, we provide an L^p-Liouville type theorem (Theorem 3.2.4) in section 3.2, and a Cheng type Liouville theorem (Theorem 3.3.4) in section 3.3. In section 3.4, we use a Harnack type inequality, which is obtained from the mean value inequality, and obtain a gradient estimate (Theorem 3.4.1). In section 3.5, we give a Cheng type inequality (Theorem 3.5.1) as related topics. We consider it natural to be interested in the analysis of eigenfunction since Munteanu-Wang [**87**] applied the Brighton type Liouville theorem to obtain the rigidity property of the Cheng type inequality. In our case, as far as we know, unlike the case in [**87**], the analysis of harmonic functions and eigenfunctions of the weighted Laplacian do not interact with each other.

Chapter 4 is devoted to the analysis of porous medium equation under lower bounds of Ric_f^N with ε-range for $N \in (-\infty, 1] \cup [n, \infty]$. In section 4.1, we obtain a local Aronson-Bénilan type gradient estimate (Theorem 4.1.1), and this allows us to obtain a global estimate on non-compact spaces. In section 4.2, we give an alternative proof of the global estimate (Theorem 4.2.1) on compact manifolds.

CHAPTER 2

Preliminaries

In this chapter, we review basic comparison geometric results under lower bounds of weighted Ricci curvature. Some of them are generalized in the subsequent chapters. Those which are not generalized are also useful to understand the backgrounds of geometric analysis on harmonic functions, eigenfunctions and porous medium equations.

2.1. Weighted Ricci curvature

In this section, we introduce the lower bounds of weighted Ricci curvature with ε-range, and provide some comparison geometric results related to the weighted Ricci curvature. These are useful throughout the rest of this thesis.

Let (M, g, f) be an n-dimensional weighted Riemannian manifold. We denote the *weighted measure* and the *weighted Laplacian* as follows:

$$m_f := \mathrm{e}^{-f}\ v_g, \quad \Delta_f := \Delta - \langle \nabla f, \nabla \cdot \rangle,$$

where v_g is the Riemannian volume measure. For $N \in (-\infty, 1] \cup [n, +\infty]$, the *$N$-weighted Ricci curvature* is defined by

$$\mathrm{Ric}_f^N := \mathrm{Ric}_g + \mathrm{Hess} f - \frac{\mathrm{d}f \otimes \mathrm{d}f}{N-n},$$

where the last term is considered to be 0 when $N = \infty$, and when $N = n$, we only consider constant functions for f.

Remark 2.1.1. In the case $N \in (n, \infty) \cap \mathbb{N}$, Lott [**76**] pointed out that Ric_f^N on (M, g, f) coincides with the Ricci curvature on a specific warped product. We consider a sphere $\mathbb{S}^{N-n}$ equipped with the canonical metric $g_{\mathbb{S}^{N-n}}$. On the product $M \times \mathbb{S}^{N-n}$, we set a warped product metric

$$G := g + \mathrm{e}^{\frac{2f}{N-n}} g_{\mathbb{S}^{N-n}}.$$

Let Ric_G denote the Ricci curvature on $(M \times \mathbb{S}^{N-n}, G)$, and $\overline{X}$ denote the horizontal lift to $M \times \mathbb{S}^{N-n}$ of a vector field X on M. Then we have

$$\mathrm{Ric}_G(\overline{X}, \overline{X}) = \mathrm{Ric}_f^N(X, X).$$

We present several known examples below for the case $N \in (-\infty, 0) \cup \{\infty\}$.

Example 2.1.2. The Euclidean space equipped with a Gaussian measure:

$$\left(\mathbb{R}^n,\ |\cdot|,\ \frac{1}{(2\pi)^{n/2}} \mathrm{e}^{-\frac{|x|^2}{2}} \mathrm{d}x \right)$$

satisfies $\mathrm{Ric}_f^\infty \geq g$.

Example 2.1.3 ([**86**])**.** For $\alpha > 0$, the Cauchy distribution $m^{n,\alpha}$ on $\mathbb{R}^n$ is defined as follows:

$$m^{n,\alpha} := (1 + |x|^2)^{-\frac{n+\alpha}{2}}\ \mathrm{d}x.$$

Then we see that $(\mathbb{R}^n, |\cdot|, m^{n,\alpha})$ satisfies $\mathrm{Ric}_f^{-\alpha} \geq 0$ (see Milman [**86**]).

Example 2.1.4 ([**82**])**.** A one-dimensional space

$$\left(\mathbb{R}, |\cdot|, \cosh\left(\sqrt{\frac{K}{1-N}}x\right)^{N-1} \mathrm{d}x\right)$$

for $K > 0$ and $N < 0$ satisfies $\mathrm{Ric}_f^N \equiv Kg$ (see Mai [**82**, Example 3.1]).

Example 2.1.5 ([**86**])**.** For $n \geq 2$, let H^n denote the Haar measure on $\mathbb{R}^{n+1}$. For $\alpha > -n$ and $x \in \mathbb{R}^{n+1}$ with $|x| < 1$, the harmonic measure $m_x^{n,\alpha}$ on $\mathbb{S}^n$ is defined as follows:

$$m_x^{n,\alpha}(y) := \frac{1}{|y-x|^{n+\alpha}} \mathrm{d}\mathrm{H}^n(y).$$

We denote the canonical metric on $\mathbb{S}^n$ by $g_{\mathbb{S}^n}$. Then the space $(\mathbb{S}^n, g_{\mathbb{S}^n}, m_x^{n,\alpha})$ satisfies $\mathrm{Ric}_f^{-\alpha} \geq \left(n - 1 - \frac{n+\alpha}{4}\right) g$ (see Milman [**86**, Theorem 1.1]).

We turn to the Bochner formulas and their applications. We have the following Bochner identity for the weighted case with $N = \infty$:

Proposition 2.1.6. *Let (M, g, f) be an n-dimensional complete weighted Riemannian manifold. For $\varphi \in C^\infty(M)$, we have*

$$\Delta_f\left(\frac{|\nabla\varphi|^2}{2}\right) - \langle \nabla\Delta_f\varphi, \nabla\varphi\rangle = \mathrm{Ric}_f^\infty(\nabla\varphi, \nabla\varphi) + \|\mathrm{Hess}\,\varphi\|^2.$$

The Bochner inequality for the weighted case with $N \in (-\infty, 0)$ is as follows (see e.g., [**93**, Theorem 4.1]):

Proposition 2.1.7 ([**93**])**.** *Let (M, g, f) be an n-dimensional complete weighted Riemannian manifold. For $\varphi \in C^\infty(M)$ and $N \in (-\infty, 0)$, we have*

$$\Delta_f\left(\frac{|\nabla\varphi|^2}{2}\right) - \langle \nabla\Delta_f\varphi, \nabla\varphi\rangle \geq \mathrm{Ric}_f^N(\nabla\varphi, \nabla\varphi) + \frac{(\Delta_f\varphi)^2}{N}.$$

Wylie [**135**] introduced the following Bochner formula (see [**135**, Lemma 2]):

Proposition 2.1.8 ([**135**])**.** *Let (M, g, f) be an n-dimensional complete weighted Riemannian manifold and $\varphi \in C^3(M)$. For $K \in \mathbb{R}$, $k \in \mathbb{N}$ and $p \in M$, we assume*

$$\mathrm{Ric}_f^{n-k} \geq Kg,$$

and $\mathrm{Hess}\,\varphi|_p$ *has at most k non-zero eigenvalues. Then we have*

$$\Delta_f\left(\frac{|\nabla\varphi|^2}{2}\right) - \mathrm{e}^{\frac{-2f}{k}}\left\langle \nabla\left(\mathrm{e}^{\frac{2f}{k}}\Delta_f\varphi\right), \nabla\varphi\right\rangle \geq K|\nabla\varphi|^2 + \frac{(\Delta_f\varphi)^2}{k}.$$

We note that $k = n-1$ in Proposition 2.1.8 if we substitute the distance function into φ. Wylie [**135**] applied this to the distance function and obtained the Riccati inequality under $\mathrm{Ric}_f^1 \geq 0$ as follows (see [**135**, Theorem 3.2] and also [**134**, Lemma 6.1]):

Theorem 2.1.9 ([**135**])**.** *Let (M, g, f) be an n-dimensional complete weighted Riemannian manifold. We assume*

$$\mathrm{Ric}_f^1 \geq 0.$$

For a fixed point $q \in M$, let x be a point that the distance function d_q is smooth at x. Also, let γ be the unique minimal geodesic from q to x parameterized by arclength. Then

$$(\Delta_f d_q)(x) \leq (n-1)\,\mathrm{e}^{\frac{-2f(x)}{n-1}} \left(\int_0^{d_q(x)} \mathrm{e}^{\frac{-2f(\gamma(t))}{n-1}}\,\mathrm{d}t \right)^{-1}. \tag{6}$$

Remark 2.1.10. Actually, this was first obtained in a more general form by Wylie [**134**, Lemma 6.1], which is a paper before [**135**].

Wylie-Yeroshkin [**136**] pointed out that the term $\int_0^{d_q(x)} \mathrm{e}^{\frac{-2f(\gamma(t))}{n-1}}\,\mathrm{d}t$ can be interpreted from the viewpoint of an affine connection. For $\varphi \in C^\infty(M)$, they introduced an affine connection:

$$\nabla^\varphi_X Y := \nabla_X Y - \mathrm{d}\varphi(X)Y - \mathrm{d}\varphi(Y)X,$$

where ∇ denotes the Levi-Civita connection. We called ∇^φ the *Wylie-Yeroshkin type connection* in the previous chapter. In the same setting as in Theorem 2.1.9, we set $\varphi := \frac{f}{n-1}$ and

$$s_\gamma(t) := \int_0^t \mathrm{e}^{\frac{-2f(\gamma(t))}{n-1}}\,\mathrm{d}t. \tag{7}$$

Let $\gamma' := \frac{\mathrm{d}\gamma}{\mathrm{d}t}$ and $\dot\gamma := \frac{\mathrm{d}\gamma}{\mathrm{d}s}$, where $s = s_\gamma(t)$. Since $\dot\gamma = \mathrm{e}^{\frac{2f}{n-1}}\gamma'$, we see

$$\nabla^\varphi_{\dot\gamma}\dot\gamma = \mathrm{e}^{\frac{2f}{n-1}}\nabla_{\gamma'}\gamma' = 0.$$

Hence, we may regard that the reparametrization $s_\gamma(t)$ gives a geodesic with respect to ∇^φ. In other words, we see that the images of a ∇^φ-geodesic and a ∇-geodesic coincide. This property actually holds for a wider class of affine connections, which are projectively equivalent to ∇, and this is guaranteed by the Weyl theorem (see e.g., [**136**]).

Remark 2.1.11. We set the Riemannian curvature tensor and Ricci curvature with respect to ∇^φ as follows:

$$R^{\nabla^\varphi}(X,Y,Z) := \nabla^\varphi_X\nabla^\varphi_Y Z - \nabla^\varphi_Y\nabla^\varphi_X Z - \nabla^\varphi_{[X,Y]}Z,$$
$$\mathrm{Ric}^{\nabla^\varphi}(X,Y) := \mathrm{tr}\left(X \to R^{\nabla^\varphi}(X,Y)Z\right).$$

If we set $\varphi = \frac{f}{n-1}$, we have $\mathrm{Ric}^1_f = \mathrm{Ric}^{\nabla^\varphi}$ (see also subsection 1.2.2).

Wylie-Yeroshkin [**136**] obtained the Riccati inequality under the curvature bound:

$$\mathrm{Ric}^1_f \geq K\mathrm{e}^{\frac{-4f}{n-1}}g.$$

After a further generalization of Kuwae-Li [**58**] to the case $N \in (-\infty, 1]$, Lu-Minguzzi-Ohta [**79**] considered the curvature bound:

$$\mathrm{Ric}^N_f \geq K\mathrm{e}^{\frac{4(\varepsilon-1)f}{n-1}}g, \tag{8}$$

for $K \in \mathbb{R}$, where ε in the *ε-range*:

$$\varepsilon = 0 \text{ for } N = 1, \quad |\varepsilon| < \sqrt{\frac{N-1}{N-n}} \text{ for } N \neq 1, n, \quad \varepsilon \in \mathbb{R} \text{ for } N = n.$$

Later, Kuwae-Sakurai [**59**] investigated more general potentials V. We note that results in [**59**] recover those in [**79**] when $V = \nabla f$ and f is bounded.

Remark 2.1.12. As is mentioned in the introduction, the curvature bound (8) coincides with $\mathrm{Ric}^N_f \geq Kg$ if we take $\varepsilon = 1$ when $N \in [n, \infty)$. In addition, it coincides with the curvature bound (2) introduced by Wylie-Yeroshkin in the case $N = 1$, and also coincides with the curvature bound (3) introduced by Kuwae-Li if we take $\varepsilon = \frac{N-1}{N-n}$ in the case $N \in (-\infty, 1]$.

Before we give comparison theorems, we prepare some notations following the line in [**59**]. We set

$$c(n,N,\varepsilon) := \frac{1}{n-1}\left(1-\varepsilon^2\frac{N-n}{N-1}\right), \tag{9}$$

and the *comparison function* as

$$\mathbf{s}_\kappa(t) = \begin{cases} \frac{1}{\sqrt{\kappa}}\sin(\sqrt{\kappa}\,t) & \text{for } \kappa>0,\\ t & \text{for } \kappa=0,\\ \frac{1}{\sqrt{-\kappa}}\sinh(\sqrt{-\kappa}\,t) & \text{for } \kappa<0. \end{cases}$$

Also, we put

$$C_\kappa := \begin{cases} \frac{\pi}{\sqrt{\kappa}} & \text{if } \kappa>0,\\ \infty & \text{otherwise,}\end{cases} \qquad \bar{\mathbf{s}}_\kappa(s) := \begin{cases} \mathbf{s}_\kappa(s) & \text{if } s\le C_\kappa,\\ 0 & \text{otherwise ,}\end{cases} \qquad \mathcal{S}_\kappa(r) := \int_0^r \bar{\mathbf{s}}_\kappa^{1/c}(s)\,\mathrm{d}s.$$

Below, we fix a point $q\in M$. For $x\in M$, we define

$$s_{f,q}(x) := \inf_\gamma \int_0^{d_q(x)} \mathrm{e}^{\frac{2(\varepsilon-1)f(\gamma(\xi))}{n-1}}\,\mathrm{d}\xi, \tag{10}$$

where γ runs all the unit speed minimal geodesics from q to x. We denote the set of unit vectors in T_qM by U_qM. For $w\in U_qM$, we set

$$\rho(w) := \sup\{t>0 \mid d_q(\gamma_w(t)) = t\},$$

where γ_w is the unit speed geodesic such that $\gamma_w(0)=q$ and $\gamma_w'(0)=w$. In the process of generalizations in [**58**, **61**, **79**], the reparametrization (7) was generalized as follows:

$$s_{f,w}(t) := \int_0^t \mathrm{e}^{\frac{2(\varepsilon-1)f(\gamma_w(\xi))}{n-1}}\,\mathrm{d}\xi \tag{11}$$

and $\rho_f(w) := s_{f,w}(\rho(w))$ for $w\in U_qM$. We denote the inverse function of $s_{f,w}$ by $t_{f,w}$. Then we have the following Laplacian comparison property (see Kuwae-Sakurai [**59**, Theorem 2.3]):

Proposition 2.1.13 ([**59**])**.** *Let (M,g,f) be an n-dimensional complete weighted Riemannian manifold, $N\in(-\infty,1]\cup[n,\infty]$ and $\varepsilon\in\mathbb{R}$ in the ε-range. For $q\in M$, $w\in U_qM$ and $t\in(0,\rho(w))$, we assume*

$$\mathrm{Ric}_f^N(\gamma_w'(t),\gamma_w'(t)) \ge K\mathrm{e}^{\frac{4(1-\varepsilon)f(\gamma_w(t))}{n-1}}.$$

Then for $c:=c(n,N,\varepsilon)$ as in (9), $t\in(0,\tau(w))$ and $s_{f,w}(t)\in(0,\min\{\rho_f(w),C_{cK}\})$, we have

$$(\Delta_f d_q)(\gamma_w(t)) \le \frac{1}{c}\frac{\mathbf{s}_{cK}'(s_{f,w}(t))}{\mathbf{s}_{cK}(s_{f,w}(t))}\,\mathrm{e}^{\frac{2(\varepsilon-1)f(\gamma_w(t))}{n-1}}. \tag{12}$$

We denote the volume element of the level surface of d_q at $\gamma_w(t)$ by $\theta(t,w)$. We set

$$\theta_f(t,w) := \mathrm{e}^{-f(\gamma_w(t))}\theta(t,w), \quad \widehat{\theta}_f(s,w) := \theta_f(t_{f,w}(s),w),$$

and

$$B_{f,r}(q) := \{x\in M \mid s_{f,q}(x) < r\}.$$

The volume comparison theorem holds as follows (see [**59**, Proposition 6.2]):

Proposition 2.1.14 ([**59**]). *Let (M, g, f) be an n-dimensional complete weighted Riemannian manifold, $N \in (-\infty, 1] \cup [n, \infty]$ and $\varepsilon \in \mathbb{R}$ in the ε-range. For $K \in \mathbb{R}$, we assume*

$$\mathrm{Ric}_f^N \geq K \mathrm{e}^{\frac{4(\varepsilon-1)f}{n-1}} g.$$

Then for $c := c(n, N, \varepsilon)$ as in (9) *and $R \geq r > 0$, we have*

$$\frac{m_{\{1+\frac{2(1-\varepsilon)}{n-1}\}f}(B_{f,R}(q))}{m_{\{1+\frac{2(1-\varepsilon)}{n-1}\}f}(B_{f,r}(q))} \leq \frac{\mathcal{S}_{cK}(R)}{\mathcal{S}_{cK}(r)}.$$

Below, we assume that f is bounded. In that case, the Laplacian comparison theorem is as follows (see Lu-Minguzzi-Ohta [**79**, Theorem 3.9]):

Proposition 2.1.15 ([**79**]). *Let (M, g, f) be an n-dimensional complete weighted Riemannian manifold, $N \in (-\infty, 1] \cup [n, \infty]$ and $\varepsilon \in \mathbb{R}$ in the ε-range. For $K \in \mathbb{R}$ and $b_2 \geq b_1 > 0$, we assume*

$$\mathrm{Ric}_f^N \geq K \mathrm{e}^{\frac{4(\varepsilon-1)f}{n-1}} g, \quad b_1 \leq \mathrm{e}^{\frac{2(1-\varepsilon)}{n-1}f} \leq b_2.$$

We set $\rho := b_1$ if $\mathbf{s}'_{cK}(d_q(x)/b_2) \geq 0$, $\rho := b_2$ if $\mathrm{s}'_{cK}(d_q(x)/b_2) < 0$, where $c := c(n, N, \varepsilon)$ as in (9)*. Then for $q \in M$, we have*

$$\Delta_f d_q(x) \leq \frac{1}{c\,\rho} \frac{\mathbf{s}'_{cK}(d_q(x)/b_2)}{\mathbf{s}_{cK}(d_q(x)/b_2)}, \tag{13}$$

on $M \backslash (\mathrm{Cut}(q) \cup \{q\})$, where $\mathrm{Cut}(q)$ denotes the cut locus of q.

Remark 2.1.16. This corresponds to Wylie-Yeroshkin [**136**, Theorem 4.4] in the case $N = 1$, and also to Kuwae-Li [**58**, Theorem 2.4] if we take $\varepsilon = \frac{N-1}{N-n}$ in the case $N \in (-\infty, 1]$.

Remark 2.1.17. The boundedness of f implies the boundedness of other quantities as follows:

$$\frac{1}{b_2} \leq \mathrm{e}^{\frac{2(\varepsilon-1)}{n-1}} \leq \frac{1}{b_1}, \quad \frac{d_q(x)}{b_2} \leq s_{f,q}(x) \leq \frac{d_q(x)}{b_1}. \tag{14}$$

Remark 2.1.18. Proposition 2.1.13 implies Proposition 2.1.15 as pointed out in [**59**]. For example, in the case $N \in (-\infty, 1)$ and $K < 0$, the right-hand side of (12) is estimated as

$$\frac{1}{c} \frac{\mathbf{s}'_{cK}(s_{f,w}(t))}{\mathbf{s}_{cK}(s_{f,w}(t))} \mathrm{e}^{\frac{2(\varepsilon-1)f(\gamma_w(t))}{n-1}} \leq \frac{1}{b_1} \sqrt{\frac{K}{c}} \coth\left(\frac{\sqrt{cK}}{b_2} d_q(x)\right),$$

where we used (14), and the right-hand side of this inequality coincides with that in (13).

We denote the ball with radius r centered at $x \in M$ by $B_r(x)$. The volume comparison theorem under an assumption that f is bounded is as follows (see [**79**, Theorem 3.11]):

Proposition 2.1.19 ([**79**]). *We assume that (M, g, f) satisfies the same condition as in Proposition* 2.1.15*. Then for $q \in M$ and $R \geq r > 0$ with $R \leq b_2\pi/\sqrt{cK}$, where $\pi/\sqrt{cK} := \infty$ when $K \leq 0$, we have*

$$\frac{m_f(B_R(q))}{m_f(B_r(q))} \leq \frac{b_2}{b_1} \frac{\int_0^{\min\{R/b_1, \pi/\sqrt{cK}\}} \mathbf{s}_{cK}(t)^{1/c}\, \mathrm{d}t}{\int_0^{r/b_2} \mathbf{s}_{cK}(t)^{1/c}\, \mathrm{d}t}.$$

Remark 2.1.20. This corresponds to Wylie-Yeroshkin [**136**, Theorem 4.5] in the case $N = 1$, and also to Kuwae-Li [**58**, Theorem 2.10] if we take $\varepsilon = \frac{N-1}{N-n}$ in the case $N \in (-\infty, 1]$.

Remark 2.1.21. Proposition 2.1.14 implies Proposition 2.1.19. For example, in the case $N \in (-\infty, 1)$ and $K < 0$, for $r > 0$, it follows from (14) that

$$B_{f,r}(q) \subset B_{b_2 r}(q), \quad B_{b_1 r}(q) \subset B_{f,r}(q).$$

This leads us to

$$m_{\left\{1+\frac{2(1-\varepsilon)}{n-1}\right\}f}\left(B_{f,R/b_1}(q)\right) \geq \frac{1}{b_2} m_f(B_R(q)), \quad m_{\left\{1+\frac{2(1-\varepsilon)}{n-1}\right\}f}\left(B_{f,r/b_2}(q)\right) \leq \frac{1}{b_1} m_f(B_r(q)).$$

Combining these, we have

$$\frac{b_1}{b_2}\frac{m_f(B_R(q))}{m_f(B_r(q))} \leq \frac{m_{\left\{1+\frac{2(1-\varepsilon)}{n-1}\right\}f}\left(B_{f,R/b_1}(q)\right)}{m_{\left\{1+\frac{2(1-\varepsilon)}{n-1}\right\}f}\left(B_{f,r/b_2}(q)\right)} \leq \frac{\mathcal{S}_{cK}\left(R/b_1\right)}{\mathcal{S}_{cK}\left(r/b_2\right)},$$

and this recovers Proposition 2.1.19.

2.2. Harmonic functions

In this section, we review Liouville type theorems and gradient estimates. These are obtained as applications of functional inequalities such as Poincaré type inequalities and Sobolev type inequalities. Some results in this section are generalized in the next chapter.

2.2.1. **Functional inequalities.**

Some functional inequalities are useful in analyzing harmonic functions. We review them in this subsection. For $q \in M$ and $r > 0$, we denote the set of compactly supported functions in $C^\infty(B_r(q))$ by $C_0^\infty(B_r(q))$. We have the following Li-Schoen type local Poincaré inequality (see Li-Schoen [**68**, Corollary 1.1], also refer to Schoen-Yau [**117**, Chapter II, Lemma 6.1]):

Theorem 2.2.1 ([**68**])**.** *Let (M, g) be an n-dimensional complete Riemannian manifold. For $K \geq 0$, we assume*

$$\mathrm{Ric}_g \geq -Kg.$$

Then, for $p > 1$, there exist positive constants C and D depending only on n and p such that

$$\int_{B_r(q)} |\varphi|^p \, \mathrm{d}v_g \leq C r^p \mathrm{e}^{D\sqrt{K}\, r} \int_{B_r(q)} |\nabla \varphi|^p \, \mathrm{d}v_g$$

for any $q \in M$, $r > 0$ and $\varphi \in {C_0}^\infty(B_r(q))$.

Remark 2.2.2. This Li-Schoen type Poincaré inequality is an important ingredient in proving the Neumann-Sobolev type inequality (Theorem 3.1.7).

In [**68**], as an application of this inequality, we arrive at the following mean value inequality (see [**68**, Theorem 1.2], also refer to [**117**, Theorem 6.2]):

Theorem 2.2.3 ([**68**])**.** *We assume that (M, g) satisfies the same condition as in Theorem* 2.2.1. *Let u be a non-negative subharmonic function, i.e., $\Delta u \geq 0$. Then there exist positive constants C and D such that*

$$\sup_{B_{(1-\theta)r}(q)} u^2 \leq C\theta^{-D(1+\sqrt{K}\, r)} \frac{1}{v_g(B_r(q))} \int_{B_r(q)} u^2 \, \mathrm{d}v_g$$

for any $\theta \in (0, 1/2)$, $r > 0$, $q \in M$.

Remark 2.2.4. As is pointed out in [**117**, Chapter II, Section 6], it leads to the Yau type Liouville theorem for bounded harmonic functions. This proof of the Yau type Liouville theorem uses a gradient estimate of harmonic functions (see e.g., [**68**, Lemma 1]), giving an alternative proof different from the original proof in Yau [**137**].

The mean value of a function φ on $B_r(q)$ is denoted by

$$\varphi_{B_r(q)} := \frac{1}{m_f(B_r(q))} \int_{B_r(q)} \varphi \, \mathrm{d}m_f.$$

The Neumann-Poincaré inequality holds as follows (see [**116**, Theorem 5.6.5]):

Theorem 2.2.5 ([**12**, **116**])**.** *Let (M, g) be an n-dimensional complete Riemannian manifold. For $K \geq 0$, we assume*

$$\mathrm{Ric}_g \geq -Kg.$$

For $p \in [1, \infty)$, there exist a constant $C > 0$ depending only on n, p and a constant $D > 0$ depending only on n such that

$$\int_{B_r(q)} \left|\varphi - \varphi_{B_r(q)}\right|^p \, \mathrm{d}m_f \leq C\, r^p \mathrm{e}^{D\sqrt{K}\, r} \int_{B_r(q)} |\nabla \varphi|^p \, \mathrm{d}m_f$$

for $q \in M$, $r > 0$ and $\varphi \in C^\infty(B_r(q))$.

Remark 2.2.6. This was obtained by Buser [**12**]. Furthermore, an alternative proof was presented by Saloff-Coste [**116**, Theorem 5.6.5] (see also [**115**, (6)]).

In the case $N \in [n, \infty)$, we possess the following local Sobolev inequality (see Wang et al. [**129**, Lemma 3.2]):

Theorem 2.2.7 ([**129**])**.** *Let (M, g, f) be an n-dimensional weighted Riemannian manifold with $n \geq 2$, and $N \in [n, \infty)$. For $K \geq 0$, we assume*

$$\mathrm{Ric}_f^N \geq -Kg.$$

Then there exists a constant $C > 0$ depending only on n and N such that

$$\left(\int_{B_r(q)} |\varphi|^{\frac{2N}{N-2}} \, \mathrm{d}m_f\right)^{\frac{N-2}{N}} \leq \mathrm{e}^{C\left(1+\sqrt{(N-1)^{-1}K}\right)r} m_f(B_r(q))\, r^2 \int_{B_r(q)} \left(|\nabla \varphi|^2 + r^{-2}\varphi^2\right) \, \mathrm{d}m_f,$$

for any $q \in M$, $r > 0$ and $\varphi \in C_0^\infty(B_r(q))$.

Below, we call this type of Sobolev inequality as the *Saloff-Coste type Sobolev inequality.*

Remark 2.2.8. This is obtained by using the Neumann-Poincaré inequality as is shown in Saloff-Coste [**114**, Theorem 2.1]. The proof is omitted in [**129**]. In the weighted case $N = \infty$, Wu [**133**, Lemma 2.4] obtained this under an additional assumption that f is bounded.

We turn to the weighted case with $N = \infty$. Munteanu-Wang obtained the Neumann-Poincaré inequality for the case $N = \infty$ as follows (see [**87**, Lemma 3.4]):

Theorem 2.2.9 ([**87**])**.** *Let (M, g, f) be an n-dimensional complete weighted Riemannian manifold. We assume*

$$\mathrm{Ric}_f^\infty \geq 0.$$

For $R > 0$ and $q \in M$, we set

$$b(R) := \sup_{B_{3R}(q)} f.$$

There exist positive constants C and D depending only on n such that we have

$$\int_{B_r(x)} \left|\varphi - \varphi_{B_r(x)}\right|^2 \, \mathrm{d}m_f \leq C\mathrm{e}^{Db(R)}\, r^2 \int_{B_r(x)} |\nabla \varphi|^2 \, \mathrm{d}m_f$$

for $x \in B_R(q)$, $0 < r < R$ and $\varphi \in C^\infty(B_r(x))$.

Remark 2.2.10. Munteanu-Wang [**87**] obtained this by applying Buser's argument [**12**].

As an application, Munteanu-Wang [**87**] obtained the following Neumann-Sobolev inequality (see [**87**, Lemma 3.5]):

Theorem 2.2.11 ([**87**])**.** *We assume that (M, g, f) satisfies the same condition as in Theorem* 2.2.9*. There exist positive constants $\nu > 2$, C and D depending only on n such that*

$$\left(\int_{B_r(q)} \left|\varphi - \varphi_{B_r(q)}\right|^{\frac{2\nu}{\nu-2}}\right)^{\frac{\nu-2}{\nu}} \leq \frac{C\, \mathrm{e}^{Db(r)} r^2}{m_f(B_r(q))^{2/\nu}} \int_{B_r(q)} |\nabla \varphi|^2 \, \mathrm{d}m_f$$

for any $q \in M$, $r > 0$ and $\varphi \in C^\infty(B_r(q))$.

Remark 2.2.12. Munteanu-Wang [**87**, Lemma 3.2] used the argument in Hajłasz-Koskela [**47**] to prove this. Their proof does not use the Saloff-Coste type Sobolev inequality.

Under the more general condition of lower bounds of Ric_f^N, Munteanu-Wang [**88**] obtained the Neumann-Poincaré inequality as follows (see [**88**, Lemma 3.2]):

Theorem 2.2.13 ([**88**])**.** *Let (M, g, f) be an n-dimensional complete weighted Riemannian manifold. For $b > 0$, we assume*

$$\mathrm{Ric}_f^\infty \geq -(n-1)g, \qquad \sup_{y \in B_1(x)} |f(x) - f(y)| \leq b$$

for any $x \in M$. Then there exists a constant $C > 0$ depending only on n and b such that

$$\int_{B_r(q)} \left|\varphi - \varphi_{B_r(q)}\right|^2 \leq C \int_{B_r(q)} |\nabla \varphi|^2 \, \mathrm{d}m_f$$

for any $q \in M$, $r > 0$ and $\varphi \in C^\infty(B_r(q))$.

Remark 2.2.14. In the weighted case $N = \infty$, Wu [**133**, Lemma 2.4] obtained this under an additional assumption that f is bounded instead of the condition $\sup_{y \in B_1(x)} |f(x) - f(y)| \leq b$.

The Neumann-Sobolev inequality was generalized as follows (see [**88**, Lemma 3.3]):

Theorem 2.2.15 ([**88**])**.** *We assume that (M, g, f) satisfies the same condition as in Proposition* 2.2.13*. Then there exist constants $\nu > 2$ and $C > 0$ depending only on n and b such that*

$$\left(\int_{B_1(q)} \left|\varphi - \varphi_{B_1(q)}\right|^{\frac{2\nu}{\nu-2}} \, \mathrm{d}m_f\right)^{\frac{\nu-2}{\nu}} \leq \frac{C}{m_f(B_1(q))^{2/\nu}} \int_{B_1(x)} |\nabla \varphi|^2 \, \mathrm{d}m_f$$

for any $q \in M$ and $\varphi \in C^\infty(B_1(q))$.

For $p > 0$, a function φ on M, $q \in M$ and $r > 0$, we set

$$\|\varphi\|_{p,r} := \left(\int_{B_r(q)} |\varphi|^p \, \mathrm{d}v_g\right)^{\frac{1}{p}}, \quad \|\varphi\|_{\infty,r} := \sup_{B_r(q)} |\varphi|.$$

We denote the $(1,2)$-Sobolev space on $B_r(q)$ by $H^{1,2}(B_r(q))$ and the set of compact support elements of it by $H_c^{1,2}(B_r(q))$. We list two mean value inequalities. The first one is as follows (see e.g., Li [**65**, Lemma 11.1]):

Proposition 2.2.16 (cf. [**65**])**.** *Let (M, g) be an n-dimensional complete Riemannian manifold, $q \in M$ and $u \in H^{1,2}(B_r(q))$ be non-negative and satisfy*

$$\Delta u \geq -\phi u,$$

for some non-negative $\phi \in C^\infty(B_r(q))$*. We set* $p_1 := n/2$ *if* $n > 2$*, and we take* p_1 *arbitrary in* $(1, \infty)$ *if* $n = 2$*, and also* $p_2 > 0$ *such that* $\frac{1}{p_1} + \frac{1}{p_2} = 1$*. We assume* $\|\phi\|_p < \infty$ *for some* $p > p_1$*, and that there exists a constant* $C^{Sob} > 0$ *such that*

$$\frac{1}{v_g(B_r(q))} \int_{B_r(q)} |\nabla \varphi|^2 \, \mathrm{d}v_g \geq \frac{C^{Sob}}{R^2} \left(\frac{1}{v_g(B_r(q))} \int_{B_r(q)} \varphi^{2p_2} \, \mathrm{d}v_g \right)^{\frac{1}{p_2}}$$

for any $\varphi \in H_c^{1,2}(B_r(q))$*. For fixed* $\theta \in (0,1)$*, let* $C^{vol} > 0$ *be a constant such that*

$$\frac{v_g(B_r(q))}{v_g(B_{\theta r}(q))} \leq C^{vol}.$$

Then for any $\sigma > 0$*, there exists a constant* $C^{\mathrm{sub}} > 0$ *depending only on* $\sigma, p_1, p, C^{Sob}, C^{vol}$ *such that we have*

$$\|u\|_{\infty, \theta r} \leq C^{\mathrm{sub}} \left((Ar^2)^{\frac{p}{p-p_1}} + (1-\theta)^{-2} \right)^{\frac{p_1}{\sigma}} \frac{\|u\|_{\sigma, r}}{v_g(B_r(q))^{1/\sigma}},$$

where we set

$$A := \begin{cases} \left(v_g(B_r(q))^{-1} \int_{B_r(q)} \phi^p \, \mathrm{d}v_g \right)^{\frac{1}{p}} & \text{if } p < \infty, \\ \|\phi\|_{\infty, r} \text{ if } p = \infty. \end{cases}$$

For $q \in M$ and $r > 0$, we denote the first Neumann eigenvalue of $B_r(q)$ by $\lambda_1^{\mathrm{Neu}}(q, r)$. The variational characterization of $\lambda_1^{\mathrm{Neu}}(q, r)$ is as follows:

$$\lambda_1^{\mathrm{Neu}}(q, r) = \inf_{\varphi \in C^\infty(B_r(q))} \left\{ \frac{\int_{B_r(q)} |\nabla \varphi|^2 \, \mathrm{d}v_g}{\int_{B_r(q)} \varphi^2 \, \mathrm{d}v_g} \;\middle|\; \int_{B_r(q)} \varphi \, \mathrm{d}v_g = 0 \right\}.$$

The second one is as follows (see e.g., [**65**, Lemma 11.2]):

Proposition 2.2.17 (cf. [**65**])**.** *Let* (M, g) *be an* n*-dimensional complete Riemannian manifold,* $q \in M$*,* $r > 0$*, and let* $u \in H^{1,2}(B_r(q))$ *be non-negative and satisfy*

$$\Delta u \leq \lambda u$$

for a constant $\lambda \geq 0$*. As in Theorem 2.2.16, we take* p_1, p_2, C^{Sob}*, and we assume that there exist the same positive constants* C^{vol} *for* $\theta := 1/16$*. Furthermore, we assume that there exists a constant* $C^{\mathrm{NP}} > 0$ *such that we have*

$$\min \left\{ \frac{r^2}{16} \lambda_1^{\mathrm{Neu}} \left(q, \frac{r}{4} \right), \frac{r^2}{4} \lambda_1^{\mathrm{Neu}} \left(q, \frac{r}{2} \right) \right\} \geq C^{\mathrm{NP}}.$$

Then for sufficiently small $\sigma > 0$*, there exists a constant* $C^{\mathrm{sup}} > 0$ *depending only on* σ, p_1, C^{Sob}*,* C^{NP}, C^{vol} *and* $\lambda r^2 + 1$ *such that*

$$\frac{\|u\|_{\sigma, r/8}}{v_g(B_{r/8}(q))^{1/\sigma}} \leq C^{\mathrm{sup}} \inf_{B_{r/16}(q)} u.$$

These mean value inequalities imply the Harnack inequality (see e.g., [**65**, Theorem 11.1]):

Theorem 2.2.18 ([**65**])**.** *Let* (M, g) *be an* n*-dimensional complete Riemannian manifold,* $q \in M$*,* $r > 0$*, and let* $u \in H^{1,2}(B_r(q))$ *be non-negative and satisfy*

$$|\Delta u| \leq \lambda u$$

for some constant $\lambda \geq 0$. As in Proposition 2.2.17, we take p_1, and we assume that there exist the same positive constants C^{Sob}, C^{vol} for $\theta = 1/16$ and C^{NP}. Then there exists a constant $C^{\mathrm{Har}} > 0$ depending only on n, p_1, $C^{Sob}, C^{vol}, C^{\mathrm{NP}}$ and $\lambda r^2 + 1$ such that

$$\sup_{B_{r/16}(q)} u \leq C^{\mathrm{Har}} \inf_{B_{r/16}(q)} u.$$

2.2.2. **Liouville type theorems and gradient estimates for harmonic functions.** We call $u \in C^\infty(M)$ an *f-harmonic function* if $\Delta_f u = 0$. Brighton [**11**] generalized the Yau type Liouville theorem for bounded harmonic functions to the weighted case $N = \infty$ as follows (see [**11**, Theorem 1]):

Theorem 2.2.19 ([**11**])**.** *Let (M, g, f) be a complete weighted Riemannian manifold. We assume*

$$\mathrm{Ric}_f^\infty \geq 0.$$

Then any bounded f-harmonic function must be a constant function.

Remark 2.2.20. This is obtained as an application of a new type of gradient estimate (see [**11**, Theorem 2]). In obtaining this gradient estimate, Brighton [**11**] modified the Yau type gradient estimate.

In [**11**], he also pointed out that we cannot obtain the Yau type Liouville theorem for positive harmonic functions by showing a counterexample:

Example 2.2.21 ([**11**])**.** For $x \in \mathbb{R}^n$, let x_1 denote the first coordinate. We set $f(x) := x_1$ and $u(x) := \mathrm{e}^{x_1}$. Then we have $\Delta_f u = 0$ and $(\mathbb{R}^n, |\cdot|, f)$ satisfies $\mathrm{Ric}_f^\infty \geq 0$.

Yau's Liouville theorem for positive harmonic functions was generalized by Wu [**131**] as follows (see [**131**, Corollary 3.4]):

Theorem 2.2.22 ([**131**])**.** *Let (M, g, f) be an n-dimensional complete non-compact weighted Riemannian manifold and $N \in [n, \infty)$. We assume*

$$\mathrm{Ric}_f^N \geq 0.$$

Then any positive f-harmonic function must be a constant function.

Remark 2.2.23. Yau [**137**, Corollary 1] obtained the unweighted case $f \equiv 0$ by a gradient estimate for harmonic functions. Wu [**131**] used the gradient estimate for a more general equation $\Delta_f u + \lambda u = 0$, which is an adaptation of arguments in [**67**, Theorem 6.1].

A function φ on M is said to have *linear growth rate of* b_1 if we have $\varphi(x) \leq b_1 d_q(x) + b_2$ for some $q \in M$ and positive constants b_1, b_2. Also, a function φ is said to be of *sublinear growth* if we have

$$\lim_{d_q(x)\to\infty} \frac{|\varphi(x)|}{d_q(x)} = 0$$

for some $q \in M$. The following gradient estimate was obtained by Munteanu-Wang [**87**] (see [**87**, Theorem 3.1]):

Theorem 2.2.24 ([**87**])**.** *Let (M, g, f) be an n-dimensional complete non-compact weighted Riemannian manifold. For $b > 0$, we assume that f has linear growth late b and*

$$\mathrm{Ric}_f^\infty \geq 0.$$

Let u be a positive f-harmonic function. Then there exists a constant $C > 0$ depending only on n such that

$$|\nabla \log u| \leq C\, b.$$

Remark 2.2.25. Munteanu-Wang [**87**] modified the arguments of the Brighton type gradient estimate to prove this theorem.

As a corollary, we possess the following Liouville property (see [**87**, Corollary 3.2]):

Corollary 2.2.26 ([**87**])**.** *Let (M, g, f) be a complete non-compact weighted Riemannian manifold. We assume that f is of sublinear growth and*

$$\mathrm{Ric}_f^\infty \geq 0.$$

Then any positive f-harmonic function must be a constant function.

An example shows that this is sharp as follows (see [**87**, Example 1.2]):

Example 2.2.27 ([**87**])**.** Let g denote the Riemannian product metric of $\mathbb{R} \times \mathbb{S}^{n-1}$. For $\alpha > 0$, $t \in \mathbb{R}$ and $w \in \mathbb{S}^{n-1}$, we set $f(t, w) := \alpha t$ and $u(t, w) := \mathrm{e}^{\alpha t}$. Then we have $\Delta_f u = 0$ and $(\mathbb{R} \times \mathbb{S}^{n-1}, g, f)$ satisfies $\mathrm{Ric}_f^\infty \geq 0$.

As for sublinear growth harmonic functions, Munteanu-Wang [**87**] showed the following Cheng type Liouville property (see [**87**, Theorem 3.2]):

Theorem 2.2.28 ([**87**])**.** *Let (M, g, f) be a complete non-compact weighted Riemannian manifold. We assume that f is bounded and*

$$\mathrm{Ric}_f^\infty \geq 0.$$

Then any sublinear growth f-harmonic function must be a constant function.

Remark 2.2.29. We refer to Theorem 1.3.2 for the unweighted case $f \equiv 0$. Munteanu-Wang [**87**] proved this by combining the De Giorgi-Nash-Moser theory and the Brighton type gradient estimate. This was a new proof even for the unweighted case $f \equiv 0$.

As a generalization of the gradient estimate in Theorem 2.2.24, Munteanu-Wang [**87**] obtained the following gradient estimate (see [**87**, Theorem 3.1]):

Theorem 2.2.30 ([**88**])**.** *Let (M, g, f) be an n-dimensional complete non-compact weighted Riemannian manifold. For $K \geq 0$ and $b > 0$, we assume*

$$\mathrm{Ric}_f^\infty \geq -Kg, \qquad \sup_{y \in B_1(x)} |f(y) - f(x)| \leq b$$

for any $x \in M$. Let u be a positive f-harmonic function. Then there exists a constant $C > 0$ depending only on n, b, K such that

$$|\nabla \log u| \leq C.$$

Remark 2.2.31. Munteanu-Wang [**88**] proved this by the De Giorgi-Nash-Moser theory.

We turn to the L^p-Liouville theorem. Wu [**133**] obtained the weighted case $N = \infty$ as follows (see [**133**, Theorem 6.1]):

Theorem 2.2.32. *Let (M, g, f) be an n-dimensional complete non-compact weighted Riemannian manifold. For $b > 0$, we assume $|f| \leq b$. Then there exists a constant $\delta > 0$ depending only on n and b such that the following assertion holds*:

We assume that there exists $q \in M$ such that

$$\mathrm{Ric}_f^\infty \geq \delta d_q^{-2} g$$

when d_q is sufficiently large. Let u be a non-negative $L^p(m_f)$-function with $p \in (0, \infty)$ with $\Delta_f u \geq 0$. Then u is identically zero.

Remark 2.2.33. For the unweighted case $f \equiv 0$, we refer to Theorem 1.3.4.

2.3. Eigenfunctions

We review various Cheng type estimates of upper bounds of eigenvalue. Especially in the weighted case $N = \infty$, we note that Munteanu-Wang applied the Liouville theorem to obtain the rigidity of Cheng type inequality.

On (M, g, f), we denote the first eigenvalue of the p-Laplacian by $\lambda_{f,p}$. The variational characterization of $\lambda_{f,p}$ is as follows:

$$\lambda_{f,p} := \inf_{\varphi \in C_0^\infty(M)} \frac{\int_M |\nabla \varphi|^p \, \mathrm{d}m_f}{\int_M \varphi^p \, \mathrm{d}m_f}.$$

This coincides with the first eigenvalue of the weighted Laplacian when $p = 2$. Wang [**128**] obtained the following Cheng type theorem (see [**128**, Theorem 3.2]):

Theorem 2.3.1 ([**128**])**.** *Let (M, g, f) be an n-dimensional complete non-compact weighted Riemannian manifold and $N \in [n, \infty)$. For $K \geq 0$, we assume*

$$\mathrm{Ric}_f^N \geq -Kg.$$

Then we have

$$\lambda_{f,p} \leq \left(\frac{\sqrt{(N-1)K}}{p} \right)^p.$$

Remark 2.3.2. Wang [**128**] used the volume comparison theorem to show this inequality. For the case $p = 2$, Wu [**131**, Theorem 1.1] and Wang [**127**, Theorem 1.1] obtained the Cheng type inequality by the gradient estimate of solutions of $\Delta_f u + \lambda u = 0$ for some constant $\lambda \geq 0$.

Remark 2.3.3. For the unweighted case $f \equiv 0$ and $p = 2$, we refer to Theorem 1.3.5. Cheng [**25**] obtained it by an explicit calculation of the eigenfunctions of the model spaces without using the gradient estimates or the volume comparison theorems. Schoen-Yau [**117**, Chapter III, Section 3] pointed out that the theory of heat kernel simplifies the proof. There is also a different proof using a gradient estimate of solutions of $\Delta u + \lambda u = 0$ for some constant $\lambda \geq 0$ (see Li [**67**, Corollary 6.4]).

We turn to the weighted case with $N = \infty$. Wang [**128**] obtained the following inequality (see [**128**, Theorem 3.3]):

Theorem 2.3.4 ([**128**])**.** *Let (M, g, f) be an n-dimensional complete non-compact weighted Riemannian manifold. For $K \geq 0$ and $b \geq 0$, we assume*

$$\mathrm{Ric}_f^\infty \geq -Kg, \quad \frac{\partial f}{\partial r} \geq -b,$$

where $r(x) := d_q(x)$ for some fixed $q \in M$. Then we have

$$\lambda_{f,p} \leq \left(\frac{\sqrt{(n-1)K} + b}{p} \right)^p.$$

For the case $p = 2$ and $N = \infty$, we list two other types below. The first one is as follows (see Munteanu-Wang [**87**, Theorem 2.2]):

Theorem 2.3.5 ([**87**])**.** *Let (M, g, f) be a complete non-compact weighted Riemannian manifold. For $b \geq 0$, we assume that f has linear growth rate b and*

$$\mathrm{Ric}_f^\infty \geq 0.$$

Then we have

$$\lambda_{f,2} \leq \frac{b^2}{4}.$$

The second one is as follows (see Munteanu-Wang [**88**, Theorem 2.2]):

Theorem 2.3.6 ([**88**])**.** *Let (M, g, f) be a complete non-compact weighted Riemannian manifold. For $b \geq 0$, we assume that f has a linear growth rate b and*

$$\mathrm{Ric}_f^\infty \geq -(n-1)g.$$

Then we have

$$\lambda_{f,2} \leq \frac{(n-1+b)^2}{4}.$$

Remark 2.3.7. Munteanu-Wang [**87**, **88**] used volume comparison theorems to prove these Cheng type inequalities, and their rigidities were also obtained. In the weighted case $N = \infty$, under the assumption that $|\nabla f|$ is bounded, Su-Zhang [**121**, Proposition 2.1] also obtained this together with its rigidity. Wu [**132**, Theorem A] also obtained a Cheng type inequality by the gradient estimate of solutions of $\Delta_f u + \lambda u = 0$.

2.4. Porous medium equation

In this section, we give the Li-Yau gradient estimate and Aronson-Bénilan gradient estimate. Li-Yau [**72**] obtained the gradient estimate as follows (see Li-Yau [**72**, Theorem 1.2]):

Theorem 2.4.1 ([**72**])**.** *Let (M, g) be an n-dimensional complete Riemannian manifold. For $K \geq 0$, $q \in M$ and $r > 0$, we assume*

$$\mathrm{Ric}_g \geq -Kg$$

on $B_r(q)$. Let u be a positive smooth solution of $\partial_t u = \Delta u$ on $B_q(2r) \times [0, T]$. Then for any $\alpha > 1$, there exists a constant $C > 0$ depending only on n such that

$$\frac{|\nabla u|^2}{u^2} - \alpha \frac{\partial_t u}{u} \leq \frac{C\,\alpha^2}{r^2} \left(\frac{\alpha^2}{\alpha - 1} + \sqrt{K}\, r \right) + \frac{n\alpha^2 K}{2(\alpha - 1)} + \frac{n\alpha^2}{2t}$$

on $B_r(q) \times (0, T]$.

By letting $r \to \infty$, we have the following estimate (see [**72**, Theorem 1.3]):

Corollary 2.4.2 ([**72**])**.** *Let (M, g) be an n-dimensional complete Riemannian manifold. For $K \geq 0$, we assume*

$$\mathrm{Ric}_g \geq -Kg.$$

Let u be a positive smooth solution to $\partial_t u = \Delta u$ on $M \times [0, T]$. Then for any $\alpha > 1$, we have

$$\frac{|\nabla u|^2}{u^2} - \alpha \frac{\partial_t u}{u} \leq \frac{n\alpha^2 K}{2(\alpha - 1)} + \frac{n\alpha^2}{2t}$$

on $M \times (0,T]$. *In particular, if* $K=0$, *we have*

$$\frac{|\nabla u|^2}{u^2} - \frac{\partial_t u}{u} \leq \frac{n}{2t} \tag{15}$$

on $M \times (0,T]$.

Aronson-Bénilan type gradient estimates for the porous medium equation can be regarded as a counterpart of Li-Yau type gradient estimates for the heat equation. Huang-Li [**50**] obtained the following Aronson-Bénilan type estimate (see [**50**, Theorem 1.6]):

Theorem 2.4.3 ([**50**]). *Let* (M,g,f) *be an* n*-dimensional complete weighted Riemannian manifold and* $N \in [n,\infty)$. *For* $K \geq 0$, $q \in M$ *and* $r > 0$, *we assume*

$$\mathrm{Ric}_f^N \geq -Kg$$

on $B_{2r}(q)$. *Let* u *be a positive smooth solution to* $\partial_t u = \Delta_f u^m$ *with* $m > 1$ *on* $B_{2r}(q) \times [0,T]$. *We set*

$$v := \frac{m}{m-1} u^{m-1}, \quad L := (m-1) \sup_{B_{2r}(p)\times[0,T]} v, \quad a := \frac{N(m-1)}{N(m-1)+2}.$$

Then for any $\alpha > 1$, *there exist positive constants* C *and* D *depending only on* N *such that*

$$\frac{|\nabla v|^2}{v} - \alpha \frac{\partial_t v}{v} \leq \Bigg[\frac{a\alpha^2 m L^{1/2}}{(\alpha-1)^{1/2}(m-1)^{1/2}} \frac{C}{r} + a^{1/2}\alpha \left\{ \frac{1}{t} + \frac{KL}{2(\alpha-1)} + \frac{D\,L}{r^2}\left(1+\sqrt{K}\,r \coth\left(\sqrt{\frac{K}{N-1}}\,r\right)\right)\right\}^{\frac{1}{2}} \Bigg]^2$$

on $B_r(q) \times (0,T]$.

Remark 2.4.4. In the unweighted case $f \equiv 0$, this was obtained by Huang-Huang-Li [**49**, Theorem 1.1], which is an improvement of Aronson-Bénilan gradient estimate by Lu-Ni-Vázquez-Villani [**78**, Theorem 3.3].

By letting $r \to \infty$, we have the following estimate (see e.g., [**50**, Theorem 1.6]):

Corollary 2.4.5 ([**50**]). *Let* (M,g,f) *be an* n*-dimensional complete non-compact weighted Riemannian manifold and* $N \in [n,\infty)$. *For* $K \geq 0$, *we assume*

$$\mathrm{Ric}_f^N \geq -Kg.$$

Let u *be a positive smooth solution to* $\partial_t u = \Delta_f u^m$ *with* $m > 1$ *on* $M \times [0,T]$. *We set*

$$v := \frac{m}{m-1} u^{m-1}, \quad L := (m-1) \sup_{M\times[0,T]} v, \quad a := \frac{N(m-1)}{N(m-1)+2}.$$

Then for any $\alpha > 1$, *we have*

$$\frac{|\nabla v|^2}{v} - \alpha \frac{\partial_t v}{v} \leq a\alpha^2 \left(\frac{1}{t} + \frac{KL}{2(\alpha-1)} \right)$$

on $M \times (0,T]$. *In particular, if* $K = 0$, *we have*

$$\frac{|\nabla v|^2}{v} - \frac{\partial_t v}{v} \leq \frac{a}{t} \tag{16}$$

on $M \times (0,T]$.

Remark 2.4.6. In the unweighted case $f \equiv 0$, a gradient estimate as in (16) was obtained by Vázquez [**125**, Proposition 11.12], and Li-Li [**73**, Theorem 7.6] generalized it on compact weighted Riemannian manifolds in the case $N \in [n, \infty)$. Their proof on compact manifolds is simpler than the case of non-compact manifolds.

Remark 2.4.7. Below, we observe that the Aronson-Bénilan type estimate (16) recovers the Li-Yau type gradient estimate when $m \searrow 1$ in the unweighted case $f \equiv 0$. We have

$$\nabla v = mu^{m-2}\nabla u, \quad \partial_t v = mu^{m-2}\left\{m(m-1)|\nabla u|^2 u^{m-2} + mu^{m-1}\Delta u\right\}.$$

Combining these, the right-hand side of the Aronson-Bénilan estimate (16) is calculated as

$$\frac{|\nabla v|^2}{v} - \frac{\partial_t v}{v} = (m-1)\left(mu^{m-3}|\nabla u|^2 - mu^{m-2}\Delta u\right).$$

Then we see that (16) implies

$$mu^{m-3}|\nabla u|^2 - mu^{m-2}\Delta u \leq \frac{n}{(n(m-1)+2)\,t}. \tag{17}$$

Letting $m \searrow 1$, we obtain

$$\frac{|\nabla u|^2}{u^2} - \frac{\Delta u}{u} \leq \frac{n}{2t}.$$

This coincides with the Li-Yau gradient estimate (15). We also remark that (17) coincides with the classical Aronson-Bénilan estimate (5) on Euclidean space.

CHAPTER 3

Analysis of harmonic functions

In this chapter, we show a Liouville theorem for harmonic functions of sublinear growth (Theorem 3.3.4), an L^p-Liouville theorem (Theorem 3.2.4) and a gradient estimate (Theorem 3.4.1). These are obtained using the mean value inequality (Theorem 3.1.8) under lower bounds of Ric_f^N with ε-range.

3.1. Functional inequalities

The purpose of this section is to obtain the mean value inequality. In order to obtain the mean value inequality, we first present a Neumann-Sobolev type inequality (Theorem 3.1.7). Although Munteanu-Wang obtained this by the argument in Hajłasz-Koskela [**47**], we take a different approach. Ingredients of our approach are the Neumann-Poincaré type inequality (Proposition 3.1.1), the Saloff-Coste type Sobolev inequality (Proposition 3.1.2) and the Li-Schoen type Poincaré inequality (Proposition 3.1.6).

3.1.1. **Sobolev inequality.** Let (M, g, f) be an n-dimensional complete weighted Riemannian manifold, $N \in (-\infty, 1] \cup [n, \infty]$ and $\varepsilon \in \mathbb{R}$ in the ε-range. For $x, y \in M$ and $r > 0$, we set

$$K_\varepsilon(x) := \max\left\{0, \sup_{w \in U_x M} \left(-\mathrm{e}^{\frac{4(1-\varepsilon)f(x)}{n-1}} \mathrm{Ric}_f^N(w, w)\right)\right\}, \quad K_\varepsilon(y, r) := \sup_{x \in B_r(y)} K_\varepsilon(x).$$

The Neumann-Poincaré type inequality holds as follows (see [**38**, Theorem 3.1]):

Proposition 3.1.1 ([**38**]). *Let (M, g, f) be an n-dimensional complete weighted Riemannian manifold, $N \in (-\infty, 1] \cup [n, \infty]$ and $\varepsilon \in \mathbb{R}$ in the ε-range. For $b_2 \geq b_1 > 0$ we assume*

$$b_1 \leq \mathrm{e}^{\frac{2(1-\varepsilon)f}{n-1}} \leq b_2.$$

Then for $c := c(n, N, \varepsilon)$ as in (9), $q \in M$, $r > 0$ and $\varphi \in C^\infty(B_r(q))$, we have

$$\int_{B_r(q)} \left|\varphi - \varphi_{B_r(q)}\right|^2 \, \mathrm{d}m_f \leq 2^{n+3} \left(\frac{2b_2}{b_1}\right) \exp\left(\frac{K_\varepsilon(q, 2r)}{c} \frac{2r}{b_1}\right) r^2 \int_{B_{2r}(q)} |\nabla\varphi|^2 \, \mathrm{d}m_f.$$

Proof. For $x, y \in M$, let $\gamma_{x,y}$ denote a minimal geodesic connecting from x to y parameterized by arclength. For $\tau \in [0,1]$, we set $l_{x,y}(\tau) := \gamma_{x,y}(\tau d(x,y))$. We have

(18)
$$\begin{aligned}
\int_{B_r(q)} |\varphi - \varphi_{B_r(q)}|^2 \, \mathrm{d}m_f &= \int_{B_r(q)} \left| \int_{B_r(q)} \frac{\varphi(x) - \varphi(y)}{m_f(B_r(q))} \, \mathrm{d}m_f(y) \right|^2 \, \mathrm{d}m_f(x) \\
&\leq \frac{1}{m_f(B_r(q))} \int_{B_r(q)} \int_{B_r(q)} \left(\int_0^1 \left| \frac{\mathrm{d}(\varphi \circ l_{x,y})}{\mathrm{d}\tau}(\tau) \right| \, \mathrm{d}\tau \right)^2 \, \mathrm{d}m_f(x) \, \mathrm{d}m_f(y) \\
&\leq \frac{1}{m_f(B_r(q))} \int_{B_r(q)} \int_{B_r(q)} \int_0^1 \left| \frac{\mathrm{d}(\varphi \circ l_{x,y})}{\mathrm{d}\tau}(\tau) \right|^2 \, \mathrm{d}\tau \, \mathrm{d}m_f(x) \, \mathrm{d}m_f(y) \\
&= \frac{2}{m_f(B_r(q))} \int_{B_r(q)} \int_{B_r(q)} \int_{1/2}^1 \left| \frac{\mathrm{d}(\varphi \circ l_{x,y})}{\mathrm{d}\tau}(\tau) \right|^2 \, \mathrm{d}\tau \, \mathrm{d}m_f(x) \, \mathrm{d}m_f(y),
\end{aligned}$$

where we used $l_{x,y}(\tau) = l_{y,x}(1-\tau)$ in the last equality.

We set the unit vector $w := \partial_t \gamma_{x,y}(t)|_{t=0}$ and $J(x,t,w)$ be the Jacobian of the map $\exp_x : T_x M \to M$ at tw with respect to m_f, i.e., we have

$$\mathrm{d}m_f = J(x,t,w) \, \mathrm{d}t \, \mathrm{d}w.$$

For $\tau \in [0,1]$, we denote the Jacobian of the map $\Phi_{x,\tau} : y \mapsto \gamma_{x,y}(\tau d(x,y))$ by $J_{x,\tau}$. First, we estimate $J_{x,\tau}$ from below. For $t := \tau d(x,y)$, we obtain

$$J_{x,\tau}(y) = \left(\frac{t}{d(x,y)} \right)^n \frac{J(x,t,w)}{J(x,d(x,y),w)}.$$

We denote the reparametrization (11) by

$$s_\gamma(t) := \int_0^t \mathrm{e}^{\frac{2(\varepsilon-1)f(\gamma(\xi))}{n-1}} \, \mathrm{d}\xi, \tag{19}$$

where $\gamma := \gamma_{x,y}$. It follows from the argument in [**79**, Theorem 3.6] that the quantity $\frac{J(x,t,w)}{\mathbf{s}_{-cK}(s_\gamma(t))}$ is non-increasing. This yields

$$\left(\frac{t}{d(x,y)} \right)^n \frac{J(x,t,w)}{J(x,d(x,y),w)} \geq \left(\frac{t}{d(x,y)} \right)^n \frac{\mathbf{s}_{-cK}(s_\gamma(t))^{1/c}}{\mathbf{s}_{-cK}(s_\gamma(d(x,y)))^{1/c}}.$$

Below, we set $K := K_\varepsilon(q, 2r)$. For $x, y \in B_r(q)$ and $t \in (d(x,y)/2, d(x,y))$, this leads us to

$$\begin{aligned}
J_{x,\tau}(y) &\geq \frac{1}{2^n} \frac{\mathbf{s}_{-cK}(s_\gamma(t))^{1/c}}{\mathbf{s}_{-cK}(s_\gamma(d(x,y)))^{1/c}} \\
&\geq \frac{1}{2^n} \left(\frac{s_\gamma(d(x,y)/2)}{s_\gamma(d(x,y))} \right) \exp \left(-\sqrt{\frac{K}{c}} s_\gamma(d(x,y)) \right) \\
&\geq \frac{1}{2^n} \left(\frac{b_1}{2b_2} \right) \exp \left(-\sqrt{\frac{K}{c}} \frac{2r}{b_1} \right).
\end{aligned}$$

Setting

$$F(r) := \left\{ \frac{1}{2^n} \left(\frac{b_1}{2b_2} \right) \exp \left(-\sqrt{\frac{K}{c}} \frac{2r}{b_1} \right) \right\}^{-1},$$

we see

$$J_{x,\tau}(y) \geq F(r)^{-1}$$

for $\tau \in [1/2, 1]$. Then the last term of (18) is estimated as follows:

$$\begin{aligned}
(18) &\leq F(r) \int_{B_r(q)} \int_{B_r(q)} \int_{1/2}^{1} \left| \frac{\mathrm{d}\varphi(l_{x,y}(\tau))}{\mathrm{d}\tau} \right| J_{x,\tau}(t)\ \mathrm{d}\tau\ \mathrm{d}m_f(x)\ \mathrm{d}m_f(y) \\
&\leq F(r) \int_{B_r(q)} \int_{B_r(q)} \int_{0}^{1} |\nabla\varphi(l_{x,y}(t))|^2 d(x,y)^2 J_{x,\tau}\ \mathrm{d}t\ \mathrm{d}m_f(x)\ \mathrm{d}m_f(y) \\
&\leq 4r^2 F(r) \int_0^1 \int_{B_r(q)} \int_{B_r(q)} |\nabla\varphi(l_{x,y}(t))|^2 J_{x,\tau}(y)\ \mathrm{d}m_f(x)\ \mathrm{d}m_f(y)\ \mathrm{d}\tau \\
&\leq 4r^2 F(r) \int_0^1 \int_{B_r(q)} \int_{\Phi_{x,\tau}(B_r(q))} |\nabla\varphi(z)|\ \mathrm{d}m_f(z)\ \mathrm{d}m_f(x)\ \mathrm{d}\tau \\
&\leq 4r^2 F(r) m_f(B_r(q)) \int_{B_{2r}(q)} |\nabla\varphi|^2\ \mathrm{d}m_f,
\end{aligned}$$

where we used $\Phi_{x,\tau}(B_r(q)) \subset B_{2r}(q)$ in the last inequality. We complete the proof. □

The Saloff-Coste type local Sobolev inequality is as follows (see [**38**, Theorem 3.2]):

Proposition 3.1.2 ([**38**])**.** *We assume that (M, g, f) satisfies the same condition as in Theorem 3.1.1. We set*

$$\nu := \begin{cases} 3 & \text{if } c = 1, \\ 1 + \frac{1}{c} & \text{if } c < 1. \end{cases} \tag{20}$$

There exist positive constants C and D depending only on n, c, b_1, b_2 such that

$$\begin{aligned}
\left(\frac{1}{m_f(B_r(q))} \int_{B_r(q)} |\varphi|^{\frac{2\nu}{\nu-2}}\ \mathrm{d}m_f \right)^{\frac{\nu-2}{\nu}} &\leq C \exp\left(D\sqrt{K_\varepsilon(q, 10r)}\, r \right) r^2 \\
&\quad \times \frac{1}{m_f(B_r(q))} \int_{B_r(q)} \left(|\nabla\varphi|^2 + r^{-2}\varphi^2 \right)\ \mathrm{d}m_f
\end{aligned}$$

for any $q \in M$, $r > 0$, and $\varphi \in C_0^\infty(B_r(q))$.

Remark 3.1.3. We refer to [**37**, Theorem 7] and [**38**, Theorem 3.1]. The difference between them is whether the term $K_\varepsilon(q, \cdot)$ appears or not.

We prepare two lemmas to prove Proposition 3.1.2. The first lemma is as follows (see [**38**, Lemma 3.1]):

Lemma 3.1.4 ([**38**])**.** *We assume that (M, g, f) satisfies the same condition as in Proposition 3.1.1. Let $q \in M$, $r > 0$, $\varphi \in C_0^\infty(B_r(q))$. For $0 < s < r$, we set*

$$\chi_s(x, z) := \frac{1}{m_f(B_s(x))} 1_{B_s(x)}(z), \quad \varphi_s(x) := \int_M \chi_s(x, z)\varphi(z)\ \mathrm{d}m_f(z).$$

Then we have

$$\|\varphi_s\|_2 \leq \left(\frac{b_2}{b_1} \right)^{2+\frac{1}{c}} 8^{\frac{1}{2}\left(1+\frac{1}{c}\right)} \left(\frac{r}{s} \right)^{\frac{1}{2}\left(1+\frac{1}{c}\right)} \exp\left(\sqrt{\frac{K_\varepsilon(q, 10r)}{c}}\, \frac{6r}{2a} \right) \frac{\|\varphi\|_1}{m_f(B_r(q))^{1/2}}.$$

Proof. We have

$$\|\varphi\|_2 \le \|\varphi_s\|_1^{\frac{1}{2}} \|\varphi_s\|_\infty^{\frac{1}{2}}. \tag{21}$$

We first estimate $\|\varphi_s\|_1$ from above. By Proposition 2.1.19, for $r_2 > r_1 > 0$ and $x \in M$, we see

$$\frac{m_f(B_{r_2}(x))}{m_f(B_{r_1}(x))} \le \left(\frac{b_2}{b_1}\right)^{2+\frac{1}{c}} \left(\frac{r_2}{r_1}\right)^{1+\frac{1}{c}} \exp\left(\sqrt{\frac{K_\varepsilon(x, r_2)}{c}} \frac{r_2}{b_1}\right). \tag{22}$$

For $x \in \operatorname{supp}\varphi_s$ and $z \in \operatorname{supp}\chi_s(x,\cdot)$, we have $d(x,z) < s$, and $B_s(x) \subset B_{10r}(q)$. Then using (22), we obtain

$$\begin{aligned} m_f(B_s(z)) &\le m_f(B_{2s}(x)) \\ &\le m_f(B_s(x)) \left(\frac{b_2}{b_1}\right)^{2+\frac{1}{c}} 2^{1+\frac{1}{c}} \exp\left(\sqrt{\frac{K_\varepsilon(x, 2s)}{c}} \frac{2s}{b_1}\right) \\ &\le m_f(B_s(x)) \left(\frac{b_2}{b_1}\right)^{2+\frac{1}{c}} 2^{1+\frac{1}{c}} \exp\left(\sqrt{\frac{K_\varepsilon(q, 10r)}{c}} \frac{2s}{b_1}\right). \end{aligned}$$

This leads us to

$$\chi_s(x,z) \le \frac{2^{1+\frac{1}{c}}}{m_f(B_s(z))} \left(\frac{b_2}{b_1}\right)^{2+\frac{1}{c}} \exp\left(\sqrt{\frac{K_\varepsilon(q, 10r)}{c}} \frac{2s}{b_1}\right) 1_{B_s(x)}(z).$$

Hence, we find

$$\begin{aligned} \|\varphi_s\|_1 &\le \int_M \mathrm{d}m_f(x) \int_M \left\{ \frac{2^{1+\frac{1}{c}}}{m_f(B_s(z))} \left(\frac{b_2}{b_1}\right)^{2+\frac{1}{c}} \exp\left(\sqrt{\frac{K_\varepsilon(q, 10r)}{c}} \frac{2s}{b_1}\right) 1_{B_s(x)}(z) |\varphi(z)| \right\} \mathrm{d}m_f(z) \\ &= \left(\frac{b_2}{b_1}\right)^{2+\frac{1}{c}} 2^{1+\frac{1}{c}} \exp\left(\sqrt{\frac{K_\varepsilon(q, 10r)}{c}} \frac{2s}{b_1}\right) \|\varphi\|_1. \end{aligned}$$

Next, we estimate $\|\varphi_s\|_\infty$. From (22), for $x \in \operatorname{supp}\varphi_s$, we have

$$\frac{m_f(B_{4r}(x))}{m_f(B_s(x))} \le \left(\frac{b_2}{b_1}\right)^{2+\frac{1}{c}} \left(\frac{4r}{s}\right)^{1+\frac{1}{c}} \exp\left(\sqrt{\frac{K_\varepsilon(x, 4r)}{c}} \frac{4r}{b_1}\right).$$

Since $B_r(q) \cap B_s(x) \neq \emptyset$, we see $K_\varepsilon(x, 4r) \le K_\varepsilon(q, 10r)$. Combining these, we see

$$\frac{1}{m_f(B_s(x))} \le \left(\frac{b_2}{b_1}\right)^{2+\frac{1}{c}} \left(\frac{4r}{s}\right)^{1+\frac{1}{c}} \exp\left(\sqrt{\frac{K_\varepsilon(q, 10r)}{c}} \frac{4r}{b_1}\right) \frac{1}{m_f(B_r(q))}.$$

This leads us to

$$\begin{aligned} \|\varphi_s\|_\infty &\le \left(\frac{b_2}{b_1}\right)^{2+\frac{1}{c}} \left(\frac{4r}{s}\right)^{1+\frac{1}{c}} \exp\left(\sqrt{\frac{K_\varepsilon(q, 10r)}{c}} \frac{4r}{b_1}\right) \frac{1}{m_f(B_r(q))} \left\| \int_M 1_{B_s(x)} \varphi \, \mathrm{d}m_f \right\|_\infty \\ &\le \left(\frac{b_2}{b_1}\right)^{2+\frac{1}{c}} \left(\frac{4r}{s}\right)^{1+\frac{1}{c}} \exp\left(\sqrt{\frac{K_\varepsilon(q, 10r)}{c}} \frac{4r}{b_1}\right) \frac{\|\varphi\|_1}{m_f(B_r(q))}. \end{aligned}$$

Finally combining these with (21), we deduce

$$\|\varphi_s\|_2 \le \left(\frac{b_2}{b_1}\right)^{2+\frac{1}{c}} 2^{\frac{1}{2}\left(1+\frac{1}{c}\right)} \left(\frac{4r}{s}\right)^{\frac{1}{2}\left(1+\frac{1}{c}\right)} \exp\left(\sqrt{\frac{K_\varepsilon(q, 10r)}{c}} \frac{4r+2s}{2b_1}\right) \frac{\|\varphi\|_1}{m_f(B_r(q))^{1/2}}.$$

We complete the proof. $\square$

The second lemma is as follows (see [**38**, Lemma 3.2]):

Lemma 3.1.5 ([**38**]). *We assume that (M, g, f) satisfies the same condition as in Proposition 3.1.1. Let $q \in M$, $r > 0$ and $\varphi \in C_0^\infty(B_r(q))$. Then we have*

$$\|\varphi - \varphi_s\|_2 \leq 2^{8+\frac{5}{c}+\frac{n}{2}} \left(\frac{b_2}{b_1}\right)^{3+\frac{2}{c}} \exp\left(\sqrt{\frac{K_\varepsilon(q, 10r)}{c}}\,\frac{11r}{a}\right) s\||\nabla\varphi|\|_2.$$

Proof. For the brevity of notations, we set C_1 and C_2 as follows:

$$C_1 := \left(\frac{b_2}{b_1}\right)^{2+\frac{1}{c}} 16^{1+\frac{1}{c}} \exp\left(\sqrt{\frac{K_\varepsilon(q, 10r)}{c}}\,\frac{8r}{b_1}\right),$$

$$C_2 := 2^{n+5} \left(\frac{2b_2}{b_1}\right)^{\frac{1}{c}} \exp\left(\sqrt{\frac{K_\varepsilon(q, 10r)}{c}}\,\frac{4r}{b_1}\right).$$

For fixed $s \in (0, r)$, we see that there exists a set X satisfying the following conditions:

- for $x, y \in X$, $B_{s/2}(x) \cap B_{s/2}(y) = \emptyset$;
- for any $x \in B_{2r}(q)$, there exists $y \in X$ such that $B_{s/2}(x) \cap B_{s/2}(y) \neq \emptyset$.

We label $X = \{x_i \in X \mid i \in I_q\}$, and we set $B_i := B_{s/2}(x_i)$ and $kB_i := B_{(ks)/2}(x_i)$ for $k > 0$. We have

$$B_{2r}(q) \subset \bigcup_{i \in I_q} 2B_i.$$

Indeed, for $x \in B_{2r}(q)$, if $x \notin X$, there exists $y \in X$ such that $B_{s/2}(x) \cap B_{s/2}(y) \neq \emptyset$. Then for $z \in B_{s/2}(x) \cap B_{s/2}(y)$, we obtain $d(x, y) \leq d(x, z) + d(z, y) < s$. This implies $x \in B_s(y)$. The other case $x \in X \cap B_{2r}(q)$ also follows immediately.

For $x \in B_{2r}(q)$, we set

$$I_q(x) := \{i \in I_q(x) \mid x \in 8B_i\}, \quad N_q(x) := \#I_q(x),$$

and let B_x be an element of $\{B_i \mid i \in I_q(x)\}$ such that $x \in 2B_x$. First, we estimate $N_q(x)$. For $i \in I_q(x)$, we have $B_x \subset 16B_i \subset B_{8r}(x_i)$. Together with Proposition 2.1.19, we obtain

$$m_f(B_x) \leq m_f(16B_i) \leq \left(\frac{b_2}{b_1}\right)^{2+\frac{1}{c}} 16^{1+\frac{1}{c}} \exp\left(\sqrt{\frac{K_\varepsilon(x_i, 8r)}{c}}\,\frac{8r}{b_1}\right) m_f(B_i) \leq C_1\, m_f(B_i).$$

This implies

$$\sum_{i \in I_q(x)} m_f(B_i) \geq \frac{N(x)\, m_f(B_x)}{C_1}. \tag{23}$$

Let $x_0 \in B_x$ be the center of B_x. We have

$$\sum_{i \in I_q(x)} m_f(B_i) \leq m_f(16B_x) \leq \left(\frac{b_2}{b_1}\right)^{2+\frac{1}{c}} 16^{1+\frac{1}{c}} \exp\left(\sqrt{\frac{K_\varepsilon(x_0, 8r)}{c}}\,\frac{8r}{b_1}\right) \leq C_1\, m_f(B_x).$$

Together with (23), this leads us to

$$\frac{N_q(x)\, m_f(B_x)}{C_1} \leq C_1\, m_f(B_x),$$

which implies $N_q(x) \leq C_1^2 =: N_0$.

We have

$$\|\varphi - \varphi_s\|_2^2 \le \sum_{i\in I_q(z)} 2\int_{2B_i} \left(|\varphi - \varphi_{4B_i}|^2 + |\varphi_{4B_i} - \varphi_s|^2\right)\, \mathrm{d}m_f. \tag{24}$$

It follows from Proposition 3.1.1 and $B_{4s}(x_i) \subset B_{10r}(q)$ that

$$\int_{4B_i} |\varphi - \varphi_{4B_i}|^2 \, \mathrm{d}m_f \le C_2\, s^2 \int_{8B_i} |\nabla\varphi|^2 \,\mathrm{d}m_f. \tag{25}$$

For any $y \in 2B_i$, since $B_{2r}(x_i) \subset B_{10r}(q)$, we find

$$m_f(B_s(x_i)) \le m_f(B_{2s}(y)) \le m_f(B_s(x)) \left(\frac{b_2}{b_1}\right)^{2+\frac{1}{c}} 2^{1+\frac{1}{c}} \exp\left(\sqrt{\frac{K_\varepsilon(q,10r)}{c}}\frac{2s}{b_1}\right).$$

Combining these, we have

$$\begin{aligned}
&\int_{2B_i} |\varphi_{4B_i} - \varphi_s|^2 \,\mathrm{d}m_f \\
&\le \int_{2B_i} \mathrm{d}m_f(y) \left|\int_{B_s(y)} \frac{1}{m_f(B_s(y))} (\varphi_{4B_i}(y) - \varphi)\,\mathrm{d}m_f\right|^2 \\
&\le \frac{1}{m_f(B_s(x_i))}\left(\frac{b_2}{b_1}\right)^{2+\frac{1}{c}} 2^{1+\frac{1}{c}} \exp\left(\sqrt{\frac{K_\varepsilon(q,10r)}{c}}\frac{2s}{b_1}\right) \int_{2B_i} \mathrm{d}m_f \int_{4B_i} |\varphi_{4B_i} - \varphi|^2 \,\mathrm{d}m_f \\
&\le \left(\frac{b_2}{b_1}\right)^{2+\frac{1}{c}} 2^{1+\frac{1}{c}} \exp\left(\sqrt{\frac{K_\varepsilon(q,10r)}{c}}\frac{2r}{b_1}\right) C_1\, s^2 \int_{8B_i} |\nabla\varphi|^2 \,\mathrm{d}m_f.
\end{aligned} \tag{26}$$

Here, also for the brevity of notations, we set C_3 as follows:

$$C_3 := 4\left(\frac{b_2}{b_1}\right)^{2+\frac{1}{c}} 2^{1+\frac{1}{c}} \exp\left(\sqrt{\frac{K_\varepsilon(q,10r)}{c}}\frac{2r}{b_1}\right) C_2.$$

It follows from (23), (24), (26) that

$$\|\varphi - \varphi_s\|_2 \le C_3\, s^2 \sum_{i\in I_q} \int_{8B_i} |\nabla\varphi|^2 \,\mathrm{d}m_f \le C_3 N_0\, s^2 \|\nabla\varphi\|_2^2;$$

i.e.,

$$\|\varphi - \varphi_s\|_2 \le \sqrt{N_0 C_3}\, s\, \|\nabla\varphi\|_2.$$

We complete the proof. □

Proof of Proposition 3.1.2. Below, for the brevity of notations, we set C_4, C_5, C_6 as follows:

$$\begin{aligned}
C_4 &:= \left(\frac{b_2}{b_1}\right)^{2+\frac{1}{c}} 8^{\frac{1}{2}\left(1+\frac{1}{c}\right)} \exp\left(\sqrt{\frac{K_\varepsilon(q,10r)}{c}}\,\frac{6r}{2a}\right),\\
C_5 &:= 2^{8+\frac{5}{c}+\frac{n}{2}} \left(\frac{b_2}{b_1}\right)^{3+\frac{2}{c}} \exp\left(\sqrt{\frac{K_\varepsilon(q,10r)}{c}}\,\frac{11r}{a}\right),\\
C_6 &:= \frac{\nu C_4}{8C_5}.
\end{aligned}$$

For $0 < s \leq r$, it follows from Lemma 3.1.4 and Lemma 3.1.5 that

$$\begin{aligned}(27)\qquad \|\varphi\|_2 &\leq \|\varphi_s\|_2 + \|\varphi - \varphi_s\|_2 \\ &\leq \frac{C_4}{m_f\left(B_r(q)\right)^{1/2}}\left(\frac{r}{s}\right)^{\frac{\nu}{2}}\|\varphi\|_1 + C_5\, s\|\nabla\varphi\|_2 \\ &\leq \frac{C_4}{m_f\left(B_r(q)\right)^{1/2}}\left(\frac{r}{s}\right)^{\frac{\nu}{2}}\|\varphi\|_1 + 4C_5\, s\left(\|\nabla\varphi\|_2 + \frac{\|\varphi\|_2}{r}\right).\end{aligned}$$

Let s_0 minimize the right-hand side of (27). We have

$$\frac{C_4\, r^{\nu/2}}{m_f\left(B_r(q)\right)^{1/2}}\left(-\frac{\nu}{2}\right)s_0^{-\frac{\nu}{2}-1}\|\varphi\|_1 + 4C_5\, s_0\left(\|\nabla\varphi\|_2 + \frac{\|\varphi\|_2}{r}\right) = 0.$$

Substituting $s = s_0$ to (27), a straightforward calculation leads us to

$$\|\varphi\|_2 \leq \left(C_4 C_6^{\frac{-\nu}{\nu+2}} + 4C_5 C_6^{\frac{\nu}{\nu+2}}\right)\left(r\,\|\nabla\varphi\|_2 + \|\varphi\|_2\right)^{\frac{\nu}{\nu+2}}\left(\frac{\|\varphi\|_1^2}{m_f(B_r(q))}\right)^{\frac{1}{\nu+2}}.$$

Then there exist positive constants C_7 and C_8 depending only on n, b_1, b_2, c such that

$$C_4 C_6^{\frac{-\nu}{\nu+2}} + 4C_5 C_6^{\frac{\nu}{\nu+2}} \leq C_7 \exp\left(C_8\sqrt{K_\varepsilon(q,10r)}\, r\right).$$

Then the desired assertion follows from [**114**, Theorem 2.2]. □

Before we give the Neumann-Sobolev type inequality in Theorem 3.1.7, we show the following Li-Schoen type Poincaré inequality (see [**38**, Lemma 3.3]):

Proposition 3.1.6 ([**38**])**.** *We assume that (M, g, f) satisfies the same condition as in Proposition 3.1.1. We fix $p \geq 1$. Then there exist positive constants C and D depending only on p, b_1, b_2, c such that*

$$\int_{B_r(q)} |\varphi|^p \,\mathrm{d}m_f \leq C\exp\left(D\sqrt{K_\varepsilon(q,5r)}\, r\right)\int_{B_r(q)} |\nabla\varphi|^p \,\mathrm{d}m_f$$

for any $q \in M$, $r > 0$ and $\varphi \in C_0^\infty(B_r(q))$.

Proof. We set $K := K_\varepsilon(q, 5r)$. We fix $y \in \partial B_{3r}(q)$. It follows from Proposition 2.1.15 that

$$(28)\qquad \Delta_f d_y \leq \frac{1}{b_1}\sqrt{\frac{K}{c}}\coth\left(\frac{\sqrt{cK}}{b_2}d_y\right) \leq \frac{1}{b_1}\sqrt{\frac{K}{c}} + \frac{b_2}{b_1 c d_y}$$

on $B_{5r}(q)$. For $x \in B_r(q)$, we have $2r \leq d_y(x) \leq 4r$. Hence,

$$\Delta_f d_y \leq \frac{1}{b_1}\sqrt{\frac{K}{c}} + \frac{b_2}{2b_1 c\, r} =: \sigma.$$

This implies

$$(29)\qquad \Delta_f \mathrm{e}^{-\sigma d_y} = \mathrm{e}^{-\sigma d_y}\left(-\sigma\Delta_f d_y + \sigma^2\right) \geq \frac{\sigma^2}{2}\mathrm{e}^{-\sigma d_y}.$$

For $\varphi \in C_0^\infty(B_r(q))$ with $\varphi \geq 0$, (29) and integration by parts yields

$$\frac{\sigma^2}{2}\int_{B_r(q)} \varphi\,\mathrm{e}^{-\sigma d_y} \leq \sigma\int_{B_r(q)} \mathrm{e}^{-\sigma d_y}\langle\nabla\varphi, \nabla d_y\rangle \,\mathrm{d}m_f \leq \sigma\int_{B_r(q)} \mathrm{e}^{-\sigma d_y}|\nabla\varphi| \,\mathrm{d}m_f.$$

From $2r \leq d_y(x) \leq 4r$, we have

$$\frac{\sigma^2}{2} \int_{B_r(q)} \varphi\, \mathrm{e}^{-4\sigma r}\ \mathrm{d}m_f \leq \sigma \int_{B_r(q)} \mathrm{e}^{-2\sigma r} |\nabla \varphi|\ \mathrm{d}m_f.$$

This leads us to

$$\int_{B_r(q)} \varphi\ \mathrm{d}m_f \leq C_9 r\, \mathrm{e}^{C_{10}\sqrt{K}\, r} \int_{B_r(q)} |\nabla \varphi|\ \mathrm{d}m_f, \tag{30}$$

where C_9 and C_{10} are positive constants depending only on c, b_1, b_2. We replace φ with $|\varphi|$ in (30), and using $|\nabla|\varphi|| = |\nabla\varphi|$, we see

$$\int_{B_r(q)} |\varphi|\ \mathrm{d}m_f \leq C_9 r\, \mathrm{e}^{C_{10}\sqrt{K}\, r} \int_{B_r(q)} |\nabla \varphi|\ \mathrm{d}m_f.$$

For $p > 1$, we replace φ with φ^p in (30), and the Hölder inequality imply

$$\begin{aligned}\int_{B_r(q)} |\varphi|^p\ \mathrm{d}m_f &\leq C_9 r\, \mathrm{e}^{C_{10}\sqrt{K}\, r} \int_{B_r(q)} p|\varphi|^{p-1} |\nabla \varphi|\ \mathrm{d}m_f \\ &\leq C_9 r\, \mathrm{e}^{C_{10}\sqrt{K}\, r} p \left(\int_{B_r(q)} |\varphi|^p\ \mathrm{d}m_f\right)^{\frac{p-1}{p}} \left(\int_{B_r(q)} |\nabla\varphi|^p\ \mathrm{d}m_f\right)^{\frac{1}{p}}.\end{aligned}$$

By dividing both sides by $\left(\int_{B_r(q)} |\varphi|^p\ \mathrm{d}m_f\right)^{\frac{p-1}{p}}$, we see that the desired assertion follows. □

We are now in a position to show the Neumann-Sobolev inequality (see [**38**, Theorme 3.3]):

Theorem 3.1.7 ([**38**]). *We assume that (M, g, f) satisfies the same condition as in Proposition 3.1.1. We set $\nu > 2$ as in (20). There exist positive constants C and D depending only on n, b_1, b_2, c such that*

$$\left(\frac{1}{m_f(B_r(q))} \int_{B_r(q)} |\varphi|^{\frac{2\nu}{\nu-2}}\ \mathrm{d}m_f\right)^{\frac{\nu-2}{\nu}} \leq C \exp\left(D\sqrt{K_\varepsilon(q, 10r)}\, r\right) \frac{r^2}{m_f(B_r(q))} \int_{B_r(q)} |\nabla\varphi|^2\ \mathrm{d}m_f$$

for any $q \in M$, $r > 0$ and $\varphi \in C_0^\infty(B_r(q))$.

Proof. By Proposition 3.1.2, there exist positive constants C_9 and C_{10} depending only on n, c, b_1, b_2 such that

$$\begin{aligned}\left(\frac{1}{m_f(B_r(q))} \int_{B_r(q)} |\varphi|^{\frac{\nu}{\nu-2}}\ \mathrm{d}m_f\right)^{\frac{\nu-2}{\nu}} &\leq C_9 \exp\left(C_{10}\sqrt{K_\varepsilon(q, 10r)}\, r\right) r^2 \\ &\quad \times \frac{1}{m_f(B_r(q))} \int_{B_r(q)} \left(|\nabla\varphi|^2 + r^{-2}\varphi^2\right)\ \mathrm{d}m_f.\end{aligned} \tag{31}$$

It follows from Proposition 3.1.6 that there exist positive constants C_{11} and C_{12} depending only on c, b_1, b_2 such that

$$\int_{B_r(q)} \varphi^2\ \mathrm{d}m_f \leq C_{11} \exp\left(C_{12}\sqrt{K_\varepsilon(q, 10r)}\, r\right) \int_{B_r(q)} |\nabla\varphi|^2\ \mathrm{d}m_f.$$

Hence, the right-hand side of (31) is estimated as

$$\begin{aligned}&C_9 \exp\left(C_{10}\sqrt{K_\varepsilon(q, 10r)}\, r\right) \frac{r^2}{m_f(B_r(q))} \int_{B_r(q)} \left(|\nabla\varphi|^2 + r^{-2}\varphi^2\right)\ \mathrm{d}m_f \\ &\leq C_9 \exp\left(C_{10}\sqrt{K_\varepsilon(q, 10r)}\, r\right) \left(1 + C_{11} \exp\left(C_{12}\sqrt{K_\varepsilon(q, 5r)}\, r\right)\right) \frac{r^2}{m_f(B_r(q))} \int_{B_r(q)} |\nabla\varphi|^2\ \mathrm{d}m_f.\end{aligned}$$

This yields the desired assertion. □

3.1.2. **Mean value inequality.** Functional inequalities in the previous subsection imply the mean value inequality as follows (see [**38**, Theorem 3.4]):

Theorem 3.1.8 ([**38**]). *We assume that (M, g, f) satisfies the same condition as in Proposition 3.1.1. For a constant $\lambda > 0$, let u be a non-negative smooth function with $\Delta_f u \geq -\lambda u$. We set $\nu > 2$ as in (20). Then there exist positive constants C and D depending only on n, p, b_1, b_2, c such that we have*

$$\|u\|_{\infty,\theta r} \leq C \left\{ \exp\left(D\sqrt{K_\varepsilon(q,10r)}\, r \right) \left(\lambda r^2 + \frac{16}{(1-\theta)^2} \right) \right\}^{\frac{\nu}{2p}} \left(\frac{1}{m_f(B_{\theta r}(q))} \int_{B_r(q)} u^p \, \mathrm{d}m_f \right)^{\frac{1}{p}}$$

for any $q \in M$, $r > 0$ and $\theta \in (0,1)$.

Proof. We fix a constant $\alpha \geq 1$, and let $\varphi \in C_0^\infty(B_r(q))$. By a straightforward calculation, we see

$$\begin{aligned}
&\int_{B_r(q)} |\nabla(\varphi u^\alpha)|^2 \, \mathrm{d}m_f \\
&= \int_{B_r(q)} \left(|\nabla\varphi|^2 u^{2\alpha} + 2\alpha \varphi u^{2\gamma-1} \langle \nabla\varphi, \nabla u\rangle + \alpha^2 \varphi^2 u^{2\alpha-2} |\nabla u|^2 \right) \mathrm{d}m_f \\
&\leq \int_{B_r(q)} \left\{ |\nabla\varphi|^2 u^{2\alpha} + \alpha \left(\varphi u^{2\alpha-1} \langle \nabla\varphi, \nabla u\rangle + (2\alpha-1)\varphi^2 u^{2\alpha-2} |\nabla u|^2 \right) \right\} \mathrm{d}m_f \\
&= \int_{B_r(q)} |\nabla\varphi|^2 u^{2\alpha} \, \mathrm{d}m_f - \alpha \int_{B_r(q)} \varphi^2 u^{2\alpha-1} \Delta_f u \, \mathrm{d}m_f \\
&\leq \int_{B_r(q)} |\nabla\varphi|^2 u^{2\alpha} \, \mathrm{d}m_f + \alpha \int_{B_r(q)} \lambda \varphi^2 u^{2\alpha} \, \mathrm{d}m_f.
\end{aligned}$$

Together with the Neumann-Sobolev inequality in Theorem 3.1.7, this leads us to

(32)
$$\frac{C^{Sob}(r)}{r^2} \left(\frac{1}{m_f(B_r(q))} \int_{B_r(q)} \left(\varphi^2 u^{2\alpha}\right)^\beta \mathrm{d}m_f \right)^{\frac{1}{\beta}} \leq \frac{1}{m_f(B_r(q))} \int_{B_r(q)} \left(\alpha\lambda\varphi^2 u^{2\alpha} + |\nabla\varphi|^2 u^{2\alpha} \right) \mathrm{d}m_f,$$

where, using positive constants C_{13} and C_{14} depending on n, b_1, b_2, c, we set

$$\beta := \frac{\nu}{\nu-2}, \quad C^{Sob}(r) := \left\{ C_{13} \exp\left(C_{14}\sqrt{K_\varepsilon(q,10r)}\, r \right) \right\}^{-1}.$$

For $\sigma, \rho > 0$ with $\sigma + \rho < r$, we take $\varphi \in C_0^\infty(B_R(q))$ satisfying the following conditions:

- $0 \leq \varphi \leq 1$;
- $\varphi \equiv 1$ on $B_\rho(q)$;
- $\varphi \equiv 0$ on $B_r(q) \setminus B_{\rho+\sigma}(q)$;
- $|\nabla\varphi| \leq 2/\sigma$ on $B_{\rho+\sigma}(q) \setminus B_\rho(q)$.

Substituting this to (32), we have

$$\begin{aligned}
\left(\frac{1}{m_f(B_r(q))} \int_{B_\rho(q)} u^{2\alpha\beta} \, \mathrm{d}m_f \right)^{\frac{1}{\beta}} &\leq \left(\frac{1}{m_f(B_r(q))} \int_{B_r(q)} \left(\varphi^2 u^{2\alpha}\right)^\beta \mathrm{d}m_f \right)^{\frac{1}{\beta}} \\
&\leq \frac{r^2}{C^{Sob}(r) m_f(B_r(q))} \int_{B_{\rho+\sigma}(q)} \left(\alpha\lambda u^{2\alpha} + \frac{4}{\sigma^2} u^{2\alpha} \right) \mathrm{d}m_f.
\end{aligned}$$

This yields

(33)

$$\left(\frac{1}{m_f(B_r(q))}\int_{B_r(q)} u^{2\alpha\beta}\,\mathrm{d}m_f\right)^{\frac{1}{2\alpha\beta}} \leq \left\{\frac{r^2}{C^{Sob}(r)}\left(\alpha\lambda+\frac{4}{\sigma^2}\right)\right\}^{\frac{1}{2\alpha}}\left(\frac{1}{m_f(B_r(q))}\int_{B_{\rho+\sigma}(q)} u^{2\alpha}\,\mathrm{d}m_f\right)^{\frac{1}{2\alpha}}.$$

First, we show the case of $p\geq 2$. For $i\geq 0$, we set

$$\alpha_i := \frac{p\,\beta^i}{2}>1,\quad \sigma_i := \frac{(1-\theta)\,r}{2^{1+i}},\quad \rho_i := r-\sum_{j=i}^{i}\sigma_j>\theta r.$$

For $i\geq 0$, we iterate the process of substituting $\alpha=\alpha_i$, $\rho=\rho_i$ and $\sigma=\sigma_i$ to (33), and we have

$$\left(\frac{1}{m_f(B_r(q))}\int_{B_{\rho_i}(q)} u^{2\alpha_i\beta}\,\mathrm{d}m_f\right)^{\frac{1}{2\alpha_i\beta}} \leq \left[\prod_{j=0}^{i}\left\{\frac{r^2}{C^{Sob}(r)}\left(\alpha_j\lambda+\frac{4}{\sigma_j^2}\right)\right\}^{\frac{1}{2\alpha_j}}\right]\left(\frac{1}{m_f(B_r(q))}\int_{B_r(q)} u^{p}\,\mathrm{d}m_f\right)^{\frac{1}{p}}. \tag{34}$$

As for the left-hand side, we possess the following estimate:

$$\lim_{i\to\infty}\left(\frac{1}{m_f(B_r(q))}\int_{B_{\rho_i}(q)} u^{2\alpha_i\beta}\,\mathrm{d}m_f\right)^{\frac{1}{2\alpha_i\beta}} \geq \lim_{i\to\infty}\left(\frac{1}{m_f(B_r(q))}\int_{B_{\theta r}(q)} u^{2\alpha_i\beta}\,\mathrm{d}m_f\right)^{\frac{1}{2\alpha_i\beta}} = \|u\|_{\infty,\theta r}$$

since $\lim_{i\to\infty}2\alpha_i\beta=\infty$. As for the right-hand side of (34), we find

$$\begin{aligned}\prod_{j=0}^{\infty}\left(\alpha_j\lambda+\frac{4}{\sigma_j^2}\right)^{\frac{1}{2\alpha_j}} &\leq \left(\frac{p\lambda}{2}+\frac{16}{(1-\theta)^2r^2}\right)^{\frac{1}{2\alpha_j}}\max\{\beta,4\}^{\frac{j}{2\alpha_j}}\\ &\leq \left(\frac{p\lambda}{2}+\frac{16}{(1-\theta)^2r^2}\right)^{\frac{\beta}{p(\beta-1)}}\max\{\beta,4\}^{\frac{S_1(\beta)}{p}},\end{aligned}$$

where we put $S_1(\beta)=\sum_{j=0}^{\infty}j\beta^{-j}<\infty$. Putting these estimates together, we see that (34) is estimated as

$$\|u\|_{\infty,\theta r}\leq C(p,r)\left(\frac{p\lambda\,r^2}{2}+\frac{16}{(1-\theta^2)}\right)^{\frac{\beta}{p(\beta-1)}}\left(\frac{1}{m_f(B_r(q))}\int_{B_r(q)} u^{p}\,\mathrm{d}m_f\right)^{\frac{1}{p}}, \tag{35}$$

where we set

$$C(p,r):=\left(\frac{1}{C^{Sob}(r)}\right)^{\frac{\beta}{p(\beta-1)}}\max\{\beta,4\}^{\frac{S_1(\beta)}{p}}.$$

Hence, we arrive at the desired assertion for the case of $p\geq 2$.

Next, we show in the case of $p<2$. For (35) with $p=2$, we see

$$\|u\|_{\infty,\theta r}\leq C(2,r)\left(\lambda r^2+\frac{16}{(1-\theta)^2}\right)^{\frac{\beta}{2(\beta-1)}}\left(\frac{1}{m_f(B_r(q))}\int_{B_r(q)} u^{2}\,\mathrm{d}m_f\right)^{\frac{1}{2}}.$$

For any $\theta r \le \rho \le r$ and $0 < \eta < 1$, this leads us to

$$
\begin{aligned}
\|u\|_{\infty,\eta\rho} &\le \frac{C(2,r)}{m_f(B_\rho(q))^{1/2}} \left(\lambda\rho^2 + \frac{16}{(1-\eta)^2}\right)^{\frac{\beta}{2(\beta-1)}} \|u\|_{2,\rho} \\
&\le \frac{C(2,r)}{m_f(B_{\theta r}(q))^{1/2}} \left(\lambda r^2 + \frac{16}{(1-\eta)^2}\right)^{\frac{\beta}{2(\beta-1)}} \|u\|_{p,\rho}^{\frac{p}{2}} \|u\|_{\infty,\rho}^{1-\frac{p}{2}},
\end{aligned} \tag{36}
$$

where we used

$$
\left(\int_{B_\rho(q)} u^2 \, \mathrm{d}m_f\right)^{\frac{1}{2}} \le \left(\int_{B_\rho(q)} u^p \, \|u\|_{\infty,\rho}^{2-p} \, \mathrm{d}m_f\right)^{\frac{1}{2}} = \left(\int_{B_\rho(q)} u^p \, \mathrm{d}m_f\right)^{\frac{1}{2}} \|u\|_{\infty,\rho}^{1-\frac{p}{2}}.
$$

For $i \ge 1$, we set

$$
\rho_0 := \theta\, r, \quad \rho_i := \theta\, r + (1-\theta)\, r \sum_{j=1}^{i} 2^{-j}, \quad \eta_i = \frac{\rho_{i-1}}{\rho_i}.
$$

We substitute $\rho = \rho_i$ and $\eta = \eta_i$ to (36), and we deduce

$$
\begin{aligned}
\|u\|_{\infty,\theta r} &= \|u\|_{\infty,\rho_0} \\
&= \|u\|_{\infty,\eta_1\rho_1} \\
&\le \frac{C(2,r)}{m_f(B_{\theta r}(q))^{1/2}} \left(\lambda\, r^2 + \frac{16}{(1-\eta_1)^2}\right)^{\frac{\beta}{2(\beta-1)}} \|u\|_{p,\rho_1}^{\frac{p}{2}} \|u\|_{\infty,\rho_1}^{1-\frac{p}{2}} \\
&= \frac{C(2,r)}{m_f(B_{\theta r}(q))^{1/2}} \left(\lambda\, r^2 + \frac{16}{(1-\eta_1)^2}\right)^{\frac{\beta}{2(\beta-1)}} \|u\|_{p,\rho_1}^{\frac{p}{2}} \|u\|_{\infty,\eta_2\rho_2}^{1-\frac{p}{2}} \\
&\le \frac{C(2,r)}{m_f(B_{\theta r}(q))^{1/2}} \left(\lambda\, r^2 + \frac{16}{(1-\eta_1)^2}\right)^{\frac{\beta}{2(\beta-1)}} \|u\|_{p,\rho_1}^{\frac{p}{2}} \\
&\qquad \times \left\{\frac{C(2,r)}{m_f(B_{\theta r}(q))^{1/2}} \left(\lambda\, r^2 + \frac{16}{(1-\eta_2)^2}\right)^{\frac{\beta}{2(\beta-1)}} \|u\|_{p,\rho_2}^{\frac{p}{2}} \|u\|_{\infty,\rho_2}^{1-\frac{p}{2}}\right\}^{1-\frac{p}{2}} \\
&\le \|u\|_{\infty,\rho_i}^{\left(1-\frac{p}{2}\right)^i} \prod_{j=1}^{i} \left\{\frac{C(2,r)}{m_f(B_{\theta r}(q))^{1/2}} \left(\lambda\, r^2 + \frac{16}{(1-\eta_j)^2}\right)^{\frac{\beta}{2(\beta-1)}} \|u\|_{p,\rho_j}^{\frac{p}{2}}\right\}^{\left(1-\frac{p}{2}\right)^{j-1}} \\
&\le \|u\|_{\infty,r}^{\left(1-\frac{p}{2}\right)^i} \prod_{j=1}^{i} \left\{\frac{C(2,r)}{m_f(B_{\theta r}(q))^{1/2}} \left(\lambda\, r^2 + \frac{16 \cdot 2^{2j}}{(1-\theta)^2}\right)^{\frac{\beta}{2(\beta-1)}} \|u\|_{p,r}^{\frac{p}{2}}\right\}^{\left(1-\frac{p}{2}\right)^{j-1}},
\end{aligned} \tag{37}
$$

where the last inequality used the following estimate:

$$
\frac{1}{1-\eta_j} = \frac{\rho_j}{\rho_j - \rho_{j-1}} = \frac{\theta + (1-\theta)\sum_{i=1}^{j} 2^{-i}}{2^{-j}(1-\theta)} \le \frac{2^j}{1-\theta}.
$$

Concerning the right-hand side of (37), we possess the following estimates:

$$\begin{aligned}\prod_{j=1}^{\infty}\left\{\left(\lambda r^2+\frac{16\cdot 2^{2j}}{(1-\theta)^2}\right)^{\frac{\beta}{2(\beta-1)}}\right\}^{\left(1-\frac{p}{2}\right)^{j-1}} &= \prod_{j=1}^{\infty}\left[\left\{2^{2j}\left(\frac{\lambda\, r^2}{2^{2j}}+\frac{16}{(1-\theta)^2}\right)\right\}^{\frac{\beta}{2(\beta-1)}}\right]^{\left(1-\frac{p}{2}\right)^{j-1}}\\ &\leq \prod_{j=1}^{\infty}\left\{\left(\lambda\, r^2+\frac{16}{(1-\theta)^2}\right)^{\frac{\beta}{2(\beta-1)}}2^{\frac{j\nu}{2}}\right\}^{\left(1-\frac{p}{2}\right)^{j-1}}\\ &= \left(\lambda\, r^2+\frac{16}{(1-\theta)^2}\right)^{\frac{\beta}{p(\beta-1)}}2^{\frac{\nu}{2}\sum_{j=1}^{\infty}j\left(1-\frac{p}{2}\right)^{j-1}},\end{aligned}$$

where we put $S_2(p) := \frac{\nu}{2}\sum_{j=1}^{\infty} j\left(1-\frac{p}{2}\right)^{j-1} < \infty$, and

$$\lim_{i\to\infty}\|u\|_{\infty,r}^{\left(1-\frac{p}{2}\right)^{i}}=1,\quad \lim_{i\to\infty}\prod_{j=1}^{i}\|u\|_{p,r}^{\frac{p}{2}\left(1-\frac{p}{2}\right)^{j-1}}=\|u\|_{p,r}.$$

Combining these estimates with (37), we obtain

$$\|u\|_{\infty,\theta\, r}\leq C(2,r)^{\frac{2}{p}}\left(\lambda\, r^2+\frac{16}{(1-\theta)^2}\right)^{\frac{\beta}{p(\beta-1)}}\frac{2^{S_2(p)}\|u\|_{p,r}}{m_f(B_{\theta\, r}(q))}.$$

Hence, we arrive at the desired assertion also for the case $p<2$. We complete the proof. □

3.2. L^p-Liouville theorem

In this section, we address an L^p-Liouville property (Theorem 3.2.4), which is a generalization of the Li-Schoen type L^p-Liouville property (Theorem 1.3.4).

We possess the following L^p-Liouville property for $p>1$ (see [**106**, Theorem 1.1]):

Lemma 3.2.1 (cf.[**106**])**.** *Let (M,g,f) be a complete weighted Riemannian manifold. For $p>1$, let u be a smooth non-negative $L^p(m_f)$-function satisfying $\Delta_f u\geq 0$. Then u is a constant function.*

Remark 3.2.2. Pigola-Rigoli-Setti [**106**, Theorem 1.1] obtained this for more general functions (see also [**106**, Remark 15]). A simple proof for the unweighted case $f\equiv 0$ can be found in [**117**, Chapter II, Theorem 6.3] (see also [**38**, Theorem 4.1]).

The relative comparison theorem is as follows (see [**38**, Theorem 4.2]):

Proposition 3.2.3 ([**38**])**.** *Let (M,g,f) be an n-dimensional complete weighted Riemannian manifold, $N\in(-\infty,1]\cup[n,\infty]$ and $\varepsilon\in\mathbb{R}$ in the ε-range. For $K\geq 0$ and $b_2\geq b_1>0$, we assume*

$$\mathrm{Ric}_f^N\geq -K\mathrm{e}^{\frac{4(\varepsilon-1)f}{n-1}}g,\quad b_1\leq \mathrm{e}^{\frac{2(1-\varepsilon)f}{n-1}}\leq b_2.$$

Then for $c:=c(n,N,\varepsilon)$ as in (9), $q\in M$, $R\geq r>0$ and $0<S\leq r$, we have

$$m_f(B_S(q))\geq\frac{b_1}{b_2}\frac{\int_0^{S/b_2}\mathrm{s}_{-cK}(t)^{1/c}\,\mathrm{d}t}{\int_{r/b_2}^{R/b_1}\mathrm{s}_{-cK}(t)^{1/c}\,\mathrm{d}t}\,m_f(B_R(q)\backslash B_r(q)).$$

Proof. For a unit tangent vector $w \in T_qM$, let $\gamma : [0,d) \to \mathbb{R}$ be the unit speed geodesic with $\dot\gamma = w$, and $\{e_i\}_{i=1}^n$ be an orthonormal basis with $\mathrm{e}_n = w$, also let $\{E_i\}_{i=1}^n$ denote the parallel vector field along γ with $E_i(0) = e_i$. We set a matrix A such that

$$A_{ij}(t) := g(E_i(t), E_j(t)),$$

and

$$J_0(t) := (\det A(t))^{\frac{1}{2(n-1)}}, \quad J(t) := \mathrm{e}^{-cf(\gamma(t))}(\det A(t))^{\frac{c}{2}}, \quad J_1(\tau) := J(s_\gamma^{-1}(\tau))$$

for $t \in [0,d)$ and $\tau \in [0, s_\gamma(d))$, where

$$s_\gamma(t) := \int_0^t \mathrm{e}^{\frac{2(\varepsilon-1)}{n-1} f(\gamma(t))}\, \mathrm{d}t. \tag{38}$$

It follows from the argument in [**79**] that

$$\mathrm{e}^{-f(\gamma)} J_0^{n-1} / \mathbf{s}_{cK}(s_\gamma)^{1/c}$$

is non-increasing. The Gromov lemma (see e.g., [**46**, Lemma 3.2]) implies

$$\begin{aligned}\int_{s_\gamma(\min\{r,\rho(w)\})}^{s_\gamma(\min\{R,\rho(w)\})} J_1(\tau)^{1/c}\, \mathrm{d}\tau \Big/ \int_0^{s_\gamma(\min\{S,\rho(w)\})} J_1(\tau)^{1/c}\, \mathrm{d}\tau \\ \leq \int_{s_\gamma(\min\{r,\rho(w)\})}^{s_\gamma(\min\{R,\rho(w)\})} \mathbf{s}_{-cK}(\tau)^{1/c}\, \mathrm{d}\tau \Big/ \int_0^{s_\gamma(\min\{S,\rho(w)\})} \mathbf{s}_{-cK}(\tau)^{1/c}\, \mathrm{d}\tau.\end{aligned} \tag{39}$$

We set

$$C(w) := \int_{s_\gamma(\min\{r,\rho(w)\})}^{s_\gamma(\min\{R,\rho(w)\})} \mathbf{s}_{-cK}(\tau)^{1/c}\, \mathrm{d}\tau \Big/ \int_0^{s_\gamma(\min\{S,\rho(w)\})} \mathbf{s}_{-cK}(\tau)^{1/c}\, \mathrm{d}\tau.$$

We note that s_γ is defined in (19). Also, we have

$$\int_{s_\gamma(\min\{r,\rho(w)\})}^{s_\gamma(\min\{R,\rho(w)\})} J_1(\tau)^{1/c}\, \mathrm{d}\tau = \int_{\min\{r,\rho(w)\}}^{\min\{R,\rho(w)\}} J(t)^{1/c} s_\gamma'(t)\, \mathrm{d}t \geq \frac{1}{b_2} \int_{\min\{r,\rho(w)\}}^{\min\{R,\rho(w)\}} J(t)^{1/c}\, \mathrm{d}t,$$

and

$$\int_0^{s_\gamma(\min\{S,\rho(w)\})} J_1(\tau)^{1/c}\, \mathrm{d}\tau \leq \frac{1}{b_1} \int_0^{\min\{S,\rho(w)\}} J(t)^{1/c}\, \mathrm{d}t.$$

Combining them with (39), we have

$$\int_{\min\{r,\rho(w)\}}^{\min\{R,\rho(w)\}} J(t)^{1/c}\, \mathrm{d}t \Big/ \int_0^{\min\{S,\rho(w)\}} J(t)^{1/c}\, \mathrm{d}t \leq \frac{b_2}{b_1} C(w).$$

Integrating with respect to $w \in U_qM$, we obtain

$$\begin{aligned} m_f\left(B_R(q) \backslash B_r(q)\right) &= \int_{U_xM} \int_{\min\{r,\rho(w)\}}^{\min\{R,\rho(w)\}} J(t)^{1/c}\, \mathrm{d}t\, \mathrm{d}\Xi(w) \\ &\leq \frac{b_2}{b_1} \int_{U_xM} C(w) \int_0^{\min\{S,\rho(w)\}} J(t)^{1/c}\, \mathrm{d}t\, \mathrm{d}\Xi(w).\end{aligned} \tag{40}$$

We estimate $C(w)$ from above. For $w \in U_qM$ with $\rho(w) > R$, we have

$$C(w) = \frac{\int_{s\gamma(r)}^{s_\gamma(R)} \mathbf{s}_{-cK}(\tau)^{1/c}\, \mathrm{d}\tau}{\int_0^{s_\gamma(S)} \mathbf{s}_{-cK}(\tau)\, \mathrm{d}\tau} \leq \frac{\int_{r/b_2}^{R/b_1} \mathbf{s}_{-cK}(\tau)^{1/c}\, \mathrm{d}\tau}{\int_0^{S/b_2} \mathbf{s}_{-cK}(\tau)^{1/c}\, \mathrm{d}\tau}.$$

For $w \in U_q M$ with $r \le \rho(w) \le R$, we find

$$C(w) = \frac{\int_{s\gamma(r)}^{s_\gamma(\rho(w))} \mathbf{s}_{-cK}(\tau)^{1/c}\, \mathrm{d}\tau}{\int_0^{s_\gamma(S)} \mathbf{s}_{-cK}(\tau)^{1/c}\, \mathrm{d}\tau} \le \frac{\int_{r/b_2}^{R/b_1} \mathbf{s}_{-cK}(\tau)^{1/c}\, \mathrm{d}\tau}{\int_0^{S/b_2} \mathbf{s}_{-cK}(\tau)^{1/c}\, \mathrm{d}\tau}.$$

For $w \in U_q M$ with $\rho(w) < r$, we obtain $C(w) = 0$. By using them, we estimate the right-hand side of (40), and we see

$$\begin{aligned} m_f\left(B_R(q)\backslash B_r(q)\right) &\le \frac{b_2}{b_1} \frac{\int_{r/b_2}^{R/b_1} \mathbf{s}_{-cK}(\tau)^{1/c}\, \mathrm{d}\tau}{\int_0^{S/b_2} \mathbf{s}_{-cK}(\tau)^{1/c}\, \mathrm{d}\tau} \int_0^{\min\{S,\rho(w)\}} J(t)^{1/c}\, \mathrm{d}t\, \mathrm{d}\Xi(w) \\ &= \frac{b_2}{b_1} \frac{\int_{r/b_2}^{R/b_1} \mathbf{s}_{-cK}(\tau)^{1/c}\, \mathrm{d}\tau}{\int_0^{S/b_2} \mathbf{s}_{-cK}(\tau)^{1/c}\, \mathrm{d}\tau} m_f(B_S(q)). \end{aligned}$$

We complete the proof. □

We are now in a position to prove the L^p-Liouville theorem (see [**38**, Theorem 4.3]):

Theorem 3.2.4. *Let (M, g, f) be an n-dimensional complete non-compact weighted Riemannian manifold, $N \in (-\infty, 1] \cup [n, \infty]$ and $\varepsilon \in \mathbb{R}$ in the ε-range. For $c := c(n, N, \varepsilon)$ as in (9) and $b_2 \ge b_1 > 0$, we assume*

$$b_1 \le \mathrm{e}^{\frac{2(1-\varepsilon)f}{n-1}} \le b_2, \quad \frac{b_2}{b_1}\left\{\left(\frac{b_2}{b_1}\right)^{1+\frac{1}{c}} - 1\right\} < \frac{1}{20^{1+\frac{1}{c}}}. \tag{41}$$

Then there exists a constant $\delta > 0$ depending only on b_1, b_2, c such that the following assertion holds:

We assume that there exists $q \in M$ such that

$$\mathrm{Ric}_f^N \ge -\delta \mathrm{e}^{\frac{4(\varepsilon-1)f}{n-1}} d_q^{-2} g \tag{42}$$

when d_q is sufficiently large. For $p > 0$, let u be a smooth non-negative $L^p(m_f)$-function with $\Delta_f u \ge 0$. Then u is identically zero.

Proof. The case $p > 1$ follows from Lemma 3.2.1. Below, we consider the case $0 < p \le 1$. If $u(x) \to 0$ as $d(q, x) \to \infty$ for some fixed $q \in M$, we have $u \in L^\infty(m_f)$. Since $L^p(m_f) \cap L^\infty(m_f) \subset L^2(m_f)$, we see $u \in L^2(m_f)$. Then Lemma 3.2.1 implies that u is constant. In what follows, we show that $u(x) \to 0$ as $d(q, x) \to \infty$.

Let $\gamma : [0, t] \to M$ be a minimal geodesic connecting q to x with $t = d(q, x)$. For fixed $\alpha > 1$, we set

$$t_0 = 0, \quad t_1 = 1 + \alpha, \quad t_i = 2\sum_{j=0}^{i} \alpha^j - 1 - \alpha^i.$$

We take $k \in \mathbb{N}$ such that $t_k \le t$ and $t_{k+1} \ge t$. For $i \le k$, we set $x_i = \gamma(t_i)$, and find

$$d(x_i, x_{i+1}) = \alpha^i + \alpha^{i+1}, \quad d(q, x_i) = t_i, \quad d(x_k, x) < \alpha^k + \alpha^{k+1}.$$

For $i \le k$, by Theorem 3.2.3, we have

$$m_f\left(B_{\alpha^i/20}(x_i)\right) \ge D_i m_f\left(B_{\alpha^i + 2\alpha^{i-1}}(x_i) \backslash B_{\alpha^i}(x_i)\right) \ge D_i m_f\left(B_{\alpha^{i-1}/20}(x_{i-1})\right),$$

where we set $K_i := K_\varepsilon(x_i, \alpha^i + 2\alpha^{i-1})$ and

$$D_i := \frac{b_1}{b_2}\left(\int_0^{\alpha^i/(20 b_2)} \mathbf{s}_{-cK}(t)^{1/c}\, \mathrm{d}t\right) \Big/ \left(\int_{\alpha^i/b_2}^{(\alpha^i + 2\alpha^{i-1})/b_1} \mathbf{s}_{-cK}(t)^{1/c}\, \mathrm{d}t\right).$$

Integrating this yields

$$m_f\left(B_{\alpha^k/20}(x_k)\right) \geq \left(\prod_{i=1}^{k} D_i\right) m_f(B_{1/20}(q)).$$

Next, we show $m_f\left(B_{\alpha^k/20}(x_k)\right) \to \infty$ when $d(q,x) \to \infty$. For the brevity of notations, we denote $B_i := B_{\alpha^i+2\alpha^{i-1}}(x_i)$. For $y \in B_j$, it follows that

$$d(q,y) \geq d(q,x_i) - d(x_i,y) \geq \left(2\sum_{j=0}^{i}\alpha^j - 1 - \alpha^i\right) - \left(\alpha^i + 2\alpha^{i-1}\right) = \frac{1-2\alpha^{i-2}+\alpha}{1-\alpha}.$$

Using this, we have

$$\alpha^i\sqrt{K_i} = \alpha^i\sqrt{\sup_{y\in B_i} K_\varepsilon(y)} \leq \alpha^i \sup_{y\in B_i}\frac{\sqrt{\delta}}{d(q,y)} \leq \frac{\alpha^2(\alpha-1)\sqrt{\delta}}{2-\alpha^{2-i}-\alpha^{3-i}}$$

for sufficiently large i. This implies that $\alpha^i\sqrt{K_i}$ can be made arbitrarily small by taking δ small enough. Then D_i is approximated as follows:

$$\frac{b_1}{b_2}\frac{\left\{\alpha^i\sqrt{K_i}/(20b_2)\right\}^{1+1/c}}{\left\{(\alpha^i+2\alpha^{i-1})\sqrt{K_i}/b_1\right\}^{1+1/c} - \left(\alpha^i\sqrt{K_i}/b_2\right)} = \frac{b_1}{b_2}\frac{1}{20^{1+1/c}}\frac{1}{\{(1+2/\alpha)\,b_2/b_1\}^{1+1/c}-1}. \tag{43}$$

Since we have the assumption (41), by taking α sufficiently large, we can assume that D_i is larger than 1. Therefore, by taking suitable α, we see

$$m_f\left(B_{\alpha^k/20}(x_k)\right) \to \infty \tag{44}$$

when $d(q,x) \to \infty$.

We divide into two cases and estimate $u(x)$ when x is far away from q.

Case 1: We first consider the case $d(x,x_k) < \alpha^k/20$. From Theorem 3.1.8, there exist positive constants C_{15} and C_{16} depending only on n, p, c, b_1, b_2 such that

$$u(x) \leq \sup_{B_{\alpha^k/20}(x_k)} u \leq C_{15}\exp\left(C_{16}\sqrt{K_\varepsilon(x_k,\alpha^k)}\,\alpha^k\right) m_f\left(B_{\alpha^k/20}(x_k)\right)^{-\frac{1}{p}}\|u\|_p. \tag{45}$$

Case 2: We consider the case $d(x,x_k) \geq \alpha^k/20$. Also, it follows from Theorem 3.1.8 that

$$u(x) \leq \sup_{B_{\alpha^k/20}(x)} u \leq C_{15}\exp\left(C_{16}\sqrt{K_\varepsilon(x,\alpha^k)}\,\alpha^k\right) m_f\left(B_{\alpha^k/20}(x_k)\right)^{-\frac{1}{p}}\|u\|_p. \tag{46}$$

We estimate the right-hand side of this inequality. We note that

$$B_{\alpha^k/20}(x_k) \subset B_{d(x,x_k)+\alpha^k/20}(x)\backslash B_{d(x,x_k)-\alpha^k/20}(x).$$

Together with the argument in Theorem 3.2.3, we have

$$\begin{aligned} m_f\left(B_{\alpha^k/20}(x_k)\right) &\leq m_f\left(B_{d(x,x_k)+\alpha^k/20}(x)\backslash B_{d(x,x_k)-\alpha^k/20}(x)\right) \\ &\leq \frac{b_2}{b_1}\int_{U_xM} C(w)\int_0^{\min\left\{\alpha^k/20,\,\rho(w)\right\}} J(t)^{1/c}\,\mathrm{d}t\,\mathrm{d}\Xi(w), \end{aligned} \tag{47}$$

where we set $K = K_\varepsilon\left(x, d(x,x_k)+\alpha^k/20\right)$ and

$$C(w) := \int_{s_\gamma\left(\min\left\{d(x,x_k)-\alpha^k/20,\rho(w)\right\}\right)}^{s_\gamma\left(\min\left\{d(x,x_k)+\alpha^k/20,\rho(w)\right\}\right)} \mathbf{s}_{-cK}(\tau)^{1/c}\,\mathrm{d}\tau \Big/ \int_0^{s_\gamma\left(\min\left\{\alpha^k/20,\rho(w)\right\}\right)} \mathrm{s}_{-cK}(\tau)\,\mathrm{d}\tau.$$

If $d(x,x_k)-\alpha^k/20<\rho(w)<\alpha^k/20$, we have $C(w)\le 1$. We consider the other cases below. From the proof of Theorem 3.2.3, we obtain

$$C(w)\le \int_{\left(d(x,x_k)-\alpha^k/20\right)/b_2}^{\left(d(x,x_k)+\alpha^k/20\right)/b_1} \mathrm{s}_{-cK}(\tau)^{1/c}\,\mathrm{d}\tau \Big/ \int_0^{\alpha^k/(20b_2)} \mathrm{s}_{-cK}(\tau)^{1/c}\,\mathrm{d}\tau.$$

Denoting $B_k:=B_{d(x,x_k)+\alpha^k}(x)$, we see

$$\sqrt{K_\varepsilon(x,d(x,x_k)+\alpha^k)}\le \sup_{y\in B_k}\frac{\sqrt{\delta}}{d(q,y)}\le \frac{\sqrt{\delta}}{1+2\alpha+\cdots+2\alpha^{k-1}}=\frac{(1-\alpha)\sqrt{\delta}}{1-2\alpha^k+\alpha}$$

and then

$$\begin{aligned}\left(d(x,x_k)+\alpha^k/20\right)\sqrt{K_\varepsilon\left(x,d(x,x_k)+\alpha^k/20\right)} &\le \left(d(x,x_k)+\alpha^k\right)\sqrt{K_\varepsilon(x,d(x,x_k)+\alpha^k)}\\ &\le \left(\alpha^{k+1}+2\alpha^k\right)\frac{(1-\alpha)\sqrt{\delta}}{1-2\alpha^k+\alpha}\\ &=\frac{(2+\alpha)(\alpha-1)\sqrt{\delta}}{2-\alpha^{1-k}-\alpha^{-k}}.\end{aligned}$$

If we take k sufficiently large, we have

$$\left(d(x,x_k)+\frac{\alpha^k}{20}\right)\sqrt{K_\varepsilon\left(x,d(x,x_k)+\frac{\alpha^k}{20}\right)}\le(2+\alpha)(\alpha-1)\sqrt{\delta}.$$

Since the right-hand side of this inequality can be made sufficiently small by taking δ small, $C(w)$ is estimated from above by

$$\frac{\left(\left(d(x,x_k)+\alpha^k/20\right)/b_1\right)^{1+1/c}-\left(\left(d(x,x_k)-\alpha^k/20\right)/b_2\right)^{1+1/c}}{\left(\alpha^k/(20b_2)\right)^{1+1/c}}. \tag{48}$$

We note that this follows from the first order approximation. Furthermore, the quantity (48) is estimated from above as follows:

$$(48)\le\frac{\left(\left(d(x,x_k)+\alpha^k\right)/b_1\right)^{1+1/c}}{\left(\alpha^k/(20b_2)\right)^{1+1/c}}\le\frac{\left(\left(\alpha^{k+1}+2\alpha^k\right)/b_1\right)^{1+1/c}}{\left(\alpha^k/(20b_2)\right)^{1+1/c}}=\left(\frac{20b_2(2+\alpha)}{b_1}\right)^{1+\frac{1}{c}}.$$

Combining these estimates of $C(w)$ with (47), we see that there exists a constant $C_{17}>0$ depending only on c,b_1,b_2,α such that

$$m_f\left(B_{\alpha^k/20}(x_k)\right)\le m_f\left(B_{d(x,x_k)+\alpha^k/20}(x)\backslash B_{d(x,x_k)-\alpha^k/20}(x)\right)\le C_{17}m_f\left(B_{\alpha^k/20}(x)\right).$$

Together with (46), we see that there exist positive constants C_{18} and C_{19} depending only on n,p,c,b_1,b_2,α such that

$$u(x)\le C_{18}\exp\left(C_{19}\sqrt{K_\varepsilon(x,\alpha^k)}\,\alpha^k\right)m_f\left(B_{\alpha^k/20}(x_k)\right)^{-\frac{1}{p}}\|u\|_p.$$

Combining this with (45) in Case 1, we see that there exist positive constants C_{20} and C_{21} depending only on n,p,c,b_1,b_2,α such that

$$u(x)\le C_{20}\exp\left(C_{21}\max\left\{\sqrt{K_\varepsilon(x_k,\alpha^k)},\sqrt{K_\varepsilon(x,\alpha^k)}\right\}\alpha^k\right)m_f\left(B_{\alpha^k/20}(x_k)\right)^{-\frac{1}{p}}\|u\|_p. \tag{49}$$

As $d(q,x)\to\infty$, since k increases, we see $m_f(B_{\alpha^k/20}(x_k))\to\infty$. Also, it follows from (42) that $\alpha^k\sqrt{K_\varepsilon(x_k,\alpha^k)}$ and $\alpha^k\sqrt{K_\varepsilon(x,\alpha^k)}$ are bounded from above as $d(q,x)\to\infty$. This implies that, when $d(q,x)\to\infty$, the right-hand side of (49) goes to 0. From the arguments at the beginning of this proof, we see that u is a constant function. It follows from (44) that $m_f(M) = \infty$. Hence, u is identically zero.

Lastly, we remark about the dependence of δ. We took α so that (43) is larger than 1, and we took δ so that approximations in (43) and (48) hold. Hence, we see that δ depends only on c, b_1, b_2. $\square$

Remark 3.2.5. In the unweighted case $f \equiv 0$, this recovers Theorem 1.3.4. Indeed, if we take $\varepsilon = 1$ and $b_1 = b_2 = 1$, the assumption (41) is satisfied.

Remark 3.2.6. The latter assumption (41) is satisfied when b_1 and b_2 are close. In that case, the former assumption implies that the reparametrization $s_\gamma(t)$ is close to t (see also Remark 2.1.17).

Remark 3.2.7. An alternative proof is also presented in [**38**, Theorem 4.3].

Below, we consider the weighted case $N = \infty$, and see that similar arguments recover Theorem 2.2.32. We use the following relative volume comparison theorem (see e.g., [**133**], see also [**38**, Theorem 4.1]):

Proposition 3.2.8 ([**133**])**.** *Let (M, g, f) be an n-dimensional complete Riemannian manifold. For $K \geq 0$ and $b > 0$, we assume*

$$\mathrm{Ric}_f^\infty \geq -Kg, \quad |f| \leq b.$$

Then for $x \in M$, $R \geq r > 0$ and $0 < S \leq r$, we have

$$m_f(B_S(x)) \geq \frac{\int_0^S \mathrm{s}_{-K/(n-1)}(t)^{n-1+4b}\, \mathrm{d}t}{\int_r^R \mathrm{s}_{-K/(n-1)}(t)^{n-1+4b}}\, m_f(B_R(x) \backslash B_r(x)).$$

We give a proof of Theorem 2.2.32 below:

Proof of Theorem 2.2.32. We set

$$K_\infty(x) := \max\left\{0, \sup_{w \in U_xM}(-\mathrm{Ric}_f^\infty(w,w))\right\}, \quad K_\infty(y,r) := \sup_{x \in B_r(y)} K_\infty(x).$$

For $x \in M$ and $\alpha > 0$, we take $\{x_i\}$ in the same way as in the proof of Theorem 3.2.4. Applying Proposition 3.2.8 instead of Proposition 3.2.3, we have

$$m_f\left(B_{\alpha^i/20}(x_i)\right) \geq D_i m_f\left(B_{\alpha^i+2\alpha^{i-1}}(x_i) \backslash B_{\alpha^i}(x_i)\right) \geq D_i m_f\left(B_{\alpha^{i-1}/20}(x_{i-1})\right),$$

where we set

$$D_i := \int_0^{\alpha^i/20} \mathrm{s}_{-K/(n-1)}(t)^{n-1+4b}\, \mathrm{d}t \Big/ \int_{\alpha^i}^{\alpha^i+2\alpha^{i-1}} \mathrm{s}_{-K/(n-1)}(t)^{n-1+4b}\, \mathrm{d}t$$

and $K = K_\infty(x_i, \alpha^i + 2\alpha^{i-1})$. For sufficiently small δ, then D_i is approximated as

$$\frac{(\alpha^i/20)^{n+4b}}{(\alpha^i + 2\alpha^{i-1})^{n+4b} - (\alpha^i)^{n+4b}} = \frac{(1/20)^{n+4b}}{(1+2/\alpha)^{n+4b} - 1}. \tag{50}$$

We take $\alpha > 0$ satisfying

$$\alpha > \frac{2}{(20^{-(n+4b)} + 1)^{1/(n+4b)} - 1} > 1.$$

This implies that, if we take δ small enough, we see $D_i > 1$ by (50). Hence, as in (44), we have

$$m_f\left(B_{\alpha^k/20}(x_k)\right) \to \infty.$$

The rest of the arguments follows by the same arguments as in the proof of Theorem 3.2.4. Indeed, we fix $\varepsilon \in (-1,1)$, and set $b_1 := \mathrm{e}^{\frac{2(\varepsilon-1)b}{n-1}}$ and $b_2 := \mathrm{e}^{\frac{2(1-\varepsilon)b}{n-1}}$. For small enough $\delta > 0$, we first assume $\mathrm{Ric}_f^\infty \geq -b_2^{-2} d_q^{-2}$, which implies $\mathrm{Ric}_f^\infty \geq -\delta \mathrm{e}^{\frac{4(\varepsilon-1)f}{n-1}} d_q^{-2}$. Hence, we may apply the same arguments as in the proof of Theorem 3.2.4 after (44) in our setting. □

Remark 3.2.9. We note that this proof is slightly different from that in Wu [**133**]. Indeed, the mean value inequality in [**133**] was obtained by using the elliptic Harnack inequality, while our proof does not.

3.3. Liouville theorem for sublinear growth functions

In this section, we give a Liouville type theorem for sublinear growth f-harmonic functions under $\mathrm{Ric}_f^N \geq 0$ (Theorem 3.3.4). We note that the variable curvature bound (4) degenerates to constant curvature bound $\mathrm{Ric}_f^N \geq 0$ if $K = 0$. This enables us to obtain several functional inequalities under $\mathrm{Ric}_f^N \geq 0$.

The Neumann-Poincaré inequality under $\mathrm{Ric}_f^N \geq 0$ is as follows (see [**38**, Lemma 5.1]):

Lemma 3.3.1 ([**38**])**.** *Let (M, g, f) be an n-dimensional complete weighted Riemannian manifold and $N \in (-\infty, 1] \cup [n, \infty]$. For $b > 0$, we assume*

$$\mathrm{Ric}_f^N \geq 0, \quad |f| \leq b.$$

Then there exists a constant $C_{22} > 0$ depending only on n, b, N such that, for $q \in M$, $r > 0$ and $\varphi \in C^\infty(M)$, we have

$$\int_{B_r(q)} \left|\varphi - \varphi_{B_r(q)}\right|^2 \, \mathrm{d}m_f \leq C_{22}\, r^2 \int_{B_{2r}(q)} |\nabla \varphi|^2 \, \mathrm{d}m_f.$$

Proof. We take $\varepsilon \in \mathbb{R}$ satisfying

$$|\varepsilon| < \min\left\{1, \sqrt{\frac{N-1}{N-n}}\right\}$$

if $N \neq 1$ and $\varepsilon = 0$ if $N = 1$. We note that this $\varepsilon \in \mathbb{R}$ is contained in the ε-range. We set $b_1 := \mathrm{e}^{\frac{2(\varepsilon-1)b}{n-1}}$, $b_2 := \mathrm{e}^{\frac{2(1-\varepsilon)b}{n-1}}$. For these ε, b_1, b_2, we apply the arguments in Proposition 3.1.1. We complete the proof. □

Remark 3.3.2. Actually, in the case $N \in [n, \infty)$, we do not need the boundedness of the weight function f since we may take $\varepsilon = 1$ and $b_1 = b_2 = 1$.

The same argument yields the following inequality (see [**38**, Theorem 5.1]):

Lemma 3.3.3 ([**38**])**.** *We assume that (M, g, f) satisfies the same condition as in Lemma 3.3.1. Let u be a smooth non-negative function satisfying $\Delta_f u \geq 0$. For $q \in M$, $\theta \in (0,1)$, $r > 0$ and $p > 0$, there exists a constant $C_{23} > 0$ depending only on n, N, b, p, θ such that*

$$\|u\|_{\infty, \theta r} \leq C_{23} \left(\frac{1}{m_f(B_{\theta r}(q))} \int_{B_r(q)} u^p \, \mathrm{d}m_f \right)^{\frac{1}{p}}.$$

These imply the following Liouville property (see [**38**, Theorem 5.2]):

Theorem 3.3.4 ([**38**])**.** *Let (M, g, f) be an n-dimensional complete non-compact weighted Riemannian manifold and $N \in (-\infty, 0) \cup [n, \infty]$. We assume that f is bounded and*

$$\mathrm{Ric}_f^N \geq 0.$$

Then any sublinear growth f-harmonic function must be a constant function.

Proof. Let u be a sublinear growth f-harmonic function. Combining $\Delta_f u = 0$ and the Bochner inequality in Proposition 2.1.7, we find

$$\Delta_f \left(\frac{|\nabla u|^2}{2} \right) \geq \mathrm{Ric}_f^N(\nabla u, \nabla u) \geq 0.$$

For $q \in M$ and $r > 0$, it follows from Theorem 3.1.8 that there exists a constant $C_{24} > 0$ depending only on n, N, b such that

$$\sup_{B_{r/2}(q)} |\nabla u|^2 \leq \frac{C_{24}}{m_f(B_{r/2}(q))} \int_{B_r(q)} |\nabla u|^2 \, \mathrm{d}m_f. \tag{51}$$

Let ϕ be the cut-off function with $\phi \equiv 1$ on $B_r(q)$ and $\phi \equiv 0$ on $M \backslash B_{2r}(q)$ and $|\nabla \phi| \leq \frac{2}{r}$. Using the integration by parts and $\Delta_f u = 0$, we see

$$\begin{aligned}
\int_M |\nabla u|^2 \phi^2 \, \mathrm{d}m_f &= -2 \int_M u\phi \langle \nabla u, \nabla \phi \rangle \, \mathrm{d}m_f \\
&\leq 2 \int_M |u||\phi| \langle \nabla u, \nabla \phi \rangle \, \mathrm{d}m_f \\
&\leq \frac{1}{2} \int_M |\nabla u|^2 \phi^2 \, \mathrm{d}m_f + 2 \int_M u^2 |\nabla \phi|^2 \, \mathrm{d}m_f.
\end{aligned}$$

Then we have

$$\frac{1}{4} \int_{B_r(q)} |\nabla u|^2 \, \mathrm{d}m_f \leq \int_M u^2 |\nabla \phi|^2 \, \mathrm{d}m_f \leq \frac{4}{r^2} \int_{B_{2r}(q) \backslash B_r(q)} u^2 \, \mathrm{d}m_f \leq \frac{4 m_f(B_{2r}(q))}{r^2} \sup_{B_{2r}(q)} u^2.$$

From Proposition 2.1.19, for a constant $C_{25} > 0$ depending on n, N, b, we have $m_f(B_{2r}(q)) \leq C_{25} m_f(B_{r/2}(q))$. Hence, we see

$$\frac{1}{m_f(B_{r/2}(q))} \int_{B_r(q)} |\nabla u|^2 \, \mathrm{d}m_f \leq \frac{4C_{25}}{r^2} \left(\sup_{B_{2r}(q)} u^2 \right),$$

where the right-hand side goes to 0 since u is of sublinear growth. Combining this with (51), we obtain

$$\lim_{r \to \infty} \sup_{B_{r/2}(q)} |\nabla u|^2 \leq 0.$$

This implies $|\nabla u| \equiv 0$. We obtain the desired assertion. □

Remark 3.3.5. In the weighted case with $N = \infty$, this recovers Theorem 2.2.28. Actually, we do not need the boundedness of f in the case $N \in [n, \infty)$ (see Remark 3.3.2), and recovers Theorem 1.3.2 in the unweighted case $f \equiv 0$.

3.4. Gradient estimates

In this section, we obtain a gradient estimate of harmonic functions under lower bounds of Ric_f^N with ε-range (Theorem 3.4.1). This is obtained as an application of a Harnack type inequality. The gradient estimate is as follows (see [**38**, Theorem 6.1]):

Theorem 3.4.1 ([**38**]). *Let (M, g, f) be an n-dimensional complete weighted Riemannian manifold, $N \in (-\infty, 0) \cup [n, \infty]$ and $\varepsilon \in \mathbb{R}$ in the ε-range. For $K \geq 0$ and $b_2 \geq b_1 > 0$, we assume*

$$\mathrm{Ric}_f^N \geq -K \mathrm{e}^{\frac{4(\varepsilon - 1)f}{n-1}} g, \quad b_1 \leq \mathrm{e}^{\frac{2(1-\varepsilon)f}{n-1}} \leq b_2.$$

Let u be a positive f-harmonic function. Then there exists a constant $C > 0$ depending only on n, K, b_1, b_2 and $c(n, N, \varepsilon)$ in (9) *such that*

$$|\nabla \log u| \leq C.$$

Proof. In this proof, for the brevity of notations, we refer to a constant that only depends on $n, c(n, N, \varepsilon), K, b_1, b_2$ just as a constant, and do not mention about the dependence for the seke of the brevity. From the Bochner inequality in Proposition 2.1.7, we have

$$\Delta_f \left(\frac{|\nabla u|^2}{2} \right) \geq \mathrm{Ric}_f^N (\nabla u, \nabla u) \geq -\frac{K}{b_1^2} |\nabla u|^2.$$

By Theorem 3.1.8, there exists a constant $C_{26} > 0$ such that

$$\sup_{B_{1/16}(q)} |\nabla u|^2 \leq \frac{C_{26}}{m_f(B_{1/16}(q))} \int_{B_(1/8)(q)} |\nabla u|^2 \, \mathrm{d}m_f \tag{52}$$

for any $q \in M$. Let ϕ be a cut-off function such that $\phi \equiv 1$ on $B_{1/8}(q)$ and $\phi \equiv 0$ on $M \backslash B_{1/4}(q)$ with $|\nabla \phi| \leq 16$. The same argument as in Theorem 3.3.4 implies

$$\int_{B_{1/16}(q)} |\nabla u|^2 \, \mathrm{d}m_f \leq 16 \cdot 8^2 m_f(B_{1/4}(q)) \left(\sup_{B_{1/4}(q)} u \right)^2. \tag{53}$$

Combining (52) and (53) with Proposition 2.1.19, we see that there exists a constant $C_{27} > 0$ such that

$$|\nabla u|(q) \leq C_{27} \sup_{B_{1/4}(q)} u. \tag{54}$$

By the argument in Proposition 2.2.17, for sufficiently small $\sigma > 0$, there exists a constant $C_{28} > 0$ such that

$$\left(\frac{1}{m_f(B_{1/2}(q))} \int_{B_{1/2}(q)} u^\sigma \, \mathrm{d}m_f \right)^{\frac{1}{\sigma}} \leq C_{28} \inf_{B_{1/4}(q)} u. \tag{55}$$

We note that the lower bounds of Neumann-Poincaré eigenvalue in Proposition 2.2.17 is guaranteed by Proposition 3.1.1. This enable us to conduct the argument in the proof of Proposition 2.2.17 even in our setting. Then Theorem 2.2.16 implies that there exists a constant $C_{29} > 0$ such that

$$\|u\|_{\infty,1/4} \leq C_{29} \left(\frac{1}{m_f(B_{1/4}(q))} \int_{B_{1/2}(q)} u^\sigma \, \mathrm{d}m_f \right)^{\frac{1}{\sigma}}.$$

Together with (55), we find that there exists a constant $C_{30} > 0$ such that

$$\sup_{B_{1/4}(q)} u \leq C_{30} \inf_{B_{1/4}(q)} u.$$

Combining this with (54), we see

$$|\nabla u|(q) \leq C_{27} C_{30} u(q).$$

This yields the desired assertion. □

As a corollary, we obtain the following assertion (see [**38**, Corollary 6.1]):

Corollary 3.4.2 ([**38**]). *Let (M, g, f) be an n-dimensional complete weighted Riemannian manifold and $N \in (-\infty, 0) \cup [n, \infty]$. For $b > 0$, we assume*

$$\mathrm{Ric}_f^N \geq 0, \quad |f| \leq b.$$

Then there exists a constant $C_{31} > 0$ depending only on n, N, b such that

$$|\nabla \log u| \leq C_{31}.$$

3.5. Related topics : analysis of eigenfunction

In this section, we give a Cheng type inequality under lower bounds of Ric_f^N with ε-range. In the classical case, Liouville type theorems and the Cheng inequality are closely related. Especially, they are both obtained by gradient estimates of the solution of $\Delta u + \lambda u = 0$ (see e.g., [**131**, Theorem 1.1] and [**67**, Corollary 6.4]). Moreover, in the weighted case $N = \infty$ the Liouville type theorem was used to obtain the rigidity of the Cheng type inequality (see [**87**]). Although we obtain a Cheng type inequality, the relation with the analysis of harmonic functions is left for future work in our setting.

Under lower bounds of weighted Ricci curvature with ε-range, the Cheng type estimate is as follows (see [**37**, Theorem 6]):

Theorem 3.5.1 ([**37**]). *Let (M, g, f) be an n-dimensional weighted complete non-compact Riemannian manifold, $N \in (-\infty, 1] \cup [n, \infty]$ and $\varepsilon \in \mathbb{R}$ in the ε-range. For $K > 0$ and $b_2 \geq b_1 > 0$, we assume*

$$\mathrm{Ric}_f^N \geq -K \mathrm{e}^{\frac{4(\varepsilon-1)f}{n-1}} g, \quad b_1 \leq \mathrm{e}^{\frac{2(1-\varepsilon)f}{n-1}} \leq b_2.$$

Then for $p > 1$ and $c := c(n, N, \varepsilon)$ as in (9), we have

$$\lambda_{f,p} \leq \left(\sqrt{\frac{K}{c}} \frac{1}{p b_1} \right)^p.$$

Proof. For an arbitrary $\alpha > 0$, we set

$$\beta := -\frac{1}{p} \left(\sqrt{\frac{K}{c}} \frac{1}{b_1} + \alpha \right),$$

and for $q \in M$, a constant $C_{32} > 0$ and $r \geq 2$, let ϕ be a cut-off function such that $\phi \equiv 1$ on $B_{r-1}(q)$ and $\phi \equiv 0$ on $M \backslash B_r(q)$ with $|\nabla \phi| \leq C_{32}$. We set

$$\varphi(x) := \exp(\beta d_q(x)) \phi(x).$$

For an arbitrary $\sigma > 0$, we obtain

$$\begin{aligned} |\nabla \varphi|^p &= \left| \beta \mathrm{e}^{\beta d_q} \phi \nabla d_q + \mathrm{e}^{\beta d_q} \nabla \phi \right|^p \\ &\leq \mathrm{e}^{p\,\beta d_q} \left(-\beta \phi + |\nabla \phi| \right)^p \\ &\leq \mathrm{e}^{p\,\beta d_q} \left\{ (1+\sigma)^{p-1} (-\beta \phi)^p + \left(\frac{1+\sigma}{\sigma} \right)^{p-1} |\nabla \phi|^p \right\}. \end{aligned}$$

We find

$$\begin{aligned} \lambda_{f,p} &\leq (1+\sigma)^{p-1} (-\beta)^p + \left(\frac{1+\sigma}{\sigma} \right)^{p-1} \frac{\int_{B_r(q) \backslash B_{r-1}(q)} \mathrm{e}^{p\,\beta d_q} |\nabla \phi|^p \, \mathrm{d}m_f}{\int_M \mathrm{e}^{p\,\beta d_q} \phi^p \, \mathrm{d}m_f} \\ &= (1+\sigma)^{p-1} (-\beta)^p + C_{32}^p \left(\frac{1+\sigma}{\sigma} \right)^{p-1} \frac{\mathrm{e}^{p\,\beta(r-1)} m_f(B_R(q))}{\mathrm{e}^{p\,\beta} m_f(B_1(q))}. \end{aligned} \tag{56}$$

From Proposition 2.1.19, we have

$$\frac{m_f(B_r(q))}{m_f(B_1(q))} \le \frac{b_2}{b_1}\frac{\int_0^{r/b_1} \mathbf{s}_{-cK}(\tau)^{1/c}\, \mathrm{d}\tau}{\int_0^{1/b_2} \mathbf{s}_{-cK}(\tau)^{1/c}\, \mathrm{d}\tau}. \tag{57}$$

By direct calculations, we obtain

$$\begin{aligned}\int_0^{r/b_1} \mathbf{s}_{-cK}(\tau)^{\frac{1}{c}}\, \mathrm{d}\tau &= (cK)^{-\frac{1}{2c}}\int_0^{r/b_1}\left\{\frac{\exp\left(\sqrt{cK}\,\tau\right)-\exp\left(-\sqrt{cK}\,\tau\right)}{2}\right\}^{\frac{1}{c}} \mathrm{d}\tau\\ &\le (cK)^{-\frac{1}{2c}}\int_0^{r/b_1}\exp\left(\sqrt{\frac{K}{c}}\,\tau\right)\,\mathrm{d}\tau\\ &= (cK)^{-\frac{1}{2c}}\sqrt{\frac{c}{K}}\left(\exp\left(\sqrt{\frac{K}{c}}\,\frac{r}{b_1}\right)-1\right).\end{aligned}$$

Together with (57), we have

$$\frac{m_f(B_r(q))}{m_f(B_1(q))} \le \frac{b_2}{b_1}\left(\int_0^{1/b_2}\mathbf{s}_{-cK}(\tau)^{1/c}\,\mathrm{d}\tau\right)^{-1}(cK)^{-\frac{1}{2c}}\sqrt{\frac{c}{K}}\exp\left(\sqrt{\frac{K}{c}}\,\frac{r}{b_1}\right).$$

Hence, there exists a constant $C_{33}>0$ depending only on c, b_1, b_2, K, α such that

$$\frac{\mathrm{e}^{p\,\beta(r-1)}m_f(B_r(q))}{\mathrm{e}^{p\,\beta}m_f(B_1(q))} \le C_{33}\exp\left(p\,\beta r+\sqrt{\frac{K}{c}}\,\frac{r}{b_1}\right) = C_{33}\exp\left(-\alpha r\right)\to 0$$

as $r\to\infty$. Combining this with (56), we see

$$\lambda_{f,p} \le (1+\sigma)^{p-1}(-\beta)^p.$$

Since σ, α are arbitrary, we arrive at the desired assertion. □

Remark 3.5.2. In the case $N\in[n,\infty)$, we take $\varepsilon=1$ and $b_1=b_2=1$, and this recovers Theorem 2.3.1.

Actually, slightly different type estimate of Cheng type is available. This is obtained as an application of the volume comparison property in Proposition 2.1.14. Indeed, we possess the following Cheng type estimate (see [**37**, Theorem 10]):

Theorem 3.5.3 ([**37**])**.** *Let (M,g,f) be an n-dimensional complete weighted Riemannian manifold. For $q\in M$, $b>0$ and $K\ge 0$, we assume $s_{f,q}$, which is defined in* (10)*, is smooth and*

$$\mathrm{Ric}_f^N \ge -K\mathrm{e}^{\frac{4(\varepsilon-1)f}{n-1}}g, \quad |\nabla s_q|\le b.$$

For $c:=c(n,N,\varepsilon)$ as in (9) *and $p>1$, we have*

$$\lambda_{\{1+2\frac{1-\varepsilon}{n-1}\},p} \le \left(\frac{b}{p}\sqrt{\frac{K}{c}}\right)^p.$$

Remark 3.5.4. As mentioned in Remark 2.3.3, there are several proofs in the unweighted case $f\equiv 0$. Although the gradient estimate is shown also in the weighted case $N\in[n,\infty)$, it seems that there is a difficulty when we adapt the gradient estimate arguments straightforwardly in the case $N\in(-\infty,1]\cup\{\infty\}$. The difficulty appears when we apply the Bochner formula.

CHAPTER 4

Analysis of porous medium equations

We give an Aronson-Bénilan type gradient estimate for porous medium equation $\partial_t u = \Delta_f u^m$ under lower bounds of Ric_f^N with ε-range (Theorems 4.1.1 and 4.2.1). It turned out that this type of estimate is available when $\left(-\infty, \frac{-2}{m-1}\right) \cup [n, \infty]$.

4.1. Non-compact case

In this section, we obtain a local Aronson-Bénilan type estimate. As an application, a global estimate is obtained. The local Aronson-Bénilan estimate was obtained as follows (see [**36**, Theorem 4]):

Theorem 4.1.1 ([**36**]). *Let (M, g, f) be an n-dimensional complete weighted Riemannian manifold, $m > 1$, $q \in M$, $r > 0$, and let u be a positive smooth solution of $\partial_t u = \Delta_f u^m$ on $B_{2r}(q) \times [0, T]$. Also, let $N \in \left(-\infty, \frac{-2}{m-1}\right) \cup [n, \infty]$, $\varepsilon \in \mathbb{R}$ in the ε-range. For $K \geq 0$, $b_2 \geq b_1 > 0$, we assume*

$$\mathrm{Ric}_f^N \geq -K \mathrm{e}^{\frac{4(\varepsilon-1)f}{n-1}} g, \quad b_1 \leq \mathrm{e}^{\frac{2(1-\varepsilon)f}{n-1}} \leq b_2$$

on $B_{2r}(q)$. We set

$$a(m, N) := \begin{cases} 1 & \text{when } N = \infty, \\ \frac{N(m-1)}{N(m-1)+2} & \text{when } \left(-\infty, \frac{-2}{m-1}\right) \cup [n, \infty), \end{cases} \tag{58}$$

and

$$v := \frac{m}{m-1} u^{m-1}, \quad L := (m-1) \sup_{B_{2r}(q) \times [0,T]} v.$$

Then for $a := a(m, N)$ and any $\alpha > 1$, there exist positive constants C and D depending only on N, ε, b_1 such that we have

$$\frac{|\nabla v|^2}{v^2} - \alpha \frac{\partial_t v}{v} \leq \left[\frac{a \alpha^2 m L^{1/2}}{(\alpha-1)^{1/2} (m-1)^{1/2}} \frac{C}{r} \right.$$
$$\left. a^{1/2} \alpha \left\{ \frac{1}{t} + \frac{K}{b_1^2} \frac{L}{2(\alpha-1)} + \frac{D}{r^2} \left(1 + \sqrt{K}\, r \coth\left(\frac{\sqrt{cK}}{b_2} r \right) \right) \right\}^{\frac{1}{2}} \right]^2$$

on $B_r(q) \times (0, T]$.

We first show the following inequality (see [**36**, Lemma 1]):

Lemma 4.1.2 ([**38**]). *Let (M, g, f) be an n-dimensional complete weighted Riemannian manifold. We use the same notation $u, m, v, N, a := a(m, N)$ as in Theorem 4.1.1. For an arbitrary $\alpha \in \mathbb{R}$, we set*

$$F_\alpha := \frac{|\nabla v|^2}{v} - \alpha \frac{\partial_t v}{v}, \quad \mathcal{L} := \frac{\partial}{\partial t} - (m-1) v \Delta_f.$$

We have

$$\mathcal{L}(F_\alpha) \le -\frac{1}{a}\{(m-1)\Delta_f v\}^2 + (1-\alpha)\left(\frac{\partial_t v}{v}\right)^2 + 2m\langle \nabla v, \nabla F_\alpha\rangle - 2(m-1)\operatorname{Ric}_f^N(\nabla v, \nabla v).$$

PROOF. It follows from the argument in [**50**, Lemma 2.1] that

$$\mathcal{L}(F_\alpha) = -2(m-1)\left(\|\operatorname{Hess} v\|^2 + \operatorname{Ric}_f^\infty(\nabla v, \nabla v)\right) + 2m\langle \nabla F_\alpha, \nabla v\rangle - (\alpha-1)\left(\frac{\partial_t v}{v}\right)^2 - F_1^2.$$

Together with Propositions 2.1.6 and 2.1.7, we see

$$\begin{aligned}
\mathcal{L}(F_\alpha) &\le -2(m-1)\left(\frac{(\Delta_f v)^2}{N} + \operatorname{Ric}_f^N(\nabla v, \nabla v)\right) + 2m\langle \nabla F_\alpha, \nabla v\rangle - (\alpha-1)\left(\frac{\partial_t v}{v}\right)^2 - F_1^2 \\
&= \left\{\frac{-2(m-1)}{N} - (m-1)^2\right\}(\Delta_f v)^2 - 2(m-1)\operatorname{Ric}_f^N(\nabla v, \nabla v) \\
&\quad + 2m\langle \nabla F_\alpha, \nabla v\rangle - (\alpha-1)\left(\frac{\partial_t v}{v}\right)^2.
\end{aligned}$$

□

We are now in a position to give a proof of Theorem 4.1.1.

PROOF OF THEOREM 4.1.1. Let ϕ_1 be a non-negative cut-off function on $[0,\infty)$ such that

$$\phi_1 = \begin{cases} 1 & \text{on } [0,1], \\ 0 & \text{on } [2,\infty), \end{cases}$$

and $0 \le \phi_1 \le 1$ on $(1,2)$ with $-C_{34}\phi_1^{1/2} \le \phi_1' \le 0$ and $\phi_1'' \ge -C_{34}$, where $C_{34} > 0$ is a constant. For $q \in M$, we set

$$\phi(x) := \phi_1\left(\frac{d_q(x)}{r}\right).$$

There exists a constant $C_{35} > 0$ depending on C_{34} such that

$$\frac{|\nabla \phi|^2}{\phi} \le \frac{C_{35}}{r^2}.$$

We only consider the case $K > 0$. By Proposition 2.1.15, we have

$$\Delta_f d_q \le \frac{\sqrt{K}}{b_1\sqrt{c}}\coth\left(\frac{\sqrt{cK}\,r}{b_2}\right).$$

Then we have

$$\begin{aligned}
\Delta_f \phi &= \frac{\phi_1'(d_q/r)\Delta_f d_q}{r} + \frac{\phi_1''(d_q/r)|\nabla d_q|^2}{r^2} \\
&\ge -\frac{C_{34}\phi_1^{1/2}(d_q/r)}{r}\frac{\sqrt{K}}{b_1\sqrt{c}}\coth\left(\frac{\sqrt{cK}}{b_2}r\right) - \frac{C_{34}}{r^2} \\
&\ge -\frac{C_{36}}{r^2}\left(1 + \sqrt{K}\,r\coth\left(\frac{\sqrt{cK}}{b_2}\,r\right)\right),
\end{aligned} \tag{59}$$

where $C_{36} > 0$ is a constant depending on N, ε, b_1. For an arbitrarily fixed $\alpha \geq 1$, we have

$$\begin{aligned}
\mathcal{L}(F_\alpha) &\leq -\frac{1}{a}\left\{(m-1)\Delta_f v\right\}^2 + 2m\langle \nabla v, \nabla F_\alpha\rangle + \frac{2KL}{b_1^2}\frac{|\nabla v|^2}{v} \\
&= -\frac{1}{a\alpha^2}\left(F_\alpha + (\alpha-1)\frac{|\nabla v|^2}{v}\right)^2 + 2m\langle \nabla v, \nabla F_\alpha\rangle + \frac{2KL}{b_1^2}\frac{|\nabla v|^2}{v},
\end{aligned} \tag{60}$$

where we used

$$(m-1)\Delta_f v = \frac{\partial_t v}{v} - \frac{|\nabla v|^2}{v}.$$

We apply the maximum principle to $G_\alpha := t\phi F_\alpha$. Let $(x_1, t_1) \in B_{2r}(q) \times [0, T]$ be a point that attains the maximum of G_α with $G_\alpha(x_1, t_1) > 0$. Since $G_\alpha(x, t) = 0$ when $t = 0$, we see $t_1 > 0$. We have

$$\nabla G_\alpha(x_1, t_1) = 0, \quad \Delta G_\alpha(x_1, t_1) \leq 0, \quad \partial_t G_\alpha(x_1, t_1) \geq 0.$$

This implies

$$\mathcal{L}(G_\alpha)(x_1, t_1) \geq 0.$$

At (x_1, t_1), using $\nabla G_\alpha(x_1, t_1) = 0$ and (60), we see

$$\begin{aligned}
0 &\leq \mathcal{L}(G_\alpha) \\
&= \phi F_\alpha + t_1\phi\,\partial_t F_\alpha - (m-1)vt_1 F_\alpha \Delta_f \phi - 2(m-1)vt_1\langle \nabla\phi, \nabla F_\alpha\rangle \\
&= \frac{G_\alpha}{t_1} + t_1\phi\mathcal{L}(F_\alpha) - (m-1)v\frac{\Delta_f\phi}{\phi}G_\alpha + 2(m-1)v\frac{|\nabla\phi|^2}{\phi^2}G_\alpha \\
&\leq \frac{G_\alpha}{t_1} + t_1\phi\left\{-\frac{1}{a\alpha^2}\left(F_\alpha + (\alpha-1)\frac{|\nabla v|^2}{v}\right)^2 + 2m\langle \nabla v, \nabla F_\alpha\rangle + \frac{2KL}{b_1^2}\frac{|\nabla v|^2}{v}\right\} \\
&\quad - (m-1)v\frac{\Delta_f\phi}{\phi}G_\alpha + 2(m-1)v\frac{|\nabla\phi|^2}{\phi^2}G_\alpha \\
&= \frac{G_\alpha}{t_1} + t_1\phi\left\{-\frac{1}{a\alpha^2}\left(F_\alpha + (\alpha-1)\frac{|\nabla v|^2}{v}\right)^2 - \frac{2mF_\alpha}{\phi}\langle \nabla v, \nabla\phi\rangle + \frac{2KL}{b_1^2}\frac{|\nabla v|^2}{v}\right\} \\
&\quad - (m-1)v\frac{\Delta_f\phi}{\phi}G_\alpha + 2(m-1)v\frac{|\nabla\phi|^2}{\phi^2}G_\alpha.
\end{aligned}$$

We set

$$\beta := \frac{|\nabla v|^2}{vF_\alpha}(x_1, t_1) \geq 0.$$

Then we deduce

$$\begin{aligned}
0 &\geq \frac{G_\alpha}{t_1} - \frac{t_1\phi F_\alpha^2}{\alpha^2}\left\{1 + (\alpha-1)\beta\right\}^2 - 2mt_1\phi F_\alpha\frac{\langle \nabla v, \nabla\phi\rangle}{\phi} + \frac{2KLt_1\phi}{b_1^2}\frac{|\nabla v|^2}{v} \\
&\quad - (m-1)v\frac{\Delta_f\phi}{\phi}G_\alpha + 2(m-1)v\frac{|\nabla\phi|^2}{\phi^2}G_\alpha \\
&= \frac{G_\alpha}{t_1} - \frac{G_\alpha^2}{a\alpha^2 t_1\phi}\left\{1 + (\alpha-1)\beta\right\}^2 - 2mG_\alpha\frac{\langle \nabla v, \nabla\phi\rangle}{\phi} + \frac{2KL\beta}{b_1^2}G_\alpha \\
&\quad - (m-1)v\frac{\Delta_f\phi}{\phi}G_\alpha + 2(m-1)v\frac{|\nabla\phi|^2}{\phi^2}G_\alpha.
\end{aligned}$$

We estimate the third term. The Cauchy-Schwarz inequality implies

$$\begin{aligned}\left|2mG_\alpha\frac{\langle\nabla v,\nabla\phi\rangle}{\phi}\right| &\le 2mG_\alpha\frac{|\nabla v|\,|\nabla\phi|}{\phi}\\ &= 2mG_\alpha\beta^{1/2}F_\alpha^{1/2}\frac{|\nabla\phi|\phi^{1/2}t_1^{1/2}}{(m-1)^{1/2}\phi^{3/2}t_1^{1/2}}(m-1)^{1/2}v^{1/2}\\ &\le 2mG_\alpha\beta^{1/2}\frac{|\nabla\phi|G_\alpha^{1/2}}{(m-1)^{1/2}\phi^{3/2}t_1^{1/2}}L^{1/2}.\end{aligned}$$

Hence,

$$\begin{aligned}0\le{}&\frac{G_\alpha}{t_1}-\frac{G_\alpha^2}{a\alpha^2t_1\phi}\{1+(\alpha-1)\beta\}^2+2mG_\alpha\beta^{1/2}L^{1/2}\frac{|\nabla\phi|G_\alpha^{1/2}}{(m-1)^{1/2}\phi^{3/2}t_1^{1/2}}+\frac{2KL\beta}{b_1^2}G_\alpha\\ &-(m-1)v\frac{\Delta_f\phi}{\phi}G_\alpha+2(m-1)v\frac{|\nabla\phi|^2}{\phi^2}G_\alpha.\end{aligned}$$

Multiplying both sides by $\frac{\phi}{G_\alpha}$ and moving some terms in the other side, we obtain

$$\begin{aligned}&\frac{\{1+(\alpha-1)\beta\}^2}{a\alpha^2t_1}G_\alpha-2m\frac{\beta^{1/2}L^{1/2}|\nabla\phi|}{(m-1)^{1/2}\phi^{1/2}t_1^{1/2}}G_\alpha^{1/2}\\ &\le\frac{\phi}{t_1}+\frac{2KL\beta}{b_1^2}\phi-(m-1)v\Delta_f\phi+2(m-1)v\frac{|\nabla\phi|^2}{\phi}G_\alpha\\ &\le\frac{\phi}{t_1}+\frac{2KL\beta}{b_1^2}\phi+L\frac{C_{36}}{r^2}\left(1+\sqrt{K}\,r\coth\left(\frac{\sqrt{cK}}{b_2}\,r\right)\right)+2L\frac{|\nabla\phi|^2}{\phi}.\end{aligned}$$

For $A_1>0, A_2, A_3\ge 0$, if $A_1x^2-A_2x\le A_3$, we have

$$x\le\frac{A_2}{A_1}+\sqrt{\frac{A_3}{A_1}}.$$

We apply this for $x:=G_\alpha^{1/2}$ and obtain

$$\begin{aligned}G_\alpha^{1/2}\le{}&\frac{2a\alpha^2mL^{1/2}t_1^{1/2}\{(\alpha-1)\beta\}^{1/2}}{(m-1)^{1/2}(\alpha-1)^{1/2}\{1+(\alpha-1)\beta\}^2}\frac{|\nabla\phi|}{\phi^{1/2}}\\ &+\frac{a^{1/2}\alpha t_1^{1/2}}{1+(\alpha-1)\beta}\left\{\frac{\phi}{t_1}+\frac{2KL\beta}{b_1^2}\phi+\frac{C_{36}L}{r^2}\left(1+\sqrt{K}\,r\coth\left(\frac{\sqrt{cK}}{b_2}r\right)\right)+2L\frac{|\nabla\phi|^2}{\phi}\right\}^{1/2}.\end{aligned}$$

Below, we estimate terms in this inequality as follows:

$$\frac{\{(\alpha-1)\beta\}^{1/2}}{\{1+(\alpha-1)\beta\}^2}\le\frac{1}{2}\frac{1+(\alpha-1)\beta}{\{1+(\alpha-1)\beta\}^2}\le\frac{1}{2},$$

and this implies

$$\begin{aligned}\frac{a\alpha^2t_1}{\{1+(\alpha-1)\beta\}^2}\left(\frac{1}{t_1}+\frac{2KL\beta}{b_1^2}\right)&\le a\alpha^2+\frac{2a\alpha^2t_1}{(\alpha-1)}\frac{KL}{b_1^2}\frac{(\alpha-1)\beta}{\{1+(\alpha-1)\beta\}^2}\\ &\le a\alpha^2\left(1+\frac{KLt_1}{2(\alpha-1)b_1^2}\right).\end{aligned}$$

For $x \in B_r(q)$, this yields

$$
\begin{aligned}
&G_\alpha^{1/2}(x,T)\\
&\leq G_\alpha^{1/2}(x_1,t_1)\\
&\leq \frac{a\alpha^2 m L^{1/2} t_1^{1/2}}{(\alpha-1)^{1/2}(m-1)^{1/2}} \frac{\sqrt{C_{35}}}{r} + \left[a\alpha^2 \left(1 + \frac{KLt_1}{2(\alpha-1)b_1^2} \right) \right.\\
&\quad \left. + \frac{a\alpha^2 t_1}{\{1+(\alpha-1)\beta\}^2} \left\{ \frac{C_{36}L}{r^2} \left(1 + \sqrt{K}\, r \coth\left(\frac{\sqrt{cK}}{b_2} r \right) \right) \right\} + \frac{2C_{35}L}{r^2} \right]^{1/2}\\
&\leq \frac{a\alpha^2 m L^{1/2} t_1^{1/2}}{(\alpha-1)^{1/2}(m-1)^{1/2}} \frac{\sqrt{C_{35}}}{r}\\
&\quad + a^{1/2} T^{1/2} \alpha \left\{ \frac{1}{T} + \frac{KL}{2(\alpha-1)b_1^2} + \frac{(C_{36}+2C_{35})L}{r^2} \left(1 + \sqrt{K}\, r \coth\left(\frac{\sqrt{cK}}{b_2} r \right) \right) \right\}^{1/2}.
\end{aligned}
$$

We divide the both sides by $T^{1/2}$ and we arrive at the desired assertion. □

As a corollary, we have the following estimate (see [**38**, Corollary 3]):

Corollary 4.1.3 ([**38**])**.** *Let (M,g,f) be an n-dimensional complete non-compact weighted Riemannian manifold, and for $m>1$, let u be a positive smooth solution to $\partial_t u = \Delta_f u^m$ on $M \times [0,T]$. Also, let $N \in \left(-\infty, \frac{-2}{m-1}\right) \cup [n,\infty]$ and $\varepsilon \in \mathbb{R}$ in the ε-range. For $K \geq 0$ and $b_2 \geq b_1 > 0$, we assume*

$$\mathrm{Ric}_f^N \geq -K \mathrm{e}^{\frac{4(\varepsilon-1)f}{n-1}} g, \quad b_1 \leq \mathrm{e}^{\frac{2(1-\varepsilon)f}{n-1}} \leq b_2.$$

We set v and L in the same way as in Theorem 4.1.1. Then for any $\alpha > 1$, we have

$$\frac{|\nabla v|^2}{v} - \alpha \frac{\partial_t v}{v} \leq a\alpha^2 \left(\frac{1}{t} + \frac{K}{b_1^2} \frac{L}{2(\alpha-1)} \right)^2$$

on $M \times (0,T]$. In particular, if $K=0$, we have

$$\frac{|\nabla v|^2}{v} - \frac{\partial_t v}{v} \leq \frac{a}{t}$$

on $M \times (0,T]$.

Remark 4.1.4. In the case $N \in [n,\infty)$, we take $\varepsilon = 1$ and $b_1 = b_2 = 1$. Then Theorem 4.1.1 recovers Theorem 2.4.3, and Corollary 4.1.3 recovers Corollary 2.4.5.

4.2. Compact case

In this section, we provide a global Aronson-Bénilan type estimate on compact manifolds under $\mathrm{Ric}_f^N \geq 0$. The proof is essentially different from the case of non-compact manifolds.

Theorem 4.2.1 ([**36**])**.** *Let (M,g,f) be an n-dimensional compact weighted Riemannian manifold, and for $m>1$, let u be a positive smooth solution to $\partial_t u = \Delta_f u^m$ on $M \times [0,T]$. Let $N \in \left(-\infty, \frac{-2}{m-1}\right) \cup [n,\infty]$. We assume*

$$\mathrm{Ric}_f^N \geq 0.$$

We set

$$v := \frac{m}{m-1} u^{m-1}.$$

Then we have

$$\frac{|\nabla v|^2}{v} - \frac{\partial_t v}{v} \leq \frac{a}{t}$$

on $M \times (0, T]$, *where we set* $a := a(m, N)$ *as in* (58).

Proof. Lemma 4.1.2 for the case $\alpha = 1$ and $\mathrm{Ric}_f^N \geq 0$ yields

$$\mathcal{L}(F_1) \leq -\frac{1}{a}\left\{(m-1)\Delta_f v\right\}^2 + 2m\langle \nabla v, \nabla F_1\rangle.$$

For $F := tF_1$, we see

$$\mathcal{L}(F) = t\mathcal{L}(F_1) + F_1 \leq -\frac{1}{a}\frac{F^2}{t} + 2m\langle \nabla F, \nabla v\rangle + \frac{F}{t}.$$

Let $(x_1, t_1) \in M \times [0, T]$ be a point that attains the maximum of F. We may assume $F(x_1, t_1) > 0$. As in the argument in Theorem 4.1.1, we have $\mathcal{L}(F)(x_1, t_1) \geq 0$. Therefore, we have

$$0 \leq \mathcal{L}(F)(x_1, t_1) = -\frac{1}{a}\frac{F(x_1, t_1)^2}{t_1} + \frac{F(x_1, t_1)}{t_1}.$$

For any $(x, t) \in M \times [0, T]$, we have

$$F(x, t) \leq F(x_1, t_1) \leq a.$$

We arrive at the desired assertion. □

Remark 4.2.2. In the case $N \in [n, \infty)$, we take $\varepsilon = 1$ and $b_1 = b_2 = 1$, and this recovers [**73**, Theorem 7.6] (see also Remark 2.4.6).

Remark 4.2.3. As mentioned in section 1.4, while gradient estimates for the heat equation under lower bounds of Ric_f^N with $N \in (-\infty, 0)$ is listed as an open question in [**93**, **95**], the range $\left(-\infty, \frac{-2}{m-1}\right)$ degenerates as $m \searrow 1$.

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