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Extensions of the cosymplectic reduction theorem

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Abstract

Albert [Alb89] studied Hamiltonian actions on cosymplectic manifolds, which are the odd-dimensional analogue of symplectic manifolds, and established a reduction theorem for the actions. A cosymplectic manifold has a Poisson structure whose symplectic foliation has codimension one. In this thesis, we extend Albert's reduction theorem to cases involving Riemannian metrics and Lie groupoid actions.

Hitchin et al. [HKLR87] proved reduction theorems for Kähler and hyperKähler manifolds, which are symplectic manifolds compatible with a Riemannian metric. In this thesis, we establish analogous results for odd-dimensional counterparts, namely coKähler manifolds and 3-cosymplectic manifolds. We further investigate the relationship between the reduction processes for these manifolds and those for Kähler and hyperKähler manifolds.

A Lie groupoid is a groupoid suitable for smooth category. This notion is a generalization of the concept of a Lie group. Mikami and Weinstein [MW88] studied actions of Lie groupoids on symplectic manifolds and established a reduction theorem for such actions. In this thesis, we introduce a new class of submanifolds, called Lagrangian-Legendrean submanifolds, within cosymplectic manifolds. This allows the definition of appropriate Lie groupoid actions on cosymplectic manifolds. We then prove an odd-dimensional analogue of Mikami and Weinstein's theorem, which is also a generalized version of Albert's theorem. The proof heavily utilized the symplectic foliation structure of cosymplectic manifolds.

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Contents

Chapter 1

Introduction

Since the pioneering work of Marsden-Weinstein and Meyer [MW74, Mey73], many types of reduction theorems have been studied for various geometric structures on manifolds (see the table below).

Geometric structures	Structures with metric	3-structures with metric
Symplectic [MW74]	Kähler [HKLR87]	HyperKähler [HKLR87]
Contact [Alb89]	Sasakian [GO01]	3-Sasakian [BGM94]
Cosymplectic [Alb89]	CoKähler	3-cosymplectic

In [HKLR87], Hitchin et al. proved the reduction theorem of Kähler manifolds. They also introduced a notion of a hyperKähler momentum map and proved the reduction theorem of hyperKähler manifolds. Albert [Alb89] studied Hamiltonian actions on contact manifolds and cosymplectic manifolds and proved the reduction theorems. Afterwards, several types of reduction theorems of contact manifolds have been studied [Gei97, Wil02, ZZ06]. In [BGM94], Boyer et al. proved the reduction theorem of 3-Sasakian manifolds via the hyperKähler reduction theorem. Afterwards, Grantcharov and Ornea [GO01] proved the reduction theorem of Sasakian manifolds.

On the other hand, Mikami-Weinstein [MW88] generalized the Marsden-Weinstein-Meyer theorem to symplectic groupoid actions, which extends a notion of a Hamiltonian action on symplectic manifolds (see the table below).

	Phase space	Symmetry
Marsden-Weinstein-Meyer	Symplectic manifold	Lie group
Mikami-Weinstein	Symplectic manifold	Symplectic groupoid
Albert	Cosymplectic manifold	Lie group
Our result	Cosymplectic manifold	Cosymplectic groupoid

In this thesis, we focus on the reduction theorem of cosymplectic manifolds proved by Albert and extend it to cases involving Riemannian metrics and Lie groupoid actions.

Firstly, we obtain reduction theorems of *coKähler manifolds* and *3-cosymplectic manifolds*. They are another odd-dimensional versions of Kähler and hyperKähler manifolds instead of Sasakian and 3-Sasakian manifolds, respectively (see [CMDNY13a] for more details). We give typical examples of coKähler/3-cosymplectic quotients by using "cylinder constructions" and "mapping torus constructions", respectively.

Secondly, we define a notion of an action of a *cosymplectic groupoid* on a cosymplectic manifold by using a notion of a *Lagrangian-Legendrean submanifold*. Afterwards, we prove a reduction theorem which is an analogue of the Mikami-Weinstein theorem. A notion of a cosymplectic groupoid is introduced by [DW15] and recently studied in [FP23]. A cosymplectic groupoid is a Lie groupoid whose space of arrows is endowed with a multiplicative cosymplectic structure.

This thesis is organized as follows.

In chapter 2, basic materials on Poisson geometry are presented. In section 2.1 through section 2.4, we introduce some fundamental concepts of this thesis, such as Poisson manifolds, cosymplectic manifolds, Lie groupoids, and symplectic groupoids. We also give some examples of these notions. In section 2.5, we recall a notion of Morita equivalence of symplectic groupoids, and in section 2.6 we explain a proof of the Mikami-Weinstein theorem by Xu [Xu91b] using Morita equivalence.

chapter 3 is devoted to our first two main theorems. In section 3.1, we recall cosymplectic structures and cosymplectic momentum maps and the proof of the reduction theorem by Albert. In we prove the following coKähler reduction theorem, which is a natural analogue of the Kähler reduction theorem.

Theorem 1.0.1 ([Yon24b]). *Let $(M, g, \varphi, \xi, \eta)$ be a coKähler manifold with the underlying cosymplectic structure (η, ω). Suppose that there is a free and proper Hamiltonian action of a Lie group G on (M, η, ω) which preserves φ. Let $\mu : M \to \mathfrak{g}^*$ be a momentum map and $\zeta \in \mathfrak{g}^*$ a central and regular value of μ. Then $M^\zeta := \mu^{-1}(\zeta)/G$ admits a coKähler structure $(g^\zeta, \varphi^\zeta, \xi^\zeta, \eta^\zeta)$. Moreover, the underlying cosymplectic manifold of $(M^\zeta, g^\zeta, \varphi^\zeta, \xi^\zeta, \eta^\zeta)$ is the cosymplectic quotient $(M^\zeta, \eta^\zeta, \omega^\zeta)$.* □

In section 3.2 we introduce a notion of a 3-cosymplectic momentum map and prove the following 3-cosymplectic reduction theorem, which is a natural analogue of the hyperKähler reduction theorem.

Theorem 1.0.2 ([Yon24b]). *Let $(M, g, (\varphi_i, \xi_i, \eta_i)_{i=1,2,3})$ be a 3-cosymplectic manifold with underlying cosymplectic structures $(\eta_i, \omega_i)_{i=1,2,3}$. Suppose that there is a free and proper action of a Lie group G on M which is Hamiltonian with respect to all three cosymplectic structures $(\eta_i, \omega_i)_{i=1,2,3}$ and preserves $(\varphi_i)_{i=1,2,3}$. Let $\mu : M \to \mathfrak{g}^* \otimes \mathrm{Im}\mathbb{H}$ be a 3-cosymplectic momentum map and $\zeta \in \mathfrak{g}^* \otimes \mathrm{Im}\mathbb{H}$ a central and regular value of μ. Then $M^\zeta := \mu^{-1}(\zeta)/G$ inherits the 3-cosymplectic structure of M.* □

In section 3.3, we study the reduction of geometric structures on cylinders. Let M be a Kähler manifold. Then its *cylinder* $C(M) := M \times \mathbb{R}$ admits a coKähler structure. Conversely, if M is a coKähler manifold, then $C(M)$ admits a Kähler structure. We show that the Kähler (resp. coKähler) quotient of $C(M)$ is the cylinder of the coKähler (resp. Kähler) quotient of M. Similarly, hyperKähler structures and 3-cosymplectic structures are also related by cylinder constructions, and we also show that hyperKähler/3-cosymplectic reduction procedures are compatible with these cylinder constructions. In section 3.4, we investigate coKähler quotients of mapping tori of Kähler manifolds. For a Kähler manifold S and a Hermitian isometry f of S, the mapping torus

$$S_f = (S \times [0, 1])/\{(p, 0) \sim (f(p), 1) \mid p \in S\}$$

admits a coKähler structure. Suppose that there is a free and proper Hamiltonian action of a Lie group on S which preserves the Kähler structure and let $\mu : S \to \mathfrak{g}^*$ be a momentum map. Let f be an equivariant Hermitian isometry of S. Then we show that the action on S is naturally lifted to a Hamiltonian action on S_f if and only if $\mu(f(p)) = \mu(p)$ holds for some $p \in S$. In this situation, we prove that the Kähler/coKähler reduction procedures are compatible with the mapping torus procedure. In section 3.5, we interpret our coKähler reduction theorem from the physical viewpoint. In short, our result suggests that we can reduce time-dependent dynamical systems preserving the property that the flows of the system are geodesics.

In chapter 4 we discuss our third main theorem. In section 4.1, we review the definition and some properties of cosymplectic groupoids. In section 4.2, we introduce a notion of a Lagrangian-Legendrean submanifold of cosymplectic manifolds, and define cosymplectic actions of cosymplectic groupoids on cosymplectic manifolds. We observe that if a cosymplectic groupoid $\mathcal{G} = (G_1 \rightrightarrows G_0)$ acts on a cosymplectic manifold M, then a symplectic groupoid $S_\mathcal{G} = (S_{G_1} \rightrightarrows G_0)$, where S_{G_1} is the symplectic leaf of G_1 that contains unit arrows, acts on each symplectic leaf of M. In section 4.3, we prove the following theorem:

Theorem 1.0.3 ([Yon24a]). *Let $\mathcal{G} = (G_1 \rightrightarrows G_0)$ be a cosymplectic groupoid and M a cosymplectic, free and proper left $\mathcal{G}$-module with respect to a momentum map $\rho : M \to G_0$. Assume that $\xi \in \rho(M)$ is a regular value of ρ. Then $(S_\mathcal{G})_\xi \backslash \rho^{-1}(\xi)$ is a cosymplectic manifold, where $(S_\mathcal{G})_\xi$ is a Lie group consisting of arrows in S_{G_1} whose source and target are both ξ.* □

In section 4.4, we give examples of Theorem 1.0.3. The main example reconstructs Albert's cosymplectic reduction theorem. In section 4.5, we mention Morita equivalence of cosymplectic groupoids.

Lastly, in chapter 5 we show potential for future research.

Chapter 2

Preliminaries from Poisson geometry

There are two purposes of this chapter. The first is to explain the fundamental concepts of Poisson geometry that will be used in the subsequent chapters. The second is to outline the proof of the reduction theorem by Mikami and Weinstein.

2.1 Poisson manifolds

Definition 2.1.1. *A Poisson structure on a manifold M is an $\mathbb{R}$-bilinear map $\{\cdot,\cdot\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$ which satisfies the following conditions:*

- $\{f, g\} = -\{g, f\}$,
- $\{\{f, g\}, h\} + \{\{h, f\}, g\} + \{\{g, h\}, f\} = 0 \qquad$ (*Jacobi rule*),
- $\{f, gh\} = g\{f, h\} + h\{f, g\} \qquad$ (*Leibniz rule*). ■

There is an alternative definition of a Poisson structure using a bivector. A bivector $\pi \in \mathfrak{X}^2(M) = \Gamma(\wedge^2 TM)$ is locally expressed as

$$\pi = \sum_{i<j} \pi_{ij} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}$$

by using local coordinates $(x_1, \cdots, x_{\dim M})$.

Definition 2.1.2. *A Poisson structure on a manifold M is a bivector $\pi \in \mathfrak{X}^2(M)$ which satisfies $[\pi, \pi]_S = 0$, where $[\pi, \pi]_S \in \mathfrak{X}^3(M)$ is a 3-vector whose coefficients of local representation are given by*

$$([\pi, \pi]_S)_{ijk} = \sum_l \left(\frac{\partial \pi_{ij}}{\partial x_l} \pi_{lk} + \frac{\partial \pi_{ki}}{\partial x_l} \pi_{lj} + \frac{\partial \pi_{jk}}{\partial x_l} \pi_{li} \right).$$

■

Two definitions of Poisson manifolds are related by the formula $\{f, g\} = \pi(df, dg)$.

Let (M, π) be a Poisson manifold. We have a map $\pi^\sharp : T^*M \to TM$ by $\beta(\pi^\sharp\alpha) = \pi(\alpha, \beta)$ for $\alpha, \beta \in T^*M$. Then $\mathrm{Im}\pi^\sharp \subset TM$ defines a (possibly singular) foliation on M. Moreover, each leaf of the foliation naturally admits a symplectic structure.

Example 2.1.3. *Any manifold M admits the* trivial Poisson structure *defined by $\{f, g\} = 0$ for any $f, g \in C^\infty(M)$. The symplectic leaves of this Poisson manifold are points of M.* ∎

Example 2.1.4. *Let (M, ω) be a symplectic manifold. Then we have an isomorphism $\omega_\sharp : TM \to T^*M$ defined by $\omega_\sharp(X) = \omega(X, -)$. Then we obtain a Poisson bivector $\pi \in \mathfrak{X}^2(M)$ induced by the map $\pi^\sharp := (\omega_\sharp)^{-1} : T^*M \to TM$. In this case, there is only one symplectic leaf, that is, M itself.* ∎

Example 2.1.5. *Let $\mathfrak{g}$ be a Lie algebra. Then the dual vector space $\mathfrak{g}^*$ admits a Poisson structure defined by*

$$\{f, g\}(\xi) = -\xi([df_\xi, dg_\xi])$$

for $f, g \in C^\infty(\mathfrak{g}^*)$ and $\xi \in \mathfrak{g}^*$, where $[\cdot, \cdot]$ is the Lie bracket of $\mathfrak{g}$ and we consider $df_\xi, dg_\xi : \mathfrak{g}^* \to \mathbb{R}$ identifying $T_\xi\mathfrak{g}^*$ with $\mathfrak{g}^*$. Symplectic leaves of this Poisson manifold are orbits of the coadjoint action $\mathrm{Ad}^* : \mathrm{G} \to \mathrm{GL}(\mathfrak{g}^*)$.

This Poisson structure is called a linear Poisson structure *since local coefficients of the corresponding bivector are linear functions. In general, for a vector space V there is a correspondence*

$$\Big\{\textit{Lie algebra structure on } V\Big\} \xleftrightarrow{\textit{1:1}} \Big\{\textit{Linear Poisson structure on } V^*\Big\}.$$

∎

2.2 Cosymplectic manifolds

In this section, we discuss an important example of Poisson manifolds in this thesis, namely cosymplectic manifolds, and describe Hamiltonian actions on them as well as Albert's reduction theorem.

An *almost cosymplectic structure* on a $(2n + 1)$-dimensional manifold M is a pair of $\eta \in \Omega^1(M)$ and $\omega \in \Omega^2(M)$ such that $\eta \wedge \omega^n \neq 0$. On an almost cosymplectic manifold (M, η, ω) there is a unique vector field ξ which satisfies

$$\omega(\xi, -) = 0, \qquad \eta(\xi) = 1.$$

ξ is called the Reeb vector field of (M, η, ω). We have an isomorphism of $C^\infty(M)$-modules $\flat : \mathfrak{X}(M) \to \Omega^1(M)$ defined by $\flat(X) = \omega(X, -) + \eta(X)\eta$. Conversely, a pair (η, ω) is an almost cosymplectic structure if and only if the map $\flat : \mathfrak{X}(M) \to \Omega^1(M)$ defined as above is an isomorphism and there is a vector field ξ which satisfies the above conditions.

An almost cosymplectic structure (η, ω) is called a contact structure when $\omega = d\eta$. On the other hand, an almost cosymplectic structure (η, ω) is called a *cosymplectic structure* when $d\eta = 0,\ d\omega = 0$. For a contact structure $\eta \in \Omega^1(M)$, the distribution $\mathrm{Ker}\eta$ is completely non-integrable. On the other hand, for a cosymplectic structure (η, ω), the distribution $\mathrm{Ker}\eta$ is integrable since η is closed. Therefore, contact structures and cosymplectic structures are two classes of almost cosymplectic structures which are polar opposites of each other.

Example 2.2.1. *The simplest example of cosymplectic manifolds is the* $(2n+1)$*-dimensional Euclidean space* $\mathbb{R}^{2n+1} = \{(x_1, \cdots, x_n, y_1, \cdots, y_n, t)\}$*, equipped with a cosymplectic structure*

$$\eta_{\mathrm{std}} = dt, \qquad \omega_{\mathrm{std}} = \sum_{i=1}^{n} dx_i \wedge dy_i.$$

In fact, any cosymplectic manifold of dimension $2n+1$ *is locally equivalent to the standard cosymplectic manifold* $(\mathbb{R}^{2n+1}, \eta_{\mathrm{std}}, \omega_{\mathrm{std}})$ *(Darboux type theorem).* ■

Example 2.2.2. *For a symplectic manifold* S *and a symplectomorphism* $f : S \to S$*, the* mapping torus

$$S_f = (S \times [0,1])/\{(p, 0) \sim (f(p), 1) \mid p \in S\}$$

naturally admits a cosymplectic structure (it will be discussed in detail in section 3.4). ■

A cosymplectic structure (η, ω) on M induces a Poisson structure $\pi \in \mathfrak{X}^2(M)$ on M by

$$\pi(\alpha, \beta) = \omega(\flat^{-1}\alpha, \flat^{-1}\beta),$$

where $\alpha, \beta \in T^*M$. This Poisson structure is regular and has corank 1. Its symplectic leaves coincide with those of the integrable distribution $\mathrm{Ker}\eta$ and the symplectic form on a symplectic leaf S is $\omega|_S$. In fact, it is known that a cosymplectic structure on M is equivalent to a corank 1 regular Poisson structure on M with a Poisson vector field which is transverse to the symplectic leaves [GMP11].

For every function $f \in C^\infty(M)$ on a cosymplectic manifold M, we associate a vector field X_f by

$$X_f = \flat^{-1}(df - \xi(f)\eta).$$

X_f is called the Hamiltonian vector field of f. This condition is equivalent to

$$\omega(X_f, -) = df - \xi(f)\eta, \qquad \eta(X_f) = 0.$$

Let (M, η, ω) be a cosymplectic manifold and G a Lie group acts on M from left. We suppose that the action preserves η, ω, i.e., $L_g^*\eta = \eta,\ L_g^*\omega = \omega$. Denote the Lie algebra of G as $\mathfrak{g}$. Albert [Alb89] defined the notion of momentum maps on cosymplectic manifolds:

Definition 2.2.3. *A smooth map* $\mu : M \to \mathfrak{g}^*$ *is called a momentum map when the following conditions are satisfied:*

- *μ is equivariant, i.e., $\mu(gp) = Ad_g^*\mu(p)$ holds for any $p \in M$ and $g \in G$.*

- *For any $A \in \mathfrak{g}$, the induced vector field $A^* \in \mathfrak{X}(M)$ is the Hamiltonian vector field of a function $\mu^A : M \to \mathbb{R}$ defined by $\mu^A(p) = (\mu(p))(A)$,*

- *For the Reeb vector field ξ and any $A \in \mathfrak{g}$, $d\mu^A(\xi) = 0$ holds.* ■

The action of G is said to be Hamiltonian if there is a momentum map. Now we assume that there is a Hamiltonian action of G on (M, η, ω) which is free and proper. Let $\zeta \in \mathfrak{g}^*$ be a regular value of a momentum map $\mu : M \to \mathfrak{g}^*$. Since μ is equivariant, the isotropy group G_ζ acts on $\mu^{-1}(\zeta)$. Let $M^\zeta := \mu^{-1}(\zeta)/G_\zeta$ and $\pi : \mu^{-1}(\zeta) \to M^\zeta$ be the natural projection.

Theorem 2.2.4 (Albert [Alb89])**.** *There is a unique cosymplectic structure $(\eta^\zeta, \omega^\zeta)$ on M^ζ which satisfies $\pi^*\eta^\zeta = \eta|_{\mu^{-1}(\zeta)}$, $\pi^*\omega^\zeta = \omega|_{\mu^{-1}(\zeta)}$.*

Proof. Since $d\mu^A(X) = \omega(A^*, X)$, for any $p \in \mu^{-1}(\zeta)$ we have

$$T_p\mu^{-1}(\zeta) = \{X_p \in T_pM \mid \omega(A_p^*, X_p) = 0,\ A \in \mathfrak{g}\}. \tag{2.2.1}$$

Let $\mathfrak{g}_\zeta$ be the Lie algebra of G_ζ. For any $p \in \mu^{-1}(\zeta)$ and $g \in G_\zeta$ we have $\mu(L_g(p)) = \zeta$, and by differentiating this we obtain

$$\mathfrak{g}_\zeta = \{B \in \mathfrak{g} \mid d\mu(B_p^*) = 0\}$$

for any $p \in \mu^{-1}(\zeta)$. Hence, if we define $\mathfrak{g}_p := \{A_p^* \mid A \in \mathfrak{g}\}$ and $\mathfrak{g}_{\zeta,p} := \{B_p^* \mid B \in \mathfrak{g}_\zeta\}$, we obtain

$$\mathfrak{g}_{\zeta,p} = \mathfrak{g}_p \cap T_p\mu^{-1}(\zeta). \tag{2.2.2}$$

We see that $\eta|_{\mu^{-1}(\zeta)}$ and $\omega|_{\mu^{-1}(\zeta)}$ are basic with respect to the fibration $\pi : \mu^{-1}(\zeta) \to M^\zeta$. For any $B \in \mathfrak{g}_\zeta$, we have $\eta|_{\mu^{-1}(\zeta)}(B^*) = 0$. In addition, since $d\eta|_{\mu^{-1}(\zeta)} = 0$,

$$\mathcal{L}_{B^*}\eta|_{\mu^{-1}(\zeta)} = d\iota_{B^*}\eta|_{\mu^{-1}(\zeta)} + \iota_{B^*}d\eta|_{\mu^{-1}(\zeta)} = 0$$

holds, and thus $\eta|_{\mu^{-1}(\zeta)}$ is basic. On the other hand, (2.2.1) implies that $\omega|_{\mu^{-1}(\zeta)}(B_p^*, X_p) = 0$ for any $B \in \mathfrak{g}_\zeta$ and $X_p \in T_p\mu^{-1}(\zeta)$, so similarly $\omega|_{\mu^{-1}(\zeta)}$ is basic. Then we obtain η^ζ and ω^ζ. Moreover, they are closed since η, ω are closed and π^* is injective.

All that remains is to prove that $(\eta^\zeta, \omega^\zeta)$ is an almost cosymplectic structure. We have $\xi_p \in T_p\mu^{-1}(\zeta)$ for any $p \in \mu^{-1}(\zeta)$, and $L_g^*\eta = \eta$, $L_g^*\omega = \omega$ implies $(L_g)_*\xi_p = \xi_{L_g(p)}$. So we can define a vector field $\xi^\zeta = d\pi(\xi|_{\mu^{-1}(\zeta)})$ on M^ζ, and we have $\eta^\zeta(\xi^\zeta) = 1$, $\omega^\zeta(\xi^\zeta, -) = 0$.

Lastly, we prove that the map $\flat^\zeta : TM^\zeta \to T^*M^\zeta;\ X^\zeta \mapsto \iota_{X^\zeta}\omega^\zeta + \iota_{X^\zeta}\eta^\zeta\eta^\zeta$ is an isomorphism. Suppose that $X^\zeta \in TM^\zeta$ satisfies $\flat^\zeta(X^\zeta) = 0$. Take $p \in \mu^{-1}(\zeta)$ and let $x = \pi(p)$. We can take $X_p \in T_p\mu^{-1}(\zeta)$ such that $d\pi(X_p) = X_x^\zeta$. Then

$$\omega(X_p, Y_p) + \eta(X_p)\eta(Y_p) = 0$$

holds for any $Y_p \in T_p\mu^{-1}(\zeta)$, and this implies

$$\eta(X_p) = 0, \qquad \omega(X_p, Y_p) = 0. \tag{2.2.3}$$

The almost cosymplectic structure (η, ω) on M gives a decomposition

$$T_pM = \mathbb{R}\langle \xi_p \rangle \oplus \mathrm{Ker}\eta_p$$

and $\omega_p|_{\mathrm{Ker}\eta_p}$ is non-degenerate. We define $E_p = \mathrm{Ker}\eta_p \cap T_p\mu^{-1}(\zeta)$. Then by (2.2.1) and (2.2.3) we obtain

$$X_p \in (E_p)^{\omega_p|_{\mathrm{Ker}\eta_p}} = ((\mathfrak{g}_p)^{\omega_p|_{\mathrm{Ker}\eta_p}})^{\omega_p|_{\mathrm{Ker}\eta_p}} = \mathfrak{g}_p,$$

where $V^{\omega_p|_{\mathrm{Ker}\eta_p}}$ denotes the orthogonal complement of $V \subset \mathrm{Ker}\eta_p$ with respect to $\omega_p|_{\mathrm{Ker}\eta_p}$. Now we can conclude that $X_p \in \mathfrak{g}_{\zeta,p}$ by (2.2.2) and thus $X_x^\zeta = d\pi(X_p) = 0$. □

2.3 Lie groupoids

A groupoid is a small category in which all arrows are invertible. This is summarized in the following diagram:

$$G_1 {}_s\!\times_t G_1 \xrightarrow{m} G_1 \underset{t}{\overset{s}{\rightrightarrows}} G_0 \xrightarrow{u} G_1, \qquad G_1 \circlearrowleft i$$

where $G_1 {}_s\!\times_t G_1 = \{(g,h) \in G_1 \times G_1 \mid s(g) = t(h)\}$[*]. G_1 is a set of arrows and G_0 is a set of objects, m, i, s, t, u (these maps are called structure maps of the groupoid) are maps of multiplication, inverse, source, target, and unit, respectively. G_1 and G_0 are sometimes called the total space and the base space, respectively. For any $\xi \in G_0$, $s^{-1}(\xi) \cap t^{-1}(\xi)$ is a group. This group is called the *isotropy group* on ξ, and denoted by $\mathcal{G}_\xi$. A subset

$$\{\zeta \in G_0 \mid \text{there is an arrow } g \in G_1 \text{ such that } s(g) = \xi,\ t(g) = \zeta\}$$

of G_0 is called the *groupoid orbit* through ξ. We denote a groupoid $\mathcal{G} = (G_1, G_0, m, i, s, t, u)$ simply by $\mathcal{G} = (G_1 \rightrightarrows G_0)$, $m(g,h)$ by gh, $u(\xi)$ by 1_ξ for $\xi \in G_0$.

A groupoid is called a *Lie groupoid* if G_1 and G_0 are smooth manifolds, s, t are smooth submersions, and m, i, u are smooth maps. A Lie groupoid $H_1 \rightrightarrows H_0$ is called a *Lie subgroupoid* of another Lie groupoid $G_1 \rightrightarrows G_0$ when $H_1 \rightrightarrows H_0$ is a subcategory of $G_1 \rightrightarrows G_0$ and H_1 is an immersed submanifold of G_1. A *morphism* between Lie groupoids is a smooth functor.

Example 2.3.1. *A Lie group G is regarded as a Lie groupoid $G \rightrightarrows \{*\}$ which has only one object.* ■

Example 2.3.2. *Let G_0 be a manifold and G a Lie group acting on G_0 from left. Then one obtains a Lie groupoid $G \times G_0 \rightrightarrows G_0$ by defining the following structure maps:*

$$s(g,\xi) = \xi, \qquad t(g,\xi) = g\xi, \qquad u(\xi) = (e,\xi),$$

$$m((g,h\xi),(h,\xi)) = (gh,\xi), \qquad i(g,\xi) = (g^{-1}, g\xi),$$

where $g, h \in G$, $\xi \in G_0$ and e is the unit of G. The Lie groupoid $G \times G_0 \rightrightarrows G_0$ is called the action groupoid *associated to the Lie group action.* ■

[*]Throughout the thesis, we will use this "fibered product" notation without explanation.

The notion of an action of a Lie groupoid on a manifold M is a generalization of the situation where an action of Lie group G on M and an equivariant map $\rho : M \to G_0$, where G_0 is another manifold on which G acts, is given (see Example 2.4.6).

Definition 2.3.3. *Let $\mathcal{G} = (G_1 \rightrightarrows G_0)$ be a Lie groupoid and M be a manifold. A* left action *of $\mathcal{G}$ on M is a pair (ρ, Φ) of smooth maps $\rho : M \to G_0$ and $\Phi : G_1 {}_s\!\times_\rho M \to M$ which satisfies the following conditions:*

1. $\rho(\Phi(g, x)) = t(g)$ *when* $(g, x) \in G_1 {}_s\!\times_\rho M$,
2. $\Phi(g, \Phi(h, x)) = \Phi(gh, x)$ *when* $(g, h) \in G_1 {}_s\!\times_t G_1$, $(h, x) \in G_1 {}_s\!\times_\rho M$,
3. $\Phi(1_{\rho(x)}, x) = x$ *for any* $x \in M$.

Hereinafter, we simply denote $\Phi(g, x)$ by gx and refer to M as left $\mathcal{G}$-module. *The map $\rho : M \to G_0$ is called a* momentum map. *A right action of $\mathcal{G}$ on M is also defined similarly, by swapping the role of the source map and the target map.* ■

A left $\mathcal{G}$-action on M (or a left $\mathcal{G}$-module M) is said to be

- *free* if $gx = x$ (for some $x \in M$ such that $(g, x) \in G_1 {}_s\!\times_\rho M$) implies $g = 1_{\rho(x)}$,
- *proper* if a map $G_1 {}_s\!\times_\rho M \to M \times M$; $(g, x) \mapsto (gx, x)$ is proper.

The orbit space $\mathcal{G}\backslash M$ of a free and proper Lie groupoid action is a smooth manifold and the quotient map $M \to \mathcal{G}\backslash M$ is a submersion. In particular, for any regular value $\xi \in G_0$ of ρ, the isotropy Lie group $\mathcal{G}_\xi = s^{-1}(\xi) \cap t^{-1}(\xi)$ smoothly acts on $\rho^{-1}(\xi)$, and the quotient map $\rho^{-1}(\xi) \to \mathcal{G}_\xi\backslash\rho^{-1}(\xi)$ is a submersion to the smooth quotient space.

2.4 Symplectic groupoids

In Poisson geometry, there is an important class of Lie groupoids, that is, symplectic groupoids. Roughly speaking, symplectic groupoids are "integration" of Poisson manifolds.

Definition 2.4.1. *A* symplectic groupoid *is a pair $(G_1 \rightrightarrows G_0, \omega_{G_1})$ of a Lie groupoid and a symplectic form on G_1 which is* multiplicative, *i.e.,*

$$m^*\omega_{G_1} = pr_1^*\omega_{G_1} + pr_2^*\omega_{G_1}$$

holds, where $pr_i : G_1 {}_s\!\times_t G_1 \to G_1$ denotes the natural projections. ■

The space of objects G_0 of a symplectic groupoid $(G_1 \rightrightarrows G_0, \omega_{G_1})$ has a unique *integrable*[†] Poisson structure such that the source map is a Poisson map. Conversely, Mackenzie and Xu [MX00] proved that for any integrable Poisson manifold G_0, there exists a unique (up to

[†] A Poisson manifold is said to be integrable when induced Lie algebroid (cotangent bundle) is integrable by a Lie groupoid. For more details, see [CF11] for example.

isomorphism) symplectic groupoid $(G_1 \rightrightarrows G_0, \omega_{G_1})$ whose s-fiber $s^{-1}(\xi)$ on each $\xi \in G_0$ is simply connected (such a Lie groupoid is said to be s-simply connected), and these operations are inverses of each other. So there is a correspondence

$$\Big\{s\text{-simply connected symplectic groupoid}\Big\} \xleftrightarrow{1:1} \Big\{\text{integrable Poisson manifold}\Big\}.$$

Under the correspondence, the connected components of the groupoid orbits of $G_1 \rightrightarrows G_0$ coincide with the symplectic leaves of G_0.

The following are examples of integrable Poisson manifolds and their corresponding symplectic groupoids:

Example 2.4.2. *For the trivial Poisson structure on a manifold M, the corresponding symplectic groupoid is $T^*M \rightrightarrows M$, where both source and target maps are the projection of the vector bundle $T^*M \to M$, groupoid multiplication is the fiberwise addition, and multiplicative symplectic structure is the canonical one on T^*M.* ■

Example 2.4.3. *For a symplectic manifold (S, ω), the corresponding symplectic groupoid is the* fundamental groupoid $\Pi(S) \rightrightarrows S$ *of S. Here, $\Pi(S)$ is the set of homotopy classes of paths in S with endpoints fixed, and source map s and target map t send a homotopy class to its endpoints. $\Pi(S)$ admits a multiplicative symplectic structure $(s \times t)^*(\omega \oplus -\omega)$, where $s \times t : \Pi(S) \to S \times S$.* ■

Example 2.4.4. *Let $\mathfrak{g}$ be the Lie algebra of a Lie group G. Consider the coadjoint action $\mathrm{Ad}^* : \mathrm{G} \to \mathrm{GL}(\mathfrak{g}^*)$ on $\mathfrak{g}^*$ and the action groupoid associated to this action. The space of arrows $G \times \mathfrak{g}^* \simeq T^*G$ has the canonical symplectic form, and $G \times \mathfrak{g}^* \rightrightarrows \mathfrak{g}^*$ is a symplectic groupoid by this symplectic form. In this case, the corresponding Poisson structure on the space of objects $\mathfrak{g}^*$ is the linear Poisson structure induced by the Lie algebra $\mathfrak{g}$.* ■

Let $(\mathcal{G}, \omega_{G_1})$ be a symplectic groupoid and (M, ω) a symplectic manifold. A left $\mathcal{G}$-action on M (or a left $\mathcal{G}$-module M) is said to be *symplectic* if the graph of the action, i.e.,

$$\{(g, x, gx) \in G_1 \times M \times M \mid (g, x) \in G_1 {}_s\!\times_\rho M\}$$

is a Lagrangian submanifold of $(G_1 \times M \times M, \omega_{G_1} + \omega_1 - \omega_2)$, where ω_i denotes the symplectic structure of i-th M.

Remark 2.4.5. *The condition that the graph is a Lagrangian submanifold is grounded in Weinstein's "symplectic creed" [Wei81] philosophy.* ■

Example 2.4.6. *Let G_0 be a manifold and G a Lie group acting on G_0 from left. Consider the action groupoid $\mathcal{G} = (G \times G_0 \rightrightarrows G_0)$. Then we obtain a correspondence*

$$\Big\{\textit{left } \mathcal{G}\textit{-action on } M\Big\} \xleftrightarrow{1:1} \Big\{\textit{left } G\textit{-action on } M \textit{ with a equivariant map } \rho : M \to G_0\Big\}$$

by a formula $gx = (g, \rho(x))x$, where the left side means the action of $g \in G$ on $x \in M$ and the right side means the action of $(g, \rho(x)) \in G \times G_0$ on x. Moreover, when $\mathcal{G}$ is $G \times \mathfrak{g}^* \rightrightarrows \mathfrak{g}^*$ and M has a symplectic form ω, we have a correspondence

$$\Big\{\textit{symplectic left } \mathcal{G}\textit{-action on } (M,\omega)\Big\} \overset{\textit{1:1}}{\longleftrightarrow} \Big\{\textit{Hamiltonian left } G\textit{-action on } (M,\omega)\Big\}$$

(see [CFM21], for example). ■

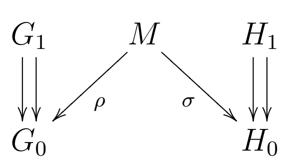
2.5 Morita equivalence

Morita equivalence of Lie groupoids is an analogy to that in ring theory or C^*-algebra theory and is defined using a concept analogous to bimodules. The concept of Morita equivalence can be readily extended to symplectic groupoids (and in section 4.5, we further extend it to cosymplectic groupoids).

Definition 2.5.1. *A Lie groupoid $\mathcal{G} = (G_1 \rightrightarrows G_0)$ is said to be* Morita equivalent *to another Lie groupoid $\mathcal{H} = (H_1 \rightrightarrows H_0)$ when there is a manifold M, a left $\mathcal{G}$-action and a right $\mathcal{H}$-action on M which satisfies the following conditions:*

1. *Momentum maps $\rho : M \to G_0$ and $\sigma : M \to H_0$ are surjective submersions;*
2. *Actions of $\mathcal{G}$ and $\mathcal{H}$ on M are both free and proper;*
3. *The two actions commute with each other;*
4. *ρ is constant on each orbit of the action of $\mathcal{H}$ and an induced map $M/\mathcal{H} \to G_0$ is a diffeomorphism; Similarly, σ is constant on each orbit of the action of $\mathcal{G}$ and an induced map $\mathcal{G}\backslash M \to H_0$ is a diffeomorphism.*

In this situation, (M, ρ, σ) is called an equivalence bimodule *from $\mathcal{G}$ to $\mathcal{H}$.* ■



A symplectic groupoid $(\mathcal{G} = (G_1 \rightrightarrows G_0), \omega_{G_1})$ is said to be Morita equivalent to another symplectic groupoid $(\mathcal{H} = (H_1 \rightrightarrows H_0), \omega_{H_1})$ when there is an equivalence bimodule (M, ρ, σ) from $\mathcal{G}$ to $\mathcal{H}$ in which M is a symplectic manifold and the actions of $\mathcal{G}, \mathcal{H}$ on M are both symplectic.

Proposition 2.5.2. *Morita equivalence is an equivalence relation among Lie (symplectic) groupoids.*

Proof. Reflexivity: Let $\mathcal{G} = (G_1 \rightrightarrows G_0)$ be a Lie groupoid and s, t its source map and target map. Then a triplet (G_1, t, s) is an equivalence bimodule from $\mathcal{G}$ to itself. In fact, right and left $\mathcal{G}$-actions on G_1 is given by composition of arrows of $\mathcal{G}$.

Symmetry: If (M, ρ, σ) is an equivalence bimodule from a Lie groupoid $\mathcal{G}$ to another Lie groupoid $\mathcal{H}$, (M, σ, ρ) is an equivalence bimodule from $\mathcal{H}$ to $\mathcal{G}$. In fact, for a left $\mathcal{G}$-action on M denoted by $(g, x) \mapsto gx$, one can define a right $\mathcal{G}$-action on M which has the same momentum map by $(x, g) \mapsto g^{-1}x$.

Transitivity: Let $\mathcal{G}_i$ $(i = 1, 2, 3)$ be Lie groupoids and $(M_1, \rho_1, \sigma_1), (M_2, \rho_2, \sigma_2)$ equivalence bimodules from $\mathcal{G}_1$ to $\mathcal{G}_2$, from $\mathcal{G}_2$ to $\mathcal{G}_3$, respectively. We denote as $g_i \in \mathcal{G}_i$ $(i = 1, 2, 3)$ and $x_i \in M_i$ $(i = 1, 2)$. Then we can define a left $\mathcal{G}_2$-action on $M_1 {}_{\sigma_1}\!\times_{\rho_2} M_2$ by $(x_1, x_2) \mapsto (x_1 g_2^{-1}, g_2 x_2)$ and obtain an equivalence bimodule (M_3, ρ_3, σ_3) from $\mathcal{G}_1$ to $\mathcal{G}_3$ by

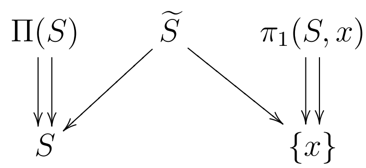
$$M_3 := \mathcal{G}_2 \backslash (M_1 {}_{\sigma_1}\!\times_{\rho_2} M_2),$$

$$\rho_3([x_1, x_2]) := \rho_1(x_1), \;\; \sigma_3([x_1, x_2]) := \sigma_2(x_2),$$

$$g_1 \cdot [x_1, x_2] := [g_1 x_1, x_2], \;\; [x_1, x_2] \cdot g_3 := [x_1, x_2 g_3].$$

□

Example 2.5.3. *Let S be a symplectic manifold and $x \in S$. Then a symplectic groupoid $\Pi(S) \rightrightarrows S$ (see Example 2.4.3) is Morita equivalent to a symplectic groupoid $\pi_1(S, x) \rightrightarrows \{x\}$, where an equivalence bimodule is universal covering space $\widetilde{S}$ of S.* ■



Example 2.5.4. *Let G_1, G_2 be Lie groups and $\mathfrak{g}_1, \mathfrak{g}_2$ their Lie algebras. Then symplectic groupoids $T^*G_1 \rightrightarrows \mathfrak{g}_1^*$ and $T^*G_2 \rightrightarrows \mathfrak{g}_2^*$ (see Example 2.4.4) are Morita equivalent if and only if G_1 and G_2 are isomorphic as Lie groups.* ■

Remark 2.5.5. *Morita equivalence classes of Lie groupoids are equivalent to the concept of* differentiable stacks *[BX11]. In particular, Morita equivalence classes of proper and étale Lie groupoids are equivalent to the concept of* orbifolds *[MP97].* ■

2.6 Mikami-Weinstein theorem

In this section, we review the proof of the Mikami-Weinstein theorem according to Xu [Xu91b]. The following is the statement of the Mikami-Weinstein theorem.

Theorem 2.6.1 ([MW88]). *Let $\mathcal{G} = (G_1 \rightrightarrows G_0)$ be a symplectic groupoid and M a symplectic, free and proper left $\mathcal{G}$-module with respect to a momentum map $\rho : M \to G_0$. Assume that $\xi \in \rho(M)$ is a regular value of ρ. Then $\mathcal{G}_\xi \backslash \rho^{-1}(\xi)$ is a symplectic manifold. Moreover, if ρ is submertive, a family of symplectic manifolds $\{\mathcal{G}_\xi \backslash \rho^{-1}(\xi)\}_{\xi \in \rho(M)}$ is precisely the symplectic foliation of the Poisson manifold $\mathcal{G} \backslash M$.* □

Example 2.6.2. *Consider the case of $\mathcal{G} = (G \times \mathfrak{g}^* \rightrightarrows \mathfrak{g}^*)$. For any $\xi \in \mathfrak{g}^*$, $\mathcal{G}_\xi \simeq \{g \in G \mid \mathrm{Ad}^*_g \xi = \xi\}$. Thus by the correspondence in Example 2.4.6, we can see that the Marsden-Weinstein-Meyer theorem is a special case of the Mikami-Weinstein theorem.* ■

Let $\mathcal{G} = (G_1 \rightrightarrows G_0)$ and $\mathcal{H} = (H_1 \rightrightarrows H_0)$ be Morita equivalent Lie groupoids and (M, ρ, σ) an equivalence bimodule. We say that $\xi \in G_0$ and $\zeta \in H_0$ are *related* when $\rho^{-1}(\xi) \cap \sigma^{-1}(\zeta) \neq \phi$ holds. Then the definition of Morita equivalence implies the following.

Lemma 2.6.3. *Let $\xi_0 \in G_0$ and $\zeta_0 \in H_0$ be related. then the followings hold.*

- *$\zeta \in H_0$ is related to ξ_0 if and only if ζ lies in the same groupoid orbit as ζ_0.*
- *$\xi \in H_0$ is related to ζ_0 if and only if ξ lies in the same groupoid orbit as ξ_0.* □

Let $\mathcal{O}, \mathcal{O}'$ be groupoid orbits of $\mathcal{G}, \mathcal{H}$, respectively. We say that $\mathcal{O}$ and $\mathcal{O}'$ are related if for some $\xi \in \mathcal{O}, \zeta \in \mathcal{O}'$ (and therefore for any $\xi \in \mathcal{O}, \zeta \in \mathcal{O}'$), ξ and ζ are related. Hence we have a correspondence

$$\Big\{\text{groupoid orbit of } \mathcal{G}\Big\} \overset{1:1}{\longleftrightarrow} \Big\{\text{groupoid orbit of } \mathcal{H}\Big\}.$$

In particular, when $\mathcal{G} = (G_1 \rightrightarrows G_0)$ and $\mathcal{H} = (H_1 \rightrightarrows H_0)$ are symplectic groupoids, we obtain a correspondence

$$\Big\{\text{symplectic leaf of } G_0\Big\} \overset{1:1}{\longleftrightarrow} \Big\{\text{symplectic leaf of } H_0\Big\}.$$

Symplectic leaves $L \subset G_0, L' \subset H_0$ are said to be related if they are related as groupoid orbits.

Lemma 2.6.4. *Let $L \subset G_0, L' \subset H_0$ be related leaves. Then $\rho^{-1}(L) = \sigma^{-1}(L')$ holds.*

Proof. Take $x \in \rho^{-1}(L)$. Then since $\rho(x) \in L$ and L is related to L', there is $y \in M$ such that $\rho(x) = \rho(y)$ and $\rho(y) \in L'$. Because $\sigma(x)$ and $\rho(x)$, $\sigma(y)$ and $\rho(y)$ are related respectively, we can see that $\sigma(x)$ and $\sigma(y)$ are related, and thus $\sigma(x) \in L'$ holds. Similarly, we can prove $\sigma^{-1}(L') \subset \rho^{-1}(L)$. □

Proposition 2.6.5. *Let $\mathcal{G} = (G_1 \rightrightarrows G_0)$ be a symplectic groupoid, M a free and proper symplectic left $\mathcal{G}$-module whose momentum map ρ is submertive. Then for any symplectic left $\mathcal{G}$-module X with momentum map σ, $\mathcal{G}\backslash(M\,{}_\rho{\times}_\sigma X)$ is a symplectic manifold, where $\mathcal{G}$ acts on $\mathcal{G}\backslash(M\,{}_\rho{\times}_\sigma X)$ diagonally.*

Proof. $M\,{}_\rho{\times}_\sigma X$ is a smooth manifold since ρ is submertive. Moreover, we can see that the left $\mathcal{G}$-action on $M\,{}_\rho{\times}_\sigma X$ is free and proper, and thus $\mathcal{G}\backslash(M\,{}_\rho{\times}_\sigma X)$ is a smooth manifold.

Let $\mathcal{F}_\mathcal{G}$ be a foliation on $M\,{}_\rho{\times}_\sigma X$ induced by the $\mathcal{G}$-action and $\mathcal{F}_\perp$ another foliation on $M\,{}_\rho{\times}_\sigma X$ defined by $T\mathcal{F}_\perp = T(M\,{}_\rho{\times}_\sigma X)^{-\omega_M \oplus \omega_X}$, where ω_M and ω_X are symplectic forms on M and X, respectively. Then we can prove that $\mathcal{F}_\mathcal{G} = \mathcal{F}_\perp$ (we omit the details).

Moreover, $M\,{}_\rho{\times}_\sigma X$ is a coisotropic submanifold of a symplectic manifold $(M \times X, -\omega_M \oplus \omega_X)$. Hence the quotient

$$T(M\,{}_\rho{\times}_\sigma X)/T\mathcal{F}_\mathcal{G} = T(M\,{}_\rho{\times}_\sigma X)/T(M\,{}_\rho{\times}_\sigma X)^{-\omega_M \oplus \omega_X}$$

naturally admits a (linear) symplectic structure. □

Now we outline the proof of the Mikami-Weinstein theorem.

Proof of Theorem 2.6.1. We may assume that ρ is surjective.

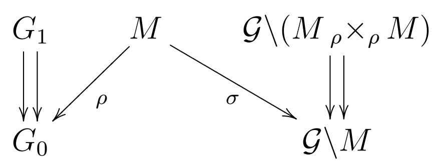
We obtain a Lie groupoid $\mathcal{G}\backslash(M\,{}_\rho\!\times_\rho M) \rightrightarrows \mathcal{G}\backslash M$ whose structure maps are

$$s([x,y]) = [x], \qquad t([x,y]) = [y],$$

$$u([x]) = [x,x], \qquad i([x,y]) = [y,x], \qquad m([x,y],[y,z]) = [x,z].$$

By Proposition 2.6.5, we can see that $\mathcal{G}\backslash(M\,{}_\rho\!\times_\rho M)$ admits a symplectic structure. Moreover the symplectic structure is multiplicative, thus $\mathcal{G}\backslash(M\,{}_\rho\!\times_\rho M) \rightrightarrows \mathcal{G}\backslash M$ is a symplectic groupoid and $\mathcal{G}\backslash M$ is a Poisson manifold.

We can also see that M is a symplectic right module on $\mathcal{G}\backslash(M\,{}_\rho\!\times_\rho M) \rightrightarrows \mathcal{G}\backslash M$, whose momentum map is the natural projection $\sigma : M \to \mathcal{G}\backslash M$ and right action is given by $x \cdot [x,y] = y$. For simplicity, we assume that ρ is submertive. Then the symplectic groupoid $\mathcal{G}\backslash(M\,{}_\rho\!\times_\rho M) \rightrightarrows \mathcal{G}\backslash M$ is Morita equivalent to $\mathcal{G} = (G_1 \rightrightarrows G_0)$ (see the diagram below).



Take $\xi \in G_0$ and let $L_\xi \subset G_0$ be the symplectic leaf through ξ. Then by Lemma 2.6.4, $\mathcal{G}\backslash\rho^{-1}(L_\xi)$ is a symplectic leaf of $\mathcal{G}\backslash M$. In addition, we obtain a diffeomorphism

$$\mathcal{G}_\xi\backslash\rho^{-1}(\xi) \simeq \mathcal{G}\backslash\rho^{-1}(L_\xi),$$

and thus $\mathcal{G}_\xi\backslash\rho^{-1}(\xi)$ naturally admits a symplectic structure and the family $\{\mathcal{G}_\xi\backslash\rho^{-1}(\xi)\}_{\xi\in G_0}$ is precisely the symplectic foliation of the Poisson manifold $\mathcal{G}\backslash M$.

In the case that ρ is not submertive, $\mathcal{G}\backslash(M\,{}_\rho\!\times_\rho M) \rightrightarrows \mathcal{G}\backslash M$ is no longer Morita equivalent to $\mathcal{G}$. However, we did not use submertiveness of momentum maps in the definition of related symplectic leaves, and thus $\mathcal{G}_\xi\backslash\rho^{-1}(\xi)$ admits a symplectic structure as long as ξ is a regular value of ρ. □

Chapter 3

Reduction of coKähler and 3-cosymplectic manifolds

3.1 CoKähler case

Let g be a Riemannian metric and (φ, ξ, η) an almost contact structure on M, i.e., a triplet of $\varphi \in \mathrm{End}(TM)$, a vector field ξ and a 1-form η, which satisfies $\varphi^2 = -\mathrm{id} + \eta \otimes \xi$, $\eta(\xi) = 1$. Then a quartets (g, φ, ξ, η) is called an almost contact metric structure if

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y)$$

holds. For almost contact structures and almost contact metric structures, refer to [YK85].

Lemma 3.1.1. *Let (g, φ, ξ, η) be an almost contact metric structure on M.*

(1) $\varphi\xi = 0$ holds.

(2) $g(\xi, X) = \eta(X)$ holds for any $X \in \mathfrak{X}(M)$.

Proof. (2) immediately follows from (1). We prove (1). Note that we have $\varphi^2\xi = -\xi + \xi = 0$. Hence

$$0 = \varphi^2(\varphi\xi) = -\varphi\xi + \eta(\varphi\xi)\xi$$

holds, and thus we have $\varphi\xi = \eta(\varphi\xi)\xi$. Suppose that $\varphi\xi \neq 0$. Then $\varphi\xi = \eta(\varphi\xi)\xi$ implies $\eta(\varphi\xi) \neq 0$. However, $\varphi\xi = \eta(\varphi\xi)\xi$ also implies

$$0 = \varphi^2\xi = \eta(\varphi\xi)\varphi\xi,$$

which contradicts $\varphi\xi \neq 0$ and $\eta(\varphi\xi) \neq 0$. Therefore $\varphi\xi = 0$ holds. □

Given an almost contact metric structure (g, φ, ξ, η), we obtain an almost cosymplectic structure (η, ω), where $\omega(X, Y) = g(X, \varphi Y)$.

Definition 3.1.2. *An almost contact metric structure* (g, φ, ξ, η) *is said to be an* almost coKähler structure *if the induced almost cosymplectic structure* (η, ω) *is a cosymplectic structure. An almost coKähler structure* (g, φ, ξ, η) *is said to be a* coKähler structure *if the almost contact structure* (φ, ξ, η) *is normal, i.e., the Nijenhuis tensor* N_φ *of* φ *satisfies* $N_\varphi = -2d\eta \otimes \xi$. ■

If we replace the cosymplectic condition $d\eta = d\omega = 0$ with $\omega = d\eta$ in the definition above, then (g, φ, ξ, η) is called a Sasakian structure.

Let $(M, g, \varphi, \xi, \eta)$ be a coKähler manifold and G a Lie group. Suppose that there is a Hamiltonian action of G on a cosymplectic manifold (M, η, ω), and let $\zeta \in \mathfrak{g}^*$ be a regular value of a momentum map $\mu : M \to \mathfrak{g}^*$. Moreover, we assume that the action preserves φ (and hence preserves the metric g) and ζ is central, i.e., $G_\zeta = G$ holds.

Lemma 3.1.3. *Let* $p \in \mu^{-1}(\zeta)$*. Then there is a subspace* $H_p \subset T_pM$ *and a decomposition*

$$T_pM = H_p \oplus \mathfrak{g}_p \oplus \varphi_p(\mathfrak{g}_p), \tag{3.1.1}$$

which is orthogonal with respect to the metric g*. Moreover,* H_p *is invariant under* φ_p*.*

Proof. For any $v \in T_p\mu^{-1}(\zeta)$ and $A_p^* \in \mathfrak{g}_p$, we have

$$g(v, \varphi_p A_p^*) = \omega(v, A_p^*) = -(d\mu^A)(v) = 0.$$

In addition, since φ_p is an isomorphism on $\mathrm{Ker}\eta_p$, we have $\dim \varphi_p(\mathfrak{g}_p) = \dim G$ and thus we obtain a decomposition

$$T_pM = T_p\mu^{-1}(\zeta) \oplus \varphi_p(\mathfrak{g}_p).$$

We define H_p as the g-orthogonal complement of $\mathfrak{g}_p$ in $T_p\mu^{-1}(\zeta)$. Then we obtain the decomposition (3.1.1). Note that $\xi_p \in H_p$ holds since $g(\xi_p, A_p^*) = \eta(A_p^*) = 0$. Therefore the decomposition (3.1.1) implies $\varphi_p(H_p) = H_p$ since $\varphi_p^2(v) = -v$ holds for $v \in \mathrm{Ker}\eta_p$. □

Lemma 3.1.4. H_p *is invariant under the action of* G*, namely,* $(L_h)_*(H_p) = H_{hp}$ *holds for any* $h \in G$ *and* $p \in \mu^{-1}(\zeta)$*.*

Proof. For any $A_p^* \in \mathfrak{g}_p$, we have

$$(L_h)_*(\varphi_p A_p^*) = \varphi_{hp}(L_h)_* A_p^* = \varphi_{hp}(\mathrm{Ad}_h A)_{hp}^*$$

and thus $(L_h)_*(\varphi_h(\mathfrak{g}_p)) = \varphi_{hp}(\mathfrak{g}_{hp})$. Hence the decomposition (3.1.1) implies $(L_h)_*v \in T_{hp}\mu^{-1}(\zeta)$ for $v \in H_p$. Moreover, $(L_h)_*v$ is orthogonal to $\mathfrak{g}_{hp}$ since the action of G preserves the metric g, therefore we obtain $(L_h)_*(H_p) = H_{hp}$. □

Now we obtain the following reduction theorem.

Theorem 3.1.5. *Let* $(M, g, \varphi, \xi, \eta)$ *be a coKähler manifold with the underlying cosymplectic structure* (η, ω)*. Suppose that there is a free and proper Hamiltonian action of a Lie group* G *on* (M, η, ω) *which preserves* φ*. Let* $\mu : M \to \mathfrak{g}^*$ *be a momentum map and* $\zeta \in \mathfrak{g}^*$ *a central and regular value of* μ*. Then* $M^\zeta := \mu^{-1}(\zeta)/G$ *admits a coKähler structure* $(g^\zeta, \varphi^\zeta, \xi^\zeta, \eta^\zeta)$*. Moreover, the underlying cosymplectic manifold of* $(M^\zeta, g^\zeta, \varphi^\zeta, \xi^\zeta, \eta^\zeta)$ *is the cosymplectic quotient* $(M^\zeta, \eta^\zeta, \omega^\zeta)$*.*

Proof. For any $p \in \mu^{-1}(\zeta)$, the map $(d\pi)_p|_{H_p} : H_p \to T_{\pi(p)}M^\zeta$ is an isomorphism. We define a Riemannian metric g^ζ on M^ζ and $\varphi^\zeta \in \mathrm{End}(TM^\zeta)$ as pushforwards by $(d\pi)_p|_{H_p}$, i.e.,

$$g^\zeta_{\pi(p)}(X_{\pi(p)}, Y_{\pi(p)}) = g_p(\widetilde{X}_p, \widetilde{Y}_p),$$

$$\varphi^\zeta_{\pi(p)}(X_{\pi(p)}) = (d\pi)_p(\varphi_p(\widetilde{X}_p)),$$

where $\widetilde{X}_p, \widetilde{Y}_p \in H_p$ are vectors which satisfies

$$(d\pi)_p(\widetilde{X}_p) = X_{\pi(p)},\ (d\pi)_p(\widetilde{Y}_p) = Y_{\pi(p)}.$$

We check that $g^\zeta_{\pi(p)}, \varphi^\zeta_{\pi(p)}$ are independent of the choice of p. For $h \in G$, we have $(d\pi)_{hp}(L_h)_*\widetilde{X}_p = (d\pi)_p\widetilde{X}_p$, and thus Lemma 3.1.4 implies $\widetilde{X}_{hp} = (L_h)_*\widetilde{X}_p$. Hence we obtain

$$g_p(\widetilde{X}_p, \widetilde{Y}_p) = g_{hp}((L_h)_*\widetilde{X}_p, (L_h)_*\widetilde{Y}_p) = g_{hp}(\widetilde{X}_{hp}, \widetilde{Y}_{hp}),$$

$$\begin{aligned}(d\pi)_p(\varphi_p(\widetilde{X}_p)) &= (d\pi)_{hp}(L_h)_*(\varphi_p(\widetilde{X}_p)) \\ &= (d\pi)_{hp}(\varphi_{hp}((L_h)_*\widetilde{X}_p)) \\ &= (d\pi)_{hp}(\varphi_{hp}(\widetilde{X}_{hp})),\end{aligned}$$

therefore $g^\zeta_{\pi(p)}, \varphi^\zeta_{\pi(p)}$ are well-defined (the metric g^ζ is called the quotient metric on M^ζ).

Let $(M^\zeta, \eta^\zeta, \omega^\zeta)$ be the cosymplectic quotient and ξ^ζ the Reeb vector field. Then $\eta^\zeta(\xi^\zeta) = 1$ holds. Moreover, since φ_p preserves H_p, we have $\widetilde{\varphi^\zeta(X)} = \varphi(\widetilde{X})$, and thus we obtain

$$\begin{aligned}\varphi^\zeta(\varphi^\zeta(X)) &= d\pi(\varphi\widetilde{\varphi^\zeta(X)}) \\ &= d\pi(\varphi\varphi(\widetilde{X})) \\ &= d\pi(-\widetilde{X} + \eta(\widetilde{X})\xi) \\ &= -X + \eta^\zeta(X)\xi^\zeta,\end{aligned}$$

so $(\varphi^\zeta, \xi^\zeta, \eta^\zeta)$ is an almost contact structure on M^ζ.

We can easily check the compatibility of g^ζ, φ^ζ with η^ζ, ω^ζ, i.e.,

$$g^\zeta(\varphi^\zeta X, \varphi^\zeta Y) = g^\zeta(X, Y) - \eta^\zeta(X)\eta^\zeta(Y),$$

$$\omega^\zeta(X, Y) = g^\zeta(X, \varphi^\zeta Y).$$

Lastly, we prove that $(g^\zeta, \varphi^\zeta, \xi^\zeta, \eta^\zeta)$ is a coKähler structure. It was proved in [Bla67] that an almost contact metric structure (g, φ, ξ, η) is coKähler if and only if $\nabla\varphi = 0$, where ∇ is the Levi-Civita connection of the metric g. Let ∇ be the Levi-Civita connection of the metric on M. Using a general property of quotient metrics, we can compute the Levi-Civita connection ∇^ζ of g^ζ by

$$\nabla^\zeta_X Y = d\pi(\mathrm{pr}_H(\nabla_{\widetilde{X}}\widetilde{Y})),$$

where $\widetilde{X}, \widetilde{Y}$ are extended to a neighborhood of $\mu^{-1}(\zeta)$ and $\mathrm{pr}_H : TM \to H$ denotes the orthogonal projection. Then since $\nabla\varphi = 0$ we have

$$\begin{aligned}\widetilde{\nabla^\zeta_X \varphi^\zeta Y} &= \mathrm{pr}_H(\nabla_{\widetilde{X}}\varphi\widetilde{Y}) = \mathrm{pr}_H(\varphi\nabla_{\widetilde{X}}\widetilde{Y}) \\ &= \varphi\mathrm{pr}_H(\nabla_{\widetilde{X}}\widetilde{Y}) = \varphi(\widetilde{\nabla^\zeta_X Y}) \\ &= \widetilde{\varphi^\zeta(\nabla^\zeta_X Y)}.\end{aligned}$$

Hence $\nabla^\zeta_X \varphi^\zeta(Y) = \varphi^\zeta(\nabla^\zeta_X Y)$ and thus $\nabla^\zeta\varphi^\zeta = 0$. □

3.2 3-cosymplectic case

In this section, we prove a reduction theorem for 3-cosymplectic manifolds. First we recall the definition of 3-cosymplectic structures.

Definition 3.2.1. *A* 3-cosymplectic structure *on a manifold M is a quartet $(g, (\varphi_i, \xi_i, \eta_i)_{i=1,2,3})$ of a Riemannian metric g and three almost contact structures $(\varphi_i, \xi_i, \eta_i)_{i=1,2,3}$ on M such that each $(g, \varphi_i, \xi_i, \eta_i)$ is coKähler and*

$$\varphi_\gamma = \varphi_\alpha\varphi_\beta - \eta_\beta \otimes \xi_\alpha = -\varphi_\beta\varphi_\alpha + \eta_\alpha \otimes \xi_\beta,$$

$$\xi_\gamma = \varphi_\alpha\xi_\beta = -\varphi_\beta\xi_\alpha,$$

$$\eta_\gamma = \varphi^*_\beta\eta_\alpha = -\varphi^*_\alpha\eta_\beta$$

holds for any even permutation (α, β, γ) of $\{1, 2, 3\}$. ■

Remark 3.2.2. *This notion should be called "3-coKähler structures", but since the name "3-cosymplectic" has become established, we will follow it here as well. If $(g, (\varphi_i, \xi_i, \eta_i)_{i=1,2,3})$ satisfies all conditions of Definition 3.2.1 except the normality of each $(\varphi_i, \xi_i, \eta_i)$, the normality of them automatically follows (see [FIP04]).* ■

Lemma 3.2.3. *Let $(M, g, (\varphi_i, \xi_i, \eta_i)_{i=1,2,3})$ be a 3-cosymplectic manifold and ω_2 a 2-form defined by $\omega_2(X, Y) = g(X, \varphi_2 Y)$. Then $\omega_2(X, \xi_1) = -\eta_3(X)$ holds.*

Proof. Since $\eta_\alpha(\xi_\gamma) = \eta_\alpha(\varphi_\alpha\xi_\beta) = 0$, we obtain

$$\begin{aligned}\omega_2(\varphi_3 X, Y) &= g(\varphi_3 X, \varphi_2 Y) \\ &= g(\varphi_3\varphi_3 X, \varphi_3\varphi_2 Y) + \eta_3(\varphi_3 X)\eta_3(\varphi_2 Y) \\ &= g(-X + \eta_3(X)\xi_3, -\varphi_2\varphi_3 Y + \eta_3(Y)\xi_2 + \eta_2(Y)\xi_3) \\ &= g(X, \varphi_2\varphi_3 Y) - \eta_3(Y)\eta_2(X) - \eta_2(Y)\eta_3(X) \\ &\quad - \eta_3(X)\eta_3(\varphi_2\varphi_3 Y) + \eta_3(X)\eta_3(Y)\eta_3(\xi_2) + \eta_3(X)\eta_2(Y)\eta_3(\xi_3) \\ &= g(X, \varphi_2\varphi_3 Y) - \eta_2(X)\eta_3(Y) - \eta_3(X)\eta_3(-\varphi_3\varphi_2 Y + \eta_3(Y)\xi_2 + \eta_2(Y)\xi_3) \\ &= g(X, \varphi_2\varphi_3 Y) - \eta_2(X)\eta_3(Y) - \eta_3(X)\eta_2(Y) \\ &= \omega_2(X, \varphi_3 Y) - \eta_2(X)\eta_3(Y) - \eta_3(X)\eta_2(Y).\end{aligned}$$

Therefore we have

$$\begin{aligned}\omega_2(X,\xi_1) &= -\omega_2(X,\varphi_3\xi_2)\\ &= -\omega_2(\varphi_3 X,\xi_2) - \eta_2(X)\eta_3(\xi_2) - \eta_3(X)\eta_2(\xi_2)\\ &= -\eta_3(X).\end{aligned}$$

□

Let G be a Lie group acting on a 3-cosymplectic manifold $(M, g, (\varphi_i, \xi_i, \eta_i)_{i=1,2,3})$. Suppose that the action is Hamiltonian with respect to all three cosymplectic structures $(\eta_i, \omega_i)_{i=1,2,3}$, and let $\mu_i : M \to \mathfrak{g}^*$ $(i = 1, 2, 3)$ be momentum maps.

Lemma 3.2.4. *Let $\zeta_2, \zeta_3 \in \mathfrak{g}^*$ be regular values of μ_2, μ_3, respectively. Suppose that submanifolds $\mu_2^{-1}(\zeta_2)$ and $\mu_3^{-1}(\zeta_3)$ intersect transversally. Then $N := \mu_2^{-1}(\zeta_2) \cap \mu_3^{-1}(\zeta_3)$ is an almost contact submanifold of $(M, \varphi_1, \xi_1, \eta_1)$.*

Proof. Using Lemma 3.2.3, we have

$$d\mu_2^A(\xi_1) = \omega_2(A^*, \xi_1) = -\eta_3(A^*) = 0,$$

and similarly $d\mu_3^A(\xi_1) = 0$ holds, thus $\xi_1 \in TN$. Next, we check that $\varphi_1(TN) \subset TN$. Suppose that $d\mu_3(X) = 0$. Then we obtain

$$\begin{aligned}0 = d\mu_3^A(X) &= \omega_3(A^*, X) = g(A^*, \varphi_3 X)\\ &= g(A^*, -\varphi_2\varphi_1 X + \eta_1(X)\xi_2)\\ &= -g(A^*, \varphi_2\varphi_1 X) + \eta_1(X)\eta_2(A^*)\\ &= -\omega_2(A^*, \varphi_1 X) = -d\mu_2^A(\varphi_1 X).\end{aligned}$$

Similarly, $d\mu_2(X) = 0$ implies $d\mu_3^A(\varphi_1 X) = 0$, hence $\varphi_1 X \in TN$ holds for any $X \in TN$. □

We define a 3-cosymplectic momentum map $\mu : M \to \mathfrak{g}^* \otimes \mathrm{Im}\mathbb{H}$ by

$$\mu = \mu_1 i + \mu_2 j + \mu_3 k,$$

where i, j, k are generators of $\mathrm{Im}\mathbb{H}$. Let G act on $\mathfrak{g}^* \otimes \mathrm{Im}\mathbb{H}$ by the tensor representation of the coadjoint action and the trivial action on $\mathrm{Im}\mathbb{H}$. Then for any $\zeta = \zeta_1 i + \zeta_2 j + \zeta_3 k \in \mathfrak{g}^* \otimes \mathrm{Im}\mathbb{H}$, the G_ζ-action preserves $\mu^{-1}(\zeta) = \mu_1^{-1}(\zeta_1) \cap \mu_2^{-1}(\zeta_2) \cap \mu_3^{-1}(\zeta_3)$.

Theorem 3.2.5. *Let $(M, g, (\varphi_i, \xi_i, \eta_i)_{i=1,2,3})$ be a 3-cosymplectic manifold with underlying cosymplectic structures $(\eta_i, \omega_i)_{i=1,2,3}$. Suppose that there is a free and proper action of a Lie group G on M which is Hamiltonian with respect to all three cosymplectic structures $(\eta_i, \omega_i)_{i=1,2,3}$ and preserves $(\varphi_i)_{i=1,2,3}$. Let $\mu : M \to \mathfrak{g}^* \otimes \mathrm{Im}\mathbb{H}$ be a 3-cosymplectic momentum map and $\zeta \in \mathfrak{g}^* \otimes \mathrm{Im}\mathbb{H}$ a central and regular value of μ. Then $M^\zeta := \mu^{-1}(\zeta)/G$ inherits the 3-cosymplectic structure of M.*

Proof. Lemma 3.2.4 implies that $(\varphi_1|_N, \xi_1|_N, \eta_1|_N)$ is an almost contact structure on N. Therefore from a result in [Lud70], we can see that $(N, g|_N, \varphi_1|_N, \xi_1|_N, \eta_1|_N)$ is a coKähler manifold.

The action of G on N preserves the coKähler structure, and $\mu_1|_N$ is a momentum map for this action. So Theorem 3.1.5 implies that there is a coKähler structure $(g_1^\zeta, \varphi_1^\zeta, \xi_1^\zeta, \eta_1^\zeta)$ on $M^\zeta = (N \cap \mu_1^{-1}(\zeta_1))/G$. Similarly, we obtain two more coKähler structures $(g_2^\zeta, \varphi_2^\zeta, \xi_2^\zeta, \eta_2^\zeta)$ and $(g_3^\zeta, \varphi_3^\zeta, \xi_3^\zeta, \eta_3^\zeta)$ on M^ζ.

Each Riemannian metric g_i^ζ $(i = 1, 2, 3)$ coincides with the quotient metric of the principal bundle $\mu^{-1}(\zeta) \to M^\zeta$, so three Riemannian metrics $g_1^\zeta, g_2^\zeta, g_3^\zeta$ are the same, and thus the three coKähler structures constitutes a 3-cosymplectic structure on M^ζ. □

3.3 Cylinder constructions

When M is endowed with a geometric structure we are studying, it induces a geometric structure on the cylinder $C(M) := M \times \mathbb{R}$ as shown in the table below (in the case that M is hyperKähler, we use $C^3(M) := M \times \mathbb{R}^3$ instead of the cylinder). In this section, we prove that reduction procedures are compatible with these constructions.

Structure on the base	Induced structure on the cylinder
Kähler	CoKähler
CoKähler	Kähler
HyperKähler	3-cosymplectic
3-cosymplectic	HyperKähler

Let (M, h, J) be a Kähler manifold (h denotes the Riemannian metric and J denotes the complex structure). Then $C(M) := M \times \mathbb{R}$ admits a natural coKähler structure (g, φ, ξ, η) defined by

$$g = h + dt^2, \; \varphi\Big(X, f\frac{\partial}{\partial t}\Big) = (JX, 0), \; \xi = \frac{\partial}{\partial t}, \; \eta = dt,$$

where t is the coordinate of $\mathbb{R}$ and $f \in C^\infty(C(M))$.

Example 3.3.1 ([FM74])**.** *Let f be a Hermitian isometry on (M, h, J). Define an action of $\mathbb{Z}$ on $C(M)$ by $k \cdot (p, t) := (f^k(p), t + k)$. This action is free and properly discontinuous, so $C(M)/\mathbb{Z}$ is a smooth manifold. Then $C(M)/\mathbb{Z}$ inherits a coKähler structure. This is a coKähler quotient for a trivial momentum map on $C(M)$. $C(M)/\mathbb{Z}$ is diffeomorphic to the mapping torus of M with respect to f, and we will discuss the coKähler structure on it in detail later.* ■

Assume that there is a Hamiltonian action of G on M preserving J, and let $\mu : M \to \mathfrak{g}^*$ be a momentum map. We define an action of G on $C(M)$ by $g \cdot (p, t) := (gp, t)$. Then this action preserves the coKähler structure on $C(M)$, and is a Hamiltonian action whose momentum map is $\widetilde{\mu} = \mu \circ \mathrm{pr}_M$.

Proposition 3.3.2. *When a Kähler quotient $\mu^{-1}(\zeta)/G$ is defined for $\zeta \in \mathfrak{g}^*$, a coKähler quotient $\widetilde{\mu}^{-1}(\zeta)/G$ is also defined. Moreover, $C(\mu^{-1}(\zeta)/G)$ and $\widetilde{\mu}^{-1}(\zeta)/G$ are equivalent as coKähler manifolds.*

Proof. Clearly $C(\mu^{-1}(\zeta)/G)$ and $\widetilde{\mu}^{-1}(\zeta)/G$ are diffeomorphic, and the diagram

$$\begin{array}{ccc} \widetilde{\mu}^{-1}(\zeta) & \xrightarrow{\ \widetilde{\pi}\ } & C(\mu^{-1}(\zeta)/G) \\ {\scriptstyle \mathrm{pr}}\downarrow & & \downarrow{\scriptstyle \mathrm{pr}} \\ \mu^{-1}(\zeta) & \xrightarrow[\ \pi\ ]{} & \mu^{-1}(\zeta)/G \end{array}$$

commutes, where pr denotes natural projections and $\widetilde{\pi}$ is the projection of coKähler reduction. We orthogonally decompose $T_p\mu^{-1}(\zeta)$ and $T_{(p,t)}\widetilde{\mu}^{-1}(\zeta)$ as

$$T_p\mu^{-1}(\zeta) = H_p \oplus \mathfrak{g}_p,$$

$$T_{(p,t)}\widetilde{\mu}^{-1}(\zeta) = H'_{(p,t)} \oplus \mathfrak{g}_{(p,t)},$$

respectively. Then we can easily check that $(\mathrm{pr})_*(\mathfrak{g}_{(p,t)}) = \mathfrak{g}_p$ and thus obtain the following commutative diagram.

$$\begin{array}{ccc} H'_{(p,t)} & \xrightarrow[\ \simeq\ ]{\ d\widetilde{\pi}\ } & T_{\widetilde{\pi}(p,t)}C(M)^\zeta \\ {\scriptstyle (\mathrm{pr})_*}\downarrow & & \downarrow{\scriptstyle (\mathrm{pr})_*} \\ H_p & \xrightarrow[\ d\pi\ ]{\ \simeq\ } & T_{\pi(p)}M^\zeta \end{array}$$

We denote the lift of $X \in T_{\pi(p)}M^\zeta$ and $(X, f\frac{\partial}{\partial r}) \in T_{\widetilde{\pi}(p,t)}C(M)^\zeta$ as $\widetilde{X}, \widetilde{(X, f\frac{\partial}{\partial r})}$, respectively. Let r be the coordinate of $\mathbb{R}$ in $C(\mu^{-1}(\zeta)/G)$. Since $d\pi(\mathrm{pr})_*\widetilde{\frac{\partial}{\partial r}} = (\mathrm{pr})_*\frac{\partial}{\partial r} = 0$, we get $(\mathrm{pr})_*\widetilde{\frac{\partial}{\partial r}} = 0$. Hence $\widetilde{\frac{\partial}{\partial r}} = a\xi$ holds for some $a \in \mathbb{R}^\times$. We normalize the coordinate r to satisfy $\widetilde{\frac{\partial}{\partial r}} = \xi$. Then we obtain $\xi^\zeta = \frac{\partial}{\partial r}$ and $\eta^\zeta = dr$.

Since $\widetilde{\frac{\partial}{\partial r}} = \xi$,

$$\varphi\widetilde{\Big(X, f\frac{\partial}{\partial r}\Big)} = \varphi\Big(\widetilde{X}, f\widetilde{\frac{\partial}{\partial r}}\Big) = (J\widetilde{X}, 0)$$

holds. Hence from the diagram above, we obtain

$$\varphi^\zeta\Big(X, f\frac{\partial}{\partial r}\Big) = d\widetilde{\pi}\varphi\widetilde{\Big(X, f\frac{\partial}{\partial r}\Big)} = (d\pi(J\widetilde{X}), 0) = (J^\zeta X, 0).$$

We can also see that $g^\zeta = h^\zeta + dr^2$, therefore the coKähler structure $(g^\zeta, \varphi^\zeta, \xi^\zeta, \eta^\zeta)$ coincides with one obtained by the cylinder construction $C(\mu^{-1}(\zeta)/G) = (\mu^{-1}(\zeta)/G) \times \mathbb{R}$. $\square$

Conversely, for a given coKähler manifold $(M, g, \varphi, \xi, \eta)$, we can define a Kähler structure (h, J) on the cylinder $C(M)$ by

$$h = g + dt^2,\ J\Big(X, f\frac{\partial}{\partial t}\Big) = \Big(\varphi X - f\xi, \eta(X)\frac{\partial}{\partial t}\Big).$$

Assume that there is a Hamiltonian action of G on M preserving the coKähler structure, and let $\mu : M \to \mathfrak{g}^*$ be a momentum map. We define an action of G on $C(M)$ in the same way as in Proposition 3.3.2. Then the action preserves the Kähler structure on $C(M)$ and $\widetilde{\mu} = \mu \circ \mathrm{pr}_M$ is a momentum map. We obtain the following.

Proposition 3.3.3. *When a coKähler quotient $\mu^{-1}(\zeta)/G$ is defined for $\zeta \in \mathfrak{g}^*$, a Kähler quotient $\widetilde{\mu}^{-1}(\zeta)/G$ is also defined. Moreover, $C(\mu^{-1}(\zeta)/G)$ and $\widetilde{\mu}^{-1}(\zeta)/G$ are equivalent as Kähler manifolds.*

Proof. We define $H_p \subset T_p\mu^{-1}(\zeta)$ and $H'_{(p,t)} \subset T_{(p,t)}\widetilde{\mu}^{-1}(\zeta)$ as in the proof of Proposition 3.3.2, and obtain the same commutative diagrams. Then we obtain

$$\begin{aligned} J^\zeta\Big(X, f\frac{\partial}{\partial r}\Big) &= d\widetilde{\pi} J\widetilde{\Big(X, f\frac{\partial}{\partial r}\Big)} = d\widetilde{\pi} J\Big(\widetilde{X}, f\frac{\partial}{\partial t}\Big) \\ &= d\widetilde{\pi}\Big(\varphi\widetilde{X} - f\xi, \eta(\widetilde{X})\frac{\partial}{\partial t}\Big) \\ &= \Big(d\pi(\varphi\widetilde{X} - f\xi), \eta(\widetilde{X})\frac{\partial}{\partial r}\Big) \\ &= \Big(\varphi^\zeta X - f\xi^\zeta, \eta^\zeta(X)\frac{\partial}{\partial r}\Big). \end{aligned}$$

We can also see that $h^\zeta = g^\zeta + dr^2$, therefore the Kähler structure (h^ζ, J^ζ) coincides with one obtained by the cylinder construction $C(\mu^{-1}(\zeta)/G) = (\mu^{-1}(\zeta)/G) \times \mathbb{R}$. □

Next we see the relationship between hyperKähler reduction and 3-cosymplectic reduction. Let (M, h, J_1, J_2, J_3) be a hyperKähler manifold. Then $C^3(M) := M \times \mathbb{R}^3$ admits a natural 3-cosymplectic structure $(g, (\varphi_i, \xi_i, \eta_i)_{i=1,2,3})$ defined by

$$g = h + \sum_{i=1}^{3} dt_i^2,\ \ \xi_i = \frac{\partial}{\partial t_i},\ \ \eta_i = dt_i,$$

$$\begin{aligned} \varphi_1\Big(X, f_1\frac{\partial}{\partial t_1}, f_2\frac{\partial}{\partial t_2}, f_3\frac{\partial}{\partial t_3}\Big) &= \Big(J_1X, 0, -f_3\frac{\partial}{\partial t_2}, f_2\frac{\partial}{\partial t_3}\Big), \\ \varphi_2\Big(X, f_1\frac{\partial}{\partial t_1}, f_2\frac{\partial}{\partial t_2}, f_3\frac{\partial}{\partial t_3}\Big) &= \Big(J_2X, f_3\frac{\partial}{\partial t_1}, 0, -f_1\frac{\partial}{\partial t_3}\Big), \\ \varphi_3\Big(X, f_1\frac{\partial}{\partial t_1}, f_2\frac{\partial}{\partial t_2}, f_3\frac{\partial}{\partial t_3}\Big) &= \Big(J_3X, -f_2\frac{\partial}{\partial t_1}, f_1\frac{\partial}{\partial t_2}, 0\Big), \end{aligned}$$

where (t_1, t_2, t_3) is the coordinate of $\mathbb{R}^3$ and $f_i \in C^\infty(C^3(M))$.

Example 3.3.4 ([CMDNY13b]). *Let f be a hyperKähler isometry on (M, h, J_1, J_2, J_3). Define an action of $\mathbb{Z}^3$ on $C^3(M)$ by*

$$(k_1, k_2, k_3) \cdot (p, t_1, t_2, t_3) := (f^{k_1+k_2+k_3}(p), t_1 + k_1, t_2 + k_2, t_3 + k_3).$$

This action is free and properly discontinuous, so $C^3(M)/\mathbb{Z}^3$ is a smooth manifold. Then $C^3(M)/\mathbb{Z}^3$ inherits a 3-cosymplectic structure. This is a 3-cosymplectic quotient for a trivial momentum map on $C^3(M)$. ■

Assume that there is an action of G on a hyperKähler manifold (M, h, J_1, J_2, J_3) which is Hamiltonian with respect to three symplectic structures and preserves h, J_1, J_2, J_3. Let $\mu : M \to \mathfrak{g}^* \otimes \mathrm{Im}\mathbb{H}$ be a hyperKähler momentum map. We define an action of G on $C^3(M)$ by $g \cdot (p, t_1, t_2, t_3) := (gp, t_1, t_2, t_3)$. Then this action preserves the 3-cosymplectic structure on $C^3(M)$, and $\widetilde{\mu} = \mu \circ \mathrm{pr}_M$ is a 3-cosymplectic momentum map.

Proposition 3.3.5. *When a hyperKähler quotient $\mu^{-1}(\zeta)/G$ is defined for $\zeta = \zeta_1 i + \zeta_2 j + \zeta_3 k \in \mathfrak{g}^* \otimes \mathrm{Im}\mathbb{H}$, a 3-cosymplectic quotient $\widetilde{\mu}^{-1}(\zeta)/G$ is also defined. Moreover, $C^3(\mu^{-1}(\zeta)/G)$ and $\widetilde{\mu}^{-1}(\zeta)/G$ are equivalent as 3-cosymplectic manifolds.*

Proof. We define $N := \mu_2^{-1}(\zeta_2) \cap \mu_3^{-1}(\zeta_3)$. Then clearly $\widetilde{\mu_2}^{-1}(\zeta_2) \cap \widetilde{\mu_3}^{-1}(\zeta_3)$ is diffeomorphic to $C^3(N)$. N is endowed with a Kähler structure $(h|_N, J_1|_N)$ (see [HKLR87]), and the coKähler structure $(g|_{C^3(N)}, \varphi_1|_{C^3(N)}, \xi_1|_{C^3(N)}, \eta_1|_{C^3(N)})$ on $C^3(N)$ is obtained by applying cylinder constructions to $(N, h|_N, J_1|_N)$ three times. Moreover, by Proposition 3.3.2 and Proposition 3.3.3,

$$C(C(C(N)))^{\zeta_1} \simeq C(C(C(N))^{\zeta_1}) \simeq C(C(C(N)^{\zeta_1})) \simeq C(C(C(N^{\zeta_1})))$$

holds as coKähler manifolds. Therefore the coKähler structure $(g^\zeta, \varphi_1^\zeta, \xi_1^\zeta, \eta_1^\zeta)$ on $(C^3(M))^\zeta$ coincides with one of three coKähler structures on $C^3(M^\zeta)$ obtained by (h^ζ, J_1^ζ).

Repeating the same argument, we can see that the reduced 3-cosymplectic structure $(g^\zeta, (\varphi_i^\zeta, \xi_i^\zeta, \eta_i^\zeta)_{i=1,2,3})$ coincides with one obtained by the cylinder construction $C^3(\mu^{-1}(\zeta)/G) = (\mu^{-1}(\zeta)/G) \times \mathbb{R}^3$. □

Conversely, for a given 3-cosymplectic manifold $(M, g, (\varphi_i, \xi_i, \eta_i)_{i=1,2,3})$, we can define a hyperKähler structure (h, J_1, J_2, J_3) on the cylinder $C(M)$ by

$$h = g + dt^2, \ J_i\Big(X, f\frac{\partial}{\partial t}\Big) = \Big(\varphi_i X - f\xi_i, \eta_i(X)\frac{\partial}{\partial t}\Big).$$

Assume that there is an action of G on a 3-cosymplectic manifold $(M, g, (\varphi_i, \xi_i, \eta_i)_{i=1,2,3})$ which is Hamiltonian with respect to three cosymplectic structures and preserves three coKähler structures. Let $\mu : M \to \mathfrak{g}^* \otimes \mathrm{Im}\mathbb{H}$ be a 3-cosymplectic momentum map. We define an action of G on $C(M)$ in the same way as in Proposition 3.3.2. Then the action preserves the hyperKähler structure on $C(M)$ and $\widetilde{\mu} = \mu \circ \mathrm{pr}_M$ is a hyperKähler momentum map. We obtain the following.

Proposition 3.3.6. *When a 3-cosymplectic quotient $\mu^{-1}(\zeta)/G$ is defined for $\zeta = \zeta_1 i + \zeta_2 j + \zeta_3 k \in \mathfrak{g}^* \otimes \mathrm{Im}\mathbb{H}$, a hyperKähler quotient $\widetilde{\mu}^{-1}(\zeta)/G$ is also defined. Moreover, $C(\mu^{-1}(\zeta)/G)$ and $\widetilde{\mu}^{-1}(\zeta)/G$ are equivalent as hyperKähler manifolds.*

Proof. $N := \mu_2^{-1}(\zeta_2) \cap \mu_3^{-1}(\zeta_3)$ is endowed with a coKähler structure $(g|_N, \varphi_1|_N, \xi_1|_N, \eta_1|_N)$, and by Proposition 3.3.3

$$(C(M)^\zeta, h^\zeta, J_1^\zeta) \simeq C(N)^{\zeta_1} \simeq C(N^{\zeta_1}) \simeq \Big(C(M^\zeta), g^\zeta + dr^2, \varphi_1^\zeta + \eta_1^\zeta \otimes \frac{\partial}{\partial r} - dr \otimes \xi_1^\zeta\Big)$$

holds as Kähler manifolds.

Repeating the same argument, we can see that the hyperKähler structure $(h^\zeta, J_1^\zeta, J_2^\zeta, J_3^\zeta)$ coincides with one obtained by the cylinder construction $C(\mu^{-1}(\zeta)/G) = (\mu^{-1}(\zeta)/G) \times \mathbb{R}$. □

3.4 CoKähler reduction of mapping tori

In general, every coKähler manifold is locally the Riemannian product of a Kähler manifold with the real line (see [CMDNY13a], for example). Therefore it is important to find coKähler manifolds which are not the global product of a Kähler manifold with $\mathbb{R}$ or S^1. Such coKähler manifolds which are compact are obtained by the mapping torus procedure. For a manifold S and a diffeomorphism $f : S \to S$, we define the *mapping torus* S_f as follows:

$$S_f = (S \times [0,1])/\{(p,0) \sim (f(p),1) \mid p \in S\}.$$

Note that there is a fibration $S \hookrightarrow S_f \overset{\mathrm{pr}}{\twoheadrightarrow} S^1$. In the case that S is endowed with a Kähler structure (G, J) and $f : S \to S$ is a Hermitian isometry, S_f admits a coKähler structure. In fact, S_f is diffeomorphic to $C(S)/\mathbb{Z}$ in Example 3.3.1.

Let Ω be the symplectic form of the Kähler manifold (S, h, J). We extend J and pullback h, Ω to $S \times [0,1]$, and they descend to S_f since $f^*h = h$ and $f_*J = Jf_*$. We denote them $\widetilde{J}, \widetilde{h}, \widetilde{\Omega}$. Then we can write the coKähler structure (g, φ, η, ξ) on S_f as follows:

$$\varphi = \widetilde{J},\ \eta = \mathrm{pr}^* d\theta,\ \xi_{[(p,t)]} = \left.\frac{d}{ds}\right|_{s=0} [(p, t+s)],$$

$$g(X, Y) = \widetilde{h}(X, Y) + \eta(X)\eta(Y),$$

where θ is the coordinate of S^1 and $[(p,t)]$ denotes the equivalence class of (p,t) with respect to the quotient $C(S)/\mathbb{Z}$. The corresponding 2-form is given by $\omega := \widetilde{\Omega}$.

It is known that any closed coKähler manifold is in fact a Kähler mapping torus:

Theorem 3.4.1 (Li [Li08])**.** *A closed manifold M admits a coKähler structure if and only if there exists a Kähler manifold (S, h, J) and a Hermitian isometry f of (S, h, J) such that M is diffeomorphic to S_f.* □

Assume that there is a free and proper Hamiltonian action of a Lie group G on a Kähler manifold (S, h, J) preserving the Kähler structure. Moreover, we suppose that a Hermitian isometry $f : S \to S$ of (S, h, J) is equivariant with respect to the action of G. Then we can define an action of G on the mapping torus S_f by $g \cdot [(p,t)] = [(gp, t)]$.

Proposition 3.4.2. *Let $\mu : S \to \mathfrak{g}^*$ be a momentum map of the Hamiltonian action of G on S. Then the action of G on S_f is Hamiltonian if and only if $\mu(f(p)) = \mu(p)$ holds for some $p \in S$.*

Proof. Let $\widetilde{\mu} : S_f \to \mathfrak{g}^*$ be a cosymplectic momentum map. A vector field on S_f is locally has the form $X + a\frac{\partial}{\partial t}$, where t is the coordinate of $\mathbb{R}$ and $a \in C^\infty(S_f)$. Then we have

$$d\widetilde{\mu}^A\Big(X + a\frac{\partial}{\partial t}\Big) = \omega\Big(A^*, X + a\frac{\partial}{\partial t}\Big) = \Omega(A^*, X) = d\mu^A(X)$$

for any $A \in \mathfrak{g}$. Hence the map $\widetilde{\mu}$ locally has the form

$$\widetilde{\mu}[(p,t)] = \mu(p) + \zeta$$

for some $\zeta \in \mathfrak{g}^*$ (since both μ and $\widetilde{\mu}$ are equivariant, ζ must be central). Since $\widetilde{\mu}$ is globally defined, $\mu(f(p)) = \mu(p)$ holds for any $p \in S$. Conversely, if $\mu(f(p)) = \mu(p)$ holds for any $p \in S$, we obtain a cosymplectic momentum map $\widetilde{\mu}$ by $\widetilde{\mu}[(p,t)] = \mu(p) + \zeta$ for any central ζ. However, we have

$$\begin{aligned} d(\mu^A \circ f)(X) &= d\mu^A(f_* X) = \Omega(A^*, f_* X) \\ &= \Omega(f_*(f_*^{-1} A^*), f_* X) \\ &= \Omega(f_*^{-1} A^*, X) = \Omega(A^*, X) \\ &= d\mu^A(X) \end{aligned}$$

for any $A \in \mathfrak{g}$ and $X \in \mathfrak{X}(S)$, hence it is sufficient that $\mu(f(p)) = \mu(p)$ holds for some $p \in S$. Two momentum maps on S differs only by a constant, thus the condition $\mu(f(p)) = \mu(p)$ is independent of the choice of a momentum map μ. □

Remark 3.4.3. *In [Ito77] it was proved that if a compact Kähler manifold (S, h, J) has positive holomorphic sectional curvature, then any Hermitian isometry f of (S, h, J) has a fixed point. In this case any Hamiltonian action on S such that f is equivariant induces a Hamiltonian action on S_f.* ■

Suppose that an equivariant Hermitian isometry f satisfies the condition in Proposition 3.4.2. We define a cosymplectic momentum map $\widetilde{\mu} : S_f \to \mathfrak{g}^*$ by $\widetilde{\mu}[(p,t)] = \mu(p)$. Note that the action of G on S_f is free and proper, and preserves the coKähler structure.

Let $\zeta \in \mathfrak{g}^*$ be a regular value of the momentum map $\widetilde{\mu} : S_f \to \mathfrak{g}^*$. Let $(S_f)^\zeta$ be the coKähler quotient. In the case that S is compact, $(S_f)^\zeta$ is a closed coKähler manifold, and thus it is a mapping torus of some Kähler manifold from Theorem 3.4.1. In the following we observe that it can be obtained by the Kähler quotient $S^\zeta = \mu^{-1}(\zeta)/G$ for the same value ζ. From the condition $\mu(f(p)) = \mu(p)$, f preserves $\mu^{-1}(\zeta)$ and it descends to a map $f^\zeta : S^\zeta \to S^\zeta$ since f is equivariant.

Lemma 3.4.4. *f^ζ is a Hermitian isometry of $(S^\zeta, h^\zeta, J^\zeta)$.*

Proof. f^ζ is a diffeomorphism since f^{-1} also descends to S^ζ and is the inverse of f^ζ. We orthogonally decompose $T_p\mu^{-1}(\zeta)$ as

$$T_p\mu^{-1}(\zeta) = H_p \oplus \mathfrak{g}_p.$$

Then for any $v \in H_p$, we have

$$d\mu^A f_*(v) = d(\mu^A f)(v) = d\mu^A(v) = 0$$

from the proof of Proposition 3.4.2, and also obtain

$$h(f_*(v), A^*) = h(v, (f^{-1})_*(A^*)) = h(v, A^*) = 0,$$

and thus $f_*(v) \in H_{f(p)}$.

Let $\pi : \mu^{-1}(\zeta) \to S^\zeta$ be the projection of reduction. Since $f^\zeta \pi = \pi f$ and $f_*(H_p) = H_{f(p)}$, we obtain

$$(f^\zeta)_* = (d\pi|_{H_{f(p)}}) f_* (d\pi|_{H_p})^{-1},$$

so $(f^\zeta)^* h^\zeta = h^\zeta$ holds from $f^* h = h$ and the definition of h^ζ. Similarly we can check that $(f^\zeta)_* J^\zeta = J^\zeta (f^\zeta)_*$. □

Theorem 3.4.5. *$(S_f)^\zeta$ is equivariant to $(S^\zeta)_{f^\zeta}$ as coKähler manifolds.*

Proof. There is a diffeomorphism $\Phi : (S_f)^\zeta \to (S^\zeta)_{f^\zeta}$ defined by $\Phi(\widetilde{\pi}[(p,t)]) = [(\pi(p), t)]$, where $\widetilde{\pi} : \widetilde{\mu}^{-1}(\zeta) \to (S_f)^\zeta$ is the projection of reduction. Since $\widetilde{\pi}[(p,t)] = \widetilde{\pi}[(q,s)]$ is equivalent to $[(\pi(p),t)] = [(\pi(q),s)]$, the map Φ is well-defined and one-to-one. Then the diagram

$$\begin{array}{ccc} \mu^{-1}(\zeta) & \xrightarrow{\pi} & \mu^{-1}(\zeta)/G \\ {\scriptstyle i_t}\downarrow & & \downarrow{\scriptstyle i_t} \\ \widetilde{\mu}^{-1}(\zeta) & \xrightarrow[\widetilde{\pi}]{} & (S_f)^\zeta \end{array}$$

commutes, where i_t denotes natural inclusions to mapping tori $p \mapsto [(p,t)]$. We orthogonally decompose $T_p \mu^{-1}(\zeta)$ and $T_{[(p,t)]} \widetilde{\mu}^{-1}(\zeta)$ as

$$T_p \mu^{-1}(\zeta) = H_p \oplus \mathfrak{g}_p,$$

$$T_{[(p,t)]} \widetilde{\mu}^{-1}(\zeta) = H'_{[(p,t)]} \oplus \mathfrak{g}_{[(p,t)]},$$

respectively. Then we can easily check that $(i_t)_*(\mathfrak{g}_p) = \mathfrak{g}_{[(p,t)]}$ and thus obtain the following commutative diagram.

$$\begin{array}{ccc} H_p & \xrightarrow[\simeq]{d\pi} & T_{\pi(p)} S^\zeta \\ {\scriptstyle (i_t)_*}\downarrow & & \downarrow{\scriptstyle (i_t)_*} \\ H'_{[(p,t)]} & \xrightarrow[d\widetilde{\pi}]{\simeq} & T_{\widetilde{\pi}[(p,t)]} (S_f)^\zeta \end{array}$$

From $\varphi = \widetilde{J}$ and the diagram above, for $X \in T_{\pi(p)} S^\zeta$

$$\begin{aligned} \varphi^\zeta((i_t)_* X) &= d\widetilde{\pi}(\varphi(\widetilde{(i_t)_* X})) \\ &= d\widetilde{\pi}(\varphi((i_t)_* \widetilde{X})) \\ &= d\widetilde{\pi}((i_t)_* J(\widetilde{X})) \\ &= (i_t)_* d\pi(J(\widetilde{X})) = (i_t)_* J^\zeta(X) \end{aligned}$$

holds, and thus $\varphi^\zeta = \widetilde{J^\zeta}$. Similarly we have $\widetilde{h}(\widetilde{X}, \widetilde{Y}) = \widetilde{h^\zeta}(X,Y)$ for $X, Y \in T_{\widetilde{\pi}[(p,t)]}(S_f)^\zeta$, hence we obtain

$$\begin{aligned} g^\zeta(X,Y) = g(\widetilde{X}, \widetilde{Y}) &= \widetilde{h}(\widetilde{X}, \widetilde{Y}) + \eta(\widetilde{X}) \eta(\widetilde{Y}) \\ &= \widetilde{h^\zeta}(X,Y) + \eta^\zeta(X) \eta^\zeta(Y). \end{aligned}$$

Let $\mathrm{pr}_1 : S_f \to S^1$ and $\mathrm{pr}_2 : (S^\zeta)_{f^\zeta} \to S^1$ be natural projections. Then the diagram

$$\begin{array}{ccc} \widetilde{\mu}^{-1}(\zeta) & \xrightarrow{\ \widetilde{\pi}\ } & (S_f)^\zeta \\ & {\scriptstyle \mathrm{pr}_1} \searrow \quad \swarrow {\scriptstyle \mathrm{pr}_2} & \\ & S^1 & \end{array}$$

commutes, and thus we have $\widetilde{\pi}^* \mathrm{pr}_2^* d\theta = \mathrm{pr}_1^* d\theta = \eta|_{\widetilde{\mu}^{-1}(\zeta)}$. Hence $\mathrm{pr}_2^* d\theta = \eta^\zeta$ holds.

From the above, the induced 2-form ω^ζ and the Reeb vector field ξ^ζ on $(S_f)^\zeta$ are the same as those on $(S^\zeta)_{f^\zeta}$, and thus the cokähler structure on $(S_f)^\zeta$ coincides with that of $(S^\zeta)_{f^\zeta}$. □

3.5 A perspective from dynamical systems

In this section, we interpret our coKähler reduction theorem from the viewpoint of dynamical systems. First, we explain why cosymplectic manifolds describe time-dependent Hamiltonian systems.

Let (M, ω) be a $2n$-dimensional symplectic manifold and $H \in C^\infty(M \times \mathbb{R})$. Then we consider a cosymplectic manifold $(M \times \mathbb{R},\ \eta := dt,\ \omega_H := \omega + dH \wedge dt)$, where t is the coordinate of $\mathbb{R}$. Using Darboux coordinates (p_i, q_i) of (M, ω), the Reeb vector field ξ_H of $(M \times \mathbb{R}, \eta, \omega_H)$ is written as

$$\xi_H = \sum_{i=1}^{n} \Big(\frac{\partial H}{\partial p_i} \frac{\partial}{\partial q_i} - \frac{\partial H}{\partial q_i} \frac{\partial}{\partial p_i} \Big) + \frac{\partial}{\partial t}$$

and its integral curves are controlled by an ODE system

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i}, \qquad \dot{t} = 1. \tag{3.5.1}$$

Therefore we can consider t as a time-parameter which parameterize ordinary Hamiltonian systems. Any cosymplectic manifold is locally the product of a symplectic manifold with the real line, so cosymplectic manifolds provide a good framework for time-dependent Hamiltonian systems.

Remark 3.5.1. *The Reeb vector field ξ_H of $(M \times \mathbb{R}, \eta, \omega_H)$ coincides with the vector field $\frac{\partial}{\partial t} + X_H$, where X_H is the Hamiltonian vector field with respect to the cosymplectic structure (η, ω). The vector field $\frac{\partial}{\partial t} + X_H$ is called the* evolution vector field *of the time-dependent system.* ■

For any almost coKähler manifold $(M, g, \varphi, \xi, \eta)$, the following is known.

Lemma 3.5.2. *Integral curves of ξ are geodesics with respect to g.*

Proof. Let ∇ be the Levi-Civita connection with respect to g. Then for any $X \in \mathfrak{X}(M)$, $g(\nabla_X \xi, \xi) = 0$ holds since $X(g(\xi,\xi)) = g(\nabla_X\xi,\xi) + g(\xi, \nabla_X \xi)$. Hence

$$\begin{aligned} 0 = 2d\eta(\xi, X) &= \xi(\eta(X)) - \eta([\xi, X]) \\ &= \xi(g(X,\xi)) - g(\nabla_\xi X - \nabla_X \xi, \xi) \\ &= \xi(g(X,\xi)) - g(\nabla_\xi X, \xi) + g(\nabla_X \xi, \xi) \\ &= \xi(g(X,\xi)) - g(\nabla_\xi X, \xi) \\ &= g(X, \nabla_\xi \xi) \end{aligned}$$

holds, and thus we obtain $\nabla_\xi \xi = 0$. □

From Lemma 3.5.2 and Theorem 3.1.5, we immediately obtain the following.

Proposition 3.5.3. *Suppose that the cosymplectic manifold $(M\times\mathbb{R}, \eta, \omega_H)$ admits a coKähler structure and a Hamiltonian action of a Lie group which preserves the coKähler structure. Then the image of a solution of (3.5.1) by the projection of reduction is a geodesic with respect to the reduced metric on M^ζ.* □

Example 3.5.4 (cf.[Alb89])**.** *The motion of a solid in $\mathbb{R}^3$ with a fixed point (its center of inertia) is described by a manifold $T^*SO(3)$ equipped with the canonical symplectic form on $T^*SO(3)$ and a Hamiltonian $H : T^*SO(3) \simeq SO(3) \times \mathfrak{so}(3)^* \to \mathbb{R}$ defined by*

$$H(A, \alpha) = \sum_{1\leq i\leq 3} \frac{\alpha_i^2}{M_i},$$

where $\alpha_1, \alpha_2, \alpha_3$ are coefficients with respect to a suitable basis of $\mathfrak{so}(3)^*$ and M_1, M_2, M_3 are coefficients of the ellipsoid of inertia of the solid.

Then the associated time-dependent Hamiltonian system is given by a cosymplectic manifold $(T^*SO(3) \times \mathbb{R},\ \eta := dt,\ \omega_H := \omega_{T^*SO(3)} + dH \wedge dt)$. We define an action of $SO(3)$ on $T^*SO(3) \times \mathbb{R}$ by

$$B \cdot (A, \alpha, t) = (BA, \alpha, t)$$

where $A, B \in SO(3)$, $\alpha \in \mathfrak{so}(3)^*$, $t \in \mathbb{R}$. This action is a Hamiltonian action whose momentum map is given by $\mu(A, \alpha, t) = Ad_A^* \alpha$. Then the reduced cosymplectic manifold at any non-zero vector $\zeta \in \mathfrak{so}(3)^*$ is $(\mathbb{S}^2 \times \mathbb{R},\ dt,\ \omega_{\mathbb{S}^2} + dH \wedge dt)$, where $\omega_{\mathbb{S}^2}$ is the standard symplectic form on $\mathbb{S}^2$ (note that the Hamiltonian H is invariant by the action of $SO(3)$ and thus descend to the quotient).

Since $SO(3)$ is compact, we can naturally construct a Kähler structure (h, I) on $T^*SO(3)$ which is compatible with the canonical symplectic form $\omega_{T^*SO(3)}$. Then $T^*SO(3) \times \mathbb{R}$ is endowed with a $SO(3)$-invariant almost coKähler structure (g, φ, ξ_H, dt) defined by

$$g(X, Y) = h(X, Y), \qquad g(X, \xi_H) = 0, \qquad g(\xi_H, \xi_H) = 1$$

$$\varphi(X) = I(X), \qquad \varphi(\xi_H) = 0$$

for any $X, Y \in \ker dt = TM$. Since the Levi-Civita connection ∇ of g satisfies $\nabla\varphi = 0$, (g, φ, ξ, η) is a coKähler structure, and the projection of time-dependent flows onto $\mathbb{S}^2 \times \mathbb{R}$ are geodesics with respect to the reduced coKähler metric. ■

Chapter 4

Reduction theorem for cosymplectic groupoid actions

4.1 Cosymplectic groupoids

The notion of a cosymplectic groupoid is defined in exactly the same way as that of a symplectic groupoid:

Definition 4.1.1. *A* cosymplectic groupoid *is a triplet* $(G_1 \rightrightarrows G_0, \eta_{G_1}, \omega_{G_1})$ *of a Lie groupoid and a cosymplectic structure on* G_1 *such that*

$$m^*\eta_{G_1} = pr_1^*\eta_{G_1} + pr_2^*\eta_{G_1} \qquad m^*\omega_{G_1} = pr_1^*\omega_{G_1} + pr_2^*\omega_{G_1}$$

holds. ■

Example 4.1.2. *Let* $G_1 \rightrightarrows G_0$ *be a Lie groupoid and* G *an abelian Lie group. Then a pair* $(P \rightrightarrows G_0, (\pi, id_{G_0}))$ *of a Lie groupoid* $P \rightrightarrows G_0$ *and a morphism* $(\pi, id_{G_0}) : (P \rightrightarrows G_0) \to (G_1 \rightrightarrows G_0)$ *is called a* central extension *of* $G_1 \rightrightarrows G_0$ *by* G *when* G *acts on* P *and the map* $\pi : P \to G_1$ *is a principal* G*-bundle.*

For any symplectic groupoid $(G_1 \rightrightarrows G_0, \omega_{G_1})$*, let us consider a central extension* $(P \rightrightarrows G_0, (\pi, id_{G_0}))$ *by* $G = \mathbb{R}$ *or* $G = \mathbb{S}^1$*. Let* η_P *be a multiplicative, flat connection form of the principal bundle* $\pi : P \to G_1$*. Then* $(P \rightrightarrows G_0, \eta_P, \omega_P)$ *is a cosymplectic groupoid, where* $\omega_P = \pi^*\omega_{G_1}$*. In particular, the trivial* $\mathbb{R}$*-central extension* $(G_1 \times \mathbb{R} \rightrightarrows G_0, pr_{\mathbb{R}}^* dt, pr_{G_1}^*\omega_{G_1})$*, where* pr *denotes the projections, is a cosymplectic groupoid.* ■

The space of arrows of a cosymplectic groupoid has a symplectic foliation defined by the distribution Kerη and there is a distinguished symplectic leaf:

Theorem 4.1.3 ([FP23])**.** *Let* $\mathcal{G} = (G_1 \rightrightarrows G_0)$ *be a cosymplectic groupoid. Then any unit arrow in* G_1 *belongs to the same symplectic leaf* S_{G_1}*. Moreover,* $S_{\mathcal{G}} := (S_{G_1} \rightrightarrows G_0)$ *is a Lie subgroupoid of* $\mathcal{G}$ *and it is a symplectic groupoid.* □

4.2 Actions of cosymplectic groupoids

In order to define the notion of a cosymplectic groupoid action, we need to consider an analogue of Lagrangian submanifolds.

Let (M, η, ω) be a cosymplectic manifold and $N \subset M$ a submanifold. Then we call N a *Lagrangian-Legendrean submanifold* or in short, *LL submanifold* if

$$T_pN \subset \mathrm{Ker}\eta_p \qquad \text{(Legendrean property)},$$

$$(T_pN)^{\omega_p|_{\mathrm{Ker}\eta_p}} = T_pN \qquad \text{(Lagrangian property)}$$

holds for any $p \in N$, where $(T_pN)^{\omega_p|_{\mathrm{Ker}\eta_p}}$ denotes the orthogonal complement of T_pN with respect to $\omega_p|_{\mathrm{Ker}\eta_p}$.

In fact, the notion of a LL submanifold is defined for *almost* cosymplectic manifolds. In the case of contact manifolds, the definition of a LL submanifold coincides with that of a Legendrean submanifold.

We can rephrase the definition of a cosymplectic groupoid by using the notion of a LL submanifold:

Proposition 4.2.1. *Let $\mathcal{G} = (G_1 \rightrightarrows G_0)$ be a Lie groupoid and (η, ω) a cosymplectic structure on G_1. Then a triplet $(\mathcal{G}, \eta, \omega)$ is a cosymplectic groupoid if and only if The graph of the multiplication, i.e.,*

$$\Gamma := \{(g, h, 1, gh, 1) \in G_1 \times G_1 \times \mathbb{R} \times G_1 \times \mathbb{R} \mid (g, h) \in G_1 {}_s\!\times_t G_1\}$$

is a LL submanifold of a cosymplectic manifold $(G_1 \times G_1 \times \mathbb{R} \times G_1 \times \mathbb{R}, \widetilde{\eta}, \widetilde{\omega})$, where

$$\widetilde{\eta} := \eta_1 + \eta_2 - \eta_3,$$

$$\widetilde{\omega} := (\omega_1 + \omega_2 + \eta_1 \wedge dt_1) - \omega_3 - (\eta_1 + \eta_2) \wedge dt_2$$

(t_i denotes the coordinate of i-th $\mathbb{R}$ and (η_i, ω_i) denotes the cosymplectic structure of i-th G_1).

Proof. Note that for two cosymplectic manifolds (M_1, η_1, ω_1) and (M_2, η_2, ω_2), a pair $(\eta_1 + \eta_2, \omega_1 + \omega_2 + \eta_1 \wedge dt)$ is a cosymplectic structure on $M_1 \times M_2 \times \mathbb{R}$.

Γ is the image of an embedding $\iota : G_1 {}_s\!\times_t G_1 \to G_1 \times G_1 \times \mathbb{R} \times G_1 \times \mathbb{R}$ given by $\iota(g, h) = (g, h, 1, gh, 1)$. Then we obtain

$$\iota^*\widetilde{\eta} = \iota^*(p_1^*\eta + p_2^*\eta - p_3^*\eta) = \pi_1^*\eta + \pi_2^*\eta - m^*\eta,$$

$$\begin{aligned}\iota^*\widetilde{\omega} &= \iota^*(p_1^*\omega + p_2^*\omega - p_3^*\omega + (p_1^*\eta) \wedge dq_1 - (p_2^*\eta) \wedge dq_2)\\ &= \pi_1^*\omega + \pi_2^*\omega - m^*\omega,\end{aligned}$$

where p_i and q_i denotes projections to i-th G_1 and i-th $\mathbb{R}$, respectively, and $\pi_i : G_1 {}_s\!\times_t G_1 \to G_1$ also denotes projections. Hence the multiplicativity of η and ω is equivalent to $\iota^*\widetilde{\eta} = 0$ and $\iota^*\widetilde{\omega} = 0$, respectively. In addition, we have

$$\dim \Gamma = \dim(G_1 {}_s\!\times_t G_1) = 2 \dim G_1 - \dim G_0,$$

and since $\dim G_1 = 2\dim G_0 + 1$ (see [FP23]), we obtain

$$\begin{aligned} 2\dim\Gamma + 1 &= 4\dim G_1 - 2\dim G_0 + 1 = 3\dim G_1 + 2 \\ &= \dim(G_1 \times G_1 \times \mathbb{R} \times G_1 \times \mathbb{R}). \end{aligned}$$

Therefore the multiplicativity condition is equivalent to Γ being a LL submanifold. $\square$

Now we can define a notion of a cosymplectic groupoid action:

Definition 4.2.2. *Let $(\mathcal{G} = (G_1 \rightrightarrows G_0), \eta_{G_1}, \omega_{G_1})$ be a cosymplectic groupoid and (M, η, ω) a cosymplectic manifold. A left $\mathcal{G}$-action on M (or a left $\mathcal{G}$-module M) is said to be* cosymplectic *if the following conditions are satisfied:*

1. *The momentum map $\rho : M \to G_0$ of the action satisfies $d\rho(R) = 0$, where R is the Reeb vector field of (M, η, ω),*
2. *The graph of the action, i.e.,*

$$\Gamma := \{(g, x, 1, gx, 1) \in G_1 \times M \times \mathbb{R} \times M \times \mathbb{R} \mid (g, x) \in G_1 {}_s\times_\rho M\}$$

is a LL submanifold of a cosymplectic manifold $(G_1 \times M \times \mathbb{R} \times M \times \mathbb{R}, \widetilde{\eta}, \widetilde{\omega})$, where

$$\widetilde{\eta} := \eta_{G_1} + \eta_1 - \eta_2,$$

$$\widetilde{\omega} := (\omega_{G_1} + \omega_1 + \eta_{G_1} \wedge dt_1) - \omega_2 - (\eta_{G_1} + \eta_1) \wedge dt_2$$

(t_i denotes the coordinate of i-th $\mathbb{R}$ and (η_i, ω_i) denotes the cosymplectic structure of i-th M).

$\blacksquare$

The following proposition is essentially used in section 4.3 for the proof of our main theorem:

Proposition 4.2.3. *Let $(\mathcal{G} = (G_1 \rightrightarrows G_0), \eta_{G_1}, \omega_{G_1})$ be a cosymplectic groupoid, (M, η, ω) a cosymplectic left $\mathcal{G}$-module and (ρ, Φ) its action maps. Let $S_\mathcal{G} = (S_{G_1} \rightrightarrows G_0)$ be the symplectic subgroupoid obtained by Theorem 4.1.3. Then any symplectic leaf S of (M, η, ω) is a symplectic left $S_\mathcal{G}$-module by action maps*

$$\rho|_S : S \to G_0,$$

$$\Phi|_{S_{G_1} {}_s\times_\rho S} : S_{G_1} {}_s\times_\rho S \to S.$$

Proof. Firstly, we see that the Legendrean property of the graph Γ of the action (ρ, Φ) implies $\Phi(S_{G_1} {}_s\times_\rho S) \subset S$. Let $(g, x) \in S_{G_1} {}_s\times_\rho S$ and $(g(t), x(t))$ be a smooth path in $S_{G_1} {}_s\times_\rho S$ whose starting point is $(1_{\rho(x)}, x)$ and ending point is (g, x). Then we obtain a smooth path $(g(t), x(t), 1, (gx)(t), 1)$ in Γ and

$$0 = \widetilde{\eta}(\dot{g}(t), \dot{x}(t), 0, (\dot{gx})(t), 0) = \eta_{G_1}(\dot{g}(t)) + \eta(\dot{x}(t)) - \eta((\dot{gx})(t)) = -\eta((\dot{gx})(t))$$

holds. Therefore two points $x = 1_{\rho(x)}x$ and gx are in the same symplectic leaf S.

Secondly, we see that the Lagrangian property of Γ implies that the restricted action $(\rho|_S, \Phi|_{S_{G_1\ s}\times_\rho S})$ is symplectic. Let $(g(t), x(t))$ be a smooth path in $S_{G_1\ s}\times_\rho S$. Then we have

$$0 = \widetilde{\omega}(\dot{g}(t), \dot{x}(t), 0, \dot{(gx)}(t), 0) = \omega_{G_1}(\dot{g}(t)) + \omega(\dot{x}(t)) - \omega(\dot{(gx)}(t)).$$

In addition to this, taking the dimension count into consideration, we can see that the graph of the $S_\mathcal{G}$-action on S is a Lagrangian submanifold. □

4.3 Mikami-Weinstein type theorem

The following is our main theorem in this chapter:

Theorem 4.3.1. *Let $\mathcal{G} = (G_1 \rightrightarrows G_0)$ be a cosymplectic groupoid and M a cosymplectic, free and proper left $\mathcal{G}$-module with respect to a momentum map $\rho : M \to G_0$. Assume that $\xi \in \rho(M)$ is a regular value of ρ. Then $(S_\mathcal{G})_\xi \backslash \rho^{-1}(\xi)$ is a cosymplectic manifold.*

Proof. We denote $(S_\mathcal{G})_\xi \backslash \rho^{-1}(\xi)$ as M^ξ and the quotient map as $\pi : \rho^{-1}(\xi) \to M^\xi$. Let $\{S_i\}_{i\in I}$ be the symplectic foliation of M. Since the Reeb vector field R of M satisfies $d\rho(R) = 0$, each S_i intersects transversely with $\rho^{-1}(\xi)$, and thus $(\rho|_{S_i})^{-1}(\xi)$ is a smooth manifold.

By Proposition 4.2.3, the symplectic groupoid $S_\mathcal{G}$ acts on each leaf S_i symplectically. Hence $\{S_i^\xi := (S_\mathcal{G})_\xi \backslash (\rho|_{S_i})^{-1}(\xi)\}_{i\in I}$ forms a foliation on M^ξ of codimension 1 (see section 1.3 of [MM03]). In addition, we can apply Theorem 2.6.1 on each leaf and thus $\{S_i^\xi\}_{i\in I}$ is a symplectic foliation on M^ξ.

Let $L_g : \rho^{-1}(\xi) \to \rho^{-1}(\xi)$ be the left action map by $g \in (S_\mathcal{G})_\xi$ and $x(t)$ a integral curve of R in $\rho^{-1}(\xi)$. Then by the Legendrean property of the graph,

$$\eta((L_g)_*R) = \eta(\dot{(gx)}(t)) = \eta_{G_1}(0) + \eta(\dot{x}(t)) = \eta(R) = 1$$

holds. Similarly, by the Lagrangian property of the graph, we have $\omega((L_g)_*R, -) = 0$ and thus R is left invariant. Hence R descends to a vector field $R^\xi := d\pi(R)$ on the quotient M^ξ. R^ξ is transverse to the symplectic foliation on M^ξ.

The reduced foliation $\{S_i^\xi\}_{i\in I}$ is coorientable since $\{S_i\}_{i\in I}$ is. We choose a defining 1-form η^ξ of the foliation $\{S_i^\xi\}_{i\in I}$ such that $\eta^\xi(R^\xi) = 1$ holds. Then we have $\pi^*\eta^\xi = \eta$. Let ω_i be the symplectic form on S_i^ξ. Then we define a 2-form ω^ξ on M^ξ by

$$\omega^\xi(R^\xi, -) = 0, \qquad \omega^\xi|_{S_i^\xi} = \omega_i.$$

Then we have $\pi^*\omega^\xi = \omega$. η^ξ, ω^ξ are closed since η, ω are closed and π is a submersion. We can easily see that $\eta^\xi \wedge (\omega^\xi)^n$ is a volume form, and thus a pair (η^ξ, ω^ξ) is a cosymplectic structure on M^ξ. □

4.4 Examples

In this section, we give two examples of Theorem 4.3.1.

Example 4.4.1. *Let $\mathcal{G} = (G_1 \rightrightarrows G_0)$ be a cosymplectic groupoid. Then $\mathcal{G}$ acts on G_1 by the multiplication of groupoid with $t : G_1 \to G_0$ as the momentum map. This action is free, proper and cosymplectic. In fact, the graph of the action is a LL submanifold because of Proposition 4.2.1, and the Reeb vector field R of G_1 satisfies $R \in \operatorname{Ker} dt$ (see [FP23]).*

For any $\xi \in G_0$, the reduced cosymplectic manifold $(S_{\mathcal{G}})_\xi \backslash t^{-1}(\xi)$ is obtained by Theorem 4.3.1. Here the symplectic leaf $(S_{\mathcal{G}})_\xi \backslash (t|_{S_{G_1}})^{-1}(\xi)$ coincides with the $S_{\mathcal{G}}$-orbit in G_0 through ξ. We can see that it is also a leaf of the symplectic foliation of a Poisson manifold G_0 by Theorem 2.6.1, and thus we have two foliated manifolds each having the orbit as a leaf. ■

Example 4.4.2. *Let G be a Lie group acts on a cosymplectic manifold (M, η, ω) freely and properly. We assume that there is a momentum map $\mu : M \to \mathfrak{g}^*$ with respect to the action. Then let us consider a cosymplectic groupoid*

$$T^*G \times \mathbb{R} \simeq G \times \mathfrak{g}^* \times \mathbb{R} \rightrightarrows \mathfrak{g}^*$$

(the trivial $\mathbb{R}$-central extension of a symplectic groupoid $T^*G \rightrightarrows \mathfrak{g}^*$).

For any $\varepsilon > 0$, we define

$$M_\varepsilon = \{ \ x \in M \mid \text{Reeb flow } \varphi_t(x) \text{ is defined in } t \in [-\varepsilon, \varepsilon] \ \},$$

$$\mathcal{G}_\varepsilon = T^*G \times (-\varepsilon, \varepsilon) \subset T^*G \times \mathbb{R}.$$

In fact, although $\mathcal{G}_\varepsilon$ is not a Lie groupoid, it is a local Lie groupoid *(where the composition of arrows is defined only in a neighborhood of the unit arrows), and the previously discussed concepts related to actions can also be applied to local Lie groupoids. We can define a cosymplectic $\mathcal{G}_\varepsilon$-action on M_ε by*

$$(g, \xi, t) \cdot x := \varphi_t(gx)$$

for $(g, \xi, t) \in G \times \mathfrak{g}^* \times (-\varepsilon, \varepsilon)$, $x \in M_\varepsilon$, with $\mu|_{M_\varepsilon} : M_\varepsilon \to \mathfrak{g}^*$ as the momentum map. In this case, Theorem 4.3.1 coincides with Theorem 2.2.4 for the G-action on $(M_\varepsilon, \eta|_{M_\varepsilon}, \omega|_{M_\varepsilon})$. ■

4.5 Morita equivalence of cosymplectic groupoids

We defined the notion of a cosymplectic groupoid action, thus we can also define the notion of Morita equivalence between cosymplectic groupoids as in the case of symplectic groupoids:

Definition 4.5.1. *A cosymplectic groupoid $\mathcal{G} = (G_1 \rightrightarrows G_0)$ is said to be* Morita equivalent *to another cosymplectic groupoid $\mathcal{H} = (H_1 \rightrightarrows H_0)$ when there is a cosymplectic manifold M, a left cosymplectic $\mathcal{G}$-action and a right cosymplectic $\mathcal{H}$-action on M which satisfies the following conditions:*

1. *Momentum maps $\rho : M \to G_0$ and $\sigma : M \to H_0$ are surjective submersions;*

2. *Actions of $\mathcal{G}$ and $\mathcal{H}$ on M are both free and proper;*

3. *The two actions commute with each other;*

4. *ρ is constant on each orbit of the action of $\mathcal{H}$ and an induced map $M/\mathcal{H} \to G_0$ is a diffeomorphism; Similarly, σ is constant on each orbit of the action of $\mathcal{G}$ and an induced map $\mathcal{G}\backslash M \to H_0$ is a diffeomorphism.*

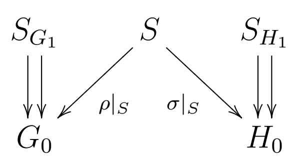
(M, ρ, σ) is called an equivalence bimodule from $\mathcal{G}$ to $\mathcal{H}$. ■

Regarding the relationship between Morita equivalence of two cosymplectic groupoids $\mathcal{G}, \mathcal{H}$ and that of their symplectic subgroupoids $S_{\mathcal{G}}, S_{\mathcal{H}}$, we obtain the following.

Proposition 4.5.2. *Let $\mathcal{G} = (G_1 \rightrightarrows G_0)$ and $\mathcal{H} = (H_1 \rightrightarrows H_0)$ be Morita equivalent cosymplectic groupoids and $S_{\mathcal{G}} = (S_{G_1} \rightrightarrows G_0)$, $S_{\mathcal{H}} = (S_{H_1} \rightrightarrows H_0)$ their symplectic subgroupoids. Let (M, ρ, σ) be an equivalence bimodule from $\mathcal{G}$ to $\mathcal{H}$ and assume that there is a symplectic leaf S of M which satisfies the following conditions:*

- *$\rho|_S : S \to G_0,\ \sigma|_S : S \to H_0$ are surjective.*
- *For any $x \in S$ and $g \in G_1$ such that gx is defined, $gx \in S$ implies $g \in S_{G_1}$.*
- *For any $x \in S$ and $h \in H_1$ such that xh is defined, $xh \in S$ implies $h \in S_{H_1}$.*

Then the triplet $(S, \rho|_S, \sigma|_S)$ is an equivalence bimodule from $S_{\mathcal{G}}$ to $S_{\mathcal{H}}$, and thus these symplectic groupoids are Morita equivalent.



Proof. First, Proposition 4.2.3 implies that actions of $S_{\mathcal{G}}$ and $S_{\mathcal{H}}$ preserves the leaf S, and these actions are both symplectic.

Since actions of $\mathcal{G}, \mathcal{H}$ are both cosymplectic, $d\rho(R) = 0,\ d\sigma(R) = 0$ holds for the Reeb vector field R of M. Hence $\rho|_S, \sigma|_S$ are submersions.

Then $\rho|_S$ is constant along each orbit of the $S_{\mathcal{H}}$-action, and it induces a diffeomorphism $S/S_{\mathcal{H}} \to G_0$ since for $x \in S$, $gx \in S$ implies $g \in S_{G_1}$ and ρ induces a diffeomorphism $M/\mathcal{H} \to G_0$. Similarly, we can see that $\sigma|_S$ induces a diffeomorphism $S_{\mathcal{G}}\backslash S \to H_0$. The other conditions can be easily verified. □

Chapter 5

Conclusion and further study

In this thesis, we proved reduction theorems for Hamiltonian actions on coKähler manifolds and 3-cosymplectic manifolds, which are the polar opposites of Sasakian manifolds and 3-Sasakian manifolds, respectively. A notion of momentum maps can also be defined for Lie group actions on contact manifolds [Alb89, Gei97, Wil02]. However, the situation is quite different from that on symplectic manifolds, such as the momentum map being uniquely determined by the action. On the other hand, Hamiltonian actions on cosymplectic manifolds have properties that are very similar to those on symplectic manifolds, and therefore, the results in chapter 3 are natural odd-dimensional analogues of the reduction theorems by Hitchin *et al* [HKLR87].

Xu [Xu91b] studied the notion of Morita equivalence of symplectic groupoids and applied it to investigate Morita equivalence of Poisson manifold [Xu91a]. In this thesis, we defined the notion of a cosymplectic groupoid action and that of Morita equivalence between cosymplectic groupoids. Regarding these, future work includes demonstrating that results parallel to those in the case of symplectic groupoids hold (e.g., whether Morita equivalence between two cosymplectic groupoids implies an equivalence of categories between their module categories).

Another possible direction of research is to define symplectic groupoid actions on differentiable stacks or orbifolds (see Remark 2.5.5) endowed with symplectic structures and to extend the Mikami-Weinstein theorem to these settings.

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