

Title	Three dualities on the integral homology of infinite cyclic coverings of manifolds
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Citation	Osaka Journal of Mathematics. 1986, 23(3), p. 633–651
Version Type	VoR
URL	https://doi.org/10.18910/10193
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THREE DUALITIES ON THE INTEGRAL HOMOLOGY OF INFINITE CYCLIC COVERINGS OF MANIFOLDS

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(Received July 15, 1985)

0. Statement of main result

We consider a compact oriented topological n-manifold M. Let γ be an element of the first integral cohomology $H^1(M)$ and \tilde{M} be the infinite cyclic covering space of M associated with γ . The covering transformation group is infinite cyclic and denoted by $\langle t \rangle$ with a generator t, specified by γ . A subboundary, A, of M is \emptyset or a compact (n-1)-submanifold of the boundary ∂M such that $A' = cl_{\partial M}(\partial M - A)$ is \emptyset or a compact (n-1)-submanifold of ∂M . The pair (A, A') is called a *splitting* of ∂M . Let \tilde{A} be the lift of A, i.e., the preimage of A under the covering $\tilde{M} \rightarrow M$. Let Λ be the integral group ring of $\langle t \rangle$. The integral homology $H_*(\tilde{M}, \tilde{A})$ forms a *finitely generated* Λ -module, because by [K/S](M, A) is homotopy equivalent to a compact polyhedral pair and Λ is Noetherian. For an abelian group H, let $e^{i}H = \operatorname{Ext}_{Z}^{i}(H;Z)$ (so that $e^{i}H = 0$ for $i \ge 2$ and $\operatorname{Hom}_{Z}(H,Z) = \mathfrak{P}(H)$, tH = the Z-torsion part of H and bH = H/tH. When H is a Λ -module, let $E^i H = \operatorname{Ext}_{\Lambda}^i(H, \Lambda)$ (so that $\operatorname{Hom}_{\Lambda}(H, \Lambda) = E^0 H$) and TH =the A-torsion part of H and BH=H/TH. Since A has the global dimension 2 (cf. MacLane [Ma, p. 205]), we have $E^i H=0$ for $i \geq 3$. The following A-submodule, DH, of H was introduced by Blanchfield [B]:

 $DH = \{x \in H \mid \exists \text{ coprime } \lambda_1, \lambda_2, \dots, \lambda_m \in \Lambda(m \ge 2) \text{ with } \lambda_i x = 0, \forall i \}.$

If H is finitely generated over Λ , then we see that DH is the (unique) maximal finite Λ -submodule of H and there are natural Λ -isomorphisms $DH \cong E^2E^2H$ and $TH/DH \cong E^1E^1H$. Further, E^0H is Λ -free and there is a natural Λ -monomorphism $BH \to E^0E^0H$ whose cokernel is finite. The purpose of this paper is to establish the Zeroth, First and Second Duality Theorems giving dual structures between $E^iE^iH_p(\tilde{M}, \tilde{A})$ and $E^iE^iH_{n-p-i}(\tilde{M}, \tilde{A}')$ for i=0, 1 and 2, respectively. It turns out that the first two are similar to the Blanchfield Dualities [B] and the third, the Farber/Levine Duality [F], [L]. Let $f: (M_1; A_1, A_1') \to (M_2; A_2, A_2')$ be a proper oriented homotopy equivalence (on each of M_1, A_1 and A_1') with $f^*(\gamma_2) =$ γ_1 for compact oriented *n*-manifolds M_i with splittings (A_i, A_i') of ∂M_i and $\gamma_i \in$ $H^1(M_i), i=1, 2$. For the covering spaces \tilde{M}_i of M_i associated with γ_i , f lifts to

a proper homotopy equivalence $\tilde{f}: (\tilde{M}_1; \tilde{A}_1, \tilde{A}_1') \to (\tilde{M}_2; \tilde{A}_2, \tilde{A}_2')$, which induces Λ -isomorphisms $E^i E^i H_*(\tilde{M}_1, \tilde{A}_1) \cong E^i E^i H_*(\tilde{M}_2, \tilde{A}_2)$ and $E^i E^i H_*(\tilde{M}_1, \tilde{A}_1') \cong E^i E^i H_*(\tilde{M}_2, \tilde{A}_2)$ $(\tilde{M}_2, \tilde{A}_2')$ denoted by \tilde{f}_* .

The Zeroth Duality Theorem. For a compact oriented n-manifold M with $\gamma \in H^1(M)$ and a splitting (A, A') of ∂M and integers p, q with p+q=n, there is a pairing

$$S \colon E^{0}E^{0}H_{p}(\tilde{M},\tilde{A}) imes E^{0}E^{0}H_{q}(\tilde{M},\tilde{A}') o \Lambda$$

such that

(1) (Homotopy invariance) A proper oriented homotopy equivalence $f: (M_1; A_1, A_1') \rightarrow (M_2; A_2, A_2')$ with $f^*(\gamma_2) = \gamma_1$ induces the identity $S(\tilde{f}_*(x), \tilde{f}_*(y)) = S(x, y)$ for $x \in E^0 E^0 H_p(\tilde{M}_1, \tilde{A}_1)$ and $y \in E^0 E^0 H_q(\tilde{M}_1, \tilde{A}_1')$,

(2) S is sesquilinear, i.e., $\lambda S(x, y) = S(\overline{\lambda}x, y) = S(x, \lambda y)$ for $x \in E^0 E^0 H_p(\tilde{M}, \tilde{A})$, $y \in E^0 E^0 H_q(\tilde{M}, \tilde{A}')$ and $\lambda \in \Lambda$, where—denotes the involution on Λ sending t to t^{-1} , (3) S is $\mathcal{E}(pq)$ -Hermitian, i.e., $S(x, y) = \mathcal{E}(pq)\overline{S(y, x)}$ for $x \in E^0 E^0 H_p(\tilde{M}, \tilde{A})$ and $y \in E^0 E^0 H_q(\tilde{M}, \tilde{A}')$, where $\mathcal{E}(m) = (-1)^m$,

(4) S is non-singular, i.e., S induces a t-anti Λ -isomorphism

$$E^{0}E^{0}H_{p}(\tilde{M},\tilde{A})\cong \operatorname{Hom}_{\Lambda}(E^{0}E^{0}H_{q}(\tilde{M},\tilde{A}'),\Lambda)$$
.

In fact, we construct S by extending the Λ -intersection pairing $\tilde{I}nt: BH_p(\tilde{M}, \tilde{A}) \times BH_q(\tilde{M}, \tilde{A}') \to \Lambda$. Blanchfield [B] has formulated a similar duality over local rings of Λ . Let $Q(\Lambda)$ be the quotient field of Λ .

The First Duality Theorem. For a compact oriented n-manifold M with $\gamma \in H^1(M)$ and a splitting (A, A') of ∂M and integers p, r with p+r+1=n, there is a pairing

$$L: E^{1}E^{1}H_{p}(\tilde{M}, \tilde{A}) \times E^{1}E^{1}H_{r}(\tilde{M}, \tilde{A}') \to Q(\Lambda)/\Lambda$$

such that

(1) (Homotopy invariance) A proper oriented homotopy equivalence $f: (M_1; A_1, A'_1) \rightarrow (M_2; A_2, A'_2)$ with $f^*(\gamma_2) = \gamma_1$ induces the identity $L(\tilde{f}_*(x), \tilde{f}_*(y)) = L(x, y)$ for $x \in E^1E^1H_p(\tilde{M}_1, \tilde{A}_1)$ and $y \in E^1E^1H_r(\tilde{M}_1, \tilde{A}'_1)$,

(2) L is sesquilinear, i.e., $\lambda L(x, y) = L(\lambda x, y) = L(x, \lambda y)$ for $x \in E^1 E^1 H_p(\tilde{M}, \tilde{A})$, $y \in E^1 E^1 H_r(\tilde{M}, \tilde{A}')$ and $\lambda \in \Lambda$,

(3) L is $\mathcal{E}(pr+1)$ -Hermitian, i.e., $L(x, y) = \mathcal{E}(pr+1)\overline{L(y, x)}$ for $x \in E^1E^1H_p(\tilde{M}, \tilde{A})$ and $y \in E^1E^1H_r(\tilde{M}, \tilde{A}')$,

(4) L is non-singular, i.e., L induces a t-anti Λ -isomorphism

 $E^{1}E^{1}H_{p}(\tilde{M},\tilde{A})\cong \operatorname{Hom}\left(E^{1}E^{1}H_{r}(\tilde{M},\tilde{A}'),Q(\Lambda)/\Lambda\right).$

When M is triangulated, we can see that our pairing L is essentially the same as (precisely, the *t*-conjugate of) a pairing of Blanchfield [B] (cf. Remark

5.5). Our next plan is to give a dual structure between $E^2 E^2 H_p(\tilde{M}, \tilde{A})$ and $E^2 E^2 H_s(\tilde{M}, \tilde{A}')$ with p+s+2=n, but it turns out that there is not in general any non-singular pairing on these whole modules [In fact, $E^2 E^2 H_p(\tilde{M}, \tilde{A}) \cong E^2 E^2 H_s(\tilde{M}, \tilde{A}')$ as abelian groups in general]. For this reason, we construct (in **6**) a *t*-anti Λ -epimorphism $\theta: E^2 E^2 H_p(\tilde{M}, \tilde{A}) \to E^1 B H_{s+1}(\tilde{M}, \tilde{A}')$ which is invariant under a proper oriented homotopy equivalence $f: (M_1; A_1, A_1') \to (M_2; A_2, A_2')$ with $f^*(\gamma_2) = \gamma_1$. Let $E^2 E^2 H_p(\tilde{M}, \tilde{A}') \to E^1 B H_{s+1}(\tilde{M}, \tilde{A})$.

The Second Duality Theorem. For a compact oriented n-manifold M with $\gamma \in H^1(M)$ and a splitting (A, A') of ∂M and integers p,s with p+s+2=n, there is a pairing

$$l: E^2 E^2 H_{\mathfrak{g}}(\widetilde{M}, \widetilde{A})^{\theta} \times E^2 E^2 H_{\mathfrak{g}}(\widetilde{M}, \widetilde{A}')^{\theta} \to Q/Z$$

such that

(1) (Homotopy invariance) A proper oriented homotopy equivalence $f: (M_1; A_1, A'_1) \rightarrow (M_2; A_2, A'_2)$ with $f^*(\gamma_2) = \gamma_1$ induces the identity $l(\tilde{f}_*(x), \tilde{f}_*(y)) = l(x, y)$ for $x \in E^2 E^2 H_p(\tilde{M}_1, \tilde{A}_1)^{\theta}$ and $y \in E^2 E^2 H_s(\tilde{M}_1, \tilde{A}'_1)^{\theta}$,

(2) *l* is t-isometric, i.e., l(tx, ty) = l(x, y) for $x \in E^2 E^2 H_p(\tilde{M}, \tilde{A})^\theta$ and $y \in E^2 E^2 H_s$ $(\tilde{M}, \tilde{A}')^\theta$,

(3) *l* is $\mathcal{E}(p_s+1)$ -symmetric, i.e., $l(x, y) = \mathcal{E}(p_s+1)l(y, x)$ for $x \in E^2 E^2 H_p(\tilde{M}, \tilde{A})^{\theta}$ and $y \in E^2 E^2 H_s(\tilde{M}, \tilde{A}')^{\theta}$,

(4) *l* is non-singular, i.e., *l* induces a *t*-anti Λ -isomorphism

 $E^{2}E^{2}H_{p}(\tilde{M}, \tilde{A})^{\theta} \cong \operatorname{Hom}_{Z}(E^{2}E^{2}H_{s}(\tilde{M}, \tilde{A}')^{\theta}, Q/Z).$

Since a finitely generated torsion-free Λ -module H is Λ -free if and only if $E^{1}H=0$ (cf. 3), it follows that $BH_{p+1}(\tilde{M}, \tilde{A})$ and $BH_{s+1}(\tilde{M}, \tilde{A}')$ are Λ -free if and and only if l defines a pairing

$$E^{2}E^{2}H_{p}(\tilde{M},\tilde{A})\times E^{2}E^{2}H_{s}(\tilde{M},\tilde{A}') \rightarrow Q/Z$$
.

Hence we see that $tH_p(\tilde{M}, \tilde{A})$, $tH_s(\tilde{M}, \tilde{A}')$ are finite and $BH_{p+1}(\tilde{M}, \tilde{A})$, $BH_{s+1}(\tilde{M}, \tilde{A}')$ are Λ -free if and only if l defines a pairing

$$tH_{\rho}(\tilde{M}, \tilde{A}) \times tH_{s}(\tilde{M}, \tilde{A}') \rightarrow Q/Z$$
,

since a finitely generated Λ -module H has $tH=DH(\cong E^2E^2H)$ if and only if tH is finite. Farber [4] and Levine [L] constructed the same pairing when $tH_p(\tilde{M}, \tilde{A}), tH_s(\tilde{M}, \tilde{A}')$ are finite and $BH_{p+1}(\tilde{M}, \tilde{A})=BH_{s+1}(\tilde{M}, \tilde{A}')=0.^{*)}$ Therefore, our pairing l may be considered as an extreme generalization of their pairing. A basic idea of proving these Duality Theorems is to examine a universal coefficient exact sequence for cohomology over Λ , which has been done by

^{*)} They also assumed that manifolds are piecewise-linear.

Levine [L] in an important special case (cf. Corollary 1.3).

In §1 we construct a universal coefficient exact sequence for chomology over a ring of global dimension ≤ 2 . In §2 we describe the Reidemeister duality on a regular covering of a (topologiacl) manifold. In §3 we note several properties of Λ -modules needed for our purpose. In §4, 5 and 6 we prove the Zeroth, First and Second Duality Theorems, respectively.

1. A universal coefficient exact sequence for chomology over a ring of global dimension ≤ 2

Let Γ be a ring with unit. Let $C = \{C_q, \partial\}$ be a left Γ -projective chain complex and F, a left Γ -module. In general, $H^*(C; F) = H^*(\operatorname{Hom}_{\Gamma}(C, F))$ and $\operatorname{Ext}_{\Gamma}^i(H_*(C), F)$ are abelian groups, but when Γ is commutative, they can be considered as Γ -modules. Let $h: H^*(C; F) \to \operatorname{Hom}_{\Gamma}(H_*(C), F)$ be the homomorphism defined by $h(\{f\})(\{z\}) = f(z)$ for $\{f\} \in H^*(C; F)$ and $\{z\} \in H_*(C)$. Let $K^*(C; F)$ be the kernel of h. We assume that Γ has the left global dimension ≤ 2 . Then $\operatorname{Ext}_{\Gamma}^i(H_*(C), F) = 0$ for $i \geq 3$ and we obtain the following Universal Coefficient Exact Sequence, referred to as UCES:

Theorem 1.1. For all q, there is a natural exact sequence

$$\begin{array}{c} 0 \to K^{q}(C;F) \xrightarrow{\subset} H^{q}(C;F) \xrightarrow{h} \operatorname{Hom}_{\Gamma}(H_{q}(C),F) \xrightarrow{\eta} \operatorname{Ext}^{2}_{\Gamma}(H_{q-1}(C),F) \\ \xrightarrow{\Delta} K^{q+1}(C;F) \xrightarrow{\rho} \operatorname{Ext}^{1}_{\Gamma}(H_{q}(C),F) \to 0 \end{array} .$$

The proof is quite elementary. The following corresponds to the usual universal coefficient theorem:

Corollary 1.2. If $Ext_{\Gamma}^{2}(H_{*}(C), F)=0$, then for all q there is a natural short exact sequence

$$0 \to \operatorname{Ext}^{1}_{\Gamma}(H_{q-1}(C), F) \xrightarrow{\rho^{-1}} H^{q}(C; F) \xrightarrow{h} \operatorname{Hom}_{\Gamma}(H_{q}(C), F) \to 0$$

The following corresponds to the case considered by Levine [L]:

Corollary 1.3. If $Hom_{\Gamma}(H_*(C), F) = 0$, then for all q there is a natural short exact sequence

$$0 \to \operatorname{Ext}^2_{\Gamma}(H_{q-2}(C), F) \xrightarrow{\Delta} H^q(C; F) \xrightarrow{\rho} \operatorname{Ext}^1_{\Gamma}(H_{q-1}(C), F) \to 0.$$

1.4 Proof of Theorem 1.1. For all $q, B_q(C) = \partial C_{q+1}$ has the Γ -projective dimension ≤ 1 , since C_q is Γ -projective and $0 \rightarrow B_q(C) \rightarrow C_q \rightarrow C_q/B_q(C) \rightarrow 0$ is Γ -exact and $C_q/B_q(C)$ has the Γ -projective dimension ≤ 2 . So, $Z_q(C) = \text{Ker}(\partial: C_q \rightarrow C_{q-1})$ is Γ -projective by the short Γ -exact sequence $0 \rightarrow Z_q(C) \rightarrow C_q \xrightarrow{\partial} B_{q-1}(C) \rightarrow 0$. This

sequence also induces an exact sequence

$$0 \to B^{q-1}(C) \to C^q \xrightarrow{j^q} Z^q(C) \to \operatorname{Ext}^1_{\Gamma}(B_{q-1}(C), F) \to 0,$$

where $B^{q-1}(C) = \operatorname{Hom}_{\Gamma}(B_{q-1}(C), F)$, $C^{q} = \operatorname{Hom}_{\Gamma}(C_{q}, F)$ and $Z^{q}(C) = \operatorname{Hom}_{\Gamma}(Z_{q}(C), F)$ and j^{q} is the map induced from the inclusion $Z_{q}(C) \subset C_{q}$. Let $Z_{1}^{q}(C) = \operatorname{Im} j^{q}$. Then we have an exact sequence $0 \to B^{q-1}(C) \to C^{q} \to Z_{1}^{q}(C) \to 0$ and an isomorphism $Z^{q}(C)/Z_{1}^{q}(C) \cong \operatorname{Ext}_{\Gamma}^{1}(B_{q-1}(C), F)$. Regarding $B^{\mathfrak{k}-1}(C) = \{B^{q-1}(C)\}$ and $Z_{1}^{\mathfrak{k}}(C) = \{Z_{1}^{q}(C)\}$ as cochain complexes with trivial coboundary operators, we obtain from the short exact sequence $0 \to B^{\mathfrak{k}-1}(C) \to C^{\mathfrak{k}} \to Z_{1}^{\mathfrak{k}}(C) \to 0$ a long cohomology exact sequence

$$\to H^{q-1}(Z^{\sharp}_{1}(C)) \stackrel{\delta}{\to} H^{q}(B^{\sharp-1}(C)) \to H^{q}(C;F) \to H^{q}(Z^{\sharp}_{1}(C)) \stackrel{\delta}{\to} H^{q+1}(B^{\sharp-1}(C)) \to .$$

Note that the coboundary map $\delta: H^q(Z_1^{\sharp}(C)) \to H^{q+1}(B^{\sharp-1}(C))$ is identical with the restriction $i_1^q: Z_1^q(C) \to B^q(C)$ of the map $i^q: Z^q(C) \to B^q(C)$, induced from the inclusion $B_q(C) \subset Z_q(C)$. We have the following four short exact sequences.

$$0 \to \operatorname{Coker} i_{1}^{q-1} \to H^{q}(C; F) \to \operatorname{Ker} i_{1}^{q} \to 0,$$

$$0 \to \operatorname{Ker} i_{1}^{q} \xrightarrow{\subset} \operatorname{Ker} i^{q} \to \operatorname{Ker} i^{q} / \operatorname{Ker} i_{1}^{q} \to 0,$$

$$0 \to (\operatorname{Ker} i^{q} + Z_{1}^{q}(C)) / Z_{1}^{q}(C) \xrightarrow{\subset} Z^{q}(C) / Z_{1}^{q}(C) \to Z^{q}(C) / (\operatorname{Ker} i^{q} + Z_{1}^{q}(C)) \to 0,$$

$$0 \to \operatorname{Im} i^{q} / \operatorname{Im} i_{1}^{q} \to \operatorname{Coker} i_{1}^{q} \to \operatorname{Coker} i^{q} \to 0.$$

Using the isomorphisms Ker $i^q/\text{Ker } i_1^q \simeq (\text{Ker } i^q + Z_1^q(C))/Z_1^q(C)$ and $Z^q(C)/(\text{Ker } i^q + Z_1^q(C)) \simeq \text{Im } i^q/\text{Im } i_1^q$, we can construct an exact sequence

$$0 \to \operatorname{Coker} i_1^{q-1} \xrightarrow{\alpha_1} H^q(C; F) \xrightarrow{\alpha_2} \operatorname{Ker} i^q \xrightarrow{\alpha_3} Z^q(C) / Z_1^q(C) \xrightarrow{\alpha_4} \operatorname{Coker} i_1^q \xrightarrow{\alpha_5} \operatorname{Coker} i^q \to 0.$$

Since $Z_q(C)$ is Γ -projective, the short exact sequence $0 \to B_q(C) \to Z_q(C) \to H_q(C)$ $\to 0$ induces an isomorphism $\operatorname{Ext}_{\Gamma}^1(B_q(C),F) \cong \operatorname{Ext}_{\Gamma}^2(H_q(C),F)$ and an exact sequence $0 \to \operatorname{Hom}_{\Gamma}(H_q(C),F) \to Z^q(C) \xrightarrow{i^q} B^q(C) \to \operatorname{Ext}_{\Gamma}^1(H_q(C),F) \to 0$, so that $\operatorname{Hom}_{\Gamma}(H_q(C),F) \cong \operatorname{Ker} i^q$ and $\operatorname{Coker} i^q \cong \operatorname{Ext}_{\Gamma}^1(H_q(C),F)$. Note that the composite $H^q(C;F) \xrightarrow{\alpha_2} \operatorname{Ker} i^q \cong \operatorname{Hom}_{\Gamma}(H_q(C),F)$ is given by h. So, α_1 induces an isomorphism $\operatorname{Coker} i_1^{q-1} \cong K^q(C;F)$. Let η be the composite $\operatorname{Hom}_{\Gamma}(H_q(C),F) \cong \operatorname{Ker} i^q \xrightarrow{\alpha_3} Z^q(C)/Z_1^q(C) \cong \operatorname{Ext}_{\Gamma}^1(B_{q-1}(C),F) \cong \operatorname{Ext}_{\Gamma}^2(H_{q-1}(C),F)$ and Δ , the composite $\operatorname{Ext}_{4}^2(H_{q-1}(C),F) \cong \operatorname{Ext}_{\Gamma}^1(B_{q-1}(C),F) \cong Z^q(C)/Z_1^q(C) \xrightarrow{\alpha_4} \operatorname{Coker} i_1^q \cong K^{q+1}(C;F)$ and ρ , the composite $K^{q+1}(C;F)$ $\operatorname{Coker} i_1^p \xrightarrow{\alpha_5} \operatorname{Coker} i^q \cong \operatorname{Ext}_{\Gamma}^1(H_q(C),F)$, where \cong denotes one of the isomorphisms constructed above or its inverse. Then we obtain the exact sequence stated in Theorem 1.1. It is easy to check from construction that a Γ -chain map between left Γ -projective chain complexes induces homomorphisms commuting the resulting two exact sequences. It is similar for a Γ -homomorphism between coefficient left Γ -modules. This completes the proof.

2. The Reidemeister duality on a regular covering of a manifold

Let X be an oriented (possibly, non-compact) *n*-manifold and $\partial_i X$, i=1, 2, be \emptyset or (n-1)-submanifolds of ∂X with $\partial_1 X = cl_{\partial X}(\partial X - \partial_2 X)$ and $\partial_2 X = cl_{\partial X}(\partial X - \partial_1 X)$. By Spanier [Sp, p. 301] the orientation of X determine determines a unique element of $H_n^c(X, \partial X) = \lim_{\epsilon} \{H_n(X, (X-K) \cup \partial X) | K \subset X, \text{ compact}\}$, which we call the *fundamental class* of X and denote by [X]. For integers p, qwith p+q=n the map $\cap [X]: H_c^p(X, \partial_1 X) \to H_q(X, \partial_2 X)$ is well defined by taking the limit of $\cap [X]_K: H^p(X, (X-K) \cup \partial_1 X) \to H_q(X, \partial_2 X)$ for all K, where $[X]_K \in$ $H_n(X, (X-K) \cup \partial X)$ denotes the projection image of [X].

2.1. The Poincaré duality theorem. The map $\cap [X]: H_c^{\flat}(X, \partial_1 X) \rightarrow H_q(X, \partial_2 X)$ is an isomorphism.

This is known (cf., for example, [Ka₃, Appendix A] for an outlined proof). Let (\tilde{M}, \tilde{A}) be a regular covering space over a compact pair (M, A) with covering transformation group G. The singular chain complex $\Delta_{\mathfrak{s}}(\tilde{M}, \tilde{A})$ forms a left ZG-free chain complex. $H_c^*(\tilde{M}, \tilde{A})$ is the cohomology of the complex $\Delta_c^*(\tilde{M}, \tilde{A})$ of all singular cochains with compact supports. Let $H_{ZG}^*(\tilde{M}, \tilde{A})$ be the cohomology of $\Delta_{ZG}^*(\tilde{M}, \tilde{A}) = \operatorname{Hom}_{ZG}(\Delta_{\mathfrak{s}}(\tilde{M}, \tilde{A}), ZG)$. We define a cochain map

$$\phi \colon \Delta^{\sharp}_{c}(\tilde{M},\tilde{A}) \to \Delta^{\sharp}_{ZG}(\tilde{M},\tilde{A})$$

by the identity $\phi(f)(x) = \sum_{g \in G} f(gx)g^{-1}$ for $f \in \Delta^{\sharp}_{c}(\tilde{M}, \tilde{A})$ and $x \in \Delta_{\sharp}(\tilde{M}, \tilde{A})$, where the sum is easily checked to be a finite sum.

Lemma 2.2. If (M, A) is homotopy equivalent to a compact polyhedral pair, then the induced map ϕ^* : $H^*_{\epsilon}(\tilde{M}, \tilde{A}) \rightarrow H^*_{ZG}(\tilde{M}, \tilde{A})$ is an isomorphism.

Proof. Since $H^*_c(\tilde{M}, \tilde{A})$ and $H^*_{Z_G}(\tilde{M}, \tilde{A})$ are proper *G*-homotopy type invariants and ϕ commutes with proper *G*-maps, it suffices to show that ϕ^* is an isomorphism when (M, A) is a compact polyhedral pair. Let (M^*, A^*) be a triangulation of (M, A) and $(\tilde{M}^*, \tilde{A}^*)$ be its lift. For a subcomplex N^* of \tilde{M}^* , let $\Delta_{\sharp}(\tilde{M}^*, \tilde{A}^* \cup N^*)$ (or $C_{\sharp}(\tilde{M}^*, \tilde{A}^* \cup N^*)$, resp.) be the ordered (or oriented, resp.) chain complex. Let $k_1: \Delta_{\sharp}(\tilde{M}^*, \tilde{A}^* \cup N^*) \to \Delta_{\sharp}(\tilde{M}, \tilde{A} \cup N), N = |N^*|$, and $k_2: \Delta_{\sharp}(\tilde{M}^*, \tilde{A}^* \cup N^*) \to C_{\sharp}(\tilde{M}^*, \tilde{A}^* \cup N^*)$ be the natural chain equivalences (cf. [Sp, 4.3.8 and 4.6.8]). Let $\Delta^{\sharp}_{I}(\tilde{M}^*, \tilde{A}^*)$ (or $C^{\sharp}_{I}(\tilde{M}^*, \tilde{A}^*)$, resp.) be the complex of all finite ordered (or oriented, resp.) cochains. Let $\Delta^{\sharp}_{Z_G}(\tilde{M}^*, \tilde{A}^*)$ =Hom_{ZG}($\Delta_{\sharp}(\tilde{M}^*, \tilde{A}^*), ZG$) and $C^{\sharp}_{Z_G}(\tilde{M}^*, \tilde{A}^*)$ =Hom_{ZG}($C_{\sharp}(\tilde{M}^*, \tilde{A}^*), ZG$). We have the following commutative diagram:

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$$\Delta^{\sharp}_{c}(\tilde{M},\tilde{A}) \xrightarrow{k_{1}^{\sharp}} \Delta^{\sharp}_{f}(\tilde{M}^{*},\tilde{A}^{*}) \xleftarrow{k_{2}^{\sharp}} C^{\sharp}_{f}(\tilde{M}^{*},\tilde{A}^{*}) \downarrow \phi \qquad \downarrow \phi_{1} \qquad \downarrow \phi_{2} \Delta^{\sharp}_{ZG}(\tilde{M},\tilde{A}) \xrightarrow{k_{1}^{\sharp}} \Delta^{\sharp}_{ZG}(\tilde{M}^{*},\tilde{A}^{*}) \xleftarrow{k_{2}^{\sharp}} C^{\sharp}_{ZG}(\tilde{M}^{*},\tilde{A}^{*}) ,$$

where ϕ_i are defined by the same rule as ϕ . Note that all of the k_i^{\sharp} 's in this diagram induce isomorphisms in cohomology. In fact, for the upper k_i^{\sharp} , it can be seen by taking the limit of the sequence

$$\Delta^{\sharp}(\tilde{M}, \tilde{A} \cup N) \xrightarrow{k_1^{\sharp}} \Delta^{\sharp}(\tilde{M}^*, \tilde{A}^* \cup N^*) \xleftarrow{k_2^{\sharp}} C^{\sharp}(\tilde{M}^*, \tilde{A}^* \cup N^*)$$

for all cofinite subcomplexes N^* of \tilde{M}^* , and for the lower k_i^* , Eilenberg [E, p. 392] proved it. Since $C_{\mathfrak{s}}(\tilde{M}^*, \tilde{A}^*)$ is ZG-free of finite rank, we see that ϕ_2 is bijective. Hence we have the isomorphism $\phi^*: H^*_{\mathfrak{c}}(\tilde{M}, \tilde{A}) \cong H^*_{\mathbb{Z}G}(\tilde{M}, \tilde{A})$, completing the proof.

2.3. The Reidemeister duality theorem. For a compact oriented *n*-manifold M and a splitting (A, A') of ∂M and integers p, q with p+q=n, there is an isomorphism \tilde{D} : $H_{ZG}^{p}(\tilde{M}, \tilde{A}) \cong H_{q}(\tilde{M}, \tilde{A}')$.

Proof. By [K/S] (M, A) is homotopy equivalent to a compact polyhedral pair. So, by Lemma 2.2 ϕ^* : $H^*_c(\tilde{M}, \tilde{A}) \to H^*_{ZG}(\tilde{M}, \tilde{A})$ is an isomorphism. We take as \tilde{D} the composite $H^*_{ZG}(\tilde{M}, \tilde{A}) \stackrel{\phi^{*-1}}{\simeq} H^*_c(\tilde{M}, \tilde{A}) \stackrel{\cap}{\simeq} H^*_q(\tilde{M}, \tilde{A}')$, where the later denotes the Poincaré duality. This completes the proof.

This duality is due to Reidemeister when M is triangulated (cf. Milnor [Mi₁]). Wall [W] also considered it from a different viewpoint. We can always give $H^*_{Z_G}(\tilde{M}, \tilde{A})$ a left ZG-module structure so that \tilde{D} is a ZG-isomorphism (cf. [Mi₁]), but in this paper we never use it to avoid making a confusion. When G is abelian, $H^*_c(\tilde{M}, \tilde{A})$ and $H^*_{Z_G}(\tilde{M}, \tilde{A})$ form ZG-modules by the action of G, so that ϕ^* is a ZG-isomorphism and \tilde{D} is a g-anti map, i.e., $g^{-1}\tilde{D}=\tilde{D}g$, for all $g \in G$. Here we used the identity $g[\tilde{M}]=[\tilde{M}]$. The following chain level version of this identity is used in 5 and 6:

Lemma 2.4. For a splitting (A, A') of ∂M , there is a cycle \tilde{z} in $\Delta_n^c(\tilde{M}; \tilde{A}, \tilde{A}')$ = $\lim_{\leftarrow} \{\Delta_n(\tilde{M})/(\Delta_n(\tilde{M}-K)+\Delta_n(\tilde{A})+\Delta_n(\tilde{A}'))|K\subset \tilde{M}, \text{ compact}\}$ representing $[\tilde{M}] \in H_n^c(\tilde{M}, \partial \tilde{M})$ such that $g\tilde{z}=\tilde{z}$ for all $g\in G$.

Proof. Let $z = \sum_{i=1}^{m} n_i \sigma_i \in \Delta_n(M)$ represent an element of $H_n(\Delta_{\mathfrak{s}}(M)/(\Delta_{\mathfrak{s}}(A) + \Delta_{\mathfrak{s}}(A')))$ coresponding to $[M] \in H_n^c(M, \partial M)$ under the natural isomorphisms $H_n(\Delta_{\mathfrak{s}}(M)/(\Delta_{\mathfrak{s}}(A) + \Delta_{\mathfrak{s}}(A'))) \simeq H_n(M, \partial M) \simeq H_n^c(M, \partial M)$ (cf. [Sp, 6.3.7]). Let $\sigma_{i,j}$, $j \in J$, be the lifts of the singular *n*-simplex σ_i to \tilde{M} . For any compact $K \subset \tilde{M}$, $\sigma_{i,j}$ are in $\Delta_n(\tilde{M}-K)$ except a finite number of *j* and we let $\tilde{z}_K = \sum_{i=1}^{m} n_i \sum_{j \in J} \sigma_{i,j} \in \Delta_n(\tilde{M})/(\Delta_n(\tilde{M}-K) + \Delta_n(\tilde{A}) + \Delta_n(\tilde{A}'))$. Then we see that \tilde{z}_K is a cycle and

 $\{\tilde{z}_{\kappa}\}_{\kappa}$ determines a cycle \tilde{z} in $\Delta_{n}^{c}(\tilde{M};\tilde{A},\tilde{A}')$ with $g\tilde{z}=\tilde{z}$ for all $g\in G$. Take $\tilde{x}\in \tilde{V}\subset \tilde{M}$ so that \tilde{V} is an open ball and the projection $\tilde{M}\to M$ sends (\tilde{V},\tilde{x}) to a pair (V,x) homeomorphically. For any cycle $z'_{x}=\sum_{i=1}^{m'}n'_{i}\sigma'_{i}\in\Delta_{n}(M,M-x)$ with $\{z'_{x}\}=\{z\}$ in $H_{n}(M,M-x)$, let $\tilde{z}'_{x}=\sum_{i=1}^{m'}n'_{i}\sum_{j\in J}\sigma'_{i,j}\in\Delta_{n}(\tilde{M},\tilde{M}-\tilde{x})$. Then \tilde{z}'_{x} is a well-defined cycle with $\{\tilde{z}'_{x}\}=\{\tilde{z}_{x}\}$ in $H_{n}(\tilde{M},\tilde{M}-\tilde{x})$. Let z'_{x} be in $\Delta_{n}(V,V-x)$. Since $\tilde{V}\cap g\,\tilde{V}=\emptyset$ for $g\neq 1$, we see from the isomorphisms

$$H_n(\tilde{M}, \tilde{M} - \tilde{x}) \stackrel{\simeq}{\leftarrow} H_n(\tilde{V}, \tilde{V} - \tilde{x}) \stackrel{\simeq}{\to} H_n(V, V - x) \stackrel{\simeq}{\to} H_n(M, M - x)$$

that $\tilde{z}'_{\tilde{x}}$ represents $[\tilde{M}]_{\tilde{x}}$ so that \tilde{z} represents $[\tilde{M}]$ (cf. [Sp, 6.3.3]). This completes the proof.

3. Several properties of Λ -modules

Let $\Lambda_0 = \Lambda \otimes_Z Q$ and for the field Z_p of prime order p, $\Lambda_p = \Lambda \otimes_Z Z_p$. For any finitely generated Λ -module H, note that E^2H is Z-torsional and E^1H is Λ torsional, since $E^2H \otimes_Z Q = E^1H \otimes_{\Lambda} Q(\Lambda) = 0$. Let $H^{(p)} = \{x \in H \mid px = 0\}$. $H^{(p)}$ is a Λ_p -module.

Lemma 3.1. $\Lambda/(m, \lambda_1, \dots, \lambda_r)$ is a finite Λ -module for coprime non-zero m, $\lambda_1, \dots, \lambda_r \in \Lambda$ $(r \ge 1)$ with m an integer.

Proof. Let $m = \pm p_1 p_2 \cdots p_s$ be a prime decomposition. $\Lambda/(p_s, \lambda_1, \cdots, \lambda_r) = \Lambda_{p_s}/(\lambda_1, \cdots, \lambda_r)$ is finite. Since

 $\Lambda/(p_1\cdots p_{s-1}, \lambda_1, \cdots, \lambda_r) \xrightarrow{\dot{P}_s} \Lambda/(m, \lambda_1, \cdots, \lambda_r) \to \Lambda/(p_s, \lambda_1, \cdots, \lambda_r)$

is exact, the induction on s shows that $\Lambda/(m, \lambda_1, \dots, \lambda_r)$ is finite, completing the proof.

Corollary 3.2. A finitely generated Λ -module H has $mH = (t^{m'} - 1)H = 0$ for some non-zero integers m, m' if and only if H is finite

Proof. The "if" part is easy. The "only if" part follows from Lemma 3 1, since H is a quotient of a direct sum of finite copies of $\Lambda/(m, t^{m'}-1)$. This completes the proof.

Corollary 3.3. For any Λ -module H, DH is the smallest Λ -submodule of H containing all finite Λ -submodules. Further, if H is finitely generated over Λ , then DH is finite.

Proof. By Corollary 3.2 *DH* contains all finite Λ -submodules. For $x \in DH$ let $\lambda_1, \dots, \lambda_r \in \Lambda$ $(r \geq 2)$ be non-zero coprime elements with $\lambda_i x = 0$ for all *i*. Since Λ_0 is PID, there are $\lambda'_1, \dots, \lambda'_r \in \Lambda$ and non-zero $m \in Z$ such that $\lambda_1 \lambda'_1 + \dots + \lambda_r \lambda'_r = m$. Then mx = 0 and x is in the image of a Λ -homomorphism $\Lambda/(m, \lambda_1, \dots, M)$

 $\lambda_r \rightarrow H$. Since $m, \lambda_1, \dots, \lambda_r$ are coprime, we see from Lemma 3.1 that x is in a finite Λ -submodule of H, showing the first half. If H is finitely generated over Λ , so is DH. Then DH is a quotient of a direct sum of a finite number of finite Λ -modules and hence is finite. This completes the proof.

Lemma 3.4. For a finitely generated Λ -module H, E^2H is finite and there are natural isomorphisms $E^2H \cong E^2DH$ and $DH \cong E^2E^2H$.

Proof. Since E^2bH is Z-torsional and finitely generated over Λ , there is an integer $m \neq 0$ with $mE^2bH = 0$. By the short exact sequence $0 \rightarrow bH \xrightarrow{m} bH \rightarrow$ $bH/mbH \rightarrow 0$, we have $E^2bH = mE^2bH = 0$. So, $E^2H \simeq E^2tH$ by the short exact sequence $0 \rightarrow tH \rightarrow H \rightarrow bH \rightarrow 0$. Let H_p be the *p*-component of tH. We show that E^2H_p is finite by induction on $n \ge 0$ with $p^nH_p=0$. The short exact sequence $0 \rightarrow pH_p \xrightarrow{\subset} H_p \rightarrow H_p/pH_p \rightarrow 0$ induces an exact sequence $E^2(H_p/pH_p) \rightarrow E^2H_p$ $\rightarrow E^2(pH_p)$. H_p/pH_p is a finitely generated Λ_p -module and splits into a free Λ_p module and a torsion (i.e., finite) Λ_p -module T_p , so that $E^2(H_p/pH_p) \simeq E^2 T_p$ is finite (by Corollary 3.2). By the inductive hypothesis, $E^2(pH_p)$ is finite. Hence E^2H_p is finite. Since tH is finitely generated over Λ , $H_p=0$ except a finite number of p. Therefore, $E^2H \simeq E^2tH \simeq \bigoplus_p E^2H_p$ is finite. Next, let H' = tH/DH. Take an integer $m' \neq 0$ with $(t^{m'}-1)E^2H'=0$. Since $0 \rightarrow H' \xrightarrow{t^{m'}-1} H' \rightarrow H'/(t^{m'}-1)$ $H' \rightarrow 0$ is exact, $t^{m'} - 1: E^2 H' \rightarrow E^2 H'$ is onto, so that $E^2 H' = 0$ and $E^2 t H \simeq E^2 D H$. Thus, $E^2H \simeq E^2tH \simeq E^2DH$. Since DH is finite and $E^1DH \simeq \operatorname{Hom}_{\Lambda}(DH, Q(\Lambda)/\Lambda)$ =0, we see from [L, (3.6)] that $DH \simeq E^2 E^2 DH$. Using $E^2 H \simeq E^2 DH$, we complete the proof.

Lemma 3.5. For a finitely generated Λ -module H, there are a natural short exact sequence $0 \rightarrow E^1BH \rightarrow E^1H \rightarrow E^1(TH/DH) \rightarrow 0$ and natural isomorphisms $E^1BH \simeq DE^1H$ and $TH/DH \simeq E^1E^1H$.

Proof. By Lemma 3.4, $E^2BH=0$. The short exact sequence $0 \to TH \to H$ $\to BH \to 0$ induces an exact sequence $(S) \ 0 \to E^1BH \to E^1H \to E^1TH \to 0$. Since $E^0DH=E^1DH=0$, $E^1(TH/DH)\cong E^1TH$. Combining it with (S), we obtain a desired sequence. Directly, $DE^1TH\cong D \operatorname{Hom}_{\Lambda}(TH, Q(\Lambda)/\Lambda)=0$. By (S), $DE^1BH\cong DE^1H$. For a free Λ -module F of finite rank containing BH (cf. Cartan/Eilenberg [C/E, p. 131]), we have $E^1BH\cong E^2(F/BH)$. By Lemma 3.4, E^1BH is finite and $E^1BH=DE^1BH\cong DE^1H$. Then $E^1E^1TH\cong E^1E^1H$ by (S). Since $E^2(TH/DH)=0$ by Lemma 3.4, TH/DH has the projective dimension ≤ 1 by [L, (3.5)]. By (L, (3.6)], we have $TH/DH\cong E^1E^1(TH/DH)$. Since $E^1(TH/DH)$

Lemma 3.6. For a finitely generated Λ -module H, $E^{\circ}H$ is Λ -free and there is a natural exact sequence $0 \rightarrow BH \rightarrow E^{\circ}E^{\circ}H \rightarrow E^{2}E^{1}BH \rightarrow 0$.

Proof. Since $E^0BH = E^0H$, we may assume that H = BH. Then H has the projective dimension ≤ 1 , for there is a Λ -free module F containing H and F/H has the projective dimension ≤ 2 . A Λ -projective (i.e., Λ -free by [Se]) resolution $0 \rightarrow F^1 \rightarrow F^0 \rightarrow H \rightarrow 0$ of H with F^i of finite rank induces an exact sequence (S^*) $0 \rightarrow E^0H \rightarrow E^0F^0 \rightarrow E^0F^1 \rightarrow E^1H \rightarrow 0$. Since E^1H has the projective dimension ≤ 2 and E^0F^i are Λ -free, E^0H is Λ -projective that is Λ -free by [Se]. By Lemma 3.5, $E^1E^1H=0$. Then (S^*) induces an exact sequence $0 \rightarrow E^0E^0F^1 \rightarrow E^0E^0F^0 \rightarrow E^0E^0H$ $\rightarrow E^2E^1H \rightarrow 0$. Using $F^i \simeq E^0E^0F^i$ and the natural injection $H \rightarrow E^0E^0H$, we obtain a natural short exact sequence $0 \rightarrow H \rightarrow E^0E^0H \rightarrow E^2E^1H \rightarrow 0$. This completes the proof.

The following is obtained from Lemmas 3.4, 3.5 and 3.6:

Corollary 3.7. A finitely generated Λ -module H is Λ -free if and only if $E^{1}H = E^{2}H = 0$.

Corollary 3.8. The following conditions on a finitely generated Λ -module H are equivalent:

- (1) $E^{2}H=0$,
- (2) DH=0,
- (3) $H^{(p)}$ is Λ_p -free for all prime p,
- (4) *H* has the projective dimension ≤ 1 .

Proof. Take a short exact sequence $0 \rightarrow H' \rightarrow F \rightarrow H \rightarrow 0$ with F, Λ -free of finite rank. Assuming (1), $E^1H' \simeq E^2H = 0$. By Lemma 3.6, $H' \simeq E^0E^0H'$ is Λ -free, showing (1) \Rightarrow (4). The others are trivial or follow from Lemma 3.4. This completes the proof.

Corollary 3.8 generalizes [L, (3.5)] and implies that a self-reciprocal Λ -module in [Ka₂] has the Λ -projective dimension ≤ 1 . The following observation is originally due to Kervaire [Ke] (when $\lambda = t-1$):

Corollary 3.9. Let $\lambda \in \Lambda$ be no unit in Λ_p for all prime p. If a finitely generated Λ -module H has $\lambda H = H$, then $\lambda : H \cong H$, H = TH and tH is finite.

Proof. The Noetherian property gives $\lambda: H \cong H$ (cf. Shinohara/Sumners [S/S]). E^0H is Λ -free by Lemma 3.6 and $\lambda: E^0H \cong E^0H$, meaning that $E^0H = 0$, i.e., H = TH. If $tH/DH \neq 0$, then there is a prime p with $(tH/DH)^{(p)} \neq 0$. $(tH/DH)^{(p)}$ is Λ_p -free by Corollary 3.8 and $\lambda: (tH/DH)^{(p)} \cong (tH/DH)^{(p)}$, meaning that $(tH/DH)^{(p)} = 0$, a contradiction. Hence tH = DH, which is finite by Corollary 3.3. This completes the proof.

4. Proof of the Zeroth Duality Theorem

For a Λ -projective chain complex C with $H_*(C)$ finitely generated over Λ ,

we see from UCES that $TH^*(C; \Lambda) = K^*(C; \Lambda)$ and $h: H^*(C; \Lambda) \to E^0H_*(C)$ induces a monomorphism $BH^*(C; \Lambda) \to E^0H_*(C)$, also denoted by h. We now return to **0** where M is a compact oriented *n*-manifold and (A, A') is a splitting of ∂M and $(\tilde{M}; \tilde{A}, \tilde{A}')$ is an infinite cyclic covering of (M; A, A'), associated with $\gamma \in H^1(M)$. We denote by $\mathcal{E}_{\tilde{M}}$ the augmentation map $H_0(\tilde{M}; G) \to G$ for any (untwisted) coefficient group G.

For integers p, q with p+q=n, the Z-intersection pairing

Int:
$$H_p(\tilde{M}, \tilde{A}) \times H_q(\tilde{M}, \tilde{A}') \to Z$$

is given by the identity $\operatorname{Int}(x, y) = \mathcal{E}_{\widetilde{M}}((u \cup v) \cap [\widetilde{M}]) = \mathcal{E}_{\widetilde{M}}(u \cap y)$ for $x \in H_p(\widetilde{M}, \widetilde{A})$, $y \in H_q(\widetilde{M}, \widetilde{A}'), u \in H^q_c(\widetilde{M}, \widetilde{A}'), v \in H^p_c(\widetilde{M}, \widetilde{A})$ with $x = u \cap [\widetilde{M}], y = v \cap [\widetilde{M}]$ (cf. [Ka₃, Appendix A]). Then the Λ -intersection pairing

$$\widetilde{\mathrm{Int}}: H_{p}(\widetilde{M}, \widetilde{A}) \times H_{a}(\widetilde{M}, \widetilde{A}') \to \Lambda$$

is given by the identity $\operatorname{Int}(x, y) = \sum_{i=-\infty}^{+\infty} \operatorname{Int}(x, t^i y) t^{-i}$. By Λ -sesquilinearity of Int, Int induces a pairing

$$\operatorname{Int}_{B}: BH_{p}(\widetilde{M}, \widetilde{A}) \times BH_{q}(\widetilde{M}, \widetilde{A}') \to \Lambda .$$

Let β be the composite *t*-anti Λ -homomorphism

$$H_p(\tilde{M}, \tilde{A}) \stackrel{\tilde{D}^{-1}}{\simeq} H^q_{\Lambda}(\tilde{M}, \tilde{A}') \stackrel{h}{\to} E^0 H_q(\tilde{M}, \tilde{A}') ,$$

where \tilde{D} denotes the Reidemeister duality in 2.

Lemma 4.1. For $x \in H_p(\tilde{M}, \tilde{A})$ and $y \in H_q(\tilde{M}, \tilde{A}')$. we have $\beta(x)(y) = \tilde{I}$ nt (x, y).

Proof. For
$$u_x = \{f_x\} \in H^q_c(\tilde{M}, \tilde{A}')$$
 with $x = u_x \cap [\tilde{M}]$ and $y = \{c_y\}, \beta(x)(y) = \phi(f_x)(c_y) = \sum_{i=-\infty}^{+\infty} f_x(t^i c_y)t^{-i} = \sum_{i=-\infty}^{+\infty} \mathcal{E}_{\tilde{M}}(u_x \cap t^i y)t^{-i} = \tilde{I}nt(x, y)$, as desired.

4.2 Proof of the Zeroth Duality Theorem. Let β_B be the composite *t*-anti Λ -monomorphism

$$BH_p(\tilde{M}, \tilde{A}) \stackrel{\tilde{D}^{-1}}{\simeq} BH^q_{\Lambda}(\tilde{M}, \tilde{A}') \stackrel{h}{\rightarrow} E^0H_q(\tilde{M}, \tilde{A}') = E^0BH_q(\tilde{M}, \tilde{A}')$$

induced from β . By UCES and Lemma 3.4, the cokernel of β_B is a finite Λ -module. By Lemma 3.5, β_B induces a *t*-anti Λ -isomorphism $\beta_B^* \colon E^0 E^0 BH_q(\tilde{M}, \tilde{A}') \cong E^0 BH_p(\tilde{M}, \tilde{A})$ and hence a *t*-anti Λ -isomorphism $\beta_B^* \colon E^0 E^0 BH_p(\tilde{M}, \tilde{A}) \cong E^0 E^0 E^0 BH_q(\tilde{M}, \tilde{A}')$. Regard $BH_p(\tilde{M}, \tilde{A}) \subset E^0 E^0 BH_p(\tilde{M}, \tilde{A})$ and $BH_q(\tilde{M}, \tilde{A}') \subset E^0 E^0 BH_q(\tilde{M}, \tilde{A}')$ in a natural way. We can see from Lemmas 3.4, 3.5 and 3.6 that $\beta_B^* \colon |BH_p(\tilde{M}, \tilde{A}) = \beta_B$ under the identification $E^0 E^0 E^0 BH_q(\tilde{M}, \tilde{A}) = E^0 BH_q(\tilde{M}, \tilde{A})$. We define a pairing

$$S: E^{0}E^{0}H_{\mathfrak{p}}(\tilde{M},\tilde{A}) \times E^{0}E^{0}H_{\mathfrak{q}}(\tilde{M},\tilde{A}') = E^{0}E^{0}BH_{\mathfrak{p}}(\tilde{M},\tilde{A}) \times E^{0}E^{0}BH_{\mathfrak{q}}(\tilde{M},\tilde{A}') \to \Lambda$$

by $S(x, y) = \beta_{k}^{*}(x)(y)$. By Lemma 4.1, S is an extension of the pairing \tilde{Int}_{B} : $BH_{p}(\tilde{M}, \tilde{A}) \times BH_{q}(\tilde{M}, \tilde{A}') \rightarrow \Lambda$. From construction, (2) and (4) are satisfied. To see (1), let $f: (M_{1}; A_{1}, A_{1}') \rightarrow (M_{2}; A_{2}, A_{2}')$ be a proper oriented homotopy equivalence with $f^{*}(\gamma_{2}) = \gamma_{1}$. The lift $\tilde{f}: (\tilde{M}_{1}; \tilde{A}_{1}, \tilde{A}_{1}') \rightarrow (\tilde{M}_{2}; \tilde{A}_{2}, \tilde{A}_{2}')$ induces Λ isomorphisms $\tilde{f}_{*}: (E^{0}E^{0}BH_{p}(\tilde{M}_{1}, \tilde{A}_{1}), BH_{p}(\tilde{M}_{1}, \tilde{A}_{1})) \simeq (E^{0}E^{0}BH_{p}(\tilde{M}_{2}, \tilde{A}_{2}), BH_{p}(\tilde{M}_{2}, \tilde{A}_{2}))$ and $\tilde{f}_{*}: (E^{0}E^{0}BH_{q}(\tilde{M}_{1}, \tilde{A}_{1}'), BH_{q}(\tilde{M}_{1}, \tilde{A}_{1}')) \simeq (E^{0}E^{0}BH_{q}(\tilde{M}_{2}, \tilde{A}_{2}'), BH_{q}(\tilde{M}_{2}, \tilde{A}_{2}'))$. For $x \in E^{0}E^{0}BH_{p}(\tilde{M}_{1}, \tilde{A}_{1}), y \in E^{0}E^{0}BH_{q}(\tilde{M}_{1}, \tilde{A}_{1}')$, there are non-zero integers m, m'such that $mx = x' \in BH_{p}(\tilde{M}_{1}, \tilde{A}_{1}), m'y = y' \in BH_{q}(\tilde{M}_{1}, \tilde{A}_{1}')$, by Lemmas 3.4 and 3.6. Since $\tilde{f}_{*}[\tilde{M}_{1}] = [\tilde{M}_{2}]$, it is easily proved that $\tilde{Int}_{B}(\tilde{f}_{*}(x'), \tilde{f}_{*}(y')) = \tilde{Int}_{B}(x', y')$. Then $mm'S(\tilde{f}_{*}(x), \tilde{f}_{*}(y)) = S(\tilde{f}_{*}(x'), \tilde{f}_{*}(y')) = Int_{b}(\tilde{f}_{*}(x'), \tilde{f}_{*}(y')) = \tilde{Int}_{B}(x', y') = S(x', y') = mm'S(x, y)$. That is, $S(\tilde{f}_{*}(x), \tilde{f}_{*}(y)) = S(x, y)$, showing (1). To see (3), let $x \in E^{0}E^{0}BH_{p}(\tilde{M}, \tilde{A})$ and $y \in E^{0}E^{0}BH_{q}(\tilde{M}, \tilde{A}')$. For $x' = mx \in BH_{p}(\tilde{M}, \tilde{A})$ $\tilde{A})$ and $y' = m'y \in BH_{q}(\tilde{M}, \tilde{A}')$ with $mm' \neq 0$, we have $mm'S(x, y) = S(x', y') = \tilde{Int}_{B}(x', y') = \tilde{Int}_{B}(x', y') = \tilde{E}(pq)\overline{S(y, x)}$, i.e., $S(x, y) = \mathcal{E}(pq)\overline{S(y, x)}$. This completes the proof.

5. Proof of the First Duality Theorem

For a Λ -module H, we have a Λ -exact sequence $\operatorname{Hom}_{\Lambda}(H, Q(\Lambda))$ $\stackrel{\nu_{\sharp}}{\to} \operatorname{Hom}_{\Lambda}(H, Q(\Lambda)/\Lambda) \to E^{1}H \to 0$ induced from the short exact sequence $0 \to \Lambda$ $\to Q(\Lambda) \stackrel{\nu}{\to} Q(\Lambda)/\Lambda \to 0$, by which we identify $E^{1}H$ with the cokernel of ν_{\sharp} . Let C be a projective Λ -chain complex with $H_{\ast}(C)$ finitely generated over Λ . For $u = \{f_u\} \in TH^{q+1}(C; \Lambda)$ we have a non-zero $\lambda \in \Lambda$ and a cochain $f_u^+: C_q \to \Lambda$ such that $\lambda f_u = \delta(f_u^+)$. Letting $\rho'(u)(c) = f_u^+(c)/\lambda \in Q(\Lambda)/\Lambda$ for $c \in Z_q(C)$, we obtain a well-defined Λ -homomorphism $\rho': TH^{q+1}(C; \Lambda) \to E^1H_q(C)$.

Lemma 5.1. For the map $\rho: TH^{q+1}(C; \Lambda) \to E^1H_q(C)$ appearing in UCES, there is a natural Λ -isomorphism $\rho'': E^1H_q(C) \cong E^1H_q(C)$ such that $\rho = \rho''\rho'$.

Proof. Recall that ρ is the composite

$$TH^{q+1}(C;\Lambda) = K^{q+1}(C;\Lambda) \stackrel{\rho_1}{\cong} \operatorname{Coker} i_1^q \stackrel{\alpha_5}{\to} \operatorname{Coker} i^q \stackrel{\rho_2}{\cong} E^1H_q(C).$$

For $u = \{f_u\} \in TH^{q+1}(C; \Lambda)$, we have $f_u(Z_{q+1}(C)) = 0$ and hence a map $f_u^B : B_q(C)$ $\stackrel{\partial^{-1}}{\cong} C_{q+1}/Z_{q+1}(C) \stackrel{f_u}{\to} \Lambda$. Then note that $\rho_1(u) = \{f_u^B\} \in \text{Coker } i_1^q$. The map α_5 is an obvious surjection. We shall construct a natural Λ -isomorphism ρ'_2 : Coker $i^q \cong E^1H_q(C)$. For $f^B \in B^q(C)$ we have a non-zero $\lambda \in \Lambda$ and $f^Z \in Z^q(C)$ such that $f^Z | B_q(C) = \lambda f^B[\text{Note that Coker } i^q \cong E^1H_q(C) \text{ is } \Lambda\text{-torsional}].$ Letting $\rho'_2(f^B)(c) = f^Z(c)/\lambda \in Q(\Lambda)/\Lambda$ for $c \in Z_q(C)$, we obtain a well-defined Λ -homomorphism ρ'_2 : Coker $i^q \to E^1H_q(C)$. The naturality of ρ'_2 is clear. Given a Λ -homomorphism $f: H_q(C) \to Q(\Lambda)/\Lambda$, we have a Λ -homomorphism $\tilde{f}: Z_q(C) \to Q(\Lambda)$ inducing f, because $Z_q(C)$ is Λ -projective. Then $\tilde{f}(B_q(C)) \subset \Lambda$ and we can see that the correspondence $\{f\} \in E^1H_q(C) \to \{\tilde{f} \mid B_q(C)\} \in \text{Coker } i^q$ is the well-defined inverse of ρ'_2 . So, ρ'_2 is a natural Λ -isomorphism. The identity $\rho'_2\alpha_5\rho_1=\rho'$ is easily checked. Letting $\rho''=\rho_2\rho'_2^{-1}$, we obtain the identity $\rho=\rho''\rho'$, completing the proof.

5.2 Proof of the First Duality Theorem. By UCES and Lemma 5.1, ρ' induces a Λ -isomorphism $TH_{\Lambda}^{r+1}(\tilde{M}, \tilde{A}')/DH_{\Lambda}^{r+1}(\tilde{M}, \tilde{A}') \cong E^{1}H_{r}(\tilde{M}, \tilde{A}')/DE^{1}H_{r}(\tilde{M}, \tilde{A}')$, also denoted by ρ' . By Lemma 3.5, the latter is identical with $E^{1}(TH_{r}(\tilde{M}, \tilde{A}')/DH_{r}(\tilde{M}, \tilde{A}')) = \text{Hom}_{\Lambda}(TH_{r}(\tilde{M}, \tilde{A}')/DH_{r}(\tilde{M}, \tilde{A}'), Q(\Lambda)/\Lambda) = \text{Hom}_{\Lambda}(E^{1}E^{1}H_{r}(\tilde{M}, \tilde{A}'), Q(\Lambda)/\Lambda)$. By the Reidemeister duality, we have a *t*-anti Λ -isomorphism \tilde{D} : $TH_{\Lambda}^{r+1}(\tilde{M}, \tilde{A}')/DH_{\Lambda}^{r+1}(\tilde{M}, \tilde{A}') \cong TH_{p}(\tilde{M}, \tilde{A})/DH_{p}(\tilde{M}, \tilde{A}) = E^{1}E^{1}H_{p}(\tilde{M}, \tilde{A})$. Then we define a pairing

$$L: E^{1}E^{1}H_{p}(\tilde{M}, \tilde{A}) \times E^{1}E^{1}H_{r}(\tilde{M}, \tilde{A}')$$

= $TH_{p}(\tilde{M}, \tilde{A})/DH_{p}(\tilde{M}, \tilde{A}) \times TH_{r}(\tilde{M}, \tilde{A}')/DH_{r}(\tilde{M}, \tilde{A}') \rightarrow Q(\Lambda)/\Lambda$

by $L(x,y) = \mathcal{E}(p+1)\rho'\tilde{D}^{-1}(x)(y)$. By construction, (2) and (4) are satisfied. To see (1), let $f: (M_1; A_1, A'_1) \to (M_2; A_2, A'_2)$ be a proper oriented homotopy equivalence with $f^*(\gamma_2) = \gamma_1$. The lift \tilde{f} induces the following commutative diagram (Use $\tilde{f}_*[\tilde{M}_1] = [\tilde{M}_2]$ for the left square):

$$\begin{split} E^{1}E^{1}H_{p}(\tilde{M}_{1},\tilde{A}_{1}) & \stackrel{D}{\leftarrow} TH_{\Lambda}^{u+1}(\tilde{M}_{1},\tilde{A}_{1}')/DH_{\Lambda}^{r+1}(\tilde{M}_{1},\tilde{A}_{1}') \stackrel{\rho'}{\to} \operatorname{Hom}_{\Lambda}(E^{1}E^{1}H_{r}(\tilde{M}_{1},\tilde{A}_{1}'),Q(\Lambda)/\Lambda) \\ & \simeq \downarrow \tilde{f}_{*} \qquad \simeq \uparrow \tilde{f}^{*} \qquad \simeq \uparrow \tilde{f}^{*} \\ E^{1}E^{1}H_{p}(\tilde{M}_{2},\tilde{A}_{2}) \stackrel{\rho'}{\leftarrow} TH_{\Lambda}^{r+1}(\tilde{M}_{2},\tilde{A}_{2}')/DH_{\Lambda}^{r+1}(\tilde{M}_{2},\tilde{A}_{2}') \stackrel{\rho'}{\to} \operatorname{Hom}_{\Lambda}(E^{1}E^{1}H_{r}(\tilde{M}_{2},\tilde{A}_{2}'),Q(\Lambda)/\Lambda). \end{split}$$

This means $L(\tilde{f}_*(x), \tilde{f}_*(y)) = L(x, y)$, showing (1). To see (3), let $x = \{c_x\} \in TH_p(\tilde{M}, \tilde{A}), y = \{c_y\} \in TH_r(\tilde{M}, \tilde{A}'), u_x = \{f_x\} \in TH_c^{r+1}(\tilde{M}, \tilde{A}')$ and $u_y = \{f_y\} \in TH_c^{p+1}(\tilde{M}, \tilde{A})$ with $u_x \cap [\tilde{M}] = x$ and $u_y \cap [\tilde{M}] = y$. Then there are non-zero λ_x , $\lambda_y \in \Lambda$ and $c_x^+ \in \Delta_{p+1}(\tilde{M}, \tilde{A})$ and $c_y^+ \in \Delta_{r+1}(\tilde{M}, \tilde{A}')$ such that $\partial c_x^+ = \lambda_x c_x$ and $\partial c_y^+ = \lambda_y c_y$. Since $\lambda_x u_x = \lambda_y u_y = 0$, there are $f_x^+ \in \Delta_c^r(\tilde{M}, \tilde{A}')$ and $f_y^+ \in \Delta_c^p(\tilde{M}, \tilde{A})$ such that $\partial (f_x^+) = \lambda_x f_x$ and $\delta (f_y^+) = \lambda_y f_y$. By definition,

$$L(x,y) = \mathcal{E}(t+1) \sum_{i=-\infty}^{+\infty} f_x^+(t^i c_y) t^{-i} / \overline{\lambda}_x \pmod{\Lambda}.$$

Assertion 5.3. $L(x,y) = \mathcal{E}((p+1)r) \sum_{i=-\infty}^{+\infty} f_y^+(t^{-i}c_x)t^{-i}/\lambda_y \pmod{\Lambda}$.

From this, we have $L(x,y) = \mathcal{E}(pr+1)L(y,x)$, showing (3), since $L(y,x) = \mathcal{E}(r+1)$ $\sum_{i=-\infty}^{i=+\infty} f_y^+(t^i c_x) t^{-i} / \lambda_y \pmod{\Lambda}$. This completes the proof of the First Duality Theorem, except for the proof of Assertion 5.3.

5.4. Proof of Assertion 5.3. By Lemma 2.4, we have a *t*-invariant cycle $\tilde{z} \in \Delta_{n}^{c}(\tilde{M}; \tilde{A}, \tilde{A}')$ representing $[\tilde{M}]$. The map $\cap \tilde{z} \colon \Delta_{c}^{k}(\tilde{M}, \tilde{A}) \to \Delta_{n-k}(\tilde{M}, \tilde{A}'), \cap \tilde{z} \colon$

 $\Delta_c^k(\tilde{M}, \tilde{A}') \to \Delta_{n-k}(\tilde{M}, \tilde{A})$ or $\cap \tilde{z}: \Delta_c^k(\tilde{M}; \tilde{A}, \tilde{A}') \to \Lambda_{n-k}(\tilde{M})$ is defined to be the limit (on K) of the cap product map $\cap \tilde{z}_{\kappa} : \Delta^k(\tilde{M}, (\tilde{M}-K) \cup \tilde{A}) \to \Delta_{n-k}(\tilde{M}, \tilde{A}'), \cap \tilde{z}_{\kappa} :$ $\Delta^{k}(\tilde{M}, (\tilde{M}-K) \cup \tilde{A}') \to \Delta_{n-k}(\tilde{M}, \tilde{A}) \text{ or } \cap \tilde{z}_{\kappa} \colon \operatorname{Hom}_{Z}(\Lambda_{k}(\tilde{M})/(\Delta_{k}(\tilde{M}-K)+\Delta_{\kappa}(\tilde{A})))$ $+\Delta_{k}(\tilde{A}'), Z) \rightarrow \Delta_{n-k}(\tilde{M})$ with respect to the Alexander/Whitney diagonal approximation, respectively. Assume that $f_x \cap \tilde{z} = c_x$ and $f_y \cap \tilde{z} = c_y$. Let $T: \tilde{M} \times \tilde{M}$ $\rightarrow \tilde{M} \times \tilde{M}$ be the map changing the factors and $T' : \Delta_{\mathfrak{s}}(\tilde{M}) \otimes \Delta_{\mathfrak{s}}(\tilde{M}) \rightarrow \Delta_{\mathfrak{s}}(\tilde{M}) \otimes \Delta_{\mathfrak{s}}(\tilde{M})$ be the chain map defined by $T'(c^p \otimes c^q) = \mathcal{E}(pq)c^q \otimes c^p$. Let $\tau: \Delta_{\sharp}(\tilde{M} \times \tilde{M}) \to \Delta_{\sharp}(\tilde{M})$ $\otimes \Delta_{\mathfrak{s}}(\widetilde{M})$ be a natural chain equivalence so that $\tau d_{\mathfrak{s}}$ is the Alexander/Whitney diagonal approximation, where $d: \tilde{M} \rightarrow \tilde{M} \times \tilde{M}$ is the diagonal map. Since there is a natural chain homotopy $D_1: \tau T_2 \simeq T' \tau$ (cf. [Sp, 5.3.8]), we have $\delta d^* D_1^* +$ $d^{\sharp}D_{1}^{\sharp}\delta = d^{\sharp}T^{\sharp}\tau^{\sharp} - d^{\sharp}\tau^{\sharp}T^{\prime}$, where each summand is regarded as a homomorphism $\Delta^{\sharp}_{c}(\tilde{M},\tilde{A}')\otimes \Delta^{\sharp}_{c}(M,A) \to \Delta^{\sharp}_{c}(\tilde{M};\tilde{A},\tilde{A}')$ of degree 0. Using that τd_{\sharp} and $\tau T_{\sharp}d_{\sharp}$ are the Alexander/Whitney diagonal approximations, we obtain $\delta d^*D_1^*(t^if_x^+\otimes f_y)+$ $d^{\sharp}D_{1}^{\sharp}\delta(t^{i}f_{x}^{+}\otimes f_{x}) = t^{i}f_{x}^{+} \cup f_{y} - \mathcal{E}((p+1)r)f_{y} \cup t^{i}f_{x}^{+}. \quad \text{But, } \delta d^{\sharp}D_{1}^{\sharp}(t^{i}f_{x}^{+}\otimes f_{y}) \cap \tilde{z} = 0$ and $\sum_{i=-\infty}^{+\infty} \mathcal{E}_{\widetilde{M}}(d^{\sharp}D_{1}^{\sharp}\delta(t^{i}f_{x}^{+}\otimes f_{y})\cap \widetilde{z})t^{-i} = \lambda_{x}\sum_{i=-\infty}^{+\infty} \mathcal{E}_{\widetilde{M}}(d^{\sharp}D_{1}^{\sharp}(t^{i}f_{x}\otimes f_{y})\cap \widetilde{z})t^{-i}.$ Hence

 $\sum_{i \stackrel{t}{\longrightarrow} 0} f_{x}^{*}(t^{i}c_{y})t^{-i}/\overline{\lambda}_{x} = \sum_{i \stackrel{t}{\longrightarrow} 0} \mathcal{E}_{\widetilde{M}}((t^{i}f_{x}^{+} \cup f_{y}) \cap \widetilde{z})t^{-i}/\overline{\lambda}_{x} \equiv \mathcal{E}((p+1)r)\sum_{i \stackrel{t}{\longrightarrow} 0} \mathcal{E}_{\widetilde{M}}((f_{y} \cup t^{i}f_{x}^{+}) \cap \widetilde{z})t^{-i}/\overline{\lambda}_{x}\lambda_{y} = \mathcal{E}((p+1)r)\sum_{i \stackrel{t}{\longrightarrow} 0} \mathcal{E}_{\widetilde{M}}((\delta(f_{y}^{+}) \cup t^{i}f_{x}^{+}) \cap \widetilde{z})t^{-i}/\overline{\lambda}_{x}\lambda_{y} = \mathcal{E}((p+1)r)\mathcal{E}(p+1)\sum_{i \stackrel{t}{\longrightarrow} 0} \mathcal{E}_{\widetilde{M}}(f_{y}^{+} \cup t^{i}\delta(f_{x}^{+})) \cap \widetilde{z})t^{-i}/\overline{\lambda}_{x}\lambda_{y} = \mathcal{E}((p+1)r)\mathcal{E}(p+1)\sum_{i \stackrel{t}{\longrightarrow} 0} f_{y}^{+}(t^{-i}c_{x})t^{-i}/\overline{\lambda}_{y}.$

The result follows.

REMARK 5.5. Assume that x, y are represented by c_x, c_y with $|c_x| \cap |t^i c_y| = \emptyset$ for all *i*. For example, if *M* is triangulable, then this assumption is satisfied. Then the intersection numbers $\operatorname{Int}(c_x^+, t^i c_y)$ are defined (cf. [Ka₃, Appendix A]) and we have

$$L(x, y) = \sum_{i=-\infty}^{+\infty} \operatorname{Int}(c_x^+, t^i c_y) t^{-i} / \overline{\lambda}_x.$$

In fact, $L(x,y) = \mathcal{E}(p+1) \sum_{i=-\infty}^{+\infty} f_x^*(t^i \partial c_y^*) t^{-i} / \overline{\lambda}_x \lambda_y = \mathcal{E}(p+1) \sum_{i=-\infty}^{+\infty} f_x(t^i \overline{\lambda}_x c_y^*) t^{-i} / \overline{\lambda}_x \lambda_y$ $= \mathcal{E}(p+1) \sum_{i=-\infty}^{+\infty} \operatorname{Int}(c_x, t^i \overline{\lambda}_x c_y^*) t^{-i} / \overline{\lambda}_x \lambda_y$ (cf. [Ka₃, A. 4]) $= \mathcal{E}(p+1) \sum_{i=-\infty}^{+\infty} \operatorname{Int}(\partial c_x^*, t^i \partial c_y^*) t^{-i} / \overline{\lambda}_x \lambda_y = \sum_{i=-\infty}^{+\infty} \operatorname{Int}(c_x^*, t^i \partial c_y^*) t^{-i} / \overline{\lambda}_x \lambda_y = \sum_{i=-\infty}^{+\infty} \operatorname{Int}(c_x^*, t^i \partial c_y^*) t^{-i} / \overline{\lambda}_x \lambda_y = \sum_{i=-\infty}^{+\infty} \operatorname{Int}(c_x^*, t^i \partial c_y^*) t^{-i} / \overline{\lambda}_x \lambda_y = \sum_{i=-\infty}^{+\infty} \operatorname{Int}(c_x^*, t^i \partial c_y^*) t^{-i} / \overline{\lambda}_x \lambda_y = \sum_{i=-\infty}^{+\infty} \operatorname{Int}(c_x^*, t^i \partial c_y^*) t^{-i} / \overline{\lambda}_x \lambda_y = \sum_{i=-\infty}^{+\infty} \operatorname{Int}(c_x^*, t^i \partial c_y^*) t^{-i} / \overline{\lambda}_x \lambda_y = \sum_{i=-\infty}^{+\infty} \operatorname{Int}(c_x^*, t^i \partial c_y^*) t^{-i} / \overline{\lambda}_x \lambda_y = \sum_{i=-\infty}^{+\infty} \operatorname{Int}(c_x^*, t^i \partial c_y^*) t^{-i} / \overline{\lambda}_x \lambda_y = \sum_{i=-\infty}^{+\infty} \operatorname{Int}(c_x^*, t^i \partial c_y^*) t^{-i} / \overline{\lambda}_x \lambda_y = \sum_{i=-\infty}^{+\infty} \operatorname{Int}(c_x^*, t^i \partial c_y^*) t^{-i} / \overline{\lambda}_x \lambda_y = \sum_{i=-\infty}^{+\infty} \operatorname{Int}(c_x^*, t^i \partial c_y^*) t^{-i} / \overline{\lambda}_x \lambda_y = \sum_{i=-\infty}^{+\infty} \operatorname{Int}(c_x^*, t^i \partial c_y^*) t^{-i} / \overline{\lambda}_x \lambda_y = \sum_{i=-\infty}^{+\infty} \operatorname{Int}(c_x^*, t^i \partial c_y^*) t^{-i} / \overline{\lambda}_x \lambda_y = \sum_{i=-\infty}^{+\infty} \operatorname{Int}(c_x^*, t^i \partial c_y^*) t^{-i} / \overline{\lambda}_x \lambda_y = \sum_{i=-\infty}^{+\infty} \operatorname{Int}(c_x^*, t^i \partial c_y^*) t^{-i} / \overline{\lambda}_x \lambda_y = \sum_{i=-\infty}^{+\infty} \operatorname{Int}(c_x^*, t^i \partial c_y^*) t^{-i} / \overline{\lambda}_x \lambda_y = \sum_{i=-\infty}^{+\infty} \operatorname{Int}(c_x^*, t^i \partial c_y^*) t^{-i} / \overline{\lambda}_x \lambda_y = \sum_{i=-\infty}^{+\infty} \operatorname{Int}(c_x^*, t^i \partial c_y^*) t^{-i} / \overline{\lambda}_x \lambda_y = \sum_{i=-\infty}^{+\infty} \operatorname{Int}(c_x^*, t^i \partial c_y^*) t^{-i} / \overline{\lambda}_x \lambda_y = \sum_{i=-\infty}^{+\infty} \operatorname{Int}(c_x^*, t^i \partial c_y^*) t^{-i} / \overline{\lambda}_x \lambda_y = \sum_{i=-\infty}^{+\infty} \operatorname{Int}(c_x^*, t^i \partial c_y^*) t^{-i} / \overline{\lambda}_x \lambda_y = \sum_{i=-\infty}^{+\infty} \operatorname{Int}(c_x^*, t^i \partial c_y^*) t^{-i} / \overline{\lambda}_x \lambda_y = \sum_{i=-\infty}^{+\infty} \operatorname{Int}(c_x^*, t^i \partial c_y^*) t^{-i} / \overline{\lambda}_x \lambda_y = \sum_{i=-\infty}^{+\infty} \operatorname{Int}(c_x^*, t^i \partial c_y^*) t^{-i} / \overline{\lambda}_x \lambda_y = \sum_{i=-\infty}^{+\infty} \operatorname{Int}(c_x^*, t^i \partial c_y^*) t^{-i} / \overline{\lambda}_x \lambda_y = \sum_{i=-\infty}^{+\infty} \operatorname{Int}(c_x^*, t^i \partial c_y^*) t^{-i} / \overline{\lambda}_x \lambda_y = \sum_{i=-\infty}^{+\infty} \operatorname{Int}(c_x^*, t^i \partial c_y^*) t^{-i} / \overline{\lambda}_x \lambda_y = \sum_{i=-\infty}^{+\infty} \operatorname{$

6. Proof of the Second Duality Theorem

Since the infinite cyclic covering $\widetilde{M} \to M$ is the pullback of the exponential covering exp: $R \to S^1$ by a map $f_{\gamma}: M \to S^1$ representing γ , the lift $\widetilde{f}_{\gamma}: \widetilde{M} \to R$ of f_{γ} is a proper map. Let $M_i^+ = \widetilde{f}_{\gamma}^{-1}(i, +\infty)$ and $M_i^- = \widetilde{f}_{\gamma}^{-1}(-\infty, -i)$. Let $H^*(\widetilde{M}, \mathcal{E}(\pm) \cup \widetilde{A}) = \lim_{i \to +\infty} H^*(\widetilde{M}, M_i^{\pm} \cup A)$. Taking the limit $i \to +\infty$ of the Mayer/Vietoris sequence for $(M; M_i^+ \cup \widetilde{A}, M_i^- \cup \widetilde{A})$, we obtain an exact sequence

$$\rightarrow H^{q}_{\varepsilon}(\tilde{M},\tilde{A}) \rightarrow H^{q}(\tilde{M},\varepsilon(+)\cup\tilde{A}) \oplus H^{q}(\tilde{M},\varepsilon(-)\cup\tilde{A}) \rightarrow H^{q}(\tilde{M},\tilde{A}) \xrightarrow{\delta_{\varepsilon}} H^{q+1}_{\varepsilon}(\tilde{M},\tilde{A}) \rightarrow .$$

Lemma 6.1. There is one and only one element μ of $H_{n-1}(\tilde{M}, \partial \tilde{M})$ such that

(1) $(t-1)\mu=0$,

(2) The map $p_*: H_{n-1}(\tilde{M}, \partial \tilde{M}) \to H_{n-1}(M, \partial M)$ sends μ to $\gamma \cap [M]$, where p denotes the covering projection.

Further, μ is given by $\delta_c(1) \cap [\tilde{M}]$ for $\delta_c \colon H^0(\tilde{M}) \to H^1_c(\tilde{M})$.

Proof. For uniqueness, let μ , μ' have (1) and (2). By the Wang exact sequence $H_{n-1}(\tilde{M}, \partial \tilde{M}) \xrightarrow{t-1} H_{n-1}(\tilde{M}, \partial \tilde{M}) \xrightarrow{p_*} H_{n-1}(M, \partial M)$ (cf. $[Mi_2]$), we have $\mu - \mu' = (t-1)x$ for an $x \in H_{n-1}(\tilde{M}, \partial \tilde{M})$. By (1), $(t-1)^2 x = 0$. By the Reidemeister duality and UCES, $TH_{n-1}(\tilde{M}, \partial \tilde{M}) \simeq TH_{\Lambda}^1(\tilde{M}) \simeq E^1H_0(\tilde{M})$ and the last is easily seen to be a direct sum of modules of type $\Lambda/(t^4-1)(q \neq 0)$ (cf. $[Ka_1, Lemma 1]$). Hence $(t-1)^2 x = 0$ means (t-1)x = 0 and $\mu = \mu'$. Next, let $\mu'' = \delta_c(1) \cap [\tilde{M}]$. Since t1=1 and $t[\tilde{M}]=[\tilde{M}], \mu''$ has (1). To see that it has (2), first assume that f_{γ} has a leaf V in M (cf. $[Ka_3]$). Regard $V \subset \tilde{M}$ and thicken $V \times I \subset \tilde{M}$ so that $\tilde{f_{\gamma}}^{-1}(0) = V$ and $\tilde{f_{\gamma}}^{-1}I = V \times I$ and $\tilde{f_{\gamma}} | V \times I : V \times I \to I$ is the projection, where $I = [0, \varepsilon]$ for a small $\varepsilon > 0$. The following commutative diagram is obtained ($\mathring{I} = I - \partial I$):

$$\begin{array}{ccc} H^{1}(I, \partial I) & \stackrel{(\tilde{f}_{\gamma} \mid V \times I)^{*}}{\longrightarrow} H^{1}(V \times I, V \times \partial I) & \stackrel{\bigcap [V \times I]}{\longrightarrow} H_{n-1}(V \times I, (\partial V) \times I) \\ & \cong \downarrow & \downarrow \\ H^{1}(R, R - \mathring{I}) & \stackrel{\tilde{f}_{\gamma}^{*}}{\longrightarrow} & H^{1}(\tilde{M}, \tilde{M} - V \times \mathring{I}) \\ & \cong \downarrow & \downarrow \\ H^{1}_{\iota}(R) & \stackrel{\tilde{f}_{\gamma}^{*}}{\longrightarrow} & H^{1}_{\iota}(\tilde{M}) & \stackrel{\bigcap [\tilde{M}]}{\longrightarrow} & H_{n-1}(\tilde{M}, \partial \tilde{M}) \end{array}$$

Since $\delta_{c}(1) = \tilde{f}_{\gamma}^{*}[R]$ (cf. [Ka₁, p. 98]), we see that $\mu'' = [V] \in H_{n-1}(\tilde{M}, \partial \tilde{M})$. So, $p_{*}(\mu'') = [V] \in H_{n-1}(M, \partial M)$, which equals $\gamma \cap [M]$. Hence μ'' has (2). If γ has no leaf, then we take $M_{P} = M \times CP^{2}$ and $\gamma_{P} = \gamma \times 1 \in H^{1}(M_{P})$. Then by $[K/S] \gamma_{P}$ has a leaf. By the identity $(\delta_{c}(1) \times 1) \cap ([\tilde{M}] \times [CP^{2}]) = (\delta_{c}(1) \cap [\tilde{M}]) \times [CP^{2}], \mu''$ has also (2). This completes the proof.

We call μ of Lemma 6.1 the fundamental class of the covering $\tilde{M} \to M$. By Lemmas 3.4 and 3.5, the epimorphism $\rho: TH_{\Lambda}^{q+1}(\tilde{M}, \tilde{A}) \to E^{1}H_{q}(\tilde{M}, \tilde{A})$ in UCES induces an epimorphism $DH_{\Lambda}^{q+1}(\tilde{M}, \tilde{A}) \to E^{1}BH_{q}(\tilde{M}, \tilde{A})$, also denoted by ρ . We define a *t*-anti epimorphism

$$\theta: DH_p(\tilde{M}, \tilde{A}) \to E^1 BH_{s+1}(\tilde{M}, \tilde{A}')$$

by the composite $DH_p(\tilde{M}, \tilde{A}) \stackrel{\tilde{D}^{-1}}{\simeq} DH_{\Lambda}^{s+2}(\tilde{M}, \tilde{A}') \stackrel{\rho}{\to} E^1 BH_{s+1}(\tilde{M}, \tilde{A}')$. Clearly, any proper oriented homotopy equivalence $f: (M_1; A_1, A_1') \to (M_2; A_2, A_2')$ with $f^*(\gamma_2) = \gamma_1$ induces the following commutative square:

$$DH_{p}(\tilde{M}_{1}, \tilde{A}_{1}) \xrightarrow{\theta} E^{1}BH_{s+1}(\tilde{M}_{1}, \tilde{A}_{1}')$$

$$\approx \downarrow \tilde{f}_{*} \qquad \approx \uparrow \tilde{f}^{*}$$

$$DH_{p}(\tilde{M}_{2}, \tilde{A}_{2}) \xrightarrow{\theta} E^{1}BH_{s+1}(\tilde{M}_{2}, \tilde{A}_{2}').$$

Let $DH_p(\tilde{M}, \tilde{A})^{\theta}$ be the kernel of θ . By identifying $DH_p(\tilde{M}, \tilde{A})$ with $E^2 E^2 H_p$ (\tilde{M}, \tilde{A}) in a natural way, we also consider θ as

$$\theta: E^2 E^2 H_p(\tilde{M}, \tilde{A}) \to E^1 B H_{s+1}(\tilde{M}, \tilde{A}')$$
.

In this case, the kernel of θ is denoted by $E^2 E^2 H_p(\tilde{M}, \tilde{A})^{\theta}$. Note that $DH_s(\tilde{M}, \tilde{A}')^{\theta}$, $E^2 E^2 H_s(\tilde{M}, \tilde{A}')^{\theta}$ are the kernels of $\theta: DH_s(\tilde{M}, \tilde{A}') \to E^1 B H_{p+1}(\tilde{M}, \tilde{A}), \theta: E^2 E^2 H_s$ $(\tilde{M}, \tilde{A}') \to E^1 B H_{p+1}(\tilde{M}, \tilde{A})$, respectively. Let $\tau: e^1 H_p(\tilde{M}, \tilde{A}) \to H^{p+1}(\tilde{M}, \tilde{A})$ be the monomorphism ρ^{-1} in Corollary 1.2 with $\Gamma = F = Z$. Then the following square is commutative:

$$e^{1}H_{p}(\tilde{M},\tilde{A}) \xrightarrow{\tau} H^{p+1}(\tilde{M},\tilde{A})$$

surjection $\uparrow \qquad h \qquad \uparrow \delta_{Q/Z}$
Hom_z($H_{p}(\tilde{M},\tilde{A}), Q/Z$) $\rightleftharpoons H^{p}(\tilde{M},\tilde{A}; Q/Z),$

where $\delta_{Q/Z}$ denotes the Bockstein coboundary map. Let $\tau H^{p+1}(\tilde{M}, \tilde{A}) = \tau e^{1}H_{p}$ $(\tilde{M}, \tilde{A}) = \delta_{Q/Z} H^{p}(\tilde{M}, \tilde{A}; Q/Z)$. By UCES with $\Gamma = F = Z$, $u = \{f\} \in H^{p+1}(\tilde{M}, \tilde{A})$ is in $\tau H^{p+1}(\tilde{M}, \tilde{A})$ iff $f \mid Z_{p+1}\Delta_{\mathbf{f}}(\tilde{M}, \tilde{A}) = 0$. Let $\tau_{\theta} H^{p+1}(\tilde{M}, \tilde{A})$ be the Λ -submodule of $\tau H^{p+1}(\tilde{M}, \tilde{A})$ consisting of all elements $u = \{f\}$ such that $f(c) \equiv 0 \pmod{d}$ for $\{c_1\} \in DH_p(\tilde{M}, \tilde{A})^{\theta}$, $c \in \Delta_{p+1}(\tilde{M}, \tilde{A})$ and $d(\pm 0) \in Z$ with $\partial c = dc_1$. Regarding $e^1(H_p(\tilde{M}, \tilde{A})/DH_p(\tilde{M}, \tilde{A})^{\theta}) \subset e^1H_p(\tilde{M}, \tilde{A})$ in a natural way, we can obtain from an argument similar to $[F, \S 1]$ the following (whose proof is omitted):

Lemma 6.2. $\tau_{\theta}H^{p+1}(\tilde{M},\tilde{A}) = \tau e^{1}(H_{p}(\tilde{M},\tilde{A})/DH_{p}(\tilde{M},\tilde{A}))$.

We consider the *t*-anti homomorphism $\cap \mu : \tau H^{p+1}(\tilde{M}, \tilde{A}) \to H_s(\tilde{M}, \tilde{A}')$.

Lemma 6.3. $\tau H^{p+1}(\tilde{M}, \tilde{A}) \cap \mu \subset DH_s(\tilde{M}, \tilde{A}')^{\theta}$.

Proof. By Lemma 6.1, $\tau H^{p+1}(\tilde{M}, \tilde{A}) \cap \mu = ((\tau H^{p+1}(\tilde{M}, \tilde{A}) \cup \delta_c(1)) \cap [\tilde{M}] = \delta_c \tau H^{p+1}(\tilde{M}, \tilde{A}) \cap [\tilde{M}]$. For $\{f\} \in \tau H^{p+1}(\tilde{M}, \tilde{A})$, there are $f^{\pm} \in \Delta^{p+1}(\tilde{M}, M_i^{\pm} \cup \tilde{A})$ $(i \geq 1)$ such that $f = f^+ - f^-$ in $\Delta^{p+1}(\tilde{M}, \tilde{A})$. Then $\delta_c\{f\} = \{\delta f^+\}$. Since $f \mid Z_{p+1} \Delta_{\mathfrak{g}}(\tilde{M}, \tilde{A}) = 0$, it follows that $f^+ = f^-$ on $Z_{p+1} \Delta_{\mathfrak{g}}(\tilde{M}, \tilde{A})$ and $\phi(f^+)$ is well-defined on it. Let $f_{\Lambda} = \phi(f^+) \mid Z_{p+1} \Delta_{\mathfrak{g}}(\tilde{M}, \tilde{A}) \in E^0 Z_{p+1} \Delta_{\mathfrak{g}}(\tilde{M}, \tilde{A})$. Noting that some multiple λf_{Λ} is extendable to $\Delta_{p+1}(\tilde{M}, \tilde{A})$ (for $E^1 B_p \Delta_{\mathfrak{g}}(\tilde{M}, \tilde{A}) \cong E^2 H_p(\tilde{M}, \tilde{A})$ is finite) and $\phi(\delta f^+) = \delta f_{\Lambda}$, we see from Lemma 5.1 that $\{\phi \delta f^+\} \in TH^{p+2}_{\Lambda}(\tilde{M}, \tilde{A})$ and $\rho' \{\phi \delta f^+\} = 0 = \rho \{\phi \delta f^+\}$. This means that $\tau H^{p+1}(\tilde{M}, \tilde{A}) \cap \mu \subset \operatorname{Ker}[TH_s(\tilde{M}, \tilde{A}') \cong TH^{p+2}_{\Lambda}(\tilde{M}, \tilde{A})]$, which equals $DH_s(\tilde{M}, \tilde{A}')^{\theta}$ by UCES. This completes the proof.

Lemma 6.4. Ker $[\cap \mu: \tau H^{p+1}(\tilde{M}, \tilde{A}) \rightarrow H_s(\tilde{M}, \tilde{A}')] \subset \tau_{\theta} H^{p+1}(\tilde{M}, \tilde{A}).$

Proof. Let $u = \{f\} \in \tau H^{p+1}(\tilde{M}, \tilde{A})$ have $u \cap \mu = 0$. Then $\delta_c u = 0$ and there are $\{f^{\pm}\} \in H^{p+1}(\tilde{M}, \mathcal{E}(\pm) \cup \tilde{A})$ with $f = f^+ - f^-$ in $\Delta^{p+1}(\tilde{M}, \tilde{A})$. Since f induces the zero map $H_{p+1}(\tilde{M}, \tilde{A}) \to Z, f^{\pm}$ induce the same map $H_{p+1}(\tilde{M}, \tilde{A}) \to Z$. Hence $\phi(f^+)|Z_{p+1}\Delta_{\sharp}(\tilde{M},\tilde{A})$ is well-defined and defines an element $f_{\Lambda} \in E^0H_{p+1}(\tilde{M},\tilde{A})$. Take an integer m > 0 so that $(t^m - 1)DH_{\mathfrak{g}}(\tilde{M}, \tilde{A}) = 0$. By UCES and Lemma 3.4, $(1-t^m)f_{\Lambda} = h\phi^*\{f^c\}$ for some $\{f^c\} \in H^{p+1}_c(\tilde{M}, \tilde{A})$. Then we have $f^+ - f^+t^m = f^c$ on $Z_{p+1}\Delta_{\mathfrak{s}}(\tilde{M},\tilde{A})$. Define $f_m, f_m^{\pm} \in \Delta^{p+1}(\tilde{M},\tilde{A})$ by $f_m(c) = \sum_{k=-\infty}^{+\infty} f^c(t^{km}c), f_m^+(c) =$ $\sum_{k=0}^{+\infty} f^c(t^{km}c) \text{ and } f_m = f_m^+ - f_m^-. \text{ We have that } \delta(f_m^{\pm}) = 0 = f_m^{\pm} |\Delta_{p+1}(M_i^{\pm} \cup \tilde{A}, \tilde{A}),$ taking i so large that f^c represents an element of $H^{p+1}(\tilde{M}, M_i^+ \cup M_i^- \cup \tilde{A})$. Moreover, for $x \in \mathbb{Z}_{p+1} \Delta_{\sharp}(\widetilde{M}, \widetilde{A}), f_m(x) = \sum_{k=-\infty}^{+\infty} f^c(t^{km}x) = \sum_{k=-\infty}^{+\infty} (f^+(t^{km}x) - f^+(t^{km+m}x))$ =0 and similarly, $f_m^+(x) = f^+(x)$, so that $f_m^-(x) = f^+(x) = f^-(x)$. Let $f_0 = f - f_m$ and $f_0^{\pm} = f^{\pm} - f_m^{\pm}$. Then $f_0 = f_0^{\pm} - f_0^{-}$ and f_0^{\pm} represent elements of $H^{p+1}(\tilde{M}, M_i^{\pm} \cup \tilde{A})$ for a large *i* and $f_0^{\pm} | Z_{p+1} \Delta_{\sharp}(\tilde{M}, \tilde{A}) = 0$. By construction, f_m and f_0 represent elements u_m and u_0 of $\tau H^{p+1}(\tilde{M}, \tilde{A})$, respectively. To prove that $u_m, u_0 \in \tau_{\theta} H^{p+1}$ (\tilde{M}, \tilde{A}) , let $x = \{c_1\} \in DH_p(\tilde{M}, \tilde{A})^{\theta}, c \in \Delta_{p+1}(\tilde{M}, \tilde{A})$ and $d(\pm 0) \in Z$ such that $\partial c = 0$ dc_1 . Since $t^m DH_b(\tilde{M}, \tilde{A})^{\theta} = DH_b(\tilde{M}, \tilde{A})^{\theta}$, we can find an element $\{c_1^+\}$ of $H_b(M_i^+)$ $\bigcup \tilde{A}, \tilde{A}$) of finite order sending to x under the natural map $H_p(M_i^+ \cup \tilde{A}, \tilde{A}) \rightarrow$ $H_p(\widetilde{M}, \widetilde{A})$. That is, there are $h^+ \in \Delta_{p+1}(\widetilde{M}, \widetilde{A}), d_1(\pm 0) \in \mathbb{Z}$ and $c^+ \in \Delta_{p+1}(M_i^+ \cup \mathbb{Z})$ $\widetilde{A},\widetilde{A}$) such that $c_1 - c_1^+ = \partial h^+$ and $\partial c^+ = d_1 c_1^+$. Let $\overline{c}^+ = d_1 c - dd_1 h^+ - dc^+ \in \Delta_{p+1}(\widetilde{M},\widetilde{A})$. Then $\partial \bar{c}^+=0$ in $\Delta_{\rho}(\bar{M},\tilde{A})$ and $f_0^+(\bar{c}^+)=d_1f_0^+(c)-dd_1f_0^+(h^+)-df_0^+(c^+)=0$. But, $f_0^+(c^+) = 0$, so that $f_0^+(c) = df_0^+(h^+) \equiv 0 \pmod{d}$. Similarly, $f_0^-(c) \equiv 0 \pmod{d}$. Thus, $f_0(c) \equiv 0 \pmod{d}$, meaning that $u_0 \in \tau_{\theta} H^{p+1}(\tilde{M}, \tilde{A})$. To see that $u_m \in \tau_{\theta} H^{p+1}$ (\tilde{M}, \tilde{A}) , it suffices to show that $f_m(c_Q) \in Z$ for some Q-chains $c_Q \in \Delta_{p+1}(\tilde{M}; \tilde{A}, Q)$ with $\partial c_Q = c_1$, where f_m is extended to a map $\Delta_{p+1}(\tilde{M}, \tilde{A}) \otimes Q \rightarrow Q$. We may take c_1 so that $c_1 = f_1^c \cap \tilde{z}$ for $\{f_1^c\} \in H_c^{s+2}(\tilde{M}, \tilde{A}')$ of finite order and \tilde{z} in Lemma 2.4. Let $f'_{Q} \in \Delta_{c}^{s+1}(\tilde{M}, \tilde{A}'; Q)$ have $\delta f'_{Q} = f_{1}^{c}$. We use the same ϕ for the Q-extension of $\phi: \Delta_c^{s+1}(\tilde{M}, \tilde{A}') \to E^0 \Delta_{s+1}(\tilde{M}, \tilde{A}')$. Then $\phi(f'_Q): \Delta_{s+1}(\tilde{M}, \tilde{A}') \to \Lambda_0$ induces a map $\hat{\phi}(f'_{Q}): H_{s+1}(\tilde{M}, \tilde{A}') \to \Lambda_{0}/\Lambda$. Since $\{c_{1}\} \in DH_{p}(\tilde{M}, \tilde{A})^{\theta}$, it follows from the definition of θ that there is a Λ -homomorphism $f'_{\Lambda} : H_{s+1}(\tilde{M}, \tilde{A}') \to \Lambda_0$ inducing $\phi(f_{\phi})$. Note that the composite

$$H^{s+1}_{c}(\tilde{M},\tilde{A}';Q) \cong H^{s+1}_{c}(\tilde{M},\tilde{A}') \otimes Q \stackrel{\phi^{*} \otimes 1}{\cong} H^{s+1}_{\Lambda}(\tilde{M},\tilde{A}') \otimes Q \stackrel{h \otimes 1}{\to} E^{0}H_{s+1}(\tilde{M},\tilde{A}') \otimes Q \stackrel{h \otimes 1}{\cong} \operatorname{Hom}_{\Lambda}(H_{s+1}(\tilde{M},\tilde{A}'),\Lambda_{0})$$

is onto by UCES. Let $\{f'_{Q}\} \in H^{s+1}_{c}(\tilde{M}, \tilde{A}'; Q)$ be a preimage of f'_{Δ} . Let $f'_{Q} = f'_{Q} - f''_{Q}$. Then f'_{Q} induces the zero map $H_{s+1}(\tilde{M}, \tilde{A}') \to Q/Z$. Let $c_{Q} = \varepsilon(p+1)$ $f'_{Q} \cap \tilde{z} \in \Delta_{p+1}(\tilde{M}, \tilde{A}; Q)$. Then $\partial c_{Q} = \varepsilon(p+1)\varepsilon(s+1-n)\delta f'_{Q} \cap \tilde{z} = c_{1}$ (cf. [Sp, p. 253]). Regarding f'_{Q} as a cocycle $\Delta_{s+1}(\tilde{M}, \tilde{A}') \to Q/Z$, we have in Q/Z

$$f_m(c_Q) = \sum_{k=-\infty}^{+\infty} f^c(t^{km}c_Q) = \mathcal{E}(p+1) \sum_{k=-\infty}^{+\infty} \mathcal{E}_{\widetilde{M}}((t^{km}f^c \cup f^c_Q) \cap \widetilde{z}) \\ = \mathcal{E}((p+1)s) \sum_{k=-\infty}^{+\infty} \mathcal{E}_{\widetilde{M}}((f^c_Q \cup t^{km}f^c) \cap \widetilde{z}) = \mathcal{E}((p+1)s) \sum_{k=-\infty}^{+\infty} f^c_Q(t^{km}f^c \cap \widetilde{z}) = 0,$$

for $t^{k_m} f^c \cap \tilde{z} \in Z_{s+1} \Delta_{\sharp}(\tilde{M}, \tilde{A}')$. Thus, $u_m \in \tau_{\theta} H^{p+1}(\tilde{M}, \tilde{A})$ and $u = u_m + u_0 \in \tau_{\theta} H^{p+1}(\tilde{M}, \tilde{A})$. This completes the proof.

Theorem 6.5. The maps $e^{1}H_{p}(\tilde{M}, \tilde{A}) \xrightarrow{\tau} \tau H^{p+1}(\tilde{M}, \tilde{A}) \xrightarrow{\cap} H_{s}(\tilde{M}, \tilde{A}')$ induce isomorphisms

$$e^1DH_p(ilde{M}, ilde{A})^ heta\stackrel{ au}{\simeq} au H^{p+1}(ilde{M}, ilde{A})/ au_ heta H^{p+1}(ilde{M}, ilde{A})\stackrel{ee \mu}{\simeq} DH_s(ilde{M}, ilde{A}')^ heta$$

Proof. Let $\tau_{\kappa} H^{p+1}$ be the kernel of $\cap \mu$. By Lemmas 6.2, 6.3 and 6.4, we obtain the following diagram:

Since $e^{1}DH_{p}(\tilde{M},\tilde{A})^{\theta} \simeq DH_{p}(\tilde{M},\tilde{A})^{\theta}$ as abelian groups, it follows that $|DH_{p}(\tilde{M},\tilde{A})^{\theta}| \le |DH_{s}(\tilde{M},\tilde{A}')^{\theta}|$. Interchanging the roles of $H_{p}(\tilde{M},\tilde{A})$ and $H_{s}(\tilde{M},\tilde{A}')$, we have $|DH_{s}(\tilde{M},\tilde{A}')^{\theta}| = |DH_{p}(\tilde{M},\tilde{A})^{\theta}|$. This means that $\tau_{K}H^{p+1} = \tau_{\theta}H^{p+1}(\tilde{M},\tilde{A})$ and $\cap \mu : \tau H^{p+1}(\tilde{M},\tilde{A})/\tau_{\theta}H^{p+1}(\tilde{M},\tilde{A}) \simeq DH_{s}(\tilde{M},\tilde{A}')^{\theta}$. This completes the proof.

6.6. Proof of the Second Duality Theorem. By Theorem 6.5, we define a pairing

$$l: E^2 E^2 H_{\mathfrak{g}}(\tilde{M}, \tilde{A})^{\theta} \times E^2 E^2 H_{\mathfrak{s}}(\tilde{M}, \tilde{A}')^{\theta} = D H_{\mathfrak{g}}(\tilde{M}, \tilde{A})^{\theta} \times D H_{\mathfrak{s}}(\tilde{M}, \tilde{A}')^{\theta} \to Q/Z$$

by $l(x, y) = \mathcal{E}(s+1)f_s(y)$ for $f_s \in e^1 DH_s(\tilde{M}, \tilde{A}')^{\theta} = \operatorname{Hom}_Z(DH_s(\tilde{M}, \tilde{A}')^{\theta}, Q/Z)$ with $\tau f_s \cap \mu = x \in DH_p(\tilde{M}, \tilde{A})^{\theta}$ and $y \in DH_s(\tilde{M}, \tilde{A}')^{\theta}$. By construction, l has (2) and (4). For any $u_s \in H^s(\tilde{M}, \tilde{A}'; Q/Z)$ and $u_y \in H^p(\tilde{M}, \tilde{A}; Q/Z)$ with $\delta_{Q/Z}(u_s) \cap \mu = x$ and $\delta_{Q/Z}(u_y) \cap \mu = y$, we also have in Q/Z

$$l(x, y) = \varepsilon(s+1)\varepsilon_{\widetilde{M}}((u_x \cup \delta_{Q/Z}(u_y)) \cap \mu) = \varepsilon_{\widetilde{M}}((\delta_{Q/Z}(u_x) \cup u_y) \cap \mu)$$

(cf. [F, Lemma 3.8]). We have $l(x, y) = \varepsilon(ps+1)l(y, x)$, showing (3). (1) is obvious, since μ is invariant under a proper oriented homotopy equivalence $f: M_1 \to M_2$ with $f^*(\gamma_2) = \gamma_1$. This completes the proof.

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