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Osaka University
THREE DUALITIES ON THE INTEGRAL HOMOLOGY OF INFINITE CYCLIC COVERINGS OF MANIFOLDS

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0. Statement of main result

We consider a compact oriented topological n-manifold M. Let γ be an element of the first integral cohomology $H^1(M)$ and $\tilde{M}$ be the infinite cyclic covering space of $M$ associated with $\gamma$. The covering transformation group is infinite cyclic and denoted by $\langle t \rangle$ with a generator $t$, specified by $\gamma$. A subboundary, $A$, of $M$ is $\emptyset$ or a compact $(n-1)$-submanifold of the boundary $\partial M$ such that $A' = cl_{\tilde{M}}(\partial M - A)$ is $\emptyset$ or a compact $(n-1)$-submanifold of $\partial M$. The pair $(A, A')$ is called a splitting of $\partial M$. Let $\tilde{A}$ be the lift of $A$, i.e., the preimage of $A$ under the covering $\tilde{M} \to M$. Let $\Lambda$ be the integral group ring of $\langle t \rangle$. The integral homology $H_*(M, A)$ forms a finitely generated $\Lambda$-module, because by $[K/S](M, A)$ is homotopy equivalent to a compact polyhedral pair and $\Lambda$ is Noetherian. For an abelian group $H$, let $e_H = Ext^1_H(H; \mathbb{Z})$ (so that $e_H = 0$ for $i \geq 2$ and $\text{Hom}_Z(H, \mathbb{Z}) = \mathbb{Z}H$), $t_H$ = the $\mathbb{Z}$-torsion part of $H$ and $b_H = H/t_H$. When $H$ is a $\Lambda$-module, let $E_i^H = Ext^i_{\Lambda}(H, \Lambda)$ (so that $\text{Hom}_\Lambda(H, \Lambda) = E^0H$ and $TH = \Lambda$-torsion part of $H$ and $b_H = H/TH$. Since $\Lambda$ has the global dimension 2 (cf. MacLane [Ma, p. 205]), we have $E_i^H = 0$ for $i \geq 3$. The following $\Lambda$-submodule, $DH$, of $H$ was introduced by Blanchfield [B]:

$$DH = \{x \in H | \exists \text{ coprime } \lambda_1, \lambda_2, \ldots, \lambda_m \in \Lambda(m \geq 2) \text{ with } \lambda_i x = 0, \forall i \}. $$

If $H$ is finitely generated over $\Lambda$, then we see that $DH$ is the (unique) maximal finite $\Lambda$-submodule of $H$ and there are natural $\Lambda$-isomorphisms $DH \approx E^2E^2H$ and $TH/DH \approx E^2E^2H$. Further, $E^2H$ is $\Lambda$-free and there is a natural $\Lambda$-isomorphism $B_H \to E^2E^2H$ whose cokernel is finite. The purpose of this paper is to establish the Zeroth, First and Second Duality Theorems giving dual structures between $E^iE^iH(\tilde{M}, \tilde{A})$ and $E^iE^iH_{n-2+i}(\tilde{M}, \tilde{A}')$ for $i = 0, 1$ and 2, respectively. It turns out that the first two are similar to the Blanchfield Dualities [B] and the third, the Farber/Levine Duality [F], [L]. Let $f: (M_1; A_1, A'_1) \to (M_2; A_2, A'_2)$ be a proper oriented homotopy equivalence (on each of $M_1, A_1$ and $A'_1$) with $f^*(\gamma_1) = \gamma_2$ for compact oriented $n$-manifolds $M_i$ with splittings $(A_i, A'_i)$ of $\partial M_i$ and $\gamma_i \in H^0(M_i), i = 1, 2$. For the covering spaces $\tilde{M}_i$ of $M_i$ associated with $\gamma_i, f$ lifts to
The Zeroth Duality Theorem. For a compact oriented $n$-manifold $M$ with $\gamma \in H^1(M)$ and a splitting $(A, A')$ of $\partial M$ and integers $p, q$ with $p+q=n$, there is a pairing

$$S: E^\bullet E^\bullet H_\bullet(\tilde{M}, A) \times E^\bullet E^\bullet H_\bullet(\tilde{M}, A') \to \Lambda$$

such that

1. (Homotopy invariance) A proper oriented homotopy equivalence $f: (M_1; A_1, A'_1) \to (M_2; A_2, A'_2)$ with $f^*(\gamma_2) = \gamma_1$ induces the identity $S(f_*(x), f_*(y)) = S(x, y)$ for $x \in E^\bullet E^\bullet H_\bullet(\tilde{M}_1, A_1)$ and $y \in E^\bullet E^\bullet H_\bullet(\tilde{M}_1, A'_1)$,

2. $S$ is sesquilinear, i.e., $\lambda S(x, y) = S(\overline{\lambda x}, \overline{\lambda y}) = S(x, \lambda y)$ for $x \in E^\bullet E^\bullet H_\bullet(\tilde{M}, A)$, $y \in E^\bullet E^\bullet H_\bullet(\tilde{M}, A')$ and $\lambda \in \Lambda$, where $-\overline{\lambda}$ denotes the involution on $\Lambda$ sending $t$ to $t^{-1}$,

3. $S$ is $\varepsilon(pr+1)$-Hermitian, i.e., $\varepsilon(x, y) = \varepsilon(pr+1)S(x, y)$ for $x \in E^\bullet E^\bullet H_\bullet(\tilde{M}, A)$ and $y \in E^\bullet E^\bullet H_\bullet(\tilde{M}, A')$,

4. $S$ is non-singular, i.e., $S$ induces a $t$-anti $\Lambda$-isomorphism $E^\bullet E^\bullet H_\bullet(\tilde{M}, A) \simeq \text{Hom}_\Lambda(E^\bullet E^\bullet H_\bullet(\tilde{M}, A'), \Lambda)$. 

In fact, we construct $S$ by extending the $\Lambda$-intersection pairing $\text{Int}: BH_\bullet(\tilde{M}, A) \times BH_\bullet(\tilde{M}, A') \to \Lambda$. Blanchfield [B] has formulated a similar duality over local rings of $\Lambda$. Let $Q(\Lambda)$ be the quotient field of $\Lambda$.

The First Duality Theorem. For a compact oriented $n$-manifold $M$ with $\gamma \in H^1(M)$ and a splitting $(A, A')$ of $\partial M$ and integers $p, r$ with $p+r+1=n$, there is a pairing

$$L: E^1E^1H_\bullet(\tilde{M}, A) \times E^1E^1H_\bullet(\tilde{M}, A') \to Q(\Lambda)/\Lambda$$

such that

1. (Homotopy invariance) A proper oriented homotopy equivalence $f: (M_1; A_1, A'_1) \to (M_2; A_2, A'_2)$ with $f^*(\gamma_2) = \gamma_1$ induces the identity $L(f_*(x), f_*(y)) = L(x, y)$ for $x \in E^1E^1H_\bullet(\tilde{M}_1, A_1)$ and $y \in E^1E^1H_\bullet(\tilde{M}_1, A'_1)$,

2. $L$ is sesquilinear, i.e., $\lambda L(x, y) = L(\overline{\lambda x}, \overline{\lambda y}) = L(x, \lambda y)$ for $x \in E^1E^1H_\bullet(\tilde{M}, A)$, $y \in E^1E^1H_\bullet(\tilde{M}, A')$ and $\lambda \in \Lambda$,

3. $L$ is $\varepsilon(pr+1)$-Hermitian, i.e., $L(x, y) = \varepsilon(pr+1)L(y, x)$ for $x \in E^1E^1H_\bullet(\tilde{M}, A)$ and $y \in E^1E^1H_\bullet(\tilde{M}, A')$, 

4. $L$ is non-singular, i.e., $L$ induces a $t$-anti $\Lambda$-isomorphism $E^1E^1H_\bullet(\tilde{M}, A) \simeq \text{Hom}(E^1E^1H_\bullet(\tilde{M}, A'), Q(\Lambda)/\Lambda)$.

When $M$ is triangulated, we can see that our pairing $L$ is essentially the same as (precisely, the $t$-conjugate of) a pairing of Blanchfield [B] (cf. Remark
Our next plan is to give a dual structure between $E^2E^3H_p(M, A)$ and $E^2E^3H_s(M, A')$ with $p+s+2=n$, but it turns out that there is not in general any non-singular pairing on these whole modules. In fact, $E^2E^3H_p(M, A) \cong E^2E^3H_s(M, A')$ as abelian groups in general. For this reason, we construct (in 6) a $\Lambda$-anti $\Lambda$-epimorphism $\theta: E^2E^3H_p(M, A) \rightarrow E^3E^3H_{p+1}(M, A')$ which is invariant under a proper oriented homotopy equivalence $f: (M_1; A_1, A'_1) \rightarrow (M_2; A_2, A'_2)$ with $f^*(\gamma_2) = \gamma_1$. Let $E^2E^3H_p(M, A)^\theta$ be the kernel of $\theta$. Similarly, $E^2E^3H_s(M, A')^\theta$ for the kernel of $\theta: E^2E^3H_s(M, A') \rightarrow E^3E^3H_{p+1}(M, A)$.

The Second Duality Theorem. For a compact oriented $n$-manifold $M$ with $\gamma \in H^i(M)$ and a splitting $(A, A')$ of $\partial M$ and integers $p, s$ with $p+s+2=n$, there is a pairing

$$l: E^2E^3H_p(M, A)^\theta \times E^2E^3H_s(M, A')^\theta \rightarrow \mathbb{Q}/\mathbb{Z}$$

such that

1. (Homotopy invariance) A proper oriented homotopy equivalence $f: (M_1; A_1, A'_1) \rightarrow (M_2; A_2, A'_2)$ with $f^*(\gamma_2) = \gamma_1$ induces the identity $l(f_*^x(x), f_*^y(y)) = l(x, y)$ for $x \in E^2E^3H_p(M_1, A_1)^\theta$ and $y \in E^2E^3H_s(M_1, A_1)^\theta$.
2. $l$ is $t$-isometric, i.e., $l(tx, ty) = l(x, y)$ for $x \in E^2E^3H_p(M, A)^\theta$ and $y \in E^2E^3H_s(M, A)^\theta$.
3. $l$ is $\varepsilon(ps+1)$-symmetric, i.e., $l(x, y) = \varepsilon(ps+1)l(y, x)$ for $x \in E^2E^3H_p(M, A)^\theta$ and $y \in E^2E^3H_s(M, A)^\theta$.
4. $l$ is non-singular, i.e., $l$ induces a $t$-anti $\Lambda$-isomorphism

$$E^2E^3H_p(M, A)^\theta \cong \text{Hom}_\mathbb{Z}(E^2E^3H_s(M, A')^\theta, \mathbb{Q}/\mathbb{Z}).$$

Since a finitely generated torsion-free $\Lambda$-module $H$ is $\Lambda$-free if and only if $E^1H=0$ (cf. 3), it follows that $BH_{p+1}(M, A)$ and $BH_{s+1}(M, A')$ are $\Lambda$-free if and only if $l$ defines a pairing

$$E^2E^3H_p(M, A) \times E^2E^3H_s(M, A') \rightarrow \mathbb{Q}/\mathbb{Z}.$$

Hence we see that $tH_p(M, A)$, $tH_s(M, A')$ are finite and $BH_{p+1}(M, A)$, $BH_{s+1}(M, A')$ are $\Lambda$-free if and only if $l$ defines a pairing

$$tH_p(M, A) \times tH_s(M, A') \rightarrow \mathbb{Q}/\mathbb{Z},$$

since a finitely generated $\Lambda$-module $H$ has $tH=\text{DH}(\cong E^2E^3H)$ if and only if $tH$ is finite. Farber [4] and Levine [L] constructed the same pairing when $tH_p(M, A)$, $tH_s(M, A')$ are finite and $BH_{p+1}(M, A)=BH_{s+1}(M, A')=0$.* Therefore, our pairing $l$ may be considered as an extreme generalization of their pairing. A basic idea of proving these Duality Theorems is to examine a universal coefficient exact sequence for cohomology over $\Lambda$, which has been done by

*) They also assumed that manifolds are piecewise-linear.
Levine [L] in an important special case (cf. Corollary 1.3).

In §1 we construct a universal coefficient exact sequence for chomology over a ring of global dimension ≤2. In §2 we describe the Reidemeister duality on a regular covering of a (topological) manifold. In §3 we note several properties of Λ-modules needed for our purpose. In §§4, 5 and 6 we prove the Zeroth, First and Second Duality Theorems, respectively.

1. A universal coefficient exact sequence for chomology over a ring of global dimension ≤2

Let Γ be a ring with unit. Let \( C = \{C_q, \partial\} \) be a left Γ-projective chain complex and \( F \), a left Γ-module. In general, \( H^*(C; F) = H^*(\text{Hom}_\Gamma(C, F)) \) and \( \text{Ext}_1^\Gamma(H^*(C), F) \) are abelian groups, but when Γ is commutative, they can be considered as Γ-modules. Let \( h: H^*(C; F) \rightarrow \text{Hom}_\Gamma(H^*(C), F) \) be the homomorphism defined by \( h(f)(z) = f(z) \) for \( f \in H^*(C; F) \) and \( z \in H^*(C) \). Let \( K^*(C; F) \) be the kernel of \( h \). We assume that Γ has the left global dimension ≤2. Then \( \text{Ext}_1^\Gamma(H^*(C), F) = 0 \) for \( i \geq 3 \) and we obtain the following Universal Coefficient Exact Sequence, referred to as UCES:

**Theorem 1.1.** For all \( q \), there is a natural exact sequence

\[
0 \rightarrow K^q(C; F) \xrightarrow{h} H^q(C; F) \xrightarrow{\eta} \text{Hom}_\Gamma(H^q(C), F) \xrightarrow{\Delta} K^{q+1}(C; F) \xrightarrow{\rho} \text{Ext}_1^\Gamma(H^q(C), F) \rightarrow 0 .
\]

The proof is quite elementary. The following corresponds to the usual universal coefficient theorem:

**Corollary 1.2.** If \( \text{Ext}_1^\Gamma(H^*(C), F) = 0 \), then for all \( q \) there is a natural short exact sequence

\[
0 \rightarrow \text{Ext}_1^\Gamma(H^q_{q-1}(C), F) \xrightarrow{\rho^{-1}} H^q(C; F) \xrightarrow{h} \text{Hom}_\Gamma(H^q(C), F) \rightarrow 0 .
\]

The following corresponds to the case considered by Levine [L]:

**Corollary 1.3.** If \( \text{Hom}_\Gamma(H^*(C), F) = 0 \), then for all \( q \) there is a natural short exact sequence

\[
0 \rightarrow \text{Ext}_1^\Gamma(H^q_{q-1}(C), F) \xrightarrow{\Delta} H^q(C; F) \xrightarrow{\rho} \text{Ext}_1^\Gamma(H^q_{q-1}(C), F) \rightarrow 0 .
\]

1.4 Proof of Theorem 1.1. For all \( q, B_q(C) = \partial C_{q+1} \) has the Γ-projective dimension ≤1, since \( C_q \) is Γ-projective and \( 0 \rightarrow B_q(C) \rightarrow C_q \rightarrow C_q/B_q(C) \rightarrow 0 \) is Γ-exact and \( C_q/B_q(C) \) has the Γ-projective dimension ≤2. So, \( Z_q(C) = \text{Ker}(\partial: C_q \rightarrow C_{q-1}) \) is Γ-projective by the short Γ-exact sequence \( 0 \rightarrow Z_q(C) \rightarrow C_q \rightarrow B_{q-1}(C) \rightarrow 0 \). This
sequence also induces an exact sequence

\[ 0 \to B^{\ast+1}(C) \to C^\ast \xrightarrow{j^\ast} Z^\ast(C) \to \text{Ext}^1(B^{\ast-1}(C), F) \to 0, \]

where \( B^{\ast+1}(C) = \text{Hom}_\Gamma(B^{\ast-1}(C), F), \ C^\ast = \text{Hom}_\Gamma(C, F) \) and \( Z^\ast(C) = \text{Hom}_\Gamma(Z_q(C), F) \) and \( j^\ast \) is the map induced from the inclusion \( Z_q(C) \subseteq C_q. \) Let \( Z^\ast(C) = \text{Im} j^\ast. \) Then we have an exact sequence \( 0 \to B^{\ast-1}(C) \to C^\ast \to Z^\ast(C) \to 0 \) and an isomorphism \( Z^\ast(C)/Z^\ast(C) \cong \text{Ext}^1(B^{\ast-1}(C), F). \) Regarding \( B^{\ast-1}(C) = \{ B^{\ast-1}(C) \} \) and \( Z^\ast(C) = \{ Z^\ast(C) \} \) as cochain complexes with trivial coboundary operators, we obtain from the short exact sequence \( 0 \to B^{\ast-1}(C) \to C^\ast \to Z^\ast(C) \to 0 \) a long cohomology exact sequence

\[ \delta \to H^{\ast+1}(Z^\ast(C)) \to H^\ast(B^{\ast-1}(C)) \to H^\ast(C; F) \to H^{\ast+1}(B^{\ast-1}(C)) \to \delta \]

Note that the coboundary map \( \delta: H^\ast(Z^\ast(C)) \to H^{\ast+1}(B^{\ast-1}(C)) \) is identical with the restriction \( i^\ast: Z^\ast(C) \to B^\ast(C) \) of the map \( i^\ast: Z^\ast(C) \to B^\ast(C), \) induced from the inclusion \( B_q(C) \subseteq Z_q(C). \) We have the following four short exact sequences.

\[ 0 \to \text{Ker} i^\ast \to H^\ast(C; F) \to \text{Ker} i^\ast \to 0, \]
\[ 0 \to \text{Ker} i^\ast \to \text{Ker} i^\ast / \text{Ker} i^\ast \to \text{Ker} i^\ast / \text{Ker} i^\ast \to 0, \]
\[ 0 \to (\text{Ker} i^\ast + Z^\ast(C))/Z^\ast(C) \to Z^\ast(C)/Z^\ast(C) \to Z^\ast(C)/(\text{Ker} i^\ast + Z^\ast(C)) \to 0, \]
\[ 0 \to \text{Im} i^\ast / \text{Im} i^\ast \to \text{Coker} i^\ast \to \text{Coker} i^\ast \to 0. \]

Using the isomorphisms \( \text{Ker} i^\ast / \text{Ker} i^\ast \cong (\text{Ker} i^\ast + Z^\ast(C))/Z^\ast(C) \) and \( Z^\ast(C)/(\text{Ker} i^\ast + Z^\ast(C)) \cong \text{Im} i^\ast / \text{Im} i^\ast \), we can construct an exact sequence

\[ 0 \to \text{Coker} i^\ast \xrightarrow{\alpha_1} H^\ast(C; F) \xrightarrow{\alpha_2} \text{Ker} i^\ast \xrightarrow{\alpha_3} Z^\ast(C)/Z^\ast(C) \xrightarrow{\alpha_4} \text{Coker} i^\ast \xrightarrow{\alpha_5} \text{Coker} i^\ast \to 0. \]

Since \( Z_q(C) \) is \( \Gamma \)-projective, the short exact sequence \( 0 \to B_q(C) \to Z_q(C) \to H_q(C) \to 0 \) induces an isomorphism \( \text{Ext}^1(B_q(C), F) \cong \text{Ext}^2(B_q(C), F) \) and an exact sequence

\[ 0 \to \text{Hom}_\Gamma(H_q(C), F) \to Z^\ast(C) \xrightarrow{\eta^\ast} B^\ast(C) \to \text{Ext}^1(H_q(C), F) \to 0, \]

so that \( \text{Hom}_\Gamma(H_q(C), F) \cong \text{Ker} i^\ast \) and \( \text{Coker} i^\ast \cong \text{Ext}^1(H_q(C), F). \) Note that the composite \( H^\ast(C; F) \xrightarrow{\alpha_2} \text{Ker} i^\ast \cong \text{Hom}_\Gamma(H_q(C), F) \) is given by \( h. \) So, \( \alpha_1 \) induces an isomorphism \( \text{Coker} i^\ast \cong K^\ast(C; F). \) Let \( \eta \) be the composite \( \text{Hom}_\Gamma(H_q(C), F) \cong \text{Ker} i^\ast \xrightarrow{\alpha_3} Z^\ast(C)/Z^\ast(C) \cong \text{Ext}^1(B_q(C), F) \cong \text{Ext}^2(B_q(C), F) \) and \( \Delta, \) the composite \( \text{Ext}^2(B_q(C), F) \cong \text{Ext}^3(B_q(C), F) \cong Z^\ast(C)/Z^\ast(C) \xrightarrow{\alpha_4} \text{Coker} i^\ast \cong K^{\ast+1}(C; F) \) and \( \rho, \) the composite \( K^{\ast+1}(C; F) \) \( \text{Coker} i^\ast \xrightarrow{\alpha_5} \text{Coker} i^\ast \cong \text{Ext}^1(H_q(C), F), \) where \( \cong \) denotes one of the isomorphisms constructed above or its inverse. Then we obtain the exact sequence stated in Theorem 1.1. It is easy to check from construction that a \( \Gamma \)-chain map between left \( \Gamma \)-projective chain complexes induces
homomorphisms commuting the resulting two exact sequences. It is similar for a \( \Gamma \)-homomorphism between coefficient left \( \Gamma \)-modules. This completes the proof.

2. The Reidemeister duality on a regular covering of a manifold

Let \( X \) be an oriented (possibly, non-compact) \( n \)-manifold and \( \partial_i X, i=1, 2, \) be \( \emptyset \) or \((n-1)\)-submanifolds of \( \partial X \) with \( \partial_1 X=\partial_2 X=\partial X, \partial_2 X=\partial X, \) and \( \partial_2 X=\partial X \). By Spanier [Sp, p. 301] the orientation of \( X \) determines a unique element of \( H_n(X, \partial X) = \lim_{\to} \{ H_n(X, (X-K) \cup \partial X) \mid K \subset X, \text{ compact} \} \), which we call the fundamental class of \( X \) and denote by \([X]\). For integers \( p, q \) with \( p+q=n \) the map \( \Pi[X]: H_p(X, \partial_1 X) \to H_q(X, \partial_2 X) \) is well defined by taking the limit of \( \Pi[X]_K: H_p(X, (X-K) \cup \partial X) \to H_q(X, \partial_2 X) \) for all \( K \), where \([X]_K \in H_n(X, (X-K) \cup \partial X) \) denotes the projection image of \([X]\).

2.1. The Poincaré duality theorem. The map \( \cap[X]: H_q(X, \partial_1 X) \to H_q(X, \partial_2 X) \) is an isomorphism.

This is known (cf., for example, [Ka9, Appendix A] for an outlined proof). Let \((\tilde{M}, \tilde{A})\) be a regular covering space over a compact pair \((M, A)\) with covering transformation group \( G \). The singular chain complex \( \Delta_t(M, \tilde{A}) \) forms a left \( ZG \)-free chain complex. \( H^t(M, \tilde{A}) \) is the cohomology of the complex \( \Delta^t(M, \tilde{A}) \) of all singular cochains with compact supports. Let \( H^*_G(M, \tilde{A}) \) be the cohomology of \( \Delta^*_G(M, \tilde{A})=\text{Hom}_{ZG}(\Delta_t(M, \tilde{A}), ZG) \). We define a cochain map

\[
\phi: \Delta^t(M, \tilde{A}) \to \Delta^*_G(M, \tilde{A})
\]

by the identity \( \phi(f)(x)=\sum_{g \in G} f(gx)g^{-1} \) for \( f \in \Delta^t(M, \tilde{A}) \) and \( x \in \Delta_t(M, \tilde{A}) \), where the sum is easily checked to be a finite sum.

Lemma 2.2. If \((M, A)\) is homotopy equivalent to a compact polyhedral pair, then the induced map \( \phi^*: H^t(M, \tilde{A}) \to H^*_G(M, \tilde{A}) \) is an isomorphism.

Proof. Since \( H^t(M, \tilde{A}) \) and \( H^*_G(M, \tilde{A}) \) are proper \( G \)-homotopy type invariants and \( \phi \) commutes with proper \( G \)-maps, it suffices to show that \( \phi^* \) is an isomorphism when \((M, A)\) is a compact polyhedral pair. Let \((M^*, \tilde{A}^*)\) be a triangulation of \((M, A)\) and \((\tilde{M}^*, \hat{A}^*)\) be its lift. For a subcomplex \( N^* \) of \( \tilde{M}^* \), let \( \Delta_t(\tilde{M}^*, \hat{A}^* \cup N^*) \) (or \( C_t(\tilde{M}^*, \hat{A}^* \cup N^*) \), resp.) be the ordered (or oriented, resp.) chain complex. Let \( k_1: \Delta_t(\tilde{M}^*, \hat{A}^* \cup N^*) \to \Delta_t(\tilde{M}, \hat{A} \cup N), N|=|N^| \), and \( k_2: \Delta_t(\tilde{M}^*, \hat{A}^* \cup N^*) \to C_t(\tilde{M}^*, \hat{A}^* \cup N^*) \) be the natural chain equivalences (cf. [Sp, 4.3.8 and 4.6.8]). Let \( \Delta^*_t(\tilde{M}^*, \hat{A}^*) \) (or \( C^*_t(\tilde{M}^*, \hat{A}^*) \), resp.) be the complex of all finite ordered (or oriented, resp.) cochains. Let \( \Delta^*_G(\tilde{M}^*, \hat{A}^*)=\text{Hom}_{ZG}(\Delta_t(\tilde{M}^*, \hat{A}^*), ZG) \) and \( C^*_G(\tilde{M}^*, \hat{A}^*)=\text{Hom}_{ZG}(C_t(\tilde{M}^*, \hat{A}^*), ZG) \). We have the following commutative diagram:
THREE DUALITIES ON THE INTEGRAL HOMOLOGY

\[ \Delta^i(\tilde{M}, \tilde{A}) \rightarrow \Delta^i(\tilde{M}^*, \tilde{A}^*) \hspace{1cm} \phi^*_1 \downarrow \phi^*_1 \downarrow \phi^*_1 \downarrow \phi^*_1 \Delta^i_c(\tilde{M}, \tilde{A}) \rightarrow \Delta^i(\tilde{M}^*, \tilde{A}^*) \rightarrow C^i_c(\tilde{M}^*, \tilde{A}^*), \]

where \( \phi \) are defined by the same rule as \( \phi \). Note that all of the \( \phi^*_i \)'s in this diagram induce isomorphisms in cohomology. In fact, for the upper \( \phi^*_i \), it can be seen by taking the limit of the sequence

\[ \Delta^i(\tilde{M}, \tilde{A} \cup N) \rightarrow \Delta^i(\tilde{M}^*, \tilde{A}^* \cup N^*) \rightarrow C^i_c(\tilde{M}^*, \tilde{A}^* \cup N^*) \]

for all cofinite subcomplexes \( N^* \) of \( \tilde{M}^* \), and for the lower \( \phi^*_i \), Eilenberg [E, p. 392] proved it. Since \( C^i_c(\tilde{M}^*, \tilde{A}^*) \) is \( \mathbb{Z}G \)-free of finite rank, we see that \( \phi^*_2 \) is bijective. Hence we have the isomorphism \( \phi^*: H^i_c(\tilde{M}, \tilde{A}) \rightarrow H^i(\tilde{M}, \tilde{A}) \), completing the proof.

2.3. **The Reidemeister duality theorem.** For a compact oriented \( n \)-manifold \( M \) and a splitting \( (A, A') \) of \( \partial M \) and integers \( p, q \) with \( p+q=n \), there is an isomorphism \( D: H^p_c(M, A) \rightarrow H^q_c(M, A') \).

**Proof.** By [K/S] \((M, A)\) is homotopy equivalent to a compact polyhedral pair. So, by Lemma 2.2 \( \phi^*: H^p_c(M, A) \rightarrow H^q_c(\tilde{M}, \tilde{A}) \) is an isomorphism. We take as \( D \) the composite \( H^p_c(\tilde{M}, \tilde{A}) \rightarrow H^q_c(\tilde{M}, \tilde{A}) \rightarrow H^q_c(M, A') \), where the later denotes the Poincaré duality. This completes the proof.

This duality is due to Reidemeister when \( M \) is triangulated (cf. Milnor [M1]). Wall [W] also considered it from a different viewpoint. We can always give \( H^p_c(\tilde{M}, \tilde{A}) \) a left \( \mathbb{Z}G \)-module structure so that \( D \) is a \( \mathbb{Z}G \)-isomorphism (cf. [M1]), but in this paper we never use it to avoid making a confusion. When \( G \) is abelian, \( H^p_c(\tilde{M}, \tilde{A}) \) and \( H^q_c(\tilde{M}, \tilde{A}) \) form \( \mathbb{Z}G \)-modules by the action of \( G \), so that \( \phi^* \) is a \( \mathbb{Z}G \)-isomorphism and \( D \) is a \( g \)-anti map, i.e., \( g^{-1}D=\bar{D}g \), for all \( g \in G \). Here we used the identity \( g[\tilde{M}]=[\tilde{M}] \). The following chain level version of this identity is used in 5 and 6:

**Lemma 2.4.** For a splitting \( (A, A') \) of \( \partial M \), there is a cycle \( z \) in \( \Delta^i(\tilde{M}; A, A') \)

\[ \Rightarrow \lim \{ \Delta^i(\tilde{M}^m)/(\Delta^i(\tilde{M}^m-K)+\Delta^i(\tilde{A})+\Delta^i(\tilde{A}')) \} \{ K \subset \tilde{M}, \text{compact} \} \]

representing \([\tilde{M}]\) such that \( g^2z=\bar{z} \) for all \( g \in G \).

**Proof.** Let \( z=\sum_{i=1}^n n_i \sigma_i \in \Delta^i_c(M) \) represent an element of \( H^i_c(M)/(\Delta^i_c(M)+\Delta^i(A)+\Delta^i(A')) \) corresponding to \([\tilde{M}]\subset H^i_c(M, \partial M) \) under the natural isomorphisms

\[ H^i_c(M)/(\Delta^i(A)+\Delta^i(A')) \approx H^i_c(M, \partial M) \approx H^i_c(M, \partial M) \] (cf. [Sp, 6.3.7]). Let \( \sigma_{i,j} \)

\( j \in J \), be the lifts of the singular \( n \)-simplex \( \sigma_i \) to \( \tilde{M} \). For any compact \( K \subset \tilde{M} \), \( \sigma_{i,j} \) are in \( \Delta^i_c(\tilde{M}-K) \) except a finite number of \( j \) and we let \( z=\sum_{i=1}^n n_i \sum_{j \in J} \sigma_{i,j} \in \Delta^i_c(\tilde{M}) \). Then we see that \( z \) is a cycle and
\{ z_\kappa \}_k \text{ determines a cycle } z \text{ in } \Delta'_S(M; \mathbb{A}, \hat{\mathbb{A}}) \text{ with } gz = z \text{ for all } g \in G. \text{ Take } \bar{x} \in \bar{V} \subset \bar{M} \text{ so that } \bar{V} \text{ is an open ball and the projection } \bar{M} \to M \text{ sends } (\bar{V}, \bar{x}) \text{ to a pair } (V, x) \text{ homeomorphically. For any cycle } z'_s = \sum_i z_i'^s n_i \sigma_i \in \Delta_n(M, M-x) \text{ with } \{ z_i'^s \} = \{ z \} \text{ in } H_* (M, M-x), \text{ let } z'_s = \sum_i z_i'^s n_i \sigma_i \in \Delta_n(\bar{M}, \bar{M}-\bar{x}). \text{ Then } 2^s z'_s \text{ is a well-defined cycle with } \{ 2^s z'_s \} = \{ z \} \text{ in } H_* (\bar{M}, \bar{M}-\bar{x}). \text{ Let } z'_s \text{ be in } \Delta_n(V, V-x). \text{ Since } \bar{V} \cap g \bar{V} = \emptyset \text{ for } g \neq 1, \text{ we see from the isomorphisms}

H_* (\bar{M}, \bar{M}-\bar{x}) \cong H_* (V, V-x) \cong H_* (M, M-x)

that } 2^s z'_s \text{ represents } [\bar{M}]_2 \text{ so that } z \text{ represents } [\bar{M}] \text{ (cf. [Sp, 6.3.3]). This completes the proof.}

3. Several properties of \Lambda-modules

Let \Lambda_0 = \Lambda \otimes \mathbb{Z} Q \text{ and for the field } Z_p \text{ of prime order } p, \Lambda_p = \Lambda \otimes \mathbb{Z} Z_p. \text{ For any finitely generated } \Lambda\text{-module } H, \text{ note that } E^H \text{ is } \mathbb{Z}\text{-torsion and } E^H \text{ is } \Lambda\text{-torsion, since } E^H \otimes \mathbb{Z} Q = E^H \otimes \Lambda Q (\Lambda) = 0. \text{ Let } H^{(s)} = \{ x \in H | px = 0 \}. \text{ } H^{(s)} \text{ is a } \Lambda_p\text{-module.}

\textbf{Lemma 3.1.} \Lambda/(m, \lambda_1, \cdots, \lambda_r) \text{ is a finite } \Lambda\text{-module for coprime non-zero } m, \lambda_1, \cdots, \lambda_r \in \Lambda \text{ with } m \text{ an integer.}

\textbf{Proof.} \text{ Let } m = \pm p_1 p_2 \cdots p_s \text{ be a prime decomposition. } \Lambda/(p_1, \lambda_1, \cdots, \lambda_r) = \Lambda_{p_1}/(\lambda_1, \cdots, \lambda_r) \text{ is finite. Since}

\Lambda/(p_1, \cdots, p_{s-1}, \lambda_1, \cdots, \lambda_r) \xrightarrow{p_1} \Lambda/(m, \lambda_1, \cdots, \lambda_r) \xrightarrow{p_1} \Lambda/(p_1, \lambda_1, \cdots, \lambda_r)

is exact, the induction on } s \text{ shows that } \Lambda/(m, \lambda_1, \cdots, \lambda_r) \text{ is finite, completing the proof.}

\textbf{Corollary 3.2.} A finitely generated \Lambda\text{-module } H \text{ has } mH = (t^m-1)H = 0 \text{ for some non-zero integers } m, m' \text{ if and only if } H \text{ is finite}

\textbf{Proof.} \text{ The “if” part is easy. The “only if” part follows from Lemma 3 1, since } H \text{ is a quotient of a direct sum of finite copies of } \Lambda/(m, t^m-1). \text{ This completes the proof.}

\textbf{Corollary 3.3.} For any \Lambda\text{-module } H, DH \text{ is the smallest } \Lambda\text{-submodule of } H \text{ containing all finite } \Lambda\text{-submodules. Further, if } H \text{ is finitely generated over } \Lambda, \text{ then } DH \text{ is finite.}

\textbf{Proof.} \text{ By Corollary 3.2 } DH \text{ contains all finite } \Lambda\text{-submodules. For } x \in DH \text{ let } \lambda_1, \cdots, \lambda_r \in \Lambda \text{ (} r \geq 2 \text{) be non-zero coprime elements with } \lambda_i x = 0 \text{ for all } i. \text{ Since } \Lambda_0 \text{ is PID, there are } \lambda'_1, \cdots, \lambda'_r \in \Lambda \text{ and non-zero } m \in \mathbb{Z} \text{ such that } \lambda_1 \lambda'_1 + \cdots + \lambda_r \lambda'_r = m. \text{ Then } mx = 0 \text{ and } x \text{ is in the image of a } \Lambda\text{-homomorphism } \Lambda/(m, \lambda_1, \cdots, m' \text{ if and only if } H \text{ is finite.
\[ \lambda_r \to H. \] Since \( m, \lambda_1, \ldots, \lambda_r \) are coprime, we see from Lemma 3.1 that \( x = 0 \) is a finite \( \Lambda \)-submodule of \( H \), showing the first half. If \( H \) is finitely generated over \( \Lambda \), so is \( DH \). Then \( DH \) is a quotient of a direct sum of a finite number of finite \( \Lambda \)-modules and hence is finite. This completes the proof.

**Lemma 3.4.** For a finitely generated \( \Lambda \)-module \( H \), \( E^2H \) is finite and there are natural isomorphisms \( E^3H \cong E^2DH \) and \( DH \cong E^3E^2H \).

Proof. Since \( E^2bH \) is \( \mathbb{Z}\)-torsional and finitely generated over \( \Lambda \), there is an integer \( m \neq 0 \) with \( mE^2bH = 0 \). By the short exact sequence \( 0 \to bH \to bH \to bH/mbH \to 0 \), we have \( E^2bH = mE^2bH = 0 \). So, \( E^2H \cong E^2bH \) by the short exact sequence \( 0 \to tH \to H \to bH \to 0 \). Let \( H_p \) be the \( p \)-component of \( tH \). We show that \( E^2H_p \) is finite by induction on \( n \geq 0 \) with \( p^nH_p = 0 \). The short exact sequence \( 0 \to pH_p \to H_p \to H_p/pH_p \to 0 \) induces an exact sequence \( E^2(H_p/pH_p) \to E^2H_p \to E^2(pH_p) \). \( H_p/pH_p \) is a finitely generated \( \Lambda \)-module and splits into a free \( \Lambda \)-module and a torsion (i.e., finite) \( \Lambda \)-module \( T_p \) so that \( E^2(H_p/pH_p) \cong E^2T_p \) is finite (by Corollary 3.2). By the inductive hypothesis, \( E^2(pH_p) \) is finite. Hence \( E^2H_p \) is finite. Since \( tH \) is finitely generated over \( \Lambda \), \( H_p = 0 \) except a finite number of \( p \). Therefore, \( E^2H \cong E^2tH \cong \bigoplus_p E^2H_p \) is finite. Next, let \( H' = tH/\text{DH} \).

Take an integer \( m' \neq 0 \) with \( (t^{m'} - 1)E^2H' = 0 \). Since \( 0 \to H' \to H'/(t^{m'} - 1)H' \to 0 \) is exact, \( t^{m'} - 1 : E^2H' \to E^2H' \) is onto, so that \( E^2H' = 0 \) and \( E^2tH \cong E^2DH \). Thus, \( E^3H \cong E^2tH \cong E^2DH \). Since \( DH \) is finite and \( E^2DH \cong \text{Hom}_\Lambda(DH, Q(\Lambda)/\Lambda) = 0 \), we see from \([L, (3.6)]\) that \( DH \cong E^2E^2H \). Using \( E^2H \cong E^2E^2H \), we complete the proof.

**Lemma 3.5.** For a finitely generated \( \Lambda \)-module \( H \), there are a natural short exact sequence \( 0 \to E^1BH \to E^1H \to E^1(TH/\text{DH}) \to 0 \) and natural isomorphisms \( E^1BH \cong DE^1H \) and \( TH/\text{DH} \cong E^1E^1H \).

Proof. By Lemma 3.4, \( E^2BH = 0 \). The short exact sequence \( 0 \to TH \to H \to BH \to 0 \) induces an exact sequence \( (S) \). Since \( E^0DH = E^0DH = 0 \) and \( E^1(TH/\text{DH}) \cong E^1TH \). Combining it with \((S)\), we obtain a desired sequence. Directly, \( DE^1TH \cong \text{Hom}_\Lambda(TH, Q(\Lambda)/\Lambda) = 0 \). By \((S)\), \( DE^1BH \cong DE^1H \). For a free \( \Lambda \)-module \( F \) of finite rank containing \( BH \) (cf. Cartan/Eilenberg \([C/E, \text{p. 131}]\)), we have \( E^1BH \cong E^1(F/BH) \). By Lemma 3.4, \( E^1BH \) is finite and \( E^2BH = DE^2BH \cong DE^2H \). Then \( E^1E^1TH \cong E^1E^1H \) by \((S)\). Since \( E^2(TH/\text{DH}) = 0 \) by Lemma 3.4, \( TH/\text{DH} \) has the projective dimension \( \leq 1 \) by \([L, (3.5)]\). By \((L, (3.6))\), we have \( TH/\text{DH} \cong E^1T(TH/\text{DH}) \). Since \( E^1T(TH/\text{DH}) \cong E^1E^1TH \), the proof is completed.

**Lemma 3.6.** For a finitely generated \( \Lambda \)-module \( H \), \( E^0H \) is \( \Lambda \)-free and there is a natural exact sequence \( 0 \to BH \to E^0E^0H \to E^2E^1BH \to 0 \).
Proof. Since $E^0BH = E^0H$, we may assume that $H = BH$. Then $H$ has the projective dimension $\leq 1$, for there is a $\Lambda$-free module $F$ containing $H$ and $F/H$ has the projective dimension $\leq 2$. A $\Lambda$-projective (i.e., $\Lambda$-free by [Se]) resolution $0 \to F^i \to F^0 \to H \to 0$ of $H$ with $F^i$ of finite rank induces an exact sequence $(S^*)$ $0 \to E^0H \to E^0F^0 \to E^0F^1 \to E^1H \to 0$. Since $E^1H$ has the projective dimension $\leq 2$ and $E^0F^i$ are $\Lambda$-free, $E^0H$ is $\Lambda$-projective that is $\Lambda$-free by [Se]. By Lemma 3.5, $E^1E^1H = 0$. Then $(S^*)$ induces an exact sequence $0 \to E^0E^0F^1 \to E^0E^0F^0 \to E^0E^0H \to E^2E^1H \to 0$. Using $F^i \cong E^0E^0F^i$ and the natural injection $H \to E^0E^0H$, we obtain a natural short exact sequence $0 \to H \to E^0E^0H \to E^2E^1H \to 0$. This completes the proof.

The following is obtained from Lemmas 3.4, 3.5 and 3.6:

**Corollary 3.7.** A finitely generated $\Lambda$-module $H$ is $\Lambda$-free if and only if $E^1H = E^2H = 0$.

**Corollary 3.8.** The following conditions on a finitely generated $\Lambda$-module $H$ are equivalent:

1. $E^2H = 0$,
2. $DH = 0$,
3. $H(p)$ is $\Lambda_p$-free for all prime $p$,
4. $H$ has the projective dimension $\leq 1$.

Proof. Take a short exact sequence $0 \to H \to F \to H \to 0$ with $F$, $\Lambda$-free of finite rank. Assuming (1), $E^1H \cong E^2H = 0$. By Lemma 3.6, $H \cong E^0E^0H$ is $\Lambda$-free, showing (1) $\Rightarrow$ (4). The others are trivial or follow from Lemma 3.4. This completes the proof.

Corollary 3.8 generalizes [L, (3.5)] and implies that a self-reciprocal $\Lambda$-module in [Ka] has the $\Lambda$-projective dimension $\leq 1$. The following observation is originally due to Kervaire [Ke] (when $\lambda = t - 1$):

**Corollary 3.9.** Let $\lambda \in \Lambda$ be no unit in $\Lambda_p$ for all prime $p$. If a finitely generated $\Lambda$-module $H$ has $\lambda H = H$, then $\lambda : H \cong H$, $H = TH$ and $tH$ is finite.

Proof. The Noetherian property gives $\lambda : H \cong H$ (cf. Shinohara/Sumners [S/S]). $E^0H$ is $\Lambda$-free by Lemma 3.6 and $\lambda : E^0H \cong E^0H$, meaning that $E^0H = 0$, i.e., $H = TH$. If $tH/\lambda H \neq 0$, then there is a prime $p$ with $(tH/\lambda H)(p) \neq 0$. $(tH/\lambda H)(p)$ is $\Lambda_p$-free by Corollary 3.8 and $\lambda : (tH/\lambda H)(p) \cong (tH/\lambda H)(p)$, meaning that $(tH/\lambda H)(p) = 0$, a contradiction. Hence $tH = DH$, which is finite by Corollary 3.3. This completes the proof.

4. **Proof of the Zeroth Duality Theorem**

For a $\Lambda$-projective chain complex $C$ with $H_\phi(C)$ finitely generated over $\Lambda$,
we see from UCES that \(TH^*(C; \Lambda) = K^*(C; \Lambda)\) and \(h: H^*(C; \Lambda) \to E^0H_*(C)\) induces a monomorphism \(BH^*(C; \Lambda) \to E^0H_*(C)\), also denoted by \(h\). We now return to 0 where \(M\) is a compact oriented \(n\)-manifold and \((\hat{M}; \hat{A}, \hat{A}')\) is an infinite cyclic covering of \((M; A, A')\), associated with \(\gamma \in H^0(M)\). We denote by \(\varepsilon_{\hat{M}}\) the augmentation map \(H_0(\hat{M}; G) \to G\) for any (untwisted) coefficient group \(G\).

For integers \(p, q\) with \(p+q=n\), the \(Z\)-intersection pairing

\[\text{Int}: H_0(\hat{M}, \hat{A}) \times H_0(\hat{M}, \hat{A}') \to Z\]

is given by the identity \(\text{Int}(x, y) = \varepsilon_{\hat{M}}((u \cup v) \cap [\hat{M}]) = \varepsilon_{\hat{M}}(u \cap Y)\) for \(x \in H_0(\hat{M}, \hat{A})\), \(y \in H_0(\hat{M}, \hat{A}'), u \in H_0(\hat{M}, \hat{A}'), v \in H_0(\hat{M}, \hat{A})\) with \(x = u \cap [\hat{M}], y = v \cap [\hat{M}]\) (cf. [Ka3, Appendix A]). Then the \(\Lambda\)-intersection pairing

\[\text{\hat{Int}}: H_0(\hat{M}, \hat{A}) \times H_0(\hat{M}, \hat{A}') \to \Lambda\]

is given by the identity \(\text{\hat{Int}}(x, y) = \sum_{i} \int_{\pi_{i}} \text{Int}(x, t'y) t'^{-1}\). By \(\Lambda\)-sesquilinearity of \(\text{\hat{Int}}, \text{Int}\) induces a pairing

\[\text{\hat{Int}}_\beta: BH_0(\hat{M}, \hat{A}) \times BH_0(\hat{M}, \hat{A}') \to \Lambda\]

Let \(\beta\) be the composite \(t\)-anti \(\Lambda\)-homomorphism

\[H_0(\hat{M}, \hat{A}) \xrightarrow{D^{-1}} H_0(\hat{M}, \hat{A}') \xrightarrow{h} E^0H_0(\hat{M}, \hat{A}')\]

where \(D\) denotes the Reidemeister duality in 2.

**Lemma 4.1.** For \(x \in H_0(\hat{M}, \hat{A})\) and \(y \in H_0(\hat{M}, \hat{A}')\), we have \(\beta(x) (y) = \text{\hat{Int}}(x, y)\).

**Proof.** For \(u_x = \{f_x\} \in H_0(\hat{M}, \hat{A}')\) with \(x = u_x \cap [\hat{M}]\) and \(y = \{c_y\}\), \(\beta(x) (y) = f_x(c_y) = \sum_{i} \varepsilon_{\hat{M}}(u_x \cap t'y) t'^{-1}\), as desired.

**4.2 Proof of the Zeroth Duality Theorem.** Let \(\beta_\beta\) be the composite \(t\)-anti \(\Lambda\)-monomorphism

\[BH_0(\hat{M}, \hat{A}) \xrightarrow{D^{-1}} BH_0(\hat{M}, \hat{A}') \xrightarrow{h} E^0H_0(\hat{M}, \hat{A}') = E^0BH_0(\hat{M}, \hat{A}')\]

induced from \(\beta\). By UCES and Lemma 3.4, the cokernel of \(\beta_\beta\) is a finite \(\Lambda\)-module. By Lemma 3.5, \(\beta_\beta\) induces a \(t\)-anti \(\Lambda\)-isomorphism \(\beta_\beta^*: E^0E^0BH_0(\hat{M}, \hat{A}') \cong E^0BH_0(\hat{M}, \hat{A})\) and hence a \(t\)-anti \(\Lambda\)-isomorphism \(\beta_\beta^*: E^0E^0BH_0(\hat{M}, \hat{A}) \cong E^0E^0BH_0(\hat{M}, \hat{A}')\). Regard \(BH_0(\hat{M}, \hat{A}) \subset E^0E^0BH_0(\hat{M}, \hat{A})\) and \(BH_0(\hat{M}, \hat{A}') \subset E^0E^0BH_0(\hat{M}, \hat{A}')\) in a natural way. We can see from Lemmas 3.4, 3.5 and 3.6 that \(\beta_\beta^* | BH_0(\hat{M}, \hat{A}) = \beta_\beta\) under the identification \(E^0E^0BH_0(\hat{M}, \hat{A}) = E^0E^0BH_0(\hat{M}, \hat{A})\). We define a pairing
\[ S: E^0E^0H_\rho(\tilde{M}, \tilde{A}) \times E^0E^0H_\rho(\tilde{M}, \tilde{A}') = E^0E^0H_\rho(\tilde{M}, \tilde{A}) \times E^0E^0H_\rho(\tilde{M}, \tilde{A}') \rightarrow \Lambda \]

by \( S(x, y) = \beta_x(y) \). By Lemma 4.1, \( S \) is an extension of the pairing \( \text{Int}_\rho: BH_\rho(\tilde{M}, \tilde{A}) \times BH_\rho(\tilde{M}, \tilde{A}') \rightarrow \Lambda \). From construction, (2) and (4) are satisfied. To see (1), let \( f^1: (M_1; A_1, A_1') \rightarrow (M_2; A_2, A_2') \) be a proper oriented homotopy equivalence with \( f^1_*\gamma_2 = \gamma_1 \). The lift \( \tilde{f}^1: (\tilde{M}_1; A_1, A_1') \rightarrow (\tilde{M}_2; A_2, A_2') \) induces a \( \Lambda \)-isomorphism \( \tilde{f}^1_*: (E^0E^0BH_\rho(M_1, A_1), BH_\rho(M_1, A_1)) \cong (E^0E^0BH_\rho(M_2, A_2), BH_\rho(M_2, A_2)) \) and \( \tilde{f}^2_*: (E^0E^0BH_\rho(M_2, A_2), BH_\rho(M_2, A_2)) \cong (E^0E^0BH_\rho(M_2, A_2'), BH_\rho(M_2, A_2')). \)

From construction, (2) and (4) are satisfied. To see (1), let \( f^1: (M_1; A_1, A_1') \rightarrow (M_2; A_2, A_2') \) be a proper oriented homotopy equivalence with \( f^1_*\gamma_2 = \gamma_1 \). The lift \( \tilde{f}^1: (\tilde{M}_1; A_1, A_1') \rightarrow (\tilde{M}_2; A_2, A_2') \) induces \( \Lambda \)-isomorphisms \( \tilde{f}^1_*: (E^0E^0BH_\rho(M_1, A_1), BH_\rho(M_1, A_1)) \cong (E^0E^0BH_\rho(M_2, A_2), BH_\rho(M_2, A_2)) \) and \( \tilde{f}^2_*: (E^0E^0BH_\rho(M_2, A_2), BH_\rho(M_2, A_2)) \cong (E^0E^0BH_\rho(M_2, A_2'), BH_\rho(M_2, A_2')). \)

For \( x \in E^0E^0BH_\rho(M_1, A_1), y \in E^0E^0BH_\rho(M_1, A_1) \), there are non-zero integers \( m, m' \) such that \( mx = x' \in E^0E^0BH_\rho(M_2, A_2) \) and \( m'y = y' \in E^0E^0BH_\rho(M_2, A_2') \), by Lemmas 3.4 and 3.6. Since \( \text{Int}^1(f^1, f^2) = \text{Int}^2(f^2, f^1) \), \( f^1\text{Int}^1(y') = f^2\text{Int}^2(y', x') = f^2\text{Int}^1(y', x') = \text{Int}^2(f^2, f^1)(y', x') = S_y(y') \).

That completes the proof.

5. Proof of the First Duality Theorem

For a \( \Lambda \)-module \( H \), we have a \( \Lambda \)-exact sequence \( \text{Hom}_\Lambda(H, Q(\Lambda)) \xrightarrow{\nu_\Lambda} \text{Hom}_\Lambda(H, Q(\Lambda)) \rightarrow Q(\Lambda) \xrightarrow{\nu_\Lambda} Q(\Lambda) \rightarrow 0 \), by which we identify \( E^1H \) with the cokernel of \( \nu_\Lambda \). Let \( C \) be a projective \( \Lambda \)-chain complex with \( H^*(C) \) finitely generated over \( \Lambda \). For \( u = \{f_a\} \in TH^{q+1}(C; \Lambda) \) we have a non-zero \( \lambda \in \Lambda \) and a cochain \( f_\lambda : C_q \rightarrow \Lambda \) such that \( \lambda f_a = \delta f_a \). Letting \( \rho(u) (c) = f_\lambda(c) \Lambda \in Q(\Lambda) \Lambda \) for \( c \in Z_q(C) \), we obtain a well-defined \( \Lambda \)-homomorphism \( \rho : TH^{q+1}(C; \Lambda) \rightarrow E^1H_q(C) \).

**Lemma 5.1.** For the map \( \rho : TH^{q+1}(C; \Lambda) \rightarrow E^1H_q(C) \) appearing in UCES, there is a natural \( \Lambda \)-isomorphism \( \rho' : E^1H_q(C) = E^1H_q(C) \) such that \( \rho = \rho' \).

**Proof.** Recall that \( \rho \) is the composite

\[ TH^{q+1}(C; \Lambda) = K^{q+1}(C; \Lambda) \xrightarrow{\rho} \text{Coker } i^1 \xrightarrow{\alpha} \text{Coker } i^2 \xrightarrow{\rho^2} E^1H_q(C). \]

For \( u = \{f_a\} \in TH^{q+1}(C; \Lambda) \), we have \( f_a(Z_{q+1}(C)) = 0 \) and hence a map \( f^a : B_q(C) \cong C_{q+1}Z_{q+1}(C) \rightarrow \text{Coker } i^1 \). Then note that \( \rho(u) = \{f^a\} \subseteq \text{Coker } i^1 \). The map \( \alpha \) is an obvious surjection. We shall construct a natural \( \Lambda \)-isomorphism \( \rho^2 : \text{Coker } i^2 \rightarrow E^1H_q(C) \).

For \( u = \{f_a\} \in TH^{q+1}(C; \Lambda) \), we have \( f_a(Z_{q+1}(C)) = 0 \) and hence a map \( f^a : B_q(C) \cong C_{q+1}Z_{q+1}(C) \rightarrow \text{Coker } i^1 \). Then note that \( \rho(u) = \{f^a\} \subseteq \text{Coker } i^1 \). The map \( \alpha \) is an obvious surjection. We shall construct a natural \( \Lambda \)-isomorphism \( \rho^2 : \text{Coker } i^2 \rightarrow E^1H_q(C) \).

**Proof.** Recall that \( \rho \) is the composite

\[ TH^{q+1}(C; \Lambda) = K^{q+1}(C; \Lambda) \xrightarrow{\rho} \text{Coker } i^1 \xrightarrow{\alpha} \text{Coker } i^2 \xrightarrow{\rho^2} E^1H_q(C). \]

For \( u = \{f_a\} \in TH^{q+1}(C; \Lambda) \), we have \( f_a(Z_{q+1}(C)) = 0 \) and hence a map \( f^a : B_q(C) \cong C_{q+1}Z_{q+1}(C) \rightarrow \text{Coker } i^1 \). Then note that \( \rho(u) = \{f^a\} \subseteq \text{Coker } i^1 \). The map \( \alpha \) is an obvious surjection. We shall construct a natural \( \Lambda \)-isomorphism \( \rho^2 : \text{Coker } i^2 \rightarrow E^1H_q(C) \).

The naturality of \( \rho^2 \) is clear. Given a \( \Lambda \)-homomorphism
Three Dualities on the Integral Homology

645

\( f: H_q(C) \to Q(\Lambda)/\Lambda, \) we have a \( \Lambda \)-homomorphism \( \tilde{f}: Z_q(C) \to Q(\Lambda) \) inducing \( f \), because \( Z_q(C) \) is \( \Lambda \)-projective. Then \( \tilde{f}(B_q(C)) \subset \Lambda \) and we can see that the correspondence \( \{ f \} \subset E^i H_q(C) \to \{ \tilde{f} \mid B_q(C) \} \subset \text{Coker} \ i^e \) is the well-defined inverse of \( \rho_i^e \). So, \( \rho_i^e \) is a natural \( \Lambda \)-isomorphism. The identity \( \rho_i^e \rho_i^e = \rho_i^e \) is easily checked. Letting \( \rho'' = \rho_i^e \rho_i^e \), we obtain the identity \( \rho = \rho'' \rho', \) completing the proof.

5.2 Proof of the First Duality Theorem. By UCES and Lemma 5.1, \( \rho' \) induces a \( \Lambda \)-isomorphism \( \text{TH}^*_\Lambda(M, A')/\text{DH}^*_\Lambda(M, A') \approx \text{E}^* \text{H}_r(M, A')/\text{DE}^* \text{H}_r(M, A') \), also denoted by \( \rho' \). By Lemma 3.5, the latter is identical with \( E^r(\text{TH}_r(M, A'))/\text{DH}_r(M, A') \) = \( \text{Hom}_\Lambda(\text{TH}_r(M, A'), \text{DH}_r(M, A')) \), \( Q(\Lambda)/\Lambda \). By the Reidemeister duality, we have a \( t \)-anti \( \Lambda \)-isomorphism \( \tilde{D}: \text{TH}^*_\Lambda(M, A')/\text{DH}^*_\Lambda(M, A') \approx \text{TH}_r(M, A')/\text{DH}_r(M, A') = \text{E}^* \text{H}_r(M, A') \). Then we define a pairing

\[
L: E^* \text{H}_r(M, A) \times E^* \text{H}_r(M, A') \to Q(\Lambda)/\Lambda
\]

by \( L(x, y) = \varepsilon(p + 1) \rho'' \tilde{D}^{-1}(x)(y) \). By construction, (2) and (4) are satisfied. To see (1), let \( f: (M_1; A_1, A_1') \to (M_2; A_2, A_2') \) be a proper oriented homotopy equivalence with \( f^*(\gamma_2) = \gamma_1 \). The lift \( \tilde{f} \) induces the following commutative diagram (Use \( f_\ast [M_1] = [M_2] \) for the left square):

This means \( L(\tilde{f}_\ast(x), \tilde{f}_\ast(y)) = L(x, y) \), showing (1). To see (3), let \( x = \{ c_x \} \in \text{TH}_r(M, A), y = \{ c_y \} \in \text{TH}_r(M, A'), u_x = \{ f_x \} \in \text{TH}^*_\Lambda(M, A') \) and \( u_y = \{ f_y \} \in \text{TH}^*_\Lambda(M, A') \) with \( u_x \cap [\widetilde{M}] = x \) and \( u_y \cap [\widetilde{M}] = y \). Then there are non-zero \( \lambda_x, \lambda_y \in \Lambda \) and \( c_x^+ \in \Delta_{\tau_1}(\widetilde{M}, A) \) and \( c_y^+ \in \Delta_{\tau_1}(\widetilde{M}, A') \) such that \( \partial c_x^+ = \lambda_x c_x \) and \( \partial c_y^+ = \lambda_y c_y \). Since \( \widetilde{X}_x u_x = \widetilde{X}_y u_y = 0 \), there are \( f_x^+ \in \Delta_{\tau}(\widetilde{M}, A') \) and \( f_y^+ \in \Delta_{\tau}(\widetilde{M}, A') \) such that \( \delta(f_x^+) = \widetilde{X}_x f_x \) and \( \delta(f_y^+) = \widetilde{X}_y f_y \). By definition,

\[
L(x, y) = \varepsilon(p + 1) \sum_i z^{-1} f_x^+ (t^i c_x) t^{-i} / \lambda_x (\text{mod } \Lambda).
\]

Assertion 5.3. \( L(x, y) = \varepsilon((p + 1)\gamma) \sum_i z^{-1} f_x^+ (t^{-i} c_x) t^{i} / \lambda_y (\text{mod } \Lambda) \). From this, we have \( L(x, y) = \varepsilon(pr + 1) \tilde{L}(y, x) \), showing (3), since \( L(y, x) = \varepsilon(r + 1) \sum_i z^{-1} f_x^+ (t^i c_x) t^{-i} / \lambda_y (\text{mod } \Lambda) \). This completes the proof of the First Duality Theorem, except for the proof of Assertion 5.3.

5.4 Proof of Assertion 5.3. By Lemma 2.4, we have a \( t \)-invariant cycle \( z \in \Delta_{\tau}(\widetilde{M}; A, A') \) representing \([\widetilde{M}]\). The map \( \cap z: \Delta_{\tau}(\widetilde{M}, A) \to \Delta_{\tau}(\widetilde{M}, A'), \cap z:\)
\[ \Delta^t(M, A') \to \Delta_{n, k}(M, A') \text{ or } \cap_{k^*} \Delta^t(M; A, A') \to \Delta_{n, k}(M, A') \ni \cap_{k^*} : \Delta^t(x, (M - K) \cup A) \to \Delta_{n, k}(M, A'), \cap_{k^*} : \Delta^t(x, (M - K) \cup A') \to \Delta_{n, k}(M, A') \text{ or } \cap_{k^*} = \text{Hom}_{K}(\Delta^t(M; A, A'))/\Delta^t(M - K) + \Delta^t(A') \to \Delta_{n, k}(M, A') \text{ with respect to the Alexander/Whitney diagonal approximation, respectively. Assume that } f_{x \cap k} = c_x \text{ and } f_{x \cap k} = c_y. \]

Let \( T : M \times M \to M \times M \) be the map changing the factors and \( T' : \Delta^t(M, A) \times \Delta^t(M, A') \to \Delta^t(M, A) \times \Delta^t(M, A') \) be the chain map defined by \( T'(c \otimes c') = \varepsilon(p) c \otimes c' \). Let \( \tau : \Delta^t(M \times M) \to \Delta^t(M \times M) \otimes \Delta^t(M) \) be a natural chain equivalence so that \( \tau \) is the Alexander/Whitney diagonal approximation, where \( d : M \to M \times M \) is the diagonal map. Since there is a natural chain homotopy \( D : \tau T - \tau T' \) (cf. [Sp, 5.3.8]), we have \( \delta \partial D + D \delta = D T T' - D T T' \), where each summand is regarded as a homomorphism \( \Delta^t(M, A) \otimes \Delta^t(M, A') \to \Delta^t(M, A, A') \) of degree 0. Using that \( \tau \partial D \) and \( \tau T' \partial D \) are the Alexander/Whitney diagonal approximations, we obtain \( \delta \partial D + D \delta = \delta \partial D + D \delta = \delta \partial D + D \delta \), and \( \sum \partial D + D \delta = \delta \partial D + D \delta = \delta \partial D + D \delta \), and \( \sum \partial D + D \delta = \delta \partial D + D \delta = \delta \partial D + D \delta \). Hence \( \sum \partial D + D \delta = \delta \partial D + D \delta = \delta \partial D + D \delta \). The result follows.

Remark 5.5. Assume that \( x, y \) are represented by \( c_x, c_y \) with \( |c_x| \cap |t^i c_y| = \emptyset \) for all \( i \). For example, if \( M \) is triangulable, then this assumption is satisfied. Then the intersection numbers \( \text{Int}(c_x^+, t^i c_y) \) are defined (cf. [Ka9, Appendix A]) and we have \( L(x, y) = \sum \text{Int}(c_x^+, t^i c_y) t^{-i} \). In fact, \( L(x, y) = \varepsilon(p + 1) \sum \text{Int}(c_x^+, t^i c_y) t^{-i} \). 6. Proof of the Second Duality Theorem

Since the infinite cyclic covering \( \tilde{M} \to M \) is the pullback of the exponential covering \( \exp : R \to S^1 \) by a map \( f_\gamma : M \to S^1 \) representing \( \gamma \), the lift \( \tilde{f}_\gamma : \tilde{M} \to R \) of \( f_\gamma \) is a proper map. Let \( \tilde{M}_i = \tilde{f}_\gamma(i, + \infty) \) and \( M_i = \tilde{f}_\gamma(- \infty, -i) \). Let \( H^*(\tilde{M}, \varepsilon(\pm) \cup A) = \lim_{i \to + \infty} H^*(\tilde{M}, M_i \cup A) \). Taking the limit \( i \to + \infty \) of the Mayer/Vietoris sequence for \( (M; M_i \cup A, M \cap A) \), we obtain an exact sequence \( \to H^t(\tilde{M}, A) \to H^t(\tilde{M}, \varepsilon(\pm) \cup A) \oplus H^t(\tilde{M}, \varepsilon(-) \cup A) \to H^t(\tilde{M}, A) \to H^t(\tilde{M}, A) \).

\[ \delta : H^t(\tilde{M}, A) \to H^{t+1}(\tilde{M}, A) \to H^{t+1}(\tilde{M}, A) \to H^{t+1}(\tilde{M}, A) \to \cdots \]
Lemma 6.1. There is one and only one element $\mu$ of $H_{n-1}(\tilde{M}, \partial \tilde{M})$ such that

1. $(t-1)\mu=0$,
2. The map $p^* : H_{n-1}(\tilde{M}, \partial \tilde{M}) \to H_{n-1}(M, \partial M)$ sends $\mu$ to $\gamma \cap [M]$, where $p$ denotes the covering projection.

Further, $\mu$ is given by $\delta_1(1) \cap [\tilde{M}]$ for $\delta_1 : H^n(\tilde{M}) \to H^n_1(\tilde{M})$.

Proof. For uniqueness, let $\mu, \mu'$ have (1) and (2). By the Wang exact sequence $H_{n-1}(\tilde{M}, \partial \tilde{M}) \to H_{n-1}(M, \partial M) \to H_{n-1}(M, \partial M)$ (cf. [Mi2]), we have $\mu - \mu' = (t-1)x$ for an $x \in H_{n-1}(\tilde{M}, \partial \tilde{M})$. By (1), $(t-1)x = 0$. By the Reidemeister duality and UCES, $TH_{n-1}(\tilde{M}, \partial \tilde{M}) \simeq TH_1(\tilde{M}) \simeq E^1H_0(\tilde{M})$ and the last is easily seen to be a direct sum of modules of type $H^n_1$ (cf. [Ka, Lemma 1]). Hence $(t-1)x = 0$ means $(t-1)x = 0$ and $\mu = \mu'$. Next, let $\mu'' = \delta_1(1) \cap [\tilde{M}]$. Since $t1 = 1$ and $t[\tilde{M}] = [\tilde{M}]$, $\mu''$ has (1). To see that it has (2), first assume that $f_\gamma$ has a leaf $V$ in $M$ (cf. [Ka2, p. 98]). Regard $V \subset \tilde{M}$ and thicken $V \times I \subset \tilde{M}$ so that $f_\gamma^{-1}(0) = V$ and $f_\gamma^{-1}I = V \times I$ and $f_\gamma | V \times I : V \times I \to I$ is the projection, where $I = [0, \varepsilon)$ for a small $\varepsilon > 0$. The following commutative diagram is obtained ($\tilde{I} = I - \partial I$):

$$
\begin{array}{ccc}
H'(I, \partial I) & \xrightarrow{(f_\gamma | V \times I)^*} & H'(V \times I, V \times \partial I) \\
\downarrow & & \downarrow \chi [V \times I] \\
H'(R, R - \hat{I}) & \xrightarrow{\hat{f}_\gamma^*} & H'_1(\tilde{M}, \tilde{M} - V \times \hat{I}) \\
\downarrow & & \downarrow \\
H'_1(R) & \xrightarrow{\hat{f}_\gamma^*} & H'_1(\tilde{M}) \\
\end{array}
$$

Since $\delta_1(1) = \hat{f}_\gamma^*[R]$ (cf. [Ka1, p. 98]), we see that $\mu'' = [V] \in H_{n-1}(\tilde{M}, \partial \tilde{M})$. So, $p^*(\mu'') = [V] \in H_{n-1}(M, \partial M)$, which equals $\gamma \cap [M]$. Hence $\mu''$ has (2). If $\gamma$ has no leaf, then we take $M_\rho = M \times CP^2$ and $\gamma_\rho = \gamma \times 1 \in H^n_1(M_\rho)$. Then by [K/S] $\gamma_\rho$ has a leaf. By the identity $(\delta_1(1) \times 1) \cap ([\tilde{M}] \times [CP^2]) = (\delta_1(1) \cap [\tilde{M}]) \times [CP^2]$, $\mu''$ has also (2). This completes the proof.

We call $\mu$ of Lemma 6.1 the fundamental class of the covering $\tilde{M} \to M$. By Lemmas 3.4 and 3.5, the epimorphism $\rho : TH^{n-1}_1(\tilde{M}, \tilde{A}) \to E^1H_1(\tilde{M}, \tilde{A})$ in UCES induces an epimorphism $DH^{n-1}_{1+1}(\tilde{M}, \tilde{A}) \to E^1BH_1(\tilde{M}, \tilde{A})$, also denoted by $\rho$. We define a $t$-anti epimorphism

$$
\theta : DH_1(\tilde{M}, \tilde{A}) \to E^1BH_{1+1}(\tilde{M}, \tilde{A}')
$$

by the composite $DH_1(\tilde{M}, \tilde{A}) \simeq DH^{n+2}(\tilde{M}, (\tilde{A}') \to E^1BH_{1+1}(\tilde{M}, \tilde{A}')$. Clearly, any proper oriented homotopy equivalence $f : (M_1, A_1, A'_1) \to (M_2, A_2, A'_2)$ with $f^*(\gamma_2) = \gamma_1$ induces the following commutative square:
\[ DH_\rho(M_1, A_1) \to E^iBH_{p+1}(M_1, A_1) \]
\[ \cong f_\ast \quad \cong f_\ast \]
\[ DH_\rho(M_2, A_2) \to E^iBH_{p+1}(M_2, A_2). \]

Let \( DH_\rho(M, A) \) be the kernel of \( \theta \). By identifying \( DH_\rho(A_1, A) \) with \( E^2 \Phi \rho_\nu (M, A) \) in a natural way, we also consider \( \theta \) as \( \theta: E^2 \Phi \rho_\nu (M, A) \to E^iBH_{p+1}(M, A) \).

In this case, the kernel of \( \theta \) is denoted by \( E^2 \Phi \rho_\nu (M, A) \). Note that \( DH_\rho(M, A') \) are the kernels of \( \theta: DH_\rho(M, A') \to E^iBH_{p+1}(M, A) \), \( \theta: E^2 \Phi \rho_\nu (M, A') \to E^iBH_{p+1}(M, A) \), respectively. Let \( c^\ast: \Phi \rho_\nu (M, A) \to H^p(M, A) \) be the monomorphism \( \rho^{-1} \) in Corollary 1.2 with \( \Gamma=F=Z \). Then the following square is commutative:

\[ e^\ast \Phi \rho_\nu (M, A) \to H^p(M, A) \]

surjection \( \uparrow \)

\[ h \uparrow \delta_{Q/Z} \]

\[ \text{Hom}_\mathbb{Z}(H^p(M, A), Q/Z) \cong H^p(M, A; Q/Z), \]

where \( \delta_{Q/Z} \) denotes the Bockstein coboundary map. Let \( \tau H^p(M, A) = \tau e^\ast H_\rho (M, A) \). By UCES with \( \Gamma=F=Z \), \( u = \{f\} \in H^p(M, A) \) is in \( \tau H^p(M, A) \) iff \( f|Z_{p+1}\Delta \) is \( 0 \). Let \( \tau e^\ast H^p(M, A) \) be the \( \Delta \)-submodule of \( \tau H^p(M, A) \) consisting of all elements \( u = \{f\} \) such that \( f(c) \equiv 0 \) (mod \( d \)) for \( c = \Delta_{p+1}(M, A) \) and \( d(\neq 0) \in Z \) with \( \partial c = dc \). Regarding \( e^\ast (H_\rho(M, A)/DH_\rho(M, A) \) in a natural way, we can obtain from an argument similar to \([F, \S 1]\) the following (whose proof is omitted):

**Lemma 6.2.** \( \tau e^\ast (H_\rho(M, A)/DH_\rho(M, A) \).

We consider the \( \tau \)-anti homomorphism \( \mu: \tau H^p(M, A) \to H_\ast(M, A) \).

**Lemma 6.3.** \( \tau H^p(M, A) \cap \mu = DH_\ast(M, A) \).

**Proof.** By Lemma 6.1, \( \tau H^p(M, A) \cap \mu = ((\tau H^p(M, A) \cup \delta_{Q/Z}(1))/M) = \delta_{\tau H^p(M, A)}(M, A) \cap [M]. \) For \( \{f\} \in \tau H^p(M, A) \), there are \( f \in \Delta_{p+1}(M, A) \) such that \( f = f^+ - f^- \) in \( \Delta_{p+1}(M, A) \). Then \( \delta_{\tau H^p(M, A)}(M, A) = 0 \). Since \( f|Z_{p+1}\Delta = 0 \), it follows that \( f^+ = f^- \) on \( Z_{p+1}\Delta(M, A) \) and \( \delta(f^+) = \delta f^+ \) is well-defined on it. Let \( f^\ast = \phi(f^+) |Z_{p+1}\Delta(M, A) \in E^2 \Phi \rho_\nu (M, A) \). Noting that some multiple \( \lambda f^\ast \) is extendable to \( \Delta_{p+1}(M, A) \) (for \( E^2 \Phi \rho_\nu (M, A) \) is finite) and \( \phi(f^+) = \delta f^+ \), we see from Lemma 5.1 that \( \Phi \rho_\nu (M, A) \in \tau H^p(M, A) \) and \( \rho \Phi \rho_\nu (M, A) = 0 = \rho \Phi \rho_\nu (M, A) \).

This means that \( \tau H^p(M, A) \cap \mu = \text{Ker}[\tau H_\ast(M, A) \to \Delta^{-1}] \)

\[ TH^p(M, A) \rho \to E^iBH_{p+1}(M, A), \]

which equals \( DH_\ast(M, A) \) by UCES. This completes the proof.
Lemma 6.4. $\text{Ker}[\cap \mu : \tau H^{p+1}(\tilde{M}, \tilde{A}) \to H_{\ast}(\tilde{M}, \tilde{A}')] \subset \tau_0 H^{p+1}(\tilde{M}, \tilde{A})$.

Proof. Let $u=\{f\} \in \tau H^{p+1}(\tilde{M}, \tilde{A})$ have $u \cap \mu = 0$. Then $\delta u=0$ and there are $\{f\} \in H^{p+1}(\tilde{M}, \varepsilon(\pm) \cup \tilde{A})$ with $f=f^+-f^-$ in $\Delta^{p+1}(\tilde{M}, \tilde{A})$. Since $f$ induces the zero map $H_{p+1}(\tilde{M}, \tilde{A}) \to Z$, $f^\pm$ induce the same map $H_{p+1}(\tilde{M}, \tilde{A}) \to Z$. Hence $\phi(f^+)\mid Z_{\pm} \Delta_{\pm}(\tilde{M}, \tilde{A})$ is well-defined and defines an element $f_\Delta \in \mathcal{E}^0 H_{p+1}(\tilde{M}, \tilde{A})$. Take an integer $m>0$ so that $(t^m-1)D\rho(\tilde{M}, \tilde{A})=0$. By UCES and Lemma 3.4, $(1-t^m)f_\Delta = \varepsilon f^+ \{f\}$ for some $\{f\} \in H^{p+1}(\tilde{M}, \tilde{A})$. Then we have $f^+-f^+t^m = f^-$ on $Z_{\pm} \Delta_{\pm}(\tilde{M}, \tilde{A})$. Define $f_\Delta, f_\Delta^\pm \in \Delta^{p+1}(\tilde{M}, \tilde{A})$ by $f_\Delta^m(c) = \sum_{h=n}^{m=n} f^m(t^m c)$, $f_\Delta^m(c) = \sum_{h=n}^{m=n} f^m(t^m c)$ and $f_\Delta = f^+ - f^-$. We have that $\delta(f_\Delta) = 0$ (mod $\varepsilon$), similarly, $f_\Delta^m(c) = f_\Delta^m(c)$, so that $f_\Delta^m(c) = f_\Delta^m(c)$. Let $f_\Delta = f_\Delta^+ - f_\Delta^-$ and $f_\Delta^\pm = f^\pm - f^\pm$. Then $f_\Delta = f_\Delta^+ - f_\Delta^-$. Let $\delta f_\Delta^+ = f_\Delta^-$ and $\delta f_\Delta^- = f_\Delta^+$. Then $f_\Delta^+ = f_\Delta^- = 0$ (mod $\varepsilon$). By UCES, it follows from the definition of $\mu$ that there is a $\mu$-homomorphism $f_\Delta^\prime : H_{p+1}(\tilde{M}, \tilde{A}) \to \mu$ inducing $\phi f_\Delta^\prime : H_{p+1}(\tilde{M}, \tilde{A}) \to \mu$ inducing $\phi f_\Delta^\prime$. Note that the composite

$$H_{p+1}(\tilde{M}, \tilde{A}); Q) \to H_{p+1}(\tilde{M}, \tilde{A}) \otimes Q \phi^\otimes \to H_{p+1}(\tilde{M}, \tilde{A}) \otimes Q \to \text{Hom}_\mu(H_{p+1}(\tilde{M}, \tilde{A}), \mu)$$

is onto by UCES. Let $\{f_\Delta^\prime\} \in H^{p+1}(\tilde{M}, \tilde{A}'; Q)$ be a preimage of $f_\Delta^\prime$. Let $f_\Delta = f_\Delta - f_\Delta^\prime$. Then $f_\Delta$ induces the zero map $H_{p+1}(\tilde{M}, \tilde{A}) \to Q/Z$. Let $c_0 = \varepsilon(p+1)$ $t^m f_\Delta^\prime \cap \tilde{Z} \in \Delta_{p+1}(\tilde{M}, \tilde{A}'; Q)$. Then $\delta c_0 = \varepsilon(p+1) \varepsilon(s+1-n) f_\Delta^\prime \cap \tilde{Z} = c_0$ (cf. [Sp, p. 253]). Regarding $f_\Delta$ as a cocycle $\Delta_{p+1}(\tilde{M}, \tilde{A}') \to Q/Z$, we have in $Q/Z$

$$f_\Delta(c_0) = \sum_{h=n}^{m=n} f^m(t^m c_0) = \varepsilon(p+1) \sum_{h=n}^{m=n} \varepsilon(t^m f^m \cup f_\Delta^m \cap \tilde{Z}) = \varepsilon(p+1) \varepsilon(s+1-n) f_\Delta^m \cap \tilde{Z} = \varepsilon(p+1) \varepsilon(s+1-n) f_\Delta^m \cap \tilde{Z} = 0,$$
for \( t^nf \cap z \in \mathbb{Z}_{+1} \Delta_\theta (\bar{M}, \bar{A}) \). Thus, \( u_m \in \tau_\theta H^{p+1} (\bar{M}, \bar{A}) \) and \( u = u_m + u_0 \in \tau_\theta H^{p+1} (\bar{M}, \bar{A}) \). This completes the proof.

**Theorem 6.5.** The maps \( e H_p (\bar{M}, \bar{A}) \to \tau H^{p+1} (\bar{M}, \bar{A}) \to H_\ast (\bar{M}, \bar{A}') \) induce isomorphisms

\[
e^\delta H_p (\bar{M}, \bar{A})^{\vartheta} \cong \tau H^{p+1} (\bar{M}, \bar{A}) \cap \mu \cong DH_\ast (\bar{M}, \bar{A}')^\vartheta.
\]

Proof. Let \( \tau_K H^{p+1} \) be the kernel of \( \cap \mu \). By Lemmas 6.2, 6.3 and 6.4, we obtain the following diagram:

\[
\begin{array}{ccc}
\tau H^{p+1} (\bar{M}, \bar{A}) \cap \mu & \xrightarrow{\text{injection}} & DH_\ast (\bar{M}, \bar{A}')^\vartheta \\
\text{surjection} & & \\
e H_p (\bar{M}, \bar{A}) \cap e (H_p (\bar{M}, \bar{A}) \cap DH_p (\bar{M}, \bar{A}))^\vartheta & \cong & \tau H^{p+1} (\bar{M}, \bar{A}) \cap \tau_\theta H^{p+1} (\bar{M}, \bar{A})
\end{array}
\]

Since \( e^\delta H_p (\bar{M}, \bar{A})^\vartheta \cong DH_p (\bar{M}, \bar{A})^\vartheta \) as abelian groups, it follows that \( |DH_p (\bar{M}, \bar{A})^\vartheta| \leq |DH_\ast (\bar{M}, \bar{A}')^\vartheta| \). Interchanging the roles of \( H_\ast (\bar{M}, \bar{A}) \) and \( H_\ast (\bar{M}, \bar{A}') \), we have \( |DH_\ast (\bar{M}, \bar{A}')^\vartheta| = |DH_p (\bar{M}, \bar{A})^\vartheta| \). This means that \( \tau_K H^{p+1} = \tau_\theta H^{p+1} (\bar{M}, \bar{A}) \) and \( \cap \mu : \tau H^{p+1} (\bar{M}, \bar{A}) \cap \tau_\theta H^{p+1} (\bar{M}, \bar{A}) \cong DH_\ast (\bar{M}, \bar{A}')^\vartheta \). This completes the proof.

6.6. Proof of the Second Duality Theorem. By Theorem 6.5, we define a pairing

\[
l: E^2 E^2 H_p (\bar{M}, \bar{A})^\vartheta \times E^2 E^2 H_p (\bar{M}, \bar{A}')^\vartheta \to Q / Z
\]

by \( l(x, y) = e(s+1)f_x (y) \) for \( f_x \in e DH_p (\bar{M}, \bar{A}) \) with \( \tau f_x \cap \mu = x \in DH_p (\bar{M}, \bar{A}) \) and \( y \in DH_\ast (\bar{M}, \bar{A})^\vartheta \). By construction, \( l \) has (2) and (4). For any \( u_\ast \in H_\ast (\bar{M}, \bar{A} ; Q / Z) \) and \( u_\ast \in H_\ast (\bar{M}, \bar{A} ; Q / Z) \) with \( \delta Q / Z (u_\ast) \cap \mu = x \) and \( \delta Q / Z (u_\ast) \cap \mu = y \), we also have in \( Q / Z \)

\[
l(x, y) = e(s+1)e (u_\ast \cup \delta Q / Z (u_\ast)) \cap \mu = e (u_\ast \cup \delta Q / Z (u_\ast)) \cap \mu
\]

(cf. [F, Lemma 3.8]). We have \( l(x, y) = e(ps+1)(y, x) \), showing (3). (1) is obvious, since \( \mu \) is invariant under a proper oriented homotopy equivalence \( f: M_1 \to M_2 \) with \( f^\ast (\gamma_2) = \gamma_1 \). This completes the proof.

**References**


Three Dualities on the Integral Homology

(1947), 378–417.


