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Osaka University
ON CERTAIN QUADRATIC FORMS RELATED TO
SYMMETRIC RIEMANNIAN SPACES

BY

SOJI KANEYUKI and TADASHI NAGANO

In his recent works [8] [9], Matsushima investigates the Betti numbers of a certain type of compact locally symmetric riemannian manifolds and he shows that lower dimensional Betti numbers of such a manifold can be determined if we know that certain quadratic forms are positive definite. In order to verify the last condition, it is sufficient to compute the minimal eigenvalue of a linear transformation $Q$ which is defined in terms of the curvature tensor of the manifold. Let $M$ be the universal covering manifold of the manifold in question. In the case that $M$ is an irreducible symmetric bounded domain in $C^n$, the eigenvalues of $Q$ were computed by Calabi-Vesentini [4] and Borel [2], and Betti numbers are given by Matsushima himself. In the paper [7], the authors have treated the case that the compact form of $M$ is a group manifold. The purpose of the present paper is to study the eigenvalues of $Q$ for the case which remains to be treated, i.e. the case that $M$ is a non-compact irreducible symmetric riemannian manifold not isomorphic to a bounded domain in $C^n$. Applying the results of Matsushima, we see that the $p$-th Betti number $b_p$ of the compact locally symmetric space treated here is equal to that of the compact form of $M$ if the ratio $p/dim M$ is sufficiently small; in particular, $b_1$ vanishes in most cases. Our results give also informations about the Betti numbers of $G/\Gamma$ where $G$ is a semi-simple Lie group without compact simple factors and $\Gamma$ is a discrete subgroup of $G$ with compact quotient space $G/\Gamma$.

It is our pleasure to acknowledge Prof. Y. Matsushima for his valuable suggestions and having given us a chance to read his manuscripts of [8] [9] before publication.

§1. Preliminaries.

The notations settled in this section will be used throughout this paper. We denote by $\mathfrak{g}$ a real semi-simple Lie algebra and by $\varphi_\mathfrak{g}$ the Killing form of $\mathfrak{g}$. Let $\mathfrak{l}$ be a subalgebra of $\mathfrak{g}$. We say that the pair
(g, ℓ) is a symmetric pair, if the vector space g admits a direct sum decomposition \( g = ℓ + m \) such that \([ℓ, m] \subseteq m\) and \([m, m] \subseteq ℓ\). By the symmetric square \( m \vee m \) we mean the vector space of the symmetric tensors belonging to \( m \otimes m \). The restriction of \( \phi_g \) to \( m \) is a non-degenerate inner product on \( m \), and this gives rise to non-degenerate inner products on \( m \otimes m \) and \( m \vee m \), which are denoted by \( \langle , \rangle_m \otimes_m \) and \( \langle , \rangle_{(g, ℓ)} \) respectively. For brevity we write \( \langle , \rangle \) instead of \( \langle , \rangle_{(g, ℓ)} \). We define a linear endomorphism \( Q \) of the tensor space \( m \otimes m \) by the formula

\[
\langle Q(x \otimes y), z \otimes u \rangle_m \otimes_m = \langle x, y \rangle_m \otimes_m \langle z, u \rangle_m
\]

for any \( X, Y, Z, U \in m \). The subspace \( m \vee m \) is stable under \( Q \), and we shall denote also by \( Q \) the restriction of \( Q \) to \( m \vee m \). It is obvious that \( Q \) is a self-adjoint operator on \( m \vee m \) with respect to the inner product \( \langle , \rangle \). Now, following Matsushima [9], we define a quadratic forms \( H^r \) \((r = 1, 2, \ldots)\) on \( m \vee m \) by putting

\[
H^r(\xi) = \frac{b_r(\xi)}{r} \langle \xi, \xi \rangle + \langle Q\xi, \xi \rangle
\]

for any \( \xi \in m \vee m \), where

\[
b_r(\xi) = \min \{ 1 + \varphi_{(g, ℓ)}(X, X) ; \varphi_g(X, X) = -1 \}
\]

and \( \varphi_{(g, ℓ)} \) is the Killing form of \( ℓ \). Matsushima [8] defines also a quadratic form \( H \) on \( m \vee m \) which coincides with \( H^1 \) in the case that \( g \) is non-compact simple and \( ℓ \) is a compact semi-simple Lie algebra. In the following, we shall write \( Q, H^r, H \) also as \( Q_{(g, ℓ)}, H^r_{(g, ℓ)}, H_{(g, ℓ)} \) respectively; if \( g \) is a non-compact simple Lie algebra, \( ℓ \) means always the subalgebra corresponding to a maximal compact subgroup of the adjoint group of \( g \). The theorems of Matsushima are now stated as follows:

**Theorem A** ([8] Theorem 1). Let \( G \) be a semi-simple Lie group, all of whose simple factors are non-compact. Suppose that, for each simple factor \( G_i \) of \( G \), the quadratic form \( H_{(g, ℓ)} \) associated to the Lie algebra \( g_i \) of \( G_i \) is positive definite. Let \( \Gamma \) be a discrete subgroup of \( G \) with compact quotient space \( G/\Gamma \). Then the first Betti number of \( G/\Gamma \) is equal to 0.

**Theorem B** ([9] Theorem 1 and § 9). Let \( M \) be a simply connected, irreducible symmetric riemannian manifold which is non-compact and non-euclidean. Let \( G \) be the identity component of the group of all isometries of \( M \). Let \( \Gamma \) be a discrete subgroup of \( G \) with compact quotient \( G/\Gamma \) and without element of finite order different from the identity, so that \( \Gamma \) is a
discontinuous group of isometries of \( M \) with compact quotient \( M/\Gamma \). Let \( M_u \) be the compact form of \( M \). If the quadratic form \( H'(g,l) \) is positive definite, then the \( r \)-th Betti number \( b_r \) of \( M/\Gamma \) is equal to the \( r \)-th Betti number \( b_r(M_u) \) of \( M_u \).

We shall see when the quadratic forms \( H(g,l) \) and \( H'(g,l) \) are positive definite for a real non-compact simple Lie algebra \( g \). In the case that the center of \( \mathfrak{f} \) is not trivial, this question is already discussed by Matsushima [8] [9]. We shall therefore confine ourselves to the case that the center of \( \mathfrak{f} \) reduces to \((0)\) so that \( \mathfrak{f} \) is a compact semi-simple algebra. We know that \( H=H' \) in this case. Thus our results on \( H' \), given in §4, are sufficient to apply Theorems A and B for this case.

In order that the quadratic form \( H' \) is positive definite, it is necessary and sufficient that the absolute value of the minimal eigenvalue of \( Q \) is strictly smaller than \( b_r(g,l) \). Now, under the assumption that \( g \) is non-compact simple and \( \mathfrak{f} \) is semi-simple, the value \( b_r(g,l) \) is computed as follows:

(a) If \( \mathfrak{f} \) is simple, then we have by [8]

(1.4) \[
    b_r(g,l) = \frac{\dim m}{4 \dim \mathfrak{f}}
\]

(b) When \( \mathfrak{f} \) is semi-simple and has the same rank as \( g \),

(1.5) \[
    2b(g,l) = \text{Min} \{1 - \varphi_g(\alpha_\mathfrak{f}, \alpha_\mathfrak{f})/\varphi_\mathfrak{f}(\alpha_\mathfrak{f}, \alpha_\mathfrak{f}) ; \alpha \text{ is a root of } \mathfrak{f}^\prime\}
\]

where \( \alpha_\mathfrak{g} \) or \( \alpha_\mathfrak{f} \) is the contravariant representative of \( \alpha \) with respect to the Killing form of \( g \) or \( \mathfrak{f} \) respectively.

(c) \((A_3, D_3)\) and \((D_1, B_q \times B_{l-q-1})\) are the only symmetric pairs for which \( \mathfrak{f} \) is semi-simple and is not of the same rank as \( g \). One finds

(1.6) \[
    2b = \frac{3}{4} \quad \text{for } (g, \mathfrak{f}) = (A_3, D_3)
\]

\[
    2b = \frac{p}{2(l-1)} \quad \text{for } (g, \mathfrak{f}) = (D_l, B_q \times B_{l-q-1}), \quad \text{and } p < l,
\]

where \( p = 2q+1 \)

§ 2. Relation with the group manifold.

Given a symmetric pair \((g, \mathfrak{f})\) with non-compact simple \( g \) and compact \( \mathfrak{f} \), we wish to compare the minimal eigenvalue of \( Q(g,l) \) with that of \( Q \) for symmetric pair \((g_u \times g_u, g_u)\). Let \( g = \mathfrak{f} + m \) be the Cartan decomposi-
tion. Denoting by $g^c$ the complexification of $g$, the space $g_m = \mathfrak{f} + \sqrt{-1} \mathfrak{m}$ is also a compact simple subalgebra of $g^c$. The pair $(g_m, \mathfrak{f})$ is also a symmetric pair, the compact form of $(g, \mathfrak{f})$. By an obvious identification $\sqrt{-1} \mathfrak{m} \vee \sqrt{-1} \mathfrak{m} = \mathfrak{m} \vee \mathfrak{m}$, one obtains the equality $Q(\mathfrak{f}, \mathfrak{f}) = -Q(g_m, t)$ directly from the definition (1.1). In order to verify that $H(\mathfrak{f}, t)$ is positive definite it is thus sufficient to show that $b(\mathfrak{f}, t)/r$ is strictly greater than the maximal eigenvalue of $Q(g_m, t)$. Let $\sigma$ be the injective homomorphism of $g_m$ into $g_m \times g_m$ defined by $\sigma(X) = (X, X)$ for any $X \in g_m$. We define another injection $\tau : g_m \rightarrow g_m \times g_m$ by $\tau(X) = (X, -X)/\sqrt{2}$; $\tau$ is isometric with respect to the Killing forms. The pair $(g_m \times g_m, \sigma(g_m))$, briefly denoted by $\mathfrak{m}$, is a symmetric pair with the Cartan decomposition $g_m \times g_m = \sigma(g_m) + \tau(g_m)$

**Lemma 2.1.** The notations being as above, if $\kappa$ (resp. $\kappa_m$) is the maximal eigenvalue of $Q(g_m, t)$ (resp. $Q_m$), then we have the inequality: $\kappa \leq 2\kappa_m$

**Proof.** If $\tau \vee \tau : g_m \vee g_m \rightarrow \tau(g_m) \vee \tau(g_m)$ denotes the mapping naturally induced by $\tau$ and if $\pi$ denotes the orthogonal projection of $(\tau \vee \tau)(g_m \vee g_m)$ onto $(\tau \vee \tau)(\mathfrak{m} \vee \mathfrak{m})$ then one obtains

$$\pi \cdot Q_m \cdot (\tau \vee \tau) = \frac{1}{2} (\tau \vee \tau) Q(g_m, t)$$

In fact, one has

$$2\langle Q_m(\tau(X) \vee \tau(Y)), \tau(Y) \vee \tau(Y) \rangle$$

$$= 2\rho_{g_m} ([\tau(X), \tau(Y)], [\tau(Y), \tau(X)])$$

$$= \rho_{g_m}(\tau([X, Y]), \tau([Y, X]))$$

$$= \rho_{g_m}([X, Y], [Y, X])$$

$$= \langle Q(g_m, t)(X \vee X), Y \vee Y \rangle$$

$$= \langle (\tau \vee \tau) \cdot Q(g_m, t)(X \vee X), \tau(Y) \vee \tau(Y) \rangle$$

for any $X, Y \in \mathfrak{m}$. For any eigenvector $\xi$ of $Q(g_m, t)$ corresponding to $\kappa$, one therefore finds

$$2\kappa_m \langle \xi, \xi \rangle = 2\kappa_m \langle \tau \vee \tau)(\xi), (\tau \vee \tau)(\xi) \rangle$$

$$\geq 2\langle Q_m((\tau \vee \tau)(\xi)), (\tau \vee \tau)(\xi) \rangle$$

$$= \langle Q(g_m, t) \xi, \xi \rangle = \kappa \langle \xi, \xi \rangle$$

**Lemma 2.2.** The number $2\kappa_m$ in lemma 2.1 equals $\rho_{g_m}(\vartheta, \vartheta)$ where $\vartheta$ is the highest root of $g_m$

For the proof, see [7].
It follows from the above arguments that $H_{(g, t)}$ is positive definite if $b_{(g, t)} > \varphi_{g_k}(\vartheta, \vartheta')$. The table I gives the values $b_{(g, t)}$ and $\varphi_{g_k}(\vartheta, \vartheta')$ for all symmetric pairs $(g, t)$ with non-compact simple $g$ and compact semi-simple $t$. We get then.

**Lemma 2.3.** The quadratic form $H_{(g, t)}$ is positive definite for all $(g_k, t)$ except for the cases AII ($l=2, 3$), BII, DII, CII ($p=1, 2$), GI, or FII.

<table>
<thead>
<tr>
<th>$(g_k, t)$</th>
<th>$b_{(g, t)}$</th>
<th>$2c_u = \varphi_{g_k}(\vartheta, \vartheta')$</th>
</tr>
</thead>
<tbody>
<tr>
<td>A I</td>
<td>$\left(A_l, B_{l/2}\right)$</td>
<td>$(l+3)/4(l+1)$</td>
</tr>
<tr>
<td>A II</td>
<td>$\left(A_{p-1}, C\right)$</td>
<td>$(l-1)/4l$</td>
</tr>
<tr>
<td>B I</td>
<td>$\left(B_l, D_{l-1}\right)$</td>
<td>Min $\left(\frac{p}{2l-1}, \frac{2l-2p+1}{2(2l-1)}\right)$</td>
</tr>
<tr>
<td>D I</td>
<td>$\left(D_p, B_{l-1}\right)$</td>
<td>$p/2(l-1)$</td>
</tr>
<tr>
<td>B II</td>
<td>$\left(B_l, B_l\right)$</td>
<td>$1/2(2l-1)$</td>
</tr>
<tr>
<td>D II</td>
<td>$\left(D_l, B_{l-1}\right)$</td>
<td>$1/4(l-1)$</td>
</tr>
<tr>
<td>C II</td>
<td>$\left(C_l, C_{l-1}\right)$</td>
<td>$p/2(l+1)$</td>
</tr>
<tr>
<td>E I</td>
<td>$\left(E_6, C_4\right)$</td>
<td>$7/24$</td>
</tr>
<tr>
<td>E II</td>
<td>$\left(E_6, A_1 \times A_3\right)$</td>
<td>$1/4$</td>
</tr>
<tr>
<td>E IV</td>
<td>$\left(E_6, F_4\right)$</td>
<td>$1/8$</td>
</tr>
<tr>
<td>E V</td>
<td>$\left(E_7, A_1\right)$</td>
<td>$5/18$</td>
</tr>
<tr>
<td>E VI</td>
<td>$\left(E_7, A_1 \times D_6\right)$</td>
<td>$2/9$</td>
</tr>
<tr>
<td>E VIII</td>
<td>$\left(E_8, D_6\right)$</td>
<td>$4/15$</td>
</tr>
<tr>
<td>E IX</td>
<td>$\left(E_8, A_1 \times E_7\right)$</td>
<td>$1/5$</td>
</tr>
<tr>
<td>G I</td>
<td>$\left(G_2, A_1 \times A_1\right)$</td>
<td>$1/4$</td>
</tr>
<tr>
<td>F I</td>
<td>$\left(F_4, A_1 \times C_3\right)$</td>
<td>$5/18$</td>
</tr>
<tr>
<td>F II</td>
<td>$\left(F_4, B_2\right)$</td>
<td>$1/9$</td>
</tr>
</tbody>
</table>

§ 3. The exceptional cases

In this section we shall determine the eigenvalues of $Q_{(g, t)}$ for the exceptional cases in lemma 2.3 by the method employed in [7]. Given a symmetric pair $(g, t)$, we write $\rho$ for the linear isotropy representation
\(\rho(X) = ad(X)|m, X \in \mathfrak{f}\). We may extend \(\rho\) and \(Q\) to a representation of \(\mathfrak{f}^c\) on \(m^c\) and to an endomorphism of \(m^c \vee m^c\) respectively, which will be denoted by the same letters. \(Q\) commutes with \(\rho \vee \rho\), the restriction of \(\rho \otimes \rho\) to \(m^c \vee m^c\). Let \(\rho \vee \rho = \rho_1 + \cdots + \rho_s\) be the decomposition of \(\rho \vee \rho\) into irreducible representations. The highest weights \(\Lambda_{(i)}\) of \(\rho_i\) are assumed to be in the order \(\Lambda_{(1)} > \cdots > \Lambda_{(s)}\) with respect to some ordering; this assumption will be satisfied in the following cases. Each subspace \(W_i\) of \(m^c \vee m^c\) spanned by the weight vectors of \(\Lambda_{(i)}\) is stable under \(Q\). We write \((TrQ)_i\) for the trace of \(Q\) restricted to \(W_i\). We know in [7] that \(Q\) is represented by a scalar matrix in the irreducible invariant subspace of \(m^c \vee m^c\) corresponding to \(\rho_i\). If \(\kappa_i\) denote the unique eigenvalue of \(Q\) restricted to the subspace, then

\[ (TrQ)_i = \sum_{1 \leq j \leq s} m_j(\Lambda_{(i)})\kappa_j; \]

where \(m_j(\Lambda_{(i)})\) is the multiplicity of the weight \(\Lambda_{(i)}\) in \(\rho_j\).

Moreover, if \(\mathfrak{g}\) is compact simple, which will be the case in the sequel, one has

\[ TrQ = (\dim m)/4, \quad \dim m = \dim \mathfrak{g} - \dim \mathfrak{f}, \]

\[ (3.3) -1/2 \text{ is an eigenvalue with multiplicity one of } Q. \]

These formulas (3.1) to (3.3) are given in [7] and will be used to compute the eigenvalues of \(Q\). Hereafter a representation and its highest weight will be denoted by the same letter.

(I) The case of \(AII: (\mathfrak{g}_\text{II}, \mathfrak{f}) = (A_{l-1}, C_l) \quad l \gg 2\)

First we try to find the highest weight of \(\rho\) and its weight vector: For an arbitrary square matrix \(X\) of degree \(2l\), we divide \(X\) into four parts

\[ X = \begin{pmatrix} A_X & B_X \\ C_X & D_X \end{pmatrix} \]

where \(A_X, B_X, C_X, D_X\) are square matrices with complex coefficients of degree \(l\). \(\mathfrak{g}^c\) is the Lie algebra \(\mathfrak{sl}(2l, \mathbb{C}) = \{X; Tr(A_X + D_X) = 0\}\). If \(J\) denotes a \((2l, 2l)\)-matrix such that \(A_J = D_J = 0\) and \(B_J = -C_J = E\) (= the unit matrix), \(\mathfrak{f}^c\) is \(\{X \in \mathfrak{sl}(2l, \mathbb{C}); JX + XJ = 0\}\). Hence the orthogonal complement \(m^c\) of \(\mathfrak{f}^c\) consists of the matrices \(Y\) such that \(JY - YJ = 0\), or equivalently, such that \(B_Y, C_Y\) are skew-symmetric matrices and one has \(A_Y = D_Y\). An element \(X\) of \(\mathfrak{f}^c\) operates on \(m^c\) by \(Y \in m^c \rightarrow \rho(X)Y = [X, Y] = XY - YX\). On the other hand, the identity representation of \(\mathfrak{f}^c\) operating
on $C^{2l}$ naturally induces a representation $\rho'$ on $C^{2l} \wedge C^{2l}$ which we take as the space of skew-symmetric matrices of degree $2l$; $\rho'(X)Z = XZ + Z'X$, $Z$=skew-symmetric. $\rho'(t')$ leaves $J$ invariant, and $\rho'$ is irreducible on the space $C^{2l}$ of skew-symmetric matrices which are orthogonal to $J$. This irreducible representation, denoted also by $\rho'$, has the highest weight $\Lambda_2 = \lambda_1 + \lambda_2$ if a Cartan subalgebra of $\mathfrak{t}^r$ consists of diagonal matrices, $H$, and a base $(H_i)$ is so chosen that $H = \sum \lambda_i H_i$ in case that the $(i, i)$-element of $A_H$ is $\lambda_i$. If $\alpha(Y)$ denotes the skew-symmetric matrix $-YJ$ for any $Y$ in $m^r$, $\alpha; m^r \rightarrow n$ is an isomorphism of $\mathfrak{t}^r$-modules. Thus the highest weight of $\rho$ is also $\Lambda_2$, as was shown by E. Cartan. Let $e_i = (0, \ldots, 1, \ldots, 0)$ be the element of $C^{2l}$ whose elements are all zero except that the $i$-th equals one, and $W \in n$ be the skew-symmetric matrix corresponding to $e_i \wedge e_i$, or equivalently such that $A_W = E_{12} - E_{21}$ and $B_W$, $C_W$, and $D_W$ are zeros. Then $W$ and therefore $WJ$ is clearly the weight vector corresponding to $\Lambda_2$. For brevity we write $X$ for $WJ$. The transposed $'X$ is a weight vector of $-\Lambda_2$. By (3.1)', we obtain $\kappa = \langle TrQ \rangle = \langle Q(X \otimes X), 'X \otimes 'X \rangle / \langle X \otimes X, 'X \otimes 'X \rangle$, which is computed with (1.1). Thus we find

$$\kappa = \varphi_{g_e}(\langle X, 'X \rangle, \langle X, X \rangle) / \varphi_{g_e}(X, 'X) = 1/4l$$

With the method in [7], $\Lambda_2 \vee \Lambda_2$ turns out to be decomposed into two (resp. three) irreducible representations for $l=2$ (resp. 3). This fact, combined with (3.1) to (3.3), gives the eigenvalues of $Q$. Assume $l=3$, for instance. The three irreducible components of $\Lambda_2 \vee \Lambda_2$ are $2\Lambda_2$, $\Lambda_2$ and $\Lambda_0$ (=the trivial representation of degree 1). The degree of $2\Lambda_2$ and $\Lambda_2$ are 90 and 14 respectively. Hence we have $7/2 = \dim m^r/4 = \langle TrQ \rangle = 90 \kappa_1 + 14 \kappa_2 + \kappa_3$, where $\kappa_1 = 1/12$ as above and $\kappa_3 = -1/2$ by (3.3). This shows $\kappa_2 = -1/4$. So $\kappa_1$ is the largest eigenvalue of $Q$ (as in the other cases). Since $b_{(g, t)}$ equals 1/6, for $(g, t)$ whose compact form is $(A_5, C_3)$, $b_{(g, t)}$ exceeds $\kappa_1$ and eventually $H_{(g, t)}$ is positive definite in this case.

(II) The case of $CII$: $(g_{sw}, t) = (C_t, C_p \times C_{1-p})$ ($p=1, 2$). $\rho$ is $\Lambda_t(C_p) \otimes \Lambda_t(C_{1-p})$ and $\rho \vee \rho$ decomposed into the irreducible components shown in the first column of the following table.
For $p = 1$, $-1/2(l+1)$ and $-(l+p+1)/2(l+1)$ must be omitted. Since $b_{(g, l+1)} = 1/2(l+1)$ for $p = 1$ and $b_{(g, l+1)} = 1/(l+1)$ for $p = 2$, we see that $H_{(g, l)}$ is non-negative but not positive definite for $p = 1$ and positive definite for $p = 2$.

(III) The case of $GI : (g, f) = (G_2, A_1 \times A_1)$

$$\rho = \Lambda_1(A_1) \otimes 3\Lambda_1(A_1)$$

$$\rho \vee \rho = 2\Lambda_1 \otimes 6\Lambda_1 + 2\Lambda_1 \otimes 2\Lambda_1 + \Lambda_1 \otimes \Lambda_1 \otimes \Lambda_1 \otimes \Lambda_1$$

and the eigenvalues of $Q_{(g, l)}$ are $1/4$, $-1/6$, $-1/4$, $-1/2$. Hence $H_{(g, l)}$ is non-negative but not positive definite.

(IV) The case of $III : (g, f) = (F_4, B_3)$

$$\rho = \Lambda_4(B_3)$$

$$\rho \vee \rho = 2\Lambda_4 + \Lambda_3 + \Lambda_0$$

The eigenvalues of $Q$ are $1/18$, $-5/18$, $-1/2$. Therefore $H_{(g, l)}$ is positive definite.

(V) The case of $BDII : (g, f) = (B_1, D_1)$ or $(D_1, B_1)$. The symmetric Riemannian space $BDII$ is of the constant negative curvature. Since $AI (l = 2) = A_2/C_2 = D_2/B_3$, $III (l = 2)$ is contained in this case. If $\dim(BDII) = n$, we know $b_{(g, l)} = 1/2(n-1)$ from Table I. On the other hand, the curvature tensor of the space of constant curvature is given by

$$R_{ijkl} = K(g_{ll}g_{jk} - g_{ik}g_{jl})$$

From this relation and the fact that the Ricci tensor is $(-1/2)$-times metric tensor, we have $K = \frac{1}{2(n-1)}$. Therefore we see from the definition of $Q_{(g, l)}$ that

$$H_{(g, l)}(\xi) = \frac{1}{2(n-1)} \sum_{i,j} \xi_{ii} \xi_{jj} + \frac{1}{2(n-1)} \left[ \left( \sum_{i,j} g_{ii} \xi_{jj} \right)^2 - \sum_{i,j} \xi_{ii} \xi_{jj} \right]$$

$$= \frac{1}{2(n-1)} \left( \sum_{i,j} g_{ii} \xi_{jj} \right)^2$$

Hence $H_{(g, l)}$ is non-negative but not positive definite.

REMARK. For each symmetric pair $(g, f)$ such that $g$ is simple and $f$ is of the same rank and semi-simple or simple, the system of the fundamental roots of $f^\circ$ is obtained from the diagram of the fundamental
simplex of $g^*$ by omitting a vertex which we denote by the black vertex (cf. Borel-Siebenthal [3]). We can check case by case that the highest weight of the linear isotropy representation of $\mathfrak{t}^*$ coincides with the root $-\alpha$ where $\alpha$ is the root which corresponds to the black vertex.

§ 4. The Main Theorem.

In the following, we shall use the notation set down by Berger [1] for each simple Lie group.

**Theorem 4.1.** The quadratic form $H^r_{\text{(s,t)}}$ is positive definite for the following values of $r$:

(i) if $G$ is a complex simple Lie group with rank $l$, then $r$ satisfies the conditions:

<table>
<thead>
<tr>
<th>type of $G$</th>
<th>$r$</th>
<th>type of $G$</th>
<th>$r$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_l$</td>
<td>$r &lt; \frac{l+1}{2}$</td>
<td>$E_6$</td>
<td>$r \leq 5$</td>
</tr>
<tr>
<td>$B_l$</td>
<td>$r \leq l-1$</td>
<td>$E_7$</td>
<td>$r \leq 8$</td>
</tr>
<tr>
<td>$C_l$</td>
<td>$r &lt; \frac{l+1}{2}$</td>
<td>$E_8$</td>
<td>$r \leq 14$</td>
</tr>
<tr>
<td>$D_l$</td>
<td>$r \leq l-2$</td>
<td>$F_4$</td>
<td>$r \leq 4$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$G_2$</td>
<td>$r \leq 1$</td>
</tr>
</tbody>
</table>

(ii) if $G$ is a real non-compact simple Lie group,

<table>
<thead>
<tr>
<th>type of $G$</th>
<th>$r$</th>
<th>type of $G$</th>
<th>$r$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\text{SL}(l, \mathbb{R})$</td>
<td>$r &lt; \frac{l+2}{4}$</td>
<td>$E_6$</td>
<td>$r \leq 4$</td>
</tr>
<tr>
<td>$\text{SU}^*(2l)$</td>
<td>$r &lt; \frac{l-1}{2}$</td>
<td>$E_7$</td>
<td>$r \leq 3$</td>
</tr>
<tr>
<td>$\text{SO}^i(2l+1)$</td>
<td>$r &lt; \text{Min}\left(\frac{i}{2}, \frac{2l-i+1}{2}\right)$</td>
<td>$E_8$</td>
<td>$r \leq 7$</td>
</tr>
<tr>
<td>$\text{SO}^i(2l)$</td>
<td>$r &lt; \frac{i}{2} \leq \frac{l}{2}$</td>
<td>$E_6$</td>
<td>$r \leq 5$</td>
</tr>
<tr>
<td>$\text{Sp}^i(l)$</td>
<td>$r &lt; i \leq \frac{l}{2}$</td>
<td>$G_2$</td>
<td>$r \leq 1$</td>
</tr>
<tr>
<td>$E_6$</td>
<td>$r \leq 3$</td>
<td>$F_4$</td>
<td>$r \leq 2$</td>
</tr>
<tr>
<td>$E_7$</td>
<td>$r \leq 2$</td>
<td>$F_4$</td>
<td>$r \leq 1$</td>
</tr>
<tr>
<td>$E_8$</td>
<td>$r \leq 1$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Proof. If $G$ is complex simple we have $b_{(g,t)} = 1/4$ by [7]. And we know by lemma 2.2 the value of $\kappa_u$. (i) is a direct consequence of these facts. Next we consider the case of more general symmetric pairs $(g, \mathfrak{t})$ with semi-simple $\mathfrak{t}$. If $\mathfrak{t}$ is of the same rank, we can easily calculate
the maximal eigenvalue $\kappa$ of $Q(g_M, \iota)$, using the method explained in §3, and we obtain $\kappa = 2\kappa_u$ for all cases other than $BII$, $DII$, $CII$ and $FII$. As for $BII$, $DII$, $CII$ and $FII$, we have $\kappa = \kappa_u$. If $\iota$ is not of the same rank, the computation of $\kappa$ is very complicated but we know by Lemma 2.1 that $(b(g, \iota)/r) - 2\kappa_u > 0$ implies $(b(g, \iota)/r) - \kappa > 0$. (ii) is an immediate consequence of these facts.

Using Theorem 4.1. and Theorem B, we have the following.

**Corollary.** Let $G$ be a non-compact simple Lie group, $K$ a maximal compact subgroup of $G$ and $\Gamma$ be a discrete subgroup of $G$ without non-trivial element of finite order. Suppose that the quotient space $G/\Gamma$ is compact. Then the $r$-th Betti number $b_r(K \backslash G/\Gamma)$ of a compact locally symmetric space $K \backslash G/\Gamma$ equals the $r$-th Betti number $b_r(M_u)$ of the compact form $M_u$ of $K \backslash G$ for the values of $r$ which satisfy the condition in Theorem 4.1.

§ 5. **Betti numbers of $G/\Gamma$.**

As a corollary of Theorem 4.1, we get

**Theorem 5.1.** Let $G$ be a connected semi-simple Lie group, each of whose simple factors is non-compact and not locally isomorphic to any of $SL(2, \mathbb{C})$, $SU(n)$, $SO(n)$, $Sp(n)$, $G^{\mathbb{R}}$. Let $\Gamma$ be a discrete subgroup with the compact quotient space $G/\Gamma$. Then the first Betti number of $G/\Gamma$ vanishes.

Proof. From the results of Matsushima [8] and of [7] and the consideration in §§ 2 and 3, the quadratic form $H$ of each simple factor of $G$ is positive definite. Therefore we see from Theorem A in §1 that $b_1(G/\Gamma) = 0$.

**Theorem 5.2.** Let $G$ be a complex simple Lie group with rank 1 and $\Gamma$ be a discrete subgroup with the compact quotient space $G/\Gamma$. Let $b_i(G/\Gamma)$ be the $i$-th Betti number of $G/\Gamma$. Then

1. $b_i(G/\Gamma) = 0$, if $l \geq 4$ or if $G = B_3$.
2. $b_2(G/\Gamma) = 2$ and $b_i(G/\Gamma) = 0$, if $G = A_l (l \geq 8)$, $B_l (l \geq 5)$, $C_l (l \geq 8)$, $D_l (l \geq 6)$, $E_6$, $E_7$, $E_8$, or $F_4$.
3. $b_2(G/\Gamma) = 1$ or 2 if $G = A_l (l \geq 10)$; $b_2(G/\Gamma) = 0$ if $G = B_l (l \geq 6)$, $C_l (l \geq 10)$, $D_l (l \geq 7)$, $E_6$, $E_7$, $E_8$, or $F_4$.

Proof. Suppose that $\Gamma$ has elements of finite order. Then we know in Selberg [10] that there exists a normal subgroup $\Gamma_0$ of $\Gamma$ with finite index which contains no non-trivial element of finite order. Since $G/\Gamma_0$ is a finite covering of $G/\Gamma$, it is sufficient for $b_1(G/\Gamma) = 0$ to show that $b_i(G/\Gamma_0) = 0$. Hence we can suppose without loss of generality that $\Gamma$
contains no non-trivial element of finite order. Let $K$ be a maximal compact subgroup of $G$. $G/\Gamma$ is a principal fibre bundle over a compact locally symmetric space $K\backslash G/\Gamma$ with structure group $K$. From the hypothesis for the rank of $G$ and from theorem 4.1, we see $b_i(K\backslash G/\Gamma) = b_i(K) = 0$ for $0 < i < 3$, since $K$ is simple. Hence, by Serre [11], the following exact sequence of real cohomology groups associated to the above principal bundle is valid for the dimension $\leq 5$, since the structure group $K$ is connected.

\[(5.1) \quad 0 \to H^i(K\backslash G/\Gamma) \to H^i(G/\Gamma) \to H^i(K) \to H^i(K\backslash G/\Gamma) \to H^i(G/\Gamma) \to H^i(K)\]

(i) is a direct consequence of the above facts.

As for (2), we see from the hypothesis for the rank of $G$ $b_i(K\backslash G/\Gamma) = b_i(K) = 0$, since $K$ is simple (For the Betti numbers of compact simple Lie groups, see for instance [5], [6]). On the other hand, $H^i(K\backslash G/\Gamma) \cong H^i(K) \cong R$. Therefore we get $H^i(G/\Gamma) \cong R + R$ from (5.1), which implies (2).

Using the above exact sequence, we know by the similar method that the later half of (2) and (3) are valid.

**Theorem 5.3.** Let $G$ be a non-compact real simple Lie group and $\Gamma$ be a discrete subgroup of $G$ with compact quotient space $G/\Gamma$. Then the second Betti number $b_2(G/\Gamma)$ of $G/\Gamma$ equals zero if the type of $G$ is $E_6^i$ ($i=1,2$), $E_7^i$ ($i=1,2,3$), $E_8^i$ ($i=1,2$) or $F_4^i$, or if $G$ is classical and satisfies the following conditions:

<table>
<thead>
<tr>
<th>type of $G$</th>
<th>$l$</th>
<th>type of $G$</th>
<th>$l$</th>
</tr>
</thead>
<tbody>
<tr>
<td>SL$(l+1, R)$</td>
<td>$l \geq 6$</td>
<td>SO$^i(2l)$</td>
<td>$\frac{l}{2} \geq i &gt; 2$</td>
</tr>
<tr>
<td>SU$(l+1)$</td>
<td>$l \geq 6$</td>
<td>SO$^i(2l)$</td>
<td>$l &gt; 7$</td>
</tr>
<tr>
<td>SU$(l+1)$</td>
<td>$\frac{l+1}{2} \geq i \geq 6$</td>
<td>Sp$(l, R)$</td>
<td>$l &gt; 7$</td>
</tr>
<tr>
<td>SO$(2l+1)$</td>
<td>Min $(\frac{i}{2}, \frac{2l+1-i}{2}) &gt; 2$</td>
<td>Sp$(l)$</td>
<td>$\frac{l}{2} &gt; l &gt; 3$</td>
</tr>
</tbody>
</table>

Proof. We can suppose without loss of generality that $\Gamma$ contains no non-trivial element of finite order. Let $K$ be a maximal compact subgroup of $G$. First we consider the case where $K\backslash G$ has no complex structure. For this case, the first and the second Betti numbers of compact form $M_u$ of $K\backslash G$ equal zero. From the hypothesis on $G$, we have $b_i(K\backslash G/\Gamma) = b_i(M_u) = 0$ for $0 < i < 3$. We have $b_i(K) = 0$ for $0 < i < 3$, since $K$ is semi-simple. Therefore we get $b_2(G/\Gamma) = 0$ by the exact
sequence (5.1). In the case where $K \setminus G$ has a complex structure, we know by Matsushima [9] that $H^i(G/\Gamma) = 0$, $H^i(K \setminus G/\Gamma) \cong H^i(M) \cong \mathbb{R}$. On the other hand, we have $H^3(K) = 0$. Therefore the exact sequence (5.1) are valid for the dimension $<3$, from which we obtain $H^3(G/\Gamma) = 0$.

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Bibliography