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ON CERTAIN QUADRATIC FORMS RELATED TO SYMMETRIC RIEMANNIAN SPACES

By

SOJI KANEYUKI and TADASHI NAGANO

In his recent works [8] [9], Matsushima investigates the Betti numbers of a certain type of compact locally symmetric riemannian manifolds and he shows that lower dimensional Betti numbers of such a manifold can be determined if we know that certain quadratic forms are positive definite. In order to verify the last condition, it is sufficient to compute the minimal eigenvalue of a linear transformation Q which is defined in terms of the curvature tensor of the manifold. Let M be the universal covering manifold of the manifold in question. In the case that M is an irreducible symmetric bounded domain in C^n , the eigenvalues of Q were computed by Calabi-Vesentini [4] and Borel [2], and Betti numbers are given by Matsushima himself. In the paper [7], the authors have treated the case that the compact form of M is a group manifold. The purpose of the present paper is to study the eigenvalues of Q for the case which remains to be treated, i.e. the case that M is a non-compact irreducible symmetric riemannian manifold not isomorphic to a bounded domain in C^n . Applying the results of Matsushima, we see that the *p*-th Betti number b_p of the compact locally symmetric space treated here is equal to that of the compact form of M if the ratio $p/\dim M$ is sufficiently small; in particular, b_1 vanishes in most cases. Our results give also informations about the Betti numbers of G/Γ where G is a semi-simple Lie group without compact simple factors and Γ is a discrete subgroup of G with compact quotient space G/Γ .

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§1. Preliminaries.

The notations settled in this section will be used throughout this paper. We denote by g a real semi-simple Lie algebra and by φ_g the Killing form of g, Let \mathfrak{k} be a subalgebra of g. We say that the pair

 $(\mathfrak{g}, \mathfrak{k})$ is a symmetric pair, if the vector space g admits a direct sum decomposition $\mathfrak{g}=\mathfrak{k}+\mathfrak{m}$ such that $[\mathfrak{k},\mathfrak{m}]\subset\mathfrak{m}$ and $[\mathfrak{m},\mathfrak{m}]\subset\mathfrak{k}$. By the symmetric square $\mathfrak{m}\vee\mathfrak{m}$ we mean the vector space of the symmetric tensors belonging to $\mathfrak{m}\otimes\mathfrak{m}$. The restriction of $\varphi_{\mathfrak{g}}$ to \mathfrak{m} is a nondegenerate inner product on \mathfrak{m} , and this gives rise to non-degenerate inner products on $\mathfrak{m}\otimes\mathfrak{m}$ and $\mathfrak{m}\vee\mathfrak{m}$, which are denoted by $\langle , \rangle_{\mathfrak{m}\otimes\mathfrak{m}}$ and $\langle , \rangle_{(\mathfrak{g},\mathfrak{k})}$ respectively. For brevity we write \langle , \rangle instead of $\langle , \rangle_{(\mathfrak{g},\mathfrak{k})}$. We define a linear endomorphism Q of the tensor space $\mathfrak{m}\otimes\mathfrak{m}$ by the formula

(1.1)
$$\langle Q(X \otimes Y), Z \otimes U \rangle_{\mathfrak{m} \otimes \mathfrak{m}} = \varphi_{\mathfrak{a}}([Y, U], [Z, X])$$

for any X, Y, Z, $U \in m$, The subspace $m \vee m$ is stable under Q, and we shall denote also by Q the restriction of Q to $m \vee m$. It is obvious that Q is a self-adjoint operator on $m \vee m$ with respect to the inner product \langle , \rangle . Now, following Matsushima [9], we define a quadratic forms H^r $(r=1, 2, \cdots)$ on $m \vee m$ by putting

for any $\xi \in \mathfrak{m} \vee \mathfrak{m}$, where

$$(1.3) 2b_{(\mathfrak{g},\mathfrak{f})} = \underset{X \in \mathfrak{f}}{\operatorname{Min}} \{1 + \varphi_{\mathfrak{f}}(X,X); \varphi_{\mathfrak{g}}(X,X) = -1\}$$

and $\varphi_{\mathfrak{k}}$ is the Killing form of \mathfrak{k} . Matsushima [8] defines also a quadratic form H on $\mathfrak{m} \vee \mathfrak{m}$ which coincides with H^1 in the case that \mathfrak{g} is non-compact simple and \mathfrak{k} is a compact semi-simple Lie algebra. In the following, we shall write Q, H^r , H also as $Q_{(\mathfrak{g},\mathfrak{k})}$, $H^r_{(\mathfrak{g},\mathfrak{k})}$, $H_{(\mathfrak{g},\mathfrak{k})}$ respectively; if \mathfrak{g} is a non-compact simple Lie algebra, \mathfrak{k} means always the subalgebra corresponding to a maximal compact subgroup of the adjoint group of \mathfrak{g} . The theorems of Matsushima are now stated as follows:

Theorem A ([8] Theorem 1). Let G be a semi-simple Lie group, all of whose simple factors are non-compact. Suppose that, for each simple factor G_i of G, the quadratic form $H_{(\mathfrak{g}_i,\mathfrak{t}_i)}$ associated to the Lie algebra \mathfrak{g}_i of G_i is positive definite. Let Γ be a discrete subgroup of G with compact quotient space G/Γ . Then the first Betti number of G/Γ is equal to 0.

Theorem B ([9] Theorem 1 and § 9). Let M be a simply connected, irreducible symmetric riemannian manifold which is non-compact and noneuclidean. Let G be the identity component of the group of all isometries of M. Let Γ be a discrete subgroup of G with compact quotient G/Γ and without element of finite order different from the identity, so that Γ is a

discontinuous group of isometries of M with compact quotient M/Γ . Let M_u be the compact form of M. If the quadratic form $H^r_{(g,t)}$ is positive definite, then the r-th Betti number b_r of M/Γ is equal to the r-th Betti number $b_r(M_u)$ of M_u .

We shall see when the quadratic forms $H_{(g,t)}$ and $H^{r}_{(g,t)}$ are positive definite for a real non-compact simple Lie algebra g. In the case that the center of \mathfrak{k} is not trivial, this question is already discussed by Matsushima [8] [9]. We shall therefore confine ourselves to the case that the center of \mathfrak{k} reduces to (0) so that \mathfrak{k} is a compact semi-simple algebra. We know that $H=H^{1}$ in this case. Thus our results on H^{r} , given in §4, are sufficient to apply Theorems A and B for this case.

In order that the quadratic form H^r is positive definite, it is necessary and sufficient that the absolute value of the minimal eigenvalue of Q is strictly smaller than $\frac{b_{(\mathfrak{g},\mathfrak{f})}}{r}$. Now, under the assumption that \mathfrak{g} is non-compact simple and \mathfrak{k} is semi-simple, the value $b_{(\mathfrak{g},\mathfrak{f})}$ is computed as follows:

(a) If f is simple, then we have by [8]

$$(1.4) b_{(\mathfrak{g},\mathfrak{k})} = \frac{\dim\mathfrak{m}}{4\dim\mathfrak{k}}$$

(b) When f is semi-simple and has the same rank as g,

$$(1.5) \qquad 2b_{(\mathfrak{g},\mathfrak{k})} = \operatorname{Min} \left\{ 1 - \varphi_{\mathfrak{g}}(\alpha_{g}, \alpha_{g}) / \varphi_{\mathfrak{f}}(\alpha_{k}, \alpha_{k}); \alpha \text{ is a root of } \mathfrak{k}^{c} \right\}$$

where α_g or α_k is the contravariant representative of α with respect to the Killing form of g or f respectively.

(c) (A_3, D_2) and $(D_l, B_q \times B_{l-q-1})$ are the only symmetric pairs for which f is semi-simple and is not of the same rank as g. One finds

(1.6)
$$2b = \frac{3}{4} \text{ for } (g, \mathfrak{k}) = (A_3, D_2)$$
$$2b = \frac{p}{2(l-1)} \text{ for } (g, \mathfrak{k}) = (D_l, B_q \times B_{l-q-1}), \text{ and } p < l,$$

where p = 2q + 1

\S 2. Relation with the group manifold.

Given a symmetric pair $(\mathfrak{g}, \mathfrak{k})$ with non-compact simple \mathfrak{g} and compact \mathfrak{k} , we wish to compare the minimal eigenvalue of $Q_{(\mathfrak{g},\mathfrak{k})}$ with that of Q for symmetric pair $(\mathfrak{g}_{\mu} \times \mathfrak{g}_{\mu}, \mathfrak{g}_{\mu})$. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$ be the Cartan decomposi-

tion. Denoting by \mathfrak{g}^c the complexification of \mathfrak{g} , the space $\mathfrak{g}_{\mathfrak{u}} = \mathfrak{k} + \sqrt{-1} \mathfrak{m}$ is also a compact simple subalgebra of \mathfrak{g}^c . The pair $(\mathfrak{g}_{\mathfrak{u}}, \mathfrak{k})$ is also a symmetric pair, the *compact form* of $(\mathfrak{g}, \mathfrak{k})$. By an obvious identification $\sqrt{-1}\mathfrak{m} \vee \sqrt{-1}\mathfrak{m} = \mathfrak{m} \vee \mathfrak{m}$, one obtains the equality $Q_{(\mathfrak{g},\mathfrak{l})} = -Q_{(\mathfrak{g}_{\mathfrak{u}},\mathfrak{l})}$ directly from the definition (1.1). In order to verify that $H^r_{(\mathfrak{g},\mathfrak{l})}$ is positive definite it is thus sufficient to show that $b_{(\mathfrak{g},\mathfrak{l})}/r$ is strictly greater than the maximal eigenvalue of $Q_{(\mathfrak{g}_{\mathfrak{u}},\mathfrak{l})}$. Let σ be the injective homomorphism of $\mathfrak{g}_{\mathfrak{u}}$ into $\mathfrak{g}_{\mathfrak{u}} \times \mathfrak{g}_{\mathfrak{u}}$ defined by $\sigma(X) = (X, X)$ for any $X \in \mathfrak{g}_{\mathfrak{u}}$. We define another injection $\tau : \mathfrak{g}_{\mathfrak{u}} \to \mathfrak{g}_{\mathfrak{u}} \times \mathfrak{g}_{\mathfrak{u}}$ by $\tau(X) = (X, -X)/\sqrt{2}$; τ is isometric with respect to the Killing forms. The pair $(\mathfrak{g}_{\mathfrak{u}} \times \mathfrak{g}_{\mathfrak{u}}, \sigma(\mathfrak{g}_{\mathfrak{u}}))$, briefly denoted by \mathfrak{u} , is a symmetric pair with the Cartan decomposition $\mathfrak{g}_{\mathfrak{u}} \times \mathfrak{g}_{\mathfrak{u}}$ $= \sigma(\mathfrak{g}_{\mathfrak{u}}) + \tau(\mathfrak{g}_{\mathfrak{u}})$

Lemma 2.1. The notations being as above, if κ (resp. κ_u) is the maximal eigenvalue of $Q_{(\mathfrak{g}_u,\mathfrak{f})}$ (resp. Q_u), then we have the inequality: $\kappa \leq 2\kappa_u$

Proof. If $\tau \lor \tau$; $g_u \lor g_u \to \tau(g_u) \lor \tau(g_u)$ denotes the mapping naturally induced by τ and if π denotes the orthogonal projection of $(\tau \lor \tau)(g_u \lor g_u)$ onto $(\tau \lor \tau)(\mathfrak{m} \lor \mathfrak{m})$ then one obtains

$$\pi ullet Q_{\mathfrak{ll}}ullet (au ee au) = rac{1}{2} (au ee au) Q_{(\mathfrak{g}_{oldsymbol{u}},\mathfrak{k})}$$

In fact, one has

$$\begin{split} & 2 \langle Q_{\mathfrak{ll}}(\tau(X) \lor \tau(X)), \ \tau(Y) \lor \tau(Y) \rangle_{\mathfrak{ll}} \\ &= 2 \varphi_{\mathfrak{g}_{\mathfrak{u}} \times \mathfrak{g}_{\mathfrak{u}}}([\tau(X), \tau(Y)], \ [\tau(Y), \ \tau(X)]) \\ &= \varphi_{\mathfrak{g}_{\mathfrak{u}} \times \mathfrak{g}_{\mathfrak{u}}}(\tau([X, Y]), \ \tau([Y, X])) \\ &= \varphi_{\mathfrak{g}_{\mathfrak{u}}}([X, Y], \ [Y,X]) \\ &= \langle Q_{(\mathfrak{g}_{\mathfrak{u}},\mathfrak{l})}(X \lor X), \ Y \lor Y \rangle_{(\mathfrak{g}_{\mathfrak{u}},\mathfrak{l})} \\ &= \langle (\tau \lor \tau) \cdot Q_{(\mathfrak{g}_{\mathfrak{u}},\mathfrak{l})}(X \lor X), \ \tau(Y) \lor \tau(Y) \rangle_{\mathfrak{u}} \end{split}$$

for any $X, Y \in \mathfrak{m}$. For any eigenvector ξ of $Q_{(\mathfrak{g}_{u},\mathfrak{k})}$ corresponding to κ , one therefore finds

$$\begin{split} & 2\kappa_{\mathbf{u}}\langle\xi,\,\xi\rangle_{(\mathfrak{g}_{\mathbf{u}},\mathfrak{f})} = 2\kappa_{\mathbf{u}}\langle(\tau\vee\tau)(\xi),\,\,(\tau\vee\tau)(\xi)\rangle_{\mathfrak{ll}} \\ & \geqslant 2\langle Q_{\mathfrak{ll}}((\tau\vee\tau)(\xi)),\,\,(\tau\vee\tau)(\xi)\rangle_{\mathfrak{ll}} \\ & = \langle Q_{(\mathfrak{g}_{\mathbf{u}},\mathfrak{f})}(\xi),\,\,\xi\rangle_{(\mathfrak{g}_{\mathbf{u}},\mathfrak{f})} = \langle\kappa\xi,\,\xi\rangle_{(\mathfrak{g}_{\mathbf{u}},\mathfrak{f})} = \kappa\langle\xi,\,\xi\rangle_{(\mathfrak{g}_{\mathbf{u}},\mathfrak{f})} \end{split}$$

Lemma 2.2. The number $2\kappa_u$ in lemma 2.1 equals $\varphi_{g_u}(\vartheta, \vartheta)$ where ϑ is the highest root of g_u

For the proof, see [7].

It follows from the above arguments that $H_{(\mathfrak{g},\mathfrak{k})}$ is positive difinite if $b_{(\mathfrak{g},\mathfrak{k})} > \varphi_{\mathfrak{g}_{\mathfrak{u}}}(\vartheta,\vartheta)$. The table I gives the values $b_{(\mathfrak{g},\mathfrak{k})}$ and $\varphi_{\mathfrak{g}_{\mathfrak{u}}}(\vartheta,\vartheta)$ for all symmetric pairs $(\mathfrak{g},\mathfrak{k})$ with non-compact simple \mathfrak{g} and compact semisimple \mathfrak{k} . We get then.

Lemma 2.3. The quadratic form $H_{(g,\mathfrak{k})}$ is positive definite for all (g_u, \mathfrak{k}) except for the cases AII (l=2, 3), BII, DII, CII (p=1, 2), GI, or FII.

	(g _u , ť)	<i>b</i> (g,t)	$2\kappa_u = \varphi_{g_u}(\vartheta, \vartheta)$	
ΑI	$\begin{array}{ccc} (A_l, \ \mathbf{B}_{l/2}) & l \! \geqslant \! 2 \\ (A_l, \ D_{(l+1)/2}) & l \! \geqslant \! 3 \end{array}$	(l+3)/4(l+1)	1/(l+1)	
A II	(A_{2l-1}, C_l)	(<i>l</i> -1)/4 <i>l</i>	1/2/	
BI	$(B_l, D_p \times B_{l-p}) \\ l-1 \geqslant p \geqslant 2$	$\overline{\operatorname{Min} \left(\frac{p}{2l-1}, \frac{2l-2p+1}{2(2l-1)} \right)}$	1/(21-1)	
DI	$(D_l, D_p imes D_{l-p}) \ l \ge 2p \ge 4$	<i>p</i> /2(<i>l</i> -1)	1/2(I-1)	
	$\begin{array}{c} (D_l, \ B_p \times B_{l-p-1}) \\ l-1 \geqslant 2p \geqslant 2 \end{array}$	(2p+1)/4(l-1)	$\int \frac{1/2(i-1)}{i}$	
BII	(B_l, D_l)	1/2(2l-1)	1/(2l-1)	
DII	(D_l, B_{l-1})	1/4(<i>l</i> -1)	1/2(l-1)	
CII	$(C_l, C_p \times C_{l-p}) 2p \leqslant l$	<i>p</i> /2(<i>l</i> +1)	1/(l+1)	
ΕI	(E_6, C_4)	7/24)	
ΕII	$(E_6, A_1 \times A_5)$	1/4	1/12	
EIV	(E_6, F_4)	1/8)	
EV	(E_7, A_7)	5/18	1 /10	
E VI	$(E_7, A_1 \times D_6)$	2/9	} 1/18	
E VIII	(E_8, D_8)	4/15) 1/20 ·	
EIX	$(E_8, A_1 \times E_7)$	1/5	} 1/30	
GI	$(G_2, A_1 \times A_1)$	1/4	1/4	
FI	$(F_4, A_1 \times C_3)$	5/18		
FII	(F ₄ , B ₄)	1/9	} 1/9	

Table I.

§ 3. The exceptional cases

In this section we shall determine the eigenvalues of $Q_{(g,t)}$ for the exceptional cases in lemma 2.3 by the method employed in [7]. Given a symmetric pair (g, f), we write ρ for the linear isotropy representation

 $\rho(X) = ad(X) | m, X \in \mathfrak{k}$. We may extend ρ and Q to a representation of \mathfrak{k}^c on \mathfrak{m}^c and to an endomorphism of $\mathfrak{m}^c \vee \mathfrak{m}^c$ respectively, which will be denoted by the same letters. Q commutes with $\rho \vee \rho$, the restriction of $\rho \otimes \rho$ to $\mathfrak{m}^c \vee \mathfrak{m}^c$. Let $\rho \vee \rho = \rho_1 + \cdots + \rho_s$ be the decomposition of $\rho \vee \rho$ into irreducible representations. The highest weights $\Lambda_{(i)}$ of ρ_i are assumed to be in the order $\Lambda_{(i)} > \cdots > \Lambda_{(s)}$ with respect to some ordering: this assumption will be satisfied in the following cases. Each subspace W_i of $\mathfrak{m}^c \vee \mathfrak{m}^c$ spanned by the weight vectors of $\Lambda_{(i)}$ is stable under Q. We write $(TrQ)_i$ for the trace of Q restricted to W_i . We know in [7] that Q is represented by a scalar matrix in the irreducible invariant subspace of $\mathfrak{m}^c \vee \mathfrak{m}^c$ corresponding to ρ_i . If κ_i denote the unique eigenvalue of Q restricted to the subspace, then

(3.1)
$$(TrQ)_i = \sum_{1 \leq j \leq i} m_j(\Lambda_{(i)}) \kappa_j;$$

where $m_j(\Lambda_{(i)})$ is the multiplicity of the weight $\Lambda_{(i)}$ in ρ_j .

$$(3.1)' m_1(\Lambda_{(1)}) = 1 \text{ and so } (TrQ)_1 = \kappa_1$$

Moreover, if g is compact simple, which will be the case in the sequel, one has

(3.2)
$$TrQ = (\dim \mathfrak{m})/4, \dim \mathfrak{m} = \dim \mathfrak{g} - \dim \mathfrak{k},$$

(3.3) -1/2 is an eigenvalue with multiplicity one of Q. These formulas (3.1) to (3.3) are given in [7] and will be used to compute the eigenvalues of Q. Hereafter a representation and its highest weight will be denoted by the same letter.

(I) The case of A II: $(\mathfrak{g}_{u}, \mathfrak{k}) = (A_{2l-1}, C_{l}) l \geq 2$

First we try to find the highest weight of ρ and its weight vector: For an arbitrary square matrix X of degree 2l, we divide X into four parts

$$X = \begin{pmatrix} A_X & B_X \\ C_X & D_X \end{pmatrix}$$

where A_X , B_X , C_X , D_X are square matrices with complex coefficients of degree l. g^c is the Lie algebra $\mathfrak{Sl}(2l, \mathbb{C}) = \{X; Tr(A_X + D_X) = 0\}$. If Jdenotes a (2l, 2l)-matrix such that $A_J = D_J = 0$ and $B_J = -C_J = E$ (= the unit matrix), \mathfrak{t}^c is $\{X \in \mathfrak{Sl}(2l, \mathbb{C}); tXJ + JX = 0\}$. Hence the orthogonal complement \mathfrak{m}^c of \mathfrak{t}^c consists of the matrices Y such that tYJ - JY = 0, or eqivalently, such that B_Y , C_Y are skew-symmetric matrices and one has $tA_Y = D_Y$. An element X of \mathfrak{t}^c operates on \mathfrak{m}^c by $Y \in \mathfrak{m}^c \to \rho(X)Y = [X, Y]$ = XY - YX. On the other hand, the identity representation of \mathfrak{t}^c operating

on C^{2l} naturally induces a representation ρ' on $C^{2l} \wedge C^{2l}$ which we take as the space of skew-symmetric matrices of degree 2l; $\rho'(X)Z = XZ + Z^{t}X$, Z=skew-symmetric. $\rho'(\mathfrak{k}^c)$ leaves J invariant, and ρ' is irreducible on the space n of skew-symmetric matrices which are orthogonal to J. This irreducible representation, denoted also by ρ' , has the highest weight $\Lambda_2 = \lambda_1 + \lambda_2$ if a Cartan subalgebra of \mathfrak{t}^c consists of diagonal matrices, H, and a base (H_i) is so chosen that $H = \sum \lambda_i H_i$ in case that the (i, i)-element of A_H is λ_i , If $\alpha(Y)$ denotes the skew-symmetric matrix -YJ for any Y in $\mathfrak{m}^c, \alpha; \mathfrak{m}^c \to \mathfrak{n}$ is an isomorphism of \mathfrak{k}^{c} modules. Thus the highest weight of ρ is also Λ_2 , as was shown by E. Cartan. Let $e_i = (0, \dots, 1, \dots, 0)$ be the element of C^{2i} whose elements are all zero except that the *i*-th equals one, and $W \in \mathfrak{n}$ be the skewsymmetric matrix corresponding to $e_1 \wedge e_2$, or equivalently such that $A_W = E_{12} - E_{21}$ and B_W , C_W . and D_W are zeros. Then W and therefore WI is clearly the weight vector corresponding to Λ_2 . For brevity we write X for WJ. The transposed ${}^{t}X$ is a weight vector of $-\Lambda_{2}$. By (3.1)', we obtain $\kappa_1 = (TrQ)_1 = \langle Q(X \otimes X), {}^tX \otimes {}^tX \rangle / \langle X \otimes X, {}^tX \otimes {}^tX \rangle$, which is computed with (1, 1). Thus we find

$$\kappa_1 = \varphi_{\mathfrak{a}c}([X, {}^tX], [{}^tX, X])/\varphi_{\mathfrak{a}c}(X, {}^tX)^2 = 1/4l$$

With the method in [7], $\Lambda_2 \vee \Lambda_2$ turns out to be decomposed into two (resp. three) irreducible representations for l=2 (resp. 3). This fact, combined with (3.1) to (3.3), gives the eigenvalues of Q. Assume l=3, for instance. The three irreducible components of $\Lambda_2 \vee \Lambda_2$ are $2\Lambda_2$, Λ_2 and Λ_0 (=the trivial representation of degree 1). The degree of $2\Lambda_2$ and Λ_2 are 90 and 14 respectively. Hence we have $7/2 = \dim m/4 = TrQ$ $=90 \kappa_1 + 14 \kappa_2 + \kappa_3$, where $\kappa_1 = 1/12$ as above and $\kappa_3 = -1/2$ by (3.3). This shows $\kappa_2 = -1/4$. So κ_1 is the largest eigenvalue of Q (as in the other cases). Since $b_{(\mathfrak{g},\mathfrak{f})}$ equals 1/6, for $(\mathfrak{g},\mathfrak{f})$ whose compact form is (A_5, C_3) , $b_{(\mathfrak{g},\mathfrak{f})}$ exceeds κ_1 and eventually $H_{(\mathfrak{g},\mathfrak{f})}$ is positive definite in this case.

(II) The case of CII: $(\mathfrak{g}_u, \mathfrak{k}) = (C_l, C_p \times C_{l-p})$ (p=1, 2). ρ is $\Lambda_1(C_p) \otimes \Lambda_1(C_{l-p})$ and $\rho \vee \rho$ decomposed into the irreducible components shown in the first column of the following table.

$2\Lambda_1(C_p)\otimes 2\Lambda_1(C_{l-p})$	$\kappa_1 = 1/2(l+1)$
$\Lambda_2(C_p)\otimes\Lambda_2(C_{l-p})$	$\kappa_2 = -1/2(l+1)$
$\Lambda_0(C_p) \otimes \Lambda_2(C_{l-p})$	$\kappa_3 = -(p+1)/2(l+1)$
$\Lambda_2(C_p)\otimes\Lambda_0(C_{l-p})$	$\kappa_4 = -(l-p+1)/2(l+1)$
$\Lambda_0(C_p)\otimes\Lambda_0(C_{l-p})$	$\kappa_5 = -1/2$

For p=1, -1/2(l+1) and -(l+p+1)/2(l+1) must be omitted. Since $b_{(\mathfrak{g},\mathfrak{k})}=1/2(l+1)$ for p=1 and $b_{(\mathfrak{g},\mathfrak{k})}=1/(l+1)$ for p=2, we see that $H_{(\mathfrak{g},\mathfrak{k})}$ is non-negative but not positive definite for p=1 and positive definite for p=2.

(III) The case of $GI: (\mathfrak{g}_{\mathfrak{u}}, \mathfrak{k}) = (G_2, A_1 \times A_1)$ $\rho = \Lambda_1(A_1) \otimes 3\Lambda_1(A_1)$ $\rho \vee \rho = 2\Lambda_1 \otimes 6\Lambda_1 + 2\Lambda_1 \otimes 2\Lambda_1 + \Lambda_0 \otimes 4\Lambda_1 + \Lambda_0 \otimes \Lambda_0$

and the eigenvalues of $Q_{(g_u,t)}$ are 1/4, -1/6, -1/4, -1/2. Hence $H_{(g,t)}$ is non-negative but not positive definite.

(IV) The case of $FII: (g_u, \mathfrak{k}) = (F_4, B_4)$

$$egin{aligned} &
ho &= \Lambda_4(B_4) \ &
ho ⅇ
ho &= 2\Lambda_4 \!+\! \Lambda_1 \!+\! \Lambda_0 \end{aligned}$$

The eigenvalues of Q are 1/18, -5/18, -1/2. Therefore $H_{(\mathfrak{g},\mathfrak{f})}$ is positive definite.

(V) The case of $BDII: (g_{\mu}, t) = (B_l, D_l)$ or (D_l, B_{l-1}) . The symmetric Riemannian space BDII is of the constant negative curvature. Since AII $(l=2)=A_3/C_2=D_3/B_2$, AII(l=2) is contained in this case. If dim (BDII)=n, we know $b_{(g,t)}=1/2(n-1)$ from Table I. On the other hand, the curvature tensor of the space of constant curvature is given by

$$R_{ijkl} = K(g_{il}g_{jk} - g_{ik}g_{jl})$$

From this relation and the fact that the Ricci tensor is (-1/2)-times metric tensor, we have $K = \frac{1}{2(n-1)}$. Therefore we see from the definition of $Q_{(g,t)}$ that

$$egin{aligned} H_{(\mathfrak{g},\mathfrak{k})}(\xi) &= rac{1}{2(n\!-\!1)} \sum_{i,l} \xi_{il} \xi^{il} + rac{1}{2(n\!-\!1)} ig[(\sum_{i,l} g_{il} \xi^{il})^2 - \sum_{i,l} \xi_{il} \xi^{il} ig] \ &= rac{1}{2(n\!-\!1)} \; (\sum_{i,l} g_{il} \xi^{il})^2 \end{aligned}$$

Hence $H_{(g,t)}$ is non-negative but not positive definite.

REMARK. For each symmetric pair (g_u, t) such that g_u is simple and t is of the same rank and semi-simple or simple, the system of the fundamental roots of t^c is obtained from the diagram of the fundamental

simplex of g_{α}^{c} by omitting a vertex which we denote by the black vertex (cf. Borel-Siebenthal [3]). We can check case by case that the highest weight of the linear isotropy representation of \mathfrak{k}^{c} coincides with the root $-\alpha$ where α is the root which corresponds to the black vertex.

§4. The Main Theorem.

In the following, we shall use the notation set down by Berger [1] for each simple Lie group.

Theorem 4.1. The quadratic form $H^{r}_{(g,t)}$ is positive definite for the following values of r:

type of G	r	type of G	r
A_l	$r < \frac{l+1}{2}$	E_6	$r \leqslant 5$
B_l	$r \leqslant l-1$	E_7	$r \leqslant 8$
C_l	$r < rac{l+1}{2}$	E_8	$r \leqslant 14$
D_l	$r \leq l-2$	F ₄	$r \leqslant 4$
		G ₂	$r\leqslant 1$

(i) if G is a complex simple Lie group with rank l,

(ii) if G is a real non-compact simple Lie group,

type of G	r	type of G	r
SL(<i>l</i> , R)	$r < \frac{l+2}{4}$	E ¹ ₇	$r \leqslant 4$
SU*(2 <i>l</i>)	$r < \frac{l-1}{2}$	E_7^2	$r \leqslant 3$
$SO^i(2l+1)$	$r < ext{Min}\left(rac{i}{2}, rac{2l-i+1}{2} ight)$	E ¹ ₈	$r \leqslant 7$
$SO^i(2l)$	$r < \frac{i}{2} \leqslant \frac{l}{2}$	E ₈ ²	$r \leqslant 5$
$\operatorname{Sp^i}(l)$	$r < i \leq \frac{l}{2}$	G ₂ *	r < 1
\mathbf{E}_6^1	<i>r</i> ≤3	$\mathbf{F_4^1}$	$r \leqslant 2$
E_6^2	<i>r</i> ≤2	F_4^2	$r \leqslant 1$
\mathbf{E}_6^4	$r\leqslant 1$		

Proof. If G is complex simple we have $b_{(\mathfrak{g},\mathfrak{k})}=1/4$ by [7]. And we know by lemma 2.2 the value of κ_u . (i) is a direct consequence of these facts. Next we consider the case of more general symmetric pairs (g, \mathfrak{k}) with semi-simple \mathfrak{k} . If \mathfrak{k} is of the same rank, we can easily calculate

the maximal eigenvalue κ of $Q_{(\mathfrak{g}_u,\mathfrak{k})}$, using the method explained in § 3, and we obtain $\kappa = 2\kappa_u$ for all cases other than *BII*, *DII*, *CII* and *FII*. As for *BII*, *DII*, *CII* and *FII*, we have $\kappa = \kappa_u$. If \mathfrak{k} is not of the same rank, the computation of κ is very complicated but we know by Lemma 2.1 that $(b_{(\mathfrak{g},\mathfrak{k})}/r)-2\kappa_u \ge 0$ implies $(b_{(\mathfrak{g},\mathfrak{k})}/r)-\kappa \ge 0$. (ii) is an immediate consequence of these facts.

Using Theorem 4.1. and Theorem B, we have the following

Corollary. Let G be a non-compact simple Lie group, K a maximal compact subgroup of G and Γ be a discrete subgroup of G without non-trivial element of finite order. Suppose that the quotient space G/Γ is compact. Then the r-th Betti number $b_r(K\setminus G/\Gamma)$ of a compact lacally symmetric space $K\setminus G/\Gamma$ equals the r-th Betti number $b_r(M_u)$ of the compact form M_u of $K\setminus G$ for the values of r which satisfy the condition in Theorem 4.1.

§ 5. Betti numbers of G/Γ .

As a corollary of Theorem 4.1, we get

Theorem 5.1. Let G be a connected semi-simple Lie group, each of whose simple factors is non-compact and not locally isomorphic to any of SL(2, C), $SU^{1}(n)$, $SO^{1}(n)$, $Sp^{1}(n)$, G_{2}^{*} . Let Γ be a discrete subgroup with the compact quotient space G/Γ . Then the first Betti number of G/Γ vanishes.

Proof. From the results of Matsushima [8] and of [7] and the consideration in §§ 2 and 3, the quadratic form H of each simple factor of G is positive definite. Therefore we see from Theorem A in §1 that $b_1(G/\Gamma)=0$.

Theorem 5.2. Let G be a complex simple Lie group with rank l and Γ be a discrete subgroup with the compact quotient space G/Γ . Let $b_i(G/\Gamma)$ be the *i*-th Betti number of G/Γ . Then

(1) $b_2(G/\Gamma)=0$, if $l \ge 4$ or if $G=B_3$.

(2) $b_3(G/\Gamma) = 2$ and $b_4(G/\Gamma) = 0$, if $G = A_1$ ($l \ge 8$), B_1 ($l \ge 5$), C_1 ($l \ge 8$), D_1 ($l \ge 6$), E_6 , E_7 , E_8 or F_4 .

(3) $b_5(G/\Gamma) = 1$ or 2 if $G = A_l$ $(l \ge 10)$; $b_5(G/\Gamma) = 0$ if $G = B_l$ $(l \ge 6)$, C_l $(l \ge 10)$, D_l $(l \ge 7)$, E_6 , E_7 , or E_8 .

Proof. Suppose that Γ has elements of finite order. Then we know in Selberg [10] that there exists a normal subgroup Γ_0 of Γ with finite index which contains no non-trivial element of finite order. Since G/Γ_0 is a finite covering of G/Γ , it is sufficient for $b_r(G/\Gamma)=0$ to show that $b_r(G/\Gamma_0)=0$. Hence we can suppose without loss of generality that Γ

contains no non-trivial element of finite order. Let K be a maximal compact subgroup of G. G/Γ is a principal fibre bundle over a compact locally symmetric space $K\backslash G/\Gamma$ with structure group K. From the hypothesis for the rank of G and from theorem 4.1, we see $b_i(K\backslash G/\Gamma) = b_i(K) = 0$ for 0 < i < 3, since K is simple. Hence, by Serre [11], the following exact sequence of real cohomology groups associated to the above principal bundle is valid for the dimension ≤ 5 , since the structure group K is connected.

$$(5.1) \quad 0 \to H^{1}(K \setminus G/\Gamma) \to H^{1}(G/\Gamma) \to H^{1}(K) \to H^{2}(K \setminus G/\Gamma) \to H^{2}(G/\Gamma) \\ \to H^{2}(K) \to \cdots \to H^{5}(K \setminus G/\Gamma) \to H^{5}(G/\Gamma) \to H^{5}(K)$$

(i) is a direct consequence of the above facts.

As for (2), we see from the hypothesis for the rank of $G \ b_4(K \setminus G/\Gamma) = b_4(K) = 0$, since K is simple (For the Betti numbers of compact simple Lie groups, see for instance [5], [6]). On the other hand, $H^3(K \setminus G/\Gamma) \cong H^3(K) \cong \mathbb{R}$. Therefore we get $H^3(G/\Gamma) \cong \mathbb{R} + \mathbb{R}$ from (5.1), which implies (2).

Using the above exact sequence, we know by the similar method that the later half of (2) and (3) are valid.

Theorem 5.3. Let G be a non-compact real simple Lie group and Γ be a discrete subgroup of G with compact quotient space G/Γ . Then the second Betti number $b_2(G/\Gamma)$ of G/Γ equals zero if the type of G is E_6^i $(i=1,2), E_7^i$ $(i=1,2,3), E_8^i$ (i=1,2) or F_4^1 , or if G is classical and satisfies the following conditions:

type of G	l	txpe of G	l
SL(<i>l</i> +1, <i>R</i>)	/≥6	SO ⁱ (2 <i>l</i>)	$rac{l}{2} \! \geqslant \! rac{i}{2} \! > \! 2$
SU*(2 <i>l</i>)	<i>l</i> ≥6	SO*(2 <i>l</i>)	<i>l</i> ≥7
SU ⁱ (<i>l</i> +1)	$rac{l+1}{2} \! \geqslant \! i \! \geqslant \! 5$	Sp(<i>l</i> , <i>R</i>)	<i>l</i> ≥7
SO ⁱ (2 <i>l</i> +1)	$Min\left(rac{i}{2},rac{2l+1-i}{2} ight)\!\!>\!2$	Sp ⁱ (<i>l</i>)	$\frac{l}{2} \geqslant i \geqslant 3$

Proof. We can suppose without loss of generality that Γ contains no non-trivial element of finite order. Let K be a maximal compact subgroup of G. First we consider the case where $K \setminus G$ has no complex structure. For this case, the first and the second Betti numbers of compact form M_u of $K \setminus G$ equal zero. From the hypothesis on G, we have $b_i(K \setminus G/\Gamma) = b_i(M_u) = 0$ for 0 < i < 3. We have $b_i(K) = 0$ for 0 < i < 3, since K is semi-simple. Therefore we get $b_2(G/\Gamma) = 0$ by the exact sequence (5.1). In the case where $K \setminus G$ has a complex structure, we know by Matsushima [9] that $H^1(G/\Gamma) = 0$, $H^2(K \setminus G/\Gamma) \simeq H^2(M_u) \simeq \mathbf{R}$. On the other hand, we have $H^2(K) = 0$. Therefore the exact sequence (5.1) are valid for the dimension ≤ 3 , from which we obtain $H^2(G/\Gamma) = 0$.

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