Theory of Thermal Elastic-Plastic Analysis with A More General Workhardening Rule†

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Abstract

It is well known that welding thermal stresses and resulting residual stresses influence the strength of welded construction, causing troubles such as brittle fracture, buckling and weld cracking.

At the instant of welding, a limited portion of the welded joint is heated up to a very high temperature and cooled down to room temperature. In the thermal cycle which takes place, the temperature distribution changes with time and the plastic deformation develops. They affect the mechanical properties of the material. In order to perform a reliable theoretical analysis, the above mentioned factors should be taken into account.

The authors extended their developed theory of thermal elastic-plastic analysis with consideration of the effects of changes (that is, temperature-dependence and plastic history-dependence) of the material properties, introducing a combined model of isotropic and kinematic rules as the workhardening rule.

KEY WORDS: (Thermal Elastic-Plastic Analysis) (Workhardening Rule) (Finite Element Method) (Welding Stresses) (Mechanical Properties)

1. Introduction

It is well known that welding produces thermal stresses which cause distortion of structures and residual stresses which influence buckling strength and brittle fracture strength of welded structures. Welding cold cracking is investigated from various points of view, including dynamical one.

For better understanding of formulation of welding residual stresses and causes of weld cracking, it is necessary to obtain more accurate information on the entire histories of stresses and strains to which the material is subjected during the process of welding. At the instant of welding, a limited portion of the welded joint such as the weld metal and the base metal in its vicinity are heated up to a very high temperature and thereafter cooled down to room temperature. In this way, the thermal cycle proceeds, temperature distribution changes with time and the mechanical properties of metal depend on temperature and plastic history. In order to perform a more reliable analysis, the above mentioned factors should be taken into account.

Based on the finite element method, the authors had developed methods of thermal elastic-plastic analysis on this kind of problems with consideration of the effects of changes of the mechanical properties[1]. And the authors and their colleague have presented various information on thermal stress and strain histories in the process of welding on many types of welded joints by using their methods[2)-9)]. In their theory, it is assumed that the material obeys isotropic workhardening rule in the plastic range. When multipass welding is applied, the material undergoes multi-thermal cycles and plastic work. For more accurate analysis, it may be necessary to adopt more general workhardening rule including the translation of the yield surface in the theory.

In this paper, the authors generalize the theory of thermal elastic-plastic analysis to take into account more accurately temperature-dependence and plastic history-dependence of the material properties, introducing a combined rule of isotropic and kinematic workhardening.

2. Stress-Total Strain Incremental Relation with Consideration of Temperature and Plastic History Dependence of Material Properties

The mechanical properties of material generally change when temperature changes or its plastic deformation progresses. Especially in the process of welding, the magnitudes of changes of the material properties are very large because the welded joint is heated up to a very high temperature and thereafter cooled down to room temperature generally with large stresses and plastic strains. Accordingly, when thermal stresses, strains or deformations produced by welding are analyzed, it is necessary to consider such temperature-dependence and plastic histo-

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ry-dependence of the material properties.

In this chapter, stress-total strain incremental relations with consideration of effects of changes of the material properties will be developed for the elastic range and plastic range respectively, introducing a combined rule of isotropic and kinematic workhardening.

2.1 Stress - total strain incremental relation in elastic range

In this paper, it is assumed that creep strain is not produced in the material, and the total increment of thermal strain including transformation strain is denoted by $\{d\varepsilon^T\}$ and it will be called simply thermal strain increment hereafter. Such a thermal strain increment can be expressed by instantaneous linear expansion coefficient $\{\alpha\}$ and temperature increment dT as

$$\{d\varepsilon^T\} = \{\alpha\} dT \tag{1}$$

The above instantaneous linear expansion coefficient $\{\alpha\}$ is usually a coefficient which indicates the magnitude of expansion or shrinkage due to temperature changes at every instant. By this coefficient, expansion or shrinkage due to both temperature change and transformation can be expressed when the material is in the temperature range of the transformation. Then, in the elastic range, total strain increment $\{d\varepsilon\}$ is represented as the summation of thermal strain increment $\{d\varepsilon^T\}$ and elastic strain increment $\{d\varepsilon^e\}$ which are produced to satisfy the conditions of compatibility, that is,

$$\{d\varepsilon\} = \{d\varepsilon^e\} + \{d\varepsilon^T\} \tag{2}$$

Elastic strains $\{\varepsilon^e\}$ have always the relation to stresses $\{\sigma\}$ as

$$\{\sigma\} = [D^e]\{\varepsilon^e\} \tag{3}$$

where $[D^e]$: elasticity matrix or elastic stress-strain matrix (the components of $[D^e]$ are generally functions of Young's modulus, Poisson's ratio etc.)

When the stresses $\{\sigma\}$, elastic strains $\{\varepsilon^e\}$ and elasticity matrix $[D^e]$ change into $\{\sigma + d\sigma\}$, $\{\varepsilon^e + d\varepsilon^e\}$ and $[D^e] + d[D^e]$ respectively after the subsequent increment of loads, new stresses, elastic strains and elasticity matrix must satisfy the same equation as Eq. (3), that is,

$$\{\sigma + d\sigma\} = ([D^e] + d[D^e])\{\varepsilon^e + d\varepsilon^e\} \tag{4}$$

In the case where the elasticity matrix $[D^e]$ (containing the material properties) is a function of only temperature, the increment $d[D^e]$ of elasticity matrix $[D^e]$ in the above equation can be represented as

$$d[D^e] = \frac{d[D^e]}{dT} dT \tag{5}$$

Substraction of Eq. (3) from Eq. (4) and introduction of Eq. (5) provide the relationship between stress increment and elastic strain increment as

$$\{d\sigma\} = [D^e_d]\{d\varepsilon^e\} + \frac{d[D^e]}{dT} dT \{\varepsilon^e\} \tag{6}$$

where $[D^e_d] = [D^e] + \frac{d[D^e]}{dT} dT$

It should be noted that as temperature history and temperature-dependence of elasticity matrix $[D^e]$ are known in advance, the second term $\frac{d[D^e]}{dT} dT$ in the above matrix $[D^e_d]$ does not produce nonlinear term with respect to unknown quantities but improves the accuracy of analyses. In the right side of the above Eq. (6), the first term indicates a part of stress increment due to an increase of elastic strain, and the second term does the rest of stress increment due to change of temperature.

Elimination of elastic strain increment $\{d\varepsilon^e\}$ from Eq. (6) by using Eq. (2) and introduction of Eq. (1) for thermal strain increment furnish the incremental relationship between stress and total strain as

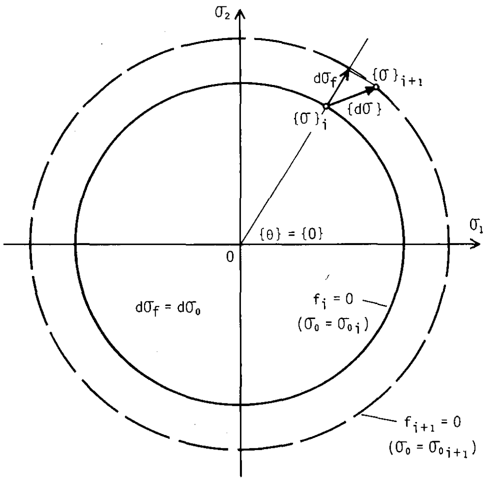
$$\{d\sigma\} = [D^e_d]\{d\varepsilon\} - [D^e_d]\left(\{\alpha\} - [D^e_d]^{-1} \frac{d[D^e]}{dT}\{\varepsilon^e\}\right) dT \tag{7}$$

2.2 Stress - total strain incremental relation in plastic range

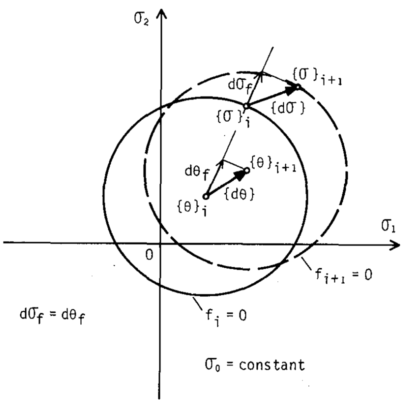
2.2.1 Yield criterion, workhardening rule and yield surface

When stresses produced at a point reach a certain magnitude, the material yields and shows complex plastic behavior for the subsequent loading. In a certain combination of stresses, a condition which defines the limit of elasticity of the material is called yield criterion. In the space (stress space) which co-ordinate axes are stress components, the yield criterion can be shown by some curved surface (yield surface). Generally the shape, size and position of the yield surface change with progress of plastic deformation of the material. That is, the yield criterion changes, being subjected to plastic work. The law for such changes of the yield criterion is called workhardening rule. Hitherto, various workhardening rules have been proposed. Isotropic workhardening rule[10] assumes that the size of the yield surface changes with increase of plastic work but the position and shape do not

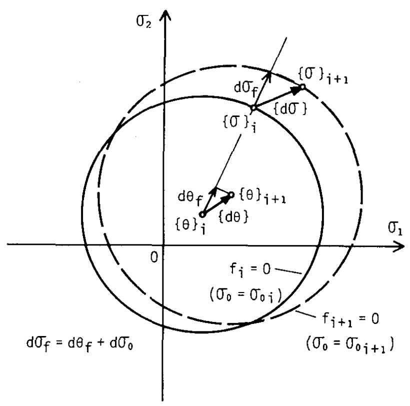
f = 0 : yield surface (in this example, circle)
{θ} : vector which indicates center of yield surface
σ_0 : a measure of size of yield surface
(in this example, radius of yield surface)
{σ} : stresses on yield surface
i, i+1 : loading step
(suffix f indicates normal component of each increment)



(a) Isotropic Workhardening Rule



(b) Kinematic Workhardening Rule



(c) Combined Workhardening Rule

Fig. 1 Schematic Illustration of Workhardening Rules

change (see **Fig. 1 (a)**). This implies that the initial yield surface expands uniformly during plastic flow. In kinematic workhardening rule[11], it is assumed that the yield surface does not change its initial size and shape but moves in the stress space like a rigid body (see **Fig. 1 (b)**). With the aid of this rule, the Bauschinger effect can be easily represented within a certain degree of accuracy. Furthermore, in order to represent plastic behavior more accurately, many complex workhardening rules have been proposed. In this study, the isotropic and kinematic workhardening rules will be combined and it is assumed that the size and position of the yield surface can be changed but its initial shape can not (see **Fig. 1 (c)**). Generally, such combined yield surface can be shown by the following expression.

$$f(\sigma_{ij} - \theta_{ij}, \sigma_o) = 0 \tag{8}$$

where $\{\theta\}$: translation vector which indicates the position of center of the yield surface in the stress space

σ_o : a measure of the size of the yield surface

The above function f which defines the yield surface (the yield criterion) is called yield function.

Next, the following hypothesis is furnished, on which changes of the size and position of the yield surface shown by Eq. (8) will be ruled. First, the size of the yield surface (σ_o) is assumed to be a function of the quantity ϵ_l^p of plastic strain and temperature T. Here,

$$\epsilon_l^p = \int d\epsilon_l^p \tag{9}$$

where $d\epsilon_l^p$: the length of the vector of the plastic strain increment (see Eq. (13))

Concerning translation of the yield surface, it is assumed that, as a general rule, the yield surface can move only when the plastic deformation increases. It should be noted that temperature change can not be the direct cause to move the yield surface. Such translation increment $\{d\theta\}$ is proportional to the magnitude $d\epsilon_l^p$ of plastic strain increment $\{d\epsilon^p\}$. The above hypotheses can be expressed as follows.

$$\sigma_o = \sigma_o(\epsilon_l^p, T) \tag{10}$$

$$d\sigma_o = \frac{\partial \sigma_o}{\partial \epsilon_l^p} d\epsilon_l^p + \frac{\partial \sigma_o}{\partial T} dT$$

$$\{d\theta\} = k \; d\epsilon_l^p \{n_\theta\} \tag{11}$$

where k : proportional coefficient
$\{n_\theta\}$: unit vector which indicates the direction of translation increment of the yield surface

In general cases, an explicit form of Eq. (10) should be determined based on the results of experiment. And, in this paper, the direction of translation increment of the yield surface is selective, that is, its direction can be selected to represent well the behavior of the material. For example, when the Ziegler rule[12] is regarded as appropriate to describe the behavior of the material based on the results of experiment, unit vector $\{n_\theta\}$ in Eq. (11) is expressed as

$$\{n_\theta\} = \{\sigma - \theta\} / \left| \{\sigma - \theta\} \right| \tag{12}$$

where $\left|\{\sigma-\theta\}\right| = (\{\sigma-\theta\}^T\{\sigma-\theta\})^{\frac{1}{2}}$

2.2.2 Plastic strain increment and workhardening modulus

Assuming that the material behaves according to incremental strain theory of plasticity (plastic flow theory) in the plastic range and introducing the yield function (f of Eq. (8)) as a plastic potential, plastic strain increment $\{d\varepsilon^p\}$ is defined as

$$\{d\varepsilon^p\} = d\varepsilon_1^p \{n\} \tag{13}$$

where $d\varepsilon_1^p$: magnitude of plastic strain increment $\{d\varepsilon^p\}$ (that is, the length of the vector $\{d\varepsilon^p\}$)
$\{n\}$: unit vector outward normal to the yield surface at the stress point

$$\{n\} = \left\{\frac{\partial f}{\partial(\sigma-\theta)}\right\} \Big/ f_1', \quad f_1' = \left| \left\{\frac{\partial f}{\partial(\sigma-\theta)}\right\} \right|$$
$$= \left(\left\{\frac{\partial f}{\partial(\sigma-\theta)}\right\}^T \left\{\frac{\partial f}{\partial(\sigma-\theta)}\right\} \right)^{\frac{1}{2}}$$

The above expression implies that the direction of plastic strain increment $\{d\varepsilon^p\}$ is shown by a vector outward normal to the yield surface at the stress point (see **Fig. 2**). When a usual yield function which is independent of hydrostatic stress is chosen as a plastic potential, plastic strain increment given by Eq. (13) satisfy the condition of incompressibility of the material automatically.

Next, a relationship between plastic strain increment $\{d\varepsilon^p\}$ and stress increment $\{d\sigma\}$ will be considered,

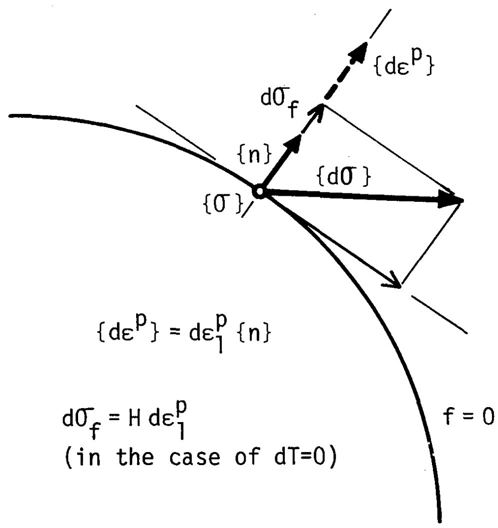


Fig. 2 Relation between Stress Increment $\{d\sigma\}$ and Plastic Strain Increment $\{d\varepsilon^p\}$ (in the Case of dT=0)

based on the following condition which must be satisfied in the case where the material is under loading in the plastic range.

$$df = 0 \tag{14}$$

First, the case where temperature does not change like usual plastic problem will be discussed. As the yield surface is defined by Eq. (8) and the changes of its size and position are calculated by Eqs. (10) and (11) respectively, the above Eq. (14) may be rewritten in the explicit form as

$$0 = df = \left\{\frac{\partial f}{\partial(\sigma-\theta)}\right\}^T \{d(\sigma-\theta)\} + \frac{\partial f}{\partial \sigma_o} d\sigma_o$$
$$= f_1' \, d\sigma_f - f_1' \, k \, n_{\theta f} \, d\varepsilon_1^p + \frac{\partial f}{\partial \sigma_o} \frac{\partial \sigma_o}{\partial \varepsilon_1^p} d\varepsilon_1^p \tag{15}$$

where $d\sigma_f = \{n\}^T \{d\sigma\}$
$n_{\theta f} = \{n\}^T \{n_\theta\}$

Rearrangement of the above expression for $d\sigma_f$ and $d\varepsilon_1^p$ may produce

$$d\sigma_f = \left(k\, n_{\theta f} - f_1'^{-1} \frac{\partial f}{\partial \sigma_o} \frac{\partial \sigma_o}{\partial \varepsilon_1^p} \right) d\varepsilon_1^p \qquad (16)$$

The above equation represents that the magnitude $d\varepsilon_1^p$ of plastic strain increment $\{d\varepsilon^p\}$ is proportional to the normal component $d\sigma_f$ of stress increment $\{d\sigma\}$ to the yield surface (see Fig. 2). Accordingly, even for the same magnitude of stress increment $\{d\sigma\}$, the magnitude of plastic strain increment $\{d\varepsilon^p\}$ is not necessarily the same, depending upon the direction of $\{d\sigma\}$. That is, nearer is the direction of $\{d\sigma\}$ to normal to the yield surface, larger is the magnitude $d\varepsilon_1^p$ of plastic strain increment $\{d\varepsilon^p\}$. Therefore, in this meaning, the normal component $d\sigma_f$ of stress increment $\{d\sigma\}$ is called effective stress increment. For further development of plastic deformation in the case of usual metals, it is necessary to increase the stresses if the temperature does not change. This phenomenon is caused by hardening of the material due to plastic work and is called workhardening (strain-hardening). As seen from Eq. (16) which represents the relationship between $d\sigma_f$ and the magnitude $d\varepsilon_1^p$ of plastic strain increment $\{d\varepsilon^p\}$ produced by $\{d\sigma\}$, the above-mentioned effective stress increment $d\sigma_f$ is the effective one to progress the plastic deformation.

Here, Eq. (16) may be rewritten as

$$d\sigma_f = H\, d\varepsilon_1^p \qquad (17)$$

where $$H = k\, n_{\theta f} - f_1'^{-1} \frac{\partial f}{\partial \sigma_o} \frac{\partial \sigma_o}{\partial \varepsilon_1^p} \qquad (18)$$

In the above equation, H is the proportional coefficient of Eq. (16) and is called workhardening modulus.

In the case where temperature changes, as the temperature change affects the mechanical properties, such as modulus of elasticity and yield stress, the size of the yield surface changes not only by the plastic deformation but also by temperature as assumed by Eq. (10). Therefore, the relationship between stress increment $\{d\sigma\}$ and plastic strain increment $\{d\varepsilon^p\}$ is influenced by the temperature change. With consideration of this effect, Eq. (17) may be rewritten in the following form, from Eqs. (14), (8), (10), (11) and (18).

$$d\sigma_f = H\, d\varepsilon_1^p - f_1'^{-1} \frac{\partial f}{\partial \sigma_o} \frac{\partial \sigma_o}{\partial T} dT \qquad (19)$$

This represents the relation among stress, plastic strain and temperature increments.

Again, Eq. (19) is solved for $d\varepsilon_1^p$ as

$$d\varepsilon_1^p = \frac{1}{H} \left(d\sigma_f + f_1'^{-1} \frac{\partial f}{\partial \sigma_o} \frac{\partial \sigma_o}{\partial T} dT \right) \qquad (20)$$

This equation represents that the effective stress increment $d\sigma_f$ must exceed the increment $\left(-f_1'^{-1} \frac{\partial f}{\partial \sigma_o} \frac{\partial \sigma_o}{\partial T} dT\right)$ due to expansion or shrinkage of the yield surface by temperature change in order to increase plastic deformation. This may be interpreted that when the yield surface expands by temperature change, the stress needs to increase beyond the expansion by the temperature change for progress of plastic deformation.

2.2.3 Stress - total strain incremental relation

In the plastic range, total strain increment $\{d\varepsilon\}$ is expressed by the summation of components as

$$\{d\varepsilon\} = \{d\varepsilon^e\} + \{d\varepsilon^p\} + \{d\varepsilon^T\} \qquad (21)$$

First, the relationship between the magnitude $d\varepsilon_1^p$ of plastic strain increment $\{d\varepsilon^p\}$ and total strain increment $\{d\varepsilon\}$ will be obtained. Equation (14) for the condition of loading in the plastic range may be reformed by introduction of Eq. (8) for the yield surface, Eqs. (10) and (11) for the changes of its size and position and Eq. (18) for the workhardening modulus.

$$\begin{aligned} 0 = df &= \left\{ \frac{\partial f}{\partial(\sigma-\theta)} \right\}^T \{d(\sigma-\theta)\} + \frac{\partial f}{\partial \sigma_o} d\sigma_o \\ &= f_1' \{n\}^T \{d\sigma\} - f_1' k\, n_{\theta f}\, d\varepsilon_1^p \\ &\quad + \frac{\partial f}{\partial \sigma_o} \frac{\partial \sigma_o}{\partial \varepsilon_1^p} d\varepsilon_1^p + \frac{\partial f}{\partial \sigma_o} \frac{\partial \sigma_o}{\partial T} dT \\ &= f_1' \{n\}^T \{d\sigma\} - f_1' H\, d\varepsilon_1^p + \frac{\partial f}{\partial \sigma_o} \frac{\partial \sigma_o}{\partial T} dT \end{aligned} \qquad (22)$$

where $$f_1' = \left| \left\{ \frac{\partial f}{\partial(\sigma-\theta)} \right\} \right| = \left(\left\{ \frac{\partial f}{\partial(\sigma-\theta)} \right\}^T \left\{ \frac{\partial f}{\partial(\sigma-\theta)} \right\} \right)^{\frac{1}{2}}$$

$$\{n\} = \left\{ \frac{\partial f}{\partial(\sigma-\theta)} \right\} / f_1'$$

$$n_{\theta f} = \{n\}^T \{n_\theta\}$$

The above equation will be further transformed according to the following procedure.

(1) To substitute Eq. (6) into stress increment $\{d\sigma\}$ in the first term of the right side and express in terms of elastic strain increment $\{d\varepsilon^e\}$ etc..

(2) To replace elastic strain increment $\{d\varepsilon^e\}$ by total strain increment $\{d\varepsilon\}$ etc., using Eq. (21).

(3) To express thermal strain increment $\{d\varepsilon^T\}$ by Eq. (1) and plastic strain increment $\{d\varepsilon^p\}$ by Eq. (13).

As a result of this manipulation, Eq. (22) is transformed into a function of only total strain increment $\{d\varepsilon\}$ and the magnitude $d\varepsilon_1^p$ of plastic strain increment $\{d\varepsilon^p\}$ as unknowns. Rearrangement of the equation provides the relationship between $\{d\varepsilon\}$ and $d\varepsilon_1^p$ as

$$d\varepsilon_1^p = \left[\{n\}^T [D_d^e]\{d\varepsilon\} - \left\{\{n\}^T [D_d^e]\left(\{\alpha\} - [D_d^e]^{-1}\frac{d[D^e]}{dT}\{\varepsilon^e\}\right) - f_1'^{-1}\frac{\partial f}{\partial \sigma_0}\frac{\partial \sigma_0}{\partial T}\right\} dT\right] \Big/ S \tag{23}$$

where $S = \{n\}^T [D_d^e]\{n\} + H$

Next, the incremental relationship between stress and total strain will be derived. Based on Eq. (6) which represents the relationship between stress increment $\{d\sigma\}$ and elastic strain increment $\{d\varepsilon^e\}$, the right side of Eq. (6) will be transformed as follows.

(1) To replace elastic strain increment $\{d\varepsilon^e\}$ by total strain increment $\{d\varepsilon\}$ etc., using Eq. (21).

(2) To express thermal strain increment $\{d\varepsilon^T\}$ by Eq. (1) and plastic strain increment $\{d\varepsilon^p\}$ by Eq. (13). Further, replace the magnitude $d\varepsilon_1^p$ of $\{d\varepsilon^p\}$ by $\{d\varepsilon\}$ etc., using Eq. (23).

As a result of the above calculation, only total strain increment $\{d\varepsilon\}$ remains as unknown on the right side of Eq. (6), and Eq. (6) becomes the incremental equation representing the relationship between stress increment $\{d\sigma\}$ and total strain increment $\{d\varepsilon\}$. Rearrangement of the right side and division of the expression into terms relating to total strain increment $\{d\varepsilon\}$ and the other terms (including temperature increment dT) furnish the following equation.

$$\{d\sigma\} = [D_d^p]\{d\varepsilon\} - \left\{[D_d^p]\left(\{\alpha\} - [D_d^e]^{-1}\frac{d[D^e]}{dT}\{\varepsilon^e\}\right) + [D_d^e]\{n\} f_1'^{-1}\frac{\partial f}{\partial \sigma_0}\frac{\partial \sigma_0}{\partial T}\Big/ S\right\} dT \tag{24}$$

where $[D_d^p] = [D_d^e] - [D_d^e]\{n\}\{n\}^T [D_d^e] / S$

Equation (24) is applied when the material is on loading in the plastic range by the subsequent loading. Then, the scalar $d\varepsilon_1^p$, the magnitude of the plastic strain increment expressed by Eq. (23), is positive. In contrast with this, if unloading occurs from the plastic range, the behavior of the material is elastic and $d\varepsilon_1^p$ becomes negative. Therefore, Eq. (7), the incremental relationship between stress and total strain in the elastic range, should be used for the analysis of the subsequent step of loading in place of Eq. (24).

In order to proceed the analysis at the new increment of loading, the resulting new size and position of the yield surface by the previous increment should be obtained. It is assumed that the size (σ_0) of the yield surface is the function of the quantity ε_1^p of plastic strain and temperature T as shown in Eq. (10). In connection with this, an explicit form of Eq. (10) should have been determined in advance based on experimental results. The position $\{\theta\}$ of the yield surface after each increment is known by summing up all translation increments $\{d\theta\}$ obtained at the preceding steps of loading. For calculation of $\{d\theta\}$, the proportional coefficient k in Eq. (11) must be determined and this can be obtained with the aid of Eq. (18) as the workhardening modulus H has been determined in advance based on experimental results*).

3. Basic Equations for Thermal Elastic - Plastic Analysis by Finite Element Method

The basic concept of the finite element method, expressed simply, is to regard a structure as an assembly of simple structural elements interconnected at a finite number of nodal points, where the equilibrium and compatibility conditions are satisfied. Accordingly, the structure under consideration should be divided into a finite number of elements at the beginning of the analysis such as triangular finite elements for plane stress or strain problems, or tetrahedral finite elements for three - dimensional stress problems.

One of typical finite elements in the continuum concerned is considered here and its mechanical behavior is represented in matrix forms in the following.

*) One of the methods to obtain the workhardening modulus H is to conduct usual uniaxial tensile tests at the temperature range which the material passes (Each test is conducted in a constant temperature). Then, the workhardening modulus H may be obtained as a function of the quantity ε_1^p of plastic strain and temperature T, from the resulting stress - plastic strain diagrams (see Eq. (17)).

3.1 Total strain - displacement incremental relation in an element

The displacements $\{s\}$ of an arbitrary point in an element will be defined as a function of the nodal displacements $\{w\}$.

$$\{s\} = [N]\{w\} = [N_i\ N_j\ \dots]\ \{w_i\ w_j\ \dots\}^T \tag{25}$$

where $[N]$: displacement function (the components of $[N]$ are generally a function of positions)

$\{w\}$: nodal displacements (suffix i, j, $\cdots$ are nodal numbers)

Introducing the co-ordinates of any point within the element into the displacement function $[N]$, the displacements of the point can be expressed as functions of the nodal displacements by Eq. (25).

The strains $\{\varepsilon\}$ in the element are obtained as functions of the nodal displacements as a result of appropriate differentiation of Eq. (25) (that is, the differentiation of $[N]$) with respect to the co-ordinates,

$$\{\varepsilon\} = [B]\{w\} \tag{26}$$

In the case of infinitesimal displacement problem, the above matrix $[B]$ can be regarded as a constant matrix. Then, the above equation may be expressed in the form of increment as

$$\{d\varepsilon\} = [B]\{dw\} \tag{27}$$

3.2 Stress - total strain incremental relation

When temperature of the element changes during an increment, the relationship between stress increment and total strain increment may be generally expressed in the following form, from Eqs. (7) and (24).

$$\{d\sigma\} = [D_d]\{d\varepsilon\} - \{C\}\, dT \tag{28}$$

where, for the elastic range

$$[D_d] = [D_d^e]$$

$$\{C\} = [D_d^e]\left(\{\alpha\} - [D_d^e]^{-1}\frac{d[D^e]}{dT}\{\varepsilon^e\}\right)$$

for the plastic range

$$[D_d] = [D_d^p]$$

$$\{C\} = [D_d^p]\left(\{\alpha\} - [D_d^e]^{-1}\frac{d[D^e]}{dT}\{\varepsilon^e\}\right) + [D_d^e]\ \{n\}\, f_1'^{-1}\ \frac{\partial f}{\partial \sigma_o}\ \frac{\partial \sigma_o}{\partial T} \Big/ S$$

3.3 Nodal force - nodal displacement incremental relation

From the basic equations which have been already shown, the relationship between nodal forces and nodal displacements will be derived by applying the principle of virtual displacement.

Here, the nodal forces $\{F\}$ are defined, which are statically equivalent to the boundary stresses acting the element.

$$\{F\} = \{F_i\ F_j \dots\}^T \tag{29}$$

Each of the forces $\{F_i\}$ must contain the same number of components as the corresponding nodal displacements $\{w_i\}$.

Imposing arbitrary virtual nodal displacements $\{w^*\}$ to the element, the external work δW_e done by the nodal forces $\{F\}$ during that displacement is

$$\delta W_e = \{w^*\}^T\{F\} \tag{30}$$

Similarly, the internal work δW_i per unit volume done by the stresses $\{\sigma\}$ is

$$\delta W_i = \{\varepsilon^*\}^T\{\sigma\} \tag{31}$$

$\{\varepsilon^*\}$ in the above equation are virtual strains due to virtual nodal displacements $\{w^*\}$, and they have the relation of Eq. (26). Thus, expressing the internal work δW_i with virtual nodal displacements $\{w^*\}$, Eq. (31) becomes

$$\delta W_i = \{w^*\}[B]^T\{\sigma\} \tag{32}$$

Equating the external work by Eq. (30) with the total internal work obtained by integrating Eq. (32) over the volume of the element, the following equation is obtained.

$$\{w^*\}^T\{F\} = \{w^*\}^T\int [B]^T\{\sigma\}\ d(\mathrm{vol}) \tag{33}$$

As this relation is valid for any value of the virtual displacements, the equality of the multipliers must exist. Therefore,

$$\{F\} = \int [B]^T\{\sigma\}\ d(\mathrm{vol}) \tag{34}$$

In the case where matrix $[B]$ can be regarded as a constant matrix, the above equation may be expressed in the form of increment as

$$\{dF\} = \int [B]^T\{d\sigma\}\ d(\mathrm{vol}) \tag{35}$$

Substitution of Eqs. (28) and (27) into Eq. (35) provides the incremental relationship between nodal forces and nodal displacements, that is,

$$\{ dF \} = [K] \{ dw \} - \{ dL \} \tag{36}$$

where $[K] = \int [B]^T [D_d] [B] \, d(vol)$: stiffness matrix of the element

$\{dL\} = \int [B]^T \{ C \} \, dT \, d(vol)$: equivalent nodal force increment for temperature change

The equilibrium state of the whole structure will be kept for satisfying the condition which is obtained as the summation of individual equilibrium equation at each node.

$$\Sigma \{ dF \} = \Sigma [K] \{ dw \} - \Sigma \{ dL \} \tag{37}$$

(Σ means summation of the appropriate matrices of all elements)

Once the above Eq. (37) is solved for nodal displacement increment $\{dw\}$, satisfying the specified boundary conditions, total strain increment $\{d\varepsilon\}$ and stress increment $\{d\sigma\}$ of each element can be evaluated from Eqs. (27) and (28).

4. Concluding Remarks

In this paper, the authors showed the theory of thermal elastic-plastic analysis with consideration of temperature-dependence and plastic history-dependence of the mechanical properties of the material. Especially to take into account more accurately the effects of plastic history like the Bauschinger effect, the combined model of isotropic and kinematic rules is introduced as the work-hardening rule. That is, it is assumed that the yield surface can change its size and position. As a result, it is possible to adopt many of verious complex work-hardening rules and more accurate information about thermal stresses and strains due to welding will be obtained by the analyses based on this developed theory.

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