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RIEMANNIAN G.O. SPACES FIBERED OVER IRREDUCIBLE SYMMETRIC SPACES

Dedicated to Professor Tadashi Nagano on his 70th birthday

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0. Introduction

We will classify the homogeneous spaces $M = G/K$ satisfying that (i) M are fibered over irreducible symmetric spaces G/H, and (ii) certain G-invariant metrics on G/K are g.o. with respect to G. Among them are many examples of weakly symmetric spaces, which were described in [4], [25] and [5]. In fact we will obtain new examples of weakly symmetric spaces. We will find very useful the theory of orbits of s-representations ([10] and [21]), and the classification of real irreducible polar representations ([6]).

A connected Riemannian manifold is called *g.o.* (geodesic orbit) if every geodesic is an orbit of a one-parameter subgroup of the isometry group. Every symmetric space is g.o. A connected Riemannian manifold is called *weakly symmetric* if, for any two points, there exists an isometry which interchanges them. Weakly symmetric spaces are also g.o. ([2]).

To study these spaces (more generally, Riemannian homogeneous spaces), the isotropy representations will play important roles. For example, W. Ziller characterized weakly symmetric spaces in terms of the isotropy representations, and provided many examples ([25]).

The isotropy representations of semi-simple symmetric spaces are called *s-representations*. One of their most interesting properties is the conjugacy of maximal abelian subspaces (see Lemma 1.5). A representation of a compact Lie group, which has the above property, is called *polar* (see below for exact definition). J. Dadok has investigated them, and classified irreducible ones ([6]). It is natural to expect that there are some relations between symmetric-like Riemannian manifolds and generalizations of s-representations.

In [7] and [20], g.o. metrics on a compact fiber bundle

$$F := H/K \longrightarrow M := G/K \longrightarrow B := G/H$$

have been investigated. A necessary and sufficient condition for certain metrics to be g.o., can be expressed in terms of the Lie algebras. Actually, the assumption that G is compact is not necessary : if G is semi-simple and (G, H) is a symmetric pair, then we can argue about this case in the same way. The aim of this paper is to classify the Riemannian g.o. spaces fibered over irreducible symmetric spaces.

The classification will be accomplished in the following steps. Let (G^*, H) be the non-compact dual of a compact symmetric pair (G, H). A triple (G, H, K) satisfies our condition if and only if (G^*, H, K) also satisfies it (Theorem 3.2). Thus we can assume G is compact. If a triple (G, H, K) satisfies our condition, then the restriction of the isotropy representation of $B = G/H$ to K, is irreducible and polar (Theorem 2.3). From the classification by J. Dadok, we can list all candidates for satisfying the condition. The classification will be done case by case. To check each of them, we have found very useful the restricted root systems and the orbit types of s-representations, which has been investigated in [10] and [21].

1. Preliminaries

Let $(M := G/K, g)$ be a connected Riemannian homogeneous space, where G is the full isometry group. Let T_oM be the tangent space of M at the origin o. The natural action of K on T_oM is called the *isotropy representation* of M.

Proposition 1.1. *$(M = G/K, g)$ is symmetric if and only if there exists x in K such that x acts on T_oM as $-id$.*

There are interesting generalizations of symmetric spaces, which have been investigated by many authors (see [14] and [3] for surveys). We mention, here, two classes of them.

One of them is a weakly symmetric space, which has been introduced by A. Selberg [18]. His original definition is not the same as above, but equivalent to it ([4]). Weakly symmetric spaces can be characterized in terms of the isotropy representations.

Proposition 1.2 ([25]). *$(M = G/K, g)$ is weakly symmetric if and only if, for any vector v in T_oM, there exists x in K such that $x \cdot v = -v$.*

By Propositions 1.1 and 1.2, it is trivial that every symmetric space is weakly symmetric.

Another generalization of a symmetric space is the class of g.o. spaces. A characterization of g.o. spaces in terms of the isotropy representations is not known. But they can be characterized in terms of the Lie algebras. Let $\mathcal{G}$ and $\mathcal{K}$ be the Lie algebras of G and K respectively, and $\mathcal{M}$ be a K-invariant complement of $\mathcal{K}$ in $\mathcal{G}$.

Proposition 1.3 ([15]). *$(M = G/K, g)$ is g.o. if and only if, for any vector v in T_oM, there exists X in $\mathcal{K}$ such that*

$$g([X + v, Y]_{\mathcal{M}}, v) = 0 \textit{ for every } Y \in \mathcal{M},$$

where the subscript $\mathcal{M}$ denotes the $\mathcal{M}$-component.

We can easily prove that every symmetric space is g.o. In fact, we have only to let $\mathcal{M}$ be the (-1)-eigenspace of the involution, and put $X := 0$. It is known that every weakly symmetric space is g.o. ([2]).

We obtain the following relations :

$$\{\text{symmetric spaces}\} \subset \{\text{weakly symmetric spaces}\} \subset \{\text{g.o. spaces}\}.$$

It is known that each inclusion is strict.

We need the polar representations, which have been investigated by J. Dadok ([6]). Let G be a compact Lie group with the Lie algebra $\mathcal{G}$, and

$$\varphi : G \longrightarrow O(V)$$

be a real representation preserving an inner product $(\ ,\ )$ on V. For $v \in V$,

$$\mathcal{A}_v := \{u \in V \mid (u, \mathcal{G} \cdot v) = 0\}$$

is called a *cross-section* at v. Each cross-section meets every G-orbit ([6]). When $\mathcal{G} \cdot v$ is of maximal possible dimension, $\mathcal{A}_v$ is said to be *minimal.*

DEFINITION 1.4. A representation φ is called *polar* if a minimal cross-section $\mathcal{A}_v$ intersects every G-orbit orthogonally (i.e., $(\mathcal{G} \cdot u, \mathcal{A}_v) = 0$ for every $u \in \mathcal{A}_v$).

A representation φ is polar if and only if every minimal cross-section is conjugate ([6]).

Lemma 1.5. *Every s-representation is polar.*

Proof. A minimal cross-section of an s-representation is the maximal abelian subspace. Since every maximal abelian subspace is conjugate, we conclude the lemma. $\square$

Theorem 1.6 ([6]). *Every real irreducible polar representation, which is not trivial, is equivalent to one of the following.*
(i) *A representation by which the group acts on the unit sphere transitively.*
(ii) *The isotropy representation of a compact irreducible symmetric space of rank ≥ 2, or the restriction to the semi-simple part of this isotropy subgroup.*
(iii) *The actions of $G_2 \times SO(2)$ on $\mathbf{R}^7 \otimes \mathbf{R}^2$, $Spin(7) \times SO(2)$ on $\mathbf{R}^8 \otimes \mathbf{R}^2$, and $Spin(7) \times SO(3)$ on $\mathbf{R}^8 \otimes \mathbf{R}^3$, where G_2 acts on $\mathbf{R}^7$ irreduciblly, and $Spin(7)$ acts on $\mathbf{R}^8$ by the spin representation.*

2. G.o. metrics on fiber bundles (compact case)

In this section we recall the results in [7] and [20], and mention the relation between g.o. spaces and polar representations.

We call (G, H, K) a *triple*, for short, if G is a connected semi-simple Lie group, H and K are compact subgroups, $G \supset H \supset K$, and with $\dim G > \dim H > \dim K$. A triple is associated with the fiber bundle

$$F := H/K \longrightarrow M := G/K \longrightarrow B := G/H.$$

Using the Killing form, we define a two-parameter family of Riemannian metrics on M, which depends on H. Let $\mathcal{G}$, $\mathcal{H}$ and $\mathcal{K}$ be the Lie algebras of G, H and K respectively. We denote by κ the Killing form of $\mathcal{G}$, which is non-degenerate. Taking the orthogonal complements with respect to κ, we obtain

$$\mathcal{G} = \mathcal{H} \oplus \mathcal{M}_B = \mathcal{K} \oplus \mathcal{M}_F \oplus \mathcal{M}_B.$$

The tangent space of $M = G/K$ at the origin can be identified with $\mathcal{M} := \mathcal{M}_F \oplus \mathcal{M}_B$.

In this section we consider the case G is compact (i.e. κ is negatve definite). For any $a, b > 0$,

$$g_{a,b} := -a \cdot \kappa \mid_{\mathcal{M}_F \times \mathcal{M}_F} - b \cdot \kappa \mid_{\mathcal{M}_B \times \mathcal{M}_B}$$

is a K-invariant inner product on $\mathcal{M}$, and thus it defines the G-invariant Riemannian metric on M (we denote the metric by the same symbol). We say that the metric g is *G-g.o.* or *g.o. with respect to G*, if every geodesic is the orbit of a one-parameter subgroup of G.

Proposition 2.1 ([7]). *The metric $g_{a,b}$ is G-g.o. for every $a, b > 0$ if and only if, for every $v_F \in \mathcal{M}_F$ and $v_B \in \mathcal{M}_B$, there exists $X \in \mathcal{K}$ such that*

$$[X, v_F] = 0 \text{ and } [X + v_F, v_B] = 0.$$

We remark that the metric $g_{a,a}$ is always g.o. and that if $g_{a,b}$ is G-g.o. for some $a \neq b$, then it holds true for every $a, b > 0$.

Proposition 2.2 ([20]). *If the metric $g_{a,b}$ is G-g.o. for every $a, b > 0$, then*
(i) *each $\mathcal{H}$-irreducible component in $\mathcal{M}_B$ is $\mathcal{K}$-irreducible, and*
(ii) *for every v in $\mathcal{M}_B$, the connected component of K-orbit $K(v)$ coincides with that of an H-orbit $H(v)$.*

A part of our aim (the compact case) is to classify triples (G, H, K) satisfying the following condition.

CONDITION I. (G, H) is a compact effective irreducible symmetric pair and the metric $g_{a,b}$ on M is G-g.o. for any $a, b > 0$.

We have only to consider the triples of the Lie algebras, since Proposition 2.1 implies that Condition I depends only on the locally isomorphism classes. The next theorem is very useful for the classification.

Theorem 2.3. *If a triple (G, H, K) satisfies Condition I, then the representation of K on $\mathcal{M}_B$ is irreducible and polar.*

Proof. The isotropy representation of $B = G/H$ is irreducible and polar by Lemma 1.5. We want to investigate its restriction to K. The irreducibility follows from Proposition 2.2. Let

$$\mathcal{A}_v^H := \{u \in \mathcal{M}_B \mid (u, \mathcal{H} \cdot v) = 0\}$$

be the minimal cross-section at a regular $v \in \mathcal{M}_B$ with respect to the action of H. By the definition of polar representations,

$$(\mathcal{H} \cdot u, \mathcal{A}_v^H) = 0 \text{ for every } u \in \mathcal{A}_v^H.$$

We obtain $\mathcal{K} \cdot v = \mathcal{H} \cdot v$ by Proposition 2.2, and thus

$$\mathcal{A}_v^H = \mathcal{A}_v^K := \{u \in \mathcal{M}_B \mid (u, \mathcal{K} \cdot v) = 0\}.$$

Now we conclude that

$$(\mathcal{K} \cdot u, \mathcal{A}_v^K) = 0 \text{ for every } u \in \mathcal{A}_v^K,$$

and the representation of K on $\mathcal{M}_B$ is polar. □

3. G.o. metrics on fiber bundles (non-compact case)

Let us consider triples (G, H, K) in the sense of Section 2.. In this section we assume that G is of non-compact type (i.e. each simple factor is non-compact), and H is maximal compact. Under these assumptions, the Killing form κ of $\mathcal{G}$ is positive definite on $\mathcal{M}_B$ and negative definite on $\mathcal{M}_F$. Thus, for any $a, b > 0$,

$$g_{a,b} := -a \cdot \kappa \,|_{\mathcal{M}_F \times \mathcal{M}_F} + b \cdot \kappa \,|_{\mathcal{M}_B \times \mathcal{M}_B}$$

is a K-invariant inner product on $\mathcal{M}$, and it also denotes the G-invariant Riemannian metric.

Proposition 3.1. *The metric $g_{a,b}$ is G-g.o. for every $a, b > 0$ if and only if, for every v_F in $\mathcal{M}_F$ and v_B in $\mathcal{M}_B$, there exists X in $\mathcal{K}$ such that*

$$[X, v_F] = 0 \text{ and } [X + v_F, v_B] = 0.$$

Proof. The proof is almost the same as that of Proposition 2.1 (see [7]). A vector $X+v_F+v_B$ is a geodesic vector (i.e., the orbit of the one-parameter subgroup generated by $X + v_F + v_B$ is a geodesic) with respect to $g_{a,b}$ if and only if

$$[X, v_F] = 0 \text{ and } [X, v_B] = \frac{a+b}{b}[v_F, v_B].$$

□

We remark that, if $g_{a,b}$ is G-g.o. for some $a, b > 0$, then it holds true for every $a, b > 0$.

CONDITION II. (G, H) is a non-compact effective irreducible symmetric pair and the metric $g_{a,b}$ on M is G-g.o. for any $a, b > 0$.

For the classification of triples which satisfy Condition II, it is sufficient to consider triples satisfying Condition I. Let (G, H) be a compact irreducible symmetric pair, and (G^*, H) be its non-compact dual.

Theorem 3.2. *A triple (G, H, K) satisfies Condition I if and only if (G^*, H, K) satisfies Condition II.*

Proof. It is obvious from Propositions 2.1 and 3.1. □

Furthermore, weakly symmetricity holds by the above duality. Since weakly symmetric spaces can be characterized in terms of the isotropy representations (Proposition 1.2), and the isotropy representations of G/K and G^*/K are equivalent.

4. The classification

In this section, we will classify the triples (G, H, K) which satisfy Condition I and II. By the duality (Theorem 3.2), it is sufficient to investigate I. Furthermore, it is enough to list triples of Lie algebras, since I and II are characterized in terms of the Lie algebras. We state the classification theorem.

Theorem 4.1. *A triple (G, H, K) satisfies Condition I if and only if the triple of Lie algebras is one of the Table 1.*

	$\mathcal{G}$	$\mathcal{H}$	$\mathcal{K}$	
(1.1)	$so(2n+1)$	$so(2n)$	$u(n)$	$n \geq 2$
(1.2)	$so(4n+1)$	$so(4n)$	$su(2n)$	$n \geq 1$
(1.3)	$so(8)$	$so(7)$	g_2	
(1.4)	$so(9)$	$so(8)$	$so(7)$	
(1.5)	$su(n+1)$	$u(n)$	$su(n)$	$n \geq 2$
(1.6)	$su(2n+1)$	$u(2n)$	$u(1) \oplus sp(n)$	$n \geq 2$
(1.7)	$su(2n+1)$	$u(2n)$	$sp(n)$	$n \geq 2$
(1.8)	$sp(n+1)$	$sp(1) \oplus sp(n)$	$u(1) \oplus sp(n)$	$n \geq 1$
(1.9)	$sp(n+1)$	$sp(1) \oplus sp(n)$	$sp(n)$	$n \geq 1$
(2.1)	$su(2r+n)$	$su(r) \oplus su(r+n) \oplus \mathbf{R}$	$su(r) \oplus su(r+n)$	$r \geq 2, n \geq 1$
(2.2)	$so(4r+2)$	$u(2r+1)$	$su(2r+1)$	$r \geq 2$
(2.3)	e_6	$\mathbf{R} \oplus so(10)$	$so(10)$	
(3.1)	$so(9)$	$so(7) \oplus so(2)$	$g_2 \oplus so(2)$	
(3.2)	$so(10)$	$so(8) \oplus so(2)$	$spin(7) \oplus so(2)$	
(3.3)	$so(11)$	$so(8) \oplus so(3)$	$spin(7) \oplus so(3)$	

Table 1: The triples satisfying Condition I

We can assume that G, H and K are connected. If (G, H, K) satisfies Condition I, then the action of K on $\mathcal{M}_B$ is irreducible and polar, and every K-orbit in $\mathcal{M}_B$ coincides with an H-orbit. All irreducible polar representations have been classified by J. Dadok (see Theorem 1.6). The proof of Theorem 4.1 is organized as follows. For each irreducible polar representation $K \longrightarrow O(\mathcal{M}_B)$, we will find every group H satisfying that H contains K, H acts on $\mathcal{M}_B$ as an s-representation, and every H-orbit in $\mathcal{M}_B$ coincides with a K-orbit. By the above way, we can classify the triples satisfying the necessary conditions which are described in Proposition 2.2. By checking whether or not these candidates satisfy Condition I case by case, we conclude the theorem.

At first we consider the case when the polar representation is of type (iii) of Theorem 1.6. In [20], it is proved that the triples (3.1), (3.2) and (3.3) satisfy Condition I. The principal orbits of the isotropy representations of the Grassmann manifolds are locally isomorphic to the Stiefel manifolds of certain dimensions. We know the classifications of transitive actions on the Stiefel manifolds ([17]), and of the principal orbit types of s-representation (e.g. [21]). Then we can check that (3.1) – (3.3) are the only triples satisfying Condition I and of type (iii).

Secondly we investigate the case (ii). Polar representations of this case are the s-representations of rank > 1, or the restrictions to their semi-simple parts.

Lemma 4.2. *Let $H \longrightarrow O(\mathcal{M}_B)$ be an s-representation, K be a subgroup of H, and the action of K on $\mathcal{M}_B$ be also an s-representation. If every H-orbit coincides with a K-orbit, then H and K are locally isomorphic.*

Proof. Let M and M' be the symmetric spaces whose isotropy representations are the actions of H and K on $\mathcal{M}_B$ respectively. By the assumption on orbits, we can take the same maximal abelian subspace $\mathcal{A}$. Then rank (M) coincides with rank (M'). Let Δ and Δ' be the restricted root systems of M and M' with respect to $\mathcal{A}$ respectively. (We refer to [8], [16] for the restricted root systems.) Since the restricted root systems determine the compact symmetric spaces locally, we have only to show $\Delta = \Delta'$ to prove the lemma. We denote the root space decompositions as follows:

$$\mathcal{H} = \mathcal{H}(0) \oplus \sum_{\alpha\in\Delta} \mathcal{H}(\alpha) \quad \text{and} \quad \mathcal{K} = \mathcal{K}(0) \oplus \sum_{\alpha\in\Delta'} \mathcal{K}(\alpha).$$

For $\alpha \in \Delta$, we take a generic vector $v \in \mathcal{A}$ which is orthogonal to α (i.e., for $\beta \in \Delta \cup \Delta'$, $\beta(v) = 0$ if and only if $\beta \in \mathbf{R}\alpha - \{0\}$). By [10], the isotropy subalgebras of $\mathcal{H}$ and $\mathcal{K}$ at v are

$$\mathcal{H}_v = \mathcal{H}(0) \oplus \sum_{\beta\in\mathbf{R}\alpha} \mathcal{H}(\beta) \quad \text{and} \quad \mathcal{K}_v = \mathcal{K}(0) \oplus \sum_{\beta\in\mathbf{R}\alpha} \mathcal{K}(\beta).$$

Since the dimension of the H-orbit through v coincides with that of the K-orbit, we have

$$\dim \mathcal{H} - \dim \mathcal{H}_v = \dim \mathcal{K} - \dim \mathcal{K}_v.$$

Furthermore, the dimension of the principal orbit with resprct to H also coincides with that with respect to K, hence

$$\dim \mathcal{H} - \dim \mathcal{H}(0) = \dim \mathcal{K} - \dim \mathcal{K}(0).$$

Thus we obtain

$$\dim \sum_{\beta\in\mathbf{R}\alpha} \mathcal{H}(\beta) = \dim \sum_{\beta\in\mathbf{R}\alpha} \mathcal{K}(\beta).$$

This implies that, for every $\alpha \in \Delta$, there exists a root of Δ' which is parallel to α (i.e. the Weyl groups coincide), and that the sum of the mutiplicities of the roots in $\Delta \cap \mathbf{R}\alpha$ coincides with that of the roots in $\Delta' \cap \mathbf{R}\alpha$. From the list of the restricted root systems of symmetric spaces (see [8] or [21]), we obtain $\Delta = \Delta'$. □

Next we investigate the latter part of (ii). Let $(\mathcal{G}, \mathcal{H})$ be an irreducible Hermitian symmetric pair, $\mathbf{R}$ be the center of $\mathcal{H}$, and $\mathcal{K}$ be the semi-simple part of $\mathcal{H}$. Then

$\mathcal{H} = \mathbf{R} \oplus \mathcal{K}$. We consider the triple $(\mathcal{G}, \mathcal{H}, \mathcal{K})$. Let Δ be the restricted root system of $(\mathcal{G}, \mathcal{H})$ with respect to $\mathcal{A}$, and

$$\mathcal{H} = \mathcal{H}(0) \oplus \sum_{\alpha \in \Delta} \mathcal{H}(\alpha)$$

be the root space decomposition. We fix a non-zero vector v_F in $\mathbf{R}$.

Lemma 4.3. *The above triple $(\mathcal{G}, \mathcal{H}, \mathcal{K})$ satisfies Condition I if and only if there exists X in $\mathcal{K}$ such that $X + v_F \in \mathcal{H}(0)$.*

Proof. For every $v_B \in \mathcal{M}_B$, there exists $g \in H$ such that $\mathrm{ad}(g)v_B \in \mathcal{A}$. Assume that there exists $X \in \mathcal{K}$ such that

$$[X + v_F, \mathrm{ad}(g)v_B] = 0.$$

Since v_F is central, we obtain

$$[\mathrm{ad}(g^{-1})X + v_F, v_B] = 0.$$

Thus we can assume $v_B \in \mathcal{A}$ and agree that Condition I holds if and only if, for every $v_B \in \mathcal{A}$, there exists $X \in \mathcal{K}$ such that $[X + v_F, v_B] = 0$. Since $\mathcal{H}(0)$ is the centralizer of certain $v_B \in \mathcal{A}$, we conclude the lemma. □

Proposition 4.4. *A triple $(\mathcal{G}, \mathbf{R} \oplus \mathcal{K}, \mathcal{K})$ satisfies Condition I if and only if the restricted root system of $(\mathcal{G}, \mathbf{R} \oplus \mathcal{K})$ is of BC-type.*

Proof. By the results of [21],

$$\mathcal{H}(0) = \sum_{\alpha \in \Delta} [\mathcal{H}(\alpha), \mathcal{H}(\alpha)]_{\mathcal{H}(0)},$$

where subscript $\mathcal{H}(0)$ denotes the $\mathcal{H}(0)$-component.

If Δ is not of BC-type, then 2α is not a root, and

$$\mathcal{H}(0) = \sum_{\alpha \in \Delta} [\mathcal{H}(\alpha), \mathcal{H}(\alpha)].$$

Since $\mathcal{K}$ is the semi-simple part of $\mathcal{H}$,

$$\mathcal{K} = [\mathbf{R} \oplus \mathcal{K}, \mathbf{R} \oplus \mathcal{K}] \supset \mathcal{H}(0).$$

This means that there exists no $X \in \mathcal{K}$ such that $X + v_F \in \mathcal{H}(0)$, and that the triple does not satisfy Condition I.

If Δ is of BC_r-type, the roots, whose multiplicities are 1, are $2\varepsilon_1, \dots, 2\varepsilon_r$ (for the notations see [1]). Thus $\sum_{j=1}^{r} \mathcal{H}(2\varepsilon_j)$ is an r-dimensional abelian subalgebra of $\mathcal{H}$. It is easy to see that $\mathcal{H}(0) \oplus \mathcal{A}$ contains a maximal abelian subalgebra of $\mathcal{G}$. Then we have

$$\text{rank } \mathcal{G} = \text{rank } \mathcal{H}(0) + \dim \mathcal{A} = \text{rank } \mathcal{H}(0) + r.$$

Since Hermitian symmetric spaces are inner, we have rank $\mathcal{G}$ = rank $\mathcal{H}$ ([16]). Thus $\mathcal{H}(0) \oplus \sum_{j=1}^{r} \mathcal{H}(2\varepsilon_j)$ contains a maximal abelian subspace of $\mathcal{H}$. Then we have $v_F \in \mathcal{H}(0) \oplus \sum_{j=1}^{r} \mathcal{H}(2\varepsilon_j)$.

Next we claim that

$$[\mathcal{H}(2\varepsilon_j), \mathcal{H}(\varepsilon_k)] = \begin{cases} 0 & \text{if } \; j \neq k, \\ \mathcal{H}(\varepsilon_j) & \text{if } \; j = k. \end{cases}$$

If $j \neq k$, $2\varepsilon_j \pm \varepsilon_k$ are not roots and the bracket products equal to 0. If $j = k$, the subset $\Delta' := \{\pm\varepsilon_j, \pm 2\varepsilon_j\}$ is a closed subsystem of BC_1-type. By [21], there exist $\mathcal{H}'(0) \subset \mathcal{H}(0)$ and $\mathcal{A}' \subset \mathcal{A}$ such that the pair

$$\left(\mathcal{H}'(0) \oplus \sum_{\alpha \in \Delta'} \mathcal{H}(\alpha) \oplus \mathcal{A}' \oplus \sum_{\alpha \in \Delta'} \mathcal{M}(\alpha),\ \mathcal{H}'(0) \oplus \sum_{\alpha \in \Delta'} \mathcal{H}(\alpha) \right)$$

is symmetric and its restricted root system is Δ'. Then it is enough to show the case of BC_1-type satisfies the claim. It can be easily checked.

The claim shows that no non-zero vector in $\sum_{j=1}^{r} \mathcal{H}(2\varepsilon_j)$ centralizes $\mathcal{H}$. Thus v_F has the non-zero $\mathcal{H}(0)$-component Y. Since $\langle v_F, Y \rangle \neq 0$, there exists $a \neq 0$ such that $\langle v_F, aY - v_F \rangle = 0$. This means $X := aY - v_F \in \mathcal{K}$ and $X + v_F \in \mathcal{H}(0)$. By Lemma 4.3, we conclude that the triple of BC_r-type satisfies Condition I. $\square$

The compact irreducible Hermitian symmeteric pairs of BC_r-types, $r \geq 2$, are

$$\begin{aligned} &(su(2r+n),\ su(r) \oplus su(r+n) \oplus \mathbf{R}), \quad r \geq 2, n \geq 1, \\ &(so(4r+2),\ u(2r+1)), \quad r \geq 1, \quad \text{and} \\ &(e_6,\ \mathbf{R} \oplus so(10)). \end{aligned}$$

Therefore we have proved the case (ii) of Theorem 4.1.

Here we remark on their isometry groups. For an irreducible Hermitian symmetric space $G/SO(2)K$, the fiber bundle $M := G/K$ is a φ-symmetric space. A φ-symmetric space is a certain circle or line bundle over a Hermitian symmetric space.

These spaces have been introduced by T. Takahashi ([19]), and classified by J. A. Jiménez and O. Kowalski ([11]). We refer to them for the exact definition of φ-symmetric spaces. It is known that the group $SO(2)G$ acts on M as an isometry group, and M is weakly symmetric ([5]). Thus its metric is g.o. with respect to $SO(2)G$. It is non-trivial, however, whether or not it is g.o. with respect to G.

Finally we prove the case (i) of theorem 4.1. All the groups, which act on spheres transitively, are well known. We list, in Table 2, the groups G acting on the unit spheres in V transitively, and the symmetric spaces whose isotropy representations are equivalent to the G-actions on V if there exist.

G	V		the symmetric spaces
$SO(n)$	$\mathbf{R}^n$	$n \geq 2$	$SO(n+1)/SO(n)$
$SU(n)$	$\mathbf{C}^n$	$n \geq 2$	none
$U(n)$	$\mathbf{C}^n$	$n \geq 1$	$SU(n+1)/U(n)$
$Sp(n)$	$\mathbf{H}^n$	$n \geq 1$	none
$U(1) \times Sp(n)$	$\mathbf{H}^n$	$n \geq 1$	none if $n \geq 2$
$Sp(1) \times Sp(n)$	$\mathbf{H}^n$	$n \geq 1$	$Sp(n+1)/Sp(1) \times Sp(n)$
G_2	$\mathbf{R}^7$		none
$Spin(7)$	$\mathbf{R}^8$		none
$Spin(9)$	$\mathbf{R}^{16}$		$F_4/Spin(9)$

Table 2: Groups acting on the unit spheres transitively

We remark that there are some overlaps in Table 2.

Let $M = G/H$ be a compact irreducible rank 1 symmetric space. For a compact subgroup K of H, every K-orbit in T_oM coincides with an H-orbit if and only if K acts on the unit sphere of T_oM transitively. From Table 2, one can find all subgroups K of H satisfying this condition. Thus, we can make up Table 3, the candidates of the triples of type (i) for satisfying Condition I.

It is known that the associated spaces with (r1), (r6), (r7) and (h1) are weakly symmetric ([4] and [25]). Thus these satisfy Condition I. (Of course we can prove these facts by Proposition 2.1.) The triple (c1) is the case of φ-symmetric spaces. The restricted root system of $(\mathcal{G}, \mathcal{H})$ of (c1) is of BC_1-type. Hence it satisfies Condition I, by Proposition 4.4. To check the remaining cases, we mention two lemmas.

Lemma 4.5. *Let $\mathcal{A}_F \subset \mathcal{M}_F$ and $\mathcal{A}_B \subset \mathcal{M}_B$ be linear subspaces. We assume that $\mathcal{A}_F \oplus \mathcal{A}_B$ meets every K-orbit in $\mathcal{M}_F \oplus \mathcal{M}_B$. Then Condition I holds if and only if, for every $v_F \in \mathcal{A}_F$ and $v_B \in \mathcal{A}_B$, there exists $X \in \mathcal{K}$ such that*

$$[X, v_F] = 0 \text{ and } [X, v_B] = [v_F, v_B].$$

	$\mathcal{G}$	$\mathcal{H}$	$\mathcal{K}$	
r1	$so(2n+1)$	$so(2n)$	$u(n)$	$n \geq 2$
r2	$so(2n+1)$	$so(2n)$	$su(n)$	$n \geq 2$
r3	$so(4n+1)$	$so(4n)$	$sp(1) \oplus sp(n)$	$n \geq 2$
r4	$so(4n+1)$	$so(4n)$	$u(1) \oplus sp(n)$	$n \geq 1$
r5	$so(4n+1)$	$so(4n)$	$sp(n)$	$n \geq 1$
r6	$so(8)$	$so(7)$	g_2	
r7	$so(9)$	$so(8)$	$spin(7)$	
r8	$so(17)$	$so(16)$	$spin(9)$	
c1	$su(n+1)$	$u(n)$	$su(n)$	$n \geq 2$
c2	$su(2n+1)$	$u(2n)$	$u(1) \oplus sp(n)$	$n \geq 2$
c3	$su(2n+1)$	$u(2n)$	$sp(n)$	$n \geq 1$
h1	$sp(n+1)$	$sp(1) \oplus sp(n)$	$u(1) \oplus sp(n)$	$n \geq 1$
h2	$sp(n+1)$	$sp(1) \oplus sp(n)$	$sp(n)$	$n \geq 1$

Table 3: The candidates of the triples of type (i) for satisfying Condition I

Proof. We have only to prove the "if part". Let $v_F \in \mathcal{M}_F$ and $v_B \in \mathcal{M}_B$. By the assumption there exists $g \in K$ such that $gv_F \in \mathcal{A}_F$ and $gv_B \in \mathcal{A}_B$. Then there exists $X \in \mathcal{K}$ such that

$$[X, gv_F] = 0 \text{ and } [X, gv_B] = [gv_F, gv_B].$$

The element that we want is $\mathrm{ad}(g^{-1})X$. □

Lemma 4.6. *Let K' be a subgroup in K. If (G, H, K') satisfies Condition I, then so does (G, H, K).*

Proof. Let us denote the decompositions of (G, H, K) and (G, H, K') by

$$\mathcal{G} = \mathcal{K} \oplus \mathcal{M}_F \oplus \mathcal{M}_B, \quad \text{and} \quad \mathcal{G} = \mathcal{K}' \oplus \mathcal{M}'_F \oplus \mathcal{M}'_B$$

respectively. By the assumption, we have $\mathcal{K}' \subset \mathcal{K}$, $\mathcal{M}'_F \supset \mathcal{M}_F$, and $\mathcal{M}_B = \mathcal{M}'_B$. Thus the lemma follows from Proposition 2.1 immediately. □

Proposition 4.7. *The triple $(so(2n+1), so(2n), su(n))$ in* (r2)*, satisfies Condition I if and only if n is even.*

Proof. We remark that $\mathcal{M}_F \cong \mathbf{R} \oplus \bigwedge^2 \mathbf{C}^n$ and $\mathcal{M}_B \cong \mathbf{C}^n$ as $su(n)$-modules. A non-zero vector in $\mathbf{R}$ acts on $\mathcal{M}_B$ by a pure imaginary scalar, since $\mathbf{R}$ is the center of $u(n)$. Let $\{e_1, \dots, e_n\}$ be an orthonormal basis for $\mathbf{C}^n$, and

$$V := \mathrm{span}_{\mathbf{C}}\{e_{2j-1} \wedge e_{2j} \mid 1 \leq j \leq \frac{n}{2}\}.$$

We remark that V is a maximal abelian subspace in $\bigwedge^2 \mathbf{C}^n \cong T_o(SO(2n)/U(n))$. Then $\mathbf{R} \oplus V \oplus \mathbf{C}^n$ meets every $SU(n)$-orbit.

Case 1: let $n = 2m$ be even. The subgroup of $SU(2m)$ which acts on $\mathbf{R} \oplus V$ trivially is $SU(2) \times \cdots \times SU(2)$, m-copies of $SU(2)$ ([21]). We denote by $su(2)^m$ its Lie algebra. Every vector $v_F \in \mathbf{R} \oplus V$ preserves $\mathrm{span}_{\mathbf{C}}\{e_{2j-1}, e_{2j}\}$ for $j = 1, \dots, m$. Each $SU(2)$ acts on certain $\mathrm{span}_{\mathbf{C}}\{e_{2j-1}, e_{2j}\}$ naturally, and on its orthogonal complement trivially. Thus, for every $v_B \in \mathbf{C}^n$, there exists $X \in su(2)^m$ such that $[X, v_B] = [v_F, v_B]$.

Case 2: let $n = 2m+1$ be odd. Let $z \in \mathbf{R}$ be non-zero and $v_F \in V$ be principal. Thus $su(2)^m$ is the Lie algebra of the isotropy subgroup of $SU(n)$ at $z + v_F$ ([21]). We can easily see that

$$[z + v_F, e_{2m+1}] \neq 0 \text{ and } [su(2)^m, e_{2m+1}] = 0.$$

Thus the triple does not satisfy Condition I in this case. □

The triples $(su(2n+1), u(2n), u(1) \oplus sp(n))$ in (c2) and $(su(2n+1), u(2n), sp(n))$ in (c3) satisfy Condition I. We will prove that they are weakly symmetric in the next section, so we omit the proof.

Proposition 4.8. *The triples* $(sp(n+1), sp(1) \oplus sp(n), u(1) \oplus sp(n))$ *in* (h1) *and* $(sp(n+1), sp(1) \oplus sp(n), sp(n))$ *in* (h2) *satisfy Condition I.*

Proof. The corresponding decomposition of the case (h2) is $sp(n+1) = sp(n) \oplus \mathbf{R}^3 \oplus \mathbf{H}^n$. Noting that $sp(n)$ acts on $\mathcal{M}_F = \mathbf{R}^3$ trivially, we can easily prove the proposition of this case. The case (h1) follows from Lemma 4.6. □

It is known that $Sp(n+1)/U(1) \times Sp(n)$ is weakly symmetric ([25]), and thus g.o. This proposition gives another proof of this fact.

The associated space $M = Sp(n+1)/Sp(n)$ with a certain metric g admits an isometry group $\cong Sp(1) \times Sp(n+1)$. It is known that (M, g) is weakly symmetric ([4]), g being g.o. with respect to $Sp(1) \times Sp(n+1)$. Despite of these known facts, the proposition is not trivial.

The remaining candidates are (r3), (r4), (r5) and (r8). In fact these do not satisfy Condition I. The following lemma gives a necessary condition for Condition I, and useful to reject the candidates.

Lemma 4.9. *Condition I implies that* $\dim K_{v_F} > 0$ *for every* v_F *in* $\mathcal{M}_F$.

Proof. Let $v_F \in \mathcal{M}_F$. Since the action of $\mathcal{H}$ on $\mathcal{M}_B$ is effective, there exists $v_B \in \mathcal{M}_B$ such that $[v_F, v_B] \neq 0$. By Proposition 2.1, there exists $X \in \mathcal{K}$ such that

$$[X, v_F] = 0 \quad \text{and} \quad [X, v_B] = [v_F, v_B].$$

The former equation means $X \in \mathcal{K}_{v_F}$, and the latter implies $X \neq 0$. □

Proposition 4.10. *The triple* $(so(17), so(16), spin(9))$ *in* (r8) *does not satisfy Condition I.*

Proof. The homogeneous space $SO(16)/Spin(9)$ is isotropy irreducible, and the isotropy representation is described in [22]. It is equivalent to the action on $\bigwedge^3 \mathbf{R}^9$, where $Spin(9)$ acts on $\mathbf{R}^9$ naturally, and the isotropy subgroup of the principal orbit is trivial ([9]). The claim follows by Lemma 4.9. □

Proposition 4.11. *The triples* $(so(4n+1), so(4n), sp(1) \oplus sp(n))$ *in* (r3), $(so(4n+1), so(4n), u(1) \oplus sp(n))$ *in* (r4), *and* $(so(4n+1), so(4n), sp(n))$ *in* (r5) *do not satisfy Condition I.*

Proof. By Lemma 4.6, it is enough to show that the case (r3) does not satisfy the Condition I. The homogeneous space $SO(4n)/Sp(1)Sp(n)$ is isotropy irreducible, and the isotropy subgroup of the principal orbit of the isotropy representation is 0-dimensional if $n > 2$ ([22] and [9]).

Thus, by virtue of Lemma 4.9, the case we have to investigate is $(so(9), so(8), sp(1) \oplus sp(2))$. The decomposition is

$$so(9) = \underbrace{sp(1) \oplus sp(2)}_{\mathcal{K}} \oplus \underbrace{\mathbf{R}^5 \otimes \mathbf{R}^3}_{\mathcal{M}_F} \oplus \underbrace{\mathbf{H}^2}_{\mathcal{M}_B}.$$

The action of $\mathcal{K} \cong so(3) \oplus so(5)$ on $\mathcal{M}_F$ is equivalent to the isotropy representation of the Grassmannian manifold $G_3(\mathbf{R}^8)$. Let $\mathcal{A}$ be a maximal abelian subspace in $\mathcal{M}_F$, and $\{H^1, H^2, H^3\}$ be the dual basis of a simple root system of the restricted root system with respect to $\mathcal{A}$. The orbit through $c_1H^1 + c_2H^2 + c_3H^3$ is principal if $c_1, c_2, c_3 > 0$, and the principal isotropy subalgebra is $so(2)$ ([10] and [21]). Let $X \in so(2)$ be non-zero.

Let us assume that the triple satisfies Condition I. Let $c_1, c_2, c_3 > 0$ and $v_B \in \mathcal{M}_B$. From Lemma 2.1 there exists $X' \in \mathcal{K}$ such that

$$[X', c_1H^1 + c_2H^2 + c_3H^3] = 0 \quad \text{and} \quad [c_1H^1 + c_2H^2 + c_3H^3, v_B] = [X', v_B].$$

The former equation means $X' \in so(2) = \mathbf{R}X$. Thus we have

$$[c_1 H^1 + c_2 H^2 + c_3 H^3, v_B] \in \mathbf{R}[X, v_B].$$

This implies that $[\mathcal{A}, v_B] \subset \mathbf{R}[X, v_B]$ for every $v_B \in \mathcal{M}_B$. Denote by $\mathcal{A}'$ a Cartan subalgebra of $so(8)$ containing $\mathcal{A}$. Let $\mathcal{A}''$ be the orthoganal complement of $\mathcal{A}$ in $\mathcal{A}'$, which is of dimension 1. Take a generic vector $v_B \in \mathbf{H}^2$, i.e., $\dim[\mathcal{A}', v_B] = 4$. Then,

$$4 = \dim[\mathcal{A}', v_B] \leq \dim[\mathcal{A}'', v_B] + \dim[\mathcal{A}, v_B] \leq 2.$$

This is a contradiction. □

Now we have completed the proof of Theorem 4.1.

5. Weakly symmetric spaces

We have classified the triples (G, H, K) satisfying Condition I. The associated homogeneous spaces $M := G/K$, with certain metrics, are good candidates for weakly symmetric spaces. In this section we find new examples of weakly symmetric spaces among them.

At first, let us consider the symmetric space $U(2n)/Sp(n)$. Let τ be the complex conjugation and

$$J := \begin{pmatrix} & -1_n \\ 1_n & \end{pmatrix} \in U(2n).$$

Then the automorphism $\tau \circ \mathrm{ad}(J)$ of $u(2n)$ is involutive, and the $(+1)$-eigenspace forms the subalgebra $sp(n)$. Let $\mathcal{M}_F$ be the (-1)-eigenspace. It is known that the subspace

$$\mathcal{A} := \left\{ \begin{pmatrix} A & \\ & A \end{pmatrix} \middle| A = \begin{pmatrix} i\varepsilon_1 & & \\ & \ddots & \\ & & i\varepsilon_n \end{pmatrix}, \ \varepsilon_1, \ldots, \varepsilon_n \in \mathbf{R} \right\}$$

is maximal abelian, and thus $\mathcal{A}$ meets every $Sp(n)$-orbit in $\mathcal{M}_F$.

Theorem 5.1. *The homogeneous spaces $SU(2n+1)/U(1) \cdot Sp(n)$ and $SU(2n+1)/Sp(n)$, associated with (1.6) and (1.7) in Table 1 respectively, are weakly symmetric with respect to any $SU(2n+1)$-invariant Riemannian metrics.*

Proof. It is enough to prove the latter case. We will investigate the isotropy representation and prove the theorem by Proposition 1.2. The isotropy representation is equivalent to the action of $Sp(n)$ on $\mathcal{M}_F \oplus \mathbf{C}^{2n}$, where $\mathcal{M}_F$ is defined above and $Sp(n)$ acts on $\mathbf{C}^{2n}$ naturally. Since the complex conjugation τ preserves $Sp(n)$, τ acts on $SU(2n+1)/Sp(n)$ as an isometry and fixes the origin.

Let $v_F \in \mathcal{M}_F$ and $v_B \in \mathbf{C}^{2n}$. We can assume $v_F \in \mathcal{A}$ without loss of generality. Then $\tau(v_F) = -v_F$ and

$$SU(2)^n := \underbrace{SU(2) \times \cdots \times SU(2)}_{n} \subset Sp(n)$$

leaves v_F fixed ([21]). It is easy to see that $SU(2)^n$ acts on $\mathbf{C}^{2n}$ naturally (i.e., the action is the direct sum of the natural actions of $SU(2)$ on $\mathbf{C}^2$). Thus there exists $g \in SU(2)^n$ such that $g(\overline{v_B}) = -v_B$. Since g fixes v_F, we obtain that $g \circ \tau$ sends $v_F + v_B$ into $-v_F - v_B$. □

Let G be the group generated by $SU(2n+1)$ and the complex conjugation τ. This theorem says that the spaces of (1.6) and (1.7) are weakly symmetric with respect to G. Thus they are G-g.o. Since $SU(2n+1)$ is the connected component of G, we have showed that they are $SU(2n+1)$-g.o.

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References

[1] N. Bourbaki: Groupes et Algèbres de Lie, Chap.4-6, Masson, Paris, 1981.

[2] J. Berndt, O. Kowalski and L. Vanhecke: *Geodesics in weakly symmetric spaces*, Ann. Global analsis and Geom. **15** (1997), 153-156.

[3] J. Berndt, F. Tricerri and L. Vanhecke: Generalized Heisenberg groups and Damek-Ricci harmonic spaces, Lecture Notes in Math. **1598**, Springer-Verlag, Berlin Heidelbelg 1995.

[4] J. Berndt and L. Vanhecke: *Geometry of weakly symmetric spaces*, J. Math. soc. Japan **48** (1996), 745-760.

[5] J. Berndt and L. Vanhecke: *Commutativity and weakly symmetry on Riemannian manifolds*, preprint.

[6] J. Dadok: *Polar coordinates induced by actions of compact Lie groups*, Trans. Amer. Math. Soc. **288** (1985), 125-137.

[7] C.S. Gordon: *Homogeneous Riemannian manifolds whose geodesics are orbits*, Topics in Geometry: In Memory of Joseph D'Atri (Ed. S. Gindikin), Progress in Nonlinear Differential Equations **20**, Birkhäuser-Verlag, Boston, Basel, Berlin, 1996, 155-174.

[8] S. Helgason: Differential Geometry, Lie Groups, and Symmetric Spaces, Academic Press, New York, 1978.

[9] W.C. Hsiang and W.Y. Hsiang: *Differentiable actions of compact connected classical groups II*, Ann. of Math. **92** (1970), 189-223.

[10] E. Heintze and C. Olmos: *Normal holonomy groups and s-representations*, Indiana Univ. Math. J. **41** (1992), 869-874.

[11] J. A. Jiménez and O. Kowalski: *The classification of φ-symmetric Sasakian manifolds*, Monatsh. Math. **115** (1993), 83-98.

[12] S. Kobayashi and N. Nomizu: Foundations of Differential Geometry, vol.1, Interscience Publishers, New York, 1963.
[13] S. Kobayashi and N. Nomizu: Foundations of Differential Geometry, vol.2, Interscience Publishers, New York, 1969.
[14] O. Kowalski, F. Prüfer and L. Vanhecke: *D'Atri spaces*, Topics in Geometry: In Memory of Joseph D'Atri (Ed. S. Gindikin), Progress in Nonlinear Differential Equations **20**, Birkhäuser-Verlag, Boston, Basel, Berlin, 1996, 241-284.
[15] O. Kowalski and L. Vanhecke: *Riemannian manifolds with homogeneous geodesics*, Bollettino U. M. I. (7) **5-B** (1991), 189-246.
[16] T. Nagano: *The involutions of symmetric spaces II*, Tokyo J. Math. **15** (1992), 39-82.
[17] A.L. Oniščik: *Transitive compact transformation groups*, Amer. Math. Soc. Transl.(2) **55** (1966), 153-194.
[18] A. Selberg: *Harmonic analysis and discontinuous groups in weakly symmetric Riemannian spaces with applications to Dirichlet series*, J. Indiana Math. Soc. **20** (1956), 47-87.
[19] T. Takahashi: *Sasakian φ-symmetric spaces*, Tohoku Math. J. **29** (1977), 91-113.
[20] H. Tamaru: *Riemannian geodesic orbit metrics on fiber bundles*, Algebras, Groups and Geometries **15** (1998), 55-67.
[21] H. Tamaru: *The local orbit types of symmetric spaces under the actions of the isotropy subgroups*, Diff. Geom. Appl., **11** (1999), 29-38.
[22] J.A. Wolf: *The geometry and structure of isotropy irreducible homogeneous spaces*, Acta Math. **120** (1968), 59-148. Correction, Acta Math. **152** (1984), 141-142.
[23] W. Ziller: *The Jacobi equation on naturally reductive compact Riemannian homogeneous spaces*, Comm. Math. Helv. **52**(1977), 573-590.
[24] W. Ziller: *Homogeneous Einstein metrics on spheres and projective spaces*, Math. Ann. **259**(1982), 351-358.
[25] W. Ziller: *Weakly symmetric spaces*, Topics in Geometry: In Memory of Joseph D'Atri (Ed. S. Gindikin), Progress in Nonlinear Differential Equations **20**, Birkhäuser-Verlag, Boston, Basel, Berlin, 1996, 355-368.

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