Muramatu, T. and Tojima, T.
Osaka J. Math.
38 (2001), 419–434

CONSTRUCTION OF THE EVOLUTION OPERATOR OF PARABOLIC TYPE

TOSINOBU MURAMATU and TAKIO TOJIMA

(Received September 16, 1999)

1. Introduction and Main Theorem

In this note we construct the evolution operator of parabolic type, or the fundamental solution of the linear ordinary differential equation

$$\frac{du(t)}{dt} + A(t)u(t) = f(t), \quad a < t < b, \tag{1.1}$$

of parabolic type in a Banach space X. The equation (1.1) is said to be "of parabolic type" if it satisfies the condition:
(A1) $-A(t)$ *is a linear operator with dense domain, and there exist constants* $\kappa > \pi/2$ *and* C_0 *such that the resolvent set of* $-A(t)$ *contains the sector* $\Sigma_\kappa := \{\lambda \in \mathbb{C};\, |\arg\lambda| \le \kappa\}$ *for any* $t \in I := [a, b]$ *and* $\|\lambda(\lambda + A(t))^{-1}\|_{X\to X} \le C_0$ *holds for any* $\lambda \in \Sigma_\kappa$ *and any* $t \in I$.

$-A(t)$ generates an analytic semi-group $\{e^{-\tau A(t)}; \tau \ge 0\}$ on X.

Our result is stated as follows:

Main Theorem. *Assume* (A1), *and the following hypotheses* (A2), (A3):
(A2) *The domain* $\mathcal{D}(A(t)) = Y$ *for any* $t \in I$ *and* $A(\cdot) \in \mathcal{C}(I; \mathcal{L}(Y, X))$, *where* Y *is a Banach space continuously imbedded in* X.
(A3) *Defining*

$$\omega(h) := \sup\{\|A(t+h) - A(t)\|_{Y\to X};\, a \le t \le b - h\}, \tag{1.2}$$

$\omega(h)/h$ *is integrable on* $(0, \delta)$ *for some positive* δ. *Then, there exists the evolution operator to the equation* (1.1), *i.e., there exists a strongly continuous* $\mathcal{L}(X)$*-valued function* $U(t, s)$, $a \le s \le t \le b$, *having the following properties*:
(a) $U(t, r)U(r, s) = U(t, s)$ *for* $a \le s \le r \le t \le b$,
(b) $U(t, t) = I$ *for* $a \le t \le b$,
(c) $(\partial/\partial t)U(t, s)x = -A(t)U(t, s)x$ *for any* $x \in X$ *and* $a \le s < t < b$,
(d) $(\partial/\partial s)U(t, s)x = U(t, s)A(s)x$ *for any* $x \in Y$ *and* $a < s < t \le b$.

2000 *Mathematics Subject Classification*. Primary 47D06.

Moreover, the evolution operator $U(t,s)$ *is uniquely determined by* $\{A(t)\}_{a\le t\le b}$*, and satisfies the estimates*

$$\|A(t)U(t,s)\|_{X\to X} \le \frac{M}{t-s}, \quad \|U(t,s)A(s)\|_{Y\to Y} \le \frac{M}{t-s} \tag{1.3}$$

for any $a \le s < t \le b$*, where* M *is a constant.*

It is well known that any strong solution $u(t)$ to (1.1) with the initial date $u(a) = u_0$ must be of the form $u(t) = U(t,a)u_0 + F(t)$, where

$$F(t) := \int_a^t U(t,s)f(s)ds. \tag{1.4}$$

It is also known that the condition $f \in \mathcal{C}(I;X)$ does not guarantee differentiability of $F(t)$. Regard to this we have

Theorem 1.1. *Assume* (A1), (A2), (A3), $f \in L^1(I;X) \cap B^0_{\infty,1}((a,b);X)_{\mathrm{loc}}$*, and define* F *by* (1.4)*. Then* $F \in \mathcal{C}(I;X) \cap \mathcal{C}^1((a,b);X)$*,* $F(t) \in \mathcal{D}(A(t))$ *for any* $t \in (a,b)$*, and* $u(t) = U(t,s)u_0 + F(t)$ *is the unique strong solution to* (1.1) *with the initial condition* $u(a) = u_0$*.*

Study of the evolution operator of parabolic type has a rather long history, but we recall here only a few articles related to our result. Tanabe [7] constructed the evolution operator under the hypotheses (A1) (A2) and

$$\omega(h) \le Ch^\theta, \quad 0 < \theta \le 1. \tag{A3$'$}$$

(i.e., $A(t)$ is a Hölder continuous $\mathcal{L}(Y,X)$-valued function.) It is easy to see that (A3) is a true improvement of (A3′). Kawatsu [2] gave also an improvement of (A3′), i.e., under the assumption that "$\omega(h)|\log h|/h$ is integrable on $(0,\delta)$" he proved the existence of the evolution operator. Our assumption is better than that of Kawatsu, and we hope that our theorem will be useful in studying non-linear problems.

Our result was announced in [3]. The proof given by one of the authors eleven years ago is based on the approximation theory of integral equations with operator-valued unknown function and it is rather long. In this note we will give a simple and straightforword proof which contains some new methods to investigate abstract differential equations in a Banach space.

The result corresponding to Theorem 1.1 for the case where $A(t)$ is independent of t has been given in [4].

NOTATION. $\|x\|_X$ denotes the norm of x in a space X.
$\mathcal{L}(X, Y)$ denotes the space of bounded linear operators from X into Y, whose norm is denoted by $\|U\|_{X\to Y}$, $\mathcal{L}(X) := \mathcal{L}(X, X)$.
$\mathcal{C}(\Omega; X)$ denotes the space of X-valued continuous functions on a domain Ω.
$L^p(\Omega; X)$ denotes the space of X-valued strongly measurable functions $f(t)$ with $\|f(t)\|_X \in L^p(\Omega)$.

2. Preliminary observation

We first observe that Main Theorem follows from the following fact:
For some small positive number δ there exists a strongly continuous $\mathcal{L}(X)$-valued function $U(t, s)$ on the area $T_\delta := \{(t, s);\ a \le s \le t \le b, t-s \le \delta\}$ satisfying the conditions (b), (c), (d) *in Main Theorem and the inequality*

$$\|U(t, s)\|_{X\to Y} \le \frac{M_1}{t-s} \quad \text{for} \quad a \le s < t \le b \quad \text{with} \quad t - s \le \delta. \tag{2.1}$$

In fact, when $(t, s) \in T_\delta$, the derivative of $U(t, r)U(r, s)$ with respect r vanishes in the interval (s, t). Therefore, $U(t, r)U(r, s)$ is independent of $r \in (s, t)$. Together with the strong continuity of $U(t, r)U(r, s)$, this implies that $U(t, r)U(r, s) = U(t, s)$ holds when $(t, s) \in T_\delta$. (1.3) follows dirctly from (2.1), since $\sup_{t\in I} \|A(t)\|_{X\to Y} < \infty$.

When $\delta \le t - s < 2\delta$, we define $U(t, s) := U(t, r)U(r, s)$, where r is a point with $\max\{s, t - \delta\} < r < \min\{s + \delta, t\}$. $U(t, s)$ is independent of the choice of r, since for any $\max\{s, t - \delta\} < r < r_1 < \min\{s + \delta, t\}$ we have $U(t, r)U(r, s) = U(t, r_1)U(r_1, r)U(r, s) = U(t, r_1)U(r_1, s)$. Thus, the evolution operator $U(t, s)$ can be defined when $t - s < 2\delta$. The fact that $U(t, s)$ has the properties (a), (b), (c) and (d) in Main Theorem is a simple consequence of the definition.

Repeating this argument, we can finally construct the evolution operator for any point (t, s) with $a \le s \le t \le b$, and we see easily that (1.3) holds for any $a \le s < t \le b$.

Finally, if $\tilde{U}(t, s)$ is another $\mathcal{L}(X)$-valued strongly continuous function satisfying (b) and (c) in Main Theorem, the derivative of $U(t, r)\tilde{U}(r, s)$ with respect r vanishes in the interval (s, t). So, $U(t, r)\tilde{U}(r, s)$ is independent of r, which implies that $U(t, s) = U(t, r)\tilde{U}(r, s) = \tilde{U}(t, s)$. This gives the uniqueness of the evolution operator, which completes the proof of Main Theorem.

3. Lemmas

Lemma 3.1. *If $f \in L^1([\alpha, \beta]; Z)$, then $\int_\alpha^\beta f(s)ds \in Z$. Here Z is a Banach space.*

Proof. See Yosida [10] p. 133. □

Lemma 3.2. *If $F(\lambda)$ is holomolphic and satisfies $\|F(\lambda)\|_X \le C|\lambda|^{\alpha}$ in $\Sigma_\kappa \cap \{\lambda \in \mathbb{C};\ |\lambda| \ge 1\}$, $\|\int_\Gamma e^{t\lambda} F(\lambda) d\lambda\|_X \le C(\alpha, c_0) C t^{-\alpha-1}$ holds for any $0 < t \le c_0 < \infty$, where $C(\alpha, c_0)$ is a constant depend only on α, c_0 and κ. Here, Γ denotes a path $\lambda = \lambda(\sigma)$ $(\sigma \in \mathbb{R})$ contained in $\Sigma_\kappa \cap \{\lambda \in \mathbb{C};\ |\lambda| \ge 1\}$ such that $|\lambda(\sigma)| \to \infty$, $0 < \varepsilon \le \pm \arg \lambda(\sigma) - \pi/2$ as $\sigma \to \pm\infty$.*

Lemma 3.3. *From* (A1) *and* (A2) *it follows that*

$$\|(\lambda + A(t))^{-1}\|_{X\to Y} \le C_1(1 + |\lambda|^{-1}), \tag{3.1}$$

$$\|\lambda(\lambda + A(t))^{-1}\|_{Y\to Y} \le C_2 \tag{3.2}$$

hold for any $\lambda \in \Sigma_\kappa$ and any $t \in I := [a, b]$. Here C_1 and C_2 are constants.

Proof. Assume (A1) and (A2). Since the identity

$$(1 + A(t))^{-1} = (1 + A(t_0))^{-1} \sum_{n=0}^{\infty} \{(A(t_0) - A(t))(1 + A(t_0))^{-1}\}^n$$

holds if $\|A(t) - A(t_0)\|_{Y\to X} < \|(1 + A(t_0))^{-1}\|_{X\to Y}^{-1}$, we see that $(1 + A(t))^{-1} \in \mathcal{C}(I; \mathcal{L}(X, Y))$, which implies that $C' := \sup_{a\le t\le b} \|(1 + A(t))^{-1}\|_{X\to Y}$ is finite. Hence, by the identity $(\lambda + A)^{-1} = \{1 + (1 - \lambda)(\lambda + A)^{-1}\}(1 + A)^{-1}$ we have (3.1). Also, by the identity $(\lambda + A)^{-1} = (1 + A)^{-1}(\lambda + A)^{-1}(1 + A)$ we have (3.2). □

Lemma 3.2, (3.1), (3.2) and the identities $e^{-\tau A(t)} = (1/(2\pi i)) \int_\Gamma e^{\lambda\tau}(\lambda + A(t))^{-1} d\lambda$,

$$(\lambda + A(t))^{-1} - (\lambda + A(s))^{-1} = (\lambda + A(t))^{-1}\{A(s) - A(t)\}(\lambda + A(s))^{-1}$$

give the following

Lemma 3.4. *Assume* (A1) *and* (A2). *Then,*

$$\|e^{-\tau A(t)}\|_{X\to X} \le M_0, \tag{3.3}$$

$$\|e^{-\tau A(t)}\|_{Y\to Y} \le M_1, \tag{3.4}$$

$$\|e^{-\tau A(t)}\|_{X\to Y} \le M\tau^{-1}, \tag{3.5}$$

$$\|e^{-\tau A(t)} - e^{-\tau A(s)}\|_{X\to X} \le P_0\|A(t) - A(s)\|_{Y\to X}, \tag{3.6}$$

$$\|e^{-\tau A(t)} - e^{-\tau A(s)}\|_{Y\to Y} \le P_1\|A(t) - A(s)\|_{Y\to X}, \tag{3.7}$$

$$\|e^{-\tau A(t)} - e^{-\tau A(s)}\|_{X\to Y} \le P\tau^{-1}\|A(t) - A(s)\|_{Y\to X}, \tag{3.8}$$

$$\|e^{-\tau A(t)} - e^{-\tau A(s)}\|_{Y\to X} \le P'\tau\|A(t) - A(s)\|_{Y\to X}, \tag{3.9}$$

hold for any $a \le s \le t \le b$ and $0 < \tau \le c_0$. Here M_0, M_1, M, P_0, P_1, P and P' are constants independent of t, s and τ.

By (3.6), (3.7), (3.8) and the strong continuity of semi-group $e^{-\tau A(s)}$ we see that $e^{-\tau A(t)} - e^{-\sigma A(s)} = \{e^{-\tau A(t)} - e^{-\tau A(s)}\} + \{e^{-\tau A(s)} - e^{-\sigma A(t)}\} \to 0$ as $(\tau, t) \to (\sigma, s)$. Hence, we have

Lemma 3.5. *Let* $0 < c < \infty$, *and assume* (A1) *and* (A2). *Then,* $e^{-\tau A(t)}$ *is an* $\mathcal{L}(X)$*-valued* (*and* $\mathcal{L}(Y)$*-valued*) *strongly continuous function of* $(\tau, t) \in [0, c] \times [a, b]$. $e^{-\tau A(t)}$ *is also an* $\mathcal{L}(X, Y)$*-valued strongly continuous function of* $(\tau, t) \in (0, c] \times [a, b]$.

4. The series giving the evolution operator

According to Tanabe [7], to construct the evolution operator $U(t, s)$ we make use of the series

$$U(t, s) = \sum_{n=0}^{\infty} W_n(t, s) := W_0(t, s) + \sum_{n=1}^{\infty} \int_s^t W_0(t, r) R_n(r, s) dr \tag{4.1}$$

where $W_0(t, s) := e^{-(t-s)A(s)}$, $R_1(t, s) := -\{A(t) - A(s)\}e^{-(t-s)A(s)}$ and

$$R_{n+1}(t, s) = \int_s^t R_1(t, r) R_n(r, s) dr \text{ for } n = 1, 2, \cdots . \tag{4.2}$$

To prove the convergence of the series (4.1) we start with

Lemma 4.1. *Let* $\omega(t)$ *be a non-negative bounded measurable function of* $t \in (0, \delta_0)$ *such that*

$$\gamma(t) := \int_0^t \omega(s) \frac{ds}{s} < \infty \tag{4.3}$$

for $0 < t \leq \delta_0$. *Then, putting* $\omega_1 = \omega$,

$$\omega_{n+1}(t) := t \int_0^t \frac{\omega(t-s)}{t-s} \frac{\omega_n(s)}{s} ds \text{ for } n = 1, 2, \cdots , \tag{4.4}$$

can be defined inductively, and

$$\int_0^t \frac{\omega_n(s)}{s} ds \leq \gamma(t)^n, \tag{4.5}$$

$$\omega_n(t) \leq n^2 M' \gamma(t)^{n-1}, \tag{4.6}$$

hold for $n = 1, 2, \cdots$ *and* $0 < t \leq \delta_0$, *where* $M' := \sup_{0 < t \leq \delta_0} \omega(t)$.

Proof. Clearly (4.5) and (4.6) hold for $n = 1$. Assume that (4.5) and (4.6) hold for n. Then, noting that γ is a increasing function, by Fubini's theorem we have

$$\int_0^t \frac{\omega_{n+1}(s)}{s} ds = \int_0^t \left\{ \int_r^t \frac{\omega(s-r)}{s-r} ds \right\} \frac{\omega_n(r)}{r} dr \leq \gamma(t)^{n+1}.$$

Also, taking $r = nt/(n+1)$, we have

$$\omega_{n+1}(t) \leq t \int_0^r \frac{M'}{t-r} \frac{\omega_n(s)}{s} ds + t \int_r^t \frac{\omega(t-s)}{t-s} \frac{M' n^2 \gamma(t)^{n-1}}{r} ds \leq M' \gamma(t)^n (n+1)^2.$$

This gives (4.6) for $n + 1$. □

In the following of this note we always assume that (A1), (A2) and (A3) hold, and by ω we denote the function defined by (1.2).

Lemma 4.2. *Let $a \leq s < t \leq b$. Then,*

$$\|R_n(t,s)\|_{X\to X} \leq M^n \frac{\omega_n(t-s)}{t-s}, \quad n = 1, 2, \cdots, \tag{4.7}$$

$$\|W_n(t,s)\|_{X\to X} \leq M_0 (M\gamma(t-s))^n, \quad n = 0, 1, \cdots. \tag{4.8}$$

Proof. As $\|A(t) - A(s)\|_{Y\to X} \leq \omega_1(t-s)$, (3.5) implies (4.7) for $n = 1$. Assume that (4.7) holds for n. Then, by (4.2) we have

$$\|R_{n+1}(t,s)\|_{X\to X} \leq M^{n+1} \int_s^t \frac{\omega(t-r)}{t-r} \frac{\omega_n(r-s)}{r-s} dr = M^{n+1} \frac{\omega_{n+1}(t-s)}{t-s}.$$

Clearly (4.8) holds for $n = 0$. (4.1), (4.5) and (4.7) imply (4.8) for $n \geq 1$. □

5. Norm of $W_n(t,s)$

We make use of the symbols: $Z_0(t,s) := e^{-(t-s)A(t)}$, $Q_1(t,s) := Z_0(t,s)\{A(t) - A(s)\}$, $H(t,\sigma,s) := \{e^{-(t-\sigma)A(t)} - e^{-(t-\sigma)A(s)}\} e^{-(\sigma-s)A(s)}$, $G(t,s) := H(t,s,s)$.

Lemma 5.1. *Let $a \leq s \leq \sigma \leq t \leq b$. Then*

$$\begin{aligned} W_1(t,s) &= \int_\sigma^t \{Q_1(t,r)W_0(r,s) - G(t,r)R_1(r,s)\} dr \\ &\quad + H(t,\sigma,s) + \int_s^\sigma W_0(t,r)R_1(r,s) dr, \end{aligned} \tag{5.1}$$

$$\begin{aligned} W_{n+1}(t,s) &= \int_\sigma^t [Q_1(t,r)W_n(r,s) + (G(t,r)\{R_n(r,s) - R_{n+1}(r,s)\}] dr \\ &\quad + \int_s^\sigma [W_0(t,r)R_{n+1}(r,s) + H(t,\sigma,r)R_n(r,s)] dr \quad (n \geq 1). \end{aligned} \tag{5.2}$$

Proof. By the formula

$$Z_0(t,r)R_1(r,s) = Q_1(t,r)W_0(r,s) - \frac{\partial}{\partial r}\left\{e^{-(t-r)A(t)}e^{-(r-s)A(s)}\right\}$$

we have

$$\int_\sigma^t Z_0(t,r)R_1(r,s)dr = \int_\sigma^t Q_1(t,r)W_0(r,s)dr + H(t,\sigma,s), \tag{5.3}$$

which implies (5.1), for $W_0(t,s) = Z_0(t,s) - G(t,s)$. By (5.3) we have

$$\begin{aligned} &\int_\sigma^t Z_0(t,r)R_{n+1}(r,s)dr \\ &= \int_\sigma^t \left[\int_\tau^t Z_0(t,r)R_1(r,\tau)dr\right] R_n(\tau,s)d\tau \\ &\quad + \int_s^\sigma \left[\int_\sigma^t Z_0(t,r)R_1(r,\tau)dr\right] R_n(\tau,s)d\tau \\ &= \int_\sigma^t \left[\int_\tau^t Q_1(t,r)W_0(r,\tau)dr + G(t,\tau)\right] R_n(\tau,s)d\tau \\ &\quad + \int_s^\sigma \left[\int_\sigma^t Q_1(t,r)W_0(r,\tau)dr + H(t,\sigma,\tau)\right] R_n(\tau,s)d\tau \\ &= \int_\sigma^t \{Q_1(t,r)W_n(r,s) + G(t,r)R_n(r,s)\}dr + \int_s^\sigma H(t,\sigma,\tau)R_n(\tau,s)d\tau, \end{aligned} \tag{5.4}$$

which gives (5.2). □

The estimate $\|W_0(t,s)\|_{X\to Y} \le M/(t-s)$ follows from (3.5). For the case $n \ge 1$ we have

Lemma 5.2. *$W_n(t,s) \in \mathcal{L}(X,Y)$ when $a \le s < t \le b$, $A(t)W_n(t,s)$ is continuous with respect to $(t,s) \in \{(t,s); a \le s < t \le b\}$, and the inequality*

$$\|W_n(t,s)\|_{X\to Y} \le \frac{Kn^3(M\gamma(t-s))^{n-1}}{t-s} \tag{5.5}$$

holds for $n = 1, 2, \cdots$. Here $\gamma(t)$ is the function given by (4.3).

Proof. Case where $n = 1$. Since it follows from (3.8) that

$$\|G(t,s)\|_{X\to Y} \le \frac{P\omega(t-s)}{t-s} \le \frac{PM'}{t-s}, \tag{5.6}$$

and $\|H(t,\sigma,s)\|_{X\to Y} \le PM'M_0(t-\sigma)^{-1}$, by (5.1) with $\sigma = (t+s)/2$, with the aid of (3.3), (3.5), and (4.7), we obtain

$$
\begin{aligned}
\|W_1(t,s)\|_{X\to Y} \le{}& \int_\sigma^t \|Q_1(t,r)W_0(r,s) - G(t,r)R_1(r,s)\|_{X\to Y}dr \\
&+\|H(t,\sigma,s)\|_{X\to Y} + \int_s^\sigma \|W_0(t,r)R_1(r,s)\|_{X\to Y}dr \\
\le{}& M\int_\sigma^t \left\{ \frac{M\omega(t-r)}{(t-r)(r-s)} + \frac{P\omega(t-r)\omega(r-s)}{(t-r)(r-s)} \right\} dr \\
&+\frac{2M'PM_0}{t-s} + \int_s^\sigma \frac{M^2\omega(r-s)}{(t-r)(r-s)}dr \\
\le{}& \frac{2M+2PM'}{t-s}M\gamma(b-a) + \frac{2M'PM_0}{t-s} + \frac{2M^2\gamma(b-a)}{t-s} \le \frac{K}{t-s}.
\end{aligned}
$$

Here we take K so that $K \ge 4M^2\gamma(b-a) + 2PM'(M\gamma(b-a) + M_0)$.

Since $A(t)$ is closed, we also see that

$$
\begin{aligned}
A(t)W_1(t,s) ={}& \int_\sigma^t A(t)\{Q_1(t,r)W_0(r,s) - G(t,r)R_1(r,s)\}dr \\
&+A(t)H(t,\sigma,s) + \int_s^\sigma A(t)W_0(t,r)R_1(r,s)dr.
\end{aligned}
$$

Hence $A(t)W_1(t,s)$ is continuous. In view of Lemma 3.1, we see that the conclusion of the lemma holds for $n = 1$.

Assume that (5.5) holds for n. Hence, taking $\sigma = (nt+s)/(n+1)$, by (5.2), (5.6), (4.7) and (4.6) we have

$$
\begin{aligned}
&\|W_{n+1}(t,s)\|_{X\to Y} \\
&\le \int_\sigma^t \frac{\omega(t-r)}{t-r}\left[\frac{KM^n\gamma^{n-1}n^3}{r-s} + P\left\{ \frac{M^n\omega_n(r-s)}{r-s} + \frac{M^{n+1}\omega_{n+1}(r-s)}{r-s} \right\}\right] dr \\
&\quad + \int_s^\sigma \Big[\frac{M}{t-r}\frac{M^{n+1}\omega_{n+1}(r-s)}{r-s} + \frac{P\omega(t-r)}{t-\sigma}M_0\frac{M^n\omega_n(r-s)}{r-s}\Big]dr \\
&\le \frac{KM^n\gamma^n n^3 + PM'M^n\gamma^n\{n^2 + (n+1)^2M\gamma\}}{\sigma-s} + \frac{M(M\gamma)^{n+1} + PM_0M'M^n\gamma^n}{t-\sigma} \\
&\le \frac{(n+1)M^n\gamma^n}{t-s}\{Kn^2 + PM'(2M\gamma+1)n + PM'(3M\gamma + M_0) + M^2\gamma\} \\
&\le \frac{(n+1)^3KM^n\gamma^n}{t-s}. \quad (\text{Here } \gamma = \gamma(t-s).)
\end{aligned}
$$

Here, we take $K := 4M^2\gamma(b-a) + PM'(3M\gamma(b-a) + 2M_0 + 1)$. This estimate gives that $W_{n+1}(t,s) \in \mathcal{L}(X,Y)$. The same argument as for W_1 gives that $A(t)W_n(t,s)$ is continuous in (t,s) when $a \le s < t \le b$. $\square$

Construction of $U(t,s)$ when $t-s$ is small. Take δ small enough so that $\gamma(\delta) < 1/M$, where $\gamma(\delta)$ is given by (4.3). Then, with help of the estimate (4.8) and (4.5), we can define $U(t,s)$ by (4.1) when $t-s\le\delta$. Since $W_n(t,s)$, $n=0,1,\cdots$ are strongly continuous function and the series (4.1) converges uniformly, we see that $U(t,s)$ is strongly continuous.

By (5.5) we see that the series (4.1) converges in $\mathcal{L}(X,Y)$ when $0<t-s\le\delta$, since $\sum_{n=1}^{\infty}(M\gamma(\delta))^{n-1}n^3<\infty$. Hence, $U(t,s)$ is a strongly continuous $\mathcal{L}(X,Y)$-valued function of $(t,s)\in\{(t,s); a\le s<t\le b, t-s\le\delta\}$, and satisfies (2.1).

6. Proof of differentiability with respect to t

Lemma 6.1. *Let $g\in\mathcal{C}([c,b];X)$, $a\le c<b$, define $G(t):=\int_c^t W_0(t,r)g(r)dr$, and assume that $G\in\mathcal{C}((c,b);Y)$. Then, $G\in\mathcal{C}^1((c,b);X)$, and*

$$\frac{dG}{dt}(t)=g(t)-\int_c^t R_1(t,r)g(r)dr-A(t)G(t). \tag{6.1}$$

Proof. Let $c<t<b$, $0<h<b-t$. Then, we have

$$\begin{aligned}\frac{1}{h}\{G(t+h)-G(t)\} &= \frac{e^{-hA(t)}-1}{h}G(t)+\int_0^1 e^{-(h-h\sigma)A(t+h\sigma)}g(t+h\sigma)d\sigma\\ &\quad+\int_s^t\frac{1}{h}\{e^{-hA(r)}-e^{-hA(t)}\}e^{-(t-r)A(r)}g(r)dr.\end{aligned}$$

Because of the fact that $G(t)\in\mathcal{D}(A(t))$, the fisrt term in the right-hand side converges to $-A(t)G(t)$ as $h\to+0$. Since $e^{-\tau A(r)}$ is a strongly continuous function of $(\tau,r)\in[0,\tau_0]\times[a,b]$ (see Lemma 3.5), it follows that $e^{-\tau A(r)}g(t)$ is a uniformly continuous function of $(\tau,r,t)\in[0,\tau_0]\times[a,b]\times[c.b]$. Hence, $e^{-(h-h\sigma)A(t+h\sigma)}g(t+h\sigma)\to g(t)$ as $h\to+0$ uniformly with respect to $\sigma\in[0,1]$, which implies that the second term in the right-hand side converges to $g(t)$. Lebesgue's dominated convergence theorem implies that the third term in the right-hand side converges to the second term of the formula (14) as $h\to+0$, since by the estimate (3.9) we have

$$\left\|\frac{1}{h}\{e^{-hA(r)}-e^{-hA(t)}\}e^{-(t-r)A(r)}g(r)\right\|_X\le C\frac{\omega(t-r)}{t-r}\|g(r)\|_X\in L^1,$$

and since

$$\frac{1}{h}\{e^{-hA(r)}-e^{-hA(t)}\}e^{-(t-r)A(r)}g(r)\to -R_1(t,r)g(r)\ \text{as}\ h\to+0$$

for any $r\in[c,t]$. Thus we can conclude that $G(t)$ is right-differentiable, and its right-derivative is strongly contimuous. From a well-known lemma (see Yosida [10], p. 239) it follows that $G(t)$ is differentiable and (14) holds, which completes the proof of Lemma 6.1. □

Lemma 6.2. *Let $a\le s<b$. Then, $U(t,s)x\in\mathcal{D}(A(t))$ is strongly differentiable in $t\in(s,\min\{s+\delta,b\})$ and its derivative is $-A(t)U(t,s)x$ for any $x\in X$.*

Proof. Assume that $a \le s < t \le \min\{s + \delta, b\}$. Then, by (5.5) we see that $\sum_{n=0}^{m} W_n(t,s)$ converges to $U(t,s)$, and $\sum_{n=0}^{m} A(t)W_n(t,s)$ converges as $m \to \infty$. As $A(t)$ is closed, it follows that $U(t,s)x \in \mathcal{D}(A(t))$ and $A(t)U(t,s)x = \sum_{n=0}^{\infty} A(t)W_n(t,s)x$ for any $x \in X$. Moreover, the estimate (5.5) implies that the above series converges uniformly with respect to $t \in [s + \varepsilon, \min\{s + \delta, b\}]$. Hence $A(t)U(t,s)x$ is continuous. On the other hand it follows from the above lemma that $W_n(t,s)x$ is differentiable in $t \in (s,b)$ and its derivative with respect t is equal to $R_n(t,s)x - R_{n+1}(t,s)x - A(t)W_n(t,s)x$ for any $x \in X$. Thus, we have

$$\frac{\partial}{\partial t}\sum_{n=0}^{m} W_n(t,s)x = -R_{n+1}(s,t)x - \sum_{n=0}^{m} A(t)W_n(t,s)x \to -A(t)U(t,s)x$$

uniformly with respect to $t \in [s + \varepsilon, \min\{s + \delta, b\})$ as $m \to \infty$, which completes the proof of the lemma. □

7. Proof of differentiability with respect to *s*

To prove differentiability of $U(t,s)$ in s we make use of another series which expresses $U(t,s)$. (See Tanabe [8].)

$$V(t,s) = \sum_{n=0}^{\infty} Z_n(t,s) := e^{-(t-s)A(t)} + \sum_{n=1}^{\infty} \int_s^t Q_n(t,r)W_0(r,s)dr, \tag{7.1}$$

where $Q_1(t,s) := e^{-(t-s)A(t)}\{A(t) - A(s)\}$ and

$$Q_{n+1}(t,s) := \int_s^t Q_n(t,r)Q_1(r,s)dr, \quad \text{for } n = 1, 2, \cdots. \tag{7.2}$$

Lemma 7.1. *If* $a \le s < t \le b$,

$$\|Q_n(t,s)\|_{Y\to Y} \le \frac{M^n\omega_n(t-s)}{t-s} \quad \textit{for} \quad n = 1, 2, \cdots, \tag{7.3}$$

$$\|Q_n(t,s)\|_{Y\to X} \le M_0M'(M\gamma(t-s))^{n-1} \quad \textit{for} \quad n = 1, 2, \cdots, \tag{7.4}$$

$$\|Z_n(t,s)\|_{Y\to Y} \le M_1M^n\gamma(t-s)^n, \quad \textit{for} \quad n = 0, 1, \cdots, \tag{7.5}$$

$$\|Z_n(t,s)\|_{X\to X} \le Kn(M\gamma(t-s))^{n-1} \quad \textit{for} \quad n = 1, 2, \cdots. \tag{7.6}$$

Proof. By (3.5) and $\|A(t)-A(s)\|_{Y\to X} \le \omega(t-s)$ we have (7.3) for $n = 1$. The inequality (7.3) can be proved in the same way as (4.7). It is clear that $\|Q_1(t,s)\|_{Y\to X} \le M_0\omega(t-s) \le M_0M'$. Assume that (7.4) holds for n. Then, by (7.2) and (7.3) we have

$$\|Q_{n+1}(t,s)\|_{Y\to X} \le \int_s^t M_0M'M^n\gamma(t-s)^{n-1}\frac{\omega(r-s)}{r-s}dr \le M_0M'(M\gamma(t-s))^n.$$

Also, (7.3) and $\|Z_0(t,s)\|_{Y\to Y} \le M_1$ implies (7.5).

Next, it follows that $\|Z_0(t,s)\|_{X\to X} \le M_0$, and it follows from the identity

$$Z_1(t,s) = \int_s^t \{Q_1(t,r)G(r,s) + Z_0(t,r)R_1(r,s)\}dr - G(t,s)$$

and the estimate

$$\|Q_1(t,r)G(r,s) + Z_0(t,r)R_1(r,s)\|_{X\to X} \le M_0\{M'P + M\}\frac{\omega(r-s)}{r-s}$$

that $Z_1(t,s) \in \mathcal{L}(X)$ and (7.6) for $n = 1$ holds. Assuming that $Z_n(t,s) \in \mathcal{L}(X)$ and (7.6) holds for n, by the identity

$$Z_{n+1}(t,s) = \int_s^t [\{Q_{n+1}(t,r) - Q_n(t,r)\}G(r,s) + Z_n(t,r)R_1(r,s)]dr$$

we see that $\|Z_{n+1}(t,s)\|_{X\to X}$ is estimated by

$$\begin{aligned}
&\int_s^t \{M_0M'(M\gamma(t-s))^{n-1}(M\gamma(t-s)+1)P + KnM^n\gamma(t-s)^{n-1}\}\frac{\omega(r-s)}{r-s}dr \\
&\le M_0M'PM^{n-1}\gamma(t-s)^n(M\gamma(t-s)+1) + KnM^n\gamma(t-s)^n \\
&\le K(n+1)(M\gamma(t-s))^n.
\end{aligned}$$

Hence $Z_{n+1}(t,s) \in \mathcal{L}(X)$ and (7.5) holds for $n+1$. Thus the lemma has been completely proved. □

Lemma 7.2. *$Z_n(t,s)y \in \mathcal{C}^1((a,t);X)$ for any $a < t \le b$ and any $y \in Y$, and its derivative with respect to s is equal to $-Q_n(t,s)y + Q_{n+1}y + Z_n(t,s)A(s)y$.*

Proof. This follows from the identity

$$\begin{aligned}
\frac{Q_n(t,s-h)y - Q_n(t,s)y}{-h} &= -\int_0^1 Q_n(t,s-h\sigma)e^{-h(1-\sigma)A(s-h\sigma)}yd\sigma \\
&\quad -\int_s^t Q_n(t,r)e^{-(r-s)A(r)}\frac{e^{-hA(r)} - e^{-hA(s)}}{h}ydr \\
&\quad -Z_n(t,s)\frac{e^{-hA(s)}-1}{h}y
\end{aligned}$$

and the argument which led to Lemma 6.1. □

In similar way as Lemma 6.2, from Lemma 7.1 and Lemma 7.2 we obtain

Lemma 7.3. *Take δ so that $M\gamma(\delta) < 1$ holds. Then, the series* (7.1) *converges when $a \le s \le t \le b$, $t - s \le \delta$, $V(t,s)y$ is differentiable with respect to s in the interval* $(\max\{t-\delta, a\}, t)$ *and its derivative is $V(t,s)A(s)y$ if $a < t \le b$ and if $y \in Y$.*

Now, the fact that the derivative of $V(t,r)U(r,s)$ with respect to r vanishes implies that $V(t,r)U(r,s)$ is independent of $r \in (s,t)$. Since $U(t,s)$ and $V(t,s)$ are strongly continuous, this gives that $V(t,s) = V(t,r)U(r,s) = U(t,s)$. Thus, by Lemma 7.3 we know that $U(t,s)y$ is differentiable with respect to s in the interval $(\max\{t-\delta, a\}, t)$ for any $t \in (a,b]$ and for any $y \in Y$.

Thus, the facts stated at the beginning of §2 have been completely proved.

8. Proof of Theorem 1.1

Lemma 8.1. *Let $f \in B^{0}_{\infty,1}((a,b);X)$, and define $F_0(t) := \int_a^t W_0(t,s)f(s)ds$. Then $F_0 \in \mathcal{C}(I;Y) \cap \mathcal{C}^1((a,b);X)$, $A(\cdot)F_0(\cdot) \in \mathcal{C}(I;X)$, and the inequalities*

$$\|A(t)F_0(t)\|_X \le C\|f\|_{B^{0}_{\infty,1}((a,b);X))}, \quad \|F_0(t)\|_Y \le \tilde{C}\|f\|_{B^{0}_{\infty,1}((a,b);X)} \tag{8.1}$$

hold for any $t \in I$, where C and $\tilde{C}$ are constants independent of f.

Proof. We first prove that $E(t) := \int_a^t Z_0(t,s)f(s)ds \in \mathcal{D}(A(t))$ for any $t \in I$, $\|A(t)E(t)\|_X \le C'\|f\|_{B^{0}_{\infty,1}((a,b);X))}$ for any $t \in I$, and $A(\cdot)E(\cdot) \in \mathcal{C}(I;X)$.

If $f \in \mathcal{C}^1(I;X)$, we have that

$$A(t)\int_a^t Z_0(t,s)f(s)ds = f(t) - Z_0(t,a)f(a) - \int_a^t Z_0(t,s)f'(s)ds \tag{8.2}$$

holds for any $t \in I$, where $f'(s) = df(s)/ds$ (see Proof of Lemma 5 in [4]). Hence, according to the theory of Besov spaces (see [4] §3), it suffices to consider the case where

$$f(t) = \int_0^c \frac{d\tau}{\tau}\int \frac{1}{\tau}\varphi\left(t, \frac{t-s}{\tau}\right)u(\tau,s)ds, \quad u \in L^1([0,c];L^\infty(I;X)).$$

Here, $\varphi(t,s) = (\partial\psi/\partial s)(t,s)$, $\psi \in \mathcal{C}^\infty(\mathbb{R}^2)$ such that $\psi(t,s) = 0$ if $s-(2t-a-b)/(b-a) \ge 1$. Let η be a C^∞-function such that

$$\eta(t) = 0 \text{ when } t \le 1,\ \eta(t) = 1 \text{ when } t \ge 2 \text{ and } 0 \le \eta(t) \le 1.$$

Then, by Fubini's theorem we have

$$\begin{aligned} E(t) &= \int_0^c \frac{d\tau}{\tau}\int \{\Phi_1(\tau,t,r) + \Phi_2(\tau,t,r)\}u(\tau,r)dr, \\ \Phi_1(\tau,t,r) &:= \int_a^t \left\{1-\eta\left(\frac{t-s}{\tau}\right)\right\}Z_0(t,s)\frac{1}{\tau}\varphi\left(s, \frac{s-r}{\tau}\right)ds, \\ \Phi_2(\tau,t,r) &:= \int_a^t \eta\left(\frac{t-s}{\tau}\right)Z_0(t,s)\frac{1}{\tau}\varphi\left(s, \frac{s-r}{\tau}\right)ds. \end{aligned} \tag{8.3}$$

As $A(t)Z_0(t,s) = (\partial/\partial s)Z_0(t,s)$, an integration by parts shows that

$$\begin{aligned} A(t)\Phi_1(\tau,t,r) = {} & \frac{1}{\tau}\varphi\left(t, \frac{t-r}{\tau}\right) - \left\{1-\eta\left(\frac{t-a}{\tau}\right)\right\} Z_0(t,a)\frac{1}{\tau}\varphi\left(a, \frac{a-r}{\tau}\right) \\ & - \sum_{j=1,2} \int_a^t \left\{1-\eta\left(\frac{t-s}{\tau}\right)\right\} Z_0(t,s)\frac{1}{\tau^j}\varphi_j\left(s, \frac{s-r}{\tau}\right) ds \\ & - \int_a^t \eta'\left(\frac{t-s}{\tau}\right) Z_0(t,s)\frac{1}{\tau^2}\varphi\left(s, \frac{s-r}{\tau}\right) ds, \end{aligned}$$

where $\varphi_1(t,s) = (\partial/\partial t)\varphi(t,s)$, $\varphi_2(t,s) = (\partial/\partial s)\varphi(t,s)$. Hence we have

$$\int \|A(t)\Phi_1(\tau,t,r)\|_{X\to X} dr \le C_0 + \sum_{j=1,2} C_j \int_{t-2\tau}^t ds\tau^{1-j} + C_3\frac{1}{\tau}\int_{t-2\tau}^{t-\tau} ds \le C_4.$$

Since

$$\varphi\left(s, \frac{s-r}{\tau}\right) = \tau\frac{\partial}{\partial s}\left\{\psi\left(s, \frac{s-r}{\tau}\right)\right\} - \tau\psi_1\left(s, \frac{s-r}{\tau}\right),$$

where $\psi_1(t,s) := (\partial\psi/\partial t)(t,s)$, we also have

$$\begin{aligned} A(t)\Phi_2(\tau,t,r) = {} & -\eta\left(\frac{t-a}{\tau}\right) A(t)Z_0(t,a)\psi\left(a, \frac{a-r}{\tau}\right) \\ & - \int_a^t \eta\left(\frac{t-s}{\tau}\right) A(t)^2 Z_0(t,s)\psi\left(s, \frac{s-r}{\tau}\right) ds \\ & + \int_a^t \frac{1}{\tau}\eta'\left(\frac{t-s}{\tau}\right) A(t)Z_0(t,s)\psi\left(s, \frac{s-r}{\tau}\right) ds \\ & - \int_a^t \eta\left(\frac{t-s}{\tau}\right) A(t)Z_0(t,s)\psi_1\left(s, \frac{s-r}{\tau}\right) ds, \end{aligned}$$

which implies, together with the fact that $\eta(t)/t \le 1$, that

$$\begin{aligned} \int \|A(t)\Phi_2(\tau,t,r)\|_{X\to X} dr \le {} & C_5\eta\left(\frac{t-a}{\tau}\right)\frac{\tau}{t-a} + C_6\tau\int_a^{t-\tau}(t-s)^{-2} ds \\ & + C_7\int_{t-2\tau}^{t-\tau}\frac{ds}{t-s} + C_8\int_a^t \eta\left(\frac{t-s}{\tau}\right)\frac{\tau}{t-s} ds \\ \le {} & C_9. \end{aligned}$$

As $\|u(\tau,\cdot)\|_{L^\infty(I;X)} \in L^1((0,c))$ and $A(t)$ is closed, these results and (8.3) give that

$$E(t) \in \mathcal{D}(A(t)),\ A(t)E(t) = \int_0^c \frac{d\tau}{\tau}\int A(t)\{\Phi_1(\tau,t,r) + \Phi_2(\tau,t,r)\}u(\tau,r) dr$$

and $\|A(t)E(t)\|_X \le C'\|f\|_{B^0_{\infty,1}((a,b);X)}$. Since this integral converges uniformly with respect to $t \in I$, we also see that $A(t)E(t)$ is continuous.

The results proved above imply, with the aid of the following lemma and the identity $F_0(t) = \int_a^t Z_0(t,s)f(s)ds - \int_a^t G(t,s)f(s)ds$, that $F_0(t) \in \mathcal{D}(A(t))$ for any $t \in I$, the first inequality in (8.1) and $A(\cdot)F_0(\cdot) \in \mathcal{C}(I;X)$. The second inequality in (8.1) and $F_0 \in \mathcal{C}(I;Y)$ follow from these facts together with the identity $F_0(t) = (I + A(t))^{-1}(I + A(t))F_0(t)$.

Finally, these facts and Lemma 6.1 imply that $F_0 \in \mathcal{C}^1((a,b);X)$. □

Lemma 8.2. *Let $f \in \mathcal{C}(I;X)$ and put $G(t) := \int_a^t G(t,s)f(s)ds$. Then $G \in \mathcal{C}(I;Y)$ and $\|G(t)\|_Y \le P\gamma(t-a)\|f\|_{L^\infty((a,t);X)}$.*

Proof. This follows from the inequality (5.6). □

Now, let us proceed *to prove* Theorem 1.1.

STEP 1. Consider the case where $f \in B^0_{\infty,1}((a,b);X)$ and $M\gamma(b-a) < 1$, where γ is the function defined by (4.3). The estimate (4.8) implies that the series $\sum_{n=0}^\infty W_n(t,s)f(s)$ converges to $U(t,s)f(s)$ in X uniformly in $(t,s) \in T := \{(t,s); a \le s \le t \le b\}$. Hence we have

$$F(t) = \int_a^t U(t,s)f(s)ds = \sum_{n=0}^\infty \int_a^t W_n(t,s)f(s)ds = \sum_{n=0}^\infty F_n(t). \tag{8.4}$$

Using Fubini's theorem, by (4.1) we have

$$F_n(t) := \int_a^t W_n(t,s)f(s)ds = \int_a^t W_0(t,s)H_n(s)ds \tag{8.5}$$

for $n = 0, 1, \cdots$, where $H_0(t) := f(t)$ and

$$H_n(t) := \int_a^t R_n(t,s)f(s)ds \quad \text{for } n = 1, 2, \cdots . \tag{8.6}$$

By Lemma 8.1 we have $F_0 \in \mathcal{C}(I;Y)$ and $\|F_0(t)\|_Y \le \tilde{C}\|f\|_{B^0_{\infty,1}(I;X)}$. Assume that

$$F_n \in \mathcal{C}(I;Y) \text{ and } \|F_n(t)\|_Y \le K(n+1)M^n\gamma(t-a)^n\|f\|_{B^0_{\infty,1}(I;X)}. \tag{8.7}$$

Here K is a constant which will be chosen later on. The identity

$$F_{n+1}(t) = \int_a^t [Q_1(t,s)F_n(s) + G(t,s)\{H_n(s) - H_{n+1}(s)\}]ds \tag{8.8}$$

for $n = 0, 1, \cdots$, which is a consequence of (5.1) and (5.2) with $\sigma = s$, together with (7.3), Lemma 8.2 and the inequality

$$\|H_n(t)\|_X \le M^n\gamma(t-a)^n\|f\|_{L^\infty(I;X)} \tag{8.9}$$

for $n = 0, 1, \cdots$, which follows from (4.7), implies that

$$\|F_{n+1}(t)\|_Y \leq \{\, K(n+1)(M\gamma(t-a))^{n+1} + 2PM^n\gamma(t-a)^{n+1} \,\}\, \|f\|_{B^{\,0}_{\infty,1}(I;X)}.$$

Taking $K = \max\{\tilde{C}, 2P/M\}$, this gives (8.7) for $n+1$. By (8.7) we see that $\sum_{n=0}^{\infty} F_n$ converges in $\mathcal{C}(I;Y)$, so that $F \in \mathcal{C}(I;Y)$.

Furthermore, by (8.5) and Lemma 6.1 we see that $F_n \in \mathcal{C}^1(I;X)$ and its derivative is $F_n'(t) = H_n(t) - H_{n+1}(t) - A(t)F_n(t)$. Since the right-hand side of the identity

$$\frac{d}{dt}\sum_{j=0}^{n} F_j(t) = f(t) - H_{n+1}(t) - A(t)\sum_{j=0}^{n} F_j(t)$$

converges to $f(t) - A(t)F(t)$ uniformly in t as $n \to \infty$, we can conclude that $F \in \mathcal{C}^1(I;X)$ and $F'(t) = f(t) - A(t)F(t)$.

STEP 2. Consider now the general case. Let $f \in L^1(I;X) \cap B^{\,0}_{\infty,1}((a,b);X)_{\rm loc}$, and let $a < t < b$. Take α and β so that $a < \alpha < t < \beta < b$ and $M\gamma(\beta-\alpha) < 1$, and put

$$F(t) = \int_a^{\alpha} U(t,s)f(s)ds + \int_{\alpha}^{t} U(t,s)f(s)ds = F_1(t) + F_2(t). \tag{8.10}$$

Since $f \in B^{\,0}_{\infty,1}((\alpha,\beta);X)$, by the results in Step 1 we see that $F_2(t) \in \mathcal{D}(A(t))$, $A(\cdot)F_2(\cdot)$ is continuous, F_2 is differentiable, and $F_2'(t) = f(t) - A(t)F_2(t)$. Since $U(t,\alpha)$ is differentiable and $\{\partial/\partial t\}U(t,\alpha) = -A(t)U(t,\alpha)$, it follows that $F_1(t) = U(t,\alpha)F(\alpha)$ is differentiable, $F_1(t) \in \mathcal{D}(A(t))$ and $A(t)F_1(t)$ is continuous. Thus, $F(t)$ is differentiable and $F'(t) = -A(t)F_1(t) + f(t) - A(t)F_2(t) = f(t) - A(t)F(t)$. This completes the proof of Theorem 1.1.

REMARK. The condition $f \in B^{\,0}_{\infty,1}((a,b);X)$ follows from $f \in \mathcal{C}(I;X)$ and

$$\rho(h;f) := \sup_{a\leq s<s+h\leq b} \|f(s+h) - f(s)\| \in L^1\Big((0,\delta), \frac{dh}{h}\Big) \tag{8.11}$$

for some δ. In fact, let $\varphi(t,s)$ be a C^{∞}-function such that $\int \varphi(t,s)ds = 0$ and $\varphi(t,s) = 0$ when $|s - (2t-a-b)/(b-a)| \geq 1$. Then we have

$$\begin{aligned}
&\int_0^c \Big\| \int \frac{1}{\tau}\varphi\Big(t, \frac{t-s}{\tau}\Big) f(s)ds \Big\|_{L^\infty(I;X)} \frac{d\tau}{\tau} \\
&= \int_0^c \Big\| \int \frac{1}{\tau}\varphi\Big(t, \frac{h}{\tau}\Big)\{f(t-h) - f(t)\}dh \Big\|_{L^\infty(I;X)} \frac{d\tau}{\tau} \\
&\leq C_0 \int_0^c \frac{d\tau}{\tau^2} \int_{|h|\leq \ell\tau, b-a} \rho(|h|;f)dh \leq 2C_0\ell \int_0^{b-a} \rho(h;f)\frac{dh}{h} < \infty.
\end{aligned}$$

Thus we have $f \in B^0_{\infty,1}((a,b);X)$ by Thoerem 1 in [4].

From this fact and Theorem 1.1 we see that $F(t)$ is differentiable if $f \in L^1((a,b);X)$ and the condition (8.11) is satisfied locally. This result has been directly proved by Tojima [9] (The case where $A(t)$ is independent of t has been discussed by Crandal-Pazy [1]).

References

[1] M. C. Crandall and A. Pazy: *On the differentiability of weak solutions of a differential equation in Banach space*, J. Math. Mech. **18** (1969), 1007–1016.

[2] S. Kawatsu: *Cauchy problem for abstract evolution equations of parabolic type*, J. Math. Kyoto, **30** (1990), 59–91.

[3] T. Muramatu: *An application of Besov spaces to the evolution equation*, Kokyuroku, RIMS, Kyoto Univ. **647** (1989), 99–124 (in Japanese).

[4] T. Muramatu: *Besov spaces and analytic semi-groups of lineat operators*, J. Math. Soc. Japan, **42** (1990), 134–146.

[5] A. Pazy: Semi-groups of Linear Operators and Applications to Partial Differential Equations, Springer-Verlag, Berlin, Heideberg, New York, 1983.

[6] H. Tanabe: *A class of the equations of evolution in a Banach Space*, Osaka J. Math. **11** (1959), 121–145.

[7] H. Tanabe: *On the equations of evolution in a Banach Space*, Osaka J. Math. **12** (1960), 363–376.

[8] H. Tanabe: Equations of Evolution, Pitman, London, 1979.

[9] T. Tojima: Solvability of Cauchy problems for the parabolic evolution equation in a Banach space, Bull. Fac. Sci. & Eng. Chuo Univ. **42** (1999), 1–9.

[10] K. Yosida: Functional Analysis, Springer-Verlag, Berlin-Heideberg-New York, 1966.

Department of Mathematics
Chuo University
1-13-27 Kasuga, Bunkyo-ku
Tokyo, 112-8551, Japan
E-mail: muramatu@math.chuo-u.ac.jp