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ON HESSIAN STRUCTURES ON THE EUCLIDEAN SPACE AND THE HYPERBOLIC SPACE

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1. Introduction

Let M be a manifold with a flat affine connection D. A Riemannian metric g on M is said to be a *Hessian metric* if g can be locally written $g = D^2u$ with a local function u. We call such a pair (D, g) a *Hessian structure* on M and a triple (M, D, g) a *Hessian manifold* ([5]). Hessian structure appears in affine differential geometry and information geometry ([1], [4]).

If (D, g) is a Hessian structure on M, then in terms of an affine coordinate system (x^i) with respect to D, g can be expressed by $g = \sum_{ij}(\partial^2 u/\partial x^i \partial x^j)dx^i dx^j$. Since a Kähler metric h on a complex manifold can be locally written $h = \sum_{i,j}(\partial^2 v/\partial z^i \partial \bar{z}^j)dz^i d\bar{z}^j$ with a local real-valued function v in terms of a complex local coordinate system (z^i), a Hessian manifold may be regarded as a real number version of a Kähler manifold. Thus we are interested in similarity between Kähler manifolds and Hessian manifolds.

Given a complex structure on a manifold, the set of Kähler metrics is infinite-dimensional. Similarly, given a flat affine connection, the set of Hessian metrics is infinite-dimensional. We next consider the converse situation. Given a Riemannian metric g, the set of almost complex structures J that makes g into a Kähler metric is finite-dimensional because J is parallel with respect to the Riemannian connection. As a Hessian structure version of this, a question arises whether the set of flat affine connections that makes a given Riemannian metric into a Hessian metric is finite-dimensional. In this paper, we shall show that in the cases of the Euclidean space $(\boldsymbol{R}^n, g_0)$ and the hyperbolic space (H^n, g_0), the set of such connections is infinite-dimensional.

We prepare the terminology and notation. Let (M, g) be a Riemannian manifold of dimension ≥ 2 and $S^3(M)$ the space of all symmetric covariant tensor fields of degree 3 on M. We denote by R and ∇ the curvature tensor and the Riemannian connection, respectively. If D is a flat affine connection of M that makes g into a Hessian metric, then the covariant tensor T corresponding to $\hat{T} = D - \nabla$ by g is an element of $S^3(M)$ satisfying $R^{\nabla+\hat{T}} = 0$ on M. Conversely, if the tensor $\hat{T}$ of type (1, 2) corresponding to $T \in S^3(M)$ by g satisfies $R^{\nabla+\hat{T}} = 0$ on M, then $D = \nabla + \hat{T}$ defines the connection above. By this relation, there is a one-to-one correspondence between the set of flat affine connections of M that makes g into a Hessian metric and the set of $T \in S^3(M)$ satisfying $R^{\nabla+\hat{T}} = 0$ on M. So we say that $T \in S^3(M)$ *generates* a Hessian structure with g on M if $R^{\nabla+\hat{T}} = 0$ on M and indicate by $\mathcal{H}(M,g)$ the set of such tensors. To consider a local problem, we also define the set $\mathcal{H}(x,g)$ by the set of symmetric covariant tensors of degree 3 defined on a neighborhood of a point $x \in M$ generating a Hessian structure with g on its domain of definition, where we identify two elements coinciding on a sufficiency small neighborhood of x.

Roughly speaking, we shall prove the following:

Theorem 1.1. *The set $\mathcal{H}(0,g_0)$ at the origin 0 of $\boldsymbol{R}^2$ has the freedom of three local functions on $\boldsymbol{R}$.*

Corollary 1.2. *The set $\mathcal{H}(\boldsymbol{R}^n,g_0)$ has at least the freedom of n functions on $\boldsymbol{R}$. In particular, the set $\mathcal{H}(T^n,g_0)$ on the n-torus T^n has at least the freedom of n periodic functions on $\boldsymbol{R}$.*

Theorem 1.3. *The set $\mathcal{H}(H^n,g_0)$ has at least the freedom of $n-1$ functions on $\boldsymbol{R}$.*

2. Euclidean case

In this section, we shall show Theorem 1.1 and Corollary 1.2.

Lemma 2.1. *Let T be an element of $S^3(M)$ with components T_{ijk}. Then, T generates a Hessian structure with g on M if and only if*

$$\nabla_k T_{ijl} = \nabla_l T_{ijk}, \tag{2.1}$$

$$R^{\nabla}_{ijkl} + \textstyle\sum_s (T_{iks}T^s{}_{jl} - T_{ils}T^s{}_{jk}) = 0. \tag{2.2}$$

Proof. By definition, T generates a Hessian structure with g on M if and only if the tensor $\hat{T}$ of type (1, 2) corresponding to T by g satisfies $R^{\nabla+\hat{T}} = 0$ on M. In terms of T_{ijk}, $R^{\nabla+\hat{T}} = 0$ may be expressed by

$$R^{\nabla}_{ijkl} + \nabla_k T_{ijl} - \nabla_l T_{ijk} + \sum_s (T_{iks}T^s{}_{jl} - T_{ils}T^s{}_{jk}) = 0.$$

Subtracting this from the one exchanged i and j in this, we get (2.2) and hence (2.1). □

Applying Lemma 2.1 to Euclidean case, we have

Lemma 2.2. *Let U be a simply connected neighborhood of the origin 0 of the Euclidean space $\boldsymbol{R}^n$ and T an element of $S^3(U)$. Let T_{ijk} be the components of T with respect to the natural coordinate system $x^1, \dots, x^n$ in $\boldsymbol{R}^n$. Then, T generates a Hessian structure on U if and only if there exists a function u on U such that*

$$T_{ijk} = \frac{\partial^3 u}{\partial x^i \partial x^j \partial x^k}, \tag{2.3}$$

$$\sum_{s=1}^{n} \frac{\partial^3 u}{\partial x^i \partial x^k \partial x^s} \frac{\partial^3 u}{\partial x^j \partial x^l \partial x^s} = \sum_{s=1}^{n} \frac{\partial^3 u}{\partial x^i \partial x^l \partial x^s} \frac{\partial^3 u}{\partial x^j \partial x^k \partial x^s}. \tag{2.4}$$

Proof. We obtain $\partial T_{ijl}/\partial x^k = \partial T_{ijk}/\partial x^l$ on U from (2.1). Thus by Poincaré's lemma, there exists a function u_{ij} on U such that $T_{ijk} = \partial u_{ij}/\partial x^k$. Moreover, because $\partial u_{ij}/\partial x^k = \partial u_{ik}/\partial x^j$ from the symmetry of T, again by Poincaré's lemma, there exists a function u_i on U such that $T_{ijk} = \partial^2 u_i/\partial x^j \partial x^k$. Once again by using the symmetry of T and Poincaré's lemma, finally we get $T_{ijk} = \partial^3 u/\partial x^i \partial x^j \partial x^k$. Substituting this to (2.2), we have (2.4). $\square$

By Lemma 2.2, we see that, up to the quadratic functions of $x^1, \dots, x^n$, there is a one-to-one correspondence between the solutions u of (2.4) on a neighborhood of $0 \in \boldsymbol{R}^n$ and $\mathcal{H}(0, g_0)$ at $0 \in \boldsymbol{R}^n$ by $u \mapsto (\partial^3 u/\partial x^i \partial x^j \partial x^k)$. So we investigate equation (2.4) in a neighborhood of $0 \in \boldsymbol{R}^n$.

In case $n = 2$, (2.4) is reduced to the only one equation:

$$u_{xxx} u_{xyy} + u_{yyy} u_{yxx} = u_{yxx}^2 + u_{xyy}^2,$$

where $x = x^1$, $y = x^2$. Then

$$\begin{aligned} 0 &= (u_{xxx} - u_{xyy}) u_{xyy} + (u_{yyy} - u_{yxx}) u_{yxx} \\ &= (u_{xx} - u_{yy})_x (u_{xy})_y - (u_{xx} - u_{yy})_y (u_{xy})_x \\ &= \begin{vmatrix} (u_{xx} - u_{yy})_x & (u_{xx} - u_{yy})_y \\ (u_{xy})_x & (u_{xy})_y \end{vmatrix}. \end{aligned}$$

This is equivalent to having a functional relation

$$F(u_{xx} - u_{yy}, u_{xy}) = 0$$

on a neighborhood of $0 \in \boldsymbol{R}^2$, where $F = F(s,t)$ is an arbitrary function satisfying $F_s^2 + F_t^2 \neq 0$. Furthermore, this can be written

$$u_{xx} - u_{yy} = f(u_{xy}) \quad \text{if } F_s \neq 0 \tag{2.5}$$

and

$$u_{xy} = \hat{f}(u_{xx} - u_{yy}) \quad \text{if } F_t \neq 0. \tag{2.6}$$

Since (2.6) is reduced to the type of (2.5): $u_{\xi\xi} - u_{\eta\eta} = \hat{f}(4u_{\xi\eta})$ by the change of variables $\xi = x + y$, $\eta = x - y$, we study (2.5).

We know by the following theorem that (2.5) has a unique solution $u(x, y)$ for any given initial data $(u(0, y), u_x(0, y))$:

Fact ([2]). *Let $u_0(y)$, $u_1(y)$ and $A(x, y, u, p, q, s, t)$ are smooth functions. Then, Cauchy problem*

$$\begin{cases} u_{xx} = A(x, y, u, u_x, u_y, u_{xy}, u_{yy}) \\ u(0, y) = u_0(y), \ \ u_x(0, y) = u_1(y) \end{cases}$$

has a unique solution $u(x, y)$ on a neighborhood of $x = 0$ if its linearized equation

$$u_{xx} - au_{xy} - bu_{yy} - (\textit{the terms of lower order}) = 0$$

with coefficients

$$\begin{aligned} a(x, y) &= A_s(x, y, U, U_x, U_y, U_{xy}, U_{yy}), \\ b(x, y) &= A_t(x, y, U, U_x, U_y, U_{xy}, U_{yy}), \end{aligned}$$

where $U(x, y) = u_0(y) + xu_1(y)$, is hyperbolic.

We check that the linearized equation of (2.5) is hyperbolic for any functions $u_0(y)$, $u_1(y)$. We need to verify that its *characteristic equation* $\lambda^2 - a\xi\lambda - b\xi^2 = 0$ has two different real roots λ_1, λ_2, i.e., its discriminant is positive for any real number $\xi \neq 0$. We get $a(x, y) = f'(u_1'(y))$ and $b(x, y) = 1$. Thus the characteristic equation is written $\lambda^2 - f'(u_1'(y))\xi\lambda - \xi^2 = 0$. Then because the discriminant is computed as $(f'(u_1'(y))\xi)^2 + 4\xi^2 = \xi^2(f'(u_1'(y))^2 + 4)$, it is positive for any $\xi \neq 0$.

Consequently we have a bijection from the solutions u of $F(u_{xx} - u_{yy}, u_{xy}) = 0$ with $F_s \neq 0$ into the triples of local functions on $\boldsymbol{R}$ by $u \mapsto (f, u(0, y), u_x(0, y))$. Therefore we obtain

Theorem 1.1. *The set $\mathcal{H}(0, g_0)$ at the origin 0 of $\boldsymbol{R}^2$ can be expressed by the union of two sets each of which is in one-to-one correspondence with the set of triples of local functions on $\boldsymbol{R}$ up to finite-dimensional factor.*

Now setting $\hat{f} = 0$ at (2.6), we get $u_{xy} = 0$ and, from this, $u = \varphi_1(x) + \varphi_2(y)$ with arbitrary functions φ_1, φ_2. If they are global functions on $\boldsymbol{R}$, this is a global solution of (2.4) on $\boldsymbol{R}^2$. Especially, if they are periodic, this is one on 2-torus T^2.

Hence we have an injection from the pairs of functions on $\boldsymbol{R}$ into $\mathcal{H}(\boldsymbol{R}^2, g_0)$ by $(\varphi_1, \varphi_2) \mapsto (\partial^3(\varphi_1(x^1)+\varphi_2(x^2))/\partial x^i \partial x^j \partial x^k)$. Restricting it on the periodic ones, we also obtain a mapping into $\mathcal{H}(T^2, g_0)$.

We prove the following lemma to generalize this:

Lemma 2.3. *If u is a solution of* (2.4) *on $\boldsymbol{R}^n$ and v is an arbitrary function on $\boldsymbol{R}$, then $u(x_1, \ldots, x_n) + v(x_{n+1})$ is a solution of* (2.4) *on $\boldsymbol{R}^{n+1}$.*

Proof. We have to establish

$$\sum_{s=1}^{n+1} \partial_i \partial_k \partial_s (u+v) \partial_j \partial_l \partial_s (u+v) = \sum_{s=1}^{n+1} \partial_i \partial_l \partial_s (u+v) \partial_j \partial_k \partial_s (u+v), \tag{2.7}$$

where we write ∂_i for $\partial/\partial x^i$. We may assume $i < j$, $k < l$ at (2.7) from symmetry. Then i, $k \neq n+1$ and

$$\begin{aligned}
&\text{the left-hand side of (2.7)} \\
&= \sum_{s=1}^{n+1} (\partial_i \partial_k \partial_s u \partial_j \partial_l \partial_s u + \partial_i \partial_k \partial_s u \partial_j \partial_l \partial_s v + \partial_i \partial_k \partial_s v \partial_j \partial_l \partial_s u + \partial_i \partial_k \partial_s v \partial_j \partial_l \partial_s v) \\
&= \left(\sum_{s=1}^{n} \partial_i \partial_k \partial_s u \partial_j \partial_l \partial_s u \right) + \partial_i \partial_k v' \partial_j \partial_l v' \\
&= \sum_{s=1}^{n} \partial_i \partial_k \partial_s u \partial_j \partial_l \partial_s u.
\end{aligned}$$

Similarly

$$\text{the right-hand side of (2.7)} = \sum_{s=1}^{n} \partial_i \partial_l \partial_s u \, \partial_j \partial_k \partial_s u.$$

Since u is a solution of (2.4) on $\boldsymbol{R}^n$ by the assumption, both sides are equal to one another. $\square$

Combining the result of 2-dimensional case and Lemma 2.3, we obtain

Corollary 1.2. *The mapping $\Phi : (\varphi_1, \ldots, \varphi_n) \mapsto (\partial^3(\varphi_1(x^1) + \cdots + \varphi_n(x^n))/\partial x^i \partial x^j \partial x^k)$ gives an injection from the set of n-tuples of functions on $\boldsymbol{R}$ into the set $\mathcal{H}(\boldsymbol{R}^n, g_0)$ up to finite-dimensional factor. Particularlly, Φ restricted on the set of periodic ones gives a mapping into the set $\mathcal{H}(T^n, g_0)$ on n-torus T^n.*

3. Hyperbolic case

In this section, we shall show Theorem 1.3.

We set

$$H^n = \{(x^1, \dots, x^n) \in \boldsymbol{R}^n | x^n > 0\} \quad \text{and} \quad g_0 = \frac{1}{(x^n)^2}\{(dx^1)^2 + \cdots + (dx^n)^2\}.$$

It is known that there exists an element $T_0 = ((T_0)_{ijk}) \in S^3(H^n)$ generating a Hessian structure with g_0 on H^n, which is given for $1 \leq i \leq j \leq k \leq n$ as follows ([3]):

$$(T_0)_{ijk} = \begin{cases} \dfrac{1}{(x^n)^3} & 1 \leq i = j \leq n-1, \ k = n \\ \dfrac{2}{(x^n)^3} & i = j = k = n \\ 0 & \text{otherwise.} \end{cases}$$

We consider the case $n = 2$ for a while.

An element X of $S^3(M)$ is called an *infinitesimal deformation* of $T \in \mathcal{H}(M, g)$ if $(d/dt)|_{t=0} R^{\nabla + T + tX} = 0$.

Lemma 3.1. *An infinitesimal deformation $X = (X_{ijk}) \in S^3(H^2)$ of $T_0 \in \mathcal{H}(H^2, g_0)$ is given by*

$$X_{111} = \frac{f''(x)y^2}{8} + \frac{g'(x)}{2} - f(x) + \frac{h(x)}{y^2}, \tag{3.1}$$

$$X_{112} = \frac{f'(x)y}{2} + \frac{g(x)}{y}, \tag{3.2}$$

$$X_{122} = f(x), \tag{3.3}$$

$$X_{222} = 0, \tag{3.4}$$

where $x = x^1$, $y = x^2$, and f, g, h are arbitrary functions.

Proof. In general, by differentiating each of ones substituted $T + tX$ for T in (2.1) and (2.2), we obtain equations for an infinitesimal deformation X of $T \in \mathcal{H}(M, g)$ as follows:

$$\nabla_k X_{ijl} - \nabla_l X_{ijk} = 0,$$

$$\textstyle\sum_s (X_{iks} T^s{}_{jl} + T_{iks} X^s{}_{jl} - X_{ils} T^s{}_{jk} - T_{ils} X^s{}_{jk}) = 0.$$

In case $(M, g) = (H^2, g_0)$ and $T = T_0$, this is reduced to

$$(X_{111})_y - (X_{112})_x + \frac{2}{y}(X_{111} + X_{122}) = 0 \tag{3.5}$$

$$(X_{112})_y - (X_{122})_x + \frac{1}{y}X_{112} = 0, \tag{3.6}$$

$$(X_{122})_y = 0, \tag{3.7}$$

$$X_{222} = 0. \tag{3.8}$$

First from (3.7), we get (3.3). Then equation (3.6) is written as

$$(X_{112})_y + \frac{1}{y}X_{112} = f'(x).$$

Solving this, we have (3.2). Finally by (3.2) and (3.3), equation (3.5) is written as

$$(X_{111})_y + \frac{2}{y}X_{111} = \frac{f''(x)y}{2} + \frac{g'(x)}{y} - 2\frac{f(x)}{y}.$$

Solving this, we obtain (3.1). □

We find out the elements of $\mathcal{H}(H^2, g_0)$ that has the form of $T_0 + X$. Since both of T_0 and X satisfy (2.1), $T_0 + X$ satisfies it. Thereby $T_0 + X$ belongs to $\mathcal{H}(H^2, g_0)$ if and only if it satisfies (2.2) in H^2. In the present case, it is reduced to the only one equation:

$$X_{111}X_{122} + X_{222}X_{112} - X_{112}^2 - X_{122}^2 = 0.$$

Substituting (3.1) $\sim$ (3.4), we get

$$\begin{aligned} 0 &= \left(\frac{f''(x)y^2}{8} + \frac{g'(x)}{2} - f(x) + \frac{h(x)}{y^2}\right)f(x) - \left(\frac{f'(x)y}{2} + \frac{g(x)}{y}\right)^2 - f(x)^2 \\ &= \left(\frac{f(x)f''(x)}{8} - \frac{f'(x)^2}{4}\right)y^2 + \frac{f(x)g'(x)}{2} \\ &\quad -f'(x)g(x) - 2f(x)^2 + (f(x)h(x) - g(x)^2)\frac{1}{y^2}. \end{aligned}$$

Hence $T_0 + X$ belongs to $\mathcal{H}(H^2, g_0)$ if and only if

$$ff'' - 2f'^2 = 0, \tag{3.9}$$

$$fg' - 2f'g - 4f^2 = 0, \tag{3.10}$$

$$fh - g^2 = 0. \tag{3.11}$$

We find the global solutions of this:

A. The case $f = 0$.

From (3.11), we have $g = 0$. So the solution is

$$\begin{cases} f = 0 \\ g = 0 \\ h\text{: an arbitrary function.} \end{cases}$$

B. The case $f \neq 0$.

By supposing $f' \neq 0$, (3.9) can be written

$$\frac{f''}{f'} = 2\frac{f'}{f}.$$

From this, we obtain $f = 1/(Ax + B)$ with arbitrary constants A, B. Then f is a global solution if and only if $A = 0$ and $B \neq 0$. But this contradicts with $f' \neq 0$. Thus $f' = 0$, i.e., f is a constant. Setting $f = C_1 (\neq 0)$, from (3.10) and (3.11), we get $g = 4C_1 x + C_2$ and $h = g^2/C_1$. So the solution is

$$\begin{cases} f = C_1 \\ g = 4C_1 x + C_2 \\ h = \dfrac{g^2}{C_1}. \end{cases}$$

Therefore we have

Proposition 3.2. *For an infinitesimal deformation $X = (X_{ijk}) \in S^3(H^2)$ of $T_0 \in \mathcal{H}(H^2, g_0)$, $T_0 + X$ belongs to $\mathcal{H}(H^2, g_0)$ if and only if X is given as follows:*

$$\begin{cases} X_{111} = \dfrac{h(x)}{y^2} \\ X_{112} = 0 \\ X_{122} = 0 \\ X_{222} = 0 \end{cases} \tag{3.12}$$

or

$$\begin{cases} X_{111} = C_1 + \dfrac{(4C_1 x + C_2)^2}{C_1 y^2} \\ X_{112} = \dfrac{4C_1 x + C_2}{y} \\ X_{122} = C_1 \\ X_{222} = 0, \end{cases} \tag{3.13}$$

where h is an arbitrary function and $C_1 \neq 0$ and C_2 are arbitrary constants.

We go back to the general case. On the analogy of (3.12), we obtain

Theorem 1.3. *Let $\tilde{X} = (\tilde{X}_{ijk}) \in S^3(H^n)$ be given by*

$$\tilde{X}_{ijk} = \begin{cases} \dfrac{f_i(x^i)}{(x^n)^2} & 1 \leq i = j = k \leq n-1 \\ 0 & \textit{otherwise,} \end{cases}$$

where f_i are arbitrary functions. Then, $T_0 + \tilde{X}$ belongs to $\mathcal{H}(H^n, g_0)$.

Proof. We prove that $T_0 + \tilde{X}$ satisfies (2.1) and (2.2). We first verify to satisfy (2.1). Because T_0 satisfies it, we need only verify that $\tilde{X}$ satisfies it, that is,

$$\partial_k \tilde{X}_{ijl} - \partial_l \tilde{X}_{ijk} + \sum_s (\Gamma_l{}^s{}_i \tilde{X}_{sjk} + \Gamma_l{}^s{}_j \tilde{X}_{isk} - \Gamma_k{}^s{}_i \tilde{X}_{sjl} - \Gamma_k{}^s{}_j \tilde{X}_{isl}) = 0, \tag{3.14}$$

where the Christoffel symbols $\Gamma_j{}^i{}_k$ of ∇ is given by

$$\Gamma_j{}^i{}_k = \begin{cases} \dfrac{1}{x^n} & i = n,\ 1 \leq j = k \leq n-1 \\ \dfrac{-1}{x^n} & 1 \leq i = j \leq n-1,\ k = n;\ \text{or } i = j = k = n \\ 0 & \text{otherwise.} \end{cases}$$

It suffices to consider (3.14) for $i \leq j,\ k < l$ by symmetry.

A. The case $i = j$.

Then

$$\begin{aligned} \text{the left-hand side of (3.14)} &= \partial_k \tilde{X}_{iil} - \partial_l \tilde{X}_{iik} + 2\sum_s (\Gamma_l{}^s{}_i \tilde{X}_{sik} - \Gamma_k{}^s{}_i \tilde{X}_{sil}) \\ &= \partial_k \tilde{X}_{iil} - \partial_l \tilde{X}_{iik} + 2\sum_s \Gamma_l{}^s{}_i \tilde{X}_{sik}. \end{aligned}$$

If $i = k$, then we get

$$\begin{aligned} \partial_k \tilde{X}_{iil} - \partial_l \tilde{X}_{iik} + 2\sum_s \Gamma_l{}^s{}_i \tilde{X}_{sik} &= -\partial_l \tilde{X}_{iii} + 2\sum_s \Gamma_l{}^s{}_i \tilde{X}_{sii} \\ &= -\partial_l \frac{f_i(x^i)}{(x^n)^2} + 2\Gamma_l{}^i{}_i \tilde{X}_{iii} \\ &= \begin{cases} 2\Gamma_l{}^i{}_i \tilde{X}_{iii} = 0 & l < n \\ 2\dfrac{f_i(x^i)}{(x^n)^3} + 2\dfrac{-1}{x^n}\dfrac{f_i(x^i)}{(x^n)^2} = 0 & l = n. \end{cases} \end{aligned}$$

If $i \neq k$, then we have

$$\partial_k \tilde{X}_{iil} - \partial_l \tilde{X}_{iik} + 2\sum_s \Gamma_l{}^s_i \tilde{X}_{sik} = \partial_k \tilde{X}_{iil} = \delta_{il}\partial_k \frac{f_i(x^i)}{(x^n)^2} = 0,$$

where δ_{ij} is Kronecker's delta.

B. The case $i < j$.

Then

$$\begin{aligned}\text{the left-hand side of (3.14)} &= \sum_s (\Gamma_l{}^s_i \tilde{X}_{sjk} + \Gamma_l{}^s_j \tilde{X}_{isk} - \Gamma_k{}^s{}_i \tilde{X}_{sjl} - \Gamma_k{}^s{}_j \tilde{X}_{isl}) \\ &= \sum_s (\Gamma_l{}^s_i \tilde{X}_{sjk} + \Gamma_l{}^s_j \tilde{X}_{isk} - \Gamma_k{}^s{}_j \tilde{X}_{isl}).\end{aligned}$$

Since (3.14) is equal to the one exchanged a pair (i,j) and a pair (k,l), we need only check the following three cases:

If $i = k$, $j = l$, then we obtain

$$\begin{aligned}\sum_s (\Gamma_l{}^s_i \tilde{X}_{sjk} + \Gamma_l{}^s_j \tilde{X}_{isk} - \Gamma_k{}^s{}_j \tilde{X}_{isl}) &= \sum_s (\Gamma_j{}^s{}_i \tilde{X}_{sji} + \Gamma_j{}^s{}_j \tilde{X}_{isi} - \Gamma_i{}^s_j \tilde{X}_{isj}) \\ &= \sum_s \Gamma_j{}^s{}_j \tilde{X}_{isi} \\ &= \Gamma_j{}^i{}_j \tilde{X}_{iii} \\ &= 0.\end{aligned}$$

If $i = k$, $j < l$, then we get

$$\begin{aligned}\sum_s (\Gamma_l{}^s_i \tilde{X}_{sjk} + \Gamma_l{}^s_j \tilde{X}_{isk} - \Gamma_k{}^s{}_j \tilde{X}_{isl}) &= \sum_s (\Gamma_l{}^s_i \tilde{X}_{sji} + \Gamma_l{}^s_j \tilde{X}_{isi} - \Gamma_i{}^s_j \tilde{X}_{isl}) \\ &= \sum_s \Gamma_l{}^s_j \tilde{X}_{isi} \\ &= \Gamma_l{}^i_j \tilde{X}_{iii} \\ &= 0.\end{aligned}$$

If $i < k$, then we have

$$\begin{aligned}\sum_s (\Gamma_l{}^s_i \tilde{X}_{sjk} + \Gamma_l{}^s_j \tilde{X}_{isk} - \Gamma_k{}^s{}_j \tilde{X}_{isl}) &= \sum_s \Gamma_l{}^s_i \tilde{X}_{sjk} \\ &= \begin{cases} \Gamma_l{}^n_i \tilde{X}_{njk} = 0 & l < n \\ \Gamma_n{}^i{}_i \tilde{X}_{ijk} = 0 & l = n. \end{cases}\end{aligned}$$

We next establish that $\tilde{T} = T_0 + \tilde{X}$ satisfies (2.2), i.e.,

$$\sum_s (\tilde{T}_{iks}\tilde{T}_{jls} - \tilde{T}_{ils}\tilde{T}_{jks}) = \frac{1}{(x^n)^6}(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}), \tag{3.15}$$

where $\tilde{T} = (\tilde{T}_{ijk})$ is given by

$$\tilde{T}_{ijk} = \begin{cases} \dfrac{1}{(x^n)^3} & 1 \le i = j \le n-1,\ k = n \\ \dfrac{f_i(x^i)}{(x^n)^2} & 1 \le i = j = k \le n-1 \\ \dfrac{2}{(x^n)^3} & i = j = k = n \\ 0 & \text{otherwise.} \end{cases}$$

It suffices to consider (3.15) in the case $i = k,\ j = l$, in the case $i = k,\ j < l$ and in the case $i < k$ under $1 \le i < j \le n,\ 1 \le k < l \le n$ from symmetry.

A. The case $i = k,\ j = l$.

Equality (3.15) is written as

$$\sum_s (\tilde{T}_{iis}\tilde{T}_{jjs} - \tilde{T}^2_{ijs}) = \frac{1}{(x^n)^6}. \tag{3.16}$$

Then

$$\begin{aligned} \text{the left-hand side of (3.16)} &= \tilde{T}_{iii}\tilde{T}_{jji} + \tilde{T}_{iin}\tilde{T}_{jjn} - \tilde{T}^2_{iji} \\ &= \tilde{T}_{iin}\tilde{T}_{jjn} - \tilde{T}^2_{iij} \\ &= \begin{cases} \dfrac{1}{(x^n)^3}\dfrac{1}{(x^n)^3} = \dfrac{1}{(x^n)^6} & j < n \\ \dfrac{1}{(x^n)^3}\dfrac{2}{(x^n)^3} - \left(\dfrac{1}{(x^n)^3}\right)^2 = \dfrac{1}{(x^n)^6} & j = n. \end{cases} \end{aligned}$$

B. The case $i = k,\ j < l$.

Equality (3.15) is simplified as

$$0 = \sum_s (\tilde{T}_{iis}\tilde{T}_{jls} - \tilde{T}_{ils}\tilde{T}_{jis}) = \sum_s \tilde{T}_{iis}\tilde{T}_{jls}. \tag{3.17}$$

Then

$$\text{the right-hand side of (3.17)} = \tilde{T}_{iii}\tilde{T}_{jli} + \tilde{T}_{iin}\tilde{T}_{jln} = 0.$$

C. The case $i < k$.

Equality (3.15) is simplified as

$$0 = \sum_s (\tilde{T}_{iks}\tilde{T}_{jls} - \tilde{T}_{ils}\tilde{T}_{jks}) = -\sum_s \tilde{T}_{ils}\tilde{T}_{jks}. \tag{3.18}$$

Then

$$\text{the right-hand side of (3.18)} = -\tilde{T}_{ili}\tilde{T}_{jki} = 0.$$ □

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