Nakauchi, N.
Osaka J. Math.
35 (1998), 303–312

A LIOUVILLE TYPE THEOREM FOR *P*-HARMONIC MAPS

Dedicated to Professor Fumiyuki Maeda on his sixtieth birthday

NOBUMITSU NAKAUCHI

(Received February 2, 1997)

1. Introduction

In this note we prove a Liouville type theorem for p-harmonic maps. Let (M, g), (N, h) be Riemannian manifolds, and let $p \geq 2$. By Nash's isometric embedding, we may assume that N is a submanifold of a Euclidean space $\mathbb{R}^d$. The Sobolev space $W^{1,p}_{loc}(M, N)$ is defined to be

$$W^{1,p}_{loc}(M, N) = \left\{ u \in W^{1,p}_{loc}(M, \mathbb{R}^d); u(x) \in N \text{ a.e. } x \in M \right\},$$

where $W^{1,p}_{loc}(M, \mathbb{R}^d)$ denotes the Sobolev space of $\mathbb{R}^d$-valued L^p_{loc}-functions on M whose derivative belong to L^p_{loc}. A *p-harmonic map* $u : M \to N$ is a weak solution of the equation

$$\text{Trace}(\nabla(\|du\|^{p-2} du)) = 0, \tag{1.1}$$

i.e., $u \in W^{1,p}_{loc}(M, N)$ satisfies

$$-\int_M \|du\|^{p-2} \nabla u \cdot \nabla \varphi + \int_M \|du\|^{p-2} A(u)(\nabla u, \nabla u) \cdot \varphi = 0 \tag{1.2}$$

for any $\varphi \in C_0^\infty(M, N)$, where A denotes the second fundamental form of N. A p-harmonic map u is characterized as a critical point of the *p-energy functional*

$$E_p(u) = \int_M \|du\|^p \tag{1.3}$$

in $W^{1,p}_{loc}(M, N)$, if the value of this functional is finite. When $p \neq 2$, the degenerate ellipticity of the equation (1.1) gives only (partial) $C^{1,\alpha}$-regularity even for minimizers of the functional (1.3), while in case $p = 2$, $C^{1,\alpha}$-regularity implies C^∞-regularity. So we are concerned with p-harmonic maps which belong to the C^1_{loc}-class, for general p.

Several studies are given for 2-harmonic maps or harmonic maps. (See Eells and Lemaire [3], [4].) For these harmonic maps, there are Liouville type theorems, which states that a harmonic map u is constant under some conditions. A typical one of such conditions is the boundedness of u, where we say that u is bounded if its image is contained in a compact set. As a result with assumptions to images of maps, a Liouville type property is known when the image is enveloped by a convex function. (See Gordon [6] for $p = 2$, L.-F.Cheung and P.-F.Leung [1] for general $p \geq 2$.) The finiteness of the energy is another typical condition; precisely speaking, a harmonic map u is constant if $E_2(u) < \infty$, when M is complete and noncompact with $\mathrm{Ric}_M \geq 0$ and $\mathrm{Sect}_N \leq 0$, where Ric_M denotes the Ricci curvature of M, and Sect_N denotes the sectional curvature of N. (See Schoen and Yau [8], Hildebrandt [7].) In this note we extend this result for general $p \geq 2$.

Theorem 1. *Let M be complete and noncompact. Assume $\mathrm{Ric}_M \geq 0$ and $\mathrm{Sect}_N \leq 0$. Let $u : M \to N$ is a p-harmonic map of C^1_{loc}-class such that $E_p(u) < \infty$. Then u is a constant map.*

In [9], Takeuchi proved Theorem 1, using Hildebrandt's argument [7], under the condition $E_{2p-2}(u) < \infty$ instead of $E_p(u) < \infty$. The exponent $2p - 2$, however, is not compatible in our case, since $2p - 2 \neq p$ when $p \neq 2$. Our proof of Theorem 1 has two steps; the first step (Section 3) for C^3_{loc}-maps and the second step (Section 4) in general case. The first step is based on a Bochner type formula (Section 2) and a standard argument of cutoff functions, and the second step depends on the approximation argument, which is used in [2] when M is an open set in a Euclidean space.

Acknowledgement. The author would like to thank Professor Shigeo Kawai for his kind comments to the Bochner type formula.

2. Bochner type formula

In this section we give the following Bochner type formula.

Lemma 1. *Let $u : M \to N$ be a map of C^3_{loc}-class. Then the following equality holds:*

(a)
$$\begin{aligned}
&\|du\|^{p-1} \triangle \|du\|^{p-1} + \langle \|du\|^{p-2}du, (d^\nabla \delta^\nabla + \delta^\nabla d^\nabla)(\|du\|^{p-2}du)\rangle \\
&= 2(\|\nabla(\|du\|^{p-2}du)\|^2 - \|\nabla\|du\|^{p-1}\|^2) \\
&\quad + \|du\|^{2p-4} \sum_{j=1}^{m} \langle \mathrm{Ric}_M(du(e_j)), du(e_j)\rangle \\
&\quad - \|du\|^{2p-4} \sum_{i,j=1}^{m} \langle \mathrm{Riem}_N(du(e_i), du(e_j))du(e_j), du(e_i)\rangle,
\end{aligned}$$

hence

(b)
$$\|du\| \triangle \|du\|^{p-1} + \langle du, (d^\nabla \delta^\nabla + \delta^\nabla d^\nabla)(\|du\|^{p-2} du)\rangle$$
$$\geq \|du\|^{p-2} \sum_{j=1}^{m} \langle \mathrm{Ric}_M(du(e_j)), du(e_j)\rangle$$
$$- \|du\|^{p-2} \sum_{i,j=1}^{m} \langle \mathrm{Riem}_N(du(e_i), du(e_j))du(e_j), du(e_i)\rangle,$$

where $\{e_j\}_{j=1}^k$ *is an orthonormal base of the tangent space of* M, *and* d^∇ [*resp.* δ^∇] *denotes the derivation with respect to* ∇ [*resp. the* L^2*-adjoint operator of* d^∇].

Therefore, when $\mathrm{Ric}_M \geq 0$ *and* $\mathrm{Sect}_N \leq 0$, *we have*

(c)
$$\|du\| \triangle \|du\|^{p-1} + \langle du, (d^\nabla \delta^\nabla + \delta^\nabla d^\nabla)(\|du\|^{p-2} du)\rangle \geq 0.$$

Proof of Lemma 1. Using the relation between the rough Laplacian $\triangle$ and the Hodge-de Rham Laplacian $d^\nabla \delta^\nabla + \delta^\nabla d^\nabla$, we have

(2.4)
$$\begin{aligned}\frac{1}{2} \triangle \|du\|^{2p-2} &= \frac{1}{2} \triangle \| \|du\|^{p-2} du\|^2 \\ &= \langle \|du\|^{p-2} du, \triangle(\|du\|^{p-2} du)\rangle + \|\nabla(\|du\|^{p-2} du)\|^2 \\ &= -\langle \|du\|^{p-2} du, (d^\nabla \delta^\nabla + \delta^\nabla d^\nabla)(\|du\|^{p-2} du)\rangle \\ &\quad + \|\nabla(\|du\|^{p-2} du)\|^2 + Q(u),\end{aligned}$$

where

$$\begin{aligned}Q(u) &= \|du\|^{2p-4} \sum_{i=1}^{n} \langle \mathrm{Ric}_M(du(e_i)), du(e_i)\rangle \\ &\quad - \|du\|^{2p-4} \sum_{i,j=1}^{n} \langle \mathrm{Riem}_N(du(e_i), du(e_j))du(e_j), du(e_i)\rangle.\end{aligned}$$

(cf. Eells and Lemaire [3, p.8, (2.20)] for $p = 2$; Note that for any harmonic map ($p = 2$), $d^\nabla(du) = \delta^\nabla(du) = 0$.) On the other hand, we see

(2.5)
$$\begin{aligned}\frac{1}{2} \triangle \|du\|^{2p-2} &= \frac{1}{2} \triangle (\|du\|^{p-1})^2 \\ &= \|du\|^{p-1} \triangle \|du\|^{p-1} + \|\nabla \|du\|^{p-1}\|^2.\end{aligned}$$

Then from (2.4) and (2.5), we have (a). From (a), we have

$$\begin{aligned}(2.6)\qquad &\|du\|^{p-1} \triangle \|du\|^{p-1} + \langle \|du\|^{p-2}du, (d^\nabla \delta^\nabla + \delta^\nabla d^\nabla)(\|du\|^{p-2}du)\rangle \\ &\geq \|du\|^{2p-4} \sum_{j=1}^{m} \langle \mathrm{Ric}_M(du(e_j)), du(e_j)\rangle \\ &\quad - \|du\|^{2p-4} \sum_{i,j=1}^{m} \langle \mathrm{Riem}_N(du(e_i), du(e_j))du(e_j), du(e_i)\rangle,\end{aligned}$$

since

$$\|\nabla \|du\|^{p-1}\| = \|\nabla\| \, \|du\|^{p-2}du\| \, \| \leq \|\nabla(\|du\|^{p-2}du)\|.$$

On any point such that $du = 0$, the inequality (b) holds trivially. On the other points, we divide the both sides of (2.6) by $\|du\|^{p-2}$, and then we have the inequality (b). □

3. Proof of Theorem 1 for C^3_{loc}-maps

Throughout this paper, all positive constants $C_1, C_2, \ldots$ depend only on p, M, N if there is no special mention. Since u is a p-harmonic map,

$$\delta^\nabla(\|du\|^{p-2}du) = 0.$$

Then by Lemma 1 (c), we have

$$(3.7)\qquad \|du\| \triangle \|du\|^{p-1} + \langle du, \delta^\nabla d^\nabla(\|du\|^{p-2}du)\rangle \geq 0.$$

since $\mathrm{Ric}_M \geq 0$, $\mathrm{Sect}_N \leq 0$. Take any point $x \in M$. Let η be a cutoff function satisfying that

$$(3.8)\qquad \eta : \begin{cases} = 1 & \text{on} \quad B_\rho(x) \\ \in [0,1] & \text{on} \quad B_{2\rho}(x) - B_\rho(x) \\ = 0 & \text{on} \quad M - B_{2\rho}(x) \end{cases}$$

and that

$$(3.9)\qquad \|\nabla \eta\|^2 \leq \frac{C_1}{\rho^2}.$$

Then from (3.7), we get

$$(3.10)\qquad \int_M \|du\|\eta^2 \triangle \|du\|^{p-1} + \int_M \langle \eta^2 du, \delta^\nabla d^\nabla(\|du\|^{p-2}du)\rangle \geq 0.$$

We have

$$
\begin{aligned}
(3.11)\quad & \int_M \|du\|\eta^2 \triangle \|du\|^{p-1} \\
&= -\int_M \nabla(\|du\|\eta^2)\cdot\nabla\|du\|^{p-1} \\
&= -(p-1)\int_M \|du\|^{p-2}\eta^2\|\nabla\|du\|\,\|^2 - 2(p-1)\int_M \|du\|^{p-2}\eta\nabla\|du\|\cdot\nabla\eta \\
&= -\frac{4(p-1)}{p^2}\int_M \|\nabla\|du\|^{p/2}\|^2\eta^2 - \frac{4(p-1)}{p}\int_M \|du\|^{p/2}\eta\nabla\|du\|^{p/2}\cdot\nabla\eta \\
&\le -\frac{4(p-1)}{p^2}\int_M \|\nabla\|du\|^{p/2}\|^2\eta^2 + \varepsilon\int_M \|\nabla\|du\|^{p/2}\|^2\eta^2 + \frac{C_2}{\varepsilon}\int_M \|du\|^p\|\nabla\eta\|^2
\end{aligned}
$$

for any $\varepsilon > 0$. We see

$$
(3.12)\quad \int_M \langle \eta^2 du, \delta^\nabla d^\nabla(\|du\|^{p-2}du)\rangle = \int_M \langle d^\nabla(\eta^2 du), d^\nabla(\|du\|^{p-2}du)\rangle.
$$

Since

$$
\|d^\nabla(\varphi du)\| \le C_3\|\nabla\varphi\|\,\|du\|,
$$

we have

$$
\begin{aligned}
(3.13)\quad & \left|\langle d^\nabla(\eta^2 du), d^\nabla(\|du\|^{p-2}du)\rangle\right| \\
&\le \|d^\nabla(\eta^2 du)\|\,\|d^\nabla(\|du\|^{p-2}du)\| \\
&\le C_4\|\nabla\eta^2\|\,\|du\|\,\|\nabla\|du\|^{p-2}\|\,\|du\| \\
&= C_5\eta\|\nabla\eta\|\,\|du\|^{p-1}\|\nabla\|du\|\,\| \\
&= C_6\eta\|\nabla\eta\|\,\|du\|^{p/2}\|\nabla\|du\|^{p/2}\| \\
&\le \varepsilon\|\nabla\|du\|^{p/2}\|^2\eta^2 + \frac{C_7}{\varepsilon}\|du\|^p\|\nabla\eta\|^2.
\end{aligned}
$$

From (3.12) and (3.13), we get

$$
\begin{aligned}
(3.14)\quad & \left|\int_M \langle \eta^2 du, \delta^\nabla d^\nabla(\|du\|^{p-2}du)\rangle\right| \\
&\le \varepsilon\int_M \|\nabla\|du\|^{p/2}\|^2\eta^2 + \frac{C_7}{\varepsilon}\int_M \|du\|^p\|\nabla\eta\|^2.
\end{aligned}
$$

Then from (3.10), (3.11) and (3.14),

$$
\left(\frac{4(p-1)}{p^2} - 2\varepsilon\right)\int_M \|\nabla\|du\|^{p/2}\|^2\eta^2 \le \frac{C_8}{\varepsilon}\int_M \|du\|^p\|\nabla\eta\|^2.
$$

Let $\varepsilon = \dfrac{p-1}{p^2}$, and then we obtain

$$\int_{B_\rho(x)} \|\nabla \|du\|^{p/2}\|^2 \le C_9 \int_M \|du\|^p \|\nabla\eta\|^2 \le \frac{C_{10}}{\rho^2}\int_M \|du\|^p. \tag{3.15}$$

Let ρ go to infinity, and then we see that $\nabla\|du\|^{p/2} \equiv 0$, i.e., $\|du\|$ is constant on M. Note that the volume of M is infinite, since $\mathrm{Ric}_M \ge 0$. Then by the condition $E_p(u) < \infty$, we conclude that the constant $\|du\|$ is zero. Therefore u is a constant map. □

4. Proof of Theorem 1

In this section we complete our proof of Theorem 1 using an approximation. We use the arguments in Duzaar and Fuchs [2]. We may assume $p > 2$, since Theorem 1 holds for $p = 2$. As mentioned in the introduction, we may assume that the target manifold N is a submanifold of a Euclidean space $\mathbb{R}^d$, and that u is a map into $\mathbb{R}^d$. Then we know

Proposition 1. *For any p-harmonic map u of C^1_{loc}-class, $\|du\|^{p/2-1}du$ belong to $W^{1,2}_{loc}(TM, \mathbb{R}^d)$.*

When M is a domain of a Euclidean space, Proposition 1 is Lemma 2.2 in Duzaar and Fuchs [2]. In the above general situation, Proposition 1 can be proved with slight modifications.

Using Proposition 1, we will complete our proof of Theorem 1. Let $M_+ := \{x \in M; \|du\|(x) \ne 0\}$. By Proposition 1, we see that du is of $W^{1,2}_{loc}$-class on M_+, since

$$\nabla du = \nabla(\|du\|^{p/2-1}du)\|du\|^{1-p/2} - \frac{p-2}{p}\nabla(\|du\|^{p/2})\|du\|^{-p/2}du,$$

hence

$$\begin{aligned}\|\nabla du\| &\le \|du\|^{1-p/2}\|\nabla(\|du\|^{p/2-1}du)\| + \frac{p-2}{p}\|du\|^{1-p/2}\|\nabla(\|du\|^{p/2})\| \\ &\le \left(1 + \frac{p-2}{p}\right)\|du\|^{1-p/2}\|\nabla(\|du\|^{p/2-1}du)\|.\end{aligned}$$

Then we can find an approximating sequence $\{u_k\}_{j=1}^\infty \subset C^\infty_{loc}(M, \mathbb{R}^d)$ such that as k goes to infinity,

(a) u_k converges to u in $C^1_{loc}(M)$,

(b) u_k converges to u weakly in $W^{1,p}_{loc}(M)$, and

(c) u_k converges to u weakly in $W^{2,2}_{loc}(M_+)$.

By Lemma 1, we have

$$\|du_k\| \triangle \|du_k\|^{p-1} + \frac{p}{2(p-1)}\langle du_k, (d^\nabla \delta^\nabla + \delta^\nabla d^\nabla)(\|du_k\|^{p-2}du_k)\rangle \geq 0,$$

since $\mathrm{Ric}_M \geq 0$, $\mathrm{Sect}_N \leq 0$. Let η be a cutoff function on M satisfying (3.8) and (3.9), and let

$$\varphi_\varepsilon(x) := \begin{cases} \dfrac{\|du\|(x)}{\max\{\|du\|(x), \varepsilon\}} & \text{on } M_+ \\ 0 & \text{on } M - M_+. \end{cases}$$

Then $\varphi_\varepsilon \in L^{1,2}_0(M_+)$, where $L^{1,2}_0(M_+)$ is the completion of $C^\infty_0(M_+)$, and $\varphi_\varepsilon \to 1$, $\nabla\varphi_\varepsilon \to 0$ in $L^{1,2}_0(M_+)$. Using the function $\varphi_\varepsilon\eta^2$ instead of η^2, we apply the estimates (3.11), (3.14) for smooth maps u_k. Then we have, instead of (3.15),

$$\begin{aligned}(4.16)\quad &\int_{M_+} \|\nabla\|du_k\|^{p/2}\|^2\varphi_\varepsilon\eta^2 + \int_{M_+} \|du_k\|^{p/2}\eta^2\nabla\|du_k\|^{p/2}\cdot\nabla\varphi_\varepsilon \\ &\leq \frac{C_{11}}{\rho^2}\int_{M_+}\|du_k\|^p\varphi_\varepsilon + C_{12}\int_{M_+}\langle\varphi_\varepsilon\eta^2 du_k, d^\nabla\delta^\nabla(\|du_k\|^{p-2}du_k)\rangle \\ &\leq \frac{C_{11}}{\rho^2}\int_{M_+}\|du_k\|^p\varphi_\varepsilon + C_{12}\int_{M_+}\delta^\nabla(\varphi_\varepsilon\eta^2 du_k)\delta^\nabla(\|du_k\|^{p-2}du_k).\end{aligned}$$

Since u_k converge to u in $C^1_{loc}(M_+) \cap W^{2,2}_{loc}(M_+)$,

$$\int_{M_+}\delta^\nabla(\varphi_\varepsilon\eta^2 du_k)\delta^\nabla(\|du_k\|^{p-2}du_k) \to \int_{M_+}\delta^\nabla(\varphi_\varepsilon\eta^2 du)\delta^\nabla(\|du\|^{p-2}du) = 0.$$

as k goes to infinity, where the last equality follows from the fact that u satisfies the p-harmonic map equation $\delta^\nabla(\|du\|^{p-2}du) = 0$. Therefore, let k goes to infinity in (4.16), and then we have

$$\int_{M_+}\|\nabla\|du\|^{p/2}\|^2\varphi_\varepsilon\eta^2 + \int_{M_+}\|du\|^{p/2}\eta^2\nabla\|du\|^{p/2}\cdot\nabla\varphi_\varepsilon \leq \frac{C_{11}}{\rho^2}\int_{M_+}\|du\|^p\varphi_\varepsilon.$$

Let ε go to zero, and then we get

$$(4.17)\quad \int_{M_+}\|\nabla\|du\|^{p/2}\|^2\eta^2 \leq \frac{C_{11}}{\rho^2}\int_{M_+}\|du\|^p$$

for any $\eta \in C_0^1(M)$ satisfing (3.8) and (3.9). Here we have used the facts that $\nabla\varphi_\varepsilon$ goes to a measure-valued tensor μ whose support is in ∂M_+, and that $\|du\|$ vanishes there. Let ρ go to infinity in (4.17), and then we have $du = 0$ on M_+, i.e., u is constant on M_+ Hence u is a constant map on M. □

5. A remark.

When M is compact, we have the following result, which is an extension of facts in harmonic map case ($p = 2$). (See Eells and Sampson [5].)

Theorem 2. *Let M be compact ($\partial M = \emptyset$). Let $u : M \to N$ be a p-harmonic map of C^1-class.*
(a) *Assume* $\mathrm{Ric}_M \geq 0$ *and* $\mathrm{Sect}_N \leq 0$. *Then u is totally geodesic.*
(b) *In addition to* (a), *if* $\mathrm{Ric}_M > 0$ *somewhere, then u is a constant map.*
(c) *In addition to* (a), *if* $\mathrm{Sect}_N < 0$, *then u is a constant map, or u maps onto a closed geodesic in N.*

Proof. We take an approximating sequence $\{u_k\}_{k=1}^{\infty}$ such as that in Section 4. Take the function φ_ε in Section 4. From Lemma 1 (b),

$$\int_{M_+} \|du_k\|\varphi_\varepsilon \triangle \|du_k\|^{p-1} + \int_{M_+} \langle \varphi_\varepsilon du_k, (d^\nabla \delta^\nabla + \delta^\nabla d^\nabla)(\|du_k\|^{p-2} du_k)\rangle \geq 0.$$

Using integration by parts, we have

$$\begin{aligned}
&\int_{M_+} \|du_k\|\varphi_\varepsilon \triangle \|du_k\|^{p-1} \\
&= -\int_{M_+} \varphi_\varepsilon \nabla\|du_k\| \cdot \nabla\|du_k\|^{p-1} - \int_{M_+} \|du_k\|\nabla\|du_k\|^{p-1} \cdot \nabla\varphi_\varepsilon \\
&= -\frac{4(p-1)}{p^2}\int_{M_+} \|\nabla\|du_k\|^{p/2}\|^2\varphi_\varepsilon - \int_{M_+} \|du_k\|\nabla\|du_k\|^{p-1} \cdot \nabla\varphi_\varepsilon.
\end{aligned}$$

Then from the above inequality, we get

$$\begin{aligned}
&\frac{4(p-1)}{p^2}\int_{M_+} \|\nabla\|du_k\|^{p/2}\|^2\varphi_\varepsilon \\
&\leq \int_{M_+} \langle \delta^\nabla(\varphi_\varepsilon du_k), \delta^\nabla(\|du_k\|^{p-2}du_k)\rangle \\
&\quad + \int_{M_+} \langle d^\nabla(\varphi_\varepsilon du_k), d^\nabla(\|du_k\|^{p-2}du_k)\rangle \\
&= \int_{M_+} \langle \delta^\nabla(\varphi_\varepsilon du_k), \delta^\nabla(\|du_k\|^{p-2}du_k)\rangle
\end{aligned}$$

$$+\int_{M_+} \|\nabla\varphi_\varepsilon\| \, \|du_k\| \, \|d^\nabla(\|du_k\|^{p-2}du_k)\|.$$

We have used the fact $\|d^\nabla(\varphi_\varepsilon du_k)\| \leq C_3 \, \|\nabla\varphi_\varepsilon\| \, \|du_k\|$. Let $k \to \infty$ and let $\varepsilon \to 0$, then we have

$$\frac{4(p-1)}{p^2}\int_{M_+} \|\nabla\|du\|^{p/2}\|^2 \leq 0.$$

Therefore $\|du\|$ is constant on M_+, hence on M, since $\|du\|$ is continuous. From Lemma 1 (a), we have

$$\begin{aligned}
&-\int_{M_+} \|\nabla(\|du_k\|^{p-2}du_k)\|^2\varphi_\varepsilon + \int_{M_+} \|du_k\|^{p/2}\nabla\|du_k\|^{p/2}\cdot\nabla\varphi_\varepsilon \\
&\quad+\int_{M_+} \langle \|du_k\|^{p-2}\varphi_\varepsilon du_k, (d^\nabla\delta^\nabla + \delta^\nabla d^\nabla)(\|du_k\|^{p-2}du_k)\rangle \\
&\geq 0.
\end{aligned}$$

We have applied the integration by parts and used the assumption $\mathrm{Ric}_M \geq 0$ and $\mathrm{Sect}_N \leq 0$ and the fact that

$$\|\nabla(\|du_k\|^{p-2}du_k)\|^2 \geq \; \|\nabla\| \, \|du_k\|^{p-2}du_k\| \, \|^2 \geq \; \|\nabla\|du_k\|^{p-1}\|^2.$$

Let $k \to \infty$, and let $\varepsilon \to 0$, then we get

$$0 \geq \; 2\int_{M_+} \|\nabla(\|du\|^{p-2}du)\|^2$$

on M_+, since $\|du\|$ is constant. Hence $\nabla du = 0$, i.e., u is totally geodesic on M_+. Since $\|du\|$ is constant, u is a harmonic map; u is totally geodesic on M. We have (a).

We know, by the proof of (a),

$$\begin{aligned}
&\|du\| \equiv \text{Const.} =: C_0 && \text{on } M \\
&\nabla du = 0 && \text{on } M_+
\end{aligned}$$

Then from Lemma 1 (a) again, we have

$$\begin{aligned}
0 &\leq C_0^{2p-4}\sum_{j=1}^{m}\langle \mathrm{Ric}_M(du(e_j)), du(e_j)\rangle \\
&= C_0^{2p-4}\sum_{i,j=1}^{m}\langle \mathrm{Riem}_N(du(e_j), du(e_j))du(e_j), du(e_i)\rangle \\
&\leq 0
\end{aligned} \tag{5.18}$$

on M_+. If $\mathrm{Ric}_M > 0$ somewhere, the inequality (5.18) implies $C_0 = 0$, or $du = 0$ at this point. Then $C_0 = 0$, and we have (b). If $\mathrm{Sect}_N < 0$ at a point $x^* \in M$, then $C_0 = 0$ or $\dim(Image(du(x^*))) \leq 1$. If $\dim(Image(du(x^*))) = 0$, then $\|du\|(x) = C_0 = \|du\|(x^*) = 0$ for any $x \in M$, i.e. u is a constant map. If $\dim(Image(du(x))) = 1$, we have (c) since u is totally geodesic. □

References

[1] L.F Cheung and P.F Leung: *A remark on convex functions and p-harmonic maps*, Geom. Dedicata, **56** (1995), 269–270.
[2] F. Duzaar and M. Fuchs: *On removable singularities of p-harmonic maps*, Ann. Inst. H. Poincaré Analyse non linéaire, **7** (1990), 385–405.
[3] J. Eells and L. Lemaire: *A report on harmonic maps*, Bull. London Math Soc. **10** (1978), 1–68.
[4] J. Eells and L. Lemaire: *Another report on harmonic maps*, Bull. London Math Soc. **20** (1988), 385–524.
[5] J. Eells and J.H. Sampson: *Harmonic mappings of Riemannian manifolds*, Amer. J. Math. **86** (1964), 109–160.
[6] W.B. Gordon: *Convex functions and harmonic maps*, Proc. Amer. Math. Soc. **33** (1972), 433–437.
[7] S. Hildebrandt: *Nonlinear elliptic systems and harmonic mappings*, Proc. Beijing Symp. Diff. Geom. and Diff. Eq. Gordon and Bread, New York, 1982, 481–615.
[8] R. Schoen and S.T. Yau: *Harmonic maps and the topology of stable hypersurfaces and manifolds of nonnegative Ricci curvature*, Comment. Math. Helv. **51** (1976), 333–341.
[9] H. Takeuchi: *Stability and Liouville theorems of P-harmonic maps*, Japan. J. Math. **17** (1991), 317–332.
[10] K. Uhlenbeck: *Regularity for a class of nonlinear elliptic systems*, Acta Math. **138** (1970), 219–240.

Department of Mathematics
Faculty of Science
Yamaguchi University
Yamaguchi 753, Japan