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A LIOUVILLE TYPE THEOREM FOR *P*-HARMONIC MAPS

Dedicated to Professor Fumiyuki Maeda on his sixtieth birthday

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1. Introduction

In this note we prove a Liouville type theorem for p-harmonic maps. Let (M, g), (N, h) be Riemannian manifolds, and let $p \geq 2$. By Nash's isometric embedding, we may assume that N is a submanifold of a Euclidean space $\mathbb{R}^d$. The Sobolev space $W^{1,p}_{loc}(M, N)$ is defined to be

$$W^{1,p}_{loc}(M,N) = \left\{u \in W^{1,p}_{loc}(M, \mathbb{R}^d); u(x) \in N \text{ a.e. } x \in M\right\},$$

where $W^{1,p}_{loc}(M, \mathbb{R}^d)$ denotes the Sobolev space of $\mathbb{R}^d$-valued L^p_{loc}-functions on M whose derivative belong to L^p_{loc}. A *p-harmonic map* $u : M \to N$ is a weak solution of the equation

$$\text{Trace}(\nabla(\|du\|^{p-2}du)) = 0, \tag{1.1}$$

i.e., $u \in W^{1,p}_{loc}(M, N)$ satisfies

$$-\int_M \|du\|^{p-2}\nabla u \cdot \nabla\varphi + \int_M \|du\|^{p-2}A(u)(\nabla u, \nabla u) \cdot \varphi = 0 \tag{1.2}$$

for any $\varphi \in C^\infty_0(M, N)$, where A denotes the second fundamental form of N. A p-harmonic map u is characterized as a critical point of the *p-energy functional*

$$E_p(u) = \int_M \|du\|^p \tag{1.3}$$

in $W^{1,p}_{loc}(M, N)$, if the value of this functional is finite. When $p \neq 2$, the degenerate ellipticity of the equation (1.1) gives only (partial) $C^{1,\alpha}$-regularity even for minimizers of the functional (1.3), while in case $p = 2$, $C^{1,\alpha}$-regularity implies C^∞-regularity. So we are concerned with p-harmonic maps which belong to the C^1_{loc}-class, for general p.

Several studies are given for 2-harmonic maps or harmonic maps. (See Eells and Lemaire [3], [4].) For these harmonic maps, there are Liouville type theorems, which states that a harmonic map u is constant under some conditions. A typical one of such conditions is the boundedness of u, where we say that u is bounded if its image is contained in a compact set. As a result with assumptions to images of maps, a Liouville type property is known when the image is enveloped by a convex function. (See Gordon [6] for $p=2$, L.-F.Cheung and P.-F.Leung [1] for general $p \geq 2$.) The finiteness of the energy is another typical condition; precisely speaking, a harmonic map u is constant if $E_2(u) < \infty$, when M is complete and noncompact with $\mathrm{Ric}_M \geq 0$ and $\mathrm{Sect}_N \leq 0$, where Ric_M denotes the Ricci curvature of M, and Sect_N denotes the sectional curvature of N. (See Schoen and Yau [8], Hildebrandt [7].) In this note we extend this result for general $p \geq 2$.

Theorem 1. *Let M be complete and noncompact. Assume $\mathrm{Ric}_M \geq 0$ and $\mathrm{Sect}_N \leq 0$. Let $u : M \to N$ is a p-harmonic map of C^1_{loc}-class such that $E_p(u) < \infty$. Then u is a constant map.*

In [9], Takeuchi proved Theorem 1, using Hildebrandt's argument [7], under the condition $E_{2p-2}(u) < \infty$ instead of $E_p(u) < \infty$. The exponent $2p-2$, however, is not compatible in our case, since $2p-2 \neq p$ when $p \neq 2$. Our proof of Theorem 1 has two steps; the first step (Section 3) for C^3_{loc}-maps and the second step (Section 4) in general case. The first step is based on a Bochner type formula (Section 2) and a standard argument of cutoff functions, and the second step depends on the approximation argument, which is used in [2] when M is an open set in a Euclidean space.

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2. Bochner type formula

In this section we give the following Bochner type formula.

Lemma 1. *Let $u : M \to N$ be a map of C^3_{loc}-class. Then the following equality holds:*

(a)
$$\begin{aligned}
&\|du\|^{p-1} \triangle \|du\|^{p-1} + \langle \|du\|^{p-2}du, (d^\nabla \delta^\nabla + \delta^\nabla d^\nabla)(\|du\|^{p-2}du)\rangle \\
&= 2(\|\nabla(\|du\|^{p-2}du)\|^2 - \|\nabla \|du\|^{p-1}\|^2) \\
&\quad + \|du\|^{2p-4} \sum_{j=1}^{m} \langle \mathrm{Ric}_M(du(e_j)), du(e_j)\rangle \\
&\quad - \|du\|^{2p-4} \sum_{i,j=1}^{m} \langle \mathrm{Riem}_N(du(e_i), du(e_j))du(e_j), du(e_i)\rangle,
\end{aligned}$$

hence

$$\text{(b)}\qquad \begin{aligned} &\|du\| \triangle \|du\|^{p-1} + \langle du, (d^\nabla \delta^\nabla + \delta^\nabla d^\nabla)(\|du\|^{p-2} du)\rangle \\ &\geq \|du\|^{p-2} \sum_{j=1}^{m} \langle \mathrm{Ric}_M(du(e_j)), du(e_j)\rangle \\ &\quad - \|du\|^{p-2} \sum_{i,j=1}^{m} \langle \mathrm{Riem}_N(du(e_i), du(e_j)) du(e_j), du(e_i)\rangle, \end{aligned}$$

where $\{e_j\}_{j=1}^{k}$ *is an orthonormal base of the tangent space of* M, *and* d^∇ [*resp.* δ^∇] *denotes the derivation with respect to* ∇ [*resp. the* L^2*-adjoint operator of* d^∇].

Therefore, when $\mathrm{Ric}_M \geq 0$ *and* $\mathrm{Sect}_N \leq 0$, *we have*

$$\text{(c)}\qquad \|du\| \triangle \|du\|^{p-1} + \langle du, (d^\nabla \delta^\nabla + \delta^\nabla d^\nabla)(\|du\|^{p-2} du)\rangle \geq 0.$$

Proof of Lemma 1. Using the relation between the rough Laplacian $\triangle$ and the Hodge-de Rham Laplacian $d^\nabla \delta^\nabla + \delta^\nabla d^\nabla$, we have

$$\text{(2.4)}\qquad \begin{aligned} \frac{1}{2} \triangle \|du\|^{2p-2} &= \frac{1}{2} \triangle \| \|du\|^{p-2} du\|^2 \\ &= \langle \|du\|^{p-2} du, \triangle(\|du\|^{p-2} du)\rangle + \|\nabla(\|du\|^{p-2} du)\|^2 \\ &= -\langle \|du\|^{p-2} du, (d^\nabla \delta^\nabla + \delta^\nabla d^\nabla)(\|du\|^{p-2} du)\rangle \\ &\quad + \|\nabla(\|du\|^{p-2} du)\|^2 + Q(u), \end{aligned}$$

where

$$\begin{aligned} Q(u) = {} & \|du\|^{2p-4} \sum_{i=1}^{n} \langle \mathrm{Ric}_M(du(e_i)), du(e_i)\rangle \\ & - \|du\|^{2p-4} \sum_{i,j=1}^{n} \langle \mathrm{Riem}_N(du(e_i), du(e_j)) du(e_j), du(e_i)\rangle. \end{aligned}$$

(cf. Eells and Lemaire [3, p.8, (2.20)] for $p = 2$; Note that for any harmonic map ($p = 2$), $d^\nabla(du) = \delta^\nabla(du) = 0$.) On the other hand, we see

$$\text{(2.5)}\qquad \begin{aligned} \frac{1}{2} \triangle \|du\|^{2p-2} &= \frac{1}{2} \triangle (\|du\|^{p-1})^2 \\ &= \|du\|^{p-1} \triangle \|du\|^{p-1} + \|\nabla \|du\|^{p-1}\|^2. \end{aligned}$$

Then from (2.4) and (2.5), we have (a). From (a), we have

$$\|du\|^{p-1} \triangle \|du\|^{p-1} + \langle \|du\|^{p-2}du, (d^\nabla \delta^\nabla + \delta^\nabla d^\nabla)(\|du\|^{p-2}du)\rangle \tag{2.6}$$
$$\geq \|du\|^{2p-4} \sum_{j=1}^{m} \langle \mathrm{Ric}_M(du(e_j)), du(e_j)\rangle$$
$$-\|du\|^{2p-4} \sum_{i,j=1}^{m} \langle \mathrm{Riem}_N(du(e_i), du(e_j))du(e_j), du(e_i)\rangle,$$

since

$$\|\nabla\|du\|^{p-1}\| = \|\nabla\| \, \|du\|^{p-2}du\| \, \| \leq \|\nabla(\|du\|^{p-2}du)\|.$$

On any point such that $du = 0$, the inequality (b) holds trivially. On the other points, we divide the both sides of (2.6) by $\|du\|^{p-2}$, and then we have the inequality (b). □

3. Proof of Theorem 1 for C^3_{loc}-maps

Throughout this paper, all positive constants $C_1, C_2, \ldots$ depend only on p, M, N if there is no special mention. Since u is a p-harmonic map,

$$\delta^\nabla(\|du\|^{p-2}du) = 0.$$

Then by Lemma 1 (c), we have

$$\|du\| \triangle \|du\|^{p-1} + \langle du, \delta^\nabla d^\nabla(\|du\|^{p-2}du)\rangle \geq 0. \tag{3.7}$$

since $\mathrm{Ric}_M \geq 0$, $\mathrm{Sect}_N \leq 0$. Take any point $x \in M$. Let η be a cutoff function satisfying that

$$\eta : \begin{cases} = 1 & \text{on} \quad B_\rho(x) \\ \in [0,1] & \text{on} \quad B_{2\rho}(x) - B_\rho(x) \\ = 0 & \text{on} \quad M - B_{2\rho}(x) \end{cases} \tag{3.8}$$

and that

$$\|\nabla\eta\|^2 \leq \frac{C_1}{\rho^2}. \tag{3.9}$$

Then from (3.7), we get

$$\int_M \|du\|\eta^2 \triangle \|du\|^{p-1} + \int_M \langle \eta^2 du, \delta^\nabla d^\nabla(\|du\|^{p-2}du)\rangle \geq 0. \tag{3.10}$$

We have

$$
\begin{aligned}
(3.11)\quad & \int_M \|du\|\eta^2 \triangle \|du\|^{p-1} \\
&= -\int_M \nabla(\|du\|\eta^2)\cdot\nabla\|du\|^{p-1} \\
&= -(p-1)\int_M \|du\|^{p-2}\eta^2\|\nabla\|du\|\,\|^2 - 2(p-1)\int_M \|du\|^{p-2}\eta\nabla\|du\|\cdot\nabla\eta \\
&= -\frac{4(p-1)}{p^2}\int_M \|\nabla\|du\|^{p/2}\|^2\eta^2 - \frac{4(p-1)}{p}\int_M \|du\|^{p/2}\eta\nabla\|du\|^{p/2}\cdot\nabla\eta \\
&\le -\frac{4(p-1)}{p^2}\int_M \|\nabla\|du\|^{p/2}\|^2\eta^2 + \varepsilon\int_M \|\nabla\|du\|^{p/2}\|^2\eta^2 + \frac{C_2}{\varepsilon}\int_M \|du\|^p\|\nabla\eta\|^2
\end{aligned}
$$

for any $\varepsilon > 0$. We see

$$
(3.12)\qquad \int_M \langle \eta^2 du, \delta^\nabla d^\nabla(\|du\|^{p-2}du)\rangle = \int_M \langle d^\nabla(\eta^2 du), d^\nabla(\|du\|^{p-2}du)\rangle.
$$

Since

$$
\|d^\nabla(\varphi du)\| \le C_3\|\nabla\varphi\|\,\|du\|,
$$

we have

$$
\begin{aligned}
(3.13)\qquad & \left|\langle d^\nabla(\eta^2 du), d^\nabla(\|du\|^{p-2}du)\rangle\right| \\
&\le \|d^\nabla(\eta^2 du)\|\,\|d^\nabla(\|du\|^{p-2}du)\| \\
&\le C_4\|\nabla\eta^2\|\,\|du\|\,\|\nabla\|du\|^{p-2}\|\,\|du\| \\
&= C_5\eta\|\nabla\eta\|\,\|du\|^{p-1}\|\nabla\|du\|\,\| \\
&= C_6\eta\|\nabla\eta\|\,\|du\|^{p/2}\|\nabla\|du\|^{p/2}\| \\
&\le \varepsilon\|\nabla\|du\|^{p/2}\|^2\eta^2 + \frac{C_7}{\varepsilon}\|du\|^p\|\nabla\eta\|^2.
\end{aligned}
$$

From (3.12) and (3.13), we get

$$
\begin{aligned}
(3.14)\qquad & \left|\int_M \langle \eta^2 du, \delta^\nabla d^\nabla(\|du\|^{p-2}du)\rangle\right| \\
&\le \varepsilon\int_M \|\nabla\|du\|^{p/2}\|^2\eta^2 + \frac{C_7}{\varepsilon}\int_M \|du\|^p\|\nabla\eta\|^2.
\end{aligned}
$$

Then from (3.10), (3.11) and (3.14),

$$
\left(\frac{4(p-1)}{p^2} - 2\varepsilon\right)\int_M \|\nabla\|du\|^{p/2}\|^2\eta^2 \le \frac{C_8}{\varepsilon}\int_M \|du\|^p\|\nabla\eta\|^2.
$$

Let $\varepsilon = \frac{p-1}{p^2}$, and then we obtain

$$\int_{B_\rho(x)} \|\nabla \|du\|^{p/2}\|^2 \leq C_9 \int_M \|du\|^p \|\nabla\eta\|^2 \leq \frac{C_{10}}{\rho^2}\int_M \|du\|^p. \tag{3.15}$$

Let ρ go to infinity, and then we see that $\nabla\|du\|^{p/2} \equiv 0$, i.e., $\|du\|$ is constant on M. Note that the volume of M is infinite, since $\mathrm{Ric}_M \geq 0$. Then by the condition $E_p(u) < \infty$, we conclude that the constant $\|du\|$ is zero. Therefore u is a constant map. □

4. Proof of Theorem 1

In this section we complete our proof of Theorem 1 using an approximation. We use the arguments in Duzaar and Fuchs [2]. We may assume $p > 2$, since Theorem 1 holds for $p = 2$. As mentioned in the introduction, we may assume that the target manifold N is a submanifold of a Euclidean space $\mathbb{R}^d$, and that u is a map into $\mathbb{R}^d$. Then we know

Proposition 1. *For any p-harmonic map u of C^1_{loc}-class, $\|du\|^{p/2-1}du$ belong to $W^{1,2}_{loc}(TM, \mathbb{R}^d)$.*

When M is a domain of a Euclidean space, Proposition 1 is Lemma 2.2 in Duzaar and Fuchs [2]. In the above general situation, Proposition 1 can be proved with slight modifications.

Using Proposition 1, we will complete our proof of Theorem 1. Let $M_+ := \{x \in M; \|du\|(x) \neq 0\}$. By Proposition 1, we see that du is of $W^{1,2}_{loc}$-class on M_+, since

$$\nabla du = \nabla(\|du\|^{p/2-1}du)\|du\|^{1-p/2} - \frac{p-2}{p}\nabla(\|du\|^{p/2})\|du\|^{-p/2}du,$$

hence

$$\begin{aligned}\|\nabla du\| &\leq \|du\|^{1-p/2}\|\nabla(\|du\|^{p/2-1}du)\| + \frac{p-2}{p}\|du\|^{1-p/2}\|\nabla(\|du\|^{p/2})\| \\ &\leq \left(1 + \frac{p-2}{p}\right)\|du\|^{1-p/2}\|\nabla(\|du\|^{p/2-1}du)\|.\end{aligned}$$

Then we can find an approximating sequence $\{u_k\}_{j=1}^\infty \subset C^\infty_{loc}(M, \mathbb{R}^d)$ such that as k goes to infinity,

(a) u_k converges to u in $C^1_{loc}(M)$,

(b) u_k converges to u weakly in $W^{1,p}_{loc}(M)$, and

(c) u_k converges to u weakly in $W^{2,2}_{loc}(M_+)$.

By Lemma 1, we have

$$\|du_k\| \triangle \|du_k\|^{p-1} + \frac{p}{2(p-1)}\langle du_k, (d^\nabla\delta^\nabla + \delta^\nabla d^\nabla)(\|du_k\|^{p-2}du_k)\rangle \geq 0,$$

since $\mathrm{Ric}_M \geq 0$, $\mathrm{Sect}_N \leq 0$. Let η be a cutoff function on M satisfying (3.8) and (3.9), and let

$$\varphi_\varepsilon(x) := \begin{cases} \dfrac{\|du\|(x)}{\max\{\|du\|(x), \varepsilon\}} & \text{on } M_+ \\ 0 & \text{on } M - M_+. \end{cases}$$

Then $\varphi_\varepsilon \in L_0^{1,2}(M_+)$, where $L_0^{1,2}(M_+)$ is the completion of $C_0^\infty(M_+)$, and $\varphi_\varepsilon \to 1$, $\nabla\varphi_\varepsilon \to 0$ in $L_0^{1,2}(M_+)$. Using the function $\varphi_\varepsilon\eta^2$ instead of η^2, we apply the estimates (3.11), (3.14) for smooth maps u_k. Then we have, instead of (3.15),

$$\begin{aligned}(4.16)\qquad &\int_{M_+} \|\nabla\|du_k\|^{p/2}\|^2\varphi_\varepsilon\eta^2 + \int_{M_+} \|du_k\|^{p/2}\eta^2\nabla\|du_k\|^{p/2}\cdot\nabla\varphi_\varepsilon \\ &\leq \frac{C_{11}}{\rho^2}\int_{M_+}\|du_k\|^p\varphi_\varepsilon + C_{12}\int_{M_+}\langle\varphi_\varepsilon\eta^2 du_k, d^\nabla\delta^\nabla(\|du_k\|^{p-2}du_k)\rangle \\ &\leq \frac{C_{11}}{\rho^2}\int_{M_+}\|du_k\|^p\varphi_\varepsilon + C_{12}\int_{M_+}\delta^\nabla(\varphi_\varepsilon\eta^2 du_k)\delta^\nabla(\|du_k\|^{p-2}du_k).\end{aligned}$$

Since u_k converge to u in $C^1_{loc}(M_+) \cap W^{2,2}_{loc}(M_+)$,

$$\int_{M_+}\delta^\nabla(\varphi_\varepsilon\eta^2 du_k)\delta^\nabla(\|du_k\|^{p-2}du_k) \to \int_{M_+}\delta^\nabla(\varphi_\varepsilon\eta^2 du)\delta^\nabla(\|du\|^{p-2}du) = 0.$$

as k goes to infinity, where the last equality follows from the fact that u satisfies the p-harmonic map equation $\delta^\nabla(\|du\|^{p-2}du) = 0$. Therefore, let k goes to infinity in (4.16), and then we have

$$\int_{M_+}\|\nabla\|du\|^{p/2}\|^2\varphi_\varepsilon\eta^2 + \int_{M_+}\|du\|^{p/2}\eta^2\nabla\|du\|^{p/2}\cdot\nabla\varphi_\varepsilon \leq \frac{C_{11}}{\rho^2}\int_{M_+}\|du\|^p\varphi_\varepsilon.$$

Let ε go to zero, and then we get

$$(4.17)\qquad \int_{M_+}\|\nabla\|du\|^{p/2}\|^2\eta^2 \leq \frac{C_{11}}{\rho^2}\int_{M_+}\|du\|^p$$

for any $\eta \in C_0^1(M)$ satisfing (3.8) and (3.9). Here we have used the facts that $\nabla\varphi_\varepsilon$ goes to a measure-valued tensor μ whose support is in ∂M_+, and that $\|du\|$ vanishes there. Let ρ go to infinity in (4.17), and then we have $du = 0$ on M_+, i.e., u is constant on M_+ Hence u is a constant map on M. □

5. A remark.

When M is compact, we have the following result, which is an extension of facts in harmonic map case ($p = 2$). (See Eells and Sampson [5].)

Theorem 2. *Let M be compact ($\partial M = \emptyset$). Let $u : M \to N$ be a p-harmonic map of C^1-class.*

(a) *Assume* $\mathrm{Ric}_M \geq 0$ *and* $\mathrm{Sect}_N \leq 0$. *Then u is totally geodesic.*
(b) *In addition to* (a), *if* $\mathrm{Ric}_M > 0$ *somewhere, then u is a constant map.*
(c) *In addition to* (a), *if* $\mathrm{Sect}_N < 0$, *then u is a constant map, or u maps onto a closed geodesic in N.*

Proof. We take an approximating sequence $\{u_k\}_{k=1}^\infty$ such as that in Section 4. Take the function φ_ε in Section 4. From Lemma 1 (b),

$$\int_{M_+} \|du_k\|\varphi_\varepsilon \triangle \|du_k\|^{p-1} + \int_{M_+} \langle \varphi_\varepsilon du_k, (d^\nabla \delta^\nabla + \delta^\nabla d^\nabla)(\|du_k\|^{p-2} du_k)\rangle \geq 0.$$

Using integration by parts, we have

$$\begin{aligned}
&\int_{M_+} \|du_k\|\varphi_\varepsilon \triangle \|du_k\|^{p-1} \\
&= -\int_{M_+} \varphi_\varepsilon \nabla\|du_k\| \cdot \nabla\|du_k\|^{p-1} - \int_{M_+} \|du_k\|\nabla\|du_k\|^{p-1}\cdot\nabla\varphi_\varepsilon \\
&= -\frac{4(p-1)}{p^2}\int_{M_+} \|\nabla\|du_k\|^{p/2}\|^2\varphi_\varepsilon - \int_{M_+} \|du_k\|\nabla\|du_k\|^{p-1}\cdot\nabla\varphi_\varepsilon.
\end{aligned}$$

Then from the above inequality, we get

$$\begin{aligned}
&\frac{4(p-1)}{p^2}\int_{M_+} \|\nabla\|du_k\|^{p/2}\|^2\varphi_\varepsilon \\
&\leq \int_{M_+} \langle \delta^\nabla(\varphi_\varepsilon du_k), \delta^\nabla(\|du_k\|^{p-2}du_k)\rangle \\
&\quad + \int_{M_+} \langle d^\nabla(\varphi_\varepsilon du_k), d^\nabla(\|du_k\|^{p-2}du_k)\rangle \\
&= \int_{M_+} \langle \delta^\nabla(\varphi_\varepsilon du_k), \delta^\nabla(\|du_k\|^{p-2}du_k)\rangle
\end{aligned}$$

$$+\int_{M_+} \|\nabla\varphi_\varepsilon\| \, \|du_k\| \, \|d^\nabla(\|du_k\|^{p-2}du_k)\|.$$

We have used the fact $\|d^\nabla(\varphi_\varepsilon du_k)\| \le C_3 \, \|\nabla\varphi_\varepsilon\| \, \|du_k\|$. Let $k \to \infty$ and let $\varepsilon \to 0$, then we have

$$\frac{4(p-1)}{p^2}\int_{M_+} \|\nabla\|du\|^{p/2}\|^2 \le 0.$$

Therefore $\|du\|$ is constant on M_+, hence on M, since $\|du\|$ is continuous. From Lemma 1 (a), we have

$$\begin{aligned}
&-\int_{M_+} \|\nabla(\|du_k\|^{p-2}du_k)\|^2\varphi_\varepsilon + \int_{M_+} \|du_k\|^{p/2}\nabla\|du_k\|^{p/2}\cdot\nabla\varphi_\varepsilon \\
&\quad + \int_{M_+} \langle \|du_k\|^{p-2}\varphi_\varepsilon du_k, (d^\nabla\delta^\nabla + \delta^\nabla d^\nabla)(\|du_k\|^{p-2}du_k)\rangle \\
&\ge 0.
\end{aligned}$$

We have applied the integration by parts and used the assumption $\mathrm{Ric}_M \ge 0$ and $\mathrm{Sect}_N \le 0$ and the fact that

$$\|\nabla(\|du_k\|^{p-2}du_k)\|^2 \ge \ \|\nabla\| \, \|du_k\|^{p-2}du_k\| \, \|^2 \ge \ \|\nabla\|du_k\|^{p-1}\|^2.$$

Let $k \to \infty$, and let $\varepsilon \to 0$, then we get

$$0 \ge \ 2\int_{M_+} \|\nabla(\|du\|^{p-2}du)\|^2$$

on M_+, since $\|du\|$ is constant. Hence $\nabla du = 0$, i.e., u is totally geodesic on M_+. Since $\|du\|$ is constant, u is a harmonic map; u is totally geodesic on M. We have (a).

We know, by the proof of (a),

$$\begin{aligned}
&\|du\| \equiv \text{Const.} =: C_0 \quad \text{on } M \\
&\nabla du = 0 \qquad\qquad\quad\ \ \text{on } M_+
\end{aligned}$$

Then from Lemma 1 (a) again, we have

$$\begin{aligned}
(5.18) \qquad 0 &\le C_0^{2p-4}\sum_{j=1}^{m}\langle \mathrm{Ric}_M(du(e_j)), du(e_j)\rangle \\
&= C_0^{2p-4}\sum_{i,j=1}^{m}\langle \mathrm{Riem}_N(du(e_j), du(e_j))du(e_j), du(e_i)\rangle \\
&\le 0
\end{aligned}$$

on M_+. If $\mathrm{Ric}_M > 0$ somewhere, the inequality (5.18) implies $C_0 = 0$, or $du = 0$ at this point. Then $C_0 = 0$, and we have (b). If $\mathrm{Sect}_N < 0$ at a point $x^* \in M$, then $C_0 = 0$ or $\dim(Image(du(x^*))) \leq 1$. If $\dim(Image(du(x^*))) = 0$, then $\|du\|(x) = C_0 = \|du\|(x^*) = 0$ for any $x \in M$, i.e. u is a constant map. If $\dim(Image(du(x))) = 1$, we have (c) since u is totally geodesic. □

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