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1. Introduction

We consider the configuration space $M'_n$ of equilateral polygon linkages with $n$ ($n \geq 3$) vertices, each edge having length 1 in the Euclidean plane $\mathbb{R}^2$ modulo isometry group. More precisely, let $C_n$ be

\begin{equation}
C_n = \{(u_1, \ldots, u_n) \in (\mathbb{R}^2)^n : |u_{i+1} - u_i| = 1 \ (1 \leq i \leq n-1), \ |u_1 - u_n| = 1\}.
\end{equation}

Note that $\text{Iso}(\mathbb{R}^2)$ (= the isometry group of $\mathbb{R}^2$, i.e., a semidirect product of $\mathbb{R}^2$ with $O(2)$), naturally acts on $C_n$. We define $M'_n$ by

\begin{equation}
M'_n = C_n/\text{Iso}(\mathbb{R}^2).
\end{equation}

We remark that $M'_n$ has the following description: We set $M_n = C_n/\text{Iso}^+(\mathbb{R}^2)$, where $\text{Iso}^+(\mathbb{R}^2)$ denotes the orientation preserving isometry group of $\mathbb{R}^2$, i.e., a semidirect product of $\mathbb{R}^2$ with $SO(2)$. Then we can write $M_n$ as

\begin{equation}
M_n = \{(u_1, \ldots, u_n) \in C_n : u_1 = (\frac{1}{2}, 0) \text{ and } u_2 = (-\frac{1}{2}, 0)\}.
\end{equation}

$M_n$ admits an involution $\sigma = \text{Iso}(\mathbb{R}^2)/\text{Iso}^+(\mathbb{R}^2)$ such that $M'_n = M_n/\sigma$. Under the identification of (1.3), $\sigma$ is given by

\begin{equation}
\sigma(u_1, \ldots, u_n) = (\bar{u}_1, \ldots, \bar{u}_n),
\end{equation}

where $\bar{u}_i = (x_i, -y_i)$ when $u_i = (x_i, y_i)$.

Many topological properties of $M_n$ are already known: First we know explicit topological type of $M_n$ ($n \leq 5$) [3],[4],[8]. Next we have the results on the smoothness of $M_n$ [5],[7],[8]. Finally $H_*(M_n; \mathbb{Z})$ are determined in [6],[7] (cf. Theorem 2.1). In particular, the natural inclusion $i_n : M_n \hookrightarrow (S^1)^{n-1}$ (cf. (1.6)) induces isomorphisms of homology groups up to a certain dimension (cf. Proposition 2.2).
On the other hand, concerning $M'_n$, what we know already are the following: First we know the following examples.

**EXAMPLES 1.5.** $M'_3 = \{1\text{-point}\}$, $M'_4 = S^1$ and $M'_5 = \#_5 \mathbb{R}P^2$, the five-times connected sum of $\mathbb{R}P^2$.

Next some assertions on the smoothness of $M'_n$ are proved in [5]. However, we have few information on $H_*(M'_n; \mathbb{Z})$, although we know $\chi(M'_n)$, the Euler characteristic of $M'_n$ [5] (cf. Proposition 3.4).

The purposes of this paper are as follows.

(i) We prove assertions on the smoothness of $M'_n$.

(ii) We determine $H_*(M'_n; \mathbb{Z}_p)$, where $p$ is an odd prime, and $H_*(M'_n; \mathbb{Q})$.

In the following (iii)-(v), we assume $n$ to be odd, and set $n = 2m + 1$. Then by the results of (i) and (ii), $M'_{2m+1}$ is a non-orientable manifold of dimension $2m - 2$.

(iii) Find a space $V_{2m}$ and an inclusion $i'_{2m+1} : M'_{2m+1} \hookrightarrow V_{2m}$ so that $i'_{2m+1}$ induces isomorphisms of homotopy groups up to a certain dimension.

(iv) As $V_{2m}$ is a natural space, we determine $H_*(V_{2m}; \mathbb{Z})$ completely. Then in particular we know $H_*(M'_{2m+1}; \mathbb{Z})$ up to some dimension by the result of (iii).

As we will see in Remark 1.9, knowing $H_*(V_{2m}; \mathbb{Z})$ is equivalent to knowing $H_*(S^{m} / \mathbb{Z}; \mathbb{Z})$.

(v) Finally we determine $H_*(M'_{2m+1}; \mathbb{Z})$ except the possibility of higher two-torsions in $H_{m-1}(M'_{2m+1}; \mathbb{Z})$ when $m$ is even.

Now we state our results. Concerning (i), we have the following:

**Theorem A.** (a) $M'_{2m+1}$ is a manifold of dimension $2m - 2$.

(b) $M'_{2m}$ is a manifold of dimension $2m - 3$ with singular points. $(u_1, \ldots, u_{2m}) \in M'_{2m}$ is a singular point iff all of $u_i$ lie on the $x$-axis, i.e., the line determined by $u_1$ and $u_2$ (cf. (1.3)). Moreover every singular point of $M'_{2m}$ is a cone-like singularity and has a neighborhood as $C(S^{m-2} \times \mathbb{Z}_2 S^{m-2})$, where $C$ denotes cone and action of $\mathbb{Z}_2$ on both factors is generated by the antipodal map.

Concerning (ii), first we prove the following:

**Theorem B.** $H_*(M'_n; \mathbb{Z})$ are odd-torsion free.

Thus in order to know $H_*(M'_n; \mathbb{Z}_p)$, we need to know $H_*(M'_n; \mathbb{Q})$, which is given by the following:
Theorem C. The Poincaré polynomials \( P_{\mathbb{Q}}(M_n') = \sum \dim_{\mathbb{Q}} H_\lambda(M_n'; \mathbb{Q}) t^\lambda \) are given by

\[
\begin{align*}
(\text{a}) \quad P_{\mathbb{Q}}(M_{2m+1}') &= \sum_{0 \leq 2a \leq m-2} \binom{2m}{2a} t^{2a} + \binom{2m}{m-1} t^{m-1} \\
&\quad + \sum_{m \leq 2b+1 \leq 2m-3} \binom{2m}{2b+3} t^{2b+1},
\end{align*}
\]

\[
\begin{align*}
(\text{b}) \quad P_{\mathbb{Q}}(M_{4l+2}') &= \sum_{0 \leq 2a \leq 2l-2} \binom{4l-1}{2a} t^{2a} + \binom{4l-1}{2l-2} t^{2l-1} \\
&\quad + \sum_{2l+1 \leq 2b+1 \leq 4l-3} \binom{4l-1}{2b+3} t^{2b+1},
\end{align*}
\]

\[
\begin{align*}
P_{\mathbb{Q}}(M_{4l+2}') &= \sum_{0 \leq 2a \leq 2l-2} \binom{4l+1}{2a} t^{2a} + \binom{4l+1}{2l+1} t^{2l} \\
&\quad + \sum_{2l+1 \leq 2b+1 \leq 4l-1} \binom{4l+1}{2b+3} t^{2b+1},
\end{align*}
\]

where \( \binom{n}{k} \) denotes the binomial coefficient.

Next we go to (iii). By setting \( z_i = u_{i+2} - u_{i+1} \) \((1 \leq i \leq n - 2)\), \( z_{n-1} = u_1 - u_n \), and identifying \( \mathbb{R}^2 \) with \( \mathbb{C} \), we can write \( M_n \) \((n \geq 3)\) as

\[
M_n \cong \{(z_1, \ldots, z_{n-1}) \in (S^1)^{n-1} : z_1 + \cdots + z_{n-1} - 1 = 0\}.
\]

Let \( i_n : M_n \hookrightarrow (S^1)^{n-1} \) be the inclusion.

As we have mentioned, \((S^1)^{n-1}\) approximates the topology of \( M_n \) up to some dimension (cf. Proposition 2.2). However, for an odd \( n = 2m+1 \), our low-dimensional computations lead us to give up the hope that \((S^1)^{2m}/\sigma\) might approximate \( M_{2m+1}' = M_{2m+1}/\sigma \), where \( \sigma \) acts on \((S^1)^{2m}\) in the same way as in (1.4). The essential reason for this is that the action of \( \sigma \) on \((S^1)^{2m}\) is not free, although on \( M_{2m+1} \) it is.

Thus we define \( V_{2m} \) by

\[
V_{2m} = \{(S^1)^{2m} - \Sigma_{2m}\} / \sigma,
\]

where we set

\[
\Sigma_{2m} = \{(z_1, \ldots, z_{2m}) \in (S^1)^{2m} : z_i = \pm 1 \ (1 \leq i \leq 2m)\}.
\]

Let \( i_{2m+1}' : M_{2m+1}' \hookrightarrow V_{2m} \) be the inclusion. Then we have the following map of covering
spaces:

\[ \begin{array}{ccc}
\mathbb{Z}_2 & \longrightarrow & \mathbb{Z}_2 \\
\downarrow & & \downarrow \\
M_{2m+1} & \xrightarrow{\iota_{2m+1}} & (S^1)^{2m} - \Sigma_{2m} \\
\downarrow & & \downarrow \\
M'_{2m+1} & \xrightarrow{\iota'_{2m+1}} & V_{2m}.
\end{array} \tag{1.8} \]

Note that \((S^1)^{2m} - \Sigma_{2m}\) is a maximal subspace of \((S^1)^{2m}\) on which \(\sigma\) acts freely. Thus it is natural to consider the topology of \(V_{2m}\).

Now concerning the relation between \(M'_{2m+1}\) and \(V_{2m}\), we have the following theorem.

**Theorem D.** \((\iota'_{2m+1})_* : \pi_q(M'_{2m+1}) \to \pi_q(V_{2m}) \) are isomorphisms for \(q \leq m - 2\), and an epimorphism for \(q = m - 1\).

Concerning (iv), we have the following:

**Theorem E.** \(H_*(V_{2m}; \mathbb{Z})\) is given by

\[
H_q(V_{2m}; \mathbb{Z}) = \begin{cases} 
\bigoplus_{q \leq (2m)} \mathbb{Z} & q \text{ : even } \leq 2m - 2 \\
\bigoplus_{(2m) \leq q \leq (2m)} \mathbb{Z}_2 & q \text{ : odd } \leq 2m - 3 \\
\bigoplus_{2m-1} \mathbb{Z} & q = 2m - 1 \\
0 & \text{otherwise},
\end{cases}
\]

where \(\bigoplus_{(2m)} \mathbb{Z}\) denotes the \((2m)\)-times direct sum of \(\mathbb{Z}\).

Note that Theorems D and E give \(H_q(M'_{2m+1}; \mathbb{Z})\) for \(q \leq m - 2\).

**Remark 1.9.** By the Poincaré-Lefschetz duality \(H^q((S^1)^{2m}/\sigma, \Sigma_{2m}; \mathbb{Z}) \cong H_{2m-q}(V_{2m}; \mathbb{Z})\), knowing \(H_*(V_{2m}; \mathbb{Z})\) is equivalent to knowing \(H_*)((S^1)^{2m}/\sigma; \mathbb{Z})\).

Concerning (v), we have the following:
Theorem F. (a) For an odd $m$, we have

$$H_q(M_{2m+1}; Z) = \begin{cases} \bigoplus \mathbb{Z} & q: \text{even } \leq m - 1 \\ \bigoplus \mathbb{Z}_2 & q: \text{odd } \leq m - 2 \\ \mathbb{Z} \bigoplus \bigoplus \mathbb{Z}_2 & q: \text{odd } \geq m - 1 \\ 0 & \text{otherwise.} \end{cases}$$

(b) For an even $m$, we have

$$H_q(M'_{2m+1}; Z) = \begin{cases} \bigoplus \mathbb{Z} & q: \text{even } \leq m - 2 \\ \bigoplus \mathbb{Z}_2 & q: \text{odd } \leq m - 3 \\ \mathbb{Z} \bigoplus Tor_{m-1} & q = m - 1 \\ \mathbb{Z} \bigoplus \bigoplus \mathbb{Z}_2 & q: \text{odd } \geq m + 1 \\ 0 & \text{otherwise,} \end{cases}$$

where $Tor_{m-1}$, the torsion submodule of $H_{m-1}(M_{2m+1}; Z)$, satisfies that

$$\dim_{\mathbb{Z}_2} Tor_{m-1} \otimes \mathbb{Z}_2 = \sum_{i \leq m-2} \binom{2m}{i}.$$ 

Thus, in particular, $H_{\text{even}}(M'_{2m+1}; Z)$ are torsion free for all $m$.

Remark 1.10. (a) By Theorems D, E and F, we see that $(i'_{2m+1})_* : H_{m-1}(M'_{2m+1}; Z) \rightarrow H_{m-1}(V_{2m}; Z)$ is an isomorphism when $m$ is odd, but not an isomorphism when $m$ is even.

(b) In order to prove Theorem F, we first determine $H_*(M'_{2m+1}; Z)$, which is given in Proposition 5.1. In particular, we see that $(i'_{2m+1})_* : H_{m-1}(M'_{2m+1}; Z) \rightarrow H_{m-1}(V_{2m}; Z)$ is an isomorphism for all $m$ (cf. Remark 5.2).

This paper is organized as follows. In §2 we recall the results of [7], then prove Theorems A, B and D. In §3 we prove Theorem C. In §4 we prove Theorem E, and in §5 we prove Theorem F.

2. Proofs of Theorems A, B and D

In [7], the following theorem is proved.
Theorem 2.1. $H_*(M_n; \mathbb{Z})$ are free $\mathbb{Z}$-modules and the Poincaré polynomials $PS(M_n) = \sum \lambda \text{rank}H_\lambda(M_n; \mathbb{Z})t^\lambda$ are given by

$$PS(M_{2m+1}) = \sum_{\lambda=0}^{m-2} \binom{2m+1}{\lambda} t^\lambda + 2 \binom{2m}{m-1} t^{m-1} + \sum_{\lambda=m}^{2m-2} \binom{2m}{\lambda+2} t^\lambda,$$

$$PS(M_{2m}) = \sum_{\lambda=0}^{m-2} \binom{2m-1}{\lambda} t^\lambda + \binom{2m}{m-1} t^{m-1} + \sum_{\lambda=m}^{2m-3} \binom{2m-1}{\lambda+2} t^\lambda.$$

The essential facts to prove Theorem 2.1 are the following three propositions.

Proposition 2.2. (i) $(z_{2m+1})_* : \pi_q(M_{2m+1}) \to \pi_q((S^1)^{2m})$ are isomorphisms for $q \leq m - 2$, and an epimorphism for $q = m - 1$.

(ii) $(z_m)_* : H_q(M_{2m}; \mathbb{Z}) \to H_q((S^1)^{2m-1}; \mathbb{Z})$ are isomorphisms for $q \leq m - 2$, and an epimorphism for $q = m - 1$.

Proposition 2.3. (i) $M_{2m+1}$ is an orientable manifold of dimension $2m - 2$. Thus the Poincaré duality homomorphisms $\cap[M_{2m+1}] : H^q(M_{2m+1}; \mathbb{Z}) \to H_{2m-2-q}(M_{2m+1}; \mathbb{Z})$ are isomorphisms for all $q$, where $[M_{2m+1}] \in H_{2m-2}(M_{2m+1}; \mathbb{Z})$ is a fundamental class.

(ii) $M_{2m}$ is a manifold of dimension $2m - 3$ with singular points. $(u_1, \ldots, u_{2m}) \in M_{2m}$ is a singular point iff all of $u_i$ lie on the $x$-axis. Moreover every singular point of $M_{2m}$ is a cone-like singularity and has a neighborhood as $C(S^{m-2} \times S^{m-2})$. Thus the Poincaré duality homomorphisms $\cap[M_{2m}] : H^q(M_{2m}; \mathbb{Z}) \to H_{2m-3-q}(M_{2m}; \mathbb{Z})$ are isomorphisms for $q \leq m - 3$ or $q \geq m$, an epimorphism for $q = m - 1$, and a monomorphism for $q = m - 2$.

Proposition 2.4. (i) $\chi(M_{2m+1}) = (-1)^{m+1}(\binom{2m}{m})$.

(ii) $\chi(M_{2m}) = (-1)^{m+1}(\binom{2m-1}{m})$.

Remark 2.5. In order to prove Theorem 2.1, the homological assertion is sufficient for Proposition 2.2 (i). But actually we can prove the homotopical assertion.

Proof of Theorem A. Since $\sigma$ acts freely on $M_{2m+1}$, and $M'_{2m} (= \text{the fixed point set of the involution})$ equals to the set of singular points in $M_{2m}$, all of the assertions except the type of the singular points of $M'_{2m}$ are deduced from Proposition 2.3.

Let $(z_1, \ldots, z_{2m-1})$ be a singular point of $M_{2m}$ in the identification of (1.6). By Proposition 2.3, we must have $z_i = \pm 1$ ($1 \leq i \leq 2m - 1$). As the symmetric group on ($2m - 1$)-letters acts on $M_{2m}$, we can assume that $z_i = 1$ ($1 \leq i \leq m$) and $z_i = -1$ ($m + 1 \leq i \leq 2m - 1$). A neighborhood of $(z_1, \ldots, z_{2m-1})$ in $(S^1)^{2m-1}$ is written...
by

$$\{(\sqrt{1 - y_1^2}/y_1, \ldots, \sqrt{1 - y_m^2}/y_m, -\sqrt{1 - y_{m+1}^2}/y_{m+1}, \ldots, -\sqrt{1 - y_{2m-1}^2}/y_{2m-1}) :$$

$$-\epsilon \leq y_i \leq \epsilon \ (1 \leq i \leq 2m - 1)\},$$

where $\epsilon > 0$ is a fixed small number. As $\epsilon$ is small, it is easy to see that we can write this neighborhood as

$$\left\{ \left( \begin{array}{c} (1 - \frac{1}{2}y_1^2)/y_1 \\ \vdots \\ (1 - \frac{1}{2}y_m^2)/y_m \\ -1 + \frac{1}{2}y_{m+1}^2/y_{m+1} \\ \vdots \\ -1 + \frac{1}{2}y_{2m-1}^2/y_{2m-1} \end{array} \right) :$$

$$-\epsilon \leq y_i \leq \epsilon \ (1 \leq i \leq 2m - 1)\right\}.$$

Thus a neighborhood of a singular point in $M_{2m}$ is written as a subspace of $\mathbb{R}^{2m-1}$ defined by two equations

$$(2.6) \quad \left\{ \begin{array}{l}
y_1^2 + \cdots + y_m^2 - y_{m+1}^2 - \cdots - y_{2m-1}^2 = 0 \\
y_1 + \cdots + y_m + y_{m+1} + \cdots + y_{2m-1} = 0.
\end{array} \right.$$  

By a linear transformation of parameters, we can write the quadratic form of (2.6), i.e.,

$$y_1^2 + \cdots + y_{m-1}^2 + (y_1 + \cdots + y_{m-1} + y_{m+1} + \cdots + y_{2m-1})^2 - y_{m+1}^2 - \cdots - y_{2m-1}^2,$$

as

$$w_1^2 + \cdots + w_{m-1}^2 - w_m^2 - \cdots - w_{2m-2}^2.$$  

Thus a singular point of $M_{2m}$ has a neighborhood $C\{(w_1, \ldots, w_{m-1}, w_m, \ldots, w_{2m-2}) : w_1^2 + \cdots + w_{m-1}^2 = 1, w_m^2 + \cdots + w_{2m-2}^2 = 1\}$, which is homeomorphic to $C(S^{m-2} \times S^{m-2}).$  

Now it is clear that $\sigma w_i = -w_i.$ Hence a singular point of $M'_{2m}$ has a neighborhood $C(S^{m-2} \times \mathbb{Z}_2, S^{m-2})$, where $\sigma(\zeta_1, \zeta_2) = (-\zeta_1, -\zeta_2)$ ($\zeta_1, \zeta_2 \in S^{m-2}).$ \[\square\]

Proof of Theorem B. For $F = \mathbb{Z}_p$ ($p$ : an odd prime) or $\mathbb{Q}$, we have that $H_*(M'_n; F) \cong H_*(M_n; F)^{\sigma}$ (= the fixed point set of $H_*(M_n; F)$ under the $\sigma$-action) (see for example [2]). As $H_*(M_n; \mathbb{Z})$ are free modules by Theorem 2.1, we have that $\dim_{\mathbb{Z}_p} H_q(M'_n; \mathbb{Z}_p) = \dim_{\mathbb{Q}} H_q(M'_n; \mathbb{Q})$. Hence Theorem B follows. \[\square\]
Proof of Theorem D. Let \( j_{2m} : (S^1)^{2m} - \Sigma_{2m} \hookrightarrow (S^1)^{2m} \) be the inclusion. Since \( \Sigma_{2m} \) is a discrete set, the general position argument shows that \( (j_{2m})_* : \pi_q((S^1)^{2m} - \Sigma_{2m}) \to \pi_q((S^1)^{2m}) \) are isomorphisms for \( q \leq 2m - 2 \). Then Proposition 2.2 (i) shows that \( (i_{2m+1})_* : \pi_q(M_{2m+1}) \to \pi_q((S^1)^{2m} - \Sigma_{2m}) \) are isomorphisms for \( q \leq m - 2 \) and an epimorphism for \( q = m - 1 \), where \( i_{2m+1} : M_{2m+1} \hookrightarrow (S^1)^{2m} - \Sigma_{2m} \) is the inclusion.

By comparing the homotopy exact sequences of two covering spaces of (1.8), we see that \((i'_{2m+1})_* : \pi_q(M'_{2m+1}) \to \pi_q(V_{2m}) \) are isomorphisms for \( q \leq m - 2 \) and an epimorphism for \( q = m - 1 \), where \( i'_{2m+1} : M'_{2m+1} \hookrightarrow V_{2m} \) is the map induced from the \( \sigma \)-equivariant inclusion \( i_{2m+1} : M_{2m+1} \hookrightarrow (S^1)^{2m} - \Sigma_{2m} \).

This completes the proof of Theorem D. \( \square \)

3. Proof of Theorem C

Let \( i_n : M_n \hookrightarrow (S^1)^{n-1} \) be the inclusion. Note that \( i_n \) is a \( \sigma \)-equivariant map. Hence \((i_n)_* : H_*(M_n; \mathbb{Q}) \to H_*((S^1)^{n-1}; \mathbb{Q}) \) is also a \( \sigma \)-equivariant homomorphism. Since \( H_*(M'_n; \mathbb{Q}) = H_*(M_n; \mathbb{Q})^\sigma \), Proposition 2.2 tells us the following:

**Proposition 3.1.**

(i) For \( q \leq m - 2 \), we have

\[
H_q(M'_{2m+1}; \mathbb{Q}) = \begin{cases} 
\bigoplus \binom{2m}{q} \mathbb{Q} & q \text{ : even} \\
0 & q \text{ : odd.}
\end{cases}
\]

(ii) For \( q \leq m - 2 \), we have

\[
H_q(M'_{2m}; \mathbb{Q}) = \begin{cases} 
\bigoplus \binom{2m-1}{q} \mathbb{Q} & q \text{ : even} \\
0 & q \text{ : odd.}
\end{cases}
\]

We assume the truth of the following Lemma for the moment. Let \([M_n] \in H_{n-3}(M_n; \mathbb{Q})\) be the fundamental class.

**Lemma 3.2.** \( \sigma_*[M_n] = (-1)^n[M_n] \).

Then we have the following:

**Proposition 3.3.**

(i) For \( q \geq m \), we have

\[
H_q(M'_{2m+1}; \mathbb{Q}) = \begin{cases} 
0 & q \text{ : even} \\
\bigoplus \binom{2m}{q+2} \mathbb{Q} & q \text{ : odd.}
\end{cases}
\]
(ii) For \( q \geq m \), we have

\[
H_q(M_{2m}; \mathbb{Q}) = \begin{cases} 
0 & q : \text{even} \\
\bigoplus_{\binom{2m-1}{q+1}} & q : \text{odd}.
\end{cases}
\]

Proof of Proposition 3.3. Take an element \( \alpha \in H_q(M_{2m+1}; \mathbb{Q}) \) \((q \geq m)\). By Proposition 2.3, there is an element \( f \in H^{2m-2-q}(M_{2m+1}; \mathbb{Q}) \) such that \( \alpha = f \cap [M_{2m+1}] \). As \( \sigma_*(f \cap [M_{2m+1}]) = \sigma*f \cap \sigma_*[M_{2m+1}] = -\sigma*f \cap [M_{2m+1}] \), we have that

\[
H_q(M_{2m+1}; \mathbb{Q})^\sigma = \{ f \in H^{2m-2-q}(M_{2m+1}; \mathbb{Q}) : \sigma*f = -f \}.
\]

Now (i) follows from Proposition 3.1. (ii) can be proved similarly.

Now in order to determine \( H_* (M'_n; \mathbb{Q}) \), we need to know only \( H_{m-1} (M'_{2m+1}; \mathbb{Q}) \) and \( H_{m-1} (M'_{2m}; \mathbb{Q}) \), which are determined if we know \( \chi(M'_n) \).

**Proposition 3.4 ([5]).**

(i) \( \chi(M'_{2m+1}) = (-1)^{m+1} \binom{2m-1}{m} \).

(ii) \( \chi(M'_{2m}) = \begin{cases} 
0 & m : \text{even} \\
\binom{2m-1}{m} & m : \text{odd}.
\end{cases} \)

Proof. By a general formula of an involution (see for example [1]), we have \( \chi(M)_n + \chi(M'_n) = 2\chi(M'_n) \). Then the result follows from Proposition 2.4.

Proof of Lemma 3.2. First we treat the case of \( n = 2m + 1 \). We define a volume element \( \omega \) of \( M_{2m+1} \) as follows. Fix \((z_1, \ldots, z_{2m}) \in M_{2m+1}\) in the identification of (1.6). It is easy to see that the tangent space \( T_{(z_1, \ldots, z_{2m})} M_{2m+1} \) is given by

\[
(3.5) \quad T_{(z_1, \ldots, z_{2m})} M_{2m+1} \cong \left\{ \left( \begin{array}{c} \xi_1 \\
\vdots \\
\xi_{2m} \end{array} \right) \in \mathbb{R}^{2m} : \xi_1 z_1 + \cdots + \xi_{2m} z_{2m} = 0 \right\}.
\]

Write \( z_i \) as \((x_i, y_i)\). Then for \( \eta_1, \ldots, \eta_{2m-2} \in T_{(z_1, \ldots, z_{2m})} M_{2m+1} \), we set

\[
(3.6) \quad \omega(\eta_1, \ldots, \eta_{2m-2}) = \det \left( \begin{array}{c} \eta_1, \ldots, \eta_{2m-2}, \\
\vdots \\
\eta_{2m-2} \end{array} \right), \begin{pmatrix} x_1 \\
\vdots \\
x_{2m} \end{pmatrix}, \begin{pmatrix} y_1 \\
\vdots \\
y_{2m} \end{pmatrix} \right).
\]

It is easy to see that \( \omega \) is nowhere zero on \( M_{2m+1} \).
For $\eta = \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_{2m} \end{pmatrix} \in T(z_1, \ldots, z_{2m})M_{2m+1}$, we see that

$$d\iota_{2m+1}(\eta) = \xi_1(\sqrt{-1}z_1) + \cdots + \xi_{2m}(\sqrt{-1}z_{2m}),$$

where $i_{2m+1} : M_{2m+1} \hookrightarrow (S^1)^{2m}$ denotes the inclusion. Hence we see that $d\sigma : T(z_1, \ldots, z_{2m})M_{2m+1} \to T(z_1, \ldots, z_{2m})M_{2m+1}$ is given by

$$d\sigma(\eta) = -\eta. \quad (3.7)$$

Now the formulae $d\sigma(\eta_i) = -\eta_i$ and $\sigma(x_i, y_i) = (x_i, -y_i)$ tell us that $(\sigma^*\omega)(\eta_1, \ldots, \eta_{2m-2}) = -\omega(\eta_1, \ldots, \eta_{2m-2}).$ Hence $\sigma^*\omega = -\omega$ and the result follows.

Next we treat the case of $n = 2m$. Let $\tilde{M}_{2m}$ be $M_{2m} - \{\text{singular points}\}$. By the same argument as in the case of $n = 2m + 1$, we see that $\sigma : \tilde{M}_{2m} \to \tilde{M}_{2m}$ preserves orientation. As $H_{2m-3}^c(M_{2m}; Q) \cong H_{2m-3}(M_{2m}; Q)$ ($H_c = \text{cohomology with compact supports}$), the result follows.

4. Proof of Theorem E

First we determine $H_{2m-1}(V_{2m}; Z)$. The Poincaré-Lefschetz duality tells us that $H_{2m-1}(V_{2m}; Z) \cong H^1((S^1)^{2m}/\sigma, \Sigma_{2m}; Z)$. As $H^1((S^1)^{2m}/\sigma; Z) = 0$, we have $H_{2m-1}(V_{2m}; Z) \cong \bigoplus_{q \leq 2m-1} Z$.

As $V_{2m}$ is a non-compact manifold of dimension $2m$, we have $H_q(V_{2m}; Z) = 0$ ($q \geq 2m$). Hence in order to complete the proof of Theorem E, we need to determine $H_q(V_{2m}; Z)$ ($q \leq 2m - 2$).

Recall that we have a fibration $(S^1)^{2m} - \Sigma_{2m} \to V_{2m} \to \mathbb{R}P^\infty$. Set $F_{2m} = (S^1)^{2m} - \Sigma_{2m}$. The local systems of this fibration of dimensions less than or equal to $2m - 2$ are easy to describe: We write the generator of $\pi_1(\mathbb{R}P^\infty)$ by $\sigma$. Then as a $\sigma$-module, we have

$$H_q(F_{2m}; Z) \cong H_q((S^1)^{2m}; Z) \quad (q \leq 2m - 2). \quad (4.1)$$

Let $\{E_{s,t}^r\}$ be the $\mathbb{Z}$-coefficient homology Serre spectral sequence of the above fibration. It is elementary to describe $E_{s,t}^2$ ($t \neq 2m - 1$) by using the following fact: We define a $\sigma$-module $S$ to be the free abelian group of rank 1 on which $\sigma$ acts by $-1$. Then have that

$$H_q(\mathbb{R}P^\infty; S) = \begin{cases} 
\mathbb{Z}_2 & q: \text{even} \\
0 & q: \text{odd}.
\end{cases} \quad (4.2)$$

**Remark 4.3.** For our reference, we give $E_{s,2m-1}^2$. Let $T$ be the free abelian group of rank 2 on which $\sigma$ acts by $\begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}$. And let $\sigma$ act on $\mathbb{Z}$ trivially. Then we can prove
that

\[ H_{2m-1}(F_{2m}; \mathbb{Z}) \cong \bigoplus_{2m} \mathbb{T} \oplus \bigoplus_{2^{2m-2m-1}} \mathbb{Z}. \]

As

\[ H_q(RP\infty; T) = \begin{cases} \mathbb{Z} & q = 0 \\ 0 & q > 0, \end{cases} \]

we can determine \( E_{s,2m-1}^2 \).

We return to \( E_{s,t}^2 \) \((t \neq 2m - 1)\). By the dimensional reason, we have the following:

**Proposition 4.4.** For \( s + t \leq 2m - 2 \), we have that \( E_{s,t}^2 \cong E_{s,t}^\infty \).

Hence in order to complete the proof of Theorem E, it suffices to determine the extensions of \( E_{s,t}^\infty \), where \( s + t \) are odd \( \leq 2m - 3 \). To do so, it is convenient to study \( H_*(V_{2m}; \mathbb{Z}_2) \).

**Proposition 4.5.** For \( q < 2m - 2 \), we have

\[ H_q(V_{2m}; \mathbb{Z}_2) = \bigoplus_{t \leq q(2m)} \mathbb{Z}_2. \]

From Proposition 4.5, we see that the extensions of \( E_{s,t}^\infty \) \((s + t \leq 2m - 2)\) are trivial. Hence Theorem E follows.

Thus in order to complete the proof of Theorem E, we need to prove Proposition 4.5, which we prove for the rest of this section.

Let \( \{ E_{s,t}^2 \} \) be the \( \mathbb{Z}_2 \)-coefficient cohomology Serre spectral sequence of the fibration \( F_{2m} \to V_{2m} \to RP\infty \). We prove the following:

**Lemma 4.6.** \( d_2 : E^{0,1}_2 \to E^{2,0}_2 \) equals to 0.

Lemma 4.6 tells us that elements of \( E_{s,t}^{s,t} \) \((t \leq 2m - 2)\) are permanent cycles. Hence Proposition 4.5 follows.

Proof of Lemma 4.6. Suppose that Lemma 4.6 fails. Then we have \( H^1(V_{2m}; \mathbb{Z}_2) = \bigoplus \mathbb{Z}_2 \). By Theorem D and the \( \mathbb{Z}_2 \)-coefficient Poincaré duality of \( M_{2m+1}' \), we have \( H_{2m-3}(M_{2m+1}'; \mathbb{Z}_2) = \bigoplus \mathbb{Z}_2 \). Since \( H_{2m-3}(M_{2m+1}'; \mathbb{Q}) = \bigoplus \mathbb{Q} \) by Theorem C (a), we have

\[ H_{2m-3}(M_{2m+1}'; \mathbb{Z}) = \bigoplus_{2m} \mathbb{Z}_2. \]
By Theorem C (a), we have $H_{2m-2}(M'_{2m+1}; \mathbb{Q}) = 0$. Hence by Theorem A (a), $M'_{2m+1}$ is a non-orientable manifold of dimension $2m - 2$. Thus we have $H_{2m-2}(M'_{2m+1}; \mathbb{Z}) = 0$. Then by (4.7), we have $H_{2m-2}(M'_{2m+1}; \mathbb{Z}) = 0$. This contradicts the fact that $H_{2m-2}(M'_{2m+1}; \mathbb{Z}) = \mathbb{Z}$, i.e., $M'_{2m+1}$ is a compact manifold of dimension $2m - 2$.

This completes the proof of Lemma 4.6, and hence also that of Theorem E.

5. Proof of Theorem F

In order to calculate $H_*(M'_{2m+1}; \mathbb{Z})$, first we need to determine $H_*(M'_{2m+1}; \mathbb{Z}_{2})$. By the Poincaré duality, it suffices to determine $H_q(M'_{2m+1}; \mathbb{Z}_{2})$ ($q \leq m - 1$), which are given by the following:

**Proposition 5.1.** For $q \leq m - 1$, we have

$$H_q(M'_{2m+1}; \mathbb{Z}_{2}) = \bigoplus_{i \leq q} (\mathbb{Z}_{2})^{(2^m)}.$$

Proof. First, $H_q(M'_{2m+1}; \mathbb{Z}_{2})$ ($q \leq m - 2$) are determined by Theorems D and E together with the universal coefficient theorem. Then $H_{m-1}(M'_{2m+1}; \mathbb{Z}_{2})$ is determined by Proposition 3.4.

**Remark 5.2.** From Theorems D, E and Proposition 5.1, we see that $(i'_{2m+1})_* : H_{m-1}(M'_{2m+1}; \mathbb{Z}_{2}) \to H_{m-1}(V_{2m}; \mathbb{Z}_{2})$ is an isomorphism for all $m$.

Now we begin to determine $H_*(M'_{2m+1}; \mathbb{Z})$.

(I) $H_{\text{even}}(M'_{2m+1}; \mathbb{Z})$.

These modules are determined from Theorem C and the following:

**Proposition 5.3.** $H_{\text{even}}(M'_{2m+1}; \mathbb{Z})$ are torsion free.

Proof. We can inductively prove this proposition from Theorem C and Proposition 5.1 together with the universal coefficient theorem.

(II) $H_{\text{odd}}(M'_{2m+1}; \mathbb{Z})$.

In order to determine these modules from Theorem C and Proposition 5.1, we need to prove the non-existence of higher two-torsions, i.e., elements of order $2^i$ ($i \geq 2$).

Let $p : M_{2m+1} \times \mathbb{R} \to M'_{2m+1}$ be the real line bundle associated to the covering space $M_{2m+1} \to M'_{2m+1}$. And let $O(M_{2m+1} \times \mathbb{R})$ denote the local system of the above vector bundle. Finally, let $O(TM'_{2m+1})$ denote the local system of $TM'_{2m+1}$, the tangent bundle of $M'_{2m+1}$.

Concerning these local systems, we have the following:
Lemma 5.4. As local systems on $M_{2m+1}'$, we have $O(M_{2m+1}' \times R) \cong O(TM_{2m+1}')$.

Proof. Let $R^2 \to \nu \to M_{2m+1}$ denote the normal bundle of $M_{2m+1}$ in $(S^1)^{2m}$ (cf. (1.6)). As $TM_{2m+1} \oplus \nu \cong T((S^1)^{2m})|M_{2m+1}$, we have

$$TM_{2m+1}' \oplus \nu/\sigma \cong TV_{2m}\mid M_{2m+1},$$

where $\nu/\sigma \to M_{2m+1}'$ denotes the vector bundle obtained from $\nu \to M_{2m+1}$ by the action of $\sigma$.

We study $\nu/\sigma$. Recall that $TM_{2m+1}$ is given by (3.5). Similarly, for $(z_1, \ldots, z_{2m}) \in M_{2m+1}$, we have

$$T((S^1)^{2m}) \cong \left\{ \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_{2m} \end{pmatrix} \in R^{2m} \right\}.$$

Hence by assigning $\left( \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_{2m} \end{pmatrix} \right) \in \nu(z_1, \ldots, z_{2m})$ to $\xi_1 z_1 + \cdots + \xi_{2m} z_{2m}$, we have

$$\nu \cong M_{2m+1} \times R^2.$$

Under this identification, the bundle homomorphism $d\sigma : \nu \to \nu$ is given by

$$d\sigma((z_1, \ldots, z_{2m}); \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}) = ((\bar{z}_1, \ldots, \bar{z}_{2m}); \begin{pmatrix} -v_1 \\ v_2 \end{pmatrix}),$$

(cf. (3.7)).

Then (5.6)-(5.7) tell us that

$$\nu/\sigma \cong M_{2m+1} \times R \oplus M_{2m+1}' \times R.$$

Now, as $V_{2m}$ is orientable, we see from (5.5) and (5.7) that

$$O(TM_{2m+1}') \otimes O(M_{2m+1} \times R) \cong \mathbb{Z},$$

where $\mathbb{Z}$ denotes the simple local system on $M_{2m+1}'$. By taking a tensor $\otimes O(TM_{2m+1}')$ on both sides of (5.9), the result follows.

Let us denote the local systems $O(M_{2m+1} \times R) \cong O(TM_{2m+1}')$ (cf. Lemma 5.4) by $\mathbb{Z}$. 

\[ \square \]
(A) The case of an odd \( m \).

We can determine \( H_q(M'_{2m+1}; \mathbb{Z}) \) (\( q : \text{odd} \leq m - 2 \)) by Theorems D and E. Thus we need to determine \( H_q(M'_{2m+1}; \mathbb{Z}) \) (\( q : \text{odd} \geq m \)). By the Poincaré duality: \( H_q(M'_{2m+1}; \mathbb{Z}) \cong H^{2m-2-q}(M'_{2m+1}; \mathbb{Z}) \), it suffices to determine \( H^r(M'_{2m+1}; \mathbb{Z}) \) (\( r : \text{odd} \leq m - 2 \)).

Consider the Gysin sequence of \( p : M_{2m+1} \times \mathbb{R} \rightarrow M'_{2m+1} : \)

\[
\cdots \rightarrow H^{r-1}(M'_{2m+1}; \mathbb{Z}) \xrightarrow{\mu} H^r(M'_{2m+1}; \mathbb{Z}) \xrightarrow{\phi} H^r(M_{2m+1}; \mathbb{Z}) \xrightarrow{\mu} \cdots.
\]

**Lemma 5.10.** For an odd \( r \leq m - 2 \), we have

1. \( H^r(M'_{2m+1}; \mathbb{Z}) = 0 \).
2. \( H^r(M_{2m+1}; \mathbb{Z}) \) is a free module.
3. The order of a torsion element of \( H^{r+1}(M'_{2m+1}; \mathbb{Z}) \) is exactly 2, i.e., \( H^{r+1}(M'_{2m+1}; \mathbb{Z}) \) does not contain higher two-torsions.

**Proof.** This lemma is an easy consequence of Theorems D, E, 2.1 and Proposition 5.3.

Now suppose that \( H^r(M'_{2m+1}; \mathbb{Z}) \) contains a higher two-torsion. Then by Lemma 5.10 (iii), \( \text{Ker }[\mu : H^r(M'_{2m+1}; \mathbb{Z}) \rightarrow H^{r+1}(M'_{2m+1}; \mathbb{Z})] \) contains a torsion element.

But by Lemma 5.10 (i)-(ii), \( \text{Im }[\phi : H^r(M_{2m+1}; \mathbb{Z}) \rightarrow H^r(M'_{2m+1}; \mathbb{Z})] \) is a free module. This is a contradiction. Thus \( H^r(M'_{2m+1}; \mathbb{Z}) \) (\( r : \text{odd} \leq m - 2 \)) does not contain higher two-torsions.

This completes the proof of Theorem F (a).

(B) The case of an even \( m \).

As in (A), it suffices to determine \( H^r(M'_{2m+1}; \mathbb{Z}) \) (\( r : \text{odd} \leq m - 1 \)). For an odd \( r \leq m - 3 \), Lemma 5.10 applies and, by the same argument as in (A), we see that \( H^r(M'_{2m+1}; \mathbb{Z}) \) does not contain higher two-torsions.

But Lemma 5.10 fails when \( r = m - 1 \). Thus our argument cannot apply in this case. This completes the proof of Theorem F (b).

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**References**


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