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Author(s)	Kamiyama, Yasuhiko
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# TOPOLOGY OF EQUILATERAL POLYGON LINKAGES IN THE EUCLIDEAN PLANE MODULO ISOMETRY GROUP

YASUHIKO KAMIYAMA

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## 1. Introduction

We consider the configuration space  $M'_n$  of equilateral polygon linkages with  $n$  ( $n \geq 3$ ) vertices, each edge having length 1 in the Euclidean plane  $\mathbf{R}^2$  modulo isometry group. More precisely, let  $\mathcal{C}_n$  be

$$(1.1) \quad \mathcal{C}_n = \{(u_1, \dots, u_n) \in (\mathbf{R}^2)^n : |u_{i+1} - u_i| = 1 \ (1 \leq i \leq n-1), \ |u_1 - u_n| = 1\}.$$

Note that  $\text{Iso}(\mathbf{R}^2)$  (= the isometry group of  $\mathbf{R}^2$ , i.e., a semidirect product of  $\mathbf{R}^2$  with  $O(2)$ ), naturally acts on  $\mathcal{C}_n$ . We define  $M'_n$  by

$$(1.2) \quad M'_n = \mathcal{C}_n / \text{Iso}(\mathbf{R}^2).$$

We remark that  $M'_n$  has the following description: We set  $M_n = \mathcal{C}_n / \text{Iso}^+(\mathbf{R}^2)$ , where  $\text{Iso}^+(\mathbf{R}^2)$  denotes the orientation preserving isometry group of  $\mathbf{R}^2$ , i.e., a semidirect product of  $\mathbf{R}^2$  with  $SO(2)$ . Then we can write  $M_n$  as

$$(1.3) \quad M_n = \{(u_1, \dots, u_n) \in \mathcal{C}_n : u_1 = (\frac{1}{2}, 0) \text{ and } u_2 = (-\frac{1}{2}, 0)\}.$$

$M_n$  admits an involution  $\sigma = \text{Iso}(\mathbf{R}^2) / \text{Iso}^+(\mathbf{R}^2)$  such that  $M'_n = M_n / \sigma$ . Under the identification of (1.3),  $\sigma$  is given by

$$(1.4) \quad \sigma(u_1, \dots, u_n) = (\bar{u}_1, \dots, \bar{u}_n),$$

where  $\bar{u}_i = (x_i, -y_i)$  when  $u_i = (x_i, y_i)$ .

Many topological properties of  $M_n$  are already known: First we know explicit topological type of  $M_n$  ( $n \leq 5$ ) [3],[4],[8]. Next we have the results on the smoothness of  $M_n$  [5],[7],[8]. Finally  $H_*(M_n; \mathbf{Z})$  are determined in [6],[7] (cf. Theorem 2.1). In particular, the natural inclusion  $i_n : M_n \hookrightarrow (S^1)^{n-1}$  (cf. (1.6)) induces isomorphisms of homology groups up to a certain dimension (cf. Proposition 2.2).

On the other hand, concerning  $M'_n$ , what we know already are the following: First we know the following examples.

EXAMPLES 1.5.  $M'_3 = \{1\text{-point}\}$ ,  $M'_4 = S^1$  and  $M'_5 = \#_5 \mathbf{R}P^2$ , the five-times connected sum of  $\mathbf{R}P^2$ .

Next some assertions on the smoothness of  $M'_n$  are proved in [5]. However, we have few information on  $H_*(M'_n; \mathbf{Z})$ , although we know  $\chi(M'_n)$ , the Euler characteristic of  $M'_n$  [5] (cf. Proposition 3.4).

The purposes of this paper are as follows.

- (i) We prove assertions on the smoothness of  $M'_n$ .
- (ii) We determine  $H_*(M'_n; \mathbf{Z}_p)$ , where  $p$  is an odd prime, and  $H_*(M'_n; \mathbf{Q})$ .

In the following (iii)-(v), we assume  $n$  to be odd, and set  $n = 2m + 1$ . Then by the results of (i) and (ii),  $M'_{2m+1}$  is a non-orientable manifold of dimension  $2m - 2$ .

(iii) Find a space  $V_{2m}$  and an inclusion  $i'_{2m+1} : M'_{2m+1} \hookrightarrow V_{2m}$  so that  $i'_{2m+1}$  induces isomorphisms of homotopy groups up to a certain dimension.

(iv) As  $V_{2m}$  is a natural space, we determine  $H_*(V_{2m}; \mathbf{Z})$  completely. Then in particular we know  $H_*(M'_{2m+1}; \mathbf{Z})$  up to some dimension by the result of (iii).

As we will see in Remark 1.9, knowing  $H_*(V_{2m}; \mathbf{Z})$  is equivalent to knowing  $H_*((S^1)^{2m}/\sigma; \mathbf{Z})$ .

(v) Finally we determine  $H_*(M'_{2m+1}; \mathbf{Z})$  except the possibility of higher two-torsions in  $H_{m-1}(M'_{2m+1}; \mathbf{Z})$  when  $m$  is even.

Now we state our results. Concerning (i), we have the following:

**Theorem A.** (a)  $M'_{2m+1}$  is a manifold of dimension  $2m - 2$ .

(b)  $M'_{2m}$  is a manifold of dimension  $2m - 3$  with singular points.  $(u_1, \dots, u_{2m}) \in M'_{2m}$  is a singular point iff all of  $u_i$  lie on the  $x$ -axis, i.e., the line determined by  $u_1$  and  $u_2$  (cf. (1.3)). Moreover every singular point of  $M'_{2m}$  is a cone-like singularity and has a neighborhood as  $C(S^{m-2} \times_{\mathbf{Z}_2} S^{m-2})$ , where  $C$  denotes cone and action of  $\mathbf{Z}_2$  on both factors is generated by the antipodal map.

Concerning (ii), first we prove the following:

**Theorem B.**  $H_*(M'_n; \mathbf{Z})$  are odd-torsion free.

Thus in order to know  $H_*(M'_n; \mathbf{Z}_p)$ , we need to know  $H_*(M'_n; \mathbf{Q})$ , which is given by the following:

**Theorem C.** *The Poincaré polynomials  $PS_{\mathbf{Q}}(M'_n) = \sum_{\lambda} \dim_{\mathbf{Q}} H_{\lambda}(M'_n; \mathbf{Q}) t^{\lambda}$  are given by*

$$(a) \quad PS_{\mathbf{Q}}(M'_{2m+1}) = \sum_{0 \leq 2a \leq m-2} \binom{2m}{2a} t^{2a} + \binom{2m}{m-1} t^{m-1} \\ + \sum_{m \leq 2b+1 \leq 2m-3} \binom{2m}{2b+3} t^{2b+1},$$

$$(b) \quad PS_{\mathbf{Q}}(M'_{4l}) = \sum_{0 \leq 2a \leq 2l-2} \binom{4l-1}{2a} t^{2a} + \binom{4l-1}{2l-2} t^{2l-1} \\ + \sum_{2l+1 \leq 2b+1 \leq 4l-3} \binom{4l-1}{2b+3} t^{2b+1},$$

$$PS_{\mathbf{Q}}(M'_{4l+2}) = \sum_{0 \leq 2a \leq 2l-2} \binom{4l+1}{2a} t^{2a} + \binom{4l+1}{2l+1} t^{2l} \\ + \sum_{2l+1 \leq 2b+1 \leq 4l-1} \binom{4l+1}{2b+3} t^{2b+1},$$

where  $\binom{a}{b}$  denotes the binomial coefficient.

Next we go to (iii). By setting  $z_i = u_{i+2} - u_{i+1}$  ( $1 \leq i \leq n-2$ ),  $z_{n-1} = u_1 - u_n$ , and identifying  $\mathbf{R}^2$  with  $\mathbf{C}$ , we can write  $M_n$  ( $n \geq 3$ ) as

$$(1.6) \quad M_n \cong \{(z_1, \dots, z_{n-1}) \in (S^1)^{n-1} : z_1 + \dots + z_{n-1} - 1 = 0\}.$$

Let  $i_n : M_n \hookrightarrow (S^1)^{n-1}$  be the inclusion.

As we have mentioned,  $(S^1)^{n-1}$  approximates the topology of  $M_n$  up to some dimension (cf. Proposition 2.2). However, for an odd  $n = 2m+1$ , our low-dimensional computations lead us to give up the hope that  $(S^1)^{2m}/\sigma$  might approximate  $M'_{2m+1} = M_{2m+1}/\sigma$ , where  $\sigma$  acts on  $(S^1)^{2m}$  in the same way as in (1.4). The essential reason for this is that the action of  $\sigma$  on  $(S^1)^{2m}$  is not free, although on  $M_{2m+1}$  is.

Thus we define  $V_{2m}$  by

$$(1.7) \quad V_{2m} = \{(S^1)^{2m} - \Sigma_{2m}\} / \sigma,$$

where we set

$$\Sigma_{2m} = \{(z_1, \dots, z_{2m}) \in (S^1)^{2m} : z_i = \pm 1 \ (1 \leq i \leq 2m)\}.$$

Let  $i'_{2m+1} : M'_{2m+1} \hookrightarrow V_{2m}$  be the inclusion. Then we have the following map of covering

spaces:

$$(1.8) \quad \begin{array}{ccc} \mathbf{Z}_2 & \xlongequal{\quad} & \mathbf{Z}_2 \\ \downarrow & & \downarrow \\ M_{2m+1} & \xrightarrow{i_{2m+1}} & (S^1)^{2m} - \Sigma_{2m} \\ \downarrow & & \downarrow \\ M'_{2m+1} & \xrightarrow{i'_{2m+1}} & V_{2m}. \end{array}$$

Note that  $(S^1)^{2m} - \Sigma_{2m}$  is a maximal subspace of  $(S^1)^{2m}$  on which  $\sigma$  acts freely. Thus it is natural to consider the topology of  $V_{2m}$ .

Now concerning the relation between  $M'_{2m+1}$  and  $V_{2m}$ , we have the following theorem.

**Theorem D.**  $(i'_{2m+1})_* : \pi_q(M'_{2m+1}) \rightarrow \pi_q(V_{2m})$  are isomorphisms for  $q \leq m-2$ , and an epimorphism for  $q = m-1$ .

Concerning (iv), we have the following:

**Theorem E.**  $H_*(V_{2m}; \mathbf{Z})$  is given by

$$H_q(V_{2m}; \mathbf{Z}) = \begin{cases} \bigoplus_{\binom{2m}{q}} \mathbf{Z} & q : \text{even} \leq 2m-2 \\ \sum_{i \leq q} \bigoplus_{\binom{2m}{i}} \mathbf{Z}_2 & q : \text{odd} \leq 2m-3 \\ \bigoplus_{2^{2m-1}} \mathbf{Z} & q = 2m-1 \\ 0 & \text{otherwise,} \end{cases}$$

where  $\bigoplus_{\binom{2m}{q}} \mathbf{Z}$  denotes the  $\binom{2m}{q}$ -times direct sum of  $\mathbf{Z}$ .

Note that Theorems D and E give  $H_q(M'_{2m+1}; \mathbf{Z})$  for  $q \leq m-2$ .

**REMARK 1.9.** By the Poincaré-Lefschetz duality  $H^q((S^1)^{2m}/\sigma, \Sigma_{2m}; \mathbf{Z}) \cong H_{2m-q}(V_{2m}; \mathbf{Z})$ , knowing  $H_*(V_{2m}; \mathbf{Z})$  is equivalent to knowing  $H_*((S^1)^{2m}/\sigma; \mathbf{Z})$ .

Concerning (v), we have the following:

**Theorem F.** (a) For an odd  $m$ , we have

$$H_q(M'_{2m+1}; \mathbf{Z}) = \begin{cases} \bigoplus_{\binom{2m}{q}} \mathbf{Z} & q : \text{even} \leq m-1 \\ \bigoplus_{\sum_{i \leq q} \binom{2m}{i}} \mathbf{Z}_2 & q : \text{odd} \leq m-2 \\ \bigoplus_{\binom{2m}{q+2}} \mathbf{Z} \oplus \bigoplus_{\sum_{i \geq q+3} \binom{2m}{i}} \mathbf{Z}_2 & q : \text{odd} \geq m \\ 0 & \text{otherwise.} \end{cases}$$

(b) For an even  $m$ , we have

$$H_q(M'_{2m+1}; \mathbf{Z}) = \begin{cases} \bigoplus_{\binom{2m}{q}} \mathbf{Z} & q : \text{even} \leq m-2 \\ \bigoplus_{\sum_{i \leq q} \binom{2m}{i}} \mathbf{Z}_2 & q : \text{odd} \leq m-3 \\ \bigoplus_{\binom{2m}{m-1}} \mathbf{Z} \oplus \text{Tor}_{m-1} & q = m-1 \\ \bigoplus_{\binom{2m}{q+2}} \mathbf{Z} \oplus \bigoplus_{\sum_{i \geq q+3} \binom{2m}{i}} \mathbf{Z}_2 & q : \text{odd} \geq m+1 \\ 0 & \text{otherwise,} \end{cases}$$

where  $\text{Tor}_{m-1}$ , the torsion submodule of  $H_{m-1}(M'_{2m+1}; \mathbf{Z})$ , satisfies that  $\dim_{\mathbf{Z}_2} \text{Tor}_{m-1} \otimes \mathbf{Z}_2 = \sum_{i \leq m-2} \binom{2m}{i}$ .

Thus, in particular,  $H_{\text{even}}(M'_{2m+1}; \mathbf{Z})$  are torsion free for all  $m$ .

**REMARK 1.10.** (a) By Theorems D, E and F, we see that  $(i'_{2m+1})_* : H_{m-1}(M'_{2m+1}; \mathbf{Z}) \rightarrow H_{m-1}(V_{2m}; \mathbf{Z})$  is an isomorphism when  $m$  is odd, but not an isomorphism when  $m$  is even.

(b) In order to prove Theorem F, we first determine  $H_*(M'_{2m+1}; \mathbf{Z}_2)$ , which is given in Proposition 5.1. In particular, we see that  $(i'_{2m+1})_* : H_{m-1}(M'_{2m+1}; \mathbf{Z}_2) \rightarrow H_{m-1}(V_{2m}; \mathbf{Z}_2)$  is an isomorphism for all  $m$  (cf. Remark 5.2).

This paper is organized as follows. In §2 we recall the results of [7], then prove Theorems A, B and D. In §3 we prove Theorem C. In §4 we prove Theorem E, and in §5 we prove Theorem F.

## 2. Proofs of Theorems A, B and D

In [7], the following theorem is proved.

**Theorem 2.1.**  $H_*(M_n; \mathbf{Z})$  are free  $\mathbf{Z}$ -modules and the Poincaré polynomials  $PS(M_n) = \sum_{\lambda} \text{rank} H_{\lambda}(M_n; \mathbf{Z}) t^{\lambda}$  are given by

$$PS(M_{2m+1}) = \sum_{\lambda=0}^{m-2} \binom{2m}{\lambda} t^{\lambda} + 2 \binom{2m}{m-1} t^{m-1} + \sum_{\lambda=m}^{2m-2} \binom{2m}{\lambda+2} t^{\lambda},$$

$$PS(M_{2m}) = \sum_{\lambda=0}^{m-2} \binom{2m-1}{\lambda} t^{\lambda} + \binom{2m}{m-1} t^{m-1} + \sum_{\lambda=m}^{2m-3} \binom{2m-1}{\lambda+2} t^{\lambda}.$$

The essential facts to prove Theorem 2.1 are the following three propositions.

**Proposition 2.2.** (i)  $(i_{2m+1})_* : \pi_q(M_{2m+1}) \rightarrow \pi_q((S^1)^{2m})$  are isomorphisms for  $q \leq m-2$ , and an epimorphism for  $q = m-1$ .

(ii)  $(i_{2m})_* : H_q(M_{2m}; \mathbf{Z}) \rightarrow H_q((S^1)^{2m-1}; \mathbf{Z})$  are isomorphisms for  $q \leq m-2$ , and an epimorphism for  $q = m-1$ .

**Proposition 2.3.** (i)  $M_{2m+1}$  is an orientable manifold of dimension  $2m-2$ . Thus the Poincaré duality homomorphisms  $\cap[M_{2m+1}] : H^q(M_{2m+1}; \mathbf{Z}) \rightarrow H_{2m-2-q}(M_{2m+1}; \mathbf{Z})$  are isomorphisms for all  $q$ , where  $[M_{2m+1}] \in H_{2m-2}(M_{2m+1}; \mathbf{Z})$  is a fundamental class.

(ii)  $M_{2m}$  is a manifold of dimension  $2m-3$  with singular points.  $(u_1, \dots, u_{2m}) \in M_{2m}$  is a singular point iff all of  $u_i$  lie on the  $x$ -axis. Moreover every singular point of  $M_{2m}$  is a cone-like singularity and has a neighborhood as  $C(S^{m-2} \times S^{m-2})$ . Thus the Poincaré duality homomorphisms  $\cap[M_{2m}] : H^q(M_{2m}; \mathbf{Z}) \rightarrow H_{2m-3-q}(M_{2m}; \mathbf{Z})$  are isomorphisms for  $q \leq m-3$  or  $q \geq m$ , an epimorphism for  $q = m-1$ , and a monomorphism for  $q = m-2$ .

**Proposition 2.4.** (i)  $\chi(M_{2m+1}) = (-1)^{m+1} \binom{2m}{m}$ .

(ii)  $\chi(M_{2m}) = (-1)^{m+1} \binom{2m-1}{m}$ .

**REMARK 2.5.** In order to prove Theorem 2.1, the homological assertion is sufficient for Proposition 2.2 (i). But actually we can prove the homotopical assertion.

**Proof of Theorem A.** Since  $\sigma$  acts freely on  $M_{2m+1}$ , and  $M_{2m}^{\sigma}$  (=the fixed point set of the involution) equals to the set of singular points in  $M_{2m}$ , all of the assertions except the type of the singular points of  $M_{2m}'$  are deduced from Proposition 2.3.

Let  $(z_1, \dots, z_{2m-1})$  be a singular point of  $M_{2m}$  in the identification of (1.6). By Proposition 2.3, we must have  $z_i = \pm 1$  ( $1 \leq i \leq 2m-1$ ). As the symmetric group on  $(2m-1)$ -letters acts on  $M_{2m}$ , we can assume that  $z_i = 1$  ( $1 \leq i \leq m$ ) and  $z_i = -1$  ( $m+1 \leq i \leq 2m-1$ ). A neighborhood of  $(z_1, \dots, z_{2m-1})$  in  $(S^1)^{2m-1}$  is written

by

$$\left\{ \left( \begin{pmatrix} \sqrt{1-y_1^2} \\ y_1 \end{pmatrix}, \dots, \begin{pmatrix} \sqrt{1-y_m^2} \\ y_m \end{pmatrix}, \begin{pmatrix} -\sqrt{1-y_{m+1}^2} \\ y_{m+1} \end{pmatrix}, \dots, \begin{pmatrix} -\sqrt{1-y_{2m-1}^2} \\ y_{2m-1} \end{pmatrix} \right) : \right. \\ \left. -\epsilon \leq y_i \leq \epsilon \ (1 \leq i \leq 2m-1) \right\},$$

where  $\epsilon > 0$  is a fixed small number. As  $\epsilon$  is small, it is easy to see that we can write this neighborhood as

$$\left\{ \left( \begin{pmatrix} 1 - \frac{1}{2}y_1^2 \\ y_1 \end{pmatrix}, \dots, \begin{pmatrix} 1 - \frac{1}{2}y_m^2 \\ y_m \end{pmatrix}, \begin{pmatrix} -1 + \frac{1}{2}y_{m+1}^2 \\ y_{m+1} \end{pmatrix}, \dots, \begin{pmatrix} -1 + \frac{1}{2}y_{2m-1}^2 \\ y_{2m-1} \end{pmatrix} \right) : \right. \\ \left. -\epsilon \leq y_i \leq \epsilon \ (1 \leq i \leq 2m-1) \right\}.$$

Thus a neighborhood of a singular point in  $M_{2m}$  is written as a subspace of  $\mathbb{R}^{2m-1}$  defined by two equations

$$(2.6) \quad \begin{cases} y_1^2 + \dots + y_m^2 - y_{m+1}^2 - \dots - y_{2m-1}^2 = 0 \\ y_1 + \dots + y_m + y_{m+1} + \dots + y_{2m-1} = 0. \end{cases}$$

By a linear transformation of parameters, we can write the quadratic form of (2.6), i.e.,

$$y_1^2 + \dots + y_{m-1}^2 + (y_1 + \dots + y_{m-1} + y_{m+1} + \dots + y_{2m-1})^2 - y_{m+1}^2 - \dots - y_{2m-1}^2,$$

as

$$w_1^2 + \dots + w_{m-1}^2 - w_m^2 - \dots - w_{2m-2}^2.$$

Thus a singular point of  $M_{2m}$  has a neighborhood  $C\{(w_1, \dots, w_{m-1}, w_m, \dots, w_{2m-2}) : w_1^2 + \dots + w_{m-1}^2 = 1, w_m^2 + \dots + w_{2m-2}^2 = 1\}$ , which is homeomorphic to  $C(S^{m-2} \times S^{m-2})$ .

Now it is clear that  $\sigma w_i = -w_i$ . Hence a singular point of  $M'_{2m}$  has a neighborhood  $C(S^{m-2} \times_{\mathbb{Z}_2} S^{m-2})$ , where  $\sigma(\zeta_1, \zeta_2) = (-\zeta_1, -\zeta_2)$  ( $\zeta_1, \zeta_2 \in S^{m-2}$ ).  $\square$

**Proof of Theorem B.** For  $F = \mathbb{Z}_p$  ( $p$  : an odd prime) or  $\mathbb{Q}$ , we have that  $H_*(M'_n; F) \cong H_*(M_n; F)^\sigma$  (= the fixed point set of  $H_*(M_n; F)$  under the  $\sigma$ -action) (see for example [2]). As  $H_*(M_n; \mathbb{Z})$  are free modules by Theorem 2.1, we have that  $\dim_{\mathbb{Z}_p} H_q(M'_n; \mathbb{Z}_p) = \dim_{\mathbb{Q}} H_q(M'_n; \mathbb{Q})$ . Hence Theorem B follows.  $\square$



**Proof of Theorem D.** Let  $j_{2m} : (S^1)^{2m} - \Sigma_{2m} \hookrightarrow (S^1)^{2m}$  be the inclusion. Since  $\Sigma_{2m}$  is a discrete set, the general position argument shows that  $(j_{2m})_* : \pi_q((S^1)^{2m} - \Sigma_{2m}) \rightarrow \pi_q((S^1)^{2m})$  are isomorphisms for  $q \leq 2m - 2$ . Then Proposition 2.2 (i) shows that  $(i_{2m+1})_* : \pi_q(M_{2m+1}) \rightarrow \pi_q((S^1)^{2m} - \Sigma_{2m})$  are isomorphisms for  $q \leq m - 2$  and an epimorphism for  $q = m - 1$ , where  $i_{2m+1} : M_{2m+1} \hookrightarrow (S^1)^{2m} - \Sigma_{2m}$  is the inclusion.

By comparing the homotopy exact sequences of two covering spaces of (1.8), we see that  $(i'_{2m+1})_* : \pi_q(M'_{2m+1}) \rightarrow \pi_q(V_{2m})$  are isomorphisms for  $q \leq m - 2$  and an epimorphism for  $q = m - 1$ , where  $i'_{2m+1} : M'_{2m+1} \hookrightarrow V_{2m}$  is the map induced from the  $\sigma$ -equivariant inclusion  $i_{2m+1} : M_{2m+1} \hookrightarrow (S^1)^{2m} - \Sigma_{2m}$ .

This completes the proof of Theorem D.  $\square$

### 3. Proof of Theorem C

Let  $i_n : M_n \hookrightarrow (S^1)^{n-1}$  be the inclusion. Note that  $i_n$  is a  $\sigma$ -equivariant map. Hence  $(i_n)_* : H_*(M_n; \mathbf{Q}) \rightarrow H_*((S^1)^{n-1}; \mathbf{Q})$  is also a  $\sigma$ -equivariant homomorphism. Since  $H_*(M'_n; \mathbf{Q}) = H_*(M_n; \mathbf{Q})^\sigma$ , Proposition 2.2 tells us the following:

**Proposition 3.1.** (i) For  $q \leq m - 2$ , we have

$$H_q(M'_{2m+1}; \mathbf{Q}) = \begin{cases} \bigoplus_{\binom{2m}{q}} \mathbf{Q} & q : \text{even} \\ 0 & q : \text{odd.} \end{cases}$$

(ii) For  $q \leq m - 2$ , we have

$$H_q(M'_{2m}; \mathbf{Q}) = \begin{cases} \bigoplus_{\binom{2m-1}{q}} \mathbf{Q} & q : \text{even} \\ 0 & q : \text{odd.} \end{cases}$$

We assume the truth of the following Lemma for the moment. Let  $[M_n] \in H_{n-3}(M_n; \mathbf{Q})$  be the fundamental class.

**Lemma 3.2.**  $\sigma_*[M_n] = (-1)^n[M_n]$ .

Then we have the following:

**Proposition 3.3.** (i) For  $q \geq m$ , we have

$$H_q(M'_{2m+1}; \mathbf{Q}) = \begin{cases} 0 & q : \text{even} \\ \bigoplus_{\binom{2m}{q+2}} \mathbf{Q} & q : \text{odd.} \end{cases}$$

(ii) For  $q \geq m$ , we have

$$H_q(M'_{2m}; \mathbf{Q}) = \begin{cases} 0 & q : \text{even} \\ \bigoplus_{\binom{2m-1}{q+2}} \mathbf{Q} & q : \text{odd}. \end{cases}$$

**Proof of Proposition 3.3.** Take an element  $\alpha \in H_q(M_{2m+1}; \mathbf{Q})$  ( $q \geq m$ ). By Proposition 2.3, there is an element  $f \in H^{2m-2-q}(M_{2m+1}; \mathbf{Q})$  such that  $\alpha = f \cap [M_{2m+1}]$ . As  $\sigma_*(f \cap [M_{2m+1}]) = \sigma^* f \cap \sigma_*[M_{2m+1}] = -\sigma^* f \cap [M_{2m+1}]$ , we have that

$$H_q(M_{2m+1}; \mathbf{Q})^\sigma = \{f \in H^{2m-2-q}(M_{2m+1}; \mathbf{Q}) : \sigma^* f = -f\}.$$

Now (i) follows from Proposition 3.1.

(ii) can be proved similarly.  $\square$

Now in order to determine  $H_*(M'_n; \mathbf{Q})$ , we need to know only  $H_{m-1}(M'_{2m+1}; \mathbf{Q})$  and  $H_{m-1}(M'_{2m}; \mathbf{Q})$ , which are determined if we know  $\chi(M'_n)$ .

**Proposition 3.4** ([5]). (i)  $\chi(M'_{2m+1}) = (-1)^{m+1} \binom{2m-1}{m}$ .

$$(ii) \chi(M'_{2m}) = \begin{cases} 0 & m : \text{even} \\ \binom{2m-1}{m} & m : \text{odd}. \end{cases}$$

**Proof.** By a general formula of an involution (see for example [1]), we have  $\chi(M_n) + \chi(M_n^\sigma) = 2\chi(M'_n)$ . Then the result follows from Proposition 2.4.  $\square$

**Proof of Lemma 3.2.** First we treat the case of  $n = 2m + 1$ . We define a volume element  $\omega$  of  $M_{2m+1}$  as follows. Fix  $(z_1, \dots, z_{2m}) \in M_{2m+1}$  in the identification of (1.6). It is easy to see that the tangent space  $T_{(z_1, \dots, z_{2m})} M_{2m+1}$  is given by

$$(3.5) \quad T_{(z_1, \dots, z_{2m})} M_{2m+1} \cong \left\{ \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_{2m} \end{pmatrix} \in \mathbf{R}^{2m} : \xi_1 z_1 + \dots + \xi_{2m} z_{2m} = 0 \right\}.$$

Write  $z_i$  as  $(x_i, y_i)$ . Then for  $\eta_1, \dots, \eta_{2m-2} \in T_{(z_1, \dots, z_{2m})} M_{2m+1}$ , we set

$$(3.6) \quad \omega(\eta_1, \dots, \eta_{2m-2}) = \det \left( \eta_1, \dots, \eta_{2m-2}, \begin{pmatrix} x_1 \\ \vdots \\ x_{2m} \end{pmatrix}, \begin{pmatrix} y_1 \\ \vdots \\ y_{2m} \end{pmatrix} \right).$$

It is easy to see that  $\omega$  is nowhere zero on  $M_{2m+1}$ .

For  $\eta = \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_{2m} \end{pmatrix} \in T_{(z_1, \dots, z_{2m})} M_{2m+1}$ , we see that

$$di_{2m+1}(\eta) = \xi_1(\sqrt{-1}z_1) + \dots + \xi_{2m}(\sqrt{-1}z_{2m}),$$

where  $i_{2m+1} : M_{2m+1} \hookrightarrow (S^1)^{2m}$  denotes the inclusion. Hence we see that  $d\sigma : T_{(z_1, \dots, z_{2m})} M_{2m+1} \rightarrow T_{(\bar{z}_1, \dots, \bar{z}_{2m})} M_{2m+1}$  is given by

$$(3.7) \quad d\sigma(\eta) = -\eta.$$

Now the formulae  $d\sigma(\eta_i) = -\eta_i$  and  $\sigma(x_i, y_i) = (x_i, -y_i)$  tell us that  $(\sigma^*\omega)(\eta_1, \dots, \eta_{2m-2}) = -\omega(\eta_1, \dots, \eta_{2m-2})$ . Hence  $\sigma^*\omega = -\omega$  and the result follows.

Next we treat the case of  $n = 2m$ . Let  $\bar{M}_{2m}$  be  $M_{2m} - \{\text{singular points}\}$ . By the same argument as in the case of  $n = 2m + 1$ , we see that  $\sigma : \bar{M}_{2m} \rightarrow \bar{M}_{2m}$  preserves orientation. As  $H_c^{2m-3}(\bar{M}_{2m}; \mathbf{Q}) \cong H^{2m-3}(M_{2m}; \mathbf{Q})$  ( $H_c$  = cohomology with compact supports), the result follows.  $\square$

#### 4. Proof of Theorem E

First we determine  $H_{2m-1}(V_{2m}; \mathbf{Z})$ . The Poincaré-Lefschetz duality tells us that  $H_{2m-1}(V_{2m}; \mathbf{Z}) \cong H^1((S^1)^{2m}/\sigma, \Sigma_{2m}; \mathbf{Z})$ . As  $H^1((S^1)^{2m}/\sigma; \mathbf{Z}) = 0$ , we have  $H_{2m-1}(V_{2m}; \mathbf{Z}) \cong \bigoplus_{2^{2m}-1} \mathbf{Z}$ .

As  $V_{2m}$  is a non-compact manifold of dimension  $2m$ , we have  $H_q(V_{2m}; \mathbf{Z}) = 0$  ( $q \geq 2m$ ). Hence in order to complete the proof of Theorem E, we need to determine  $H_q(V_{2m}; \mathbf{Z})$  ( $q \leq 2m - 2$ ).

Recall that we have a fibration  $(S^1)^{2m} - \Sigma_{2m} \rightarrow V_{2m} \rightarrow \mathbf{R}P^\infty$ . Set  $F_{2m} = (S^1)^{2m} - \Sigma_{2m}$ . The local systems of this fibration of dimensions less than or equal to  $2m - 2$  are easy to describe: We write the generator of  $\pi_1(\mathbf{R}P^\infty)$  by  $\sigma$ . Then as a  $\sigma$ -module, we have

$$(4.1) \quad H_q(F_{2m}; \mathbf{Z}) \cong H_q((S^1)^{2m}; \mathbf{Z}) \quad (q \leq 2m - 2).$$

Let  $\{E_{s,t}^r\}$  be the  $\mathbf{Z}$ -coefficient homology Serre spectral sequence of the above fibration. It is elementary to describe  $E_{s,t}^2$  ( $t \neq 2m - 1$ ) by using the following fact: We define a  $\sigma$ -module  $\mathcal{S}$  to be the free abelian group of rank 1 on which  $\sigma$  acts by  $-1$ . Then have that

$$(4.2) \quad H_q(\mathbf{R}P^\infty; \mathcal{S}) = \begin{cases} \mathbf{Z}_2 & q : \text{even} \\ 0 & q : \text{odd.} \end{cases}$$

REMARK 4.3. For our reference, we give  $E_{s,2m-1}^2$ . Let  $\mathcal{T}$  be the free abelian group of rank 2 on which  $\sigma$  acts by  $\begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}$ . And let  $\sigma$  act on  $\mathbf{Z}$  trivially. Then we can prove

that

$$H_{2m-1}(F_{2m}; \mathbf{Z}) \cong \bigoplus_{2m} \mathcal{T} \oplus \bigoplus_{2^{2m}-2m-1} \mathbf{Z}.$$

As

$$H_q(\mathbf{R}P^\infty; \mathcal{T}) = \begin{cases} \mathbf{Z} & q = 0 \\ 0 & q > 0, \end{cases}$$

we can determine  $E_{s,2m-1}^2$ .

We return to  $E_{s,t}^2$  ( $t \neq 2m-1$ ). By the dimensional reason, we have the following:

**Proposition 4.4.** *For  $s+t \leq 2m-2$ , we have that  $E_{s,t}^2 \cong E_{s,t}^\infty$ .*

Hence in order to complete the proof of Theorem E, it suffices to determine the extensions of  $E_{s,t}^\infty$ , where  $s+t$  are odd  $\leq 2m-3$ . To do so, it is convenient to study  $H_*(V_{2m}; \mathbf{Z}_2)$ .

**Proposition 4.5.** *For  $q \leq 2m-2$ , we have*

$$H_q(V_{2m}; \mathbf{Z}_2) = \bigoplus_{\sum_{i \leq q} \binom{2m}{i}} \mathbf{Z}_2.$$

From Proposition 4.5, we see that the extensions of  $E_{s,t}^\infty$  ( $s+t \leq 2m-2$ ) are trivial. Hence Theorem E follows.

Thus in order to complete the proof of Theorem E, we need to prove Proposition 4.5, which we prove for the rest of this section.

Let  $\{E_r^{s,t}\}$  be the  $\mathbf{Z}_2$ -coefficient cohomology Serre spectral sequence of the fibration  $F_{2m} \rightarrow V_{2m} \rightarrow \mathbf{R}P^\infty$ . We prove the following:

**Lemma 4.6.**  $d_2 : E_2^{0,1} \rightarrow E_2^{2,0}$  equals to 0.

Lemma 4.6 tells us that elements of  $E_2^{s,t}$  ( $t \leq 2m-2$ ) are permanent cycles. Hence Proposition 4.5 follows.

**Proof of Lemma 4.6.** Suppose that Lemma 4.6 fails. Then we have  $H^1(V_{2m}; \mathbf{Z}_2) = \bigoplus_{2m} \mathbf{Z}_2$ . By Theorem D and the  $\mathbf{Z}_2$ -coefficient Poincaré duality of  $M'_{2m+1}$ , we have  $H_{2m-3}(M'_{2m+1}; \mathbf{Z}_2) = \bigoplus_{2m} \mathbf{Z}_2$ . Since  $H_{2m-3}(M'_{2m+1}; \mathbf{Q}) = \bigoplus_{2m} \mathbf{Q}$  by Theorem C (a), we have

$$(4.7) \quad H_{2m-3}(M'_{2m+1}; \mathbf{Z}) = \bigoplus_{2m} \mathbf{Z}.$$

By Theorem C (a), we have  $H_{2m-2}(M'_{2m+1}; \mathbf{Q}) = 0$ . Hence by Theorem A (a),  $M'_{2m+1}$  is a non-orientable manifold of dimension  $2m - 2$ . Thus we have  $H_{2m-2}(M'_{2m+1}; \mathbf{Z}) = 0$ . Then by (4.7), we have  $H_{2m-2}(M'_{2m+1}; \mathbf{Z}_2) = 0$ . This contradicts the fact that  $H_{2m-2}(M'_{2m+1}; \mathbf{Z}_2) = \mathbf{Z}_2$ , i.e.,  $M'_{2m+1}$  is a compact manifold of dimension  $2m - 2$ .

This completes the proof of Lemma 4.6, and hence also that of Theorem E. □

## 5. Proof of Theorem F

In order to calculate  $H_*(M'_{2m+1}; \mathbf{Z})$ , first we need to determine  $H_*(M'_{2m+1}; \mathbf{Z}_2)$ . By the Poincaré duality, it suffices to determine  $H_q(M'_{2m+1}; \mathbf{Z}_2)$  ( $q \leq m - 1$ ), which are given by the following:

**Proposition 5.1.** *For  $q \leq m - 1$ , we have*

$$H_q(M'_{2m+1}; \mathbf{Z}_2) = \bigoplus_{\substack{\Sigma \\ i \leq q}} \binom{2m}{i} \mathbf{Z}_2.$$

*Proof.* First,  $H_q(M'_{2m+1}; \mathbf{Z}_2)$  ( $q \leq m - 2$ ) are determined by Theorems D and E together with the universal coefficient theorem. Then  $H_{m-1}(M'_{2m+1}; \mathbf{Z}_2)$  is determined by Proposition 3.4. □

**REMARK 5.2.** From Theorems D, E and Proposition 5.1, we see that  $(i'_{2m+1})_* : H_{m-1}(M'_{2m+1}; \mathbf{Z}_2) \rightarrow H_{m-1}(V_{2m}; \mathbf{Z}_2)$  is an isomorphism for all  $m$ .

Now we begin to determine  $H_*(M'_{2m+1}; \mathbf{Z})$ .

(I)  $H_{\text{even}}(M'_{2m+1}; \mathbf{Z})$ .

These modules are determined from Theorem C and the following:

**Proposition 5.3.**  *$H_{\text{even}}(M'_{2m+1}; \mathbf{Z})$  are torsion free.*

*Proof.* We can inductively prove this proposition from Theorem C and Proposition 5.1 together with the universal coefficient theorem. □

(II)  $H_{\text{odd}}(M'_{2m+1}; \mathbf{Z})$ .

In order to determine these modules from Theorem C and Proposition 5.1, we need to prove the non-existence of higher two-torsions, i.e., elements of order  $2^i$  ( $i \geq 2$ ).

Let  $p : M_{2m+1} \times_{\sigma} \mathbf{R} \rightarrow M'_{2m+1}$  be the real line bundle associated to the covering space  $M_{2m+1} \rightarrow M'_{2m+1}$ . And let  $O(M_{2m+1} \times_{\sigma} \mathbf{R})$  denote the local system of the above vector bundle. Finally, let  $O(TM'_{2m+1})$  denote the local system of  $TM'_{2m+1}$ , the tangent bundle of  $M'_{2m+1}$ .

Concerning these local systems, we have the following:

**Lemma 5.4.** *As local systems on  $M'_{2m+1}$ , we have  $O(M_{2m+1} \times_{\sigma} \mathbf{R}) \cong O(TM'_{2m+1})$ .*

**Proof.** Let  $\mathbf{R}^2 \rightarrow \nu \rightarrow M_{2m+1}$  denote the normal bundle of  $M_{2m+1}$  in  $(S^1)^{2m}$  (cf. (1.6)). As  $TM_{2m+1} \oplus \nu \cong T((S^1)^{2m})|_{M_{2m+1}}$ , we have

$$(5.5) \quad TM'_{2m+1} \oplus \nu/\sigma \cong TV_{2m}|_{M'_{2m+1}},$$

where  $\nu/\sigma \rightarrow M'_{2m+1}$  denotes the vector bundle obtained from  $\nu \rightarrow M_{2m+1}$  by the action of  $\sigma$ .

We study  $\nu/\sigma$ . Recall that  $TM_{2m+1}$  is given by (3.5). Similarly, for  $(z_1, \dots, z_{2m}) \in M_{2m+1}$ , we have

$$T_{(z_1, \dots, z_{2m})}((S^1)^{2m}) \cong \left\{ \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_{2m} \end{pmatrix} \in \mathbf{R}^{2m} \right\}.$$

Hence by assigning  $\begin{pmatrix} \xi_1 \\ \vdots \\ \xi_{2m} \end{pmatrix} \in \nu_{(z_1, \dots, z_{2m})}$  to  $\xi_1 z_1 + \dots + \xi_{2m} z_{2m}$ , we have

$$(5.6) \quad \nu \cong M_{2m+1} \times \mathbf{R}^2.$$

Under this identification, the bundle homomorphism  $d\sigma : \nu \rightarrow \nu$  is given by

$$(5.7) \quad d\sigma((z_1, \dots, z_{2m}); \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}) = ((\bar{z}_1, \dots, \bar{z}_{2m}); \begin{pmatrix} -v_1 \\ v_2 \end{pmatrix}),$$

(cf. (3.7)).

Then (5.6)-(5.7) tell us that

$$(5.8) \quad \nu/\sigma \cong M_{2m+1} \times_{\sigma} \mathbf{R} \oplus M'_{2m+1} \times \mathbf{R}.$$

Now, as  $V_{2m}$  is orientable, we see from (5.5) and (5.8) that

$$(5.9) \quad O(TM'_{2m+1}) \otimes O(M_{2m+1} \times_{\sigma} \mathbf{R}) \cong \mathbf{Z},$$

where  $\mathbf{Z}$  denotes the simple local system on  $M'_{2m+1}$ . By taking a tensor  $\otimes O(TM'_{2m+1})$  on both sides of (5.9), the result follows.  $\square$

Let us denote the local systems  $O(M_{2m+1} \times_{\sigma} \mathbf{R}) \cong O(TM'_{2m+1})$  (cf. Lemma 5.4) by  $\mathcal{Z}$ .

(A) *The case of an odd  $m$ .*

We can determine  $H_q(M'_{2m+1}; \mathbf{Z})$  ( $q : \text{odd} \leq m-2$ ) by Theorems D and E. Thus we need to determine  $H_q(M'_{2m+1}; \mathbf{Z})$  ( $q : \text{odd} \geq m$ ). By the Poincaré duality:  $H_q(M'_{2m+1}; \mathbf{Z}) \cong H^{2m-2-q}(M'_{2m+1}; \mathbf{Z})$ , it suffices to determine  $H^r(M'_{2m+1}; \mathbf{Z})$  ( $r : \text{odd} \leq m-2$ ).

Consider the Gysin sequence of  $p : M_{2m+1} \times_{\sigma} \mathbf{R} \rightarrow M'_{2m+1}$ :

$$\begin{aligned} \cdots \xrightarrow{\psi} H^{r-1}(M'_{2m+1}; \mathbf{Z}) \xrightarrow{\mu} H^r(M'_{2m+1}; \mathbf{Z}) \xrightarrow{p^*} H^r(M_{2m+1}; \mathbf{Z}) \\ \xrightarrow{\psi} H^r(M'_{2m+1}; \mathbf{Z}) \xrightarrow{\mu} \cdots \end{aligned}$$

**Lemma 5.10.** *For an odd  $r \leq m-2$ , we have*

- (i)  $H^r(M'_{2m+1}; \mathbf{Z}) = 0$ .
- (ii)  $H^r(M_{2m+1}; \mathbf{Z})$  is a free module.
- (iii) *The order of a torsion element of  $H^{r+1}(M'_{2m+1}; \mathbf{Z})$  is exactly 2, i.e.,  $H^{r+1}(M'_{2m+1}; \mathbf{Z})$  does not contain higher two-torsions.*

**Proof.** This lemma is an easy consequence of Theorems D, E, 2.1 and Proposition 5.3. □

Now suppose that  $H^r(M'_{2m+1}; \mathbf{Z})$  contains a higher two-torsion. Then by Lemma 5.10 (iii),  $\text{Ker} [\mu : H^r(M'_{2m+1}; \mathbf{Z}) \rightarrow H^{r+1}(M'_{2m+1}; \mathbf{Z})]$  contains a torsion element.

But by Lemma 5.10 (i)-(ii),  $\text{Im} [\psi : H^r(M_{2m+1}; \mathbf{Z}) \rightarrow H^r(M'_{2m+1}; \mathbf{Z})]$  is a free module. This is a contradiction. Thus  $H^r(M'_{2m+1}; \mathbf{Z})$  ( $r : \text{odd} \leq m-2$ ) does not contain higher two-torsions.

This completes the proof of Theorem F (a).

(B) *The case of an even  $m$ .*

As in (A), it suffices to determine  $H^r(M'_{2m+1}; \mathbf{Z})$  ( $r : \text{odd} \leq m-1$ ). For an odd  $r \leq m-3$ , Lemma 5.10 applies and, by the same argument as in (A), we see that  $H^r(M'_{2m+1}; \mathbf{Z})$  does not contain higher two-torsions.

But Lemma 5.10 fails when  $r = m-1$ . Thus our argument cannot apply in this case. This completes the proof of Theorem F (b). □

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Department of Mathematics  
 University of the Ryukyus  
 Nishihara-Cho, Okinawa 903-01, Japan



