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# TOPOLOGY OF EQUILATERAL POLYGON LINKAGES IN THE EUCLIDEAN PLANE MODULO ISOMETRY GROUP

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## 1. Introduction

We consider the configuration space $M'_n$ of equilateral polygon linkages with $n$ ($n \geq 3$) vertices, each edge having length 1 in the Euclidean plane $\mathbf{R}^2$ modulo isometry group. More precisely, let $\mathcal{C}_n$ be

$$(1.1) \quad \mathcal{C}_n = \{(u_1, \ldots, u_n) \in (\mathbf{R}^2)^n : |u_{i+1} - u_i| = 1 \ (1 \leq i \leq n-1), \ |u_1 - u_n| = 1\}.$$

Note that $\mathrm{Iso}(\mathbf{R}^2)$ (= the isometry group of $\mathbf{R}^2$, i.e., a semidirect product of $\mathbf{R}^2$ with $O(2)$), naturally acts on $\mathcal{C}_n$. We define $M'_n$ by

$$(1.2) \quad M'_n = \mathcal{C}_n / \mathrm{Iso}(\mathbf{R}^2).$$

We remark that $M'_n$ has the following description: We set $M_n = \mathcal{C}_n / \mathrm{Iso}^+(\mathbf{R}^2)$, where $\mathrm{Iso}^+(\mathbf{R}^2)$ denotes the orientation preserving isometry group of $\mathbf{R}^2$, i.e., a semidirect product of $\mathbf{R}^2$ with $SO(2)$. Then we can write $M_n$ as

$$(1.3) \quad M_n = \{(u_1, \ldots, u_n) \in \mathcal{C}_n : u_1 = (\tfrac{1}{2}, 0) \ \text{and} \ u_2 = (-\tfrac{1}{2}, 0)\}.$$

$M_n$ admits an involution $\sigma = \mathrm{Iso}(\mathbf{R}^2)/\mathrm{Iso}^+(\mathbf{R}^2)$ such that $M'_n = M_n/\sigma$. Under the identification of (1.3), $\sigma$ is given by

$$(1.4) \quad \sigma(u_1, \ldots, u_n) = (\bar{u}_1, \ldots, \bar{u}_n),$$

where $\bar{u}_i = (x_i, -y_i)$ when $u_i = (x_i, y_i)$.

Many topological properties of $M_n$ are already known: First we know explicit topological type of $M_n$ ($n \leq 5$) [3],[4],[8]. Next we have the results on the smoothness of $M_n$ [5],[7],[8]. Finally $H_*(M_n; \mathbf{Z})$ are determined in [6],[7] (cf. Theorem 2.1). In particular, the natural inclusion $i_n : M_n \hookrightarrow (S^1)^{n-1}$ (cf. (1.6)) induces isomorphisms of homology groups up to a certain dimension (cf. Proposition 2.2).

On the other hand, concerning $M_n'$, what we know already are the following: First we know the following examples.

EXAMPLES 1.5.   $M_3' = \{1\text{-point}\}, M_4' = S^1$ and $M_5' = \underset{5}{\sharp} \mathbf{R}P^2$, the five-times connected sum of $\mathbf{R}P^2$.

Next some assertions on the smoothness of $M_n'$ are proved in [5]. However, we have few information on $H_*(M_n'; \mathbf{Z})$, although we know $\chi(M_n')$, the Euler characteristic of $M_n'$ [5] (cf. Proposition 3.4).

The purposes of this paper are as follows.

(i) We prove assertions on the smoothness of $M_n'$.

(ii) We determine $H_*(M_n'; \mathbf{Z}_p)$, where $p$ is an odd prime, and $H_*(M_n'; \mathbf{Q})$.

In the following (iii)-(v), we assume $n$ to be odd, and set $n = 2m + 1$. Then by the results of (i) and (ii), $M_{2m+1}'$ is a non-orientable manifold of dimension $2m - 2$.

(iii) Find a space $V_{2m}$ and an inclusion $i_{2m+1}' : M_{2m+1}' \hookrightarrow V_{2m}$ so that $i_{2m+1}'$ induces isomorphisms of homotopy groups up to a certain dimension.

(iv) As $V_{2m}$ is a natural space, we determine $H_*(V_{2m}; \mathbf{Z})$ completely. Then in particular we know $H_*(M_{2m+1}'; \mathbf{Z})$ up to some dimension by the result of (iii).

As we will see in Remark 1.9, knowing $H_*(V_{2m}; \mathbf{Z})$ is equivalent to knowing $H_*((S^1)^{2m}/\sigma; \mathbf{Z})$.

(v) Finally we determine $H_*(M_{2m+1}'; \mathbf{Z})$ except the possibility of higher two-torsions in $H_{m-1}(M_{2m+1}'; \mathbf{Z})$ when $m$ is even.

Now we state our results. Concerning (i), we have the following:

**Theorem A.**   (a) $M_{2m+1}'$ is a manifold of dimension $2m - 2$.
(b) $M_{2m}'$ is a manifold of dimension $2m - 3$ with singular points. $(u_1, \ldots, u_{2m}) \in M_{2m}'$ is a singular point iff all of $u_i$ lie on the $x$-axis, i.e., the line determined by $u_1$ and $u_2$ (cf. (1.3)). Moreover every singular point of $M_{2m}'$ is a cone-like singularity and has a neighborhood as $C(S^{m-2} \times_{\mathbf{Z}_2} S^{m-2})$, where $C$ denotes cone and action of $\mathbf{Z}_2$ on both factors is generated by the antipodal map.

Concerning (ii), first we prove the following:

**Theorem B.**   $H_*(M_n'; \mathbf{Z})$ are odd-torsion free.

Thus in order to know $H_*(M_n'; \mathbf{Z}_p)$, we need to know $H_*(M_n'; \mathbf{Q})$, which is given by the following:

**Theorem C.** *The Poincaré polynomials $PS_\mathbf{Q}(M_n') = \Sigma_\lambda \dim_\mathbf{Q} H_\lambda(M_n'; \mathbf{Q})t^\lambda$ are given by*

(a) $PS_\mathbf{Q}(M_{2m+1}') = \sum\limits_{0 \le 2a \le m-2} \binom{2m}{2a}t^{2a} + \binom{2m}{m-1}t^{m-1}$

$$+ \sum\limits_{m \le 2b+1 \le 2m-3}\binom{2m}{2b+3}t^{2b+1},$$

(b) $PS_\mathbf{Q}(M_{4l}') = \sum\limits_{0 \le 2a \le 2l-2}\binom{4l-1}{2a}t^{2a} + \binom{4l-1}{2l-2}t^{2l-1}$

$$+ \sum\limits_{2l+1 \le 2b+1 \le 4l-3}\binom{4l-1}{2b+3}t^{2b+1},$$

$$PS_\mathbf{Q}(M_{4l+2}') = \sum\limits_{0 \le 2a \le 2l-2}\binom{4l+1}{2a}t^{2a} + \binom{4l+1}{2l+1}t^{2l}$$

$$+ \sum\limits_{2l+1 \le 2b+1 \le 4l-1}\binom{4l+1}{2b+3}t^{2b+1},$$

*where $\binom{a}{b}$ denotes the binomial coefficient.*

Next we go to (iii). By setting $z_i = u_{i+2} - u_{i+1}$ $(1 \le i \le n-2)$, $z_{n-1} = u_1 - u_n$, and identifying $\mathbf{R}^2$ with $\mathbf{C}$, we can write $M_n$ $(n \ge 3)$ as

(1.6) $$M_n \cong \{(z_1, \ldots, z_{n-1}) \in (S^1)^{n-1} : z_1 + \cdots + z_{n-1} - 1 = 0\}.$$

Let $i_n : M_n \hookrightarrow (S^1)^{n-1}$ be the inclusion.

As we have mentioned, $(S^1)^{n-1}$ approximates the topology of $M_n$ up to some dimension (cf. Proposition 2.2). However, for an odd $n = 2m+1$, our low-dimensional computations lead us to give up the hope that $(S^1)^{2m}/\sigma$ might approximate $M_{2m+1}' = M_{2m+1}/\sigma$, where $\sigma$ acts on $(S^1)^{2m}$ in the same way as in (1.4). The essential reason for this is that the action of $\sigma$ on $(S^1)^{2m}$ is not free, although on $M_{2m+1}$ is.

Thus we define $V_{2m}$ by

(1.7) $$V_{2m} = \left\{(S^1)^{2m} - \Sigma_{2m}\right\}/\sigma,$$

where we set

$$\Sigma_{2m} = \{(z_1, \ldots, z_{2m}) \in (S^1)^{2m} : z_i = \pm 1 \ (1 \le i \le 2m)\}.$$

Let $i_{2m+1}' : M_{2m+1}' \hookrightarrow V_{2m}$ be the inclusion. Then we have the following map of covering

spaces:

$$
\begin{array}{ccc}
\mathbf{Z}_2 & \!\!=\!\!\!=\!\!\!=\!\! & \mathbf{Z}_2 \\
\downarrow & & \downarrow \\
M_{2m+1} & \xrightarrow{\ i_{2m+1}\ } & (S^1)^{2m} - \Sigma_{2m} \\
\downarrow & & \downarrow \\
M'_{2m+1} & \xrightarrow{\ i'_{2m+1}\ } & V_{2m}.
\end{array}
$$

(1.8)

Note that $(S^1)^{2m} - \Sigma_{2m}$ is a maximal subspace of $(S^1)^{2m}$ on which $\sigma$ acts freely. Thus it is natural to consider the topology of $V_{2m}$.

Now concerning the relation between $M'_{2m+1}$ and $V_{2m}$, we have the following theorem.

**Theorem D.** $(i'_{2m+1})_* : \pi_q(M'_{2m+1}) \to \pi_q(V_{2m})$ *are isomorphisms for* $q \le m-2$, *and an epimorphism for* $q = m - 1$.

Concerning (iv), we have the following:

**Theorem E.** $H_*(V_{2m}; \mathbf{Z})$ *is given by*

$$
H_q(V_{2m}; \mathbf{Z}) = \begin{cases}
\underset{\binom{2m}{q}}{\oplus} \mathbf{Z} & q : even \ \le 2m - 2 \\[2mm]
\underset{\underset{i \le q}{\Sigma} \binom{2m}{i}}{\oplus} \mathbf{Z}_2 & q : odd \ \le 2m - 3 \\[2mm]
\underset{2^{2m}-1}{\oplus} \mathbf{Z} & q = 2m - 1 \\[2mm]
0 & otherwise,
\end{cases}
$$

*where* $\underset{\binom{2m}{q}}{\oplus} \mathbf{Z}$ *denotes the* $\binom{2m}{q}$-*times direct sum of* $\mathbf{Z}$.

Note that Theorems D and E give $H_q(M'_{2m+1}; \mathbf{Z})$ for $q \le m - 2$.

REMARK 1.9. By the Poincaré-Lefschetz duality $H^q((S^1)^{2m}/\sigma, \Sigma_{2m}; \mathbf{Z}) \cong H_{2m-q}(V_{2m}; \mathbf{Z})$, knowing $H_*(V_{2m}; \mathbf{Z})$ is equivalent to knowing $H_*((S^1)^{2m}/\sigma; \mathbf{Z})$.

Concerning (v), we have the following:

**Theorem F.** (a) *For an odd $m$, we have*

$$H_q(M'_{2m+1}; \mathbf{Z}) = \begin{cases} \underset{\binom{2m}{q}}{\oplus} \mathbf{Z} & q : even \ \leq m - 1 \\ \underset{\underset{i \leq q}{\Sigma}\binom{2m}{i}}{\oplus} \mathbf{Z}_2 & q : odd \ \leq m - 2 \\ \underset{\binom{2m}{q+2}}{\oplus} \mathbf{Z} \oplus \underset{\underset{i \geq q+3}{\Sigma}\binom{2m}{i}}{\oplus} \mathbf{Z}_2 & q : odd \ \geq m \\ 0 & otherwise. \end{cases}$$

(b) *For an even $m$, we have*

$$H_q(M'_{2m+1}; \mathbf{Z}) = \begin{cases} \underset{\binom{2m}{q}}{\oplus} \mathbf{Z} & q : even \ \leq m - 2 \\ \underset{\underset{i \leq q}{\Sigma}\binom{2m}{i}}{\oplus} \mathbf{Z}_2 & q : odd \ \leq m - 3 \\ \underset{\binom{2m}{m-1}}{\oplus} \mathbf{Z} \oplus Tor_{m-1} & q = m - 1 \\ \underset{\binom{2m}{q+2}}{\oplus} \mathbf{Z} \oplus \underset{\underset{i \geq q+3}{\Sigma}\binom{2m}{i}}{\oplus} \mathbf{Z}_2 & q : odd \ \geq m + 1 \\ 0 & otherwise, \end{cases}$$

*where $Tor_{m-1}$, the torsion submodule of $H_{m-1}(M'_{2m+1}; \mathbf{Z})$, satisfies that*
$\dim_{\mathbf{Z}_2} Tor_{m-1} \otimes \mathbf{Z}_2 = \underset{i \leq m-2}{\Sigma}\binom{2m}{i}$.

*Thus, in particular, $H_{even}(M'_{2m+1}; \mathbf{Z})$ are torsion free for all $m$.*

REMARK 1.10. (a) By Theorems D, E and F, we see that $(i'_{2m+1})_* : H_{m-1}(M'_{2m+1}; \mathbf{Z}) \to H_{m-1}(V_{2m}; \mathbf{Z})$ is an isomorphism when $m$ is odd, but not an isomorphism when $m$ is even.

(b) In order to prove Theorem F, we first determine $H_*(M'_{2m+1}; \mathbf{Z}_2)$, which is given in Proposition 5.1. In particular, we see that $(i'_{2m+1})_* : H_{m-1}(M'_{2m+1}; \mathbf{Z}_2) \to H_{m-1}(V_{2m}; \mathbf{Z}_2)$ is an isomorphism for all $m$ (cf. Remark 5.2).

This paper is organized as follows. In §2 we recall the results of [7], then prove Theorems A, B and D. In §3 we prove Theorem C. In §4 we prove Theorem E, and in §5 we prove Theorem F.

## 2. Proofs of Theorems A, B and D

In [7], the following theorem is proved.

**Theorem 2.1.** $H_*(M_n; \mathbf{Z})$ *are free* $\mathbf{Z}$-*modules and the Poincaré polynomials* $PS(M_n) = \sum_\lambda \mathrm{rank} H_\lambda(M_n; \mathbf{Z}) t^\lambda$ *are given by*

$$PS(M_{2m+1}) = \sum_{\lambda=0}^{m-2} \binom{2m}{\lambda} t^\lambda + 2 \binom{2m}{m-1} t^{m-1} + \sum_{\lambda=m}^{2m-2} \binom{2m}{\lambda+2} t^\lambda,$$

$$PS(M_{2m}) = \sum_{\lambda=0}^{m-2} \binom{2m-1}{\lambda} t^\lambda + \binom{2m}{m-1} t^{m-1} + \sum_{\lambda=m}^{2m-3} \binom{2m-1}{\lambda+2} t^\lambda.$$

The essential facts to prove Theorem 2.1 are the following three propositions.

**Proposition 2.2.** (i) $(i_{2m+1})_* : \pi_q(M_{2m+1}) \to \pi_q((S^1)^{2m})$ *are isomorphisms for* $q \leq m-2$, *and an epimorphism for* $q = m-1$.

(ii) $(i_{2m})_* : H_q(M_{2m}; \mathbf{Z}) \to H_q((S^1)^{2m-1}; \mathbf{Z})$ *are isomorphisms for* $q \leq m-2$, *and an epimorphism for* $q = m-1$.

**Proposition 2.3.** (i) $M_{2m+1}$ *is an orientable manifold of dimension* $2m-2$. *Thus the Poincaré duality homomorphisms* $\cap[M_{2m+1}] : H^q(M_{2m+1}; \mathbf{Z}) \to H_{2m-2-q}(M_{2m+1}; \mathbf{Z})$ *are isomorphisms for all* $q$, *where* $[M_{2m+1}] \in H_{2m-2}(M_{2m+1}; \mathbf{Z})$ *is a fundamental class.*

(ii) $M_{2m}$ *is a manifold of dimension* $2m-3$ *with singular points.* $(u_1, \ldots, u_{2m}) \in M_{2m}$ *is a singular point iff all of* $u_i$ *lie on the x-axis. Moreover every singular point of* $M_{2m}$ *is a cone-like singularity and has a neighborhood as* $C(S^{m-2} \times S^{m-2})$. *Thus the Poincaré duality homomorphisms* $\cap[M_{2m}] : H^q(M_{2m}; \mathbf{Z}) \to H_{2m-3-q}(M_{2m}; \mathbf{Z})$ *are isomorphisms for* $q \leq m-3$ *or* $q \geq m$, *an epimorphism for* $q = m-1$, *and a monomorphism for* $q = m-2$.

**Proposition 2.4.** (i) $\chi(M_{2m+1}) = (-1)^{m+1} \binom{2m}{m}$.

(ii) $\chi(M_{2m}) = (-1)^{m+1} \binom{2m-1}{m}$.

REMARK 2.5.    In order to prove Theorem 2.1, the homological assertion is sufficient for Proposition 2.2 (i). But actually we can prove the homotopical assertion.

Proof of Theorem A.    Since $\sigma$ acts freely on $M_{2m+1}$, and $M_{2m}^\sigma$ (=the fixed point set of the involution) equals to the set of singular points in $M_{2m}$, all of the assertions except the type of the singular points of $M_{2m}'$ are deduced from Proposition 2.3.

Let $(z_1, \ldots, z_{2m-1})$ be a singular point of $M_{2m}$ in the identification of (1.6). By Proposition 2.3, we must have $z_i = \pm 1$ $(1 \leq i \leq 2m-1)$. As the symmetric group on $(2m-1)$-letters acts on $M_{2m}$, we can assume that $z_i = 1$ $(1 \leq i \leq m)$ and $z_i = -1$ $(m+1 \leq i \leq 2m-1)$. A neighborhood of $(z_1, \ldots, z_{2m-1})$ in $(S^1)^{2m-1}$ is written

by

$$\left\{ \left( \begin{pmatrix} \sqrt{1-y_1^2} \\ y_1 \end{pmatrix}, \ldots, \begin{pmatrix} \sqrt{1-y_m^2} \\ y_m \end{pmatrix}, \begin{pmatrix} -\sqrt{1-y_{m+1}^2} \\ y_{m+1} \end{pmatrix}, \ldots, \begin{pmatrix} -\sqrt{1-y_{2m-1}^2} \\ y_{2m-1} \end{pmatrix} \right) : \right.$$
$$\left. -\epsilon \leq y_i \leq \epsilon \ (1 \leq i \leq 2m-1) \right\},$$

where $\epsilon > 0$ is a fixed small number. As $\epsilon$ is small, it is easy to see that we can write this neighborhood as

$$\left\{ \left( \begin{pmatrix} 1 - \frac{1}{2}y_1^2 \\ y_1 \end{pmatrix}, \ldots, \begin{pmatrix} 1 - \frac{1}{2}y_m^2 \\ y_m \end{pmatrix}, \begin{pmatrix} -1 + \frac{1}{2}y_{m+1}^2 \\ y_{m+1} \end{pmatrix}, \ldots, \begin{pmatrix} -1 + \frac{1}{2}y_{2m-1}^2 \\ y_{2m-1} \end{pmatrix} \right) : \right.$$
$$\left. -\epsilon \leq y_i \leq \epsilon \ (1 \leq i \leq 2m-1) \right\}.$$

Thus a neighborhood of a singular point in $M_{2m}$ is written as a subspace of $\mathbf{R}^{2m-1}$ defined by two equations

(2.6)
$$\begin{cases} y_1^2 + \cdots + y_m^2 - y_{m+1}^2 - \cdots - y_{2m-1}^2 = 0 \\ y_1 + \cdots + y_m + y_{m+1} + \cdots + y_{2m-1} = 0. \end{cases}$$

By a linear transformation of parameters, we can write the quadratic form of (2.6), i.e.,

$$y_1^2 + \cdots + y_{m-1}^2 + (y_1 + \cdots + y_{m-1} + y_{m+1} + \cdots + y_{2m-1})^2 - y_{m+1}^2 - \cdots - y_{2m-1}^2,$$

as

$$w_1^2 + \cdots + w_{m-1}^2 - w_m^2 - \cdots - w_{2m-2}^2.$$

Thus a singular point of $M_{2m}$ has a neighborhood $C\{(w_1, \ldots, w_{m-1}, w_m, \ldots, w_{2m-2}) : w_1^2 + \cdots + w_{m-1}^2 = 1, w_m^2 + \cdots + w_{2m-2}^2 = 1\}$, which is homeomorphic to $C(S^{m-2} \times S^{m-2})$.

Now it is clear that $\sigma w_i = -w_i$. Hence a singular point of $M'_{2m}$ has a neighborhood $C(S^{m-2} \times_{\mathbf{Z}_2} S^{m-2})$, where $\sigma(\zeta_1, \zeta_2) = (-\zeta_1, -\zeta_2)$ $(\zeta_1, \zeta_2 \in S^{m-2})$. $\qquad\square$

Proof of Theorem B.   For $F = \mathbf{Z}_p$ ($p$ : an odd prime) or $\mathbf{Q}$, we have that $H_*(M'_n; F) \cong H_*(M_n; F)^\sigma$ (= the fixed point set of $H_*(M_n; F)$ under the $\sigma$-action) (see for example [2]). As $H_*(M_n; \mathbf{Z})$ are free modules by Theorem 2.1, we have that $\dim_{\mathbf{Z}_p} H_q(M'_n; \mathbf{Z}_p) = \dim_{\mathbf{Q}} H_q(M'_n; \mathbf{Q})$. Hence Theorem B follows. $\qquad\square$

Proof of Theorem D.  Let $j_{2m} : (S^1)^{2m} - \Sigma_{2m} \hookrightarrow (S^1)^{2m}$ be the inclusion. Since $\Sigma_{2m}$ is a discrete set, the general position argument shows that $(j_{2m})_* : \pi_q((S^1)^{2m} - \Sigma_{2m}) \to \pi_q((S^1)^{2m})$ are isomorphisms for $q \leq 2m - 2$. Then Proposition 2.2 (i) shows that $(i_{2m+1})_* : \pi_q(M_{2m+1}) \to \pi_q((S^1)^{2m} - \Sigma_{2m})$ are isomorphisms for $q \leq m - 2$ and an epimorphism for $q = m-1$, where $i_{2m+1} : M_{2m+1} \hookrightarrow (S^1)^{2m} - \Sigma_{2m}$ is the inclusion.

By comparing the homotopy exact sequences of two covering spaces of (1.8), we see that $(i'_{2m+1})_* : \pi_q(M'_{2m+1}) \to \pi_q(V_{2m})$ are isomorphisms for $q \leq m - 2$ and an epimorphism for $q = m - 1$, where $i'_{2m+1} : M'_{2m+1} \hookrightarrow V_{2m}$ is the map induced from the $\sigma$-equivariant inclusion $i_{2m+1} : M_{2m+1} \hookrightarrow (S^1)^{2m} - \Sigma_{2m}$.

This completes the proof of Theorem D.                                    □

## 3.  Proof of Theorem C

Let $i_n : M_n \hookrightarrow (S^1)^{n-1}$ be the inclusion. Note that $i_n$ is a $\sigma$-equivariant map. Hence $(i_n)_* : H_*(M_n; \mathbf{Q}) \to H_*((S^1)^{n-1}; \mathbf{Q})$ is also a $\sigma$-equivariant homomorphism. Since $H_*(M'_n; \mathbf{Q}) = H_*(M_n; \mathbf{Q})^\sigma$, Proposition 2.2 tells us the following:

**Proposition 3.1.**   (i) *For* $q \leq m - 2$, *we have*

$$H_q(M'_{2m+1}; \mathbf{Q}) = \begin{cases} \bigoplus\limits_{\binom{2m}{q}} \mathbf{Q} & q : even \\ 0 & q : odd. \end{cases}$$

(ii) *For* $q \leq m - 2$, *we have*

$$H_q(M'_{2m}; \mathbf{Q}) = \begin{cases} \bigoplus\limits_{\binom{2m-1}{q}} \mathbf{Q} & q : even \\ 0 & q : odd. \end{cases}$$

We assume the truth of the following Lemma for the moment. Let $[M_n] \in H_{n-3}(M_n; \mathbf{Q})$ be the fundamental class.

**Lemma 3.2.**   $\sigma_*[M_n] = (-1)^n [M_n]$.

Then we have the following:

**Proposition 3.3.**   (i) *For* $q \geq m$, *we have*

$$H_q(M'_{2m+1}; \mathbf{Q}) = \begin{cases} 0 & q : even \\ \bigoplus\limits_{\binom{2m}{q+2}} \mathbf{Q} & q : odd. \end{cases}$$

(ii) *For $q \geq m$, we have*

$$
H_q(M'_{2m}; \mathbf{Q}) = \begin{cases} 0 & q : \text{even} \\ \underset{\binom{2m-1}{q+2}}{\oplus} \mathbf{Q} & q : \text{odd}. \end{cases}
$$

Proof of Proposition 3.3. Take an element $\alpha \in H_q(M_{2m+1}; \mathbf{Q})$ $(q \geq m)$. By Proposition 2.3, there is an element $f \in H^{2m-2-q}(M_{2m+1}; \mathbf{Q})$ such that $\alpha = f \cap [M_{2m+1}]$. As $\sigma_*(f \cap [M_{2m+1}]) = \sigma^* f \cap \sigma_*[M_{2m+1}] = -\sigma^* f \cap [M_{2m+1}]$, we have that

$$
H_q(M_{2m+1}; \mathbf{Q})^\sigma = \left\{ f \in H^{2m-2-q}(M_{2m+1}; \mathbf{Q}) : \sigma^* f = -f \right\}.
$$

Now (i) follows from Proposition 3.1.
(ii) can be proved similarly. $\qquad\square$

Now in order to determine $H_*(M'_n; \mathbf{Q})$, we need to know only $H_{m-1}(M'_{2m+1}; \mathbf{Q})$ and $H_{m-1}(M'_{2m}; \mathbf{Q})$, which are determined if we know $\chi(M'_n)$.

**Proposition 3.4** ([5]). (i) $\chi(M'_{2m+1}) = (-1)^{m+1} \binom{2m-1}{m}$.

(ii) $\chi(M'_{2m}) = \begin{cases} 0 & m : \text{even} \\ \binom{2m-1}{m} & m : \text{odd}. \end{cases}$

Proof. By a general formula of an involution (see for example [1]), we have $\chi(M_n) + \chi(M_n^\sigma) = 2\chi(M'_n)$. Then the result follows from Proposition 2.4. $\qquad\square$

Proof of Lemma 3.2. First we treat the case of $n = 2m + 1$. We define a volume element $\omega$ of $M_{2m+1}$ as follows. Fix $(z_1, \ldots, z_{2m}) \in M_{2m+1}$ in the identification of (1.6). It is easy to see that the tangent space $T_{(z_1,\ldots,z_{2m})} M_{2m+1}$ is given by

$$
(3.5) \qquad T_{(z_1,\ldots,z_{2m})} M_{2m+1} \cong \left\{ \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_{2m} \end{pmatrix} \in \mathbf{R}^{2m} : \xi_1 z_1 + \cdots + \xi_{2m} z_{2m} = 0 \right\}.
$$

Write $z_i$ as $(x_i, y_i)$. Then for $\eta_1, \ldots, \eta_{2m-2} \in T_{(z_1,\ldots,z_{2m})} M_{2m+1}$, we set

$$
(3.6) \qquad \omega(\eta_1, \ldots, \eta_{2m-2}) = \det \left( \eta_1, \ldots, \eta_{2m-2}, \begin{pmatrix} x_1 \\ \vdots \\ x_{2m} \end{pmatrix}, \begin{pmatrix} y_1 \\ \vdots \\ y_{2m} \end{pmatrix} \right).
$$

It is easy to see that $\omega$ is nowhere zero on $M_{2m+1}$.

For $\eta = \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_{2m} \end{pmatrix} \in T_{(z_1,\ldots,z_{2m})}M_{2m+1}$, we see that

$$di_{2m+1}(\eta) = \xi_1(\sqrt{-1}z_1) + \cdots + \xi_{2m}(\sqrt{-1}z_{2m}),$$

where $i_{2m+1} : M_{2m+1} \hookrightarrow (S^1)^{2m}$ denotes the inclusion. Hence we see that $d\sigma : T_{(z_1,\ldots,z_{2m})}M_{2m+1} \to T_{(\bar{z}_1,\ldots,\bar{z}_{2m})}M_{2m+1}$ is given by

$$(3.7) \qquad\qquad\qquad d\sigma(\eta) = -\eta.$$

Now the formulae $d\sigma(\eta_i) = -\eta_i$ and $\sigma(x_i, y_i) = (x_i, -y_i)$ tell us that $(\sigma^*\omega)(\eta_1, \ldots, \eta_{2m-2}) = -\omega(\eta_1, \ldots, \eta_{2m-2})$. Hence $\sigma^*\omega = -\omega$ and the result follows.

Next we treat the case of $n = 2m$. Let $\bar{M}_{2m}$ be $M_{2m} - \{$singular points$\}$. By the same argument as in the case of $n = 2m + 1$, we see that $\sigma : \bar{M}_{2m} \to \bar{M}_{2m}$ preserves orientation. As $H_c^{2m-3}(\bar{M}_{2m}; \mathbf{Q}) \cong H^{2m-3}(M_{2m}; \mathbf{Q})$ ($H_c =$ cohomology with compact supports), the result follows. $\qquad\qquad\qquad\qquad\qquad\qquad\square$

## 4. Proof of Theorem E

First we determine $H_{2m-1}(V_{2m}; \mathbf{Z})$. The Poincaré-Lefschetz duality tells us that $H_{2m-1}(V_{2m}; \mathbf{Z}) \cong H^1((S^1)^{2m}/\sigma, \Sigma_{2m}; \mathbf{Z})$. As $H^1((S^1)^{2m}/\sigma; \mathbf{Z}) = 0$, we have $H_{2m-1}(V_{2m}; \mathbf{Z}) \cong \underset{2^{2m}-1}{\oplus} \mathbf{Z}$.

As $V_{2m}$ is a non-compact manifold of dimension $2m$, we have $H_q(V_{2m}; \mathbf{Z}) = 0$ ($q \geq 2m$). Hence in order to complete the proof of Theorem E, we need to determine $H_q(V_{2m}; \mathbf{Z})$ ($q \leq 2m - 2$).

Recall that we have a fibration $(S^1)^{2m} - \Sigma_{2m} \to V_{2m} \to \mathbf{R}P^\infty$. Set $F_{2m} = (S^1)^{2m} - \Sigma_{2m}$. The local systems of this fibration of dimensions less than or equal to $2m - 2$ are easy to describe: We write the generator of $\pi_1(\mathbf{R}P^\infty)$ by $\sigma$. Then as a $\sigma$-module, we have

$$(4.1) \qquad\qquad H_q(F_{2m}; \mathbf{Z}) \cong H_q((S^1)^{2m}; \mathbf{Z}) \ (q \leq 2m - 2).$$

Let $\{E_{s,t}^r\}$ be the $\mathbf{Z}$-coefficient homology Serre spectral sequence of the above fibration. It is elementary to describe $E_{s,t}^2$ ($t \neq 2m - 1$) by using the following fact: We define a $\sigma$-module $\mathcal{S}$ to be the free abelian group of rank 1 on which $\sigma$ acts by $-1$. Then have that

$$(4.2) \qquad\qquad H_q(\mathbf{R}P^\infty; \mathcal{S}) = \begin{cases} \mathbf{Z}_2 & q : \text{ even} \\ 0 & q : \text{ odd.} \end{cases}$$

REMARK 4.3. For our reference, we give $E_{s,2m-1}^2$. Let $\mathcal{T}$ be the free abelian group of rank 2 on which $\sigma$ acts by $\begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}$. And let $\sigma$ act on $\mathbf{Z}$ trivially. Then we can prove

that

$$H_{2m-1}(F_{2m}; \mathbf{Z}) \cong \underset{2m}{\oplus} \mathcal{T} \oplus \underset{2^{2m}-2m-1}{\oplus} \mathbf{Z}.$$

As

$$H_q(\mathbf{R}P^\infty; \mathcal{T}) = \begin{cases} \mathbf{Z} & q = 0 \\ 0 & q > 0, \end{cases}$$

we can determine $E_{s,2m-1}^2$.

We return to $E_{s,t}^2$ ($t \neq 2m - 1$). By the dimensional reason, we have the following:

**Proposition 4.4.**   *For $s + t \leq 2m - 2$, we have that $E_{s,t}^2 \cong E_{s,t}^\infty$.*

Hence in order to complete the proof of Theorem E, it suffices to determine the extensions of $E_{s,t}^\infty$, where $s + t$ are odd $\leq 2m - 3$. To do so, it is convenient to study $H_*(V_{2m}; \mathbf{Z}_2)$.

**Proposition 4.5.**   *For $q \leq 2m - 2$, we have*

$$H_q(V_{2m}; \mathbf{Z}_2) = \underset{\underset{i \leq q}{\Sigma}\binom{2m}{i}}{\oplus} \mathbf{Z}_2.$$

From Proposition 4.5, we see that the extensions of $E_{s,t}^\infty$ ($s + t \leq 2m - 2$) are trivial. Hence Theorem E follows.

Thus in order to complete the proof of Theorem E, we need to prove Proposition 4.5, which we prove for the rest of this section.

Let $\{E_r^{s,t}\}$ be the $\mathbf{Z}_2$-coefficient cohomology Serre spectral sequence of the fibration $F_{2m} \to V_{2m} \to \mathbf{R}P^\infty$. We prove the following:

**Lemma 4.6.**   $d_2 : E_2^{0,1} \to E_2^{2,0}$ *equals to 0.*

Lemma 4.6 tells us that elements of $E_2^{s,t}$ ($t \leq 2m - 2$) are permanent cycles. Hence Proposition 4.5 follows.

Proof of Lemma 4.6.   Suppose that Lemma 4.6 fails. Then we have $H^1(V_{2m}; \mathbf{Z}_2) = \underset{2m}{\oplus}\mathbf{Z}_2$. By Theorem D and the $\mathbf{Z}_2$-coefficient Poincaré duality of $M_{2m+1}'$, we have $H_{2m-3}(M_{2m+1}'; \mathbf{Z}_2) = \underset{2m}{\oplus}\mathbf{Z}_2$. Since $H_{2m-3}(M_{2m+1}'; \mathbf{Q}) = \underset{2m}{\oplus}\mathbf{Q}$ by Theorem C (a), we have

(4.7) $$H_{2m-3}(M_{2m+1}'; \mathbf{Z}) = \underset{2m}{\oplus}\mathbf{Z}.$$

By Theorem C (a), we have $H_{2m-2}(M'_{2m+1}; \mathbf{Q}) = 0$. Hence by Theorem A (a), $M'_{2m+1}$ is a non-orientable manifold of dimension $2m - 2$. Thus we have $H_{2m-2}(M'_{2m+1}; \mathbf{Z}) = 0$. Then by (4.7), we have $H_{2m-2}(M'_{2m+1}; \mathbf{Z}_2) = 0$. This contradicts the fact that $H_{2m-2}(M'_{2m+1}; \mathbf{Z}_2) = \mathbf{Z}_2$, i.e., $M'_{2m+1}$ is a compact manifold of dimension $2m - 2$.

This completes the proof of Lemma 4.6, and hence also that of Theorem E.        □

## 5.  Proof of Theorem F

In order to calculate $H_*(M'_{2m+1}; \mathbf{Z})$, first we need to determine $H_*(M'_{2m+1}; \mathbf{Z}_2)$. By the Poincaré duality, it suffices to determine $H_q(M'_{2m+1}; \mathbf{Z}_2)$ $(q \leq m - 1)$, which are given by the following:

**Proposition 5.1.**  *For $q \leq m - 1$, we have*

$$H_q(M'_{2m+1}; \mathbf{Z}_2) = \bigoplus_{\sum_{i \leq q} \binom{2m}{i}} \mathbf{Z}_2.$$

Proof.    First, $H_q(M'_{2m+1}; \mathbf{Z}_2)$ $(q \leq m - 2)$ are determined by Theorems D and E together with the universal coefficient theorem. Then $H_{m-1}(M'_{2m+1}; \mathbf{Z}_2)$ is determined by Proposition 3.4.        □

REMARK 5.2.    From Theorems D, E and Proposition 5.1, we see that $(i'_{2m+1})_* :$ $H_{m-1}(M'_{2m+1}; \mathbf{Z}_2) \to H_{m-1}(V_{2m}; \mathbf{Z}_2)$ is an isomorphism for all $m$.

Now we begin to determine $H_*(M'_{2m+1}; \mathbf{Z})$.

(I) $H_{even}(M'_{2m+1}; \mathbf{Z})$.

These modules are determined from Theorem C and the following:

**Proposition 5.3.**    $H_{even}(M'_{2m+1}; \mathbf{Z})$ *are torsion free.*

Proof.    We can inductively prove this proposition from Theorem C and Proposition 5.1 together with the universal coefficient theorem.        □

(II) $H_{odd}(M'_{2m+1}; \mathbf{Z})$.

In order to determine these modules from Theorem C and Proposition 5.1, we need to prove the non-existence of higher two-torsions, i.e., elements of order $2^i$ $(i \geq 2)$.

Let $p : M_{2m+1} \times \mathbf{R} \underset{\sigma}{\to} M'_{2m+1}$ be the real line bundle associated to the covering space $M_{2m+1} \to M'_{2m+1}$. And let $O(M_{2m+1} \underset{\sigma}{\times} \mathbf{R})$ denote the local system of the above vector bundle. Finally, let $O(TM'_{2m+1})$ denote the local system of $TM'_{2m+1}$, the tangent bundle of $M'_{2m+1}$.

Concerning these local systems, we have the following:

**Lemma 5.4.** *As local systems on $M'_{2m+1}$, we have $O(M_{2m+1} \underset{\sigma}{\times} \mathbf{R}) \cong O(TM'_{2m+1})$.*

Proof. Let $\mathbf{R}^2 \to \nu \to M_{2m+1}$ denote the normal bundle of $M_{2m+1}$ in $(S^1)^{2m}$ (cf. (1.6)). As $TM_{2m+1} \oplus \nu \cong T((S^1)^{2m})|M_{2m+1}$, we have

$$(5.5) \qquad TM'_{2m+1} \oplus \nu/\sigma \cong TV_{2m}|M'_{2m+1},$$

where $\nu/\sigma \to M'_{2m+1}$ denotes the vector bundle obtained from $\nu \to M_{2m+1}$ by the action of $\sigma$.

We study $\nu/\sigma$. Recall that $TM_{2m+1}$ is given by (3.5). Similarly, for $(z_1, \ldots, z_{2m}) \in M_{2m+1}$, we have

$$T_{(z_1,\ldots,z_{2m})}((S^1)^{2m}) \cong \left\{ \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_{2m} \end{pmatrix} \in \mathbf{R}^{2m} \right\}.$$

Hence by assigning $\begin{pmatrix} \xi_1 \\ \vdots \\ \xi_{2m} \end{pmatrix} \in \nu_{(z_1,\ldots,z_{2m})}$ to $\xi_1 z_1 + \cdots + \xi_{2m} z_{2m}$, we have

$$(5.6) \qquad \nu \cong M_{2m+1} \times \mathbf{R}^2.$$

Under this identification, the bundle homomorphism $d\sigma : \nu \to \nu$ is given by

$$(5.7) \qquad d\sigma((z_1, \ldots, z_{2m}); \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}) = ((\bar{z}_1, \ldots, \bar{z}_{2m}); \begin{pmatrix} -v_1 \\ v_2 \end{pmatrix}),$$

(cf. (3.7)).

Then (5.6)-(5.7) tell us that

$$(5.8) \qquad \nu/\sigma \cong M_{2m+1} \underset{\sigma}{\times} \mathbf{R} \oplus M'_{2m+1} \times \mathbf{R}.$$

Now, as $V_{2m}$ is orientable, we see from (5.5) and (5.8) that

$$(5.9) \qquad O(TM'_{2m+1}) \otimes O(M_{2m+1} \underset{\sigma}{\times} \mathbf{R}) \cong \mathbf{Z},$$

where $\mathbf{Z}$ denotes the simple local system on $M'_{2m+1}$. By taking a tensor $\otimes O(TM'_{2m+1})$ on both sides of (5.9), the result follows. $\qquad\square$

Let us denote the local systems $O(M_{2m+1} \underset{\sigma}{\times} \mathbf{R}) \cong O(TM'_{2m+1})$ (cf. Lemma 5.4) by $\mathcal{Z}$.

(A) *The case of an odd $m$.*

We can determine $H_q(M'_{2m+1}; \mathbf{Z})$ ($q$ : odd $\leq m - 2$) by Theorems D and E. Thus we need to determine $H_q(M'_{2m+1}; \mathbf{Z})$ ($q$ : odd $\geq m$). By the Poincaré duality: $H_q(M'_{2m+1}; \mathbf{Z}) \cong H^{2m-2-q}(M'_{2m+1}; \mathcal{Z})$, it suffices to determine $H^r(M'_{2m+1}; \mathcal{Z})$ ($r$ : odd $\leq m - 2$).

Consider the Gysin sequence of $p : M_{2m+1} \underset{\sigma}{\times} \mathbf{R} \to M'_{2m+1}$:

$$\cdots \xrightarrow{\psi} H^{r-1}(M'_{2m+1}; \mathcal{Z}) \xrightarrow{\mu} H^r(M'_{2m+1}; \mathbf{Z}) \xrightarrow{p^*} H^r(M_{2m+1}; \mathbf{Z})$$
$$\xrightarrow{\psi} H^r(M'_{2m+1}; \mathcal{Z}) \xrightarrow{\mu} \cdots .$$

**Lemma 5.10.** *For an odd $r \leq m - 2$, we have*

(i) $H^r(M'_{2m+1}; \mathbf{Z}) = 0$.

(ii) $H^r(M_{2m+1}; \mathbf{Z})$ *is a free module.*

(iii) *The order of a torsion element of $H^{r+1}(M'_{2m+1}; \mathbf{Z})$ is exactly 2, i.e., $H^{r+1}(M'_{2m+1}; \mathbf{Z})$ does not contain higher two-torsions.*

Proof.    This lemma is an easy consequence of Theorems D, E, 2.1 and Proposition 5.3.                                                                                    □

Now suppose that $H^r(M'_{2m+1}; \mathcal{Z})$ contains a higher two-torsion. Then by Lemma 5.10 (iii), Ker $[\mu : H^r(M'_{2m+1}; \mathcal{Z}) \to H^{r+1}(M'_{2m+1}; \mathbf{Z})]$ contains a torsion element.

But by Lemma 5.10 (i)-(ii), Im $[\psi : H^r(M_{2m+1}; \mathbf{Z}) \to H^r(M'_{2m+1}; \mathcal{Z})]$ is a free module. This is a contradiction. Thus $H^r(M'_{2m+1}; \mathcal{Z})$ ($r$ : odd $\leq m - 2$) does not contain higher two-torsions.

This completes the proof of Theorem F (a).

(B) *The case of an even $m$.*

As in (A), it suffices to determine $H^r(M'_{2m+1}; \mathcal{Z})$ ($r$ : odd $\leq m - 1$). For an odd $r \leq m - 3$, Lemma 5.10 applies and, by the same argument as in (A), we see that $H^r(M'_{2m+1}; \mathcal{Z})$ does not contain higher two-torsions.

But Lemma 5.10 fails when $r = m - 1$. Thus our argument cannot apply in this case. This completes the proof of Theorem F (b).                                       □

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