Kamiyama, Y.
Osaka J. Math.
36 (1999), 731–745

TOPOLOGY OF EQUILATERAL POLYGON LINKAGES IN THE EUCLIDEAN PLANE MODULO ISOMETRY GROUP

YASUHIKO KAMIYAMA

(Received September 24, 1997)

1. Introduction

We consider the configuration space M_n' of equilateral polygon linkages with n $(n \geq 3)$ vertices, each edge having length 1 in the Euclidean plane $\mathbf{R}^2$ modulo isometry group. More precisely, let $\mathcal{C}_n$ be

$$(1.1)\quad \mathcal{C}_n = \{(u_1, \dots, u_n) \in (\mathbf{R}^2)^n : |u_{i+1} - u_i| = 1 \ \ (1 \leq i \leq n-1), \ \ |u_1 - u_n| = 1\}.$$

Note that $\mathrm{Iso}(\mathbf{R}^2)$ (= the isometry group of $\mathbf{R}^2$, i.e., a semidirect product of $\mathbf{R}^2$ with $O(2)$), naturally acts on $\mathcal{C}_n$. We define M_n' by

$$(1.2)\qquad M_n' = \mathcal{C}_n / \mathrm{Iso}(\mathbf{R}^2).$$

We remark that M_n' has the following description: We set $M_n = \mathcal{C}_n/\mathrm{Iso}^+(\mathbf{R}^2)$, where $\mathrm{Iso}^+(\mathbf{R}^2)$ denotes the orientation preserving isometry group of $\mathbf{R}^2$, i.e., a semidirect product of $\mathbf{R}^2$ with $SO(2)$. Then we can write M_n as

$$(1.3)\qquad M_n = \{(u_1, \dots, u_n) \in \mathcal{C}_n : u_1 = (\frac{1}{2}, 0) \ \text{ and } \ u_2 = (-\frac{1}{2}, 0)\}.$$

M_n admits an involution $\sigma = \mathrm{Iso}(\mathbf{R}^2)/\mathrm{Iso}^+(\mathbf{R}^2)$ such that $M_n' = M_n/\sigma$. Under the identification of (1.3), σ is given by

$$(1.4)\qquad \sigma(u_1, \dots, u_n) = (\bar{u}_1, \dots, \bar{u}_n),$$

where $\bar{u}_i = (x_i, -y_i)$ when $u_i = (x_i, y_i)$.

Many topological properties of M_n are already known: First we know explicit topological type of M_n $(n \leq 5)$ [3],[4],[8]. Next we have the results on the smoothness of M_n [5],[7],[8]. Finally $H_*(M_n; \mathbf{Z})$ are determined in [6],[7] (cf. Theorem 2.1). In particular, the natural inclusion $i_n : M_n \hookrightarrow (S^1)^{n-1}$ (cf. (1.6)) induces isomorphisms of homology groups up to a certain dimension (cf. Proposition 2.2).

On the other hand, concerning M'_n, what we know already are the following: First we know the following examples.

EXAMPLES 1.5. $M'_3 = \{1\text{-point}\}, M'_4 = S^1$ and $M'_5 = \underset{5}{\sharp}\mathbf{R}P^2$, the five-times connected sum of $\mathbf{R}P^2$.

Next some assertions on the smoothness of M'_n are proved in [5]. However, we have few information on $H_*(M'_n; \mathbf{Z})$, although we know $\chi(M'_n)$, the Euler characteristic of M'_n [5] (cf. Proposition 3.4).

The purposes of this paper are as follows.

(i) We prove assertions on the smoothness of M'_n.

(ii) We determine $H_*(M'_n; \mathbf{Z}_p)$, where p is an odd prime, and $H_*(M'_n; \mathbf{Q})$.

In the following (iii)-(v), we assume n to be odd, and set $n = 2m+1$. Then by the results of (i) and (ii), M'_{2m+1} is a non-orientable manifold of dimension $2m-2$.

(iii) Find a space V_{2m} and an inclusion $i'_{2m+1} : M'_{2m+1} \hookrightarrow V_{2m}$ so that i'_{2m+1} induces isomorphisms of homotopy groups up to a certain dimension.

(iv) As V_{2m} is a natural space, we determine $H_*(V_{2m}; \mathbf{Z})$ completely. Then in particular we know $H_*(M'_{2m+1}; \mathbf{Z})$ up to some dimension by the result of (iii).

As we will see in Remark 1.9, knowing $H_*(V_{2m}; \mathbf{Z})$ is equivalent to knowing $H_*((S^1)^{2m}/\sigma; \mathbf{Z})$.

(v) Finally we determine $H_*(M'_{2m+1}; \mathbf{Z})$ except the possibility of higher two-torsions in $H_{m-1}(M'_{2m+1}; \mathbf{Z})$ when m is even.

Now we state our results. Concerning (i), we have the following:

Theorem A. (a) *M'_{2m+1} is a manifold of dimension $2m-2$.*

(b) *M'_{2m} is a manifold of dimension $2m-3$ with singular points. $(u_1, \dots, u_{2m}) \in M'_{2m}$ is a singular point iff all of u_i lie on the x-axis, i.e., the line determined by u_1 and u_2* (cf. (1.3)). *Moreover every singular point of M'_{2m} is a cone-like singularity and has a neighborhood as $C(S^{m-2} \times_{\mathbf{Z}_2} S^{m-2})$, where C denotes cone and action of $\mathbf{Z}_2$ on both factors is generated by the antipodal map.*

Concerning (ii), first we prove the following:

Theorem B. *$H_*(M'_n; \mathbf{Z})$ are odd-torsion free.*

Thus in order to know $H_*(M'_n; \mathbf{Z}_p)$, we need to know $H_*(M'_n; \mathbf{Q})$, which is given by the following:

Theorem C. *The Poincaré polynomials $PS_{\mathbf{Q}}(M'_n) = \Sigma_\lambda \dim_{\mathbf{Q}} H_\lambda(M'_n; \mathbf{Q}) t^\lambda$ are given by*

$$\text{(a) } PS_{\mathbf{Q}}(M'_{2m+1}) = \sum_{0 \le 2a \le m-2} \binom{2m}{2a} t^{2a} + \binom{2m}{m-1} t^{m-1} + \sum_{m \le 2b+1 \le 2m-3} \binom{2m}{2b+3} t^{2b+1},$$

$$\text{(b) } PS_{\mathbf{Q}}(M'_{4l}) = \sum_{0 \le 2a \le 2l-2} \binom{4l-1}{2a} t^{2a} + \binom{4l-1}{2l-2} t^{2l-1} + \sum_{2l+1 \le 2b+1 \le 4l-3} \binom{4l-1}{2b+3} t^{2b+1},$$

$$PS_{\mathbf{Q}}(M'_{4l+2}) = \sum_{0 \le 2a \le 2l-2} \binom{4l+1}{2a} t^{2a} + \binom{4l+1}{2l+1} t^{2l} + \sum_{2l+1 \le 2b+1 \le 4l-1} \binom{4l+1}{2b+3} t^{2b+1},$$

where $\binom{a}{b}$ denotes the binomial coefficient.

Next we go to (iii). By setting $z_i = u_{i+2} - u_{i+1}$ $(1 \le i \le n-2)$, $z_{n-1} = u_1 - u_n$, and identifying $\mathbf{R}^2$ with $\mathbf{C}$, we can write M_n $(n \ge 3)$ as

$$M_n \cong \{(z_1, \dots, z_{n-1}) \in (S^1)^{n-1} : z_1 + \cdots + z_{n-1} - 1 = 0\}. \tag{1.6}$$

Let $i_n : M_n \hookrightarrow (S^1)^{n-1}$ be the inclusion.

As we have mentioned, $(S^1)^{n-1}$ approximates the topology of M_n up to some dimension (cf. Proposition 2.2). However, for an odd $n = 2m+1$, our low-dimensional computations lead us to give up the hope that $(S^1)^{2m}/\sigma$ might approximate $M'_{2m+1} = M_{2m+1}/\sigma$, where σ acts on $(S^1)^{2m}$ in the same way as in (1.4). The essential reason for this is that the action of σ on $(S^1)^{2m}$ is not free, although on M_{2m+1} is.

Thus we define V_{2m} by

$$V_{2m} = \left\{(S^1)^{2m} - \Sigma_{2m}\right\}/\sigma, \tag{1.7}$$

where we set

$$\Sigma_{2m} = \{(z_1, \dots, z_{2m}) \in (S^1)^{2m} : z_i = \pm 1 \ (1 \le i \le 2m)\}.$$

Let $i'_{2m+1} : M'_{2m+1} \hookrightarrow V_{2m}$ be the inclusion. Then we have the following map of covering spaces:

$$
(1.8)\qquad
\begin{array}{ccc}
\mathbf{Z}_2 & = & \mathbf{Z}_2 \\
\downarrow & & \downarrow \\
M_{2m+1} & \xrightarrow{i_{2m+1}} & (S^1)^{2m} - \Sigma_{2m} \\
\downarrow & & \downarrow \\
M'_{2m+1} & \xrightarrow{i'_{2m+1}} & V_{2m}.
\end{array}
$$

Note that $(S^1)^{2m} - \Sigma_{2m}$ is a maximal subspace of $(S^1)^{2m}$ on which σ acts freely. Thus it is natural to consider the topology of V_{2m}.

Now concerning the relation between M'_{2m+1} and V_{2m}, we have the following theorem.

Theorem D. $(i'_{2m+1})_* : \pi_q(M'_{2m+1}) \to \pi_q(V_{2m})$ *are isomorphisms for* $q \leq m-2$, *and an epimorphism for* $q = m-1$.

Concerning (iv), we have the following:

Theorem E. $H_*(V_{2m}; \mathbf{Z})$ *is given by*

$$
H_q(V_{2m}; \mathbf{Z}) = \begin{cases} \bigoplus_{\binom{2m}{q}} \mathbf{Z} & q: \text{ even } \leq 2m-2 \\ \bigoplus_{\sum_{i\leq q}\binom{2m}{i}} \mathbf{Z}_2 & q: \text{ odd } \leq 2m-3 \\ \bigoplus_{2^{2m}-1} \mathbf{Z} & q = 2m-1 \\ 0 & \text{otherwise,} \end{cases}
$$

where $\bigoplus_{\binom{2m}{q}} \mathbf{Z}$ *denotes the* $\binom{2m}{q}$*-times direct sum of* $\mathbf{Z}$.

Note that Theorems D and E give $H_q(M'_{2m+1}; \mathbf{Z})$ for $q \leq m-2$.

REMARK 1.9. By the Poincaré-Lefschetz duality $H^q((S^1)^{2m}/\sigma, \Sigma_{2m}; \mathbf{Z}) \cong H_{2m-q}(V_{2m}; \mathbf{Z})$, knowing $H_*(V_{2m}; \mathbf{Z})$ is equivalent to knowing $H_*((S^1)^{2m}/\sigma; \mathbf{Z})$.

Concerning (v), we have the following:

Theorem F. (a) *For an odd m, we have*

$$H_q(M'_{2m+1}; \mathbf{Z}) = \begin{cases} \bigoplus_{\binom{2m}{q}} \mathbf{Z} & q: \text{ even } \leq m-1 \\ \bigoplus_{\sum_{i \leq q} \binom{2m}{i}} \mathbf{Z}_2 & q: \text{ odd } \leq m-2 \\ \bigoplus_{\binom{2m}{q+2}} \mathbf{Z} \oplus \bigoplus_{\sum_{i \geq q+3} \binom{2m}{i}} \mathbf{Z}_2 & q: \text{ odd } \geq m \\ 0 & \text{otherwise.} \end{cases}$$

(b) *For an even m, we have*

$$H_q(M'_{2m+1}; \mathbf{Z}) = \begin{cases} \bigoplus_{\binom{2m}{q}} \mathbf{Z} & q: \text{ even } \leq m-2 \\ \bigoplus_{\sum_{i \leq q} \binom{2m}{i}} \mathbf{Z}_2 & q: \text{ odd } \leq m-3 \\ \bigoplus_{\binom{2m}{m-1}} \mathbf{Z} \oplus Tor_{m-1} & q = m-1 \\ \bigoplus_{\binom{2m}{q+2}} \mathbf{Z} \oplus \bigoplus_{\sum_{i \geq q+3} \binom{2m}{i}} \mathbf{Z}_2 & q: \text{ odd } \geq m+1 \\ 0 & \text{otherwise,} \end{cases}$$

where Tor_{m-1}, the torsion submodule of $H_{m-1}(M'_{2m+1}; \mathbf{Z})$, satisfies that $\dim_{\mathbf{Z}_2} Tor_{m-1} \otimes \mathbf{Z}_2 = \sum_{i \leq m-2} \binom{2m}{i}$.

Thus, in particular, $H_{even}(M'_{2m+1}; \mathbf{Z})$ are torsion free for all m.

REMARK 1.10. (a) By Theorems D, E and F, we see that $(i'_{2m+1})_* : H_{m-1}(M'_{2m+1}; \mathbf{Z}) \to H_{m-1}(V_{2m}; \mathbf{Z})$ is an isomorphism when m is odd, but not an isomorphism when m is even.

(b) In order to prove Theorem F, we first determine $H_*(M'_{2m+1}; \mathbf{Z}_2)$, which is given in Proposition 5.1. In particular, we see that $(i'_{2m+1})_* : H_{m-1}(M'_{2m+1}; \mathbf{Z}_2) \to H_{m-1}(V_{2m}; \mathbf{Z}_2)$ is an isomorphism for all m (cf. Remark 5.2).

This paper is organized as follows. In §2 we recall the results of [7], then prove Theorems A, B and D. In §3 we prove Theorem C. In §4 we prove Theorem E, and in §5 we prove Theorem F.

2. Proofs of Theorems A, B and D

In [7], the following theorem is proved.

Theorem 2.1. *$H_*(M_n;\mathbf{Z})$ are free $\mathbf{Z}$-modules and the Poincaré polynomials $PS(M_n)=\sum_\lambda \operatorname{rank} H_\lambda(M_n;\mathbf{Z})t^\lambda$ are given by*

$$PS(M_{2m+1})=\sum_{\lambda=0}^{m-2}\binom{2m}{\lambda}t^\lambda+2\binom{2m}{m-1}t^{m-1}+\sum_{\lambda=m}^{2m-2}\binom{2m}{\lambda+2}t^\lambda,$$

$$PS(M_{2m})=\sum_{\lambda=0}^{m-2}\binom{2m-1}{\lambda}t^\lambda+\binom{2m}{m-1}t^{m-1}+\sum_{\lambda=m}^{2m-3}\binom{2m-1}{\lambda+2}t^\lambda.$$

The essential facts to prove Theorem 2.1 are the following three propositions.

Proposition 2.2. (i) *$(i_{2m+1})_*:\pi_q(M_{2m+1})\to\pi_q((S^1)^{2m})$ are isomorphisms for $q\le m-2$, and an epimorphism for $q=m-1$.*

(ii) *$(i_{2m})_*:H_q(M_{2m};\mathbf{Z})\to H_q((S^1)^{2m-1};\mathbf{Z})$ are isomorphisms for $q\le m-2$, and an epimorphism for $q=m-1$.*

Proposition 2.3. (i) *M_{2m+1} is an orientable manifold of dimension $2m-2$. Thus the Poincaré duality homomorphisms $\cap[M_{2m+1}]:H^q(M_{2m+1};\mathbf{Z})\to H_{2m-2-q}(M_{2m+1};\mathbf{Z})$ are isomorphisms for all q, where $[M_{2m+1}]\in H_{2m-2}(M_{2m+1};\mathbf{Z})$ is a fundamental class.*

(ii) *M_{2m} is a manifold of dimension $2m-3$ with singular points. $(u_1,\dots,u_{2m})\in M_{2m}$ is a singular point iff all of u_i lie on the x-axis. Moreover every singular point of M_{2m} is a cone-like singularity and has a neighborhood as $C(S^{m-2}\times S^{m-2})$. Thus the Poincaré duality homomorphisms $\cap[M_{2m}]:H^q(M_{2m};\mathbf{Z})\to H_{2m-3-q}(M_{2m};\mathbf{Z})$ are isomorphisms for $q\le m-3$ or $q\ge m$, an epimorphism for $q=m-1$, and a monomorphism for $q=m-2$.*

Proposition 2.4. (i) $\chi(M_{2m+1})=(-1)^{m+1}\binom{2m}{m}$.

(ii) $\chi(M_{2m})=(-1)^{m+1}\binom{2m-1}{m}$.

REMARK 2.5. In order to prove Theorem 2.1, the homological assertion is sufficient for Proposition 2.2 (i). But actually we can prove the homotopical assertion.

Proof of Theorem A. Since σ acts freely on M_{2m+1}, and M_{2m}^σ (=the fixed point set of the involution) equals to the set of singular points in M_{2m}, all of the assertions except the type of the singular points of M'_{2m} are deduced from Proposition 2.3.

Let $(z_1,\dots,z_{2m-1})$ be a singular point of M_{2m} in the identification of (1.6). By Proposition 2.3, we must have $z_i=\pm1$ $(1\le i\le 2m-1)$. As the symmetric group on $(2m-1)$-letters acts on M_{2m}, we can assume that $z_i=1$ $(1\le i\le m)$ and $z_i=-1$ $(m+1\le i\le 2m-1)$. A neighborhood of $(z_1,\dots,z_{2m-1})$ in $(S^1)^{2m-1}$ is written by

$$\Big\{\Big(\begin{pmatrix}\sqrt{1-y_1^2}\\ y_1\end{pmatrix},\dots,\begin{pmatrix}\sqrt{1-y_m^2}\\ y_m\end{pmatrix},\begin{pmatrix}-\sqrt{1-y_{m+1}^2}\\ y_{m+1}\end{pmatrix},\dots,\begin{pmatrix}-\sqrt{1-y_{2m-1}^2}\\ y_{2m-1}\end{pmatrix}\Big):\\ -\epsilon\le y_i\le\epsilon\ (1\le i\le 2m-1)\Big\},$$

where $\epsilon > 0$ is a fixed small number. As ϵ is small, it is easy to see that we can write this neighborhood as

$$\Big\{\Big(\begin{pmatrix}1-\frac{1}{2}y_1^2\\ y_1\end{pmatrix},\dots,\begin{pmatrix}1-\frac{1}{2}y_m^2\\ y_m\end{pmatrix},\begin{pmatrix}-1+\frac{1}{2}y_{m+1}^2\\ y_{m+1}\end{pmatrix},\dots,\begin{pmatrix}-1+\frac{1}{2}y_{2m-1}^2\\ y_{2m-1}\end{pmatrix}\Big):\\ -\epsilon\le y_i\le\epsilon\ (1\le i\le 2m-1)\Big\}.$$

Thus a neighborhood of a singular point in M_{2m} is written as a subspace of $\mathbf{R}^{2m-1}$ defined by two equations

$$\begin{cases} y_1^2+\cdots+y_m^2-y_{m+1}^2-\cdots-y_{2m-1}^2=0\\ y_1+\cdots+y_m+y_{m+1}+\cdots+y_{2m-1}=0.\end{cases} \tag{2.6}$$

By a linear transformation of parameters, we can write the quadratic form of (2.6), i.e.,

$$y_1^2+\cdots+y_{m-1}^2+(y_1+\cdots+y_{m-1}+y_{m+1}+\cdots+y_{2m-1})^2-y_{m+1}^2-\cdots-y_{2m-1}^2,$$

as

$$w_1^2+\cdots+w_{m-1}^2-w_m^2-\cdots-w_{2m-2}^2.$$

Thus a singular point of M_{2m} has a neighborhood $C\{(w_1,\dots,w_{m-1},w_m,\dots,w_{2m-2}) : w_1^2+\cdots+w_{m-1}^2=1, w_m^2+\cdots+w_{2m-2}^2=1\}$, which is homeomorphic to $C(S^{m-2}\times S^{m-2})$.

Now it is clear that $\sigma w_i = -w_i$. Hence a singular point of M'_{2m} has a neighborhood $C(S^{m-2}\times_{\mathbf{Z}_2} S^{m-2})$, where $\sigma(\zeta_1,\zeta_2)=(-\zeta_1,-\zeta_2)$ $(\zeta_1,\zeta_2\in S^{m-2})$. □

Proof of Theorem B. For $F=\mathbf{Z}_p$ (p : an odd prime) or $\mathbf{Q}$, we have that $H_*(M'_n;F)\cong H_*(M_n;F)^\sigma$ (= the fixed point set of $H_*(M_n;F)$ under the σ-action) (see for example [2]). As $H_*(M_n;\mathbf{Z})$ are free modules by Theorem 2.1, we have that $\dim_{\mathbf{Z}_p} H_q(M'_n;\mathbf{Z}_p)=\dim_{\mathbf{Q}} H_q(M'_n;\mathbf{Q})$. Hence Theorem B follows. □

Proof of Theorem D. Let $j_{2m} : (S^1)^{2m} - \Sigma_{2m} \hookrightarrow (S^1)^{2m}$ be the inclusion. Since Σ_{2m} is a discrete set, the general position argument shows that $(j_{2m})_* : \pi_q((S^1)^{2m} - \Sigma_{2m}) \to \pi_q((S^1)^{2m})$ are isomorphisms for $q \leq 2m-2$. Then Proposition 2.2 (i) shows that $(i_{2m+1})_* : \pi_q(M_{2m+1}) \to \pi_q((S^1)^{2m} - \Sigma_{2m})$ are isomorphisms for $q \leq m-2$ and an epimorphism for $q = m-1$, where $i_{2m+1} : M_{2m+1} \hookrightarrow (S^1)^{2m} - \Sigma_{2m}$ is the inclusion.

By comparing the homotopy exact sequences of two covering spaces of (1.8), we see that $(i'_{2m+1})_* : \pi_q(M'_{2m+1}) \to \pi_q(V_{2m})$ are isomorphisms for $q \leq m-2$ and an epimorphism for $q = m-1$, where $i'_{2m+1} : M'_{2m+1} \hookrightarrow V_{2m}$ is the map induced from the σ-equivariant inclusion $i_{2m+1} : M_{2m+1} \hookrightarrow (S^1)^{2m} - \Sigma_{2m}$.

This completes the proof of Theorem D. □

3. Proof of Theorem C

Let $i_n : M_n \hookrightarrow (S^1)^{n-1}$ be the inclusion. Note that i_n is a σ-equivariant map. Hence $(i_n)_* : H_*(M_n; \mathbf{Q}) \to H_*((S^1)^{n-1}; \mathbf{Q})$ is also a σ-equivariant homomorphism. Since $H_*(M'_n; \mathbf{Q}) = H_*(M_n; \mathbf{Q})^\sigma$, Proposition 2.2 tells us the following:

Proposition 3.1. (i) *For $q \leq m-2$, we have*

$$H_q(M'_{2m+1}; \mathbf{Q}) = \begin{cases} \bigoplus_{\binom{2m}{q}} \mathbf{Q} & q : \textit{even} \\ 0 & q : \textit{odd}. \end{cases}$$

(ii) *For $q \leq m-2$, we have*

$$H_q(M'_{2m}; \mathbf{Q}) = \begin{cases} \bigoplus_{\binom{2m-1}{q}} \mathbf{Q} & q : \textit{even} \\ 0 & q : \textit{odd}. \end{cases}$$

We assume the truth of the following Lemma for the moment. Let $[M_n] \in H_{n-3}(M_n; \mathbf{Q})$ be the fundamental class.

Lemma 3.2. $\sigma_*[M_n] = (-1)^n[M_n]$.

Then we have the following:

Proposition 3.3. (i) *For $q \geq m$, we have*

$$H_q(M'_{2m+1}; \mathbf{Q}) = \begin{cases} 0 & q : \textit{even} \\ \bigoplus_{\binom{2m}{q+2}} \mathbf{Q} & q : \textit{odd}. \end{cases}$$

(ii) *For $q \geq m$, we have*

$$H_q(M'_{2m};\mathbf{Q}) = \begin{cases} 0 & q: \text{ even} \\ \bigoplus_{\binom{2m-1}{q+2}} \mathbf{Q} & q: \text{ odd.} \end{cases}$$

Proof of Proposition 3.3. Take an element $\alpha \in H_q(M_{2m+1};\mathbf{Q})$ $(q \geq m)$. By Proposition 2.3, there is an element $f \in H^{2m-2-q}(M_{2m+1};\mathbf{Q})$ such that $\alpha = f \cap [M_{2m+1}]$. As $\sigma_*(f \cap [M_{2m+1}]) = \sigma^* f \cap \sigma_*[M_{2m+1}] = -\sigma^* f \cap [M_{2m+1}]$, we have that

$$H_q(M_{2m+1};\mathbf{Q})^\sigma = \left\{ f \in H^{2m-2-q}(M_{2m+1};\mathbf{Q}) : \sigma^* f = -f \right\}.$$

Now (i) follows from Proposition 3.1.

(ii) can be proved similarly. □

Now in order to determine $H_*(M'_n;\mathbf{Q})$, we need to know only $H_{m-1}(M'_{2m+1};\mathbf{Q})$ and $H_{m-1}(M'_{2m};\mathbf{Q})$, which are determined if we know $\chi(M'_n)$.

Proposition 3.4 ([5]). (i) $\chi(M'_{2m+1}) = (-1)^{m+1}\binom{2m-1}{m}$.

(ii) $\chi(M'_{2m}) = \begin{cases} 0 & m: \text{ even} \\ \binom{2m-1}{m} & m: \text{ odd.} \end{cases}$

Proof. By a general formula of an involution (see for example [1]), we have $\chi(M_n) + \chi(M_n^\sigma) = 2\chi(M'_n)$. Then the result follows from Proposition 2.4. □

Proof of Lemma 3.2. First we treat the case of $n = 2m+1$. We define a volume element ω of M_{2m+1} as follows. Fix $(z_1, \ldots, z_{2m}) \in M_{2m+1}$ in the identification of (1.6). It is easy to see that the tangent space $T_{(z_1,\ldots,z_{2m})}M_{2m+1}$ is given by

$$T_{(z_1,\ldots,z_{2m})}M_{2m+1} \cong \left\{ \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_{2m} \end{pmatrix} \in \mathbf{R}^{2m} : \xi_1 z_1 + \cdots + \xi_{2m} z_{2m} = 0 \right\}. \tag{3.5}$$

Write z_i as (x_i, y_i). Then for $\eta_1, \ldots, \eta_{2m-2} \in T_{(z_1,\ldots,z_{2m})}M_{2m+1}$, we set

$$\omega(\eta_1, \ldots, \eta_{2m-2}) = \det \left(\eta_1, \ldots, \eta_{2m-2}, \begin{pmatrix} x_1 \\ \vdots \\ x_{2m} \end{pmatrix}, \begin{pmatrix} y_1 \\ \vdots \\ y_{2m} \end{pmatrix} \right). \tag{3.6}$$

It is easy to see that ω is nowhere zero on M_{2m+1}.

For $\eta = \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_{2m} \end{pmatrix} \in T_{(z_1,\dots,z_{2m})}M_{2m+1}$, we see that

$$di_{2m+1}(\eta) = \xi_1(\sqrt{-1}z_1) + \cdots + \xi_{2m}(\sqrt{-1}z_{2m}),$$

where $i_{2m+1} : M_{2m+1} \hookrightarrow (S^1)^{2m}$ denotes the inclusion. Hence we see that $d\sigma : T_{(z_1,\dots,z_{2m})}M_{2m+1} \to T_{(\bar{z}_1,\dots,\bar{z}_{2m})}M_{2m+1}$ is given by

$$d\sigma(\eta) = -\eta. \tag{3.7}$$

Now the formulae $d\sigma(\eta_i) = -\eta_i$ and $\sigma(x_i, y_i) = (x_i, -y_i)$ tell us that $(\sigma^*\omega)(\eta_1, \dots, \eta_{2m-2}) = -\omega(\eta_1, \dots, \eta_{2m-2})$. Hence $\sigma^*\omega = -\omega$ and the result follows.

Next we treat the case of $n = 2m$. Let $\bar{M}_{2m}$ be $M_{2m} - \{\text{singular points}\}$. By the same argument as in the case of $n = 2m+1$, we see that $\sigma : \bar{M}_{2m} \to \bar{M}_{2m}$ preserves orientation. As $H_c^{2m-3}(\bar{M}_{2m}; \mathbf{Q}) \cong H^{2m-3}(M_{2m}; \mathbf{Q})$ (H_c= cohomology with compact supports), the result follows. $\square$

4. Proof of Theorem E

First we determine $H_{2m-1}(V_{2m}; \mathbf{Z})$. The Poincaré-Lefschetz duality tells us that $H_{2m-1}(V_{2m}; \mathbf{Z}) \cong H^1((S^1)^{2m}/\sigma, \Sigma_{2m}; \mathbf{Z})$. As $H^1((S^1)^{2m}/\sigma; \mathbf{Z}) = 0$, we have $H_{2m-1}(V_{2m}; \mathbf{Z}) \cong \bigoplus_{2^{2m}-1} \mathbf{Z}$.

As V_{2m} is a non-compact manifold of dimension $2m$, we have $H_q(V_{2m}; \mathbf{Z}) = 0$ $(q \geq 2m)$. Hence in order to complete the proof of Theorem E, we need to determine $H_q(V_{2m}; \mathbf{Z})$ $(q \leq 2m-2)$.

Recall that we have a fibration $(S^1)^{2m} - \Sigma_{2m} \to V_{2m} \to \mathbf{R}P^\infty$. Set $F_{2m} = (S^1)^{2m} - \Sigma_{2m}$. The local systems of this fibration of dimensions less than or equal to $2m-2$ are easy to describe: We write the generator of $\pi_1(\mathbf{R}P^\infty)$ by σ. Then as a σ-module, we have

$$H_q(F_{2m}; \mathbf{Z}) \cong H_q((S^1)^{2m}; \mathbf{Z}) \ (q \leq 2m-2). \tag{4.1}$$

Let $\{E^r_{s,t}\}$ be the $\mathbf{Z}$-coefficient homology Serre spectral sequence of the above fibration. It is elementary to describe $E^2_{s,t}$ $(t \neq 2m-1)$ by using the following fact: We define a σ-module $\mathcal{S}$ to be the free abelian group of rank 1 on which σ acts by -1. Then have that

$$H_q(\mathbf{R}P^\infty; \mathcal{S}) = \begin{cases} \mathbf{Z}_2 & q: \ \text{even} \\ 0 & q: \ \text{odd}. \end{cases} \tag{4.2}$$

REMARK 4.3. For our reference, we give $E^2_{s,2m-1}$. Let $\mathcal{T}$ be the free abelian group of rank 2 on which σ acts by $\begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}$. And let σ act on $\mathbf{Z}$ trivially. Then we can prove that

$$H_{2m-1}(F_{2m};\mathbf{Z}) \cong \underset{2m}{\oplus}\mathcal{T} \oplus \underset{2^{2m}-2m-1}{\oplus} \mathbf{Z}.$$

As

$$H_q(\mathbf{R}P^\infty;\mathcal{T}) = \begin{cases} \mathbf{Z} & q=0 \\ 0 & q>0, \end{cases}$$

we can determine $E^2_{s,2m-1}$.

We return to $E^2_{s,t}$ $(t \neq 2m-1)$. By the dimensional reason, we have the following:

Proposition 4.4. *For $s+t \leq 2m-2$, we have that $E^2_{s,t} \cong E^\infty_{s,t}$.*

Hence in order to complete the proof of Theorem E, it suffices to determine the extensions of $E^\infty_{s,t}$, where $s+t$ are odd $\leq 2m-3$. To do so, it is convenient to study $H_*(V_{2m};\mathbf{Z}_2)$.

Proposition 4.5. *For $q \leq 2m-2$, we have*

$$H_q(V_{2m};\mathbf{Z}_2) = \underset{\sum_{i\leq q}\binom{2m}{i}}{\oplus} \mathbf{Z}_2.$$

From Proposition 4.5, we see that the extensions of $E^\infty_{s,t}$ $(s+t \leq 2m-2)$ are trivial. Hence Theorem E follows.

Thus in order to complete the proof of Theorem E, we need to prove Proposition 4.5, which we prove for the rest of this section.

Let $\{E_r^{s,t}\}$ be the $\mathbf{Z}_2$-coefficient cohomology Serre spectral sequence of the fibration $F_{2m} \to V_{2m} \to \mathbf{R}P^\infty$. We prove the following:

Lemma 4.6. $d_2 : E_2^{0,1} \to E_2^{2,0}$ *equals to* 0.

Lemma 4.6 tells us that elements of $E_2^{s,t}$ $(t \leq 2m-2)$ are permanent cycles. Hence Proposition 4.5 follows.

Proof of Lemma 4.6. Suppose that Lemma 4.6 fails. Then we have $H^1(V_{2m};\mathbf{Z}_2) = \underset{2m}{\oplus}\mathbf{Z}_2$. By Theorem D and the $\mathbf{Z}_2$-coefficient Poincaré duality of M'_{2m+1}, we have $H_{2m-3}(M'_{2m+1};\mathbf{Z}_2) = \underset{2m}{\oplus}\mathbf{Z}_2$. Since $H_{2m-3}(M'_{2m+1};\mathbf{Q}) = \underset{2m}{\oplus}\mathbf{Q}$ by Theorem C (a), we have

$$H_{2m-3}(M'_{2m+1};\mathbf{Z}) = \underset{2m}{\oplus}\mathbf{Z}. \tag{4.7}$$

By Theorem C (a), we have $H_{2m-2}(M'_{2m+1};\mathbf{Q}) = 0$. Hence by Theorem A (a), M'_{2m+1} is a non-orientable manifold of dimension $2m-2$. Thus we have $H_{2m-2}(M'_{2m+1};\mathbf{Z}) = 0$. Then by (4.7), we have $H_{2m-2}(M'_{2m+1};\mathbf{Z}_2) = 0$. This contradicts the fact that $H_{2m-2}(M'_{2m+1};\mathbf{Z}_2) = \mathbf{Z}_2$, i.e., M'_{2m+1} is a compact manifold of dimension $2m-2$.

This completes the proof of Lemma 4.6, and hence also that of Theorem E. □

5. Proof of Theorem F

In order to calculate $H_*(M'_{2m+1};\mathbf{Z})$, first we need to determine $H_*(M'_{2m+1};\mathbf{Z}_2)$. By the Poincaré duality, it suffices to determine $H_q(M'_{2m+1};\mathbf{Z}_2)$ $(q \leq m-1)$, which are given by the following:

Proposition 5.1. *For $q \leq m-1$, we have*

$$H_q(M'_{2m+1};\mathbf{Z}_2) = \bigoplus_{\sum_{i\leq q}\binom{2m}{i}} \mathbf{Z}_2.$$

Proof. First, $H_q(M'_{2m+1};\mathbf{Z}_2)$ $(q \leq m-2)$ are determined by Theorems D and E together with the universal coefficient theorem. Then $H_{m-1}(M'_{2m+1};\mathbf{Z}_2)$ is determined by Proposition 3.4. □

REMARK 5.2. From Theorems D, E and Proposition 5.1, we see that $(i'_{2m+1})_* : H_{m-1}(M'_{2m+1};\mathbf{Z}_2) \to H_{m-1}(V_{2m};\mathbf{Z}_2)$ is an isomorphism for all m.

Now we begin to determine $H_*(M'_{2m+1};\mathbf{Z})$.

(I) $H_{even}(M'_{2m+1};\mathbf{Z})$.

These modules are determined from Theorem C and the following:

Proposition 5.3. *$H_{even}(M'_{2m+1};\mathbf{Z})$ are torsion free.*

Proof. We can inductively prove this proposition from Theorem C and Proposition 5.1 together with the universal coefficient theorem. □

(II) $H_{odd}(M'_{2m+1};\mathbf{Z})$.

In order to determine these modules from Theorem C and Proposition 5.1, we need to prove the non-existence of higher two-torsions, i.e., elements of order 2^i $(i \geq 2)$.

Let $p : M_{2m+1}\underset{\sigma}{\times}\mathbf{R} \to M'_{2m+1}$ be the real line bundle associated to the covering space $M_{2m+1} \to M'_{2m+1}$. And let $O(M_{2m+1}\underset{\sigma}{\times}\mathbf{R})$ denote the local system of the above vector bundle. Finally, let $O(TM'_{2m+1})$ denote the local system of TM'_{2m+1}, the tangent bundle of M'_{2m+1}.

Concerning these local systems, we have the following:

Lemma 5.4. *As local systems on M'_{2m+1}, we have $O(M_{2m+1} \underset{\sigma}{\times} \mathbf{R}) \cong O(TM'_{2m+1})$.*

Proof. Let $\mathbf{R}^2 \to \nu \to M_{2m+1}$ denote the normal bundle of M_{2m+1} in $(S^1)^{2m}$ (cf. (1.6)). As $TM_{2m+1} \oplus \nu \cong T((S^1)^{2m})|M_{2m+1}$, we have

$$TM'_{2m+1} \oplus \nu/\sigma \cong TV_{2m}|M'_{2m+1}, \tag{5.5}$$

where $\nu/\sigma \to M'_{2m+1}$ denotes the vector bundle obtained from $\nu \to M_{2m+1}$ by the action of σ.

We study ν/σ. Recall that TM_{2m+1} is given by (3.5). Similarly, for $(z_1, \ldots, z_{2m}) \in M_{2m+1}$, we have

$$T_{(z_1,\ldots,z_{2m})}((S^1)^{2m}) \cong \left\{ \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_{2m} \end{pmatrix} \in \mathbf{R}^{2m} \right\}.$$

Hence by assigning $\begin{pmatrix} \xi_1 \\ \vdots \\ \xi_{2m} \end{pmatrix} \in \nu_{(z_1,\ldots,z_{2m})}$ to $\xi_1 z_1 + \cdots + \xi_{2m} z_{2m}$, we have

$$\nu \cong M_{2m+1} \times \mathbf{R}^2. \tag{5.6}$$

Under this identification, the bundle homomorphism $d\sigma : \nu \to \nu$ is given by

$$d\sigma((z_1, \ldots, z_{2m}); \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}) = ((\bar{z}_1, \ldots, \bar{z}_{2m}); \begin{pmatrix} -v_1 \\ v_2 \end{pmatrix}), \tag{5.7}$$

(cf. (3.7)).

Then (5.6)-(5.7) tell us that

$$\nu/\sigma \cong M_{2m+1} \underset{\sigma}{\times} \mathbf{R} \oplus M'_{2m+1} \times \mathbf{R}. \tag{5.8}$$

Now, as V_{2m} is orientable, we see from (5.5) and (5.8) that

$$O(TM'_{2m+1}) \otimes O(M_{2m+1} \underset{\sigma}{\times} \mathbf{R}) \cong \mathbf{Z}, \tag{5.9}$$

where $\mathbf{Z}$ denotes the simple local system on M'_{2m+1}. By taking a tensor $\otimes O(TM'_{2m+1})$ on both sides of (5.9), the result follows. $\square$

Let us denote the local systems $O(M_{2m+1} \underset{\sigma}{\times} \mathbf{R}) \cong O(TM'_{2m+1})$ (cf. Lemma 5.4) by $\mathcal{Z}$.

(A) *The case of an odd m.*

We can determine $H_q(M'_{2m+1};\mathbf{Z})$ (q : odd $\leq m-2$) by Theorems D and E. Thus we need to determine $H_q(M'_{2m+1};\mathbf{Z})$ (q : odd $\geq m$). By the Poincaré duality: $H_q(M'_{2m+1};\mathbf{Z}) \cong H^{2m-2-q}(M'_{2m+1};\mathcal{Z})$, it suffices to determine $H^r(M'_{2m+1};\mathcal{Z})$ (r : odd $\leq m-2$).

Consider the Gysin sequence of $p : M_{2m+1}\underset{\sigma}{\times}\mathbf{R} \to M'_{2m+1}$:

$$\cdots \xrightarrow{\psi} H^{r-1}(M'_{2m+1};\mathcal{Z}) \xrightarrow{\mu} H^r(M'_{2m+1};\mathbf{Z}) \xrightarrow{p^*} H^r(M_{2m+1};\mathbf{Z})$$
$$\xrightarrow{\psi} H^r(M'_{2m+1};\mathcal{Z}) \xrightarrow{\mu} \cdots .$$

Lemma 5.10. *For an odd $r \leq m-2$, we have*

(i) *$H^r(M'_{2m+1};\mathbf{Z}) = 0$.*

(ii) *$H^r(M_{2m+1};\mathbf{Z})$ is a free module.*

(iii) *The order of a torsion element of $H^{r+1}(M'_{2m+1};\mathbf{Z})$ is exactly 2, i.e., $H^{r+1}(M'_{2m+1};\mathbf{Z})$ does not contain higher two-torsions.*

Proof. This lemma is an easy consequence of Theorems D, E, 2.1 and Proposition 5.3. □

Now suppose that $H^r(M'_{2m+1};\mathcal{Z})$ contains a higher two-torsion. Then by Lemma 5.10 (iii), Ker $[\mu : H^r(M'_{2m+1};\mathcal{Z}) \to H^{r+1}(M'_{2m+1};\mathbf{Z})]$ contains a torsion element.

But by Lemma 5.10 (i)-(ii), Im $[\psi : H^r(M_{2m+1};\mathbf{Z}) \to H^r(M'_{2m+1};\mathcal{Z})]$ is a free module. This is a contradiction. Thus $H^r(M'_{2m+1};\mathcal{Z})$ (r : odd $\leq m-2$) does not contain higher two-torsions.

This completes the proof of Theorem F (a).

(B) *The case of an even m.*

As in (A), it suffices to determine $H^r(M'_{2m+1};\mathcal{Z})$ (r : odd $\leq m-1$). For an odd $r \leq m-3$, Lemma 5.10 applies and, by the same argument as in (A), we see that $H^r(M'_{2m+1};\mathcal{Z})$ does not contain higher two-torsions.

But Lemma 5.10 fails when $r = m-1$. Thus our argument cannot apply in this case.

This completes the proof of Theorem F (b). □

References

[1] G. Bredon: Introduction to compact transformation groups, Academic Press, 1972.

[2] P. Conner: *Concerning the action of a finite group*, Proc. Nat. Acad. Sci. U.S.A. **42** (1956), 349-351.

[3] T. Havel: *The use of distances as coordinates in computer-aided proofs of theorems in Euclidean geometry*, Journal of Symbolic Computation **11** (1991), 579-593.

[4] Y. Kamiyama: *An elementary proof of a theorem of T. F. Havel*, Ryukyu Math. J. **5** (1992), 7-12.

[5] Y. Kamiyama: *Topology of equilateral polygon linkages*, Top. and its Applications **68** (1996), 13-31.

[6] Y. Kamiyama, M. Tezuka and T. Toma: *Homology of the configuration spaces of quasi-equilateral polygon linkages*, Trans. Amer. Math. Soc. **350** (1998), 4869-4896.
[7] Y. Kamiyama and M. Tezuka: *Topology and geometry of equilateral polygon linkages in the Euclidean plane*, Quart. J. Math. (to appear).
[8] M. Kapovich and J. Millson: *On the moduli space of polygons in the Euclidean plane*, Journal of Diff. Geometry **42** (1995), 133-164.
[9] I. Schoenberg: *Linkages and distance geometry, I. Linkages*, Indag. Math. **31** (1969), 42-52.
[10] E. Spanier: Algebraic topology, McGraw-Hill, 1966.

Department of Mathematics
University of the Ryukyus
Nishihara-Cho, Okinawa 903-01, Japan