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On Diffeomorphic Approximations of Polyhedral Surfaces in 4-Space

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Let f be a semilinear homeomorphism of a connected closed surface F, with or without boundary, in 4-space E^4. Hence $f(F)$ is polyhedral. Then f is said to be *diffeomorphically approximable* if for each positive real number ε there exists a diffeomorphism g of F into E^4 such that $|f(x)-g(x)|<\varepsilon$ for each $x \in F$[1].

First suppose that F is a connected closed surface without boundary. Let $p_1, \cdots, p_n$ be all the *singularities*[2] of $f(F)$ in E^4 and $\tilde{k}_1, \cdots, \tilde{k}_n$ the knot types of these singularities, respectively. Further let $\tilde{k}$ be the ***knot product*** of $\tilde{k}_1, \cdots, \tilde{k}_n$. Then the purpose of this note is to prove the following

Theorem 1. *f is diffeomorphically approximable if $\tilde{k}$ is null-equivalent*[2].

It is easy to see that if F is the 2-sphere, then $\tilde{k}$ is null-equivalent. Therefore, as a special case of Theorem 1, we have

Theorem 2. *If F is the 2-sphere, then f has a diffeomorphic approximation.*

Now let F be a connected closed surface with boundary. In this case, by the same method of proof, we have the following

Theorem 3. *f can be diffeomorphically approximated.*

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The proof of Theorem 1 is divided into two steps.

(1) *To construct a semilinear homeomorphism h of F in 4-space E^4, which is an approximation of f such that $h(F)$ is locally flat*[4] *at every point of $h(F)$.*

1) For the case of polyheral $(n-1)$-manifold in n-space, see [3] [5].
2) See [1] [2].
3) See [1] [6].
4) See [1] [2].

(2) *To construct a diffeomorphism g of F in 4-space E^4, which is an approximation of h.*

In the second step there occurs no difficulty, since $h(F)$ is a locally flat closed surface in 4-space. Hence we shall omit the proof of it.

Now we shall prove the first step. As far as this step is concerned we may take the semilinear point of view. By definition, for each of the singularities $p_1, \cdots, p_n$ we can take sufficiently small 4-cells $C_1, \cdots, C_n$ such that $C_i \cap f(F)$ is a cone whose base is the knot k_i, where the knot type of k_i is $\tilde{k}_i$, in the bdry C_i and that the interior of the 2-cell $f(F) \cap C_i$ is contained in Int C_i, respectively.

Further let $D_i (i=2, \cdots, n)$ be a sufficiently narrow 4-cell satisfying the following conditions:

(i) $D_i \cap D_j = 0$ $(i \neq j)$,

(ii) $D_i \cap C_i$ is a 3-cell,

(iii) $D_i \cap C_i \cap f(F)$ is an arc which represents a trivial knot in $D_i \cap C_i$[5],

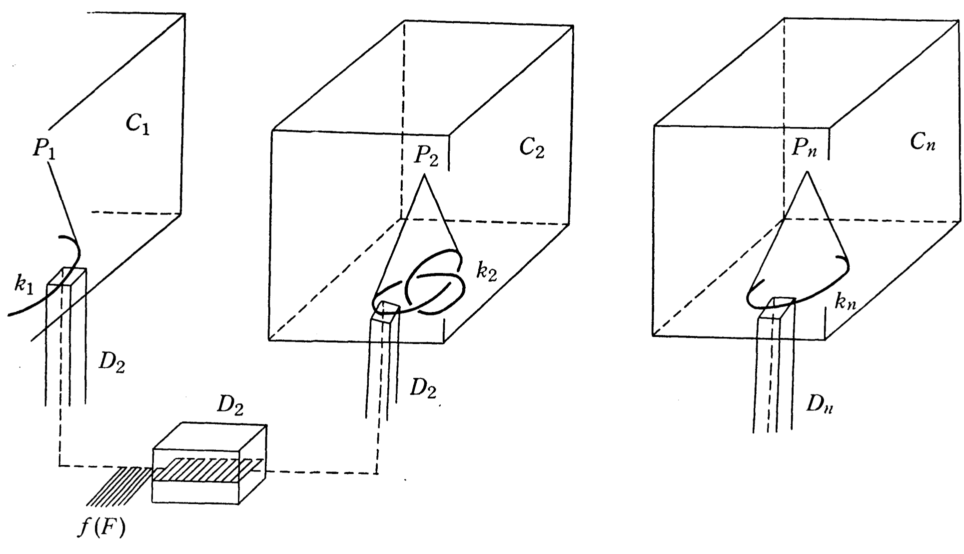
(iv) $D_i \cap f(F)$ is a narrow 2-cell which is imbedded trivially in D_i,

(v) $D_i \cap C_1$ is a 3-cell, and

(vi) $D_i \cap C_1 \cap f(F)$ is an arc which represents a trivial knot in $D_i \cap C_1$.

The existence of such D_i is obvious, since $f(F)$ is locally flat at every point of $F - \bigcup_{i=1}^{n} p_i$.

Now we shall construct the semilinear homeomorphism h in question. We shall only construct the image $h(F)$, for from this it will be easy to define the required h.



5) See [4].

First, for each i $(i=2, \cdots, n)$ we construct a knot k'_i in bdry C_i such that

(i) $k'_i = k_i \times \tilde{k}_i^{-1}$ and

(ii) $k'_i \cap (\text{bdry } C_i - D_i) = f(F) \cap (\text{bdry } C_i - D_i)$.

Since the knot type of the singularity p_i is $\tilde{k}_i$, this is possible. Then the arc $k'_i \cap D_i \cap C_i$ represents the knot type $\tilde{k}_i^{-1}$ in the 3-cell $D_i \cap C_i$. From (i) follows the existence of a locally flat 2-cell d_i such that $d_i \subset C_i$, the interior of the 2-cell d_i is contained in Int C_i and the boundary of the 2-cell d_i is k'_i.

Next we construct a knot $k = k_1 \times \cdots \times k_n$ in bdry C_1 such that

(i) $k \cap (\text{bdry } C_1 - \bigcup_{i=2}^{n} D_i) = f(F) \cap (\text{bdry } C_1 - \bigcup_{i=2}^{n} D_i)$ and

(ii) the arc $k'_i \cap D_i \cap C_i$ represents the knot type $\tilde{k}_i$ in the 3-cell $D_i \cap C_i$ $(i=2, \cdots, n)$.

Since the knot type of the singularity p_1 is $\tilde{k}_1$, this is possible. As remarked before, the arc $k'_i \cap D_i \cap C_i$ $(i=2, \cdots, n)$ represents the knot type $\tilde{k}_i^{-1}$ in the 3-cell $D_i \cap C_i$. Since $D_i \cap C_i$ and $D_i \cap C_1$ are opposite faces of the 4-cell D_i, there exists a locally flat 2-cell e_i in D_i whose boundary is

$$(k'_i \cap D_i) \cup (k \cap D_i) \cup \{(f(F) - \bigcup_{j=1}^{n} C_j) \cap \text{bdry } D_i\} .$$

By our assumption that k is null-equivalent there exists a locally flat 2-cell d_1 such that $d_1 \subset C_1$, the interior of the 2-cell d_1 is contained in Int C_1 and the boundary of the 2-cell d_1 is k.

Then

$$h(F) = \bigcup_{i=1}^{n} d_i \cup \bigcup_{i=2}^{n} e_i \cup \{f(F) \cap (E^4 - \bigcup_{i=1}^{n} C - \bigcup_{i=2}^{n} D_i)\}$$

is a locally flat connected surface homeomorphic to $f(F)$. It is easy to define a homeomorphism h as an approximation of f. This completes the proof of the first step and hence the proof of Theorem 1.

As remarked before, Theorem 2 is a special case of Theorem 1. But Theorem 2 can be proved directly using the method of the proof of Theorem 1. For, in this case, the existence of d_1, in the proof, is almost obvious and our assumption is used only to verify it.

The proof of Theorem 3 is done by the same way. In this case, since $f(F)$ is a connected surface with boundary, we can draw the narrow 4-cell D_i $(i=1, 2, \cdots, n)$, in the proof of Theorem 1, from the neighborhood of the singularity p_i to the boundary of $f(F)$. No special reference to the singularity p_1 is necessary.

As far as the converse of Theorem 1 is concerned the problem is still open.

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References

[1] R. H. Fox and J. W. Milnor: Singularities of 2-spheres in 4-space and equivalence of knots, Bull. Amer. Math. Soc. **63** (1957), 406.

[2] V.K.A.M. Gugenheim: Piecewise linear isotopy and embedding of elements and spheres (I), Proc. London Math. Soc. **3** (1953), 29-53, (II), ibid. 129-152.

[3] H. Noguchi: The smoothing of combinatorial n-manifolds in $(n+1)$-space, (unpublished).

[4] H. Schubert: Die eindeutige Zerlegbarkeit eines Knotens in Primknoten, Sitzber. Heidelberger Akad. Wiss. **3** (1949), 57-104.

[5] J. Tao: Some properties of $(n-1)$-manifolds in the Euclidean n-space, Osaka Math. J. **10** (1958), 137-148.

[6] H. Terasaka: On null-equivalent knots, Osaka Math. J. **11** (1959), 95-113.