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ON THE GROUP ALGEBRAS OF METABELIAN GROUPS OVER ALGEBRAIC NUMBER FIELDS I

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1. Introduction

In a previous paper [5], we investigated the group algebra $\boldsymbol{Q}[G]$ over the rational number field $\boldsymbol{Q}$ and Schur indices of a metacyclic group G. Here G is assumed to contain a cyclic normal subgroup A of order m with a cyclic factor group G/A of order s such that $(m, s)=1$. We showed that every simple component of $\boldsymbol{Q}[G]$ is explicitly written as a cyclic algebra. Consequently, the formulae for the Schur indices of all the irreducible representations of G were obtained.

In this paper, we pursue the same matter for a metacyclic group which does not necessarily satisfy the condition $(m, s)=1$, or more generally for a metabelian group G with an abelian normal subgroup A such that G/A is cyclic. In the first place, we refine the well known fact that every irreducible representation of a metabelian group is monomial (Theorem 1). By this Theorem 1, we find all the irreducible representations of a metabelian group G which is a semi-direct product of an abelian normal subgroup A and a cyclic subgroup $\langle\sigma\rangle$, and satisfies a certain condition. (This condition is fulfilled if G is metacyclic.) If an irreducible representation U of the above metabelian group G satisfies the assumption ($\natural$) of Theorem 2, then the enveloping algebra $\mathrm{env}_{\boldsymbol{Q}}(U)$ of U is expressed as a cyclic algebra. In Theorem 3, we give the formula for the Schur index of the above irreducible representation U.

To some extent, our argument is applicable to a non-split extension G of an abelian normal subgroup by a cyclic group. For simplicity, we shall discuss the case that G is metacyclic (§5). Finally we consider several examples and determine group algebras and Schur indices of them (§6).

Notation and Terminology As usual $\boldsymbol{Z}$, $\boldsymbol{Q}$, $\boldsymbol{C}$ denote respectively the ring of rational integers, the rational number field, the complex number field. For a set M, $\#M$ is the cardinality of M. $\langle\omega, \sigma, \cdots\rangle$ is the group generated by ω, σ, $\cdots$. An irreducible representation of a finite group G always means an absolute one. If ψ is a representation of a subgroup H of G, ψ^G denotes the representation of G induced from ψ. If χ is a character of G, $\boldsymbol{Q}(\chi)$ denotes the field obtained from $\boldsymbol{Q}$ by adjunction of all values $\chi(g)$, $g\in G$. For a natural number n, the multiplicative group of integers modulo n is denoted by $\boldsymbol{Z} \bmod^{\times} n$, and for $r\in\boldsymbol{Z}$, $(r, n)=1$, $r \bmod^{\times} n$ always means an element of $\boldsymbol{Z} \bmod^{\times} n$. If K is an extension field of k, then $N_{K/k}$ is the norm of K over k. If K is a Galois extension of k, $\mathfrak{G}(K/k)$ is its Galois group.

2. Irreducible representations of metabelian groups

In the first place we quote from [3, p. 348] Blichfeldt's theorem.

Theorem. *Let G be a finite subgroup of $GL(M)$ for some finite dimensional vector space M over an algebraically closed field K such that* char $K \nmid [G: 1]$, *and let M be an irreducible $K[G]$-module. Suppose that G contains an abelian normal subgroup A not contained in the center of G. Then there exist a proper subgroup H^* of G which contains A, and an irreducible $K[H^*]$-submodule L of M, such that $M=L^G$.*

REMARK. It is not stated in [3] that H^* can be taken so as to contain A.

The following theorem implies that, in order to give all the irreducible representations of a metabelian group G, we may fix a maximal abelian normal subgroup A such that G/A is abelian, and find all the subgroups H such that $G\supset H\supset A$, and decide all the linear characters of H.

Theorem 1. *Let G be a metabelian group with an abelian normal subgroup A such that G/A is abelian. Let K be an algebraically closed field whose characteristic does not divide $[G: 1]$. Then for every irreducible K-representation U of G, there exists a linear character ψ of a certain subgroup H which contains A, such that $U=\psi^G$.*

Proof. Since any subgroup or homomorphic image of a metabelian group is metabelian, we use induction about the order of G. Since the result is clear if G is abelian, we may assume that G is not abelian and that the theorem is true for any metabelian group of smaller order than $\#G$. Let M be any irreducible $K[G]$-module. The mapping $g\mapsto g_L$, where g_L is the linear transformation $m\mapsto gm$ of M, is a homomorphism of G onto a metabelian subgroup G_L of $GL(M)$, and M is an irreducible $K[G_L]$-module. The image A_L of A is an abelian normal subgroup of G_L such that G_L/A_L is abelian. If $g\mapsto g_L$ has a non-trivial kernel, then $[G_L: 1]<[G: 1]$, and by the induction hypothesis, there exist a subgroup H_L of G_L containing A_L, and a one-dimensional $K[H_L]$-submodule P of M such that $M=P^{G_L}$. If H is the subgroup of G consisting of all $h\in G$ such that $h_L\in H_L$, then $H\supset A$. It is easily seen that P is a one-dimensional $K[H]$-module and $M=P^G$.

We may therefore assume that $g \mapsto g_L$ is an isomorphism of G onto G_L, and we shall identify G with G_L. Let C be the center of G. If $A \not\subset C$, then by Blichfeldt's theorem, there exist a proper subgroup F of G containing A, and an irreducible $K[F]$-submodule W of M such that $M = W^G$. Since F/A and A are both abelian and $[F: 1] < [G: 1]$, the induction hypothesis implies that there exist a subgroup $H \supset A$ and a one-dimensional $K[H]$-submodule V of W such that $W = V^F$. Then we have $M = V^G$. Now we assume $A \subset C$. Since $[G, G] \subset A$, any subgroup containing C is normal in G. As G is not abelian, we can find a subgroup $E \supset C$ such that E/C is cyclic and not equal to $\langle 1 \rangle$. Then E is an abelian normal subgroup not contained in the center, and G/E is abelian. Therefore we find a subgroup $H(\supset E \supset A)$ and a one-dimensional $K[H]$-submodule V of M such that $M = V^G$. The theorem is proved.

Now let us consider a metabelian group G which is the semi-direct product of an abelian normal subgroup A and a cyclic subgroup $\langle \sigma \rangle$ of order s:

$$G = A \cdot \langle \sigma \rangle . \tag{1}$$

If $\{p_1, \cdots, p_n\}$ is the set of primes dividing the order of A, then

$$A = \langle \omega_{11} \rangle \times \cdots \times \langle \omega_{1c(1)} \rangle \times \cdots \times \langle \omega_{n1} \rangle \times \cdots \times \langle \omega_{nc(n)} \rangle , \tag{2}$$

where the order of ω_{ij} is $p_i^{a_{ij}}$ $(1 \leqslant i \leqslant n,\ 1 \leqslant j \leqslant c(i))$. In the following we assume that

$$\sigma^{-1} \omega_{ij} \sigma = \omega_{ij}^{r_{ij}} \qquad (1 \leqslant i \leqslant n,\ 1 \leqslant j \leqslant c(i)) . \tag{3}$$

Let u_{ij} be the order of $r_{ij} \bmod^{\times} p_i^{a_{ij}}$ and u be the L.C.M. of u_{ij} $(1 \leqslant i \leqslant n,$ $1 \leqslant j \leqslant c(i))$. Then $A \cdot \langle \sigma^u \rangle$ is a maximal abelian normal subgroup of G, so that by Theorem 1, any irreducible representation U of G is induced from a linear character ψ of some subgroup $H_t = A \cdot \langle \sigma^t \rangle$, $t \mid u$. H_t is a normal subgroup of G and

$$[H_t, H_t] = \prod_{i,j} \langle \omega_{ij}^{r_{ij}^t - 1} \rangle .$$

If we set

$$d_{tij} = (r_{ij}^t - 1,\ p_i^{a_{ij}}) , \tag{4}$$

all the linear characters of H_t are given by

$$\psi^{(t)}_{\alpha_{11} \cdots \alpha_{1c(1)} \cdots \alpha_{n1} \cdots \alpha_{nc(n)} \beta} , \quad 0 \leqslant \alpha_{ij} \leqslant d_{tij} - 1, \quad 0 \leqslant \beta \leqslant \frac{s}{t} - 1 , \tag{5}$$

such that

$$\begin{cases} \psi^{(t)}_{\alpha_{11} \cdots \alpha_{1c(1)} \cdots \alpha_{n1} \cdots \alpha_{nc(n)} \beta}(\omega_{ij}) = \exp \dfrac{2\pi\sqrt{-1}\,\alpha_{ij}}{d_{tij}} & \begin{pmatrix} 1 \leqslant i \leqslant n \\ 1 \leqslant j \leqslant c(i) \end{pmatrix} \\ \psi^{(t)}_{\alpha_{11} \cdots \alpha_{1c(1)} \cdots \alpha_{n1} \cdots \alpha_{nc(n)} \beta}(\sigma^t) = \exp \dfrac{2\pi\sqrt{-1}\,t\beta}{s} . & \end{cases} \tag{6}$$

For simplicity, we write them as

$$(7)\qquad \psi_{\alpha\beta}^{(t)} = \psi_{\alpha_{11}\cdots\alpha_{1c(1)}\cdots\alpha_{n1}\cdots\alpha_{nc(n)}\beta}^{(t)}.$$

The representation of G induced from $\psi_{\alpha\beta}^{(t)}$ is denoted by $U_{\alpha\beta}^{(t)}$:

$$(8)\qquad U_{\alpha\beta}^{(t)} = (\psi_{\alpha\beta}^{(t)})^{G}.$$

It is readily verified that

$$(9)\qquad U_{\alpha\beta}^{(t)}(\omega_{ij}) = \begin{pmatrix} \zeta_{tij}^{\alpha_{ij}} & & & 0 \\ & \zeta_{tij}^{\alpha_{ij}r_{ij}} & & \\ & & \ddots & \\ 0 & & & \zeta_{tij}^{\alpha_{ij}r_{ij}^{t-1}} \end{pmatrix}, \quad \zeta_{tij} = \exp\frac{2\pi\sqrt{-1}}{d_{tij}},$$

$$(10)\qquad U_{\alpha\beta}^{(t)}(\sigma) = \begin{pmatrix} 0 & \cdots & 0 & \xi_t^{\beta} \\ 1 & & & 0 \\ & \ddots & & \vdots \\ 0 & & 1 & 0 \end{pmatrix}, \quad \xi_t = \exp\frac{2\pi\sqrt{-1}\,t}{s},$$

$$(11)\qquad U_{\alpha\beta}^{(t)}(\sigma^t) = \xi_t^{\beta}\cdot 1_t,$$

where 1_t is the identity in the full matrix algebra $M_t(\boldsymbol{C})$.

Proposition 1. *$U_{\alpha\beta}^{(t)}$ is irreducible if and only if for every $\mu \not\equiv 0 \pmod{t}$, there exist i and j such that*

$$(12)\qquad \zeta_{tij}^{\alpha_{ij}} \neq \zeta_{tij}^{\alpha_{ij}r_{ij}^{\mu}}.$$

Proof. For any element $x=\omega\sigma^{\mu}$ of G where $\omega\in A$,

$$x^{-1}\omega_{ij}x = \sigma^{-\mu}\omega_{ij}\sigma^{\mu} = \omega_{ij}^{r_{ij}^{\mu}},$$
$$x^{-1}\sigma^t x = \omega_1^{-1}\sigma^t\omega_1, \quad \omega_1 = \sigma^{-\mu}\omega\sigma^{\mu}\in H_t,$$

so that

$$\psi_{\alpha\beta}^{(t)}(x^{-1}\omega_{ij}x) = \zeta_{tij}^{\alpha_{ij}r_{ij}^{\mu}},$$
$$\psi_{\alpha\beta}^{(t)}(x^{-1}\sigma^t x) = \psi_{\alpha\beta}^{(t)}(\sigma^t).$$

Then by [5, Lemma 2] we have

$U_{\alpha\beta}^{(t)}$ is irreducible

$\Leftrightarrow$ for every $x\notin H_t$, $\psi_{\alpha\beta}^{(t)} \neq (\psi_{\alpha\beta}^{(t)})^{(x)}$

$\Leftrightarrow$ for every $\mu \not\equiv 0 \pmod{t}$, there exist i and j such that $\zeta_{tij}^{\alpha_{ij}} \neq \zeta_{tij}^{\alpha_{ij} r_{ij}^{\mu}}$. q.e.d.

Proposition 2. *Let $U_{\alpha\beta}^{(t)}$ and $U_{\alpha'\beta'}^{(t)}$ be irreducible. Then $U_{\alpha\beta}^{(t)}$ and $U_{\alpha'\beta'}^{(t)}$ are inequivalent if and only if $\beta \neq \beta'$ or for every μ $(0 \leqslant \mu \leqslant t-1)$ there exist i and j such that $\zeta_{tij}^{\alpha_{ij} r_{ij}^{\mu}} \neq \zeta_{tij}^{\alpha'_{ij}}$.*

Proof. By [5, Lemma 3], we have

$U_{\alpha\beta}^{(t)}$ and $U_{\alpha'\beta'}^{(t)}$ are inequivalent

$\Leftrightarrow$ for every $x \in G$, $(\psi_{\alpha\beta}^{(t)})^{(x)} \neq \psi_{\alpha'\beta'}^{(t)}$

$\Leftrightarrow$ for every $x \in G$, $\psi_{\alpha\beta}^{(t)}(x^{-1}\sigma^t x) \neq \psi_{\alpha'\beta'}^{(t)}(\sigma^t)$, or $\psi_{\alpha\beta}^{(t)}(x^{-1}\omega_{ij}x) \neq \psi_{\alpha'\beta'}^{(t)}(\omega_{ij})$ for some i and j

$\Leftrightarrow$ $\xi_t^{\beta} \neq \xi_t^{\beta'}$ or for every μ, there exist i and j such that $\zeta_{tij}^{\alpha_{ij} r_{ij}^{\mu}} \neq \zeta_{tij}^{\alpha'_{ij}}$. q.e.d.

Proposition 3. *Let t and t' be any divisor of u, such that $t \neq t'$. Then $U_{\alpha\beta}^{(t)}$ and $U_{\alpha'\beta'}^{(t')}$ are not equivalent.*

Proof. Since $[G: H_t] = t$ and $[G: H_{t'}] = t'$, the assertion is obvious.

3. The structure of group algebra $\boldsymbol{Q}[G]$

The purpose of this section is to prove

Theorem 2. *Let G be the metabelian group discussed in the latter part of §2. (The defining relations are given by §2, (1)–(3).) Let $U_{\alpha\beta}^{(t)}$ $(t \neq 1)$ be any irreducible representation of G and $\chi_{\alpha\beta}^{(t)}$ its character. Set*

$$(1) \qquad K = \boldsymbol{Q}(\xi_t^{\beta}, \zeta_{tij}^{\alpha_{ij}}, 1 \leqslant i \leqslant n, 1 \leqslant j \leqslant c(i)) .$$

Assume that there exists an automorphism τ of K over $\boldsymbol{Q}$ such that

$$(\natural) \qquad \tau(\xi_t^{\beta}) = \xi_t^{\beta}, \ \tau(\zeta_{tij}^{\alpha_{ij}}) = \zeta_{tij}^{\alpha_{ij} r_{ij}}, \ 1 \leqslant i \leqslant n, \ 1 \leqslant j \leqslant c(i) .$$

Then the enveloping algebra of $U_{\alpha\beta}^{(t)}$ over $\boldsymbol{Q}$ is isomorphic to the cyclic algebra of center $\boldsymbol{Q}(\chi_{\alpha\beta}^{(t)})$:

$$\mathrm{env}_{\boldsymbol{Q}}(U_{\alpha\beta}^{(t)}) \simeq (\xi_t^{\beta}, K, \tau)_{\boldsymbol{Q}(\chi_{\alpha\beta}^{(t)})} .$$

EXAMPLE I. Notation being the same as before, let G be such that $a_{i1} = \cdots = a_{ic(i)}$ and $r_{i1} = \cdots = r_{ic(i)}$ for all $i\,(1 \leqslant i \leqslant n)$, and that $(p_1 p_2 \cdots p_n, s) = 1$. Then every (not one dimensional) irreducible representation of G satisfies the assumption $(\natural)$. In particular, every metacyclic group G with cyclic normal subgroup A and cyclic factor group G/A such that $([A: 1], [G: A]) = 1$ comes under this case.

EXAMPLE II. Let G be a hyperelementary group (at a prime p) generated by $\omega, \omega_1, \cdots, \omega_l, \sigma$ with defining relations:

$$\omega^m = 1,\ \sigma^{-1}\omega\sigma = \omega^r,\ \sigma^{p^b} = 1,\ \omega^{p^{b_i}} = 1\ (1 \leqslant i \leqslant l),\ (m, p) = 1,$$
$$\langle \omega, \omega_1, \cdots, \omega_l \rangle \text{ and } \langle \omega_1, \cdots, \omega_l, \sigma \rangle \text{ are abelian.}$$

Then every irreducible representation of G satisfies the assumption ($\natural$).

Theorem 2 can be proved almost in the same way as [5, Theorem 2], so that we give the proof concisely. At first, note that the order of the automorphism τ of K is equal to t. Indeed, since $U_{\alpha\beta}^{(t)}$ is irreducible, Proposition 1 implies that for any $\mu \not\equiv 0 \pmod t$, there exist i and j such that $\tau^{\mu}(\zeta_{tij}^{\alpha_{ij}}) \neq \zeta_{tij}^{\alpha_{ij}}$. On the other hand, by (4) of §2 we have $\tau^t(\zeta_{tij}^{\alpha_{ij}}) = \zeta_{tij}^{\alpha_{ij} r_{ij}^t} = \zeta_{tij}^{\alpha_{ij}}$ for all i and j. Hence the order of τ is t. For simplicity, set

$$U = U_{\alpha\beta}^{(t)} \quad \text{and} \quad \chi = \chi_{\alpha\beta}^{(t)}. \tag{2}$$

Lemma 1. *Let $U \mid H_t$ denote the restriction of U to the subgroup H_t. Then the enveloping algebra* $\mathrm{env}_{\boldsymbol{Q}}(U \mid H_t)$ *is a (commutative) field, and in fact,*

$$\mathrm{env}_{\boldsymbol{Q}}(U \mid H) \simeq K. \tag{3}$$

Proof. Denote by $[\theta_1, \theta_2, \cdots, \theta_t]$ the diagonal matrix of degree t with the diagonal elements $\theta_1, \theta_2, \cdots, \theta_t$. Then it follows from the assumption ($\natural$) that $\mathrm{env}_{\boldsymbol{Q}}(U \mid H_t)$ is just the set:

$$\mathrm{env}_{\boldsymbol{Q}}(U \mid H_t) = \{[\theta, \theta^{\tau}, \cdots, \theta^{\tau^{t-1}}];\ \theta \in K\}. \tag{4}$$

This proves our lemma.

As H_t is a normal subgroup of G, $\chi(g) = 0$ for every $g \notin H_t$. Hence we have

$$\boldsymbol{Q}(\chi) = \boldsymbol{Q}(\theta + \theta^{\tau} + \cdots + \theta^{\tau^{t-1}};\ \theta \in K). \tag{5}$$

From this we see easily that the field K is cyclic extension of $\boldsymbol{Q}(\chi)$ of degree t whose Galois group is generated by τ. By the isomorphism of K onto $\mathrm{env}_{\boldsymbol{Q}}(U \mid H_t)$, the subfield $\boldsymbol{Q}(\chi)$ is mapped onto $\boldsymbol{Q}(\chi) \cdot 1_t$, so that

$$[\mathrm{env}_{\boldsymbol{Q}}(U \mid H_t) : \boldsymbol{Q}(\chi) \cdot 1_t] = t. \tag{6}$$

Meanwhile it is well known that $[\mathrm{env}_{\boldsymbol{Q}}(U) : \boldsymbol{Q}(\chi) \cdot 1_t]$ is equal to the square of the degree of U:

$$[\mathrm{env}_{\boldsymbol{Q}}(U) : \boldsymbol{Q}(\chi) \cdot 1_t] = t^2. \tag{7}$$

Therefore, $\mathrm{env}_{\boldsymbol{Q}}(U \mid H_t)$ is a maximal subfield of $\mathrm{env}_{\boldsymbol{Q}}(U)$. The generating automorphism T of $\mathrm{env}_{\boldsymbol{Q}}(U \mid H_t)$ over $\boldsymbol{Q}(\chi) \cdot 1_t$, which corresponds to τ, is evidently given by

$$(8)\qquad T: [\theta, \theta^{\tau}, \cdots, \theta^{\tau^{t-1}}] \mapsto [\theta^{\tau}, \theta^{\tau^2}, \cdots, \theta], \qquad \theta \in K.$$

On the other hand, it is easily verified that

$$(9)\qquad U(\sigma)^{-1}[\theta, \theta^{\tau}, \cdots, \theta^{\tau^{t-1}}]U(\sigma) = [\theta^{\tau}, \theta^{\tau^2}, \cdots, \theta].$$

Hence

$$(10)\qquad U(\sigma)^{-\nu}\Theta U(\sigma)^{\nu} = T^{\nu}(\Theta),\ \Theta \in \mathrm{env}_{\boldsymbol{Q}}(U\,|\,H_t),\ 0 \leqslant \nu \leqslant t-1,$$

and so 1_t, $U(\sigma), \cdots, U(\sigma)^{t-1}$ are linearly independent over the field $\mathrm{env}_{\boldsymbol{Q}}(U\,|\,H_t)$. Recall that

$$(11)\qquad U(\sigma)^t \in \mathrm{env}_{\boldsymbol{Q}}(U\,|\,H_t).$$

Thus we see that $\mathrm{env}_{\boldsymbol{Q}}(U)$ is the cyclic algebra with the defining relation (10):

$$\begin{aligned}\mathrm{env}_{\boldsymbol{Q}}(U) &= 1_t\cdot \mathrm{env}(U\,|\,H_t) + U(\sigma)\cdot \mathrm{env}(U\,|\,H_t) + \cdots + U(\sigma)^{t-1}\cdot \mathrm{env}(U\,|\,H_t)\\ &= (U(\sigma^t), \mathrm{env}_{\boldsymbol{Q}}(U\,|\,H_t), T)_{\boldsymbol{Q}(\chi)\cdot 1_t}\\ &\simeq (\xi_t^{\beta}, K, \tau)_{\boldsymbol{Q}(\chi)}.\end{aligned}$$

This completes the proof of Theorem 2.

4. The Schur index

We shall calculate with the Schur index of the irreducible representation $U_{\alpha\beta}^{(t)}$ of the metabelian group G which appeared in Theorem 2. For this it suffices to compute the orders of norm residue symbols

$$(1)\qquad \left(\frac{\xi_t^{\beta}, K/k}{\mathfrak{p}}\right) = (\xi_t^{\beta}, K_{\mathfrak{P}}/k_{\mathfrak{p}})$$

at all places $\mathfrak{p}$ of

$$(2)\qquad k = \boldsymbol{Q}(\chi_{\alpha\beta}^{(t)}),$$

where $K_{\mathfrak{P}}/k_{\mathfrak{p}}$ represents the isomorphy type of the completion of K/k for $\mathfrak{P}\,|\,\mathfrak{p}$. Recall that

$$(3)\qquad \zeta_{tij}^{\alpha_{ij}} = \exp\frac{2\pi\sqrt{-1}\,\alpha_{ij}}{d_{tij}}, \quad d_{tij} = (p_i^{a_{ij}}, r_{ij}^t - 1).$$

If we set

$$(4)\qquad \frac{d_{tij}}{(d_{tij}, \alpha_{ij})} = p_i^{b_{ij}},$$

$$(5)\qquad a_i = \mathrm{Max}\,\{b_{i1}, b_{i2}, \cdots, b_{ic(i)}\} = b_{ij_i} \quad \text{for some} \quad j_i\ (1 \leqslant j_i \leqslant c(i)),$$

$$(6)\qquad r_i = r_{ij_i},$$

then it follows that

$$K = \boldsymbol{Q}\left(\xi_t^\beta,\ \exp\frac{2\pi\sqrt{-1}}{p_1^{a_1}},\ \cdots,\ \exp\frac{2\pi\sqrt{-1}}{p_n^{a_n}}\right), \tag{7}$$

$$\tau(\xi_t^\beta) = \xi_t^\beta, \quad \tau\left(\exp\frac{2\pi\sqrt{-1}}{p_i^{a_i}}\right) = \exp\frac{2\pi\sqrt{-1}\,r_i}{p_i^{a_i}}, \quad 1\leqslant i\leqslant n. \tag{8}$$

Recall that

$$\xi_t^\beta = \exp\frac{2\pi\sqrt{-1}\beta t}{s}. \tag{9}$$

So ξ_t^β is a primitive $v_{t,\beta}$-th root of unity, where

$$v_{t,\beta} = \frac{s/t}{(s/t,\ \beta)}. \tag{10}$$

(Case I) $\mathfrak{p}\nmid p_1p_2\cdots p_n$. Then $\mathfrak{p}$ is not ramified in $K_{\mathfrak{P}}/k_{\mathfrak{p}}$, so

$$(\xi_t^\beta,\ K_{\mathfrak{P}}/k_{\mathfrak{p}}) = 1. \tag{11}$$

(Case II) $\mathfrak{p}\mid p_i$ for some i ($1\leqslant i\leqslant n$). Set

$$p = p_i, \quad v_{t,\beta} = p^b z, \quad (p,\ z) = 1. \tag{12}$$

Then for some primitive p^b-th (resp. z-th) root of unity η_{p^b} (resp. η_z),

$$\xi_t^\beta = \eta_{p^b}\eta_z, \tag{13}$$

so that

$$(\xi_t^\beta,\ K_{\mathfrak{P}}/k_{\mathfrak{p}}) = (\eta_{p^b},\ K_{\mathfrak{P}}/k_{\mathfrak{p}})\cdot(\eta_z,\ K_{\mathfrak{P}}/k_{\mathfrak{p}}). \tag{14}$$

Consequently the order of $(\xi_t^\beta,\ K_{\mathfrak{P}}/k_{\mathfrak{p}})$ is that of $(\eta_{p^b},\ K_{\mathfrak{P}}/k_{\mathfrak{p}})$ multiplied by that of $(\eta_z,\ K_{\mathfrak{P}}/k_{\mathfrak{p}})$. Let $e_{\mathfrak{p}}$ be the ramification exponent of $K_{\mathfrak{P}}/k_{\mathfrak{p}}$ and Π be a prime element of $K_{\mathfrak{P}}$. Set $\psi = N_{K_{\mathfrak{P}}/k_{\mathfrak{p}}}(\Pi)$, $N_{k/\boldsymbol{Q}}(\mathfrak{p}) = q$, and $N_{K/\boldsymbol{Q}}(\mathfrak{P}) = q^h$, q being a power of p. Then by the same argument as in [5, §4] we have $N_{K_{\mathfrak{P}}/k_{\mathfrak{p}}}(K_{\mathfrak{P}}^\times) = \left\{\psi^n\eta_{q-1}{}^{c\lambda}N_{K_{\mathfrak{P}}/k_{\mathfrak{p}}}(\gamma);\ n\in\boldsymbol{Z},\ 1\leqslant\lambda\leqslant\frac{q-1}{c},\ \gamma\text{: principal unit of } K_{\mathfrak{P}}\right\}$ where

$$c_{\mathfrak{p}} = c = (e_{\mathfrak{p}},\ q-1). \tag{15}$$

Note that $c_{\mathfrak{p}}$ is the exponent of tame ramification of $K_{\mathfrak{P}}/k_{\mathfrak{p}}$. Since $\mathfrak{p}\nmid z$, we may assume $\eta_z = \eta_{q-1}{}^{(q-1)/z}$. Then for an integer x, $\eta_z^x\in N_{K_{\mathfrak{P}}/k_{\mathfrak{p}}}(K_{\mathfrak{P}}^\times)$ if and only if $c\Big/\left(c,\frac{q-1}{z}\right)$ divides x. Hence the order of the norm residue symbol $(\eta_z,\ K_{\mathfrak{P}}/k_{\mathfrak{p}})$ is equal to

$$(16) \qquad \frac{c_{\mathfrak{p}}}{\left(c_{\mathfrak{p}}, \frac{q-1}{z}\right)}.$$

It now remains to compute the order of $(\eta_{p^b}, K_{\mathfrak{P}}/k_{\mathfrak{p}})$, $\mathfrak{p} \mid p$.(*) Hereafter for a positive integer x, η_x denotes a primitive x-th root of unity. Set

$$\Omega = \begin{cases} \boldsymbol{Q}_p(\eta_p) & p \neq 2 \\ \boldsymbol{Q}_2(\eta_4) & p=2,\ b \geqslant 2. \end{cases}$$

Then, if $p \neq 2$ or $p=2$, $b \geqslant 2$, it follows that

$$k_{\mathfrak{p}} \supset \boldsymbol{Q}_p(\eta_{p^b}) \supset \Omega \supset \boldsymbol{Q}_p,$$

so that

$$(17) \qquad (\eta_{p^b}, K_{\mathfrak{P}}/k_{\mathfrak{p}}) = (N_{\Omega/\boldsymbol{Q}_p}(N_{k_{\mathfrak{p}}/\Omega}(\eta_{p^b})), K_{\mathfrak{P}}/\boldsymbol{Q}_p).$$

Clearly $N_{k_{\mathfrak{p}}/\Omega}(\eta_{p^b})$ is equal to η_p^{ν} (resp. η_4^{ν}) for some ν in the case $p \neq 2$ (resp. $p=2$, $b \geqslant 2$). As

$$N_{\boldsymbol{Q}_p(\eta_p)/\boldsymbol{Q}_p}(\eta_p) = 1, \text{ resp. } N_{\boldsymbol{Q}_2(\eta_4)/\boldsymbol{Q}_2}(\eta_4) = 1,$$

we have, in the case $p \neq 2$ or $p=2$, $b \geqslant 2$,

$$(18) \qquad (\eta_{p^b}, K_{\mathfrak{P}}/k_{\mathfrak{p}}) = (1, K_{\mathfrak{P}}/\boldsymbol{Q}_p) = 1.$$

Lastly the case $p=2$, $b=1$ remains. That is, we must compute the norm residue symbol

$$(19) \qquad (-1, K_{\mathfrak{P}}/k_{\mathfrak{p}}), \quad \mathfrak{p} \mid 2.$$

The field K can be expressed as

$$(20) \qquad K = \boldsymbol{Q}\left(\exp \frac{2\pi\sqrt{-1}}{2^a}, \exp \frac{2\pi\sqrt{-1}}{w}\right), \ (2, w) = 1.$$

Then

$$(21) \qquad \mathfrak{G}(K/\boldsymbol{Q}) = (\boldsymbol{Z} \bmod^{\times} 2^a) \times (\boldsymbol{Z} \bmod^{\times} w),$$

and the automorphism $\tau \in \mathfrak{G}(K/\boldsymbol{Q})$ is of the form:

$$(22) \qquad \tau = (\rho_1 \bmod^{\times} 2^a, \rho_2 \bmod^{\times} w).$$

Of course $\rho_1 \bmod^{\times} 2^a$ and $\rho_2 \bmod^{\times} w$ are uniquely determined by (8). If $a \geqslant 3$, then the group $\boldsymbol{Z} \bmod^{\times} 2^a$ is not cyclic. On the other hand K/k is cyclic, so

(*) The author is indebted to Professor Y. Akagawa for kind advice in the following argument.

that $\boldsymbol{Q}(\eta_{2^a})\cap k\neq\boldsymbol{Q}$. This implies that the degree $[k_{\mathfrak{p}}:\boldsymbol{Q}_2]$ is divisible by 2. Consequently in the case $a\geqslant 3$, we have

$$(23)\qquad (-1,\ K_{\mathfrak{P}}/k_{\mathfrak{p}})=(1,\ K_{\mathfrak{P}}/\boldsymbol{Q}_2)=1\,.$$

If $a=2$, $\rho_1\equiv 1 \pmod{2^2}$, then $k\supset\boldsymbol{Q}(\eta_4)\supset\boldsymbol{Q}$, and so $[k_{\mathfrak{p}}:\boldsymbol{Q}_2]$ is divisible by 2. Consequently

$$(24)\qquad (-1,\ K_{\mathfrak{P}}/k_{\mathfrak{p}})=1\,.$$

We come to the case $a=2$, $\rho_1\equiv -1 \pmod{2^2}$. Let the order of $\rho_2 \bmod^\times w$ be $2^\nu\cdot l$, $(2,l)=1$. If $2^\nu\neq 1$, then it can easily be shown that $\mathfrak{p}(\mathfrak{p}\,|\,2)$ is not ramified in K/k, so that

$$(25)\qquad (-1,\ K_{\mathfrak{P}}/k_{\mathfrak{p}})=1\,.$$

If $2^\nu=1$, then $[K:k]=2l$, and the ramification exponent of $\mathfrak{p}(\mathfrak{p}\,|\,2)$ in K/k is equal to 2. Meanwhile the degree $[K_{\mathfrak{P}}:\boldsymbol{Q}_2]$ is $2f$, where f is the smallest positive integer satisfying

$$(26)\qquad 2^f\equiv 1 \pmod{w}\,.$$

If f is even, it follows that $[k_{\mathfrak{p}}:\boldsymbol{Q}_2]$ is also even, so that

$$(27)\qquad (-1,\ K_{\mathfrak{P}}/k_{\mathfrak{p}})=(1,\ K_{\mathfrak{P}}/\boldsymbol{Q}_2)=1\,.$$

If f is odd, it follows that $[k_{\mathfrak{p}}:\boldsymbol{Q}_2]$ is also odd, so that

$$(28)\qquad (-1,\ K_{\mathfrak{P}}/k_{\mathfrak{p}})=(-1,\ K_{\mathfrak{P}}/\boldsymbol{Q}_2)\,.$$

However, as $-1\notin N_{\boldsymbol{Q}_2(\eta_4)/\boldsymbol{Q}_2}(\boldsymbol{Q}_2(\eta_4)^\times)$, we have

$$(29)\qquad -1\notin N_{K_{\mathfrak{P}}/\boldsymbol{Q}_2}(K_{\mathfrak{P}}^\times)\,.$$

Therefore, in this case, the order of $(-1,\ K_{\mathfrak{P}}/k_{\mathfrak{p}})$ is equal to 2.

Now we shall compute explicitly the ramification exponent $e_{\mathfrak{p}}$ and the absolute norm $N_{k/\boldsymbol{Q}}(\mathfrak{p})$ for every $\mathfrak{p}\,|\,p$, $p=p_i$ $(1\leqslant i\leqslant n)$. We have the expressions:

$$(30)\qquad K=\boldsymbol{Q}\Big(\exp\frac{2\pi\sqrt{-1}}{p^a},\ \exp\frac{2\pi\sqrt{-1}}{w}\Big),\ (p,w)=1\,,$$

$$(31)\qquad \mathfrak{G}(K/\boldsymbol{Q})=(\boldsymbol{Z} \bmod^\times p^a)\times(\boldsymbol{Z} \bmod^\times w)\,,$$

$$(32)\qquad \tau=(r \bmod^\times p^a,\ r \bmod^\times w)\,.$$

Of course, a, w, and $r \bmod^\times p^a w$ are uniquely determined from (7), (8). Let

$$(33)\qquad t_w=\text{the order of } r \bmod^\times w\,.$$

Then it can easily be shown that

$$e_{\mathfrak{p}} = \frac{t}{t_w} . \tag{34}$$

Let

$$\tilde{f} = \text{the order of } p \bmod w^{\times} , \tag{35}$$

$$f = \#[\langle r \bmod^{\times} w \rangle \cap \langle p \bmod^{\times} w \rangle] . \tag{36}$$

Then it is verified without difficulty that the relative degree of $\mathfrak{p}$ in K/k is equal to f, so that the absolute degree of $\mathfrak{p}$ is equal to $\tilde{f}/f$. Hence we have

$$N_{k/\boldsymbol{Q}}(\mathfrak{p}) = p^{\tilde{f}/f} . \tag{37}$$

(For the above argument, see [5, §4].) Thus we have completely decided the order of $(\xi_t^\beta, K_{\mathfrak{P}}/k_{\mathfrak{p}})$ for every finite prime $\mathfrak{p} \subset k$.

Finally we consider infinite prime spots $\mathfrak{p}_\infty$ of k. In the same way as in [5, §4], the following results are easily obtained. If $\xi_t^\beta = -1$ and k is real, then the local index at any $\mathfrak{p}_\infty$ of the cyclic algebra $(\xi_t^\beta, K, \tau)_k$ is equal to 2. Otherwise, the local index at any $\mathfrak{p}_\infty$ of k is equal to 1. The condition $\xi_t^\beta = -1$ amounts to $2\beta = \frac{s}{t}$, and k is real if and only if $2 | t$ and $r_i^{t/2} \equiv -1 \pmod{p_i^{a_i}}$, $1 \leqslant i \leqslant n$, where a_i and r_i are defined by (5) and (6).

Summarizing the results, we have

Theorem 3. *Let G be the metabelian group and $U_{\alpha\beta}^{(t)}$ be the irreducible representation of G which appeared in Theorem 2. Denote by $\Lambda_{\mathfrak{p}}$ the local index at $\mathfrak{p}$ of* $\mathrm{env}_{\boldsymbol{Q}}(U_{\alpha\beta}^{(t)})$, *where $\mathfrak{p}$ is a place of $k = \boldsymbol{Q}(\chi_{\alpha\beta}^{(t)})$. Then we have the following results.*

(I) *If $\mathfrak{p}$ is a prime ideal such that $\mathfrak{p} \nmid p_1 p_2 \cdots p_n$, then*

$$\Lambda_{\mathfrak{p}} = 1 .$$

(II) *$\mathfrak{p} | p_i$ for some i $(1 \leqslant i \leqslant n)$. Set $p = p_i$, $v_{t,\beta} = p^b z$, $(p, z) = 1$. Then we have*

$$\Lambda_{\mathfrak{p}} = \frac{c_{\mathfrak{p}}}{\left(c_{\mathfrak{p}}, \dfrac{q-1}{z}\right)} ,$$

except the case that $\mathfrak{p} | 2$, $v_{t,\beta} = 2z$, $(2, z) = 1$, $K = \boldsymbol{Q}\left(\exp \frac{2\pi\sqrt{-1}}{4}, \exp \frac{2\pi\sqrt{-1}}{w}\right)$, $(2, w) = 1$, $\tau = (-1 \bmod^{\times} 4, \rho \bmod^{\times} w)$, the order of $\rho \bmod^{\times} w$ is odd, and the order of $2 \bmod^{\times} w$ is odd. For this exceptional case, we have $\Lambda_{\mathfrak{p}} = 2$. In the above,

$$c_{\mathfrak{p}} = (e_{\mathfrak{p}}, q-1), \quad e_{\mathfrak{p}} = \frac{t}{t_w}, \quad q = N_{k/\boldsymbol{Q}}(\mathfrak{p}) = p^{\tilde{f}/f} ,$$

where t_w, $\tilde{f}$ *and* f *are given by* (30)-(36).

(III) *For any infinite prime spot* $\mathfrak{p}_\infty$ *of* k, *we have*

$$\Lambda_{\mathfrak{p}_\infty} = 1$$

except the case that $2\beta = \frac{s}{t}$, $2 \mid t$, $r_i^{t/2} \equiv -1 \pmod{p_i^{a_i}}$, $1 \leqslant i \leqslant n$, *where* a_i *and* r_i *are defined by* (5) *and* (6). *In this case we have, for any* $\mathfrak{p}_\infty$, $\Lambda_{\mathfrak{p}_\infty} = 2$.

Thus we have found the Schur index of the irreducible representation $U_{\alpha\beta}^{(t)}$ of G, as it is the L.C.M. of all the local indices $\Lambda_{\mathfrak{p}}$.

5. Non-split cyclic extension

Until now, we have assumed that G is a split extension of an abelian normal subgroup A by a cyclic group. The methods used are applicable to non-split extension to some extent. ("Non-split" means "not necessarily split".) Here we shall discuss the case that G is metacyclic. It has been shown in [5, §2] that, if G is a split extension of a cyclic normal subgroup with a cyclic factor group, then all the irreducible representations of G are explicitly obtained and their number is counted. Now by virtue of Theorem 1, we can definitely give all the irreducible representations of any non-split extension G.

Proposition 4. *Let* $G = \langle \omega, \sigma \rangle$ *be a metacyclic group with defining relations*

$$(1) \qquad \omega^m = 1\,, \quad \sigma^{-1}\omega\sigma = \omega^r, \quad \sigma^s = \omega^h\,.$$

Then

$$(2) \qquad (m, r) = 1\,, \ m \mid h(r-1), \ u = \text{order of } r \bmod^{\times} m, \ u \mid s\,.$$

Let U *be any irreducible representation of* G. *Then there exist a positive divisor* t *of* u *and a linear character* ψ *of the subgroup* $H_t = \langle \omega, \sigma^t \rangle$ *such that* $U = \psi^G$.

Proposition 5. *Notation being as in Prop.* 4, *all the linear characters of* H_t *are given by* $\psi_{\alpha\beta}^{(t)}$, $0 \leqslant \alpha \leqslant d_t - 1$, $0 \leqslant \beta \leqslant \frac{s}{t} - 1$, *such that*

$$(3) \qquad \psi_{\alpha\beta}^{(t)}(\omega) = \exp\frac{2\pi\sqrt{-1}\,\alpha}{d_t}\,, \quad \psi_{\alpha\beta}^{(t)}(\sigma^t) = \exp\frac{2\pi\sqrt{-1}\,\alpha h}{\frac{s}{t}d_t}\exp\frac{2\pi\sqrt{-1}\,\beta}{\frac{s}{t}}\,,$$

where

$$(4) \qquad d_t = (m, r^t - 1)\,.$$

The induced representation $U_{\alpha\beta}^{(t)} = (\psi_{\alpha\beta}^{(t)})^G$ *is irreducible if and only if*

$$(5) \qquad \alpha r^{\mu} \not\equiv \alpha \pmod{d_t}\,, \quad 1 \leqslant \mu \leqslant t-1\,.$$

Proposition 6. *Let $U_{\alpha\beta}^{(t)}$ and $U_{\alpha'\beta'}^{(t)}$ be irreducible. Then $U_{\alpha\beta}^{(t)}$ and $U_{\alpha'\beta'}^{(t)}$ are inequivalent if and only if we have*

$$(6) \qquad (\alpha-\alpha')h+(\beta-\beta')d_t \not\equiv 0 \left(\mathrm{mod}\, \frac{s}{t} d_t\right),$$

or

$$(7) \qquad \alpha r^{\mu} \not\equiv \alpha' (\mathrm{mod}\, d_t), \quad 0 \leqslant \mu \leqslant t-1 .$$

Proposition 7. *Let t, t' be distinct divisors of u. Then $U_{\alpha\beta}^{(t)}$ and $U_{\alpha'\beta'}^{(t')}$ are inequivalent for any α, β, α', β'.*

The proofs of Prop. 4–7 are performed in the same way as in §2, so that they are omitted. Here we note that the matrix representation $U_{\alpha\beta}^{(t)}$ is given by

$$(8) \qquad U_{\alpha\beta}^{(t)}(\omega)=\begin{pmatrix} \zeta_t^{\alpha} & & & 0 \\ & \zeta_t^{\alpha r} & & \\ & & \ddots & \\ 0 & & & \zeta_t^{\alpha r^{t-1}} \end{pmatrix}, \quad \zeta_t = \exp \frac{2\pi\sqrt{-1}}{d_t},$$

$$(9) \qquad U_{\alpha\beta}^{(t)}(\sigma)=\begin{pmatrix} 0 & \cdots & 0 & \rho_t^{\alpha h+\beta d_t} \\ 1 & & & 0 \\ & \ddots & & \vdots \\ 0 & & 1 & 0 \end{pmatrix}, \quad \rho_t = \exp \frac{2\pi\sqrt{-1}}{\frac{s}{t} d_t}.$$

Now we shall consider group algebras and Schur indices.

Proposition 8. *Notation being the same as before, assume that $u=s$. Namely, the centralizer of $\langle\omega\rangle$ in G coincides with $\langle\omega\rangle$ itself. Then the enveloping algebra of the irreducible representation $U_{\alpha 0}^{(s)}$ induced from a linear character $\psi_{\alpha 0}^{(s)}$ of $\langle\omega\rangle$ is isomorphic to the cyclic algebra with center $\boldsymbol{Q}(\chi_{\alpha 0}^{(s)})$:*

$$\mathrm{env}_{\boldsymbol{Q}}(U_{\alpha 0}^{(s)}) \simeq (\zeta_m^{\alpha h}, \boldsymbol{Q}(\zeta_m^{\alpha}), \tau), \quad \zeta_m = \exp \frac{2\pi\sqrt{-1}}{m}$$

where $\chi_{\alpha 0}^{(s)}$ is the character of $U_{\alpha 0}^{(s)}$ and τ is an automorphism of $\boldsymbol{Q}(\zeta_m^{\alpha})/\boldsymbol{Q}$ defined by

$$\tau(\zeta_m^{\alpha}) = \zeta_m^{\alpha r} .$$

Proposition 9. *Denote by $\Lambda_{\mathfrak{p}}$ the local index of $\mathrm{env}_{\boldsymbol{Q}}(U_{\alpha 0}^{(s)})$ at a place $\mathfrak{p}$ of $k=\boldsymbol{Q}(\chi_{\alpha 0}^{(s)})$, and set*

$$d_{\alpha} = \frac{m}{(m, \alpha)}, \quad v_{\alpha} = \frac{m}{(m, \alpha h)} .$$

Then we have the following results.

(I) *If a finite prime* $\mathfrak{p}$ *of* k *does not divide* d_α, *then*

$$\Lambda_{\mathfrak{p}} = 1 .$$

(II) *If* $\mathfrak{p} \mid d_\alpha$, *we put* $v_\alpha = p^b z$, $(p, z)=1$, $\mathfrak{p} \mid p$.
Then we have

$$\Lambda_{\mathfrak{p}} = \frac{c_{\mathfrak{p}}}{\left(c_{\mathfrak{p}}, \dfrac{q-1}{z}\right)},$$

except the case $p=2$, $b=1$, 2^2 *is the highest power of* 2 *dividing* d_α, $r \equiv -1 \pmod 4$, *the order of* $r \bmod^\times \dfrac{d_\alpha}{2^2}$ *is odd, and the order of* $2 \bmod^\times \dfrac{d_\alpha}{2^2}$ *is odd. In this exceptional case we have* $\Lambda_{\mathfrak{p}}=2$. *In the above,* $c_{\mathfrak{p}}=(e_{\mathfrak{p}}, q-1)$, $e_{\mathfrak{p}}=\dfrac{s}{t'}$, $q=p^{\tilde{f}/f}$, t' *is the order of* $r \bmod^\times \dfrac{d_\alpha}{p^a}$, $\tilde{f}$ *is the order of* $p \bmod^\times \dfrac{d_\alpha}{p^a}$, $f = \#\left[\left\langle r \bmod^\times \dfrac{d_\alpha}{p^a}\right\rangle \cap \left\langle p \bmod^\times \dfrac{d_\alpha}{p^a}\right\rangle\right]$, *and* p^a *is the highest power of* p *dividing* d_α.

(III) *For any infinite prime* $\mathfrak{p}_\infty$ *of* k, *we have*

$$\Lambda_{\mathfrak{p}_\infty} = 1 ,$$

except the case $\alpha h \equiv \dfrac{m}{2} \pmod m$, $2 \mid s$ *and* $r^{s/2} \equiv -1 \pmod{d_\alpha}$. *In this case we have, for any* $\mathfrak{p}_\infty$ *of* k, $\Lambda_{\mathfrak{p}_\infty}=2$.

The proofs of Propositions 8, 9 are almost the same as those of Theorems 2, 3, so that they are omitted. Here we only note that $\exp \dfrac{2\pi\sqrt{-1}\alpha h}{m}$ is fixed by the automorphism τ, as follows from the fact $m \mid h(r-1)$.

REMARK 1. Going back to Prop. 5, let $U_{\alpha\beta}^{(t)}$ be irreducible. We assume that there exists an automorphism τ of $\boldsymbol{Q}(\zeta_t^\alpha, \rho_t^{\alpha h+\beta d_t})/\boldsymbol{Q}$ such that

$$\tau(\zeta_t^\alpha) = \zeta_t^{\alpha r}, \quad \tau(\rho_t^{\alpha h+\beta d_t}) = \rho_t^{\alpha h+\beta d_t},$$

where ζ_t and ρ_t are defined by (8) and (9), respectively. Then the enveloping algebra of $U_{\alpha\beta}^{(t)}$ is isomorphic to the cyclic algebra with center $\boldsymbol{Q}(\chi_{\alpha\beta}^{(t)})$:

$$\mathrm{env}_{\boldsymbol{Q}}(U_{\alpha\beta}^{(t)}) \simeq (\rho_t^{\alpha h+\beta d_t}, \boldsymbol{Q}(\zeta_t^\alpha, \rho_t^{\alpha h+\beta d_t}), \tau) .$$

Hence the Schur index of $U_{\alpha\beta}^{(t)}$ can be computed.

REMARK 2. About the metacyclic groups satisfying the assumption in Prop. 8, we quote the following fact from [3, §47].

Proposition 10. *Notation is the same as in Prop.* 4. *Assume that the* *centralizer of $\langle\omega\rangle$ in G is exactly $\langle\omega\rangle$ itself. Then, every irreducible representation of G is either one-dimensional or equivalent to one induced from a linear character of $\langle\omega\rangle$ if and only if, for each i and j, $1\leqslant i\leqslant m$, $1\leqslant j\leqslant s-1$,*

$$(*)\qquad r^j i\equiv i(\text{mod m}) \quad \textit{implies} \quad ri\equiv i(\text{mod } m)\,.$$

In particular, when s is a prime, the condition $(*)$ is fulfilled.

6. Examples

1) The dihedral group D_m. The defining relations are

$$\omega^m=1\,,\quad \sigma^{-1}\omega\sigma=\omega^{-1}\,,\quad \sigma^2=1\,.$$

We use notation of Prop. 5. The one-dimensional representations of D_m are $\psi^{(1)}_{0\beta}$ ($\beta=0$, 1) if m is odd, and $\psi^{(1)}_{\alpha\beta}$ ($\alpha=0$, 1, $\beta=0$, 1) if m is even. Here

$$\psi^{(1)}_{0\beta}(\omega)=1\,,\quad \psi^{(1)}_{0\beta}(\sigma)=(-1)^\beta,\quad \text{if } m \text{ is odd},$$
$$\psi^{(1)}_{\alpha\beta}(\omega)=(-1)^\alpha,\quad \psi^{(1)}_{\alpha\beta}(\sigma)=(-1)^\beta,\quad \text{if } m \text{ is even}.$$

The other (inequivalent) irreducible representations of D_m are induced from linear characters of $\langle\omega\rangle$ and given by $U^{(2)}_{\alpha 0}\left(1\leqslant\alpha\leqslant\dfrac{m-1}{2}\right)$ if m is odd, and $U^{(2)}_{\alpha 0}\left(1\leqslant\alpha\leqslant\dfrac{m-2}{2}\right)$ if m is even. In both cases, each $U^{(2)}_{\alpha 0}$ is defined by

$$U^{(2)}_{\alpha 0}(\omega)=\begin{pmatrix}\zeta_m^\alpha & 0\\ 0 & \zeta_m^{-\alpha}\end{pmatrix},\quad \zeta_m=\exp\frac{2\pi\sqrt{-1}}{m}\,,\quad U^{(2)}_{\alpha 0}(\sigma)=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}.$$

The enveloping algebra of $U^{(2)}_{\alpha 0}$ is isomorphic to the quaternion algebra:

$$\text{env}_{\boldsymbol{Q}}(U^{(2)}_{\alpha 0})\simeq(1,\,\boldsymbol{Q}(\zeta_m^\alpha),\,\tau)\,,\quad \tau(\zeta_m^\alpha)=\zeta_m^{-\alpha}.$$

Hence the Schur index of $U^{(2)}_{\alpha 0}$ is equal to 1.

REMARK. In [4, §11], the Schur indices of any dihedral group whose order is 2-power, are discussed.

2) Let G be a split extension of a cyclic normal subgroup $\langle\omega\rangle$ by a cyclic group. Assume that the centralizer of $\langle\omega\rangle$ in G coincides with $\langle\omega\rangle$ itself. Then the Schur indices of the irreducible representations of G induced from linear characters of $\langle\omega\rangle$ are all equal to 1. (See Prop. 10).

3) The generalized quaternion group Q_m. In this case, we have for the generators ω, σ,

$$\omega^{2m}=1\,,\quad \sigma^{-1}\omega\sigma=\omega^{-1}\,,\quad \sigma^2=\omega^m\,.$$

(The integer m is *not necessarily* 2*-power* [3, p. 23]) The one-dimensional representations of Q_m are $\psi^{(1)}_{\alpha\beta}$ (α, $\beta=0$, 1) such that

$$\psi_{\alpha\beta}^{(1)}(\omega) = (-1)^{\alpha}, \quad \psi_{\alpha\beta}^{(1)}(\sigma) = (-1)^{\beta} \exp \frac{2\pi\sqrt{-1}\,\alpha m}{4}.$$

The other (inequivalent) irreducible representations of Q_m are induced from linear characters of $\langle\omega\rangle$ and given by $U_{\alpha 0}^{(2)}$ $(1\leqslant\alpha\leqslant m-1)$ such that

$$U_{\alpha 0}^{(2)}(\omega) = \begin{pmatrix} \zeta_{2m}^{\alpha} & 0 \\ 0 & \zeta_{2m}^{-\alpha} \end{pmatrix}, \quad \zeta_{2m} = \exp \frac{2\pi\sqrt{-1}}{2m}, \quad U_{\alpha 0}^{(2)}(\sigma) = \begin{pmatrix} 0 & (-1)^{\alpha} \\ 1 & 0 \end{pmatrix}.$$

The enveloping algebra of each $U_{\alpha 0}^{(2)}$ is isomorphic to the quaternion algebra with the center $\boldsymbol{Q}(\chi_{\alpha 0}^{(2)})=\boldsymbol{Q}(\zeta_{2m}^{\alpha}+\zeta_{2m}^{-\alpha})$:

$$\mathrm{env}_{\boldsymbol{Q}}(U_{\alpha 0}^{(2)}) \simeq ((-1)^{\alpha}, \boldsymbol{Q}(\zeta_{2m}^{\alpha}), \tau), \quad \tau(\zeta_{2m}^{\alpha}) = \zeta_{2m}^{-\alpha}.$$

Hence, if α is even, the Schur index of $U_{\alpha 0}^{(2)}$ is equal to 1. However, if α is odd, the local index of $\mathrm{env}_{\boldsymbol{Q}}(U_{\alpha 0}^{(2)})$ at every infinte prime spot of $\boldsymbol{Q}(\chi_{\alpha 0}^{(2)})$ is equal to 2, as follows from the fact that the center $\boldsymbol{Q}(\chi_{\alpha 0}^{(2)})$ is totally real. Consequently, if α is odd, the Schur index of $U_{\alpha 0}^{(2)}$ is equal to 2.

REMARK. The Schur indices of the generalized quaternion groups of 2-power orders are discussed in [4, §11].

4) Let G be a metacyclic group with two generators ω, σ satisfying

$$\omega^{52} = 1, \quad \sigma^{-1}\omega\sigma = \omega^{3}, \quad \sigma^{6} = 1.$$

(Note that $[\langle\omega\rangle: 1]$ and $[G: \langle\omega\rangle]$ are not relatively prime. This example appears in [3, p. 340].) We can easily find all the irreducible representations of G.

degree	number	representation
1	12	$\psi_{\alpha\beta}^{(1)}$, $\alpha=0, 1$, $\beta=0, 1, 2, 3$
2	3	$U_{10}^{(2)}$, $U_{11}^{(2)}$
3	16	$U_{10}^{(3)}$, $U_{11}^{(3)}$, $U_{20}^{(3)}$, $U_{21}^{(3)}$
6	4	$U_{10}^{(6)}$

In this table the second row, for instance, means that the number of the irreducible representations of degree two is equal to 3 and the representations $U_{10}^{(2)}$ and $U_{11}^{(2)}$ are the representatives of the algebraically conjugate classes of the irreducible representations. Here, the definitions of $U_{\alpha\beta}^{(t)}$ and $\psi_{\alpha\beta}^{(1)}$ are the same as those of Prop. 5. We can easily show that the Schur index of every irreducible representation of G is equal to 1.

5) Let G be a hyperelementary group (at 3), whose 3-Sylow group is abelian of type $(3, 3, 3^2)$, with defining relations

$$\omega^7 = 1\,,\quad \sigma_1^3 = \sigma_2^3 = \sigma^{3^2} = 1\,,\quad \sigma^{-1}\omega\sigma = \omega^2\,,$$
$$\omega\sigma_i = \sigma_i\omega\,,\quad \sigma\sigma_i = \sigma_i\sigma \qquad (i=1,2)\,.$$

The inequivalent not one dimensional irreducible representations of G are given by $U_{\alpha\alpha_1\alpha_2\beta}(\alpha=1,3,\ 0\leqslant\alpha_1,\ \alpha_2,\ \beta\leqslant 2)$ such that

$$U_{\alpha\alpha_1\alpha_2\beta}:\ \omega\mapsto\begin{pmatrix}\zeta_7^{\alpha} & & 0\\ & \zeta_7^{2\alpha} & \\ 0 & & \zeta_7^{4\alpha}\end{pmatrix},\quad \sigma_i\mapsto\zeta_3^{\alpha_i}\cdot 1_3\,,\quad i=1,2\,,$$
$$\sigma\mapsto\begin{pmatrix}0 & 0 & \zeta_3^{\beta}\\ 1 & 0 & 0\\ 0 & 1 & 0\end{pmatrix},\quad \zeta_7=\exp\frac{2\pi\sqrt{-1}}{7}\,,\quad \zeta_3=\exp\frac{2\pi\sqrt{-1}}{3}\,.$$

If $\beta=0$, then the Schur index of every $U_{\alpha\alpha_1\alpha_2 0}(\alpha=1,3,\ 0\leqslant\alpha_1,\ \alpha_2\leqslant 2)$ is equal to 1. However, if $\beta=1$ or 2, the enveloping algebra of $U_{\alpha\alpha_1\alpha_2\beta}$ is isomorphic to the cyclic algebra with the center $\boldsymbol{Q}(\zeta_3,\sqrt{-7})$:

$$\mathrm{env}_{\boldsymbol{Q}}(U_{\alpha\alpha_1\alpha_2\beta})\simeq(\zeta_3,\ \boldsymbol{Q}(\zeta_3,\zeta_7),\ \tau)\,,\quad \tau(\zeta_7)=\zeta_7^2\,,\quad \tau(\zeta_3)=\zeta_3\,.$$

From this we can conclude that for any α ($\alpha=1,3$), α_1, α_2 ($0\leqslant\alpha_1,\ \alpha_2\leqslant 2$), β ($\beta=1,2$), the Schur index of $U_{\alpha\alpha_1\alpha_2\beta}$ is equal to 3.

6) (Brauer) We fix a positive integer $s\geqslant 2$. Let p be a prime such that $p\equiv 1 \pmod s$ and $\left(\frac{p-1}{s},\ s\right)=1$. (There exist infinitely many primes satisfying this condition.) Let j, $(j,p)=1$, be an integer whose order ($\mathrm{mod}^{\times}\ p$) is equal to s. Determine a positive integer r from the congruences: $r\equiv 1 \pmod s$, $r\equiv j \pmod p$. Let G be a metacyclic group with two generators ω, σ, satisfying

$$\omega^{ps}=1\,,\quad \sigma^{-1}\omega\sigma=\omega^r\,,\quad \sigma^s=\omega^p\,.$$

We consider the irreducible representation U of G defined by

$$U(\omega)=\begin{pmatrix}\zeta & & & & 0\\ & \zeta^r & & & \\ & & \ddots & & \\ & & & \ddots & \\ 0 & & & & \zeta^{r^{s-1}}\end{pmatrix},\quad U(\sigma)=\begin{pmatrix}0 & \cdots & \cdots & 0 & \zeta^p\\ 1 & & & & 0\\ & \ddots & & & \vdots\\ & & \ddots & & \vdots\\ 0 & & & 1 & 0\end{pmatrix}$$

where $\zeta=\exp\frac{2\pi\sqrt{-1}}{ps}$. The enveloping algebra of U is known from Prop. 8:

$$\mathrm{env}_{\boldsymbol{Q}}(U)\simeq(\zeta^p,\ \boldsymbol{Q}(\zeta),\ \tau),\quad \tau(\zeta)=\zeta^r\,.$$

By Prop. 9, it is readily verified that the Schur index of U is equal to s. Thus for each positive integer $s \geqslant 2$, there exists an irreducible representation U whose degree and Schur index are both equal to s.

This result was found by Brauer [2, §5]. Berman [1] has shown the same result by giving another examples, which are also metacyclic groups.

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