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A MATHEMATICAL JUSTIFICATION FOR KORTEWEG-DE VRIES EQUATION AND BOUSSINESQ EQUATION OF WATER SURFACE WAVES

Dedicated to Professor Sigeru Mizohata on the occasion of his sixtieth birthday

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1 Introduction

1. In this paper, we give a mathematically rigorous derivation of Korteweg—de Vries equation[1] and of Boussinesq equation from the Euler equation for surface wave of water in irrotational motion.

In deriving these equations, we begin with noticing the following two facts: (i) we have solved in [6] the Cauchy problem for two space dimensional water surface wave equations in a class of analytic functions locally in time, in the dimensionless form, (ii) the surface wave and the velocity potential depend on the dimensionless parameter δ introduced in this dimensionless problem as the ratio of the water depth to wave length in such a way that they are infinitely many times differentiable with respect to δ [6].

We can expand[2] equations with respect to δ^2. In dropping the terms of order $O(\delta^4)$ in this expansion we get K-dV equation and "Boussinesq equation" and we prove that their solutions approximate the original water surface wave up to the same order of errors as the dropped terms with respect to δ.

Our "Boussinesq equation" has not the same form as the original equation given by Boussinesq himself in 1871 [3]. If one substitutes the "first approximation" $\varphi_t=\varphi_x+O(\delta^2)$ in the terms of order $O(\delta^2)$ of our "Boussinesq equation", one can immediately recover the original one. However this substitution is not justified generally. We remark later that this substitution would be rather destructive for good approximation even for waves for which it is justified, see §2. In the last part of the paragraph 2, we compare our derivation of "Boussinesq" equation with the original study of Boussinesq.

1) Mentioned simply K-dV equation hereafter.

2) This is not Friedrichs expansion [7]. The precise meaning of this expansion is given in §2.

2. In the following study, a crucial remark is in order: the shallow water wave of finite amplitude is not necessarily "long" wave.

More precisely, it is not sufficient in the study of the "long" wave of finite amplitude of water surface to consider the ratio δ of the water depth h to the wave length λ. It is also necessary to take into account the ratio ε of the amplitude κ of wave to the water depth, the amplitude κ being the maximum displacement of water surface from the mean level of still water. We prove, in fact, that K-dV equation and "Boussinesq equation" are concerned with the water surface wave in the physical conditions expressed in the following relation:

$$\delta^2 = \left(\frac{\text{water depth } h}{\text{wave length } \lambda}\right)^2 \sim \frac{\text{amplitude } \kappa}{\text{water depth } h} = \varepsilon \tag{1.1}$$

as wave length $\gg$ water depth and amplitude $\ll$ water depth[3].

We notice here that the famous soiltary wave solution for K-dV and Boussinesq equation[4] of the type

$$u(t, x) = 2A^2 \operatorname{sech}^2 A(x \pm ct) \tag{1.2}$$

with the progressing (or regressing) speed c reflects the relation (1.1) although the wave length of which is infinity. In fact, if the amplitude[5] is doubled in (1.2), then A is replaced by $\sqrt{2}A$ and (1.2) becomes now

$$U(t, x) = 4A^2 \operatorname{sech}^2 A(\sqrt{2}(x \pm ct))\,. \tag{1.3}$$

It means that the elevation at (t, x) of $U(t, x)$ is equal to 2 times the elevation of $u(t, x)$ at $(t, \sqrt{2}x)$, but not at (t, x). Thus the wave shrinks by $\frac{1}{\sqrt{2}}$ in remounting doublly in height.

On the other hand, if κ is replaced by 2κ, λ should be replaced by $\frac{1}{\sqrt{2}}\lambda$ to keep unchanged the relation $\varepsilon = \delta^2$. Although the relation $\varepsilon = \delta^2$ is not equivalent to (1.1), we actually carry out our theory in this paper in supposing $\varepsilon = \delta^2$ for the simplicity.

It would be worth noticing in this regard that J. Scott Russell [19] observed that "the length, therefore, increases with the depth of fluid directly, being equal to about 6.28 times the depth. The length does not, like the velocity of the wave, increases with the height of the wave in a given depth of fluid. On the contrary, the length appears to diminish as the height of the wave is increased" (p. 340). "This extension of length is attended with a diminution

3) Actually our theory is valid for $\delta \in [0, 1]$. If we insist here that δ^2 and ε are of the same order *as infinitesimals*, it is to emphasize the specific feature as "long" wave of water surface approximated by solutions of K-dV equation and "Boussinesq equation".

4) Our "Boussinesq equation" has, of course, this type of particular solution (see §2).

5) The highest elevation of the solitary wave.

of height, and the diminution of length with an increase of height of the wave, so that the change of length and height attend and indicate changes of depth" (p. 352).

If we take into account only the ratio water depth/wave length, the first approximation up to order $O(\delta^2)$ gives the shallow water wave equation as we proved in [6] and Ovsjannikov for periodic case in [14], [15]. We repeat here that shallow water wave contains the wave for which $\varepsilon \sim 1$, since the shallow water wave equation is derived for "long" wave of finite amplitude such that, with $\delta \in [0, 1]$, $\varepsilon/\delta^2 \gg 1$ as $\lambda \gg h$ and $h \gg \kappa$. We notice also that Scott Russell observed in fact that waves begin to break as the height approaches the water depth, page 352 in [19] describing long series of his experiments.

3. It is F. Ursell who suggested in [23] the importance to distinguish three physical conditions

$$\frac{\varepsilon}{\delta^2} \gg 1, \quad \sim 1 \text{ or } \ll 1 \quad \text{as} \quad \lambda \gg h, \quad h \gg \kappa$$

in the study of "long" surface wave of finite amplitude.

In fact, G.B. Airy, giving a nonlinear second approximation equation, claimed that "long" wave of finite amplitude of water surface could not propagate without deformation and would break down in finite time [1], [11, §§173, 187–188]. Airy denied simply and categorically the possibility of solitary wave[6] observed and studied profoundly by J. Scott Russell, and reported in 1838 and 1844 in [18], [19]. And he added: "We are not disposed to recognize this wave[7] as deserving the epithets 'great' or 'primary', ⋯, and we conceive that ever since it was known that the theory of shallow waves of great length was contained in the equation $\frac{d^2X}{dt^2} = g\kappa\frac{d^2X}{dx^2} \cdots$ the theory of the solitary wave has been perfectly well known" as cited in Rayleigh [17], page 256. Airy's "long" wave assumption means $\delta^2 \ll 1$ which contains the waves for which $\varepsilon \sim 1$.

G.G. Stokes [21], however, suggested already in 1847 another possibility of approximation supposing[8] not only "long" wave but also $\varepsilon \ll \delta^2$. He had thus a second approximation of velocity which is independent of the height of waves. Finally in this century, the existence of the solitary wave on the water of finite depth is proved mathematically as stationnary "long" wave of finite amplitude by Struik [22], Lavrentiev [12] and Friedrichs-Hyers [5] long after the proposition of Boussinesq equation [3] and K-dV equation [8].

6) Appellation by Scott Russell of "long" surface wave of water of finite amplitude propagating without changing form.
7) Wave called "great" or "primary" by Scott Russell.
8) Ursell is inspired by this study.

In this "paradoxical situation"[9] concerning "long" wave of finite amplitude of water surface, Ursell claimed the necessity to distinguish certain "long" waves from shallow water wave; and he derived formally, in Lagrangian coordinates, as the second approximate equation (i) nonlinear hyperbolic equation of Airy corresponding to shallow water wave, (ii) "Boussinesq equation"[10] and (iii) linearized "long" wave equation:

$$\frac{\partial^2 u}{\partial t^2}-\frac{\partial^2 u}{\partial x^2}=\frac{\partial^4 u}{\partial x^4},$$

corresponding, respectively, to (i) $\varepsilon \gg \delta^2$ (ii) $\varepsilon \sim \delta^2$ and (iii) $\varepsilon \ll \delta^2$.

4. The outline of this paper is as follows. In §2, "Boussinesq equation" is derived. Comparing our derivation with the original study of Boussinesq, it is pointed out that our "Boussinesq equation" is already given by Boussinesq although it is not stated explicitely. In §3, K-dV equation is derived. We also derive and discuss, in this paragraph, a new type of equation using the values of potential on the water bottom. This equation is a sort of mixed type of K-dV and so-called BBM equation and it gives a nice approximation. Finally, in appendix, we give some remarks concerning the global existence of solutions of approximate equations.

Our derivation of K-dV equatoin and "Boussinesq equation" is mathematically rigorous for finite time interval in a class of analytic functions.

2 Boussinesq equation

5. In two space dimensional case, the surface wave of water in irrotational motion is described in dimensionless form [6] by the velocity potential $\Phi=\Phi(t,x,y)$ and the free surface $y=\Gamma(t,x)$ satisfying

$$\delta^2\Phi_{xx}+\Phi_{yy}=0 \qquad \text{in} \quad \Omega(t), \tag{2.1}$$

$\Omega(t)=\{(x,y);\, x\in\boldsymbol{R},\, 0<y<\Gamma(t,x)\}$ being the domain occupied by the water, and by the boundary conditions:

$$\Phi_y=0, \qquad \text{on } y=0, \quad x\in\boldsymbol{R} \tag{2.2}$$

and

$$\begin{cases} \delta^2(\Phi_t+\frac{1}{2}\Phi_x^2+y)+\frac{1}{2}\Phi_y^2=0 \\ \delta^2(\Gamma_t+\Phi_x\Gamma_x)-\Phi_y=0, \qquad \text{on } y=\Gamma(t,x), \quad x\in\boldsymbol{R}, \end{cases} \tag{2.3}$$

with the dimensionless parameter δ defined by

9) Expression of Ursell [23]. Garett Birkhoff [2, p. 23] noted it as a consequence of the Earnshaw paradox. For Stoker, however, it is anywise not paradoxical [20, p. 342].
10) See also Keulegan-Patterson [9, pp. 72–74].

$$\delta = \frac{h}{\lambda} = \frac{\text{water depth}}{\text{wave length}}.$$

In [6], we have solved the Cauchy problem for (2.1)–(2.3) with the initial data

$$\Gamma(0, x) = \Gamma_0(x) > 0\,, \quad \Phi(0, x, y) = \Phi_0(x, y) \tag{2.4}$$

in a scale $S = \bigcup_{\rho>0} X_\rho$ of Banach spaces consisting of analytic functions (For notations, we refer readers to [6]).

By the mapping $(x, y) = (x, y)(t, \xi, \eta; \delta)$ defined by thc conformal mapping $z + x + i\delta y$ of $\zeta + \xi + i\delta\eta$ from $\Omega_1 = \{(\xi, \eta); \xi \in \boldsymbol{R}, 0 < \eta < 1\}$ to $\Omega(t)$, and by the image $\varphi = \varphi(t, \xi, \eta; \delta)$ of Φ by this mapping, our problem is reduced to solve the following on $\eta = 1$ [6]:

$$\begin{cases} x_t = wA_\delta x_\xi A_\delta \varphi_\xi + x_\xi A_\delta(wA_\delta \varphi_\xi) - x_\xi C_\delta(wA_\delta \varphi_\xi) \\ \varphi_t = -\dfrac{A_\delta}{\delta} x + \dfrac{1}{2} w\{(A_\delta \varphi_\xi)^2 - \varphi_\xi^2\} + \varphi_\xi A_\delta(wA_\delta \varphi_\xi) - \varphi_\xi C_\delta(wA_\delta \varphi_\xi) \end{cases} \tag{2.6}$$

where $w = (x_\xi^2 + A_\delta x_\xi)^2)^{-1}$ and operators A_δ and C_δ are defined by

$$\begin{cases} (A_\delta u)(\xi_0) = \dfrac{1}{2\delta} \displaystyle\int_{-\infty}^{\infty} \frac{u(\xi)}{\sinh \frac{\pi}{2\delta}(\xi - \xi_0)} d\xi \\ (C_\delta v)(\xi_0) = \dfrac{1}{2\delta} \displaystyle\int_{-\infty}^{\infty} (1 - \tanh \frac{\pi}{4\delta}(\xi - \xi_0)) v(\xi) d\xi\,. \end{cases} \tag{2.7}$$

We have actually solved the Cauchy problem for the system of integro-differential equations for $(v, u)(t) = (x_\xi, \phi_\xi)(t, \xi, 1; \delta)$ derived from (2.6) with Cauchy data $(v, u)(0) = (v_0(\xi), u_0(\xi)) \in X_{\rho_0}$, in the scale $S = \bigcup_{\rho>0} X_\rho$ of Banach spaces of analytic functions such that $(v, u)(t) \in X_\rho$ for $|t| < a(\rho_0 - \rho), \forall \rho < \rho_0$, $a > 0$. From this solution $(v, u)(t, \xi; \delta)$, we got φ and (x, y) on Ω_1, and $y = \dfrac{A_\delta}{\delta} x$ on $\eta = 1$ satisfying the Cauchy data φ_0 and y_0 which are the images of Φ_0 and Γ_0 by the mapping $(x, y) = (x, y)(0, \xi, \eta; \delta)$.

6. Let us now introduce the second dimensionless parameter ε defined by

$$\varepsilon = \frac{\kappa}{h} = \frac{\text{amplitude of wave}}{\text{water depth}}.$$

We recall first that the dimensionless equations (2.1)–(2.4) are obtained in [6] by the following change of variables

$$x = \lambda x', \quad y = hy', \quad t = \frac{\lambda}{\sqrt{gh}} t',$$

defining the dimensionless variables x', y' and t'. Let us denote now the amplitude by y^1 and replace the transformation $y=hy'$ by

$$y = h+y^1 = h+\kappa y^{1'} = h(1+\varepsilon y^{1'}) \tag{2.8}$$

defining thus the dimensionless amplitude $y^{1'}$ by the equation.

Omitting "$'$" of the dimensionless variables as before, let us define the following transformation in Ω_1:

$$\begin{cases} x = \xi+\varepsilon x^1\,, \quad y = \eta+\varepsilon y^1\,, \\ \varphi = -t+\varepsilon\varphi^1\,. \end{cases} \tag{2.9}$$

Let us simplify the condition (1.1) putting barely $\delta^2=\varepsilon$. Then by (2.6) and (2.9), x^1, y^1 and φ^1 on $\eta=1$ satisfy the following system

$$\begin{cases} x_t^1 = \varepsilon w^1 A_\delta x_\xi^1 A_\delta \varphi_\xi^1 + A_\delta(w^1 A_\delta \varphi_\xi^1) - C_\delta(w^1 A_\delta \varphi_\xi^1) + \\ \qquad\qquad + \varepsilon x_\xi^1 A_\delta(w^1 A_\delta \varphi_\xi^1) - \varepsilon x_\xi^1 C_\delta(w^1 A_\delta \varphi_\xi^1)\,, \\ \varphi_t^1 = -\dfrac{A_\delta}{\delta}\,x^1 + \dfrac{\varepsilon}{2}\,w^1\{(A_\delta\varphi_\xi^1)^2 - (\varphi_\xi^1)^2\} + \\ \qquad\qquad + \varepsilon\varphi_\xi^1 A_\delta(w^1 A_\delta\varphi_\xi^1) - \varepsilon\varphi_\xi^1 C_\delta(w^1 A_\delta\varphi_\xi^1)\,, \\ y^1 = \dfrac{A_\delta}{\delta}\,x^1\,, \qquad \text{for } t,\ \xi\in\boldsymbol{R} \end{cases} \tag{2.10}$$

where $w^1=((1+\varepsilon x_\xi^1)^2+\varepsilon^2(A_\delta x_\xi^1)^2)^{-1}$,

Since we can apply the same method as [6] [7] for (2.6) to solve the Cauchy problem for the system on $(v^1, u^1)=(x_\xi^1, \varphi_\xi^1)\,(t, \xi; \delta)$ derived from (2.10) with the Cauchy data

$$(v^1, u^1)\,(0) = (v_0^1, u_0^1)\,(\xi)\in X_{\rho_0} \tag{2.11}$$

satisfying $1+\varepsilon v_0^1=v_0$, $\varepsilon u_0^1=u_0$, we have a unique solution $(v^1, u^1)\,(t, \xi; \delta)$ in $S=\bigcup_{\rho>0} X_0$, $\delta\in[0, 1]$, which is infinitely many times differentiable with respect to $\delta\in[0, 1]$ in X_ρ, for any $|t|<a(\rho_0-\rho)$, $\rho<\rho_0$. Thus we have a one to one conformal mapping $x+i\delta y$ of $\xi+i\delta\eta$ from Ω_1 to $\Omega(t)$ and harmonic functions x^1 and ϕ^1 of $(\xi, \delta\eta)$ in Ω_1.

7. Let us define the functions $\phi=\phi(t, x, y; \delta)$ on $\Omega(t)$ and $\gamma=\gamma(t, x; \delta)$ on $\boldsymbol{R}$ using our mapping (x, y) from Ω_1 to $\Omega(t)$ by

$$\begin{cases} \gamma(t, x; \delta) = y^1(t, \xi, 1; \delta) \qquad \text{on} \quad \eta = 1, \quad \xi\in\boldsymbol{R}\,, \\ \phi(t, x, y; \delta) = \phi^1(t, \xi, \eta; \delta) \qquad \text{on} \quad \Omega_1\,. \end{cases} \tag{2.12}$$

We have then

$$\Gamma = 1+\varepsilon\gamma \tag{2.13}$$

analytic in $x \in \boldsymbol{R}$ for $|t| < a(\rho_0 - \rho)$, $\rho < \rho_0$, and

$$\Phi = -t + \varepsilon\phi \tag{2.14}$$

harmonic in $(x, \delta y)$ in $\Omega(t)$, for $|t| < a(\rho_0 - \rho)$, $\rho < \rho_0$.

Set

$$\phi(t, x, \Gamma; \delta) = \phi(t, x, 1+\delta^2\gamma(t, x; \delta); \delta) = \tilde{\phi}(t, x; \delta), \text{ i.e.,}$$
$$\Phi(t, x, \Gamma; \delta) = -t + \delta^2\tilde{\phi}(t, x; \delta).$$

Then we have the

Proposition 2.1. *The solution* $(\gamma, \tilde{\phi})$ *satisfies the following equations for* $|t| < a(\rho_0' - \rho')$, $\forall \rho' < \rho_0'$,

$$\begin{cases} \gamma_t + \tilde{\phi}_{xx} + \varepsilon\left(\dfrac{1}{3}\tilde{\phi}_{xxxx} + (\gamma\tilde{\phi}_x)_x\right) = O(\varepsilon^2), \\ \tilde{\phi}_t + \gamma + \dfrac{\varepsilon}{2}\tilde{\phi}_x^2 = O(\varepsilon^2), \quad \textit{in} \quad X_{\rho'}. \end{cases} \tag{2.15}$$

Here $X_{\rho'}$ *is a Banach space of analytic functions of* x *similar to* X_ρ *with the radius of convergence* ρ' *determined by* ρ *of* ϕ *and* x, *and that of* $\xi = \xi_1(t, x; \delta)$ *which is the inverse function of* $x = x(t, \xi, 1, \delta)$. *The right hand side of* (2.15) *means that the remainder is bounded by* $C\varepsilon^2$ *with respect to the norm* $|\ |_{\rho'}$.

Proof. We have from (2.12)

$$\begin{cases} \gamma_x = \dfrac{y_\xi^1}{1+\varepsilon x_\xi^1}, \quad \gamma_t = y_t^1 - \dfrac{\varepsilon y_\xi^1}{1+\varepsilon x_\xi^1} x_t^1, \\ \phi_x = \dfrac{\varphi_\xi^1}{1+\varepsilon x_\xi^1}, \quad \phi_t = \varphi_t^1 - \dfrac{\varepsilon \varphi_\xi^1}{1+\varepsilon x_\xi^1} x_t^1. \end{cases} \tag{2.16}$$

On the other hand, using the properties of operators A_δ and C_δ analyzed in [7], we have from (2.10) in X_ρ:

$$\begin{cases} x_t^1 = -\varphi_\xi^1 - \varepsilon\left(x_\xi^1\varphi_\xi^1 - 2\displaystyle\int_{-\infty}^{\xi} x_\xi^1\varphi_{\xi\xi}^1 d\xi\right) + O(\varepsilon^2), \\ \varphi_t^1 = -x_\xi^1 - \varepsilon\left(\dfrac{1}{3}x_{\xi\xi\xi}^1 + \dfrac{3}{2}(\varphi_\xi^1)^2\right) + O(\varepsilon^2), \\ \gamma = y^1 = \dfrac{A_\delta}{\delta}x^1 = x_\xi^1 + \dfrac{\varepsilon}{3}x_{\xi\xi\xi}^1 + O(\varepsilon^2), \\ y_t^1 = x_{t\xi}^1 + \dfrac{\varepsilon}{3}x_{t\xi\xi\xi}^1 + O(\varepsilon^2) = \\ \quad = -\varphi_{\xi\xi}^1 - \varepsilon\left(\dfrac{1}{3}\varphi_{\xi\xi\xi\xi}^1 + \varphi_\xi^1 x_{\xi\xi}^1 - x_\xi^1\varphi_{\xi\xi}^1\right) + O(\varepsilon^2), \end{cases} \tag{2.17}$$

and also in $X_{\rho'}$:

$$
(2.18)\qquad \begin{cases} \varphi^1_\xi = \tilde{\phi}_x(1+\varepsilon x^1_\xi) = \tilde{\phi}_x+\varepsilon\gamma\tilde{\phi}_x+O(\varepsilon^2)\,, \\ \varphi^1_{\xi\xi} = \tilde{\phi}_{xx}+\varepsilon(\gamma\tilde{\phi}_{xx}+(\gamma\tilde{\phi}_x)_x)+O(\varepsilon^2)\,, \\ \varphi^1_{\xi\xi\xi\xi\xi} = \tilde{\phi}_{xxxx}+O(\varepsilon)\,, \\ x^1_\xi = \gamma+O(\varepsilon)\,, \\ x^1_{\xi\xi} = \gamma_x(1+\varepsilon x^1_\xi)+O(\varepsilon) = \gamma_x+O(\varepsilon)\,. \end{cases}
$$

Substituting (2.17)–(2.18) into (2.16), we get (2.15). Q.E.D.

REMARK 1. Notice that the terms on the left hand side of (2.15) are not the first two terms of Friedrichs expansion of $\tilde{\phi}$ and γ (cf. [7, §3]).

REMARK 2. If we drop the terms of $O(\varepsilon^2)$ on the right hand side of system (2.15) for small ε, we obtain

$$
(2.19)\qquad \begin{cases} \gamma_t+\phi_{xx}+\varepsilon(\frac{1}{3}\phi_{xxxx}+(\gamma\phi_x)_x) = 0\,, \\ \phi_t+\gamma+\frac{\varepsilon}{2}\phi_x^2 = 0\,. \end{cases}
$$

But this procedure of neglecting the terms of $O(\varepsilon^2)$ is hopeless to be justified as an approximation of general solutions of Cauchy problem except for some special solutions. In fact on one hand the system (2.19) has solitary wave solutions $(\gamma, \phi)\,(x-ct)$ similarly to those of water wave equation (2.1)–(2.3). On the other hand the linearized equation of (2.19) is not well posed for the Cauchy problem in H^∞. Namely it ha s the linear dispersion relation:

$$
p = \pm i\xi\sqrt{1-\frac{\varepsilon}{3}\xi^2}\,, \quad |\xi|<\sqrt{3/\varepsilon}
$$

$$
p = \pm\xi\sqrt{\frac{\varepsilon}{3}-\frac{1}{\xi^2}}\,, \quad |\xi|>\sqrt{3/\varepsilon}
$$

See also point 12 and a derivatoin of Korteweg-de Vries equation from (2.15) in §3.

REMARK 3. Since we discuss in this paper only Boussinesq equation and Korteweg-de Vries equation, we stop the approximation (2.15) at the order of ε^2. We can get actually approximate equations to any order of accuracy with respect to ε and discuss the solution ofhigher order approximation. For example, we have the third one

$$
(2.20)\qquad \begin{cases} \gamma_t+\tilde{\phi}_{xx}+\varepsilon(\frac{1}{3}\tilde{\phi}_{xxxx}+(\gamma\tilde{\phi}_x)_x)+ \\ \qquad +\varepsilon^2(\frac{2}{15}\tilde{\phi}_{xxxxxx}+\gamma\tilde{\phi}_{xxxx}+2\gamma_x\tilde{\phi}_{xxx}+\gamma_{xx}\tilde{\phi}_{xx}) = O(\varepsilon^3)\,, \\ \tilde{\phi}_t+\gamma+\frac{\varepsilon}{2}\tilde{\phi}_x^2-\frac{\varepsilon^2}{2}\tilde{\phi}_{xx}^2 = O(\varepsilon^3)\,. \end{cases}
$$

This has the linear dispersion relation:

$$p = \pm i\xi\sqrt{1-\frac{1}{3}\varepsilon\xi^2+\frac{2}{15}\varepsilon^2\xi^4}\,.$$

In particular et us notice the first order approximation. Hereafter we omit "$'$" in $X_{\rho'}$.

Proposition 2.2. *The solution* $(\gamma, \tilde{\phi})$ *satisfies*

$$\gamma_t+\tilde{\phi}_{xx} = O(\varepsilon),\ \tilde{\phi}_t+\gamma = O(\varepsilon) \tag{2.21}$$

in X_ρ, $\forall\rho<\rho_1$, *for* $|t|<a(\rho_1-\rho)$.

Let $\bar{\gamma}$ *and* $\bar{\phi}$ *be the solution in* X_ρ, $\rho<\rho_1$, $|t|<a\ (\rho_1-\rho)$ *of*

$$\bar{\gamma}_t+\bar{\phi}_{xx} = 0,\ \bar{\phi}_t+\bar{\gamma} = 0 \tag{2.22}$$

for the same initial data as γ, $\tilde{\phi}_x$. *Then it follows*

$$|\tilde{\phi}_x-\bar{\phi}_x|_\rho = O(\varepsilon),\ |\gamma-\bar{\gamma}|_\rho = O(\varepsilon)\,. \tag{2.23}$$

Thus (2.23) justify (2.22) as the first approximation of water surfaces wave under the physical condition (1.1). These equations (2.22) were obtained long ago as the first approximation for "long" waves of infinitesimal amplitude (Lamb [11, §169]; see also Lagrange [10, §11, n^0 35]). At this first order approximation we can not have solitary wave solutions. We note for later use a further simplification concerning the uni-directional motion as follows: If the initial data satisfy

$$\gamma(0)-\tilde{\phi}_x(0) = O(\varepsilon),\ \gamma(0)+\tilde{\phi}_x(0) = O(1) \quad \text{in} \quad X_{\rho_1}\,,$$

then by (2.21) the solution satisfies

$$\gamma(t)-\tilde{\phi}_x(t) = O(\varepsilon),\ \tilde{\phi}_t(t)+\tilde{\phi}_x(t) = O(\varepsilon),\ \gamma(t)+\tilde{\phi}_x(t) = O(1)\,, \tag{2.24}$$

in X_ρ, $|t|<a(\rho_1-\rho)$, $\forall\rho<\rho_1$.

Similarly if

$$\gamma(0)+\tilde{\phi}_x(0) = O(\varepsilon),\ \gamma(0)-\tilde{\phi}_x(0) = O(1), \quad \text{in} \quad X_{\rho_1}\,,$$

then

$$\gamma+\tilde{\phi}_x = O(\varepsilon),\ \tilde{\phi}_t-\tilde{\phi}_x = O(\varepsilon) \quad \text{in} \quad X_\rho,\ |t|<a(\rho_1-\rho)\,.$$

8. Let us now show that we have a good approximation for γ and ϕ instead of (2.19) if we make use of values of the velocity potential $\Phi=-t+\varepsilon\phi(t, x, y; \delta)$ on the water bottom $y=0$: $\phi^0(t, x; \delta)=\phi(t, x, 0; \delta)$.

Since we have (cf. [7, §3, (4.7)])

$$\phi(t, x, y; \delta) = \phi^0(t, x; \delta) + \frac{\varepsilon y^2}{2} \phi^0_{xx}(t, x; \delta) + O(\varepsilon^2),$$

$$\tilde{\phi}(t, x; \delta) = \phi^0(t, x; \delta) + \frac{\varepsilon}{2} \phi^0_{xx}(t, x; \delta) + O(\varepsilon^2),$$

our equation (2.15) can be rewritten as follows:

$$\text{(2.25)} \qquad \begin{cases} \gamma_t + \phi^0_{xx} - \dfrac{\varepsilon}{6} \phi^0_{xxxx} + \varepsilon(\gamma\phi^0_x)_x = O(\varepsilon^2), \\ \phi^0_t + \gamma - \dfrac{\varepsilon}{2} \phi^0_{txx} + \dfrac{\varepsilon}{2} (\phi^0_x)^2 = O(\varepsilon^2) \end{cases}$$

in X_ρ, $\forall\rho<\rho_1$, for $|t|<a(\rho_1-\rho)$.

Then we have the

Theorem 2.3. *The solution (γ, ϕ^0) is approximated as*

$$\text{(2.26)} \qquad |\phi^0_x - \bar{\phi}^0_x|_\rho, \ |\gamma-\bar{\gamma}|_\rho = O(\varepsilon^2)$$

for $|t|<a(\rho_2-\rho)$, $\forall\rho<\forall\rho_2<\rho_1$, by the solution of the system

$$\text{(2.27)} \qquad \begin{cases} \bar{\gamma}_t + \bar{\phi}^0_{xx} - \dfrac{\varepsilon}{6} \bar{\phi}^0_{xxxx} + \varepsilon(\bar{\gamma}\bar{\phi}^0_x)_x = 0, \\ \bar{\phi}^0_t + \bar{\gamma} - \dfrac{\varepsilon}{2} + \bar{\phi}^0_{txx} + \dfrac{\varepsilon}{2} (\bar{\phi}^0_x)^2 = 0. \end{cases}$$

with the initial data satisfying

$$\text{(2.28)} \qquad \gamma(0)-\bar{\gamma}(0), \ \phi^0_x(0)-\bar{\phi}^0_x(0) = O(\varepsilon^2) \qquad \text{in } X_{\rho_1}.$$

Proof. Since the abstract Cauchy-Kowalevski theorem [6, appendix] proves the continuous dependence of solution on the inhomogeneous term by the norm

$$\text{(2.29)} \qquad M^1[u] = \sup_{\substack{0<\rho<\rho_1 \\ 0<t<a(\rho_1-\rho)}} |u(t)|_\sigma \left(1 - \frac{t}{a(\rho_1-\rho)}\right),$$

this theorem is a direct consequence of the following lemma:

Lemma 2.4. *The Cauchy problem for $\bar{\gamma}$ and $\bar{u}=\bar{\phi}^0_x$ of* (2.27)

$$\text{(2.30)} \qquad \begin{cases} \bar{\gamma}_t + \bar{u}_x - \dfrac{\varepsilon}{6} \bar{u}_{xxx} + \varepsilon(\bar{\gamma}\bar{u})_x = 0 \\ \bar{u}_t + \bar{\gamma}_x - \dfrac{\varepsilon}{2} \bar{u}_{txx} + \varepsilon\bar{u}\bar{u}_x = 0 \end{cases}$$

with the initial data $(\bar{\gamma}, \bar{u})(0) \in X_{\rho_1} \times X_{\rho_1}$, $\rho_1>0$, has a unique solution $(\gamma, u)(t) \in$

$\in X_\rho \times X_\rho$ *for any* $\rho < \rho_2$ *and for* $|t| < a(\rho_2 - \rho)$, $\forall \rho_2 < \rho_1$.

Proof. Set

$$v = \left(1 - \frac{\varepsilon}{2} \frac{\partial^2}{\partial x^2}\right) \bar{u} ,$$

then we have

$$\bar{u} = K_\varepsilon * v, \quad K_\varepsilon(x) = \pi \exp\left(-\sqrt{\frac{2}{\varepsilon}} |x|\right), \tag{2.31}$$

and (2.30) is equivalent to

$$\begin{cases} \bar{\gamma}_t + \frac{1}{3} v_x + \frac{2}{3} (K_\varepsilon * v)_x + \varepsilon(\bar{\gamma} K_\varepsilon * v)_x = 0 , \\ v_t + \bar{\gamma}_x + \varepsilon (K_\varepsilon * v)(K_\varepsilon * v)_x = 0 , \\ (\bar{\gamma}, v)(0) \in X_{\rho_2} \times X_{\rho_2}, \ \forall \rho_2 < \rho_1 . \end{cases} \tag{2.32}$$

Since $K_\varepsilon(x)$ is summable for any $\varepsilon > 0$, we see that $\bar{u} = K_\varepsilon * v \in X_\rho$ if $v \in X_\rho$ and thus we can apply the abstract Cauchy-Kowalevski theorem to (2.32) in order to get the solution $(\bar{\gamma}, v)(t)$ in X_ρ for any $\rho < \rho_2$, $|t| < a(\rho_2 - \rho)$, $\forall \rho_2 < \rho_1$. Q.E.D.

Now let $(\bar{\gamma}, \bar{\phi}^0)$ be the solution of (2.30) with the Cauchy data satisfying (2.28)

$$\gamma(0) - \bar{\gamma}(0), \ \phi_x^0(0) - \bar{\phi}_x^0(0) = O(\varepsilon^2) \qquad \text{in } X_{\rho_1} .$$

The estimate (2.26) is proved by the continuous dependence in the norm (2.29) on the inhomogeneous terms of the solution $(G, F)(t) = (\gamma(t) - \bar{\gamma}(t), \ \phi_x^0(t) - \bar{\phi}_x^0(t))$ of the following Cauchy problem:

$$\begin{cases} G_t + F_x - \frac{\varepsilon}{6} F_{xxx} + \varepsilon(\bar{u} G + \gamma F)_x = O(\varepsilon^2) , \\ F_t + G_x - \frac{\varepsilon}{2} F_{txx} + \frac{\varepsilon}{2} ((u + \bar{u}) F)_x = O(\varepsilon^2) , \\ (G, F)(0) = O(\varepsilon^2) \qquad \text{in } X_{\rho_1} , \end{cases} \tag{2.33}$$

which can be solved similarly to the Cauchy problem (2.32).

REMARK 1. The linear dispersion relation of the system (2.27) is "good" for $k^2 \gg 1$ as well as for $k^2 \ll 1$:

$$p = \pm ik \sqrt{\frac{1 + \varepsilon k^2/6}{1 + \varepsilon k^2/2}} . \tag{2.34}$$

REMARK 2. For the solution $(\bar{\gamma}, \bar{\phi}_x^0)$ of (2.27) such that $(\bar{\gamma}, \bar{\phi}_x^0)(0) \in X_\rho \cap L_\rho^\sigma$ [11],

---

11) See [6] for the notation.

we have also

$$|\phi_x^0(t)-\bar{\phi}_x^0(t)|_{L_\rho^\sigma},\ |\gamma(t)-\bar{\gamma}(t)|_{L_\rho^\sigma} = O(\varepsilon^2) \tag{2.35}$$

for $|t|<a(\rho_2-\rho)$, $\forall\rho<\forall\rho_2<\rho_1$, if $\phi^0(0)=\bar{\phi}^0(0)$ and $\gamma(0)=\bar{\gamma}(0)$.

9. Equation (2.25) gives

$$\gamma = -\phi_t^0+\frac{\varepsilon}{2}\phi_{txx}^0-\frac{\varepsilon}{2}(\phi_x^0)^2+O(\varepsilon^2)\,. \tag{2.36}$$

Eliminating γ in the second equation of (2.25), we get the single equation for ϕ^0:

$$\phi_{tt}^0-\phi_{xx}^0-\frac{\varepsilon}{2}\phi_{ttxx}^0+\frac{\varepsilon}{6}\phi_{xxxx}^0+2\varepsilon\phi_x^0\phi_{tx}^0+\varepsilon\phi_t^0\phi_{xx}^0 = O(\varepsilon^2)\,. \tag{2.37}$$

Let $\phi^0(t, x)$ be the solution in X_ρ of the Cauchy problem for

$$\begin{cases} \bar{\bar{\phi}}_{tt}^0-\bar{\bar{\phi}}_{xx}^0-\frac{\varepsilon}{2}\bar{\bar{\phi}}_{ttxx}^0+\frac{\varepsilon}{6}\bar{\bar{\phi}}_{xxxx}^0+2\varepsilon\bar{\bar{\phi}}_x^0\bar{\bar{\phi}}_{tx}^0+\varepsilon\bar{\bar{\phi}}_t^0\bar{\bar{\phi}}_{xx}^0 = 0\,. \\ \bar{\bar{\gamma}} = -\bar{\bar{\phi}}_t^0+\frac{\varepsilon}{2}\bar{\bar{\phi}}_{txx}^0-\frac{\varepsilon}{2}(\bar{\bar{\phi}}_x^0)^2 \end{cases} \tag{2.38}$$

with the Cauchy data satisfying

$$|\phi_x^0(0)-\bar{\bar{\phi}}_x^0(0)|_{\rho_1},\ |\phi_t^0(0)-\bar{\bar{\phi}}_t^0(0)|_{\rho_1} = O(\varepsilon^2)\,.$$

Then we have the

Theorem 2.5. *The solution (γ, ϕ^0) is approximated by $(\bar{\bar{\gamma}}, \bar{\bar{\phi}}^0)$ in such a way that we have*

$$|\phi_x^0(t)-\bar{\bar{\phi}}_x^0(t)|_\rho\,,\quad |\phi_t^0(t)-\bar{\bar{\phi}}_t^0(t)|_\rho\,,\quad |\gamma(t)-\bar{\bar{\gamma}}(t)|_\rho = O(\varepsilon^2) \tag{2.39}$$

for $|t|<a(\rho_2-\rho)$, $\forall\rho<\forall\rho_2<\rho_1$.

Proof. First in order to prove the existence theorem for (2.38), set

$$\psi = \left(1-\frac{\varepsilon}{2}\frac{\partial^2}{\partial x^2}\right)\bar{\bar{\phi}}^0$$

and apply the abstract Cauchy-Kowalevski theorem to the system for (ψ_t, ψ_x) obtained from (2.38). Next apply the abstract Cauchy-Kowalevski theorem to the system for

$$U(t) = \left(1-\frac{\varepsilon}{2}\frac{\partial^2}{\partial x^2}\right)(\phi_x^0(t)-\bar{\bar{\phi}}_x^0(t))$$

and

$$V(t)=\left(1-\frac{\varepsilon}{2}\frac{\partial^2}{\partial x^2}\right)(\phi_t^0(t)-\bar{\bar{\phi}}_t^0(t)),$$

then the continuous dependence of solution on the inhomogeneous term proves the estimate (2.39). Q.E.D.

REMARK 1. The linear dispersion relation for (2.37) is the same as (2.34):

$$p=\pm ik\sqrt{\frac{1+\varepsilon k^2/6}{1+\varepsilon k^2/2}}.$$

REMARK 2. As before, for the solution such that

$$(\gamma,\phi_x^0)(0)\in X_\rho\cap L_\rho^\sigma,$$

we have also

$$|\phi_x^0(t)-\bar{\bar{\phi}}_x^0(t)|_{L_\rho^\sigma},\ |\phi_t^0(t)-\bar{\bar{\phi}}_t^0(t)|_{L_\rho^\sigma},\ |\gamma(t)-\bar{\bar{\gamma}}(t)|_{L_\rho^\sigma}=O(\varepsilon^2)$$

for $|t|<a(\rho_2-\rho)$, $\forall\rho<\forall\rho_2<\rho_1$.

10. In previous theorems 2.3 and 2.5 we obtained good approximations (2.27) and (2.38) to water wave equations (2.25) and (2.36)–(2.37) respectively. Here let us examine further approximations to (2.37) following Boussinesq (1872) and (1877). Similar approximation and arguments to (2.25) can be done but we omit it here. If we consider the uni-directional motion i.e., assume the initial data for equation (2.37) satisfy

$$\phi_t^0(0)+\phi_x^0(0)=O(\varepsilon),\ \phi_t^0(0)-\phi_x^0(0)=O(1)\quad \text{in}\quad X_{\rho_1}, \tag{2.40}$$

then by using similar argument to obtain (2.24) it is easy to get the following property of the solutions

$$\phi_t^0(t)+\phi_x^0(t)=O(\varepsilon),\ \phi_t^0(t)-\phi_x^0(t)=O(1)\ \text{ in } X_\rho,\ |t|<a(\rho_1-\rho). \tag{2.41}$$

Therefore if we substitute (2.41) in the nonlinear terms of order $O(\varepsilon)$ in equation (2.37), we have

$$\phi_{tt}^0-\phi_{xx}^0-\frac{\varepsilon}{2}\phi_{ttxx}^0+\frac{\varepsilon}{6}\phi_{xxxx}^0-3\varepsilon\phi_x^0\phi_{xx}^0=O(\varepsilon^2). \tag{2.42}$$

Thus it suggests an approximate equation

$$\begin{cases}\tilde{\phi}_{tt}^0-\tilde{\phi}_{xx}^0-\frac{\varepsilon}{2}\tilde{\phi}_{ttxx}^0+\frac{\varepsilon}{6}\tilde{\phi}_{xxxx}^0-3\varepsilon\tilde{\phi}_x^0\tilde{\phi}_{xx}^0=0, \\ \tilde{\gamma}=-\tilde{\phi}_t^0+\frac{\varepsilon}{2}\tilde{\phi}_{txx}^0-\frac{\varepsilon}{2}(\tilde{\phi}_x^0)^2\end{cases} \tag{2.43}$$

(cf. Boussinesq (1872)).

Proposition 2.6. *If the initial data satisfy* (2.40) *and*

$$|\phi_t^0(0)-\tilde{\phi}_t^0(0)|_{\rho_0}=O(\varepsilon^2),\quad |\phi_x^0(0)-\tilde{\phi}_x^0(0)|_{\rho_0}=O(\varepsilon^2)\,,$$

then we have

$$|\phi_x^0(t)-\tilde{\phi}_x^0(t)|_\rho\,,\ |\phi_t^0(t)-\tilde{\phi}_t^0(t)|_\rho\,,\ |\gamma(t)-\tilde{\gamma}(t)|_\rho=O(\varepsilon^2) \tag{2.44}$$

for $|t|<a(\rho_2-\rho)$, $\forall\rho<\forall\rho_2<\rho_1$.

The proof is similar to that of Theorem 2.5.

REMARK. Equation (2.43) has the solitary wave solutions as follows:

$$\phi_x=\frac{c^2-1}{\varepsilon}\left(\cosh\sqrt{\frac{c^2-1}{2\varepsilon(c^2-1/3)}}(x-ct)\right)^{-2},\quad |c|>1\,.$$

This gives a good approximation for $\gamma(x-ct)\sim-\phi_t\sim c\phi_x$ with $c>1$ but not for $\gamma(x-ct)\sim-\phi_t\sim c\phi_x$ with $c<-1$. The latter γ has negative values and is not proper as a solitary wave solution. In fact the latter is excluded by the assumption (2.40) of Proposition 2.6. Compare the solitary wave solutions of (2.38).

Furthermore if we use the first order approximation

$$\phi_{tt}^0=\phi_{xx}^0+O(\varepsilon)$$

in the linear term ϕ_{ttxx}^0 of $O(\varepsilon)$ in the equation (2.42), then we have

$$\phi_{tt}^0-\phi_{xx}^0-\frac{\varepsilon}{3}\phi_{xxxx}^0-3\varepsilon\phi_x^0\phi_{xx}^0=O(\varepsilon^2)\,.$$

This gives formally another approximate equation

$$\tilde{\tilde{\phi}}{}_{tt}^0-\tilde{\tilde{\phi}}{}_{xx}^0-\frac{\varepsilon}{3}\tilde{\tilde{\phi}}{}_{xxxx}^0-3\varepsilon\tilde{\tilde{\phi}}{}_x^0\tilde{\tilde{\phi}}{}_{xx}^0=0\,, \tag{2.45}$$

(cf. Boussinesq (1872) and point 11).

Also on the surface, if we start with (2.15) assuming the uni-directional motion (2.24), we can get the so-called Boussinesq equation:

$$\gamma_{tt}-\gamma_{xx}-\frac{\varepsilon}{3}\gamma_{xxxx}-\frac{3}{2}\varepsilon(\gamma^2)_{xx}=0\,,$$

$$\gamma\sim\phi_x\,.$$

It has solitary wave solutions $\phi^0(x-ct)$, $\gamma(x-ct)$, $|c|>1$, but we do not know a justification of these approximations for the Cauchy problem as the same reason as Remark 2 in point 7. In these context, we call (2.27) and (2.38) "Boussinesq equation".

11. Before closing this paragraph, we compare our procedure of derivation of Boussinesq equation with the original study of Boussinesq [3] and one would find that the equation (2.27) has been obtained in fact by Boussinesq.

Let H be the depth of water at rest (or mean depth), $z=H+h(t,x)$ the water surface at t and $\phi(t,x,z)$ the velocity potential, $\phi^0(t, x)=\phi(t, x, 0)$ being its values on the water bottom.

Boussinesq assumes (not prove) [3, p. 72] that "la partie variable h de la profondeur et, par suite, la vitesse au fond $u_0=\phi_x^0(t, x)$ seront supposées très petites et leurs dérivées successives en x de plus en plus petites, de manière que la série (18) soit rapidement convergente

$$\phi = -\int_x^\infty u_0 dx - \frac{z^2}{2!}\frac{du_0}{dx} + \frac{z^4}{4!}\frac{d^3u_0}{dx^3} + \cdots, \quad 0<z<H+h\text{.''} \tag{18}$$

The boundary conditions on $z=H+h$ are given by

$$\begin{cases} \phi_t + \dfrac{1}{2}(\phi_x^2+\phi_z^2)+gh=0\,, \\ h_t+h_x\phi_x-\phi_z=0\,, \end{cases}$$

where g is the constant of gravity. He derived from this

$$\begin{cases} \phi_t^0+gh=0\,, \\ h_t+H\phi_{xx}^0=0\,, \end{cases} \tag{2.46}$$

as "the first approximatoin" ((20), p. 72, see also our proposition 2.2). His "second approximate equation" is given by the following, if an approximation of uni-directional wave motion at this level is not made:

$$\begin{cases} gh-\displaystyle\int_x^\infty \frac{du_0}{dt}dx-\frac{H^2}{2!}\frac{d^2u_0}{dxdt}+\frac{1}{2}u_0^2=0\,, \\ \dfrac{dh}{dt}+u_0\dfrac{dh}{dx}+H\dfrac{du_0}{dx}+h\dfrac{du_0}{dx}-\dfrac{H^3}{3!}\dfrac{d^3u_0}{dx^3}=0\,. \end{cases} \tag{2.47}$$

These equations are not printed explicitely on page 72 but are explained in lines 9–5 from bottom. Considering the uni-directional motion (cf. (2.36), (2.40), (2.41)) and using "the first approximation" (2.46) he gets

$$\phi_x^0=\sqrt{\frac{g}{H}}\,h\,.$$

Substituting this approximation he gets

$$\begin{cases} gh+\phi_t^0+\dfrac{g}{2}\left(\dfrac{h^2}{H}+H^2h_{xx}\right)=0\,, \\ h_t+H\phi_{xx}^0+\sqrt{\dfrac{g}{H}}\,h_x^2-\dfrac{1}{6}\sqrt{gH^5}h_{xxx}=0\,, \end{cases} \tag{2.48}$$

or eliminating ϕ^0, the following:

$$h_{tt}-gHh_{xx}=gH(\frac{3}{2H}h_{xx}^2+\frac{H^2}{3}h_{xxxx}).$$

The system (2.47) is essentially the same as (2.27) whose solution gives an approximation for water surface wave under the condition (1.1) (Theorem 2.3).

In fact (2.47) can be written as

$$\text{(2.49)}\qquad \begin{cases} gh+\phi_t^0-\dfrac{H^2}{2}\phi_{txx}^0-\dfrac{1}{2}(\phi_x^0)^2=0 \\ h_t+H\phi_{xx}^0-\dfrac{H^3}{6}\phi_{xxxx}^0+(h\phi_x^0)_x=0 \end{cases}$$

and by the transformation: $h=\kappa h'$, κ being the amplitude, $\phi^0=\varepsilon\sqrt{gh}\lambda\phi^0$, λ wave length, $t=\sqrt{gh}\,t'/\lambda$, $x=\lambda x'$, $\varepsilon=\kappa/H=(H/\lambda)^2=\delta^2$, (2.49) is transformed into the dimensionless system (2.27) with $\gamma=h$ after omitting "$'$".

3 Korteweg-de Vries equation

12. Let us consider the surface waves of water flowing with a constant velocity ω. As in §2, we set $\varepsilon=\delta^2$, ε being the ratio of the amplitude to the water depth: $\varepsilon=\kappa/h$.

Consider the following transformation

$$\text{(3.1)}\qquad \begin{cases} x=\xi+\varepsilon x^1, \quad y=\eta+\varepsilon y^1, \\ \varphi=-t+\omega\xi+\varepsilon\varphi^1, \end{cases}$$

and solve (2.6) for (x, φ) as (2.10) with the initial data (2.11) in $X_{\rho_0}\cap L_{\rho_0}^\sigma$, and define (γ, ϕ) in the same way as for $\gamma=\gamma(t, x; \delta)$ and $\phi=\phi(t, x, y; \delta)$ in §2. Then we have the solution of (2.1)–(2.3):

$$\text{(3.2)}\qquad \begin{cases} \Phi=-t+\omega x+\varepsilon\phi(t, x, y; \delta), \\ \Gamma=1+\varepsilon\gamma(t, x; \delta). \end{cases}$$

Set

$$\phi(t, x, \Gamma; \delta)=\phi(t, x, 1+\delta^2\gamma(t, x; \delta); \delta)=\tilde{\phi}(t, x; \delta),$$
$$\Phi(t, x, \Gamma; \delta)=-t+\omega x+\delta^2\tilde{\phi}(t, x; \delta)=\tilde{\Phi}(t, x; \delta),$$

then we have similarly to Proposition 2.1:

$$\text{(3.3)}\qquad \begin{cases} \gamma_t+\tilde{\phi}_{xx}+\omega\gamma_x+\dfrac{\varepsilon}{3}\tilde{\phi}_{xxxx}+\varepsilon(\gamma\tilde{\phi}_x)_x=O(\varepsilon^2), \\ \tilde{\phi}_t+\gamma+\omega\tilde{\phi}_x+\dfrac{\varepsilon}{2}\tilde{\phi}_x^2=O(\varepsilon^2), \end{cases}$$

in $X_\rho\cap L_\rho^\sigma$, for any $\rho<\rho_1$, $|t|<a(\rho_1-\rho)$.

13. We shall prove first that some of the surface waves described by $(\gamma, \tilde{\phi}_x) \in \in X_\rho \cap L_\rho^\sigma$ are approximated by solutions of Korteweg-de Vries equation. Let $(\gamma, \tilde{\phi}_x)$ be the solution of (3.3) such that $(\gamma, \tilde{\phi}_x) \in X_\rho \cap L_\rho^\sigma$, for any $\rho < \rho_1$ and for $|t| < a(\rho_1 - \rho)$, with the Cauchy data satisfying $(\gamma, \tilde{\phi}_x)(0) \in X_{\rho_1} \cap L_{\rho_1}^\sigma$. Setting $\tilde{\phi}_x = u$, (3.3) can be written as follows:

$$
\begin{cases} \gamma_t + u_x + \omega\gamma_x + \dfrac{\varepsilon}{3} u_{xxx} + \varepsilon(\gamma u)_x = O(\varepsilon^2)\,, \\ u_t + \gamma_x + \omega u_x + \varepsilon u u_x = O(\varepsilon^2)\,, \end{cases} \tag{3.4}
$$

in $X_\rho \cap L_\rho^\sigma$, for any $\rho < \rho_1$, $|t| < a(\rho_1 - \rho)$.

Let us introduce the quantities:

$$
f = \frac{1}{2}(\gamma + u)\,, \quad g = \frac{1}{2}(\gamma - u)\,. \tag{3.5}
$$

Then we have the

Proposition 3.1. *If the initial data satisfy the uni-directional condition*

$$
f(0) = \frac{1}{2}(\gamma(0) + u(0)) = O(1)\,, \quad g(0) = \frac{1}{2}(\gamma(0) - u(0) = O(\varepsilon) \tag{3.6}
$$

in $X_{\rho_1} \cap L_{\rho_1}^\sigma$, *then we have*

$$
f_t + (\omega+1) f_x + \frac{\varepsilon}{6} f_{xxx} + \frac{3}{2}\varepsilon f f_x = O(\varepsilon^2)\,, \tag{3.7}
$$

$$
g_t + (\omega-1) g_x - \frac{\varepsilon}{6} g_{xxx} - \frac{3}{2}\varepsilon g g_x = -\varepsilon\left(\frac{1}{6} f_{xxx} + \frac{1}{2} f f_x\right) + O(\varepsilon^2) \tag{3.8}
$$

in $X_\rho \cap L_\rho^\sigma$, *for any* $\rho < \rho_1$, $|t| < a(\rho_1 - \rho)$.

Proof. First we recall the property of uni-directional motion (2.24) by the condition (3.6) on the initial data:

$$
f(t) = \frac{1}{2}(\gamma(t) + u(t)) = O(1)\,, \quad g(t) = \frac{1}{2}(\gamma(t) - u(t)) = O(\varepsilon) \tag{3.9}
$$

in $X_\rho \cap L_\rho^\sigma$, $|t| < a(\rho_1 - \rho)$.

Diagonalizing (3.4) by (3.5), we have by (3.9)

$$
f_t + (\omega+1) f_x + \frac{\varepsilon}{6} f_{xxx} + \frac{3}{2}\varepsilon f f_x = \varepsilon\left(\frac{1}{6} g_{xxx} + \frac{1}{2} g g_x + \frac{1}{2}(fg)_x\right) + O(\varepsilon^2) = O(\varepsilon^2)
$$

and

$$
\begin{aligned} g_t + (\omega-1) g_x - \frac{\varepsilon}{6} g_{xxx} - \frac{3}{2}\varepsilon g g_x &= \varepsilon\left(-\frac{1}{6} f_{xxx} - \frac{1}{2} f f_x - \frac{1}{2}(gf)_x\right) + O(\varepsilon^2) = \\ &= \varepsilon\left(-\frac{1}{6} f_{xxx} - \frac{1}{2} f f_x\right) + O(\varepsilon^2) \end{aligned}
$$

in $X_\rho \cap L^\sigma_\rho$. Q.E.D.

If one drops the term on the right-hand side of order $O(\varepsilon^2)$ in (3.7), we get the Korteweg-de Vries equation [8], which corresponds to the uni-directional waves moving right-ward i.e.,

$$F_t+(\omega+1)F_x+\frac{\varepsilon}{6}F_{xxx}+\frac{3}{2}\varepsilon FF_x=0\,, \tag{3.10}$$

$$F(0)=f(0);$$

$$G_t+(\omega-1)G_x-\frac{\varepsilon}{6}G_{xxx}-\frac{3}{2}\varepsilon GG_x=\varepsilon(-\frac{1}{6}F_{xxx}-\frac{1}{2}FF_x)\,, \tag{3.11}$$

$$G(0)=g(0);$$

where F is the solution of (3.10).

If we consider the initial data satisfying

$$\gamma(0)-u(0)=O(1)\,,\quad \gamma(0)+u(0)=O(\varepsilon)\,,$$

instead of (3.6), then we have the following equations for the uni-directional motion left-ward:

$$g_t+(\omega-1)g_x-\frac{\varepsilon}{6}g_{xxx}-\frac{3}{2}\varepsilon gg_x=O(\varepsilon^2)\,, \tag{3.12}$$

$$G_t+(\omega-1)G_x-\frac{\varepsilon}{6}G_{xxx}-\frac{3}{2}\varepsilon GG_x=0\,. \tag{3.13}$$

In order to compare (3.7)–(3.8) and (3.10)–(3.11) let us now solve the Cauchy problem for (3.10) and (3.11) in the scale $\Sigma=\bigcup_{\rho>0}B_\rho$ of Banach spaces, consisting of analytic functions defined by

$$B_\rho=\left\{\begin{array}{c} v(z)\text{: holomorphic function in } \Omega_\rho=\{z=x+iy\colon x\in\boldsymbol{R},\ |y|<\rho\}\,, \\ \text{such that } |(1+|k|)e^{\rho|k|}\hat{v}(k)|_{L^2(\boldsymbol{R})}<\infty \end{array}\right\},$$

where $\hat{v}(k)$ is the Fourier transform of $v(x)$.

Proposition 3.2. *The Cauchy problem for* (3.10) ((3.11), *resp.*) *with the Cauchy data* $F(0)\in B_{\rho_1}$ ($G(0)\in B_{\rho_1}$, *resp.*) *has a unique solution* $F(t)\in B_\rho$ ($G(t)\in B_\rho$, *resp.*) *for any* $\rho<\rho_1$, $|t|<a(\rho_1-\rho)$, *a being a positive constant.*

Proof. We get by the Fourier transformation from (3.10)

$$\frac{d}{dt}\hat{F}(t)+i((\omega+1)k+\frac{\varepsilon}{6}k^3)\hat{F}(t)=-\frac{3\varepsilon}{4}\widehat{(F^2)_x}(t)\,,$$

i.e.,

$$\hat{F}(t) = \exp\{-it((\omega+1)k+\frac{\varepsilon}{6}k^3)\}\hat{F}(0)+$$

$$+\int_0^t \exp\{-i(t-s)((\omega+1)k+\frac{\varepsilon}{6}k^3)\}(-\frac{3\varepsilon}{4})(F^2)_x^{\wedge}(s)ds\,.$$

Apply the lemma 3.3 and the abstrcat Cauchy-Kowalevski theorem ([6, appendix]), we get the proposition. Q.E.D.

Lemma 3.3. *For any $G, F \in B_\rho$ and for any $\rho' < \rho$, the following inequality holds*:

$$|(G^2-F^2)_x|_{B_{\rho'}} < \frac{C(|G|_{B_\rho}+|F|_{B_\rho})}{\rho-\rho'}|G-F|_{B_\rho}\,, \tag{3.14}$$

where C is a constant independent of ρ, ρ'.

Proof. It follows from the inequality

$$|k|e^{\rho'|k|} < Ce^{\rho|k|}/(\rho-\rho')$$

and the Hausdorf-Young's inequality in the Fourier transform.

Now we can give a justification for Korteweg-de Vries equation as follows: By virtue of Proposition 3.2 we have the

Theorem 3.4. *If the Cauchy data $(\gamma, u)(0) \in X_{\rho_1} \cap L^\sigma_{\rho_1}$ satisfy (3.6), then the solution $(\gamma, u)(t) \in X_\rho \cap L^\sigma_\rho$ of (3.3) has the following approximation by the solution of Korteweg-de Vries equation (3.10) and (3.11):*

$$\begin{cases} |f(t)-F(t)|_{B_\rho} = O(\varepsilon^2)\,, \\ |g(t)-G(t)|_{B_\rho} = O(\varepsilon^2) \end{cases} \tag{3.15}$$

for any $\rho < \rho_2$, $|t| < a(\rho_2-\rho)$, $\forall \rho_2 < \rho_1$, i.e.,

$$|\gamma(t)-(F+G)(t)|_{B_\rho} = O(\varepsilon^2) \tag{3.16}$$

for any $\rho < \rho_2$, $|t| < a(\rho_2-\rho)$; $\forall \rho_2 < \rho_1$. In particular

$$|\gamma(t)-F(t)|_{B_\rho} = O(\varepsilon)\,.$$

Proof. It follows from Proposition 3.1 and 3.2 that the difference $H(t) = f(t)-F(t)$ is the solution of the Cauchy problem

$$H_t+(\omega+1)H_x+\frac{\varepsilon}{6}H_{xxx}+\frac{3\varepsilon}{2}HH_x+\frac{3\varepsilon}{2}(fH)_x = h_\varepsilon\,, \tag{3.17}$$

in B_ρ, for any $\rho < \rho_1$, $|t| < a(\rho_1-\rho)$, with the initial data $H(0)=0$, where for any $\rho < \rho_1$, $|t| < a(\rho_1-\rho)$, $|h_\varepsilon(t)|_{B_\rho} < C\varepsilon^2$.

Applying the abstract Cauchy-Kowalewski theorem, we have the first estimate of (3.15). Since (3.11) can be solved similarly, the second one is easy to see if we note that

$$|\frac{1}{6}f_{xxx}+\frac{1}{2}ff_x-(\frac{1}{6}F_{xxx}+\frac{1}{2}FF_x)|_{B_\rho}<C\varepsilon^2$$

for any $\rho<\rho_2$, $|t|<a(\rho_2-\rho)$, $\forall\rho_2<\rho_1$. Q.E.D.

14. If we use instead of $\tilde{\Phi}$ the values of the velocity potential on the bottom of water: $\phi^0(t, x; \delta)=\phi(t, x, 0; \delta)$, our equation (3.3) becomes for $v=\phi^0_x$ (cf. 2.):

$$(3.18)\qquad \begin{cases} \gamma_t+v_x+\omega\gamma_x-\frac{\varepsilon}{6}v_{xxx}+\varepsilon(\gamma v)_x = O(\varepsilon^2) \\ v_t+\gamma_x+\omega v_x-\frac{\omega}{2}\varepsilon v_{xxx}-\frac{\varepsilon}{2}v_{txx}+\varepsilon vv_x = O(\varepsilon^2)\,, \end{cases}$$

in X_ρ, $\forall\rho<\rho_1$, $|t|<a(\rho_1-\rho)$.

Let us diagonalize (3.18) by

$$(3.19)\qquad m = \frac{1}{2}(\gamma+v),\quad n = \frac{1}{2}(\gamma-v)\,,$$

then we have, by a similar reasoning to get (3.7)–(3.8), the

Proposition 3.5. *For the solution (γ, ϕ^0_x) which satisfies the condition on the initial data*:

$$(3.20)\qquad m(0) = \gamma(0)+u(0) = O(1)\,,\quad n(0) = \gamma(0)-u(0) = O(\varepsilon)$$

in X_{ρ_1}, then we have

$$(3.21)\qquad m_t+(\omega+1)m_x-\frac{\varepsilon(1+3\omega)}{12}m_{xxx}-\frac{\varepsilon}{4}m_{txx}+\frac{3}{2}\varepsilon mm_x = O(\varepsilon^2)\,,$$

and

$$(3.22)\qquad n_t+(\omega-1)n_x+\frac{\varepsilon(1-3\omega)}{12}n_{xxx}-\frac{\varepsilon}{4}n_{txx}-\frac{3}{2}\varepsilon nu_x = \\ = \varepsilon(\frac{1-3\omega}{12}m_{xxx}-\frac{1}{4}m_{txx}-mm_x)+O(\varepsilon^2)\,.$$

Now we prove that m and n are approximated by solutions of equations

$$(3.23)\qquad \begin{cases} M_t+(\omega+1)M_x-\frac{\varepsilon(1+3\omega)}{12}M_{xxx}-\frac{\varepsilon}{4}M_{txx}+\frac{3}{2}\varepsilon MM_x = 0\,. \\ N_t+(\omega-1)N_x+\frac{\varepsilon(1-3\omega)}{12}N_{xxx}-\frac{\varepsilon}{4}N_{txx}-\frac{3}{2}\varepsilon NN_x = \\ \qquad = \varepsilon(\frac{1-3\omega}{12}M_{xxx}-\frac{1}{4}M_{txx}-MM_x)\,. \end{cases}$$

Similarly if we consider the case $\gamma(0)-u(0)=O(1)$ and $\gamma(0)+u(0)=O(\varepsilon)$ we have

$$N_t+(\omega-1)N_x+\frac{\varepsilon(1-3\omega)}{12}N_{xxx}-\frac{\varepsilon}{4}N_{txx}-\frac{3}{2}\varepsilon NN_x=0 \tag{3.24}$$

and a similar equation for M. But we omit the argument for this case. Concerning (3.23) we have first the

Proposition 3.6. (i) *The Cauchy problem for* (3.23) *with the Cauchy data* $M(0)\in X_{\rho_1}$ $(X_{\rho_1}\cap L^{\sigma}_{\rho_1}$, *resp.*) *has a unique solution* $M(t)\in X_\rho$ $(X_\rho\cap L^{\sigma}_{\rho}$, *resp.*), $\forall\rho<\rho_1$, *for* $|t|<a(\rho_1-\rho)$.

Proof. Setting

$$\mathcal{M}(t)=\left(1-\frac{\varepsilon}{4}\frac{\partial^2}{\partial x^2}\right)M(t)\,, \tag{3.25}$$

apply the abstract Cauchy-Kowalevski's theorem to the equation for $\mathcal{M}(t)$ derived from (3.23). Q.E.D.

By the same reasoning as for the theorem 3.4, we have then

Theorem 3.7. *For* $m(t)$ *and* $n(t)$ *satisfying* (3.21) *and* (3.22) *in* X_ρ, *with the Cauchy data satisfying* (3.20) *in* X_{ρ_1}, *we have*

$$|M(t)-m(t)|_\rho=O(\varepsilon^2)\,,\quad |N(t)-n(t)|_\rho=O(\varepsilon^2) \tag{3.26}$$

for $|t|<a(\rho_2-\rho)$, $\forall\rho<\forall\rho_2<\rho_1$. *In other words, water surface waves given by* $(\gamma,\phi^0_x)(t)$ *satisfying* (3.20) *are approximated by solutions of the Cauchy problem for* (3.23) *as follows*:

$$\begin{cases} |\gamma(t)-(N(t)+M(t))|_\rho=O(\varepsilon^2) \\ |\phi^0_x(t)-(N(t)-M(t))|_\rho=O(\varepsilon^2)\,, \end{cases} \tag{3.27}$$

for $|t|<a(\rho_2-\rho)$, $\forall\rho<\rho_2$.

17. REMARKS. (i) Equations (3.23) and (3.24) have soiltary wave solutions and also cnoidal wave solutions as K-dV equation.
(ii) The linear dispersion relation for (3.21) is

$$p=i(\omega+1)k\left(\frac{1+(1+3\omega)\varepsilon k^2/12(1+\omega)}{1+\varepsilon k^2/4}\right),$$

and it is known the golbal existence theorem of the Cauchy problem in a Sobolev space.
(iii) In 1964, Long [13] and Broer [4] gave the following system:

$$\begin{cases} u_t+\eta_x+uu_x=\dfrac{1}{2}u_{txx} \\ \eta_t+((1+\eta)u)_x=\dfrac{1}{6}u_{xxx} \end{cases}$$

and in 1966 Peregrine [16]:

$$\begin{cases} u_t+u_x+\dfrac{3}{2}uu_x=\dfrac{1}{6}u_{txx} \\ \eta=u+O(\varepsilon)\,. \end{cases}$$

Our theorem 3.7 would give in some sense a justification for these formal derivation by Long, Broer and Peregrine.

Appendix. Global existence theorems for approximate equations

18. We shall prove first the global existence theorem for the Cauchy problem for (2.37) which we called "Boussinesq equation". Let us write it here as follows:

$$\varphi_{tt}-\varphi_{xx}-\frac{\varepsilon}{2}\varphi_{ttxx}+\frac{\varepsilon}{6}\varphi_{xxxx}+2\varepsilon\varphi_x\varphi_{tx}+\varepsilon\varphi_t\varphi_{xx}=0\,. \tag{A.1}$$

Let $\mathcal{H}(\boldsymbol{R})$ be defined by

$$\mathcal{H}(\boldsymbol{R})=\{u\colon u\in\mathcal{E}_t^0(H^4)\cap\mathcal{E}_t^1(H^3)\cap\mathcal{E}_t^2(H^2)\}\,, \tag{A.2}$$

where $u\in\mathcal{E}_t^j(H^{4-j})$ is j-times continuously differentiable with respect to t in the Sobolev space $H^{4-j}(\boldsymbol{R})$. We have the

Theorem A.1. *The Cauchy problem for* (A.1) *with initial data*

$$\varphi(0)\in H^4(\boldsymbol{R})\,,\quad \varphi_t(0)\in H^3(\boldsymbol{R}) \tag{A.3}$$

has a unique solution $\varphi(t,\cdot)\in\mathcal{H}(\boldsymbol{R})$ *for* $t\in[0,\infty)$.

The following a priori estimates in Lemma A.2 and the successive approximation prove the theorem.

Lemma A.2 (a priori estimates).
(i) *There is a positive constant* M *such that we have for any solution* $\varphi(t)\in$ $\in\mathcal{H}(\boldsymbol{R})$:

$$\sup_{(x,t)\in\boldsymbol{R}\times[0,\infty)}\{|\varphi_t(t,x)|+|\varphi_x(t,x)|\}<M\,. \tag{A.4}$$

(ii) *There is a positive constant* C *independent of* t *such that we have for any solution* $\varphi(t)\in\mathcal{H}(\boldsymbol{R})$:

$$E(t)\leqq E(0)e^{Ct}\,,\quad t\geqq 0 \tag{A.5}$$

where $E(t)=E_1(t)+E_2(t)$, *and*

$$E_1(t)=\|\varphi_t(t)\|_{L^2}^2+\|\varphi_x(t)\|_{L^2}^2+\frac{\varepsilon}{2}\|\varphi_{tx}(t)+\|_{L^2}^2+\frac{\varepsilon}{6}\|\varphi_{xx}(t)\|_{L^2}^2\,,$$

$$E_2(t)=\|\varphi_{tx}(t)\|^2_{L^2}+\|\varphi_{xx}(t)\|^2_{L^2}+\frac{\varepsilon}{2}\|\varphi_{txx}(t)\|^2_{L^2}+\frac{\varepsilon}{6}\|\varphi_{xxx}(t)\|^2_{L^2}.$$

Proof. (i) Integrating (A.1)$\times\varphi_t$ on $\boldsymbol{R}_x$, we have

$$E_1(t)=\text{constant for}\quad t\geqq 0.$$

The Sobolev's lemma proves (A.4).

(ii) Integrating[12] $[\frac{\partial}{\partial x}(\text{A.1})]\times\varphi_{tx}$ on $\boldsymbol{R}_x$, we have

$$\frac{d}{dt}E_2(t)\leqq 8\varepsilon\int_{-\infty}^{\infty}|\varphi_x(t)\varphi_{tx}(t)\varphi_{txx}(t)|\,dx+2\varepsilon\int_{-\infty}^{\infty}|\varphi_t(t)\varphi_{tx}(t)\varphi_{xxx}(t)|\,dx.$$

Then, using (A.4), we have

(A.6) $$\frac{d}{dt}E_2(t)\leqq 4\sqrt{3\varepsilon}\,ME_2(t).$$

Since $\frac{d}{dt}E_1(t)=0$, we get (A.5) by (A.6). Q.E.D.

Proof of the Theorem A.1. Let us notice that (A.1) is equivalent to

(A.7) $$\psi_{tt}-\psi_{xx}+2\varepsilon(K_\varepsilon*\psi_t)_x(K_\varepsilon*\psi_x)+\{\varepsilon(K_\varepsilon*\psi_t)-\frac{2}{3}\}(K_\varepsilon*\psi_x)_x=0$$

by the transformation of unknown:

(A.8) $$\psi=\left(1-\frac{\varepsilon}{2}\frac{\partial^2}{\partial x^2}\right)\varphi,$$

i.e.,

$$\varphi(t,x)=(K_\varepsilon*\psi)(t,x)=\int_{-\infty}^{\infty}\pi e^{-\sqrt{2/\varepsilon}|x-y|}\psi(t,y)dy.$$

By virtue of the Lemma A.2, we have the following a priori estimate for (A.7):

(A.9) $$E_0(t)\leqq CE(0)e^{C't},\ t\geqq 0,$$

where

$$E_0(t)=\|\psi_t(t)\|^2_{L^2}+\|\psi_x(t)\|^2_{L^2},$$

C and C' being positive constants independnet of t.

Thus we can prove the global existence of solution of the Cauchy problem for (A.7) and consequently for (A.1). Q.E.D.

12) For the rigorous proof, we apply the Friedrichs' mollifier to (A.1) and estimate the commutators.

A parallel discussion for (3.23) is also valid, but we omit it here.

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