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STABILITY OF TRAVELING WAVE SOLUTIONS FOR NON-CONVEX EQUATIONS OF BREAK BAROTROPIC VISCOUS GAS

Dedicated to Professor Ying-Kun Xiao on his 60th birthday

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1. Introduction

Our purpose in this paper is to investigate the stability of traveling wave solutions with shock profile for one-dimensional equations of barotropic viscous gas with non-convex nonlinearity in the form

$$v_t - u_x = 0, \tag{1.1}$$

$$u_t + p(v)_x = \left(\frac{\mu u_x}{v}\right)_x, \quad x \in \boldsymbol{R}, \quad t \geq 0, \tag{1.2}$$

with the initial data

$$(v,u)|_{t=0} = (v_0,u_0)(x), \qquad x \in \boldsymbol{R}, \tag{1.3}$$

which tend toward the given constant states $(v_\pm, u_\pm)$. Here, $x \in R^1$, $t \geq 0$, and $0 < v_- < v_+$, v is the specific volume, u the velocity, $\mu(>0)$ the viscous constant, $p(v)$ the smooth nonlinear pressure function satisfying

$$p'(v) < 0, \tag{1.4}$$

$$p''(v) \lesseqgtr 0 \quad \text{for} \quad v \gtreqless v_*, \tag{1.5}$$

so that $p(v)$ is neither convex nor concave, and has a point of inflection at $v = v_*$, where v_* is a point in the interval (v_-, v_+). We can easily see that the system (1.1), (1.2) with $\mu = 0$ is strictly hyperbolic, and both characteristic fields are neither genuinely nonlinear nor linearly degenerate in the neighborhood of $v = v_*$ due to (1.5).

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The traveling wave solutions with shock profile are defined as the solutions of the form

$$(1.6) \qquad (v,u)(t,x)=(V,U)(\xi), \quad \xi=x-st,$$

which must satisfy

$$(1.7) \qquad \begin{cases} -sV'-U'=0 \\ -sU'+p(V)'=\mu\left(\dfrac{U'}{V}\right)' \end{cases}$$

and

$$(1.8) \qquad (V,U)(\xi)\to(v_\pm,u_\pm), \qquad \xi\to\pm\infty,$$

where s is the shock speed and $(v_\pm,u_\pm)$ are constant states at $\xi=\pm\infty$. Corresponding to [6], we can easily see that there exists a traveling wave solution with shock profile for (1.1) and (1.2) under both of the Rankine-Hugoniot condition

$$(1.9) \qquad \begin{cases} -s(v_+-v_-)-(u_+-u_-)=0 \\ -s(u_+-u_-)+(p(v_+)-p(v_-))=0 \end{cases}$$

and the generalized shock condition

$$(1.10) \qquad \frac{1}{s}h(v)v\equiv-\frac{1}{s}[s^2(v-v_\pm)+p(v)-p(v_\pm)]v>0, \quad \text{for} \quad v_-<v<v_+,$$

which implies two kinds of shock conditions

$$(1.11) \qquad \text{non-degenerate shock:} \quad -p'(v_+)<s^2<-p'(v_-),$$

$$(1.12) \qquad \text{degenerate shock:} \quad -p'(v_+)=s^2<-p'(v_-).$$

To study the stability of traveling wave solutions with shock profile for a single equation or a system arising in dynamics of gas and fluid is one of hot spots in mathematicial physics. The important progress and interesting results have recently been made by many authors (see [1–15]). Among them, in the case of convex nonlinearity for the systems of viscoelastical models, Hoff and Liu [3], Kawashima and Matsumura [5], Liu [7, 8], Matsumura and Nishihara [10] showed that the traveling wave solutions with non-degenerate shock profiles are stable. In the case of non-convex nonlinearity, the stability of traveling wave solutions with non-degenerate shock profile has been otained by Kawashima and Matsumura [6] at the first time. Moreover, another interesting question is to study the stability of traveling wave solutions with the degenerate shock profile, or say, contact shock. For the scalar conservation law, the stability of the degenerate shock has been showed by the author [12] at the first time. And then, much better results including the time decay rates have been obtained by Matsumura and Nishihara [11]. For the system case, very recently, Nishihara [14] showed the stability of the degenerate shock for a model of viscoelatical system at the first time. Later then, Mei and Nishihara [13] succeeded to improve the stability results in [6, 14] with weaker conditions on nonlinear stress function, initial disturbance, and weight function. Concerning with our problem (1.1), (1.2), when the nonlinear term is non-convex, the stability result with the non-degenerate shock condition (1.11) can be known from Kawashima and Matsumura [6], see also a good survey by Matsumura [9]. However, the stability of traveling wave solutions with the degenerate shock condition (1.12) for system (1.1), (1.2), up to now, is not treated yet as the author knows.

The main purpose of this paper is to show the stability of traveling wave solution with the degenerate shock profile for (1.1), (1.2). Our scheme is due to an elementary but technical weighted energy method. Here, the condition $p'''(v) < 0$ and the smallness of both the shock strength and the initial disturbance are assumed.

This paper is organized as follows. After stating the notations and the stability theorem, we will reformulate our problem into another new system and will prove our stability theorem based on the *a priori* estimates in Section 2; The *a priori* estimates will be proved in Section 3; Finally, in Section 4, we will apply our stability theorem to the model of van der Waals fluid.

NOTATIONS. L^2_w denotes the space of measurable functions on R which satisfy $w(x)^{1/2} f \in L^2$, where $w(x) > 0$ is a called weight function, with the norm

$$|f|_w = \left(\int w(x) |f(x)|^2 dx \right)^{1/2}$$

H^l_w $(l \geq 0)$ denotes the weighted Sobolev space of L^2_w-functions f on R whose derivatives $\partial^j_x f, j = 1, \cdots, l$, are also L^2_w-functions, with the norm

$$|f|_{l,w} = \left(\sum_{j=0}^{l} |\partial^j_x f|^2_w \right)^{1/2}$$

Denoting

$$\langle x \rangle_+ = \begin{cases} \sqrt{1+x^2}, & \text{if} \quad x \geq 0 \\ 1, & \text{if} \quad x < 0, \end{cases}$$

we will make use of the space $L^2_{\langle x \rangle_+}$ and $H^l_{\langle x \rangle_+}$ $(l = 1, 2)$. We also denote $f(x) \sim g(x)$ as $x \to a$ when $C^{-1} g \leq f \leq Cg$ in a neighborhood of a, here and after, C always denote some positive constants without confusion. When $C^{-1} \leq w(x) \leq C$ for $x \in R$, we note that $L^2 = H^0 = L^2_w = H^0_w$ and $\|\cdot\| = \|\cdot\|_0 \sim |\cdot|_w = |\cdot|_{0,w}$.

Without loss of generality, throughout this paper, we restrict our problem to this case $s > 0$. We note from (1.7)

$$-s^2 V_\xi - p(V)_\xi = \mu s \left(\frac{V_\xi}{V}\right)_\xi . \tag{1.13}$$

Similarly to [6], integrating (1.13) over $(\pm\infty, \xi)$, noting (1.8) and Rankine-Hugoniot condition (1.9), we find that $v_+ > V > v_- > 0$ and

$$\mu s V_\xi = -V(s^2(V - v_\pm) + p(V) - p(v_\pm)) \equiv V h(V) > 0,$$

i.e., $h(V) > 0$ and

$$h(V) \sim |V - v_+|^2 \sim |\xi|^{-2}, \quad \text{as} \quad \xi \to +\infty, \tag{1.14}$$

$$h(V) \sim |V - v_-| \sim \exp(-c_-|\xi|), \quad \text{as} \quad \xi \to -\infty, \tag{1.15}$$

which is due to the degenerate shock condition (1.12), where $c_- = -v_-\ (p'(v_-) + s^2)/\mu s$ is a positive constant.

We also suppose that

$$\int_{-\infty}^{\infty} (v_0 - V, u_0 - U)(x) dx = 0 \tag{1.16}$$

for some pair of traveling wave solutions, and define

$$(\phi_0, \psi_0)(x) = \int_{-\infty}^{x} (v_0 - V, u_0 - U)(y) dy. \tag{1.17}$$

Our stability theorem is as follows.

Theorem 1.1. *Suppose that* (1.4), (1.5), (1.9), (1.12) *hold. When* $(v_0, u_0)(x)$ *and* $(V, U)(\xi)$ *satisfy* (1.16), *assume that*

$$p'''(v) < 0 \quad \textit{for} \quad v \in [v_-, v_+], \tag{1.18}$$

and $(\phi_0, \psi_0) \in H^2 \cap L^2_{\langle x \rangle_+}$ *and* $(\phi_{0,x}, \psi_{0,x}) \in L^2_{\langle x \rangle_+^{3/4}}$. *Then there exists a positive constant* δ *such that if* $\|(\phi_0, \psi_0)\|_2 + |(\phi_0, \psi_0)|_{\langle x \rangle_+} + |(\phi_{0,x}, \psi_{0,x})|_{\langle x \rangle_+^{3/4}} + |v_+ - v_-| \leq \delta$, *then* (1.1)–(1.3) *has a unique global solution* $(v, u)(t, x)$ *satisfying*

$$v - V \in C^0([0,\infty); H^1 \cap L^2_{\langle x \rangle_+^{3/4}}) \cap L^2([0,\infty); H^1 \cap L^2_{\langle x \rangle_+^{3/4}}),$$

$$\int_{-\infty}^{x}(v(y,t)-V(y-st))dy \in C^0([0,\infty); H^2\cap L^2_{\langle x\rangle_+}),$$
$$u-U\in C^0([0,\infty); H^1\cap L^2_{\langle x\rangle_+^{3/4}})\cap L^2([0,\infty); H^2\cap L^2_{\langle x\rangle_+^{3/4}}),$$
$$\int_{-\infty}^{x}(u(y,t)-U(y-st))dy \in C^0([0,\infty); H^2\cap L^2_{\langle x\rangle_+}).$$

Furthermore, the solution verifies the following asymptotic stability

$$\sup_{x\in R}|(v,u)(t,x)-(V,U)(x-st)|\to 0 \quad as \quad t\to\infty. \tag{1.19}$$

2. Reformulation of Problem

We seek the solution (v,u) of (1.1)–(1.3) in the form

$$(v,u)(t,x)=(V,U)(\xi)+(\phi_\xi,\psi_\xi)(t,\xi),\quad \xi=x-st,$$

where (ϕ,ψ) satisfies the following "integrated" system

$$\begin{cases}\phi_t-s\phi_\xi-\psi_\xi=0\\ \psi_t-s\psi_\xi+p(V+\phi_\xi)-p(V)=\mu\left(\dfrac{\psi_{\xi\xi}}{V+\phi_\xi}+\dfrac{sV_\xi\phi_\xi}{V(V+\phi_\xi)}\right)\\ (\phi,\psi)(0,\xi)=(\phi_0,\psi_0)(\xi)\end{cases} \tag{2.1}$$

which can be rewritten as

$$\begin{cases}\phi_t-s\phi_\xi-\psi_\xi=0\\ \psi_t-s\psi_\xi-a(V)\phi_\xi-\mu\dfrac{\psi_{\xi\xi}}{V}=F\\ (\phi,\psi)(0,\xi)=(\phi_0,\psi_0)(\xi)\end{cases} \tag{2.2}$$

with

$$a(V)=-p'(V)+\frac{\mu sV_\xi}{V^2}=-p'(V)+\frac{h(V)}{V}, \tag{2.3}$$

$$F=-\{p(V+\phi_\xi)-p(V)-p'(V)\phi_\xi\}-(\mu\psi_{\xi\xi}+h(V)\phi_\xi)\left(\frac{1}{V+\phi_\xi}-\frac{1}{V}\right). \tag{2.4}$$

It is well known that

$$|F|\le O(1)(|\phi_\xi|^2+|\phi_\xi||\psi_{\xi\xi}|). \tag{2.5}$$

We define the solution space of (2.2) as the following

$$X(0,T)=\{(\phi,\psi)\in C^0([0,T];H^2\cap L^2_{\langle\xi\rangle_+}),(\phi_\xi,\psi_\xi)\in C^0([0,T];H^1\cap L^2_{\langle\xi\rangle_+^{3/4}}),$$
$$\phi_\xi\in L^2([0,T];H^1\cap L^2_{\langle\xi\rangle_+^{3/4}}),\psi_\xi\in L^2([0,T];H^2\cap L^2_{\langle\xi\rangle_+^{3/4}})\},$$

with $0<T\le\infty$. Let

$$N(t)=\sup_{0\le\tau\le t}(\|(\phi,\psi)(\tau)\|_2+|(\phi,\psi)(\tau)|_{\langle\xi\rangle_+}+|(\phi_\xi,\psi_\xi)(\tau)|_{\langle\xi\rangle_+^{3/4}}),$$
$$N_0=\|(\phi_0,\psi_0)\|_2+|(\phi_0,\psi_0)|_{\langle\xi\rangle_+}+|(\phi_{0,\xi},\psi_{0,\xi})|_{\langle\xi\rangle_+^{3/4}},$$

we have $\sup_{\xi\in R}|(\phi,\psi)(t,\xi)|\le CN(t)$ which will be used to prove the *a priori* estimates in Section 3.

After stating the following Theorem 2.1, we shall easily know that Theorem 1.1 can be treated from Theorem 2.1. So, to prove Theorem 2.1 will be our main purpose.

Theorem 2.1. *In addition to the assumptions in Theorem* 1.1. *Then there exists a positive constant* δ_1 *such that if the initial disturbance and the shock strength satisfy* $N_0+|(v_+-v_-,u_+-u_-)|\le\delta_1$, *then* (2.2) *has a unique global solution* $(\phi,\psi)\in X(0,\infty)$ *satisfying*

$$\begin{aligned}(2.6)\qquad &\|(\phi,\psi)(t)\|_2^2+|(\phi,\psi)(t)|^2_{\langle\xi\rangle_+}+|(\phi_\xi,\psi_\xi)(t)|^2_{\langle\xi\rangle_+^{3/4}}\\ &\quad+\int_0^t\{\|\phi_\xi(\tau)\|_1^2+|\phi_\xi(\tau)|^2_{\langle\xi\rangle_+^{3/4}}+\|\psi_\xi(\tau)\|_2^2+|\psi_\xi(\tau)|^2_{\langle\xi\rangle_+^{3/4}}\}d\tau\\ &\le CN_0,\end{aligned}$$

for any $t\ge0$. *Moreover, the asymptotic stability of traveling wave holds*

$$\sup_{\xi\in R}|(\phi_\xi,\psi_\xi)(\xi,t)|\to0\quad as\quad t\to\infty. \tag{2.7}$$

Theorem 2.1 will be proved by the local existence result and the *a priori* estimates as follows.

Proposition 2.2 (Local Existence). *For any* $\delta_0>0$, *there exists a positive constant* T_0 *depending on* δ_0 *such that, if* $(\phi_0,\psi_0)\in H^2\cap L^2_{\langle\xi\rangle_+}$, $(\phi_{0,\xi},\psi_{0,\xi})\in L^2_{\langle\xi\rangle_+^{3/4}}$, *and* $N_0\le\delta_0$, *then the problem* (2.2) *has a unique solution* $(\phi,\psi)\in X(0,T_0)$ *satisfying* $N(t)\le2\delta_0$ *for* $0\le t\le T_0$.

Proposition 2.3 (*A Priori* Estimates). *Let $(\phi, \psi) \in X(0,T)$ be a solution for a positive T. Then there exists a positive constant δ_2 such that if $N(T) + |(v_+ - v_-, u_+ - u_-)| < \delta_2$, then (ϕ, ψ) satisfies the a priori estimate* (2.6) *for* $0 \le t \le T$.

We here omit the proof of Proposition 2.2 because it is easily showed in the standard way. The proof of Proposition 2.3 is a key for Theorem 2.1 and will be showed in the next section.

Proof of Theorem 2.1. From Propositions 2.2 and 2.3, by the standard continuation argument, we can obtain a unique global solution $(\phi, \psi)(t, \xi)$ satisfying (2.2) and (2.6) for all $t \in [0, \infty)$.

To prove (2.7), we consider the function $(\Phi, \Psi)(t) = \|(\phi_\xi, \psi_\xi)(t)\|^2$. By virtue of the uniform estimate (2.6), and $\langle\xi\rangle_+ \ge C$, using equations (2.2), we see that both $(\Phi, \Psi)(t)$ and $(\Phi'(t), \Psi'(t))$ are integrable over $t \ge 0$. So, it means that $(\Phi, \Psi)(t) \to 0$, i.e., $\|(\phi_\xi, \psi_\xi)(t)\| \to 0$ as $t \to \infty$. Furthermore, $\|(\phi_{\xi\xi}, \psi_{\xi\xi})(t)\|$ is uniformly bounded in $t \ge 0$ due to (2.6) and $\langle\xi\rangle_+ \ge C$. By the Sobolev inequality, we then obtain

$$\sup_{\xi \in R} |(\phi_\xi, \psi_\xi)(\xi, t)|^2 \le 2\{\|\phi_\xi\| \, \|\phi_{\xi\xi}\| + \|\psi_\xi\| \, \|\psi_{\xi\xi}\|\} \to 0$$

as $t \to \infty$. This completes the proof of Theorem 2.1. □

3. A Priori Estimates

Denote $w_1(V) > 0$ in $C^1(v_-, v_+)$ as a weight function which will be selected below. Multiplying the first equation of (2.2) by $a(V)w_1(V)\phi$ and the second one by $w_1(V)\psi$, respectively, then taking the sum of them, we obtain

$$\begin{aligned} &\frac{1}{2}\{(w_1 a)(V)\phi^2 + w_1(V)\psi^2\}_t - \{\cdots\}_\xi + \frac{\mu w_1(V)}{V}\psi_\xi^2 \\ &\quad + \frac{sV_\xi}{2} y(V)\left(\phi + \frac{\psi}{s}\right)^2 - \frac{V_\xi}{2s} z(V)\psi^2 \\ &= F w_1(V)\psi, \end{aligned} \tag{3.1}$$

where $\{\cdots\}_\xi$ denots the terms which will disappear after intergral on ξ, and

$$y(V) = (aw_1)'(V), \tag{3.2}$$

$$z(V) = (w_1 h)''(V). \tag{3.3}$$

Now, we are going to choose our desired weight function such that $y(V) \ge 0$ and $z(V) \le 0$.

Same to Nishihara [14], let us choose $w_1(v)$ as

$$w_1(v) = \begin{cases} k\dfrac{(v_+ - v)(v - A)}{h(v)}, & v \in [v_*, v_+), \\ \dfrac{v - v_-}{h(v)}, & v \in (v_-, v_*], \end{cases} \tag{3.4}$$

where v_* is the point mentioned as in (1.5), and

$$A = \frac{v_*(v_* - v_-) + v_-(v_+ - v_*)}{v_+ - v_-} > 0, \quad v_* > A,$$

$$k = (v_+ - v_-)/(v_+ - v_*)^2 > 0.$$

Thus, $w_1(V) > 0$, $w_1(V) \in C^1(v_-, v_+)$, and $w_1(V(\xi)) \sim \langle\xi\rangle_+$, and satisfies the following lemma.

Lemma 3.1. *It holds*

$$y(v) \geq 0, \quad z(v) \leq 0 \tag{3.5}$$

for $v \in [v_-, v_+]$.

Proof. We are going to prove (3.5) on the two intervals of $[v_-, v_*]$ and $[v_*, v_+]$, respectively.

Case 1. On the interval $[v_*, v_+]$, i.e., $-p''(v) > 0$. Thanks to (3.4), we first have $z(v) = -2k < 0$. In order to prove $y(v) \geq 0$, let us see the following facts.

Setting $G(v) \equiv -p''(v)v - p'(v) + h'(v)$, due to (1.5), (1.18) and $v > 0$, we obtain $G'(v) = -p'''(v)v - 3p''(v) > 0$, i.e., $G(v)$ is increasing on $[v_*, v_+]$. So, $G(v) \geq G(v_*) = -2p'(v_*) - s^2$. Noting

$$-p'(v_*) = -p'(v_+) - p''(\tilde{v})(v_* - v_+) = s^2 - p''(\tilde{v})(v_* - v_+)$$

for some $\tilde{v} \in (v_*, v_+)$, then we have

$$G(v) \geq G(v_*) = s^2\left(1 - \frac{2p''(\tilde{v})}{s^2}(v_* - v_+)\right) > 0 \tag{3.6}$$

for $|v_+ - v_-| \ll 1$. Similarly, noting for some $\hat{v} \in (v_*, v_+)$,

$$0 = h(v_+) = h(v) + h'(v)(v_+ - v) + \frac{h''(\hat{v})}{2}(v_+ - v)^2,$$

then we also have

$$-\frac{h(v)v}{v_+ - v} - h'(v)v = -p''(\hat{v})v(v_+ - v)/2 > 0. \tag{3.7}$$

On the other hand, we can easily check that $(h(v)v/(v-A))-h(v) = Ah(v)/(v-A) \geq 0$ for $v \in [v_*, v_+]$ due to $h(u) > 0$ and $v > A$. Thus, by these facts, we have

$$
\begin{aligned}
y(v) &= (aw_1)'(v) \\
&= \frac{k(v_+ - v)(v-A)}{h(v)^2 v^2}\Bigg(G(v)vh(v) \\
&\quad + a(v)v\left(-\frac{h(v)v}{v_+ - v} - h'(v)v + \frac{h(v)v}{v-A} - h(v)\right)\Bigg) \\
&\geq 0,
\end{aligned} \tag{3.8}
$$

where $a(v) > 0$ is the above (2.3).

Case 2. On the interval $[v_-, v_*]$, i.e., $-p''(v) < 0$. By (3.4), we have $z(v) = 0$. For some $\bar{v} \in (v_-, v_*)$, we also have

$$
0 = h(v_-) = h(v) + h'(v)(v_- - v) + \frac{h''(\bar{v})}{2}(v_- - v)^2, \tag{3.9}
$$

and $h''(\bar{v}) = -p''(\bar{v}) < 0$, which ensure that

$$
\begin{aligned}
y(v) &= (aw_1)'(v) \\
&= \frac{a(v)(v-v_-)}{h(v)^2}\left(\frac{h(v) - h'(v)(v-v_-)}{v - v_-} + \frac{a'(v)}{a(v)}h(v)\right) \\
&= \frac{a(v)(v-v_-)^2}{h(v)^2}\Bigg\{\frac{p''(\bar{v})}{2}(v-v_-) \\
&\quad + \frac{a'(v)}{a(v)}\left[h'(v)(v-v_-) + \frac{p''(\bar{v})}{2}(v-v_-)^2\right]\Bigg\} \\
&\geq 0,
\end{aligned} \tag{3.10}
$$

due to the facts of $a(v) > 0$, $p''(v) > 0$ and

$$
H(v) \equiv \frac{p''(\bar{v})}{2}(v-v_-) + \frac{a'(v)}{a(v)}\left[h'(v)(v-v_-) + \frac{p''(\bar{v})}{2}(v-v_-)^2\right] \geq 0. \tag{3.11}
$$

About the proof of (3.11), since $a'(v) = -p''(v) + (h'(v)v - h(v))/v^2$, substituting (3.9) into (3.11), and setting

$$
q(v) \equiv p''(v) - \frac{h'(v)v - h(v)}{v^2} + \frac{h'(v)}{v^2}(v - v_-),
$$

it is easily to see that $q(v)$ and $a(v)$ are bounded on $[v_-, v_*]$, then we have

$$
\begin{aligned}
H(v) &= \frac{p''(v)}{2}\left(1 - \frac{1}{a(v)}(-p'(v) - s^2) - \frac{q(v)}{2a(v)}(v - v_-)\right) \\
&\quad + \frac{p''(\bar{v}) - p''(v)}{2}\left(1 - \frac{q(v)}{2a(v)}(v - v_-)\right) + \frac{h'(v)^2 v_-}{a(v)v^2} \\
&\geq 0
\end{aligned}
$$

because of $p''(v) > 0$ on $v \in [v_-, v_*]$, $p'''(v) < 0$.

Combining Case 1 and Case 2, we have proved (3.5). □

Let ξ_* be the unique number such that $V(\xi_*) = v_*$, this uniqueness is due to the monotonity of $V(\xi)$. Integrating (3.1) over $R \times [0,t]$, by Lemma 3.1, $w_1(V(\xi)) \sim \langle\xi\rangle_+$, $L^2_{w_1} = L^2_{\langle\xi\rangle_+}$, and (2.5), we obtain a estimate as follows.

Lemma 3.2. *It holds*

$$(3.12)\quad |(\phi,\psi)(t)|^2_{\langle\xi\rangle_+} + \int_0^t |\psi_\xi(\tau)|^2_{\langle\xi\rangle_+} d\tau + \int_0^t \int_{\xi_*}^{+\infty} |V_\xi| \psi(\tau,\xi)^2 d\xi d\tau$$
$$\le C\{|(\phi_0,\psi_0)|^2_{\langle\xi\rangle_+} + N(t)\int_0^t (|\phi_\xi(\tau)|^2_{\langle\xi\rangle_+^{3/4}} + |\psi_{\xi\xi}(\tau)|^2_{\langle\xi\rangle_+^{3/4}}) d\tau\}.$$

The next step is to estimate $\int_0^t |\phi_\xi(\tau)|^2_{\langle\xi\rangle_+^{3/4}} d\tau$ and $\int_0^t |\psi_{\xi\xi}(\tau)|^2_{\langle\xi\rangle_+^{3/4}} d\tau$.

Lemma 3.3. *It holds*

$$(3.13)\quad |\phi_\xi(t)|^2_{\langle\xi\rangle_+^{3/4}} + (1 - CN(t))\int_0^t |\phi_\xi(\tau)|^2_{\langle\xi\rangle_+^{3/4}} d\tau$$
$$\le C\{|(\phi_0,\psi_0)|^2_{\langle\xi\rangle_+} + |\phi_{0,\xi}|^2_{\langle\xi\rangle_+^{3/4}} + N(t)\int_0^t |\psi_{\xi\xi}(\tau)|^2_{\langle\xi\rangle_+^{3/4}} d\tau\}$$
$$\textit{for small}\quad |v_+ - v_-|.$$

Proof. From equations (2.2), we have

$$(3.14)\qquad \mu\phi_{\xi t} - s\mu\phi_{\xi\xi} + a(V)V\phi_\xi + sV\psi_\xi - V\psi_t = -FV.$$

Let us choose our another weight function as

$$(3.15)\qquad w_2(\xi) = \begin{cases} 1 + \xi - \xi_*, & \text{as } \xi \ge \xi_* \\ 1, & \text{as } \xi \le \xi_*. \end{cases}$$

It is not hard to see that $w_2(\xi) \sim w_1(\xi) \sim \langle\xi\rangle_+$, $L^2_{w_2} = L^2_{w_1} = L^2_{\langle\xi\rangle_+}$, and $w_2(\xi)$ is in $C^0(R)$ but not in $C^1(R)$. We now restrict our problem in the weighted space $L^2_{w_2(\xi)}$.

<u>*Case* (i)</u>. On the interval $[\xi_*, +\infty) = R_+$, *i.e.*, $V(\xi) \in [v_*, v_+]$, multiplying (3.14) by $w_2(\xi)^{\frac{3}{4}}\phi_\xi$ (here $w_2(\xi) = 1 + \xi - \xi_*$), we get

$$(3.16)\qquad \frac{\mu}{2}\{w_2(\xi)^{\frac{3}{4}}\phi_\xi^2\}_t - \frac{s\mu}{2}\{w_2(\xi)^{\frac{3}{4}}\phi_\xi^2\}_\xi + \frac{3s\mu}{8} w_2(\xi)^{-\frac{1}{4}}\phi_\xi^2$$

$$+ w_2(\xi)^{\frac{3}{4}}a(V)V\phi_\xi^2 + sVw_2(\xi)^{\frac{3}{4}}\phi_\xi\psi_\xi - w_2(\xi)^{\frac{3}{4}}V\psi_t\phi_\xi$$
$$= -FVw_2(\xi)^{\frac{3}{4}}\phi_\xi.$$

By the first equation in (2.2), we note that

$$
\begin{aligned}
(3.17)\qquad -Vw_2(\xi)^{\frac{3}{4}}\psi_t\phi_\xi &= -\{Vw_2(\xi)^{\frac{3}{4}}\psi\phi_\xi\}_t + Vw_2(\xi)^{\frac{3}{4}}\psi\phi_{\xi t} \\
&= -\{Vw_2(\xi)^{\frac{3}{4}}\psi\phi_\xi\}_t + Vw_2(\xi)^{\frac{3}{4}}\psi(s\phi_\xi + \psi_\xi)_\xi \\
&= -\{Vw_2(\xi)^{\frac{3}{4}}\psi\phi_\xi\}_t + \{Vw_2(\xi)^{\frac{3}{4}}\psi(s\phi_\xi + \psi_\xi)\}_\xi \\
&\quad - sVw_2(\xi)^{\frac{3}{4}}\psi_\xi\phi_\xi - Vw_2(\xi)^{\frac{3}{4}}\psi_\xi^2 \\
&\quad - s\left[V_\xi w_2(\xi)^{\frac{3}{4}} + \frac{3}{4}Vw_2(\xi)^{-\frac{1}{4}}\right]\psi\phi_\xi \\
&\quad - \left\{\frac{1}{2}\left[V_\xi w_2(\xi)^{\frac{3}{4}} + \frac{3}{4}Vw_2(\xi)^{-\frac{1}{4}}\right]\psi^2\right\}_\xi \\
&\quad + \left[\frac{V_{\xi\xi}}{2}w_2(\xi)^{\frac{3}{4}} + \frac{3}{8}V_\xi w_2(\xi)^{-\frac{1}{4}} - \frac{3}{32}Vw_2(\xi)^{-\frac{5}{4}}\right]\psi^2,
\end{aligned}
$$

and

$$(3.18)\qquad \left|-s\left[V_\xi w_2(\xi)^{\frac{3}{4}} + \frac{3}{4}Vw_2(\xi)^{-\frac{1}{4}}\right]\psi\phi_\xi\right| \le \frac{1}{2}a(V)Vw_2(\xi)^{\frac{3}{4}}\phi_\xi^2 + \frac{1}{2}b(\xi)\psi^2,$$

where

$$(3.19)\qquad b(\xi) \equiv \frac{s^2[V_\xi w_2(\xi)^{\frac{3}{4}} + \frac{3}{4}Vw_2(\xi)^{-\frac{1}{4}}]^2}{a(V)Vw_2(\xi)^{\frac{3}{4}}} > 0.$$

Substituting (3.17), (3.18) into (3.16), and integrating it over $R_+ = [\xi_*, +\infty)$, we have

$$
\begin{aligned}
(3.20)\qquad &\frac{\mu}{2}\frac{d}{dt}\int_{R_+} w_2(\xi)^{\frac{3}{4}}\phi_\xi^2 d\xi + \{\cdots\}|_{\xi=\xi_*} - \frac{d}{dt}\int_{R_+} Vw_2(\xi)^{\frac{3}{4}}\phi_\xi\psi d\xi \\
&\quad + \frac{3\mu s\mu}{8}\int_{R_+} w_2(\xi)^{-\frac{1}{4}}\phi_\xi^2 d\xi + \frac{1}{2}\int_{R_+} w_2(\xi)^{\frac{3}{4}}a(V)V\phi_\xi^2 d\xi \\
&\quad - \frac{1}{2}\int_{R_+} r_1(\xi)\psi^2 d\xi - \int_{R_+} Vw_2(\xi)^{\frac{3}{4}}\psi_\xi^2 d\xi + \frac{1}{2}r_2(\xi_*)\psi(t,\xi_*)^2 \\
&\le -\int_{R_+} w_2(\xi)^{\frac{3}{4}}\phi_\xi VF d\xi,
\end{aligned}
$$

where $\{\cdots\}$ denotes the term which is produced from the integral by parts and does not contain any derivative of $w_2(\xi)$, $r_i(\xi)(i = 1, 2)$ denote as follows:

$$(3.21)\qquad r_2(\xi) = (Vw_2(\xi))_\xi = V_\xi w_2(\xi)^{\frac{3}{4}} + \frac{3}{4}Vw_2(\xi)^{-\frac{1}{4}},$$

$$(3.22)\qquad r_1(\xi) = b(\xi) - r_2'(\xi),$$

with $b(\xi)$ in (3.19). However, by Cauchy's inequality, we have

$$\begin{aligned}\frac{1}{2}r_2(\xi_*)\psi(t,\xi_*)^2 &= -\frac{1}{2}V_\xi(\xi_*)w_2(\xi_*)\int_{\xi_*}^{+\infty}\frac{\partial}{\partial\xi}(w_2(\xi)^{-\frac{1}{4}}\psi(t,\xi)^2)d\xi \\ &\quad -\frac{3}{8}V(\xi_*)w_2(\xi_*)^{-\frac{1}{8}}\int_{\xi_*}^{+\infty}\frac{\partial}{\partial\xi}(w_2(\xi)^{-\frac{1}{8}}\psi(t,\xi)^2)d\xi \qquad (3.23)\\ &= -\frac{1}{2}\Bigg[V_\xi(\xi_*)\int_{\xi_*}^{+\infty}\left(-\frac{1}{4}w_2(\xi)^{-\frac{5}{4}}\psi^2+2w_2(\xi)^{-\frac{1}{4}}\psi\psi_\xi\right)d\xi \\ &\quad +\frac{3}{4}V(\xi_*)\int_{\xi_*}^{+\infty}\left(-\frac{1}{8}w_2(\xi)^{-\frac{9}{8}}\psi^2+2w_2(\xi)^{-\frac{1}{8}}\psi\psi_\xi\right)d\xi\Bigg] \\ &\ge \frac{1}{2}\Bigg[V_\xi(\xi_*)\int_{\xi_*}^{+\infty}\left(\frac{1}{4}w_2(\xi)^{-\frac{5}{4}}\psi^2-w_2(\xi)^{\frac{-3}{2}}\psi^2-w_2(\xi)\psi_\xi^2\right)d\xi \\ &\quad +\frac{3}{4}V(\xi_*)\int_{\xi_*}^{+\infty}\bigg(\frac{1}{8}w_2(\xi)^{-\frac{9}{8}}\psi^2-w_2(\xi)^{-\frac{4}{5}}\psi^2 \\ &\qquad -w_2(\xi)\psi_\xi^2\bigg)d\xi\Bigg],\end{aligned}$$

where $w_2(\xi_*)=1$.

Substituting (3.23) into (3.20), we have

$$\begin{aligned}&\frac{\mu}{2}\frac{d}{dt}\int_{\xi_*}^{+\infty}w_2(\xi)^{\frac{3}{4}}\phi_\xi^2d\xi+\{\cdots\}|_{\xi=\xi_*}-\frac{d}{dt}\int_{\xi_*}^{+\infty}w_2(\xi)^{\frac{3}{4}}V\phi_\xi\psi d\xi \qquad (3.24)\\ &+\frac{3s\mu}{8}\int_{\xi_*}^{+\infty}w_2(\xi)^{-\frac{1}{4}}\phi_\xi^2+\frac{1}{2}\int_{\xi_*}^{+\infty}w_2(\xi)^{\frac{3}{4}}a(V)V\phi_\xi^2d\xi+\frac{1}{2}\int_{\xi_*}^{+\infty}c(\xi)\psi^2d\xi \\ &\le CN(t)\int_{\xi_*}^{+\infty}w_2(\xi)^{\frac{3}{4}}(\phi_\xi^2+\psi_{\xi\xi}^2)d\xi+C\int_{\xi_*}^{+\infty}w_2(\xi)\psi_\xi^2d\xi,\end{aligned}$$

where

$$\begin{aligned}c(\xi) &= \frac{1}{4}V_\xi(\xi_*)w_2(\xi)^{-\frac{5}{4}}-V_\xi(\xi_*)w_2(\xi)^{-\frac{3}{2}} \qquad (3.25)\\ &\quad \frac{3}{32}V(\xi_*)w_2(\xi)^{-\frac{9}{8}}-\frac{3}{4}V(\xi_*)w_2(\xi)^{-\frac{5}{4}}-b(\xi)+r_2'(\xi).\end{aligned}$$

Since $V(\xi)>0$, $V_\xi(\xi)>0$, $a(V)$ and $V(\xi)$ are bounded, and $|V_\xi|=O(|\xi|^{-2})$, $|V_{\xi\xi}|=O(|\xi|^{-3})$, $w_2(\xi)=O(|\xi|)$, as $\xi\to\infty$, we then claim that $b(\xi)=O(|\xi|^{-\frac{5}{4}})$, $|r_2'(\xi)|=O(|\xi|^{-\frac{5}{4}})$ as $\xi\to+\infty$. Therefore, there is a larger number $\xi_{**}\in(\xi_*,+\infty)$ such that, when $\xi>\xi_{**}$, it holds

$$c(\xi)\ge\frac{3}{32}V(\xi_*)w_2(\xi)^{-\frac{9}{8}}\{1-O(|\xi|^{-\frac{1}{8}})-O(|\xi|^{-\frac{3}{8}})\}\ge 0, \qquad (3.26)$$

for $\xi \in [\xi_{**}, +\infty)$. On the other hand, according to the boundness of V, V_ξ, $V_{\xi\xi}$ and $w_2(\xi)$ on $[\xi_*, \xi_{**}]$, we obtain

$$|c(\xi)| \le C, \quad \text{on} \quad [\xi_*, \xi_{**}]. \tag{3.27}$$

According to above mentioned facts, then we can rewrite (3.24) as follows

$$\begin{aligned}
&\frac{\mu}{2}\frac{d}{dt}\int_{\xi_*}^{+\infty} w_2(\xi)^{\frac{3}{4}}\phi_\xi^2 d\xi + \{\cdots\}|_{\xi=\xi_*} - \frac{d}{dt}\int_{\xi_*}^{+\infty} w_2(\xi)^{\frac{3}{4}} V\phi_\xi\psi d\xi \\
&\quad + \frac{3s\mu}{8}\int_{\xi_*}^{+\infty} w_2(\xi)^{-\frac{1}{4}}\phi_\xi^2 + \frac{1}{2}\int_{\xi_*}^{+\infty} w_2(\xi)^{\frac{3}{4}} a(V)V\phi_\xi^2 d\xi \\
&\le \frac{1}{2}\int_{\xi_*}^{\xi_{**}} c(\xi)\psi^2 d\xi + CN(t)\int_{\xi_*}^{+\infty} w_2(\xi)^{\frac{3}{4}}(\phi_\xi^2 + \psi_{\xi\xi}^2)d\xi \\
&\quad + C\int_{\xi_*}^{+\infty} w_2(\xi)\psi_\xi^2 d\xi.
\end{aligned} \tag{3.28}$$

Case (ii). On the another interval $(-\infty, \xi_*]$, namely, $V(\xi) \in [v_-, v_*]$. Mutilplying (3.14) by ϕ_ξ, and integrating it over $(-\infty, \xi_*]$ (here $w_2(\xi) = 1$), we have

$$\begin{aligned}
&\frac{\mu}{2}\frac{d}{dt}\int_{-\infty}^{\xi_*} \phi_\xi^2 d\xi - \{\cdots\}|_{\xi=\xi_*} - \frac{d}{dt}\int_{-\infty}^{\xi_*} V\phi_\xi\psi d\xi + \int_{-\infty}^{\xi_*} a(V)V\phi_\xi^2 d\xi \\
&\le CN(t)\int_{-\infty}^{\xi_*} \phi_\xi^2 d\xi + C\int_{-\infty}^{\xi_*} \psi_\xi^2 d\xi.
\end{aligned} \tag{3.29}$$

By the continuity of $w_2(\xi)$, adding (3.28) and (3.29), and integrating it over $[0, t]$, noting

$$\int_{-\infty}^{+\infty} V(\xi)w_2(\xi)^{\frac{3}{4}}|\phi_\xi\psi|d\xi \le \frac{\mu}{4}|\phi_\xi|^2_{w_2(\xi)^{3/4}} + C(\mu)|\psi|^2_{w_2(\xi)},$$

where $C(\mu)$ is a positive constant dependent on μ, we have proved (3.13) by making use of the boundness of $c(\xi)$ in $[\xi_*, \xi_{**}]$, Lemma 3.2, and $\langle\xi\rangle_+ \sim w_2(\xi)$. □

Multiplying the second equation of (2.2) by $\langle\xi\rangle_+^{\frac{3}{4}}\psi_{\xi\xi}$, and integrating it over $R \times [0, t]$, by the Cauchy's inequality and Lemmas 3.2 and 3.3, we obtain

Lemma 3.4. *It holds*

$$\begin{aligned}
&|\psi_\xi(t)|^2_{\langle\xi\rangle_+^{3/4}} + (1 - CN(t))\int_0^t |\psi_{\xi\xi}(\tau)|^2_{\langle\xi\rangle_+^{3/4}} d\tau \\
&\le C(|(\phi_0, \psi_0)|^2_{\langle\xi\rangle_+} + |(\phi_{0,\xi}, \psi_{0,\xi})|^2_{\langle\xi\rangle_+^{3/4}}) \qquad \text{for small} \quad |v_+ - v_-|.
\end{aligned} \tag{3.30}$$

When we differentiate (3.14) in ξ and multiply it by $\phi_{\xi\xi}$ and integrate the resultant equality over $[0,t] \times R$ due to Lemmas 3.2–3.4, then we obtain

Lemma 3.5. *It holds*

$$\|\phi_{\xi\xi}(t)\|^2 + \int_0^t \|\phi_{\xi\xi}(\tau)\|^2 d\tau \le C(\|\phi_{0,\xi\xi}\|^2 + |(\phi_0, \psi_0)|^2_{\langle\xi\rangle_+} + |(\phi_{0,\xi}, \psi_{0,\xi})|^2_{\langle\xi\rangle_+^{3/4}}) \tag{3.31}$$

for stuitably small $N(t)$ *and* $|v_+ - v_-|$.

Now we differentiate the second equation of (2.2) in ξ and multiply it by $-\psi_{\xi\xi\xi}$, and integrat the resultant equality over $[0,t] \times R$. Then, using Lemmas 3.2–3.5, we obtain

Lemma 3.6. *It holds*

$$\|\psi_{\xi\xi}(t)\|^2 + \int_0^t \|\psi_{\xi\xi\xi}(\tau)\|^2 d\tau \le C(\|(\phi_{0,\xi\xi}, \psi_{0,\xi\xi}\|^2 + |(\phi_0, \psi_0)|^2_{\langle\xi\rangle_+} + |(\phi_{0,\xi}, \psi_{0,\xi})|^2_{\langle\xi\rangle_+^{3/4}}) \tag{3.32}$$

for stuitably small $N(t)$ *and* $|v_+ - v_-|$.

Proof of Proposition 2.3. Combining Lemma 3.3–Lemma 3.6, we have completed the proof of Proposition 2.3 for the smallness of $N(t) + |v_+ - v_-|$, say, $N(t) + |v_+ - v_-| < \delta_2$. □

4. Application to van der Waals Fluid

In this section, we will give an application of our stability theorem to van der Waals fluid in the form of (1.1), (1.2). The pressure $p(v)$ is given explicitly as

$$p(v) = \frac{R\theta}{v-b} - \frac{a}{v^2}, \quad \text{for} \quad v > b, \tag{4.1}$$

where $R > 0$ is the gas constant, $\theta > 0$ the absolute temperature (assumed to be constant), and a and b are positive constants.

When the constants a, b, R and θ satisfy

$$m_1 a < bR\theta < m_2 a, \tag{4.2}$$

where $m_1 = (2/3)^3$ and $m_2 = (3/4)^4$, then we have

$$p'(v) < 0 \quad \text{for} \quad v > b, \tag{4.3}$$

i.e., system (1.1), (1.2) with $\mu = 0$ is strict hyperbolic. Moreover, there are constants $\underline{v}$ and $\overline{v}$ with $3b < \underline{v} < 4b < \overline{v}$ such that $p''(\underline{v}) = p''(\overline{v}) = 0$, $p''(v) < 0$ on $(\underline{v}, \overline{v})$, and $p''(v) > 0$ otherwise. Therefore, $p(v)$ is strictly decreasing and has two points of inflection at $v = \underline{v}$ and $\overline{v}$. Furthermore, there are v_1 and v_2 with $\underline{v} < v_1 < \overline{v} < v_2$ and with $v_2 > 5b$ such that $p'''(v_1) = p'''(v_2) = 0$, $p'''(v) > 0$ on (v_1, v_2), and $p'''(v) < 0$ otherwise. Consequently, we have that

$$p''(v) \lesseqgtr 0 \quad \text{for} \quad v \gtrless \underline{v}, \quad \text{and} \quad p'''(v) < 0 \quad \text{for all} \quad v \in (b, v_1), \tag{4.4}$$

$$p''(v) \gtreqless 0 \quad \text{for} \quad v \gtrless \overline{v}, \quad \text{and} \quad p'''(v) > 0 \quad \text{for all} \quad v \in (v_1, v_2). \tag{4.5}$$

In the region $b < v < v_1$, under some conditions mentioned as in Sections 1 and 2, according to the stability theory developed in previous sections, the traveling wave solution with degenerate shock can be proved to be stable as $t \to \infty$, provided that the initial disturbances and the shock strength are small.

For another region $v_1 < v < v_2$, some result on the stability of traveling wave will appear in future.

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