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## SPHERICAL MEANS ON RIEMANNIAN MANIFOLDS

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1. Let  $X$  be a compact Riemannian manifold of dimension  $n$ ,  $TX$  its tangent bundle and  $SX$  its unit sphere bundle. Denote by  $p: SX \rightarrow X$  the canonical projection. Let  $G_t: SX \rightarrow SX$  ( $t \in \mathbf{R}$ ) be the geodesic flow.

The spherical mean (of radius  $t$ )  $L_t: C^\infty(X) \rightarrow C^\infty(X)$  is defined by the following commutative diagram:

$$\begin{array}{ccc}
 C^\infty(X) & \xrightarrow{L_t} & C^\infty(X) \\
 p^* \downarrow & & \uparrow p_! \\
 C^\infty(SX) & \xrightarrow{G_t^*} & C^\infty(SX)
 \end{array}$$

Here  $p^*$  and  $G_t^*$  denote the maps induced, respectively, by  $p$  and  $G_t$ , and  $p_!$  is the fibre integral defined by

$$p_! f(x) = \int_{p^{-1}x} f \omega_F, \quad f \in C^\infty(SX),$$

$\omega_F$  being the volume element on the fibre of  $p$  defined naturally by the Riemannian metric on  $X$ .

In this paper we prove the following

**Theorem I.** *For sufficiently small positive  $t$ ,  $L_t$  is a Fourier integral operator of order  $-\frac{1}{2}(n-1)$ , which belongs to the class determined by the conormal bundle  $\Lambda \subset T^*(X \times X) \setminus 0$  of  $\Delta_t = \{(x, y); d(x, y) = t\} \subset X \times X$ ,  $d$  being the metric induced by the Riemannian metric.*

The author would like to express his gratitude to T. Sunada for suggesting the above result.

2. For convenience sake, we consider all the operators as acting on the spaces of half densities. Let  $\Omega_{\frac{1}{2}}(X)$  denote the bundle of half densities on  $X$  and  $C^\infty\Omega_{\frac{1}{2}}(X)$  the space of smooth cross-sections of  $\Omega_{\frac{1}{2}}(X)$ . The Riemannian metric of  $X$  induces canonical densities  $\omega_X$  and  $\omega_{SX}$ , respectively, on  $X$  and  $SX$ , which allow us to identify  $C^\infty(X)$  with  $C^\infty\Omega_{\frac{1}{2}}(X)$ ,  $C^\infty(SX)$  with  $C^\infty\Omega_{\frac{1}{2}}(SX)$ , respectively,

by the isomorphisms  $f \mapsto f\sqrt{\omega_X}$  and  $f \mapsto f\sqrt{\omega_{SX}}$ ,  $\sqrt{\omega_X}$  and  $\sqrt{\omega_{SX}}$  being the half densities that are the square roots of  $\omega_X$  and  $\omega_{SX}$ , respectively. Under these identifications, the operators of §1 are transformed into the operators on the spaces of half densities:

$$\begin{CD} C^\infty\Omega_{1/2}(X) @>\tilde{L}_t>> C^\infty\Omega_{1/2}(X) \\ @V\tilde{p}^*VV @AA\tilde{p}_tA \\ C^\infty\Omega_{1/2}(SX) @>\tilde{G}_t^*>> C^\infty\Omega_{1/2}(SX) . \end{CD}$$

3. Let  $K \in \mathcal{D}'(SX \times X, 1 \boxtimes \Omega(X))$  be the distribution kernel of  $\tilde{p}^*: C^\infty(X) \rightarrow C^\infty(SX)$ . Here  $\Omega(X)$  denotes the bundle of densities on  $X$ . We define  $\tilde{K} \in \mathcal{D}'(SX \times X, \Omega_{\frac{1}{2}}(SX \times X))$  by

$$\tilde{K}(x, y) = \frac{K(x, y)\sqrt{\omega_{SX}(x)}}{\sqrt{\omega_X(y)}}.$$

Then, obviously, we have

**Lemma 1.** *The operators  $\tilde{p}^*$  and  $\tilde{p}_t$  have  $\tilde{K}$  and  $\tilde{K}'$  as the distribution kernels, respectively. Here  $\tilde{K}' \in \mathcal{D}'(X \times SX, \Omega_{\frac{1}{2}}(X \times SX))$  is the distribution corresponding to  $\tilde{K}$  under the transposition map  $X \times SX \cong SX \times X$ .*

Moreover, we have

**Lemma 2.**  $\tilde{K}' \in I^{-\frac{1}{2}(n-1)}(SX \times X, \Lambda)$ , where  $\Lambda \subset T^*(SX \times X) \setminus 0$  is the conormal bundle of the graph of  $p$ , that is,  $\Lambda = \{(x, p^*\eta) \times (px, -\eta); x \in SX, \eta \in T_{px}^*X \setminus 0\}$ .

REMARK 1. As to the notation  $I^m(X, \Lambda) = I_1^m(X, \Lambda)$ , see [3].

REMARK 2. We denote a point  $e$  of a bundle  $p: E \rightarrow B$  by  $(x, e)$ , where  $x = pe$ .

This lemma follows from the following

**Lemma 3.** *Let  $M$  and  $N$  be manifolds of dimension  $m$  and  $n$ , respectively. Let  $g: M \rightarrow N$  be a smooth mapping. Fixing non-vanishing half densities on  $M$  and  $N$ , we get  $\tilde{g}^*: C^\infty\Omega_{\frac{1}{2}}(N) \rightarrow C^\infty\Omega_{\frac{1}{2}}(M)$  induced by  $g$ . Then*

$$\tilde{g}^* \in I^{\frac{1}{2}(n-m)}(M \times N, \Lambda_g),$$

where  $\Lambda_g = \{(x, g^*\eta) \times (gx, -\eta); x \in M, \eta \in T_{gx}^*N \setminus 0\}$ .

(See, for example, [1].)

This lemma also implies the following

**Lemma 4.**  $\tilde{G}_t^* \in I^0(SX \times SX, \Lambda_t)$ , where  $\Lambda_t = \{(x, G_t^*\xi) \times (G_t x, -\xi); x \in SX, \xi \in T_{G_t x}^*SX \setminus 0\}$ .

4. Now we quote a theorem concerning the composition of Fourier integral operators.

Let  $X$  and  $Y$  be manifolds. For any subset  $\Lambda \subset T^*X \times T^*Y$ , we define  $\Lambda' = \{(x, \xi) \times (y, -\eta); (x, \xi) \times (y, \eta) \in \Lambda\} \subset T^*X \times T^*Y$ . When  $\Lambda \subset (T^*X \setminus 0) \times (T^*Y \setminus 0)$  is a conic Lagrangean submanifold,  $\Lambda'$  is nothing but a homogeneous canonical relation from  $T^*Y$  to  $T^*X$  in the sense of [3].

**Theorem A** ([3] Theorem 4.2.2). *Let  $X, Y$  and  $Z$  be smooth manifolds. Let  $C_1 \subset T^*X \times T^*Y, C_2 \subset T^*Y \times T^*Z$  be homogeneous canonical relations which satisfy the following conditions:*

- i)  $C_1 \times C_2$  and  $T^*X \times \Delta(T^*Y) \times T^*Z$  intersect transversally,  $\Delta(T^*Y)$  being the diagonal of  $T^*Y \times T^*Y$ ;
- ii) the restriction  $\pi$  of the projection  $T^*X \times T^*Y \times T^*Y \times T^*Z \rightarrow T^*X \times T^*Z$  to  $C_1 \times C_2 \cap T^*X \times \Delta(T^*Y) \times T^*Z$  is injective and proper.

Denote the image of  $\pi$  by  $C_1 \circ C_2$ .

Then  $C_1 \circ C_2$  is a homogeneous canonical relation from  $T^*Z$  to  $T^*X$ . Moreover, for any  $A_1 \in I_\rho^{m_1}(X \times Y, C_1'), A_2 \in I_\rho^{m_2}(Y \times Z, C_2')$ , which are properly supported and  $\rho > \frac{1}{2}$ , we have

$$A_1 \circ A_2 \in I_\rho^{m_1+m_2}(X \times Z; (C_1 \circ C_2)').$$

**Lemma 5.**  $\tilde{G}_t^* \circ \tilde{p}^* \in I^{-k(n-1)}(SX \times X, C_2')$ , where  $C_2 = \{(x, G_t^* p^* \eta) \times (pG_t x, \eta); x \in SX, \eta \in T_{pG_t x}^* X \setminus 0\}$ .

Proof.  $\Lambda_t'$  and  $\Lambda_p'$  are obviously homogeneous canonical relations, respectively, from  $T^*SX$  to  $T^*SX$  and from  $T^*X$  to  $T^*SX$ . The following sublemma shows that the conditions of Theorem A are satisfied in this case.

**Sublemma.** *Let  $X, Y$  and  $Z$  be manifolds,  $g: X \rightarrow Y$  a diffeomorphism and  $C \subset Y \times Z$  a submanifold. Denote by  $C_g \subset X \times Y$  the graph of  $g$ . Then*

- i)  $C_g \times C$  and  $X \times \Delta(Y) \times Z$  intersect transversally;
- ii)  $p: C_g \times C \cap X \times \Delta(Y) \times Z \rightarrow C_g \circ C$  is a homeomorphism,  $p$  being the restriction of the projection  $X \times Y \times Y \times Z \rightarrow X \times Z$ .

Proof. First, we show the assertion i). Fix  $(x_0, y_0, y_0, z_0) \in C_g \times C \cap X \times \Delta(Y) \times Z$  and let  $x=(x^i), y=(y^j)$  and  $z=(z^k)$  be local charts around  $x_0, y_0$  and  $z_0$ , respectively. Since  $C_g$  has the parametrization  $y \mapsto (g^{-1}y, y)$ ,  $y \circ \pi$  gives a local chart of  $C_g$  around  $(x_0, y_0)$ , where  $\pi: X \times Y \rightarrow Y$  is the natural projection. Define  $y_1 = y \circ \pi_2$  and  $y_2 = y \circ \pi_3$ , where  $\pi_j$  denotes the projection of  $X \times Y \times Y \times Z$  on the  $j$ -th factor. In order to prove the assertion i), it suffices to show that the local equations  $y_1^j - y_2^j = 0$  of  $X \times \Delta(Y) \times Z$  in  $X \times Y \times Y \times Z$  restricted to  $C_g \times C$  are independent near  $(x_0, y_0, y_0, z_0)$ . But this follows trivially from the fact that the differentials  $dy^j$  are independent on  $C_g$ .

The assertion ii) follows from the fact that  $p$  has the inverse:  $(x, z) \mapsto (x, gx, gx, z)$ . Q.E.D.

This completes the proof of Lemma 5.

5. Let  $t_0$  be a positive real number which satisfies the following conditions:

- a) for each  $x \in X$ , the exponential mapping  $T_x X \rightarrow X$  maps  $\{\xi \in T_x X; \|\xi\| < 3t_0\}$  diffeomorphically into  $X$ , whose image is denoted by  $B(x; 3t_0)$ ;
- b) for all  $y, z \in B(x; 3t_0)$ , there is a unique geodesic curve in  $B(x; 3t_0)$  joining  $y$  and  $z$ .

Since  $X$  is compact, such  $t_0$  exists. (cf [2].)

**Theorem I.** For  $0 < t \leq t_0$ , we have  $\tilde{L}_t \in I^{-\frac{1}{2}(n-1)}(X \times X, C')$ . Here  $C'$  is the conormal bundle of  $\Delta_t$  minus the zero section.

Note that  $\Delta_t$  is a submanifold of  $X \times X$ , since  $t \leq t_0$ .

*Proof.* We shall apply Theorem A in the situation where  $X = X, Y = SX, Z = X, C_1 = \{(px, \eta) \times (x, p^*\eta); x \in SX, \eta \in T_{px}^* X \setminus 0\}, C_2 = \{(x, G_t^* p^* \eta) \times (pG_t x, \eta); x \in SX, \eta \in T_{pG_t x}^* X \setminus 0\}, A_1 = \tilde{g}_t \in I^{-\frac{1}{2}(n-1)}(X \times SX, C_1'), A_2 = \tilde{G}_t^* \circ \tilde{p}^* \in I^{-\frac{1}{2}(n-1)}(SX \times X, C_2')$ .

1) First we determine the set  $C = C_1 \circ C_2$ . Let  $(x, \xi) \times (y, \eta) \in C$ . By definition, there is a point  $(z, \zeta) \in T^* SX$  such that  $pz = x, pG_t z = y, p^* \xi = \zeta = G_t^* p^* \eta$ . From  $p^* \xi = G_t^* p^* \eta$ , it follows that  $\langle G_t^* p^* \eta, \delta x \rangle = \langle p^* \xi, \delta x \rangle = \langle \xi, p_* \delta x \rangle = 0$  for any  $\delta x \in T_z(p^{-1}x)$ . Hence  $\langle \eta, (pG_t)_* \delta x \rangle = 0$  for all  $\delta x \in T_z(p^{-1}x)$ . Since  $t \leq t_0, (pG_t)_*|_{T_z(p^{-1}x)}$  is injective, whence  $(pG_t)_*(T_z(p^{-1}x))$  is a hyperplane of  $T_y X$ . Denote by  $\hat{\eta}$ , the element of  $T_y X$  corresponding to  $\eta$  under the isomorphism  $T_y X \cong T_y^* X$  defined by the Riemannian metric. Then  $\hat{\eta}$  is orthogonal to the hyperplane  $(pG_t)_*(T_z(p^{-1}x))$ , in other words,  $\hat{\eta}$  is a normal vector at  $y$  of the geodesic sphere  $pG_t(p^{-1}x)$  of radius  $t$  with center  $x$ .  $G_t z \in S_y X \subset T_y X$  being also a normal vector of this geodesic sphere, we have  $\hat{\eta} = cG_t z$  for some  $c \in \mathbf{R} - \{0\}$ . Starting from  $G_t^* p^* \xi = p^* \eta$ , we can argue in the same way to show that  $\hat{\xi} = c'z$  for some  $c' \in \mathbf{R} - \{0\}$ . Now we shall show  $c = c'$ . Let  $V$  be the vector field on  $SX$  which generates the flow  $G_t$ . Recall that  $p_* V(z) = z (z \in SX)$ . Then

$$\begin{aligned} c &= (G_t z, cG_t z) \\ &= \langle G_t z, \eta \rangle \\ &= \langle V(G_t z), p^* \eta \rangle \\ &= \langle G_{t*} V(z), p^* \eta \rangle \\ &= \langle V(z), G_t^* p^* \eta \rangle \\ &= \langle V(z), p^* \xi \rangle \\ &= \langle z, \xi \rangle \end{aligned}$$

$$= (z, c'z) = c'.$$

Hence if  $c$  is positive, then  $\hat{\eta} = cG_t z = \tilde{G}_t(cz) = \tilde{G}_t(\frac{c\xi}{\|c\xi\|})$ . Here  $\tilde{G}_t: TX \setminus 0 \rightarrow TX \setminus 0$  is the map defined by

$$\tilde{G}_t(\xi) = \|\xi\| G_t \left( \frac{\xi}{\|\xi\|} \right).$$

If  $c$  is negative, then  $\hat{\eta} = (-c)(-G_t z) = (-c)G_{-t}(-z) = \tilde{G}_{-t}(cz) = \tilde{G}_{-t}(\frac{c\xi}{\|c\xi\|})$ . Thus  $C = \Gamma_t \cup \Gamma_{-t}$ , where  $\Gamma_t$  is the graph of the diffeomorphism of  $T^*X \setminus 0$  which is induced from  $\tilde{G}_t$  by the usual isomorphism:  $T^*X \cong TX$ . Note that  $\Gamma_t \cap \Gamma_{-t} = \phi$ , since  $t \leq t_0$ .

2) Next we show that the condition i) of Theorem A is satisfied in the present case. Fix  $P_0 = (x_0, \xi_0) \times (z_0, \zeta_0) \times (z_0, \zeta_0) \times (y_0, \eta_0) \in C_1 \times C_2 \cap T^*X \times \Delta(T^*SX) \times T^*X$ . Let  $x = (x^i), y = (y^i)$  be local charts of  $X$  around  $x_0$  and  $y_0$ , respectively, and  $(x, \xi), (y, \eta)$  the local charts of  $T^*X$  induced by them. Furthermore let  $z = (z^k)$  be a local chart of  $SX$  around  $z_0$  and  $(z, \zeta)$  the local chart of  $T^*SX$  induced by  $z$ . We denote the functions  $x \circ \pi_1, \xi \circ \pi_1, z \circ \pi_2, \zeta \circ \pi_2, z \circ \pi_3, \zeta \circ \pi_3, y \circ \pi_4, \eta \circ \pi_4$ , respectively, by  $x, \xi, z_1, \zeta_1, z_2, \zeta_2, y, \eta, \pi_j$  being the natural projection of  $W = T^*X \times T^*SX \times T^*SX \times T^*X$  onto the  $j$ -th factor. Since  $C_1 \times C_2$  has a local parametrization:  $(z_1, \xi, z_2, \eta) \mapsto (pz_1, \xi) \times (z_1, p^*\xi) \times (z_2, G_t^*p^*\eta) \times (pG_t z_2, \eta)$ , we can take  $(z_1, \xi, z_2, \eta)$  as a local chart of  $C_1 \times C_2$  around  $P_0$ .

Now the local equations of  $T^*X \times \Delta(T^*SX) \times T^*X$  in  $W$  is given by

$$\begin{cases} z_1^k - z_2^k = 0 & 1 \leq k \leq 2n-1 \\ \zeta_1^k - \zeta_2^k = 0 & 1 \leq k \leq 2n-1. \end{cases}$$

In order to verify the condition i), it suffices to show that these  $2(2n-1)$  equations remain independent after restricted to  $C_1 \times C_2$ . Obviously  $dz_1^k - dz_2^k (1 \leq k \leq 2n-1)$  are linearly independent on  $C_1 \times C_2$  at  $P_0$ . Thus it suffices to see that  $d(\zeta_1^k - \zeta_2^k) = d((p^*\xi)^k - (G_t^*p^*\eta)^k) (1 \leq k \leq 2n-1)$  are linearly independent modulo  $dz_1^k, dz_2^k (1 \leq k \leq 2n-1)$ . We can write locally  $(p^*\xi)^k = \sum_j a_j^k(z_1) \xi^j, (G_t^*p^*\eta)^k = \sum_j b_j^k(z_2) \eta^j$ . Then

$$\begin{aligned} & d((p^*\xi)^k - (G_t^*p^*\eta)^k) \\ &= d(\sum_j (a_j^k(z_1) \xi^j - b_j^k(z_2) \eta^j)) \\ &\equiv \sum_j (a_j^k(z_1) d\xi^j - b_j^k(z_2) d\eta^j) \pmod{dz_1, dz_2}. \end{aligned}$$

Hence  $d((p^*\xi)^k - (G_t^*p^*\eta)^k) (1 \leq k \leq 2n-1)$  are linearly independent modulo  $dz_1, dz_2$  on  $C_1 \times C_2$  at  $P_0$  if and only if the rank of the matrix  $(a_j^k(z_0), b_j^k(z_0))$  is  $2n-1$ , which is equivalent to say that the dimension of the subspace  $U = \{p^*\xi + G_t^*p^*\eta; \xi \in T_{x_0}^*X, \eta \in T_{y_0}^*X\} \subset T_{z_0}^*SX$  is  $2n-1$ . On the other hand, in 1), we have shown that the pair  $(\xi, \eta) \in T_{x_0}^*X \times T_{y_0}^*X$  such that  $p^*\xi = G_t^*p^*\eta$  in  $T_{z_0}^*SX$  is de-

terminated by  $z_0$  up to scalar multiplications. This, together with the injectivity of  $G_t^*p^*: T_{z_0}^*X \rightarrow T_{z_0}^*SX$ , implies that the dimension of  $U$  equals  $2n-1$ . Thus the condition i) is verified in the present case.

3) Now we check the condition ii) of Theorem A. For any  $(x, \xi) \times (y, \eta) \in C$ , we have either  $\hat{\xi} = \tilde{G}_t(\hat{\eta})$  or  $\hat{\xi} = \tilde{G}_{-t}(\hat{\eta})$ . Let  $(z, \zeta) \in T^*SX$  be such that  $(x, \xi) \times (z, \zeta) \in C_1$  and  $(z, \zeta) \times (y, \eta) \in C_2$ . Then  $\zeta = p^*\xi$ , and from 1) it follows immediately that  $z = \hat{\xi}/\|\hat{\xi}\|$  if  $\hat{\xi} = \tilde{G}_t(\hat{\eta})$  and  $z = -\hat{\xi}/\|\hat{\xi}\|$  if  $\hat{\xi} = \tilde{G}_{-t}(\hat{\eta})$ . Thus, in view of  $\Gamma_t \cap \Gamma_{-t} = \emptyset$ ,  $C_1 \times C_2 \cap T^*X \times \Delta(T^*SX) \times T^*X \rightarrow C_1 \circ C_2$  is a diffeomorphism.

4) Finally we show that  $\Gamma_t \cup \Gamma_{-t}$  is the conormal bundle minus the zero section of  $\Delta_t \subset X \times X$ . It is obvious that the projection  $T^*(X \times X) \rightarrow X \times X$  maps  $\Gamma_t \cup \Gamma_{-t}$  onto  $\Delta_t$ . The fibre of  $\Gamma_t \cup \Gamma_{-t} \rightarrow \Delta_t$  is easily seen to be  $\mathbf{R} - \{0\}$ . Since  $\Gamma_t \cup \Gamma_{-t}$  is a Lagrangean submanifold of  $T^*(X \times X)$ , it follows that  $\Gamma_t \cup \Gamma_{-t}$  is contained as an open set in the conormal bundle of  $\Delta_t$ , whence  $\Gamma_t \cup \Gamma_{-t}$  must be the conormal bundle of  $\Delta_t$  minus the zero section.

This completes the proof of Theorem I'.

6. Using Theorem I', we get some information about the regularity of  $\tilde{L}_t$ . We quote the following

**Theorem B** ([3] Theorem 4.3.2). *Let  $X$  and  $Y$  be smooth manifolds and  $C \subset T^*X \times T^*Y$  a homogeneous canonical relation satisfying the following conditions:*

- i) *the projections  $C \rightarrow X, C \rightarrow Y$  have surjective differentials;*
- ii) *the differentials of the projections  $C \rightarrow T^*X$  and  $C \rightarrow T^*Y$  have rank at least  $k + \dim X$  and  $k + \dim Y$ , respectively.*

*Let  $m \leq \frac{1}{4}(2k - \dim X - \dim Y)$ . Then every  $A \in I^m(X \times Y, C')$  is a continuous operator:  $L_c^2(Y, \Omega_{\frac{1}{2}}) \rightarrow L_{loc}^2(X, \Omega_{\frac{1}{2}})$ .*

From this, it follows immediately the following

**Corollary.** *Under the assumptions of Theorem B,  $A$  is continuous:  $H_c^s(Y, \Omega_{\frac{1}{2}}) \rightarrow H_{loc}^{s+r}(X, \Omega_{\frac{1}{2}})$ , where  $r = -m + \frac{1}{4}(2k - \dim X - \dim Y)$ .*

In the case of the operator  $\tilde{L}_t$ , the condition i) is trivially satisfied and the condition ii) is valid with  $k = n$ , since  $\Gamma_t$  and  $\Gamma_{-t}$  are the graphs of diffeomorphisms:  $T^*X \setminus 0 \rightarrow T^*X \setminus 0$ . Furthermore,  $r = \frac{1}{2}(n-1) - \frac{1}{4}(2n - n - n) = \frac{1}{2}(n-1)$ .

Hence we have

**Theorem II.** *The spherical mean  $\tilde{L}_t(0 < t \leq t_0)$  is a continuous operator:  $H^s(X, \Omega_{\frac{1}{2}}) \rightarrow H^{s+\frac{1}{2}(n-1)}(X, \Omega_{\frac{1}{2}})$  for each  $s \in \mathbf{R}$ .*

**Corollary.** *If  $n \geq 2$ , then the operator  $\tilde{L}_t: L^2(X, \Omega_{\mathbb{H}^n}) \rightarrow L^2(X, \Omega_{\mathbb{H}^n})$  is compact for  $0 < t \leq t_0$ .*

**Corollary.** *If  $n \geq 2$ , then the eigenfunctions of non-zero eigenvalues of the operator  $\tilde{L}_t: L^2(X, \Omega_{\mathbb{H}^n}) \rightarrow L^2(X, \Omega_{\mathbb{H}^n})$  are smooth functions, for  $0 < t \leq t_0$ .*

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