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## COHEN-MACAULAY LOCAL RINGS OF EMBEDDING DIMENSION $e + d - k$

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### Abstract

In this paper, we prove the following. Let  $(R, \mathfrak{m})$  be a  $d$ -dimensional Cohen-Macaulay local ring with multiplicity  $e$  and embedding dimension  $v = e + d - k$ , where  $k \geq 3$  and  $e - k > 1$ . If  $\lambda(\mathfrak{m}^3/J\mathfrak{m}^2) = 1$  and  $\mathfrak{m}^3 \subseteq J\mathfrak{m}$ , where  $J$  is a minimal reduction of  $\mathfrak{m}$ , then  $3 \leq s \leq \tau + k - 1$ , where  $s$  is the degree of the  $h$ -polynomial of  $R$  and  $\tau$  is the Cohen-Macaulay type of  $R$ .

### 1. Introduction

Let  $(R, \mathfrak{m})$  be a  $d$ -dimensional Noetherian local ring of multiplicity  $e$ . The Hilbert function of  $R$  is by definition the Hilbert function of the associated graded ring of  $R$ :

$$G := \bigoplus_{n \geq 0} \mathfrak{m}^n / \mathfrak{m}^{n+1},$$

i.e.,

$$H_R(n) = \dim_{R/\mathfrak{m}} \mathfrak{m}^n / \mathfrak{m}^{n+1}.$$

The Hilbert series of  $R$  is the power series

$$P_R(z) = \sum_{n \geq 0} H_R(n)z^n.$$

It is known that there is a polynomial  $h(z) \in \mathbb{Z}[z]$  such that  $P_R(z) = h(z)/(1-z)^d$  and  $h(1) = e$ . This polynomial  $h(z) = h_0 + h_1z + \cdots + h_s z^s$  is called the  $h$ -polynomial of  $R$ .

Let  $(R, \mathfrak{m})$  be a  $d$ -dimensional Cohen-Macaulay local ring with embedding dimension  $v = e + d - k$ , where  $k \geq 3$ . Let  $J$  be a minimal reduction of  $\mathfrak{m}$ . Let  $\tau$  be the Cohen-Macaulay type of  $R$ ,  $h = v - d$  and  $v_i = \lambda(\mathfrak{m}^{i+1}/J\mathfrak{m}^i)$  for every  $i$ ; then there are at least two possible Hilbert series of  $R/J$ :  $P_{R/J}(z) = 1 + hz + z^2 + \cdots + z^k$  and  $P_{R/J}(z) = 1 + hz + (k-1)z^2$ . In the first case,  $R$  is stretched (cf. definition below) and we have  $\mathfrak{m}^k \not\subseteq J\mathfrak{m}$ ; in the second case, following [3], we say that  $R$  is short and we have  $\mathfrak{m}^3 \subseteq J\mathfrak{m}$  and  $v_1 = k - 1$ .

Let  $(R, \mathfrak{m})$  be a  $d$ -dimensional local Cohen-Macaulay ring of multiplicity  $e$  and embedding dimension  $v$ . If  $d = 0$ , then  $R$  is called *stretched* if  $e - v$  is the least integer  $i$  such that  $\mathfrak{m}^{i+1} = 0$ . If  $d > 0$ , then  $R$  is *stretched* if there is a minimal reduction  $J$  of  $\mathfrak{m}$  such that  $R/J$  is stretched (cf. [6]), or equivalently,  $(\mathfrak{m}^2 + J)/J$  is principal. Regular local rings are not stretched since fields are not stretched. However, for any  $d$ -dimensional local Cohen-Macaulay ring  $(R, \mathfrak{m})$  having infinite residue field, if  $v = e + d - 1$  with  $e > 1$  or  $v = e + d - 2$  with  $e > 2$ , then  $R$  is stretched. Moreover, if  $v = e + d - 3$  and  $R$  is Gorenstein, then  $R$  is stretched. These stretched rings have been studied in [6], [7] and [8]. In [4], Rossi and Valla extended the notion *stretched*. There they defined, for each  $\mathfrak{m}$ -primary ideal  $I$ ,  $I$  is *stretched* if there is a minimal reduction  $J$  of  $I$  such that  $I^2 \cap J = IJ$  and  $\lambda(I^2/(JI + I^3)) = 1$ .

In [6], Sally studied the structure of stretched local Gorenstein rings, and use it to show in [8] that if  $(R, \mathfrak{m})$  is a  $d$ -dimensional Gorenstein local ring with embedding dimension  $v = e + d - 3$ , then the associated graded ring of  $R$  is Cohen-Macaulay. This result has been generalized by Rossi and Valla in [3] as follows.

**Theorem 1.1** ([3, Theorem 2.6]). *If  $(R, \mathfrak{m})$  is a  $d$ -dimensional Cohen-Macaulay local ring of multiplicity  $e = h + 3$  and  $\lambda(\mathfrak{m}^3/J\mathfrak{m}^2) = 1$ , then  $s \leq \tau + 2$ , where  $s$  is the degree of the  $h$ -polynomial of  $R$ .*

In [4], Rossi and Valla generalized Theorem 1.1 to stretched  $\mathfrak{m}$ -primary ideals. In this note, we are able to generalize Theorem 1.1 in a different manner in Section 4 as follows. In which, we do not assume  $R$  is stretched. In stead, we assume that  $R$  is short and  $v_2 = 1$ .

**Theorem 1.2.** *Let  $(R, \mathfrak{m})$  be a  $d$ -dimensional Cohen-Macaulay local ring of multiplicity  $e = h + k$ , where  $k \geq 3$  and  $e - k > 1$ . If  $\lambda(\mathfrak{m}^3/J\mathfrak{m}^2) = 1$  and  $\mathfrak{m}^3 \subseteq J\mathfrak{m}$ , where  $J$  is a minimal reduction of  $\mathfrak{m}$ , then  $3 \leq s \leq \tau + k - 1$ , where  $s$  is the degree of the  $h$ -polynomial of  $R$ .*

In the final section, we provide several examples to answer some questions raised by Rossi and Valla in [3].

## 2. One dimensional local Cohen-Macaulay ring

We state several facts of one dimensional local Cohen-Macaulay rings. These results can be derived easily from [1] and [5].

**Lemma 2.1.** *Let  $(R, \mathfrak{m})$  be a one dimensional local Cohen-Macaulay ring; then  $\lambda(\mathfrak{m}^n/\mathfrak{m}^{n+1}) = e - \lambda(\mathfrak{m}^{n+1}/J\mathfrak{m}^n)$ , where  $J$  is any minimal reduction of  $\mathfrak{m}$ .*

**Lemma 2.2.** *Let  $(R, \mathfrak{m})$  be a one dimensional Cohen-Macaulay local ring with embedding dimension 2. Then  $G(R)$  is Gorenstein.*

**Corollary 2.3.** *Let  $(R, \mathfrak{m})$  be a  $d$ -dimensional Cohen-Macaulay local ring with embedding dimension  $d + 1$ . Then  $G(R)$  is Gorenstein.*

**3. Cohen-Macaulay local rings of embedding dimension  $e + d - k$**

Let  $(R, \mathfrak{m})$  be a  $d$ -dimensional Cohen-Macaulay local ring with embedding dimension  $v = e + d - k$ , where  $k \geq 3$  and  $e - k > 1$ . Let  $\tau$  be the Cohen-Macaulay type of  $R$ ,  $h = v - d$  and  $v_i = \lambda(\mathfrak{m}^{i+1}/J\mathfrak{m}^i)$  for every  $i$ . Let  $J$  be a minimal reduction of  $\mathfrak{m}$ ; then one of the possible Hilbert series of  $R/J$  is  $1 + hz + (k - 1)z^2$ . In this case,  $\mathfrak{m}^3 \subseteq J\mathfrak{m}$  and  $v_1 = k - 1$ . If  $k = 3$ , it is shown in [3, Theorem 2.6] that if  $v_2 = 1$  then  $s \leq \tau + 2$ , where  $s$  is the degree of the  $h$ -polynomial of  $R$ . We are able to generalize this result in this section.

**Theorem 3.1.** *Let  $(R, \mathfrak{m})$  be a  $d$ -dimensional Cohen-Macaulay local ring of multiplicity  $e = h + k$ , where  $k \geq 3$  and  $e - k > 1$ . If  $\lambda(\mathfrak{m}^3/J\mathfrak{m}^2) = 1$  and  $\mathfrak{m}^3 \subseteq J\mathfrak{m}$ , where  $J$  is a minimal reduction of  $\mathfrak{m}$ , then  $3 \leq s \leq \tau + k - 1$ , where  $s$  is the degree of the  $h$ -polynomial of  $R$ .*

REMARK 3.2. (i) Notice that the assumption  $v_2 = 1$  ensures that the depth of  $G$  is at least  $d - 1$  (cf. [3]). Therefore to show Theorem 3.1, we need only to consider the case when  $d = 1$ .

(ii) If  $d = 1$ , then  $s$  is the least integer for which  $\lambda(\mathfrak{m}^s/\mathfrak{m}^{s+1}) = e$ .

(iii) Notice that  $\lambda(\mathfrak{m}^2/J\mathfrak{m}) = k - 1$ . Moreover, if  $\mathfrak{m}^2 = J\mathfrak{m} + (u_1, \dots, u_{k-1})$ , then  $\{u_1, \dots, u_{k-1}\}$  is part of a generating set of the socle of  $R$ .

By Remark 3.2, we may assume from now on that  $d = 1$  and  $v_2 = 1$ .

**Lemma 3.3.** *Let  $r$  be the reduction number of  $\mathfrak{m}$  with respect to  $J$ . If  $r \leq 3$ , then Theorem 3.1 holds.*

Proof. If  $r \leq 3$ , then  $\mathfrak{m}^4 = J\mathfrak{m}^3$ , so that  $\lambda(\mathfrak{m}^3/\mathfrak{m}^4) = e$ , it follows that  $s \leq 3 \leq \tau + k - 1$  by the choice of  $s$ . □

By Lemma 3.3, we may assume in the sequel that  $r \geq 4$ .

**Lemma 3.4.** *The following hold for  $R$ :*

(i) *If  $\mathfrak{m}^3 = J\mathfrak{m}^2 + (ab)$  for some  $b \in \text{mathfrac}{\mathfrak{m}^2}$  and  $a \in \mathfrak{m}$ , then  $\mathfrak{m}^{i+1} = J\mathfrak{m}^i + (a^{i-1}b)$  for every  $i \geq 2$ .*

(ii) *If  $y\mathfrak{m}^2 \not\subseteq J\mathfrak{m}^2$  for some  $y \in \mathfrak{m}$ , then  $y^3 \notin J\mathfrak{m}^2$ . In particular, there is an element  $y \in \mathfrak{m}$  such that  $\mathfrak{m}^{i+1} = J\mathfrak{m}^i + (y^{i+1})$  for every  $i \geq 2$ .*

Proof. (i) If  $\mathfrak{m}^{i+1} = J\mathfrak{m}^i + (a^{i-1}b)$  for some  $i \geq 2$ , then  $\mathfrak{m}^{i+2} = J\mathfrak{m}^{i+1} + a^{i-1}b\mathfrak{m} \subseteq J\mathfrak{m}^{i+1} + a^{i-1}\mathfrak{m}^3 = J\mathfrak{m}^{i+1} + (a^i b) \subseteq \mathfrak{m}^{i+2}$ .

(ii) Suppose that  $ym^2 \not\subseteq Jm^2$ . Then there are  $u, v \in \mathfrak{m}$  such that  $uvy \notin Jm^2$  and  $m^3 = Jm^2 + (yuv)$ . Therefore,  $m^4 = Jm^3 + (y^2uv)$ . It follows that  $y^2u \notin Jm^2$  and  $m^3 = Jm^2 + (y^2u)$ . Thus,  $m^4 = Jm^3 + (y^3u)$  and then  $y^3 \notin Jm^2$ . Now, choose  $y \in \mathfrak{m}$  such that  $ym^2 \not\subseteq Jm^2$ , then  $m^{i+1} = Jm^i + (y^{i+1})$  for every  $i \geq 2$ .  $\square$

**Lemma 3.5.** *Let  $J = (x)$  be a minimal reduction of  $\mathfrak{m}$ . If there is an element  $y \in \mathfrak{m}$  such that  $m^{i+1} = Jm^i + (y^{i+1})$  for every  $i \geq 2$ , then  $y^l x^t$  is a generator of the module  $(J^l m^l + m^{l+t+1})/(J^{t+1} m^{l-1} + m^{l+t+1})$  whenever  $2 \leq l < r$ , where  $r$  is the reduction number of  $\mathfrak{m}$  with respect to  $J$ .*

*Proof.* If not,  $y^l x^t \in J^{t+1} m^{l-1} + m^{l+t+1}$ , so that  $y^r x^t \in x^{t+1} m^{r-1}$ , it follows that  $y^r \in Jm^{r-1}$ , a contradiction. Therefore, the conclusion holds.  $\square$

**Theorem 3.6.** *Let  $(R, \mathfrak{m})$  be a one dimensional Cohen-Macaulay local ring of multiplicity  $e = h + k$ , where  $k \geq 3$  and  $e - k > 1$ . Assume that  $\lambda(m^3/Jm^2) = \lambda(m^4/Jm^3) = 1$  and  $m^3 \subseteq Jm$ , where  $J = (x)$  is a minimal reduction of  $\mathfrak{m}$ . Then there is a basis  $\{x, y_1, \dots, y_\tau, z_1, \dots, z_{e-\tau-k}\}$  of  $\mathfrak{m}$ , elements  $u_{t+1}, \dots, u_{k-1}$  contained in  $\mathfrak{m}$  and elements  $\{c_{ij} \mid i = 1, \dots, k-1, j = 1, \dots, j_i\}$  contained in the ideal  $(y_2, \dots, y_{k-1}, z_1, \dots, z_{e-\tau-k})$  with  $\sum_{i=1}^{k-1} j_i(k-i) = e - \tau - k$  such that  $J = (x)$  and the following hold:*

- (i)  $m^{i+1} = Jm^i + (y_1^{i+1})$  for every  $i \geq 2$ .
- (ii)  $m^2 = Jm + (y_1^2, y_1 y_2, \dots, y_1 y_t, y_{t+1} u_{t+1}, \dots, y_{k-1} u_{k-1})$ , where  $t = \lambda((y_1 m + Jm)/Jm)$ .
- (iii)  $\{y_1^2, y_1 y_2, \dots, y_1 y_t, y_{t+1} u_{t+1}, \dots, y_{k-1} u_{k-1}, y_k, \dots, y_\tau\}$  is a generating set of the socle of  $R$ .
- (iv)  $y_1 y_i \in Jm$  for  $i \geq t+1$  and  $y_1 z_i \in Jm$  for every  $i$ .
- (v)  $y_i m^3 \subseteq Jm^3$  for every  $i \geq 2$  and  $z_i m^3 \subseteq Jm^3$  for every  $i \geq 1$ .
- (vi)  $\{z_1, \dots, z_{e-\tau-k}\} = \bigcup_{i,j,k} \{z_{ij}^{(l)}\}$ ,  $\lambda((c_{ij} m + Jm)/Jm) = k - i$  and  $m^2 = Jm + \sum_{l=1}^{k-i} z_{ij}^{(l)} c_{ij}$  for every  $i = 1, \dots, k-1$  and  $j = 1, \dots, j_i$ .
- (vii)  $c_{ij} z_{i'j'}^{(l)} \in Jm$  if  $i < i'$  or  $i = i'$  but  $j < j'$ .
- (viii)  $y_1^3 \notin J(z_1, \dots, z_{e-\tau-k}) + Jm^2$ .

*Proof.* By Lemma 3.4, there is an element  $y_1 \in \mathfrak{m}$  such that (i) hold. Let  $t = \lambda((y_1 m + Jm)/Jm)$ ; then there are  $y_2, \dots, y_{k-1}, u_{t+1}, \dots, u_{k-1} \in \mathfrak{m}$  such that  $m^2 = Jm + (y_1^2, y_1 y_2, \dots, y_1 y_t, y_{t+1} u_{t+1}, \dots, y_{k-1} u_{k-1})$  and  $y_1 m + Jm = (y_1^2, y_1 y_2, \dots, y_1 y_t) + Jm$ . We may assume that  $y_1^2 y_i \in Jm^2$  for  $2 \leq i \leq t$  by replacing  $y_i$  by  $y_i + \lambda y_1$  if necessary, and assume that  $y_1 y_j \in Jm$  for  $t+1 \leq j \leq k-1$  by replacing  $y_j$  by  $y_j + \lambda_1 y_1 + \dots + \lambda_t y_t$  if necessary. It follows that  $y_i m^3 = (y_i y_1^3) + Jm^3 = Jm^3$  for every  $i \leq k-1$ . Since the Cohen-Macaulay type of  $R$  is  $\tau$  and  $\{y_1^2, y_1 y_2, \dots, y_1 y_t, y_{t+1} u_{t+1}, \dots, y_{k-1} u_{k-1}\}$  is part of a generating set of the socle of  $R$ , we may choose  $y_k, \dots, y_\tau, z_1, \dots, z_{e-\tau-k} \in \mathfrak{m}$  such that  $\{y_k, \dots, y_\tau, z_1, \dots, z_{e-\tau-k}\}$  is part of a generating set of  $\mathfrak{m}$  and  $\{y_1^2, y_1 y_2, \dots, y_1 y_t, y_{t+1} u_{t+1}, \dots, y_{k-1} u_{k-1}, y_k, \dots, y_\tau\}$  is a generating set of the socle of  $R$ . If  $z_i y_1 \notin Jm$  for some  $i$ , then we may replace  $z_i$  by

$z_i + \alpha_1 y_1 + \dots + \alpha_t y_t$  if necessary and assume that  $z_i y_1 \in J\mathfrak{m}$  for every  $i$ . Therefore  $z_i \mathfrak{m}^3 \subseteq J\mathfrak{m}^3 + z_i y_1 \mathfrak{m}^2 \subseteq J\mathfrak{m}^3$ . Hence, the basis  $\{x, y_1, \dots, y_t, z_1, \dots, z_{e-\tau-k}\}$  of  $\mathfrak{m}$  satisfies (i) to (v) so far.

**Claim.** For any integer  $i = 1, \dots, k-1$ , there is an integer  $j_i$ , a basis  $\{x, y_1, \dots, y_t, z_1, \dots, z_{e-\tau-k}\}$  of  $\mathfrak{m}$  and elements  $\{c_{ij} \mid j = 1, \dots, j_i\}$  contained in the ideal  $(y_2, \dots, y_{k-1}, z_1, \dots, z_{e-\tau-k})$  such that not only (i) to (v) but also the following hold:

- (a)  $\lambda((c_{ij}\mathfrak{m} + J\mathfrak{m})/J\mathfrak{m}) = k - i$ ,  $\mathfrak{m}^2 = J\mathfrak{m} + (z_{ij}^{(1)} c_{ij}, \dots, z_{ij}^{(k-i)} c_{ij})$ .
- (b)  $c_{ij} z_{ij}^{(l)} \in J\mathfrak{m}$  for every  $l$  if  $j < j'$  and  $c_{ij} z \in J\mathfrak{m}$  for every generator of the ideal generated by  $S_i$ , where  $S_i = \{z_1, \dots, z_{e-\tau-k}\} - \{z_{i'j}^{(l)} \mid 1 \leq i' \leq i, 1 \leq j \leq j_i, 1 \leq l \leq k - i\}$ .

Note that (vi) and (vii) follows from the Claim.

**Proof of the Claim.** We proceed by induction on  $i$ . Let  $z$  be any generator of the ideal  $(z_1, \dots, z_{e-\tau-k})$ . Since  $y_1 z, y_i z \in J\mathfrak{m}$  for every  $i \geq k$ , there is an element  $c \in (y_2, \dots, y_{k-1}, z_1, \dots, z_{e-\tau-k})$  such that  $c z \notin J\mathfrak{m}$ . If for any generating set  $\{z'_1, \dots, z'_{e-\tau-k}\}$  of the ideal  $(z_1, \dots, z_{e-\tau-k})$  there is no element  $c \in (y_2, \dots, y_{k-1}, z_1, \dots, z_{e-\tau-k})$  such that  $\mathfrak{m}^2 = (c z'_1, \dots, c z'_{k-1}) + J\mathfrak{m}$ , then the Claim holds for  $i = 1$ . If not, we may assume that  $\mathfrak{m}^2 = (c_{11} z_1, \dots, c_{11} z_{k-1}) + J\mathfrak{m}$  for some  $c_{11} \in (y_2, \dots, y_{k-1}, z_1, \dots, z_{e-\tau-k})$ . Set  $z_{11}^{(l)} = z_l$ . Let  $z$  be any generator of the ideal  $(z_k, \dots, z_{e-\tau-k})$ . If  $c_{11} z \notin J\mathfrak{m}$ , then there are elements  $\alpha_i$  such that  $c_{11} z - (\sum_{i=1}^{k-1} c_{11} z_{11}^{(i)}) \in J\mathfrak{m}$ , so that we may replace  $z$  by  $\sum_{i=1}^{k-1} z_{11}^{(i)}$  if necessary and assume that  $c_{11} z \in J\mathfrak{m}$ . If for any generating set  $\{z'_k, \dots, z'_{e-\tau-k}\}$  of the ideal  $(z_k, \dots, z_{e-\tau-k})$  there is no element  $c \in (y_2, \dots, y_{k-1}, z_1, \dots, z_{e-\tau-k})$  such that  $\mathfrak{m}^2 = (c z'_k, \dots, c z'_{2k-2}) + J\mathfrak{m}$ , then again the Claim holds for  $i = 1$ . If not, we may use the same trick to find  $c_{12}, c_{13}, \dots$  so that the Claim holds for  $i = 1$ .

Suppose now we have shown that the Claim holds for any integer  $i' \leq i$  for some  $i \geq 1$ . Let  $m = \sum_{i'=1}^i j_{i'}(k - i')$  and  $S_i = \{z_{m+1}, \dots, z_{e-\tau-k}\}$ . If for any generating set  $\{z'_{m+1}, \dots, z'_{e-\tau-k}\}$  of the ideal generated by  $S_i$  there is no element  $c \in (y_2, \dots, y_{k-1}, z_1, \dots, z_{e-\tau-k})$  such that  $\mathfrak{m}^2 = (c z'_{m+1}, \dots, c z'_{m+k-i-1}) + J\mathfrak{m}$ , then the Claim holds for  $i + 1$ . If not, we may assume that for some  $c_{i+1,1} \in (y_2, \dots, y_{k-1}, z_1, \dots, z_{e-\tau-k})$ ,  $\mathfrak{m}^2 = (c_{i+1,1} z_{m+1}, \dots, c_{i+1,1} z_{m+k-i-1}) + J\mathfrak{m}$ . Set  $z_{i+1,1}^{(l)} = z_{m+l}$ . As before, we may assume that  $c_{i+1,1} z \in J\mathfrak{m}$  for every generator  $z$  of the ideal  $(z_{m+k-i}, \dots, z_{e-\tau-k})$ . If for any generating set  $\{z'_{m+k-i}, \dots, z'_{e-\tau-k}\}$  of the ideal  $(z_{m+k-i}, \dots, z_{e-\tau-k})$  there is no element  $c \in (y_2, \dots, y_{k-1}, z_1, \dots, z_{e-\tau-k})$  such that  $\mathfrak{m}^2 = (c z'_{m+k-i}, \dots, c z'_{m+2k-2i-2}) + J\mathfrak{m}$ , then again the Claim holds for  $i + 1$ . If not, we may use the same trick to find  $c_{i+1,2}, c_{i+1,3}, \dots$  so that the Claim holds for  $i + 1$ . The Claim is now fulfilled.  $\square$

To finish the proof, assume that  $y_1^3 \in J(z_1, \dots, z_{e-\tau-k}) + J\mathfrak{m}^2$ . Then there are  $\delta_i \in R$  not all in  $\mathfrak{m}$  such that  $y_1^3 - \sum_{i=1}^{e-\tau-k} \delta_i z_i x \in J\mathfrak{m}^2$ . Let  $t$  be the smallest integer

for which  $\delta_t$  is a unit; then  $y_1^3 - \sum_{i=t}^{e-\tau-k} \delta_i z_i x \in Jm^2$ . Let  $z = c_{ij}$  if  $z_t = z_{ij}^{(l)}$  for some  $l$ ; then  $z \cdot (\sum_{i=t}^{e-\tau-k} \delta_i z_i) x - zy_1^3 \in Jm^3$ , so that  $z \cdot (\sum_{i=t}^{e-\tau-k} \delta_i z_i) \in m^3 \subseteq Jm$  as  $z \in (y_2, \dots, y_{k-1}, z_1, \dots, z_{e-\tau-k})$ . However,  $z \cdot (\sum_{i=t}^{e-\tau-k} \delta_i z_i) \notin Jm$  by the Claim, a contradiction. Therefore (viii) holds.  $\square$

Now, we are ready for:

Proof of Theorem 3.1. From the above, we may assume that  $d = 1$ ,  $\tau \geq 2$  and  $r \geq 4$ , where  $r$  is the reduction number of some minimal reduction  $J$  of  $m$ . By Theorem 3.6, there is a basis  $\{x, y_1, \dots, y_\tau, z_1, \dots, z_{e-\tau-k}\}$  of  $m$ , elements  $u_{t+1}, \dots, u_{k-1}$  contained in  $m$  and elements  $\{c_{ij} \mid i = 1, \dots, k-1, j = 1, \dots, j_i\}$  contained in the ideal  $(y_2, \dots, y_{k-1}, z_1, \dots, z_{e-\tau-k})$  with  $\sum_{i=1}^{k-1} j_i(k-i) = e - \tau - k$  such that  $J = (x)$  and the following hold:

- (i)  $m^{i+1} = Jm^i + (y_1^{i+1})$  for every  $i \geq 2$ .
- (ii)  $m^2 = Jm + (y_1^2, y_1 y_2, \dots, y_1 y_t, y_{t+1} u_{t+1}, \dots, y_{k-1} u_{k-1})$ , where  $t = \lambda((y_1 m + Jm)/Jm)$ .
- (iii)  $\{y_1^2, y_1 y_2, \dots, y_1 y_t, y_{t+1} u_{t+1}, \dots, y_{k-1} u_{k-1}, y_k, \dots, y_\tau\}$  is a generating set of the socle of  $R$ .
- (iv)  $y_1 y_i \in Jm$  for  $i \geq t+1$  and  $y_1 z_i \in Jm$  for every  $i$ .
- (v)  $y_i m^3 \subseteq Jm^3$  for every  $i \geq 2$  and  $z_i m^3 \subseteq Jm^3$  for every  $i \geq 1$ .
- (vi)  $\{z_1, \dots, z_{e-\tau-k}\} = \bigcup_{i,j,k} \{z_{ij}^{(l)}\}$ ,  $\lambda((c_{ij} m + Jm)/Jm) = k - i$  and  $m^2 = Jm + \sum_{l=1}^{k-i} z_{ij}^{(l)} c_{ij}$  for every  $i = 1, \dots, k-1$  and  $j = 1, \dots, j_i$ .
- (vii)  $c_{ij} z_{i'j'}^{(l)} \in Jm$  if  $i < i'$  or  $i = i'$  but  $j < j'$ .
- (viii)  $y_1^3 \notin J(z_1, \dots, z_{e-\tau-k}) + Jm^2$ .

If  $\tau \geq h$ , then  $s \leq e - 1 = h + k - 1 \leq \tau + k - 1$  by [2] and we are done. Therefore, we may assume that  $\tau < h$ . To show that  $s \leq \tau + k - 1$ , it is enough to show that  $\lambda(m^{\tau+k-1}/m^{\tau+k}) = e$  by Remark 3.2 (ii). Moreover, by Lemma 3.5,  $\{y_1^{\tau+k-1}, y_1^{\tau+k-2} x, \dots, y_1^2 x^{\tau+k-3}\}$  are generators of the module  $m^{\tau+k-1}/(J^{\tau+k-2} m + m^{\tau+k})$ , therefore to show that  $\lambda(m^{\tau+k-1}/m^{\tau+k}) = e$  it is enough to show that

$$\{y_1 x^{\tau+k-2}, x^{\tau+k-1}, z_1 x^{\tau+k-2}, \dots, z_{e-\tau-k} x^{\tau+k-2}\}$$

is a linearly independent set in  $(x^{\tau+k-2} m + m^{\tau+k})/m^{\tau+k}$ .

Suppose not, there are  $\alpha, \beta, \delta_i$  in  $R$  not all in  $m$  such that

$$\alpha y_1 x^{\tau+k-2} + \beta x^{\tau+k-1} + \sum_{i=1}^{e-\tau-k} \delta_i z_i x^{\tau+k-2} \in m^{\tau+k}.$$

Then

$$\alpha y_1^r x^{\tau+k-2} + \beta y_1^{r-1} x^{\tau+k-1} + \sum_{i=1}^{e-\tau-k} \delta_i z_i x^{\tau+k-2} y_1^{r-1} \in m^{\tau+r+k-1},$$

so that  $\alpha y_1^r x^{\tau+k-2} \in x^{\tau+k-1} m^{r-1}$  as  $y_1 z_i \in Jm$ , it follows that  $\alpha \in m$  by the choice of  $r$ . Therefore  $\beta x^{\tau+k-1} + \sum_{i=1}^{e-\tau-k} \delta_i z_i x^{\tau+k-2} \in m^{\tau+k}$ . If  $\delta_i \in m$  for every  $i$ , then  $x^{\tau+k-1} \in m^{\tau+k}$ , which is impossible. So, there is an integer  $i$  such that  $\delta_i$  is a unit. By replacing  $z_i$  by  $z_i + \beta/\delta_i x$ , we may assume that  $\beta \in m$ . Hence  $\sum_{i=1}^{e-\tau-k} \delta_i z_i x^{\tau+k-2} \in m^{\tau+k}$ . Let  $t$  be the smallest integer for which  $\delta_t$  is a unit; then  $\sum_{i=t}^{e-\tau-k} \delta_i z_i x^{\tau+k-2} \in m^{\tau+k}$ .

Let  $\alpha \leq \tau+k$  be the integer such that  $\sum_{i=t}^{e-\tau-k} \delta_i z_i x^{\tau+k-2} \in J^{\tau+k-\alpha} m^\alpha - J^{\tau+k+1-\alpha} m^{\alpha-1}$ . If  $\sum_{i=t}^{e-\tau-k} \delta_i z_i x^{\tau+k-2} \in J^{\tau+k-3} m^3$ , then  $\sum_{i=t}^{e-\tau-k} \delta_i z_i x \in m^3 = (y_1^3) + Jm^2$ , so that  $\sum_{i=t}^{e-\tau-k} \delta_i z_i x \in Jm^2$  by (viii), it follows that  $\sum_{i=t}^{e-\tau-k} \delta_i z_i \in m^2$ , a contradiction. Therefore,  $\alpha \geq 4$ . Since  $m^\alpha = (y^\alpha) + Jm^{\alpha-1}$  and  $\lambda(m^\alpha/Jm^{\alpha-1}) = 1$ , there is a unit  $\lambda_1$  such that

$$(1) \quad \sum_{i=t}^{e-\tau-k} \delta_i z_i x^{\alpha-2} - \lambda_1 y_1^\alpha \in Jm^{\alpha-1}.$$

Let  $z = c_{ij}$  if  $z_t = z_{ij}^{(l)}$ ; then  $z \cdot (\sum_{i=t}^{e-\tau-k} \delta_i z_i) \notin Jm$  by (vi) and (vii). Moreover,

$$z \left( \sum_{i=t}^{e-\tau-k} \delta_i z_i x^{\alpha-2} \right) - \lambda_1 y_1^\alpha z \in Jm^\alpha.$$

Furthermore,  $y_1^3 z \in Jm^3$  by (v), we have  $z(\sum_{i=t}^{e-\tau-k} \delta_i z_i x^{\alpha-3}) \in m^\alpha$ . Therefore, there is an element  $\lambda_2$  of  $R$  such that

$$(2) \quad z \left( \sum_{i=t}^{e-\tau-k} \delta_i z_i x^{\alpha-3} \right) - \lambda_2 y_1^\alpha \in Jm^{\alpha-1}.$$

From (1) and (2), we see that there is an element  $\lambda_3$  of  $R$  such that

$$(z - \lambda_3 x) \left( \sum_{i=t}^{e-\tau-k} \delta_i z_i \right) x^{\alpha-4} \in m^{\alpha-1}.$$

Let  $\beta \leq \alpha - 4 \leq \tau + k - 4$  be the non-negative integer such that

$$(z - \lambda_3 x) \left( \sum_{i=t}^{e-\tau-k} \delta_i z_i \right) x^\beta \in m^{\beta+3} \setminus Jm^{\beta+2}.$$

Since  $z \cdot (\sum_{i=t}^{e-\tau-k} \delta_i z_i) \notin Jm$ ,  $(z - \lambda_3 x)(\sum_{i=t}^{e-\tau-k} \delta_i z_i) \notin Jm^2$ ,  $\beta$  exists. Moreover, there is a unit  $\lambda_4$  of  $R$  such that

$$(3) \quad (z - \lambda_3 x) \left( \sum_{i=t}^{e-\tau-k} \delta_i z_i \right) x^\beta - \lambda_4 y_1^{\beta+3} \in Jm^{\beta+2}.$$



On the other hand, from (1), we have

$$y_1^{r-\alpha+1} \left( \sum_{i=t}^{e-\tau-k} \delta_i z_i \right) x^{\alpha-2} - \lambda_1 y_1^{r+1} \in J\mathfrak{m}^r,$$

or equivalently,

$$y_1^{r-\alpha+1} \left( \sum_{i=t}^{e-\tau-k} \delta_i z_i \right) x^{\alpha-3} \in \mathfrak{m}^r.$$

Since  $\mathfrak{m}^r = (y_1^r) + J\mathfrak{m}^{r-1}$ , there is an element  $\lambda_5$  of  $R$  such that

$$(4) \quad y_1^{r-\alpha+1} \left( \sum_{i=t}^{e-\tau-k} \delta_i z_i \right) x^{\alpha-3} - \lambda_5 y_1^r \in J\mathfrak{m}^{r-1}.$$

However, from (1), we have

$$(5) \quad y_1^{r-\alpha} \left( \sum_{i=t}^{e-\tau-k} \delta_i z_i \right) x^{\alpha-2} - \lambda_1 y_1^r \in J\mathfrak{m}^{r-1}$$

Thus, from (4) and (5), we obtain that

$$(6) \quad y_1^{r-\alpha} (y_1 - \lambda_6 x) \left( \sum_{i=t}^{e-\tau-k} \delta_i z_i \right) x^{\alpha-4} \in \mathfrak{m}^{r-1},$$

for some element  $\lambda_6$  of  $R$ . Now, if we can show that

$$(7) \quad \widetilde{y_1^{r-\beta-3}} \left( \sum_{i=t}^{e-\tau-k} \delta_i z_i \right) x^\beta \in \mathfrak{m}^{r-1}$$

for some element  $\widetilde{y_1^{r-\beta-3}} \in \mathfrak{m}^{r-\beta-3} \setminus J\mathfrak{m}^{r-\beta-4}$ , then from (3) and (7), we see that

$$(z - \lambda_3 x) \widetilde{y_1^{r-\beta-3}} \left( \sum_{i=t}^{e-\tau-k} \delta_i z_i \right) x^\beta - \lambda_4 y_1^{\beta+3} \widetilde{y_1^{r-\beta-3}} \in J\mathfrak{m}^{r-1}$$

and  $(z - \lambda_3 x) \widetilde{y_1^{r-\beta-3}} \left( \sum_{i=t}^{e-\tau-k} \delta_i z_i \right) x^\beta \in z\mathfrak{m}^{r-1} + J\mathfrak{m}^{r-1} = J\mathfrak{m}^{r-1}$  by (v), therefore  $\lambda_4 y_1^{\beta+3} \widetilde{y_1^{r-\beta-3}} \in J\mathfrak{m}^{r-1}$ , which contradicts to the choice of  $r$ . Hence, we conclude that  $\{y_1 x^{\tau+k-2}, x^{\tau+k-1}, z_1 x^{\tau+k-2}, \dots, z_{e-\tau-k} x^{\tau+k-2}\}$  is a linearly independent set in  $(x^{\tau+k-2}\mathfrak{m} + \mathfrak{m}^{\tau+k})/\mathfrak{m}^{\tau+k}$ .

Finally, by (6), we may prove (7) by reverse induction. Suppose we have shown that for some  $\delta, \beta < \delta \leq \alpha - 4$ ,

$$\widetilde{y_1^{r-\delta-3}} \left( \sum_{i=t}^{e-\tau-k} \delta_i z_i \right) x^\delta \in \mathfrak{m}^{r-1}$$

for some element  $\widetilde{y_1^{r-\delta-3}} \in \mathfrak{m}^{r-\delta-3} \setminus J\mathfrak{m}^{r-\delta-4}$ . Then there is an element  $\lambda_6 \in R$  such that

$$(8) \quad y_1 \widetilde{y_1^{r-\delta-3}} \left( \sum_{i=t}^{e-\tau-k} \delta_i z_i \right) x^\delta - \lambda_6 y_1^r \in J\mathfrak{m}^{r-1}.$$

From (5) and (8), we see that

$$\widetilde{y_1^{r-\delta-2}} \left( \sum_{i=t}^{e-\tau-k} \delta_i z_i \right) x^\delta \in J\mathfrak{m}^{r-1}$$

for some element  $\widetilde{y_1^{r-\delta-2}} \in \mathfrak{m}^{r-\delta-2} \setminus J\mathfrak{m}^{r-\delta-3}$ , it follows that

$$\widetilde{y_1^{r-\delta-2}} \left( \sum_{i=t}^{e-\tau-k} \delta_i z_i \right) x^{\delta-1} \in \mathfrak{m}^{r-1}. \quad \square$$

We end this section by providing the following example.

EXAMPLE 3.7. Let  $K$  be a field and  $R = K[[x, y, z_1, \dots, z_{k-1}]]/I$ , where  $I$  is the ideal of  $R$  generated by the set

$$\{z_1^3 - xy, y^2, yz_1, \dots, yz_{k-1}, z_1z_2, \dots, z_1z_{k-1}\} \cup \{z_i z_j \mid 2 \leq i \leq j \leq k-1\}.$$

The it is easy to see the following hold:

- (i)  $R$  is a 1-dimensional Cohen-Macaulay local ring with maximal ideal  $\mathfrak{m} = (x, y, z_1, \dots, z_{k-1})/I$ .
- (ii)  $x$  is a regular element of  $R$  and  $xR$  is a minimal reduction of  $\mathfrak{m}$ .
- (iii)  $v = k + 1, h = k$  and  $e = 2k$ .
- (iv)  $\mathfrak{m}^3 \subseteq x\mathfrak{m}$ ,  $\{z_1^3\}$  is a basis of  $\mathfrak{m}^3/x\mathfrak{m}^2$  and  $\{z_1^2, z_1z_2, \dots, z_1z_{k-1}\}$  is a basis of  $\lambda(\mathfrak{m}^2/x\mathfrak{m})$ .
- (v)  $H_R(z) = 1 + (k + 1)z + (2k - 1)z^2 + \sum_{i=3}^\infty 2kz^i = (1 + kz + (k - 2)z^2 + z^3)/(1 - z)$  and  $H_{R/xR}(z) = 1 + kz + (k - 1)z^2$ .
- (vi)  $s = r = 3$ .
- (vii)  $\text{depth } G = 0$ .

#### 4. Examples

In [3], Rossi and Valla raised the following questions:

QUESTION 1. Let  $(R, \mathfrak{m})$  be a  $d$ -dimensional Cohen-Macaulay local ring with embedding dimension  $v = e + d - 3$ . If  $\tau \geq h$ , then is  $\text{depth } G \geq d - 1$ ?

QUESTION 2. If  $(R, \mathfrak{m})$  is a  $d$ -dimensional Cohen-Macaulay local stretched domain with multiplicity  $e = h + 3$  and  $\tau = 2$ , then is  $G$  Cohen-Macaulay?

We give counterexamples to these questions as follows.

EXAMPLE 4.1. Let  $K$  be a field and  $R = K[[x, y, z, u, v]]/(u^3 - xz, v^3 - yz, u^4, v^4, uv, z^2, zu, zv)$ ; then  $(R, \mathfrak{m})$  is a 2-dimensional Cohen-Macaulay local ring and  $x, y$  is a regular sequence of  $\mathfrak{m}$ , where  $\mathfrak{m} = (x, y, z, u, v)R$ . Moreover,  $h = 3$ ,  $e = 6$  and  $\tau = 3$  as  $\{u^2, v^2, z\}$  generates the socle of  $R$ . However,  $z \in (\mathfrak{m}^3 : (x, y))$  and  $z \notin \mathfrak{m}^2$ , therefore the depth of  $G$  is 0.

EXAMPLE 4.2. Let  $K$  be a field and  $R = K[[t^5, t^6, t^{14}]]$ ; then  $(R, \mathfrak{m})$  is a one-dimensional Cohen-Macaulay local domain, where  $\mathfrak{m} = (t^5, t^6, t^{14})R$ . Let  $x = t^5$ ,  $y = t^6$  and  $z = t^{14}$ ; then  $h = 2$ ,  $e = 5 = h + 3$  and  $\tau = 2$  as  $\{z, y^3\}$  generates the socle of  $R$ . Moreover,

$$P_{R/xR}(z) = 1 + 2z + z^2 + z^3$$

and

$$P_R(z) = \frac{1 + 2z + z^2 + z^4}{1 - z}.$$

Hence  $R$  is stretched and  $G$  is not Cohen-Macaulay. In fact,  $zx \in (\mathfrak{m}^4 : x)$  and  $zx \notin \mathfrak{m}^3$ .

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