

Title	Cohen-Macaulay local rings of embedding dimension $e+d-k$
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Citation	Osaka Journal of Mathematics. 44(4) p.817-p.827
Issue Date	2007-12
oaire:version	VoR
URL	https://doi.org/10.18910/10242
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COHEN-MACAULAY LOCAL RINGS OF EMBEDDING DIMENSION $e + d - k$

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(Received July 11, 2006, revised November 13, 2006)

Abstract

In this paper, we prove the following. Let (R, \mathfrak{m}) be a d -dimensional Cohen-Macaulay local ring with multiplicity e and embedding dimension $v = e + d - k$, where $k \geq 3$ and $e - k > 1$. If $\lambda(\mathfrak{m}^3/J\mathfrak{m}^2) = 1$ and $\mathfrak{m}^3 \subseteq J\mathfrak{m}$, where J is a minimal reduction of \mathfrak{m} , then $3 \leq s \leq \tau + k - 1$, where s is the degree of the h -polynomial of R and τ is the Cohen-Macaulay type of R .

1. Introduction

Let (R, \mathfrak{m}) be a d -dimensional Noetherian local ring of multiplicity e . The Hilbert function of R is by definition the Hilbert function of the associated graded ring of R :

$$G := \bigoplus_{n \geq 0} \mathfrak{m}^n / \mathfrak{m}^{n+1},$$

i.e.,

$$H_R(n) = \dim_{R/\mathfrak{m}} \mathfrak{m}^n / \mathfrak{m}^{n+1}.$$

The Hilbert series of R is the power series

$$P_R(z) = \sum_{n \geq 0} H_R(n)z^n.$$

It is known that there is a polynomial $h(z) \in \mathbb{Z}[z]$ such that $P_R(z) = h(z)/(1-z)^d$ and $h(1) = e$. This polynomial $h(z) = h_0 + h_1z + \cdots + h_s z^s$ is called the h -polynomial of R .

Let (R, \mathfrak{m}) be a d -dimensional Cohen-Macaulay local ring with embedding dimension $v = e + d - k$, where $k \geq 3$. Let J be a minimal reduction of \mathfrak{m} . Let τ be the Cohen-Macaulay type of R , $h = v - d$ and $v_i = \lambda(\mathfrak{m}^{i+1}/J\mathfrak{m}^i)$ for every i ; then there are at least two possible Hilbert series of R/J : $P_{R/J}(z) = 1 + hz + z^2 + \cdots + z^k$ and $P_{R/J}(z) = 1 + hz + (k-1)z^2$. In the first case, R is stretched (cf. definition below) and we have $\mathfrak{m}^k \not\subseteq J\mathfrak{m}$; in the second case, following [3], we say that R is short and we have $\mathfrak{m}^3 \subseteq J\mathfrak{m}$ and $v_1 = k - 1$.

Let (R, \mathfrak{m}) be a d -dimensional local Cohen-Macaulay ring of multiplicity e and embedding dimension v . If $d = 0$, then R is called *stretched* if $e - v$ is the least integer i such that $\mathfrak{m}^{i+1} = 0$. If $d > 0$, then R is *stretched* if there is a minimal reduction J of \mathfrak{m} such that R/J is stretched (cf. [6]), or equivalently, $(\mathfrak{m}^2 + J)/J$ is principal. Regular local rings are not stretched since fields are not stretched. However, for any d -dimensional local Cohen-Macaulay ring (R, \mathfrak{m}) having infinite residue field, if $v = e + d - 1$ with $e > 1$ or $v = e + d - 2$ with $e > 2$, then R is stretched. Moreover, if $v = e + d - 3$ and R is Gorenstein, then R is stretched. These stretched rings have been studied in [6], [7] and [8]. In [4], Rossi and Valla extended the notion *stretched*. There they defined, for each \mathfrak{m} -primary ideal I , I is *stretched* if there is a minimal reduction J of I such that $I^2 \cap J = IJ$ and $\lambda(I^2/(JI + I^3)) = 1$.

In [6], Sally studied the structure of stretched local Gorenstein rings, and use it to show in [8] that if (R, \mathfrak{m}) is a d -dimensional Gorenstein local ring with embedding dimension $v = e + d - 3$, then the associated graded ring of R is Cohen-Macaulay. This result has been generalized by Rossi and Valla in [3] as follows.

Theorem 1.1 ([3, Theorem 2.6]). *If (R, \mathfrak{m}) is a d -dimensional Cohen-Macaulay local ring of multiplicity $e = h + 3$ and $\lambda(\mathfrak{m}^3/J\mathfrak{m}^2) = 1$, then $s \leq \tau + 2$, where s is the degree of the h -polynomial of R .*

In [4], Rossi and Valla generalized Theorem 1.1 to stretched \mathfrak{m} -primary ideals. In this note, we are able to generalize Theorem 1.1 in a different manner in Section 4 as follows. In which, we do not assume R is stretched. In stead, we assume that R is short and $v_2 = 1$.

Theorem 1.2. *Let (R, \mathfrak{m}) be a d -dimensional Cohen-Macaulay local ring of multiplicity $e = h + k$, where $k \geq 3$ and $e - k > 1$. If $\lambda(\mathfrak{m}^3/J\mathfrak{m}^2) = 1$ and $\mathfrak{m}^3 \subseteq J\mathfrak{m}$, where J is a minimal reduction of \mathfrak{m} , then $3 \leq s \leq \tau + k - 1$, where s is the degree of the h -polynomial of R .*

In the final section, we provide several examples to answer some questions raised by Rossi and Valla in [3].

2. One dimensional local Cohen-Macaulay ring

We state several facts of one dimensional local Cohen-Macaulay rings. These results can be derived easily from [1] and [5].

Lemma 2.1. *Let (R, \mathfrak{m}) be a one dimensional local Cohen-Macaulay ring; then $\lambda(\mathfrak{m}^n/\mathfrak{m}^{n+1}) = e - \lambda(\mathfrak{m}^{n+1}/J\mathfrak{m}^n)$, where J is any minimal reduction of \mathfrak{m} .*

Lemma 2.2. *Let (R, \mathfrak{m}) be a one dimensional Cohen-Macaulay local ring with embedding dimension 2. Then $G(R)$ is Gorenstein.*

Corollary 2.3. *Let (R, \mathfrak{m}) be a d -dimensional Cohen-Macaulay local ring with embedding dimension $d + 1$. Then $G(R)$ is Gorenstein.*

3. Cohen-Macaulay local rings of embedding dimension $e + d - k$

Let (R, \mathfrak{m}) be a d -dimensional Cohen-Macaulay local ring with embedding dimension $v = e + d - k$, where $k \geq 3$ and $e - k > 1$. Let τ be the Cohen-Macaulay type of R , $h = v - d$ and $v_i = \lambda(\mathfrak{m}^{i+1}/J\mathfrak{m}^i)$ for every i . Let J be a minimal reduction of \mathfrak{m} ; then one of the possible Hilbert series of R/J is $1 + hz + (k - 1)z^2$. In this case, $\mathfrak{m}^3 \subseteq J\mathfrak{m}$ and $v_1 = k - 1$. If $k = 3$, it is shown in [3, Theorem 2.6] that if $v_2 = 1$ then $s \leq \tau + 2$, where s is the degree of the h -polynomial of R . We are able to generalize this result in this section.

Theorem 3.1. *Let (R, \mathfrak{m}) be a d -dimensional Cohen-Macaulay local ring of multiplicity $e = h + k$, where $k \geq 3$ and $e - k > 1$. If $\lambda(\mathfrak{m}^3/J\mathfrak{m}^2) = 1$ and $\mathfrak{m}^3 \subseteq J\mathfrak{m}$, where J is a minimal reduction of \mathfrak{m} , then $3 \leq s \leq \tau + k - 1$, where s is the degree of the h -polynomial of R .*

REMARK 3.2. (i) Notice that the assumption $v_2 = 1$ ensures that the depth of G is at least $d - 1$ (cf. [3]). Therefore to show Theorem 3.1, we need only to consider the case when $d = 1$.

(ii) If $d = 1$, then s is the least integer for which $\lambda(\mathfrak{m}^s/\mathfrak{m}^{s+1}) = e$.

(iii) Notice that $\lambda(\mathfrak{m}^2/J\mathfrak{m}) = k - 1$. Moreover, if $\mathfrak{m}^2 = J\mathfrak{m} + (u_1, \dots, u_{k-1})$, then $\{u_1, \dots, u_{k-1}\}$ is part of a generating set of the socle of R .

By Remark 3.2, we may assume from now on that $d = 1$ and $v_2 = 1$.

Lemma 3.3. *Let r be the reduction number of \mathfrak{m} with respect to J . If $r \leq 3$, then Theorem 3.1 holds.*

Proof. If $r \leq 3$, then $\mathfrak{m}^4 = J\mathfrak{m}^3$, so that $\lambda(\mathfrak{m}^3/\mathfrak{m}^4) = e$, it follows that $s \leq 3 \leq \tau + k - 1$ by the choice of s . □

By Lemma 3.3, we may assume in the sequel that $r \geq 4$.

Lemma 3.4. *The following hold for R :*

(i) *If $\mathfrak{m}^3 = J\mathfrak{m}^2 + (ab)$ for some $b \in \text{mathfrac}{\mathfrak{m}^2}$ and $a \in \mathfrak{m}$, then $\mathfrak{m}^{i+1} = J\mathfrak{m}^i + (a^{i-1}b)$ for every $i \geq 2$.*

(ii) *If $y\mathfrak{m}^2 \not\subseteq J\mathfrak{m}^2$ for some $y \in \mathfrak{m}$, then $y^3 \notin J\mathfrak{m}^2$. In particular, there is an element $y \in \mathfrak{m}$ such that $\mathfrak{m}^{i+1} = J\mathfrak{m}^i + (y^{i+1})$ for every $i \geq 2$.*

Proof. (i) If $\mathfrak{m}^{i+1} = J\mathfrak{m}^i + (a^{i-1}b)$ for some $i \geq 2$, then $\mathfrak{m}^{i+2} = J\mathfrak{m}^{i+1} + a^{i-1}b\mathfrak{m} \subseteq J\mathfrak{m}^{i+1} + a^{i-1}\mathfrak{m}^3 = J\mathfrak{m}^{i+1} + (a^i b) \subseteq \mathfrak{m}^{i+2}$.

(ii) Suppose that $ym^2 \not\subseteq Jm^2$. Then there are $u, v \in \mathfrak{m}$ such that $uvy \notin Jm^2$ and $m^3 = Jm^2 + (yuv)$. Therefore, $m^4 = Jm^3 + (y^2uv)$. It follows that $y^2u \notin Jm^2$ and $m^3 = Jm^2 + (y^2u)$. Thus, $m^4 = Jm^3 + (y^3u)$ and then $y^3 \notin Jm^2$. Now, choose $y \in \mathfrak{m}$ such that $ym^2 \not\subseteq Jm^2$, then $m^{i+1} = Jm^i + (y^{i+1})$ for every $i \geq 2$. \square

Lemma 3.5. *Let $J = (x)$ be a minimal reduction of \mathfrak{m} . If there is an element $y \in \mathfrak{m}$ such that $m^{i+1} = Jm^i + (y^{i+1})$ for every $i \geq 2$, then $y^l x^t$ is a generator of the module $(J^l m^l + m^{l+t+1})/(J^{l+1} m^{l-1} + m^{l+t+1})$ whenever $2 \leq l < r$, where r is the reduction number of \mathfrak{m} with respect to J .*

Proof. If not, $y^l x^t \in J^{l+1} m^{l-1} + m^{l+t+1}$, so that $y^r x^t \in x^{t+1} m^{r-1}$, it follows that $y^r \in Jm^{r-1}$, a contradiction. Therefore, the conclusion holds. \square

Theorem 3.6. *Let (R, \mathfrak{m}) be a one dimensional Cohen-Macaulay local ring of multiplicity $e = h + k$, where $k \geq 3$ and $e - k > 1$. Assume that $\lambda(m^3/Jm^2) = \lambda(m^4/Jm^3) = 1$ and $m^3 \subseteq Jm$, where $J = (x)$ is a minimal reduction of \mathfrak{m} . Then there is a basis $\{x, y_1, \dots, y_\tau, z_1, \dots, z_{e-\tau-k}\}$ of \mathfrak{m} , elements u_{t+1}, \dots, u_{k-1} contained in \mathfrak{m} and elements $\{c_{ij} \mid i = 1, \dots, k-1, j = 1, \dots, j_i\}$ contained in the ideal $(y_2, \dots, y_{k-1}, z_1, \dots, z_{e-\tau-k})$ with $\sum_{i=1}^{k-1} j_i(k-i) = e - \tau - k$ such that $J = (x)$ and the following hold:*

- (i) $m^{i+1} = Jm^i + (y_1^{i+1})$ for every $i \geq 2$.
- (ii) $m^2 = Jm + (y_1^2, y_1 y_2, \dots, y_1 y_t, y_{t+1} u_{t+1}, \dots, y_{k-1} u_{k-1})$, where $t = \lambda((y_1 m + Jm)/Jm)$.
- (iii) $\{y_1^2, y_1 y_2, \dots, y_1 y_t, y_{t+1} u_{t+1}, \dots, y_{k-1} u_{k-1}, y_k, \dots, y_\tau\}$ is a generating set of the socle of R .
- (iv) $y_1 y_i \in Jm$ for $i \geq t+1$ and $y_1 z_i \in Jm$ for every i .
- (v) $y_i m^3 \subseteq Jm^3$ for every $i \geq 2$ and $z_i m^3 \subseteq Jm^3$ for every $i \geq 1$.
- (vi) $\{z_1, \dots, z_{e-\tau-k}\} = \bigcup_{i,j,k} \{z_{ij}^{(l)}\}$, $\lambda((c_{ij} m + Jm)/Jm) = k - i$ and $m^2 = Jm + \sum_{l=1}^{k-i} z_{ij}^{(l)} c_{ij}$ for every $i = 1, \dots, k-1$ and $j = 1, \dots, j_i$.
- (vii) $c_{ij} z_{i'j'}^{(l)} \in Jm$ if $i < i'$ or $i = i'$ but $j < j'$.
- (viii) $y_1^3 \notin J(z_1, \dots, z_{e-\tau-k}) + Jm^2$.

Proof. By Lemma 3.4, there is an element $y_1 \in \mathfrak{m}$ such that (i) hold. Let $t = \lambda((y_1 m + Jm)/Jm)$; then there are $y_2, \dots, y_{k-1}, u_{t+1}, \dots, u_{k-1} \in \mathfrak{m}$ such that $m^2 = Jm + (y_1^2, y_1 y_2, \dots, y_1 y_t, y_{t+1} u_{t+1}, \dots, y_{k-1} u_{k-1})$ and $y_1 m + Jm = (y_1^2, y_1 y_2, \dots, y_1 y_t) + Jm$. We may assume that $y_1^2 y_i \in Jm^2$ for $2 \leq i \leq t$ by replacing y_i by $y_i + \lambda y_1$ if necessary, and assume that $y_1 y_j \in Jm$ for $t+1 \leq j \leq k-1$ by replacing y_j by $y_j + \lambda_1 y_1 + \dots + \lambda_t y_t$ if necessary. It follows that $y_i m^3 = (y_i y_1^3) + Jm^3 = Jm^3$ for every $i \leq k-1$. Since the Cohen-Macaulay type of R is τ and $\{y_1^2, y_1 y_2, \dots, y_1 y_t, y_{t+1} u_{t+1}, \dots, y_{k-1} u_{k-1}\}$ is part of a generating set of the socle of R , we may choose $y_k, \dots, y_\tau, z_1, \dots, z_{e-\tau-k} \in \mathfrak{m}$ such that $\{y_k, \dots, y_\tau, z_1, \dots, z_{e-\tau-k}\}$ is part of a generating set of \mathfrak{m} and $\{y_1^2, y_1 y_2, \dots, y_1 y_t, y_{t+1} u_{t+1}, \dots, y_{k-1} u_{k-1}, y_k, \dots, y_\tau\}$ is a generating set of the socle of R . If $z_i y_1 \notin Jm$ for some i , then we may replace z_i by

$z_i + \alpha_1 y_1 + \dots + \alpha_t y_t$ if necessary and assume that $z_i y_1 \in J\mathfrak{m}$ for every i . Therefore $z_i \mathfrak{m}^3 \subseteq J\mathfrak{m}^3 + z_i y_1 \mathfrak{m}^2 \subseteq J\mathfrak{m}^3$. Hence, the basis $\{x, y_1, \dots, y_t, z_1, \dots, z_{e-\tau-k}\}$ of \mathfrak{m} satisfies (i) to (v) so far.

Claim. For any integer $i = 1, \dots, k-1$, there is an integer j_i , a basis $\{x, y_1, \dots, y_t, z_1, \dots, z_{e-\tau-k}\}$ of \mathfrak{m} and elements $\{c_{ij} \mid j = 1, \dots, j_i\}$ contained in the ideal $(y_2, \dots, y_{k-1}, z_1, \dots, z_{e-\tau-k})$ such that not only (i) to (v) but also the following hold:

- (a) $\lambda((c_{ij}\mathfrak{m} + J\mathfrak{m})/J\mathfrak{m}) = k - i$, $\mathfrak{m}^2 = J\mathfrak{m} + (z_{ij}^{(1)} c_{ij}, \dots, z_{ij}^{(k-i)} c_{ij})$.
- (b) $c_{ij} z_{ij}^{(l)} \in J\mathfrak{m}$ for every l if $j < j'$ and $c_{ij} z \in J\mathfrak{m}$ for every generator of the ideal generated by S_i , where $S_i = \{z_1, \dots, z_{e-\tau-k}\} - \{z_{i'j}^{(l)} \mid 1 \leq i' \leq i, 1 \leq j \leq j_i, 1 \leq l \leq k - i\}$.

Note that (vi) and (vii) follows from the Claim.

Proof of the Claim. We proceed by induction on i . Let z be any generator of the ideal $(z_1, \dots, z_{e-\tau-k})$. Since $y_1 z, y_i z \in J\mathfrak{m}$ for every $i \geq k$, there is an element $c \in (y_2, \dots, y_{k-1}, z_1, \dots, z_{e-\tau-k})$ such that $c z \notin J\mathfrak{m}$. If for any generating set $\{z'_1, \dots, z'_{e-\tau-k}\}$ of the ideal $(z_1, \dots, z_{e-\tau-k})$ there is no element $c \in (y_2, \dots, y_{k-1}, z_1, \dots, z_{e-\tau-k})$ such that $\mathfrak{m}^2 = (c z'_1, \dots, c z'_{k-1}) + J\mathfrak{m}$, then the Claim holds for $i = 1$. If not, we may assume that $\mathfrak{m}^2 = (c_{11} z_1, \dots, c_{11} z_{k-1}) + J\mathfrak{m}$ for some $c_{11} \in (y_2, \dots, y_{k-1}, z_1, \dots, z_{e-\tau-k})$. Set $z_{11}^{(l)} = z_l$. Let z be any generator of the ideal $(z_k, \dots, z_{e-\tau-k})$. If $c_{11} z \notin J\mathfrak{m}$, then there are elements α_i such that $c_{11} z - (\sum_{i=1}^{k-1} c_{11} z_{11}^{(i)}) \in J\mathfrak{m}$, so that we may replace z by $\sum_{i=1}^{k-1} z_{11}^{(i)}$ if necessary and assume that $c_{11} z \in J\mathfrak{m}$. If for any generating set $\{z'_k, \dots, z'_{e-\tau-k}\}$ of the ideal $(z_k, \dots, z_{e-\tau-k})$ there is no element $c \in (y_2, \dots, y_{k-1}, z_1, \dots, z_{e-\tau-k})$ such that $\mathfrak{m}^2 = (c z'_k, \dots, c z'_{2k-2}) + J\mathfrak{m}$, then again the Claim holds for $i = 1$. If not, we may use the same trick to find c_{12}, c_{13}, \dots so that the Claim holds for $i = 1$.

Suppose now we have shown that the Claim holds for any integer $i' \leq i$ for some $i \geq 1$. Let $m = \sum_{i'=1}^i j_{i'}(k - i')$ and $S_i = \{z_{m+1}, \dots, z_{e-\tau-k}\}$. If for any generating set $\{z'_{m+1}, \dots, z'_{e-\tau-k}\}$ of the ideal generated by S_i there is no element $c \in (y_2, \dots, y_{k-1}, z_1, \dots, z_{e-\tau-k})$ such that $\mathfrak{m}^2 = (c z'_{m+1}, \dots, c z'_{m+k-i-1}) + J\mathfrak{m}$, then the Claim holds for $i + 1$. If not, we may assume that for some $c_{i+1,1} \in (y_2, \dots, y_{k-1}, z_1, \dots, z_{e-\tau-k})$, $\mathfrak{m}^2 = (c_{i+1,1} z_{m+1}, \dots, c_{i+1,1} z_{m+k-i-1}) + J\mathfrak{m}$. Set $z_{i+1,1}^{(l)} = z_{m+l}$. As before, we may assume that $c_{i+1,1} z \in J\mathfrak{m}$ for every generator z of the ideal $(z_{m+k-i}, \dots, z_{e-\tau-k})$. If for any generating set $\{z'_{m+k-i}, \dots, z'_{e-\tau-k}\}$ of the ideal $(z_{m+k-i}, \dots, z_{e-\tau-k})$ there is no element $c \in (y_2, \dots, y_{k-1}, z_1, \dots, z_{e-\tau-k})$ such that $\mathfrak{m}^2 = (c z'_{m+k-i}, \dots, c z'_{m+2k-2i-2}) + J\mathfrak{m}$, then again the Claim holds for $i + 1$. If not, we may use the same trick to find $c_{i+1,2}, c_{i+1,3}, \dots$ so that the Claim holds for $i + 1$. The Claim is now fulfilled. \square

To finish the proof, assume that $y_1^3 \in J(z_1, \dots, z_{e-\tau-k}) + J\mathfrak{m}^2$. Then there are $\delta_i \in R$ not all in \mathfrak{m} such that $y_1^3 - \sum_{i=1}^{e-\tau-k} \delta_i z_i x \in J\mathfrak{m}^2$. Let t be the smallest integer

for which δ_t is a unit; then $y_1^3 - \sum_{i=t}^{e-\tau-k} \delta_i z_i x \in Jm^2$. Let $z = c_{ij}$ if $z_t = z_{ij}^{(l)}$ for some l ; then $z \cdot (\sum_{i=t}^{e-\tau-k} \delta_i z_i) x - zy_1^3 \in Jm^3$, so that $z \cdot (\sum_{i=t}^{e-\tau-k} \delta_i z_i) \in m^3 \subseteq Jm$ as $z \in (y_2, \dots, y_{k-1}, z_1, \dots, z_{e-\tau-k})$. However, $z \cdot (\sum_{i=t}^{e-\tau-k} \delta_i z_i) \notin Jm$ by the Claim, a contradiction. Therefore (viii) holds. \square

Now, we are ready for:

Proof of Theorem 3.1. From the above, we may assume that $d = 1$, $\tau \geq 2$ and $r \geq 4$, where r is the reduction number of some minimal reduction J of m . By Theorem 3.6, there is a basis $\{x, y_1, \dots, y_\tau, z_1, \dots, z_{e-\tau-k}\}$ of m , elements u_{t+1}, \dots, u_{k-1} contained in m and elements $\{c_{ij} \mid i = 1, \dots, k-1, j = 1, \dots, j_i\}$ contained in the ideal $(y_2, \dots, y_{k-1}, z_1, \dots, z_{e-\tau-k})$ with $\sum_{i=1}^{k-1} j_i(k-i) = e - \tau - k$ such that $J = (x)$ and the following hold:

- (i) $m^{i+1} = Jm^i + (y_1^{i+1})$ for every $i \geq 2$.
- (ii) $m^2 = Jm + (y_1^2, y_1 y_2, \dots, y_1 y_t, y_{t+1} u_{t+1}, \dots, y_{k-1} u_{k-1})$, where $t = \lambda((y_1 m + Jm)/Jm)$.
- (iii) $\{y_1^2, y_1 y_2, \dots, y_1 y_t, y_{t+1} u_{t+1}, \dots, y_{k-1} u_{k-1}, y_k, \dots, y_\tau\}$ is a generating set of the socle of R .
- (iv) $y_1 y_i \in Jm$ for $i \geq t+1$ and $y_1 z_i \in Jm$ for every i .
- (v) $y_i m^3 \subseteq Jm^3$ for every $i \geq 2$ and $z_i m^3 \subseteq Jm^3$ for every $i \geq 1$.
- (vi) $\{z_1, \dots, z_{e-\tau-k}\} = \bigcup_{i,j,k} \{z_{ij}^{(l)}\}$, $\lambda((c_{ij} m + Jm)/Jm) = k - i$ and $m^2 = Jm + \sum_{l=1}^{k-i} z_{ij}^{(l)} c_{ij}$ for every $i = 1, \dots, k-1$ and $j = 1, \dots, j_i$.
- (vii) $c_{ij} z_{i'j'}^{(l)} \in Jm$ if $i < i'$ or $i = i'$ but $j < j'$.
- (viii) $y_1^3 \notin J(z_1, \dots, z_{e-\tau-k}) + Jm^2$.

If $\tau \geq h$, then $s \leq e - 1 = h + k - 1 \leq \tau + k - 1$ by [2] and we are done. Therefore, we may assume that $\tau < h$. To show that $s \leq \tau + k - 1$, it is enough to show that $\lambda(m^{\tau+k-1}/m^{\tau+k}) = e$ by Remark 3.2 (ii). Moreover, by Lemma 3.5, $\{y_1^{\tau+k-1}, y_1^{\tau+k-2} x, \dots, y_1^2 x^{\tau+k-3}\}$ are generators of the module $m^{\tau+k-1}/(J^{\tau+k-2} m + m^{\tau+k})$, therefore to show that $\lambda(m^{\tau+k-1}/m^{\tau+k}) = e$ it is enough to show that

$$\{y_1 x^{\tau+k-2}, x^{\tau+k-1}, z_1 x^{\tau+k-2}, \dots, z_{e-\tau-k} x^{\tau+k-2}\}$$

is a linearly independent set in $(x^{\tau+k-2} m + m^{\tau+k})/m^{\tau+k}$.

Suppose not, there are α, β, δ_i in R not all in m such that

$$\alpha y_1 x^{\tau+k-2} + \beta x^{\tau+k-1} + \sum_{i=1}^{e-\tau-k} \delta_i z_i x^{\tau+k-2} \in m^{\tau+k}.$$

Then

$$\alpha y_1^r x^{\tau+k-2} + \beta y_1^{r-1} x^{\tau+k-1} + \sum_{i=1}^{e-\tau-k} \delta_i z_i x^{\tau+k-2} y_1^{r-1} \in m^{\tau+r+k-1},$$

so that $\alpha y_1^r x^{\tau+k-2} \in x^{\tau+k-1} \mathfrak{m}^{r-1}$ as $y_1 z_i \in J\mathfrak{m}$, it follows that $\alpha \in \mathfrak{m}$ by the choice of r . Therefore $\beta x^{\tau+k-1} + \sum_{i=1}^{e-\tau-k} \delta_i z_i x^{\tau+k-2} \in \mathfrak{m}^{\tau+k}$. If $\delta_i \in \mathfrak{m}$ for every i , then $x^{\tau+k-1} \in \mathfrak{m}^{\tau+k}$, which is impossible. So, there is an integer i such that δ_i is a unit. By replacing z_i by $z_i + \beta/\delta_i x$, we may assume that $\beta \in \mathfrak{m}$. Hence $\sum_{i=1}^{e-\tau-k} \delta_i z_i x^{\tau+k-2} \in \mathfrak{m}^{\tau+k}$. Let t be the smallest integer for which δ_t is a unit; then $\sum_{i=t}^{e-\tau-k} \delta_i z_i x^{\tau+k-2} \in \mathfrak{m}^{\tau+k}$.

Let $\alpha \leq \tau+k$ be the integer such that $\sum_{i=t}^{e-\tau-k} \delta_i z_i x^{\tau+k-2} \in J^{\tau+k-\alpha} \mathfrak{m}^\alpha - J^{\tau+k+1-\alpha} \mathfrak{m}^{\alpha-1}$. If $\sum_{i=t}^{e-\tau-k} \delta_i z_i x^{\tau+k-2} \in J^{\tau+k-3} \mathfrak{m}^3$, then $\sum_{i=t}^{e-\tau-k} \delta_i z_i x \in \mathfrak{m}^3 = (y_1^3) + J\mathfrak{m}^2$, so that $\sum_{i=t}^{e-\tau-k} \delta_i z_i x \in J\mathfrak{m}^2$ by (viii), it follows that $\sum_{i=t}^{e-\tau-k} \delta_i z_i \in \mathfrak{m}^2$, a contradiction. Therefore, $\alpha \geq 4$. Since $\mathfrak{m}^\alpha = (y^\alpha) + J\mathfrak{m}^{\alpha-1}$ and $\lambda(\mathfrak{m}^\alpha/J\mathfrak{m}^{\alpha-1}) = 1$, there is a unit λ_1 such that

$$(1) \quad \sum_{i=t}^{e-\tau-k} \delta_i z_i x^{\alpha-2} - \lambda_1 y_1^\alpha \in J\mathfrak{m}^{\alpha-1}.$$

Let $z = c_{ij}$ if $z_t = z_{ij}^{(l)}$; then $z \cdot (\sum_{i=t}^{e-\tau-k} \delta_i z_i) \notin J\mathfrak{m}$ by (vi) and (vii). Moreover,

$$z \left(\sum_{i=t}^{e-\tau-k} \delta_i z_i x^{\alpha-2} \right) - \lambda_1 y_1^\alpha z \in J\mathfrak{m}^\alpha.$$

Furthermore, $y_1^3 z \in J\mathfrak{m}^3$ by (v), we have $z(\sum_{i=t}^{e-\tau-k} \delta_i z_i x^{\alpha-3}) \in \mathfrak{m}^\alpha$. Therefore, there is an element λ_2 of R such that

$$(2) \quad z \left(\sum_{i=t}^{e-\tau-k} \delta_i z_i x^{\alpha-3} \right) - \lambda_2 y_1^\alpha \in J\mathfrak{m}^{\alpha-1}.$$

From (1) and (2), we see that there is an element λ_3 of R such that

$$(z - \lambda_3 x) \left(\sum_{i=t}^{e-\tau-k} \delta_i z_i \right) x^{\alpha-4} \in \mathfrak{m}^{\alpha-1}.$$

Let $\beta \leq \alpha - 4 \leq \tau + k - 4$ be the non-negative integer such that

$$(z - \lambda_3 x) \left(\sum_{i=t}^{e-\tau-k} \delta_i z_i \right) x^\beta \in \mathfrak{m}^{\beta+3} \setminus J\mathfrak{m}^{\beta+2}.$$

Since $z \cdot (\sum_{i=t}^{e-\tau-k} \delta_i z_i) \notin J\mathfrak{m}$, $(z - \lambda_3 x)(\sum_{i=t}^{e-\tau-k} \delta_i z_i) \notin J\mathfrak{m}^2$, β exists. Moreover, there is a unit λ_4 of R such that

$$(3) \quad (z - \lambda_3 x) \left(\sum_{i=t}^{e-\tau-k} \delta_i z_i \right) x^\beta - \lambda_4 y_1^{\beta+3} \in J\mathfrak{m}^{\beta+2}.$$

On the other hand, from (1), we have

$$y_1^{r-\alpha+1} \left(\sum_{i=t}^{e-\tau-k} \delta_i z_i \right) x^{\alpha-2} - \lambda_1 y_1^{r+1} \in J\mathfrak{m}^r,$$

or equivalently,

$$y_1^{r-\alpha+1} \left(\sum_{i=t}^{e-\tau-k} \delta_i z_i \right) x^{\alpha-3} \in \mathfrak{m}^r.$$

Since $\mathfrak{m}^r = (y_1^r) + J\mathfrak{m}^{r-1}$, there is an element λ_5 of R such that

$$(4) \quad y_1^{r-\alpha+1} \left(\sum_{i=t}^{e-\tau-k} \delta_i z_i \right) x^{\alpha-3} - \lambda_5 y_1^r \in J\mathfrak{m}^{r-1}.$$

However, from (1), we have

$$(5) \quad y_1^{r-\alpha} \left(\sum_{i=t}^{e-\tau-k} \delta_i z_i \right) x^{\alpha-2} - \lambda_1 y_1^r \in J\mathfrak{m}^{r-1}$$

Thus, from (4) and (5), we obtain that

$$(6) \quad y_1^{r-\alpha} (y_1 - \lambda_6 x) \left(\sum_{i=t}^{e-\tau-k} \delta_i z_i \right) x^{\alpha-4} \in \mathfrak{m}^{r-1},$$

for some element λ_6 of R . Now, if we can show that

$$(7) \quad \widetilde{y_1^{r-\beta-3}} \left(\sum_{i=t}^{e-\tau-k} \delta_i z_i \right) x^\beta \in \mathfrak{m}^{r-1}$$

for some element $\widetilde{y_1^{r-\beta-3}} \in \mathfrak{m}^{r-\beta-3} \setminus J\mathfrak{m}^{r-\beta-4}$, then from (3) and (7), we see that

$$(z - \lambda_3 x) \widetilde{y_1^{r-\beta-3}} \left(\sum_{i=t}^{e-\tau-k} \delta_i z_i \right) x^\beta - \lambda_4 y_1^{\beta+3} \widetilde{y_1^{r-\beta-3}} \in J\mathfrak{m}^{r-1}$$

and $(z - \lambda_3 x) \widetilde{y_1^{r-\beta-3}} \left(\sum_{i=t}^{e-\tau-k} \delta_i z_i \right) x^\beta \in z\mathfrak{m}^{r-1} + J\mathfrak{m}^{r-1} = J\mathfrak{m}^{r-1}$ by (v), therefore $\lambda_4 y_1^{\beta+3} \widetilde{y_1^{r-\beta-3}} \in J\mathfrak{m}^{r-1}$, which contradicts to the choice of r . Hence, we conclude that $\{y_1 x^{\tau+k-2}, x^{\tau+k-1}, z_1 x^{\tau+k-2}, \dots, z_{e-\tau-k} x^{\tau+k-2}\}$ is a linearly independent set in $(x^{\tau+k-2}\mathfrak{m} + \mathfrak{m}^{\tau+k})/\mathfrak{m}^{\tau+k}$.

Finally, by (6), we may prove (7) by reverse induction. Suppose we have shown that for some $\delta, \beta < \delta \leq \alpha - 4$,

$$\widetilde{y_1^{r-\delta-3}} \left(\sum_{i=t}^{e-\tau-k} \delta_i z_i \right) x^\delta \in \mathfrak{m}^{r-1}$$

for some element $\widetilde{y_1^{r-\delta-3}} \in \mathfrak{m}^{r-\delta-3} \setminus J\mathfrak{m}^{r-\delta-4}$. Then there is an element $\lambda_6 \in R$ such that

$$(8) \quad y_1 \widetilde{y_1^{r-\delta-3}} \left(\sum_{i=t}^{e-\tau-k} \delta_i z_i \right) x^\delta - \lambda_6 y_1^r \in J\mathfrak{m}^{r-1}.$$

From (5) and (8), we see that

$$\widetilde{y_1^{r-\delta-2}} \left(\sum_{i=t}^{e-\tau-k} \delta_i z_i \right) x^\delta \in J\mathfrak{m}^{r-1}$$

for some element $\widetilde{y_1^{r-\delta-2}} \in \mathfrak{m}^{r-\delta-2} \setminus J\mathfrak{m}^{r-\delta-3}$, it follows that

$$\widetilde{y_1^{r-\delta-2}} \left(\sum_{i=t}^{e-\tau-k} \delta_i z_i \right) x^{\delta-1} \in \mathfrak{m}^{r-1}. \quad \square$$

We end this section by providing the following example.

EXAMPLE 3.7. Let K be a field and $R = K[[x, y, z_1, \dots, z_{k-1}]]/I$, where I is the ideal of R generated by the set

$$\{z_1^3 - xy, y^2, yz_1, \dots, yz_{k-1}, z_1z_2, \dots, z_1z_{k-1}\} \cup \{z_i z_j \mid 2 \leq i \leq j \leq k-1\}.$$

The it is easy to see the following hold:

- (i) R is a 1-dimensional Cohen-Macaulay local ring with maximal ideal $\mathfrak{m} = (x, y, z_1, \dots, z_{k-1})/I$.
- (ii) x is a regular element of R and xR is a minimal reduction of \mathfrak{m} .
- (iii) $v = k + 1, h = k$ and $e = 2k$.
- (iv) $\mathfrak{m}^3 \subseteq x\mathfrak{m}$, $\{z_1^3\}$ is a basis of $\mathfrak{m}^3/x\mathfrak{m}^2$ and $\{z_1^2, z_1z_2, \dots, z_1z_{k-1}\}$ is a basis of $\lambda(\mathfrak{m}^2/x\mathfrak{m})$.
- (v) $H_R(z) = 1 + (k + 1)z + (2k - 1)z^2 + \sum_{i=3}^\infty 2kz^i = (1 + kz + (k - 2)z^2 + z^3)/(1 - z)$ and $H_{R/xR}(z) = 1 + kz + (k - 1)z^2$.
- (vi) $s = r = 3$.
- (vii) $\text{depth } G = 0$.

4. Examples

In [3], Rossi and Valla raised the following questions:

QUESTION 1. Let (R, \mathfrak{m}) be a d -dimensional Cohen-Macaulay local ring with embedding dimension $v = e + d - 3$. If $\tau \geq h$, then is $\text{depth } G \geq d - 1$?

QUESTION 2. If (R, \mathfrak{m}) is a d -dimensional Cohen-Macaulay local stretched domain with multiplicity $e = h + 3$ and $\tau = 2$, then is G Cohen-Macaulay?

We give counterexamples to these questions as follows.

EXAMPLE 4.1. Let K be a field and $R = K[[x, y, z, u, v]]/(u^3 - xz, v^3 - yz, u^4, v^4, uv, z^2, zu, zv)$; then (R, \mathfrak{m}) is a 2-dimensional Cohen-Macaulay local ring and x, y is a regular sequence of \mathfrak{m} , where $\mathfrak{m} = (x, y, z, u, v)R$. Moreover, $h = 3$, $e = 6$ and $\tau = 3$ as $\{u^2, v^2, z\}$ generates the socle of R . However, $z \in (\mathfrak{m}^3 : (x, y))$ and $z \notin \mathfrak{m}^2$, therefore the depth of G is 0.

EXAMPLE 4.2. Let K be a field and $R = K[[t^5, t^6, t^{14}]]$; then (R, \mathfrak{m}) is a one-dimensional Cohen-Macaulay local domain, where $\mathfrak{m} = (t^5, t^6, t^{14})R$. Let $x = t^5$, $y = t^6$ and $z = t^{14}$; then $h = 2$, $e = 5 = h + 3$ and $\tau = 2$ as $\{z, y^3\}$ generates the socle of R . Moreover,

$$P_{R/xR}(z) = 1 + 2z + z^2 + z^3$$

and

$$P_R(z) = \frac{1 + 2z + z^2 + z^4}{1 - z}.$$

Hence R is stretched and G is not Cohen-Macaulay. In fact, $zx \in (\mathfrak{m}^4 : x)$ and $zx \notin \mathfrak{m}^3$.

ACKNOWLEDGMENT. I am grateful to the referee for a number of valuable suggestions that improved the paper a lot.

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