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TWO-TONE COLORINGS AND SURJECTIVE DIHEDRAL REPRESENTATIONS FOR LINKS

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Abstract

It is well-known that a knot is Fox n-colorable for a prime n if and only if the knot group admits a surjective homomorphism to the dihedral group of degree n. However, this is not the case for links with two or more components. In this paper, we introduce a two-tone coloring on a link diagram, and give a condition for links so that the link groups admit surjective representations to the dihedral groups. In particular, it is shown that the link group of any link with at least 3 components admits a surjective homomorphism to the dihedral group of arbitrary degree.

1. Introduction

One of the most well-known invariants of knots in 3-space must be the Fox's 3-colorablity. (See Remark 1 for the definition of the Fox n-coloring.) In general, it is known that a knot is Fox n-colorable for a prime $n \geq 3$ if and only if the knot group admits a surjective homomorphism to the dihedral group D_n of degree n. For instance, it is stated in [3, Chap. VI, Exercises, 6, pp.92–93]. However, this is not the case for links with two or more components. In fact, some examples are given in [6] for D_3-coloring, which is the coloring by the symmetric group of degree three. For example, by the results in [6, Theorem 1.2], the link group of the torus link $T(2, q)$ admits a surjective homomorphism to D_3 if $q \equiv 0 \pmod 4$. On the other hand, $T(2, q)$ is Fox 3-colorable if and only if $q \equiv 0 \pmod 3$.

We remark that, although there are numerous papers studying the Fox colorings (cf. [10, 2]), it seems that the relationship between the Fox colorings on links with two or more components and the surjective homomorphisms of the link groups to the dihedral groups has not been discussed, as far as the authors know.

In this paper, we introduce a two-tone coloring on a link diagram, and give a condition for links which guarantees that the link groups admit surjective homomorphisms to the dihedral groups. In particular, we show that the link group of any link with at least 3 components admits a surjective homomorphism to the dihedral group of arbitrary degree.

Remark 1. Recall that a *Fox n-coloring* on a link diagram D is defined as a map Γ : {arcs of D} $\to \{0, 1, \ldots, n-1\}$, satisfying $2\Gamma(x) \equiv \Gamma(y) + \Gamma(z) \pmod n$ at each crossing of D with the over arc x and the under arcs y and z. It is well-known that, for $n \geq 3$, a link is Fox n-colorable, i.e., a diagram of the link admits a non-trivial Fox n-coloring (a coloring with at least two colors), if and only if $\det(L) = 0$ or $(n, \det(L)) \neq 1$, where $\det(L)$ denotes

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the determinant of the link. See [7, Proposition 2.1] for example. Also a condition for knot groups to admit a surjective homomorphism to the dihedral groups in terms of the homology of the double branched covering is known. See [1, 14.8] for example.

To state our results, we prepare some notations. Let D_n be the dihedral group of degree n. It is well-known that D_n has the following presentation with e the identity element:

$$D_n = \left\langle a, b \mid a^2 = b^n = (ab)^2 = e \right\rangle.$$

Note that any element in D_n is represented as $a^x b^y$ ($x = 0, 1$ and $0 \leq y \leq n-1$). Thus, by setting $a_i := ab^i$ ($0 \leq i \leq n-1$) and $b_j := b^j$ ($1 \leq j \leq n-1$), we see that $D_n = \{e, a_0, a_1, \ldots, a_{n-1}, b_1, \ldots, b_{n-1}\}$ as a set. In geometric viewpoint, the a_i's represent reflections and b_j's represent rotations as the symmetries of a regular polygon (n-gon).

In the following, let L be an oriented link in the 3-sphere S^3 with a link diagram D. We call a map $\Gamma : \{\text{arcs on } D\} \to D_n$ a *D_n-coloring* on D if it satisfies $\Gamma(x)\Gamma(z) = \Gamma(y)\Gamma(x)$ (respectively, $\Gamma(z)\Gamma(x) = \Gamma(x)\Gamma(y)$) in D_n at each positive (resp. negative) crossing on D, where x denotes the over arc, y and z the under arcs at the crossing supposing y is the under arc before passing through the crossing and z is the other. (See Figure 1.)

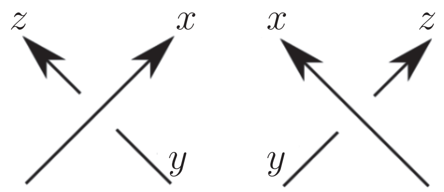

Fig. 1. Positive and negative crossings

Remark 2. The D_n-colorings and the Fox n-colorings are related in terms of representations of link groups to D_n as follows. For a link diagram D with c crossings of a link L, set $g_1, \ldots, g_c$ the Wirtinger generators of the link group G_L, i.e., $G_L = \pi_1(S^3 - L)$. Then a D_n-coloring on D corresponds to a map $\{g_1, \ldots, g_c\} \to D_n$ which extends to a homomorphism of G_L to D_n. When a D_n-coloring sends g_k's to a_i's (reflections, $0 \leq i \leq n-1$) in D_n, it induces a map $\{\text{arcs of } D\} \to \{0, 1, \ldots, n-1\}$, which gives a Fox n-coloring. Note that even if a link admits a nontrivial Fox n-coloring, it may not induce a surjective homomorphism from G_L to D_n. See the example illustrated in Figure 2. In this case, the image of the Wirtinger generators by the homomorphism induced by the Fox 4-coloring is the set $\{a_0, a_2\} \subset D_4$, but the elements a_0 and a_2 do not generate D_4. Thus the induced homomorphism is not surjective.

The following is our key definition.

Definition 1. Let Γ be a D_n-coloring on a link diagram D of an oriented link L. We say that Γ is *two-tone* if $\mathrm{Im}(\Gamma)$ does not contain the trivial element, i.e. $e \notin \mathrm{Im}(\Gamma)$, and $\mathrm{Im}(\Gamma) \cap \{a_0, \ldots, a_{n-1}\} \neq \emptyset$ and $\mathrm{Im}(\Gamma) \cap \{b_1, \ldots, b_{n-1}\} \neq \emptyset$, that is, the coloring uses colors from both $\{a_i\}$ and $\{b_j\}$. We say that a link is *two-tone D_n-colorable* if, with some orientation, it has a diagram D admitting a two-tone D_n-coloring.

Note that two-tone D_n-colorability is independent of the choice of orientations for links.

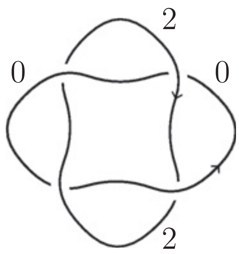

Fig. 2. Fox 4-colorable link

An example of a two-tone D_n-colorable link is the pretzel link $P(6,6,6)$ which admits a two-tone D_m-coloring if $m \geq 4$. See Figure 3 for the case where $m \geq 8$.

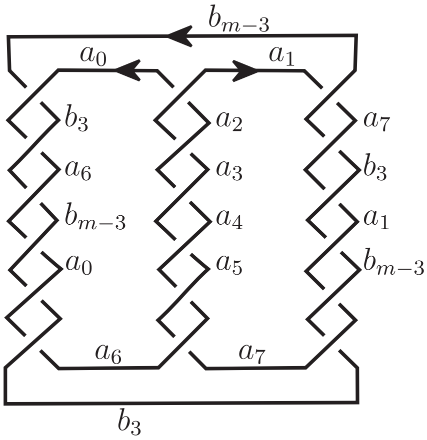

Fig. 3. A two-tone D_m-coloring on $P(6,6,6)$ for $m \geq 8$

Now the following are our main results. Here D_∞ denotes the group presented by $\langle a, b \mid a^2 = (ab)^2 = e \rangle$, and a two-tone D_∞-coloring for a link is defined in the same way as above.

Theorem 1.1. *For a 2-component link $L = \ell_1 \cup \ell_2$, the following are equivalent.*

(i) *$lk(\ell_1, \ell_2)$ is even.*
(ii) *L is two-tone D_n-colorable for some odd $n \geq 3$.*
(iii) *L is two-tone D_∞-colorable.*
(iv) *The link group G_L admits a surjective homomorphism to D_n for every $n \geq 3$.*
(v) *The link group G_L admits a surjective homomorphism to D_∞.*

Remark 3. By considering the natural embedding of D_n into D_{2n}, we see that the condition (ii) in Theorem 1.1 is equivalent to that L admits a two-tone D_n-coloring for some even $n \geq 3$ that assigns b_i with $i \neq n/2$ to some arcs. We also remark that (iii) in Theorem 1.1 does not imply that L is two-tone D_n-colorable for every odd $n \geq 3$. Actually even if there is a two-tone D_∞-coloring on a diagram of a link L, the coloring may not give a two-tone D_n-coloring for some n, but a Fox n-coloring on a sub-diagram of L. For example, pretzel links of type $(m, 2, m, 2)$ with odd m admit a two-tone D_∞-coloring on a diagram, but no two-tone D_m-colorings.

On the other hand, for 2-component links with odd linking numbers, we have the following.

Theorem 1.2. *Let $L = \ell_1 \cup \ell_2$ be a 2-component link with $lk(\ell_1, \ell_2)$ odd. Then the following hold.*

(i) *The link L admits no two-tone D_n-colorings for any odd $n \geq 3$.*
(ii) *If the link group G_L admits a surjective homomorphism to D_n for $n \geq 3$, then the homomorphism is induced from a Fox n-coloring on ℓ_1, ℓ_2 or L, i.e., the homomorphism sends a meridional element in G_L to the trivial element or a reflection in D_n.*

For the links with at least 3 components, interestingly, the following holds.

Theorem 1.3. *Let L be a link with at least 3 components. Then the link group G_L admits a surjective homomorphism to D_n for every $n \geq 3$.*

We remark that even if the link group G_L admits a surjective homomorphism to D_n for every $n \geq 3$, the link L may not be two-tone D_n-colorable for every $n \geq 3$. For example, pretzel links of type $(2m, 2m, 2m)$ with odd m admit no two-tone D_m-colorings.

As a corollary of the theorems, we have the following.

Corollary 1.4. *If a link L is two-tone D_m-colorable for some odd m, then G_L admits a surjective homomorphism to D_n for every $n \geq 3$. If G_L admits a surjective homomorphism to D_n for some n, then L contains a two-tone D_n-colorable sub-link or a Fox n-colorable sub-link.*

On the other hand, even if a link L is known to be two-tone D_n-colorable for some n, finding a two-tone D_n-coloring on a given diagram of L, or, finding a surjective homomorphism of G_L to D_n, is a tedious task in general. The next proposition and its proof give a simple way to find a two-tone D_n-coloring on some link diagrams for any $n \geq 3$.

Proposition 1.5. *Suppose that there exists a trivial component ℓ_0 of a link L and that $lk(\ell_0, \ell)$ is even for every component $\ell \subset L - \ell_0$. Then any diagram of L admits a two-tone D_n-coloring for every odd $n \geq 3$ which assigns the arcs on ℓ_0 to a_i's and the other arcs to b_j's.*

2. Properties of D_n-coloring

In this section, we study some properties of D_n-colorings, and give lemmas which will be used in the remaining sections. In the following, we set $A_n := \{a_i\}$ and $B_n := \{b_j\}$ for D_n.

Lemma 2.1. *Let Γ be a D_n-coloring on a diagram D of an oriented link L in S^3. At a crossing on D, x denotes the over arc, and y and z the under arcs at the crossing supposing that y (resp. z) is the under arc before (resp. after) passing through the crossing. Then the following hold.*

(1) *$\Gamma(y)$ and $\Gamma(z)$ are both in A_n or both in B_n.*
(2) *If $\Gamma(x) \in B_n$ and $\Gamma(y) \in B_n$, then $\Gamma(z) = \Gamma(y)$.*
(3) *If $\Gamma(x) = a_i$ and $\Gamma(y) = a_{i'}$, then $\Gamma(z) = a_k$ and $k \equiv 2i' - i \pmod{n}$.*
(4) *If $\Gamma(x) = a_i$ and $\Gamma(y) = b_j$, then $\Gamma(z) = b_k$ and $k \equiv n - j \pmod{n}$.*
(5) *If $\Gamma(x) = b_j$ and $\Gamma(y) = a_j$, then $\Gamma(z) = a_k$ and $k \equiv i + 2j$ (resp. $k \equiv i - 2j$) $\pmod{n}$ if the crossing is a positive (resp. negative) crossing.*

Proof. By definition of a D_n-coloring, $\Gamma(y)$ and $\Gamma(z)$ are conjugate in D_n, and from this, (1) holds. We give a proof of the case (4) when the crossing is a positive crossing. The others (2), (3), (5) are proved in the same way. Suppose that $\Gamma(x) = a_i$ and $\Gamma(y) = b_j$. By definition of a D_n-coloring, we have the following.

$$\Gamma(z) = (a_i)^{-1}b_j a_i = b^{n-i}a^{-1}b^j ab^i = ab^{i+j-n}ab^i = b^{n-i-j+i} = b^{n-j} = b_{n-j}$$

Thus $\Gamma(z) = b_k$ and $k \equiv n - j \pmod n$ holds. □

Remark 4. Note that (1) in the lemma implies that all the strands on a diagram of a particular component must be colored by a_i's or by b_j's for a given D_n-coloring. Also note that the tone of the colors for a particular component is independent of the choice of a diagram: If all the strands on a diagram for a particular component are colored by b_j's by a D_n-coloring, then all the strands for the component are also colored by b_j's on any diagram by the D_n-coloring obtained by performing Reidemeister moves. We will use these facts in the rest of the paper repeatedly.

Lemma 2.2. *Let Γ be a D_n-coloring on a diagram D of an oriented link L in S^3. Let x, y, z, w be either the arcs depicted in* Figure 4 *(left), or the arcs depicted in* Figure 4 *(right). If $\Gamma(x) = a_i$ and $\Gamma(y) = b_j$, then $\Gamma(z) = a_k$ with $k \equiv i-2j \pmod n$ and $\Gamma(w) = b_l$ with $l \equiv n-j \pmod n$.*

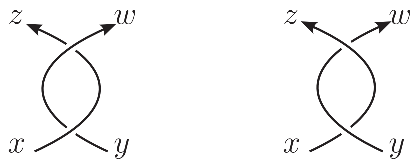

Fig. 4. A positive full twist (left). A negative full twist (right)

Proof. We only give a proof for the positive full twist case. A proof for the other case is similar. In that case, by Lemma 2.1(4), $\Gamma(w) = b_l$ with $l \equiv n - j \pmod n$ since $\Gamma(x) = a_i$ and $\Gamma(y) = b_j$. Then, by Lemma 2.1(5), $\Gamma(z) = a_k$ and $k \equiv i + 2(n - j) \equiv i - 2j \pmod n$ since $\Gamma(w) = b_l$ with $l \equiv n - j \pmod n$ and $\Gamma(x) = a_i$. □

3. Two-tone colorings and surjective homomorphisms to D_∞

In this section, we give a key proposition to prove the theorems.

In the following, let $lk(L, L')$ denote the (total) linking number of oriented links L, L', i.e., $lk(L, L') = \sum_{\ell \subset L, \ell' \subset L'} lk(\ell, \ell')$. The linking number is calculated for the link with arbitrarily chosen orientations. Note that the parity of such a linking number is independent of the choice of orientations.

Proposition 3.1. *Suppose that a link L contains a component ℓ_0 with $lk(\ell_0, L')$ even and $\det(L') \neq 0$, where $L' = L - \ell_0$. Then L admits a two-tone D_∞-coloring that induces a surjective homomorphism from G_L to D_∞.*

Proof. Let $p : X \to S^3 - L'$ be the double covering on the total linking number with L', and $\bar{p} : M \to S^3$ the double branched covering. Let $\tilde{K} = K_1 \cup K_2$ denote the inverse image $p^{-1}(\ell_0) \subset X$; because $lk(\ell_0, L')$ is even, $\tilde{K}$ is a 2-component link in X (or in M). We shall construct a surjective group homomorphism $\pi_1(M - \tilde{K}) \to \mathbb{Z}$ and extend the composition $\pi_1(X - \tilde{K}) \to \pi_1(M - \tilde{K}) \to \mathbb{Z}$ to obtain a D_∞-coloring.

Taking a regular neighborhood N of $\tilde{K}$, we consider the Mayer-Vietoris exact sequence for $M = N \cup (M - \tilde{K})$:

$$H_2(M) \to H_1(N - \tilde{K}) \to H_1(N) \oplus H_1(M - \tilde{K}) \to H_1(M) \to H_0(N - \tilde{K})$$

is exact. The rightmost map is zero as usual and the leftmost one is also zero because $\det(L') \neq 0$ (hence $|H_1(M)| = |\det(L')| < \infty$); by the Poincaré duality $H_2(M) \cong H^1(M; \mathbb{Z}) = 0$. Thus, we obtain a short exact sequence

$$0 \to H_1(N - \tilde{K}) \to H_1(N) \oplus H_1(M - \tilde{K}) \to H_1(M) \to 0.$$

Take a meridional disc $D_1 \subset M$ of K_1 and let D_2 denote $\varphi(D_1)$, where $\varphi : M \to M$ is the nontrivial covering transformation of the branched covering $\bar{p} : M \to S^3$; the covering transformation group is $\mathbb{Z}_2 = \{\mathrm{id}_M, \varphi\}$. We denote $D_1 \cup D_2$ by $\tilde{D}$. Because the kernel of the surjective homomorphism $H_1(N - \tilde{K}) \to H_1(N)$ is the image of the injective map $H_1(\partial\tilde{D}) \to H_1(N - \tilde{K})$, the short exact sequence above shows that

$$0 \to H_1(\partial\tilde{D}) \to H_1(M - \tilde{K}) \to H_1(M) \to 0 \tag{1}$$

is also exact. We should remark that the involution φ induces automorphisms φ_* on the homology groups in (1). Since the homomorphisms in (1) are induced by the inclusions, (1) is compatible with φ_*; i.e., the maps are $\mathbb{Z}_2$-equivariant.

Let $x \in H_1(\partial D_1)$ be a generator and set $y = \varphi_*(x) \in H_1(\partial D_2)$. We use the same symbols x, y for their images in $H_1(\partial D)$ or $H_1(M - \tilde{K})$. We take the quotient of (1) by the φ_*-invariant part of $H_1(\partial\tilde{D})$ to obtain an exact sequence

$$0 \to H_1(\partial\tilde{D})/(x + y) \to H_1(M - \tilde{K})/(x + y) \to H_1(M) \to 0.$$

Since $H_1(\partial\tilde{D})/(x + y) \cong \mathbb{Z}$ and $|H_1(M)| < \infty$, the rank of $H_1(M - \tilde{K})/(x + y)$ equals 1, i.e., $(H_1(M - \tilde{K})/(x + y))/\mathrm{Tor}(H_1(M - \tilde{K})/(x + y)) \cong \mathbb{Z}$. Hence there exists a surjective homomorphism $f : H_1(M - \tilde{K})/(x + y) \to \mathbb{Z}$, which satisfies $f \circ \varphi_* = -f$. Let $\bar{f} : \pi_1(X - \tilde{K}) \to \mathbb{Z}$ denote the composition

$$\pi_1(X - \tilde{K}) \to \pi_1(M - \tilde{K}) \to H_1(M - \tilde{K}) \to H_1(M - \tilde{K})/(x + y) \to \mathbb{Z}.$$

Let $m \in G_L$ be a meridian of a component of L'. Identifying $\langle b \rangle \subset D_\infty$ with $\mathbb{Z}$, we define $\tilde{f} : G_L \to D_\infty$ by

$$\tilde{f}(g) = \begin{cases} \bar{f}(g) & (g \in \pi_1(X - \tilde{K})), \\ a\bar{f}(m^{-1}g) & (g \notin \pi_1(X - \tilde{K})). \end{cases}$$

Since $a^2 = e$, $\tilde{f}$ is well-defined as a map. Furthermore, we have $\bar{f}(mgm^{-1}) = f \circ \varphi_*(g) = f(g)^{-1} = \bar{f}(g)^{-1} \in D_\infty$ for $g \in \pi_1(X - \tilde{K})$. By this equality, we can easily check that $\tilde{f}$ is a group homomorphism. Because $\bar{f}$ is surjective and $\tilde{f}(m) = a$, the homomorphism $\tilde{f} : G_L \to D_\infty$ is surjective. □

The following is an immediate corollary of the proposition above, since any knot has an odd determinant.

Corollary 3.2. *Let $L = \ell_1 \cup \ell_2$ be a 2-component link. If $lk(\ell_1, \ell_2)$ is even, then L admits a two-tone D_∞-coloring that induces a surjective homomorphism from G_L to D_∞.* □

4. Proof of theorems

In this section, we give proofs of the theorems stated in Introduction. To prove the theorems, we prepare the following two lemmas.

Lemma 4.1. *If a 2-component link $L = \ell_1 \cup \ell_2$ is two-tone D_n-colorable for some odd $n \geq 3$, then $lk(\ell_1, \ell_2)$ is even.*

Proof. Take a two-tone D_n-coloring γ on a diagram of L for some $n \geq 3$. Since γ is two-tone, one component of L is colored by a_i's, and the other by b_j's. Let ℓ_b be the component of L such that each arc in a diagram of ℓ_b is colored by b_j's by Γ. This ℓ_b is well-defined for Γ independent of the choice of a diagram. See Remark 4.

We can easily see that L admits a diagram as depicted in Figure 5, where D_b is a sub-diagram corresponding to ℓ_b, D_a is the remaining sub-diagram, and each box between D_a and D_b contains a vertical full twist (Figure 5 (right)). For this $D_a \cup D_b$, we consider the arcs β and β' which are connected in D_b as in Figure 5 (left).

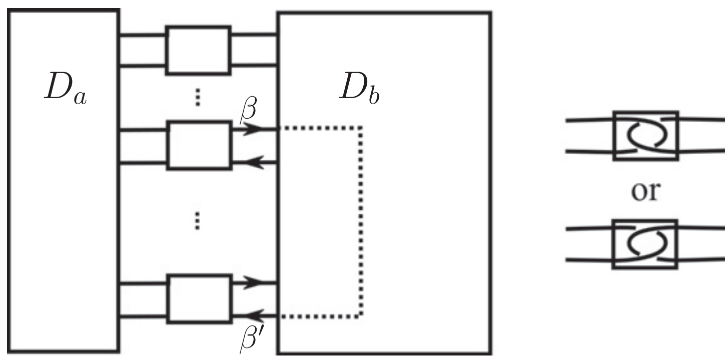

Fig. 5. The diagram of L

Since β and β' are connected in D_b, we see $\Gamma(\beta) = \Gamma(\beta')$ by Lemma 2.1(1). On the other hand, letting N be the number of the boxes (full twists) which ℓ_b runs through, if $\Gamma(\beta) = \Gamma(\beta')$, then N has to be even. This is shown by applying Lemma 2.2 repeatedly for each box (full twist) together with n is odd. The number N is congruent to $lk(\ell_a, \ell_b)$ modulo 2, and so the lemma holds. □

REMARK 5. The lemma above can be extended as follows. If L is two-tone D_n-colorable for some odd $n \geq 3$, then the sublink L_b of L consisting of those components which are colored by b_j's satisfies that, for every component $\ell \subset L_b$, $lk(\ell, L - L_b)$ is even.

Lemma 4.2. *Let $L = \ell_1 \cup \ell_2$ be a 2-component link. If* $\det(L) = 0$*, then $lk(\ell_1, \ell_2)$ is even.*

Proof. Let D be a diagram of L. Since $\det(L) = 0$, there exists a Fox 4-coloring Γ on D which induces a surjective group homomorphism to D_4. By definition of Fox colorings, if $\Gamma(x)$ equals a_0 or a_2 (resp. a_1 or a_3) for an arc x belonging to ℓ_i $(i = 1, 2)$, it holds for any arc x of ℓ_i. Then, we may assume

$$\Gamma(\{\text{arcs of } \ell_1\}) \subset \{a_1, a_3\} \quad \text{and} \quad \Gamma(\{\text{arcs of } \ell_2\}) \subset \{a_0, a_2\}.$$

For a crossing point of D, let x be the over arc and y, z the under arcs. Again, by definition of Fox colorings, we find that $\Gamma(y) = \Gamma(z)$ holds if and only if x and y belong to the same component. In particular, the colors of the under arcs at the crossing are changed if x belongs to ℓ_1 and y to ℓ_2. This implies that D has an even number of such crossings, and hence the linking number $lk(\ell_1, \ell_2)$ is even. □

Proof of Theorem 1.1. Let $L = \ell_1 \cup \ell_2$ a 2-component link. We show that all the following are equivalent.

(i) $lk(\ell_1, \ell_2)$ is even.
(ii) L is two-tone D_n-colorable for some odd $n \geq 3$.
(iii) L is two-tone D_∞-colorable.
(iv) The link group G_L admits a surjective homomorphism to D_n for every $n \geq 3$.
(v) The link group G_L admits a surjective homomorphism to D_∞.

We see that (i)⇒(iii) follows from Corollary 3.2 and (ii)⇒(i) follows from Lemma 4.1.
<u>(iii)⇒(ii):</u> Suppose that L is two-tone D_∞-colorable, that is, a diagram of L admits a two-tone D_∞-coloring. Since there is a surjection from D_∞ to D_n for every $n \geq 3$ defined by $a \in D_\infty \mapsto a \in D_n$ and $b \in D_\infty \mapsto b \in D_n$, this implies that the diagram of L admits a D_n-coloring for every n. By taking odd n sufficiently large, the D_n-coloring uses at least two colors from a_i's. Furthermore, by retaking n to satisfy $(n, \det(L)) = 1, (n, \det(\ell_1)) = 1$, and $(n, \det(\ell_2)) = 1$ if necessary, the coloring cannot come from Fox n-colorings on L, ℓ_1, or ℓ_2. Thus the coloring has to be two-tone, and so, L is two-tone D_n-colorable for some odd $n \geq 3$.

We also see that (i)⇒(v) follows from Corollary 3.2.
<u>(v)⇒(iv):</u> By the surjection from D_∞ to D_n for every $n \geq 3$ defined as above, if the link group G_L admits a surjective homomorphism to D_∞, then the link group G_L admits a surjective homomorphism to D_n for every $n \geq 3$.
<u>(iv)⇒(i) or (ii):</u> Suppose that the link group G_L admits a surjective homomorphism to D_n for every $n \geq 3$. Such a surjective homomorphism induces a D_n-coloring on a diagram of L for $n \geq 3$ by considering the Wirtinger generators for the diagram. If $\det(L) = 0$, then $lk(\ell_1, \ell_2)$ is even by Lemma 4.2, and so (i) holds. If $\det(L) \neq 0$, then for some odd n which is coprime to $\det(L), \det(\ell_1), \det(\ell_2)$, the D_n-coloring does not come from a Fox n-coloring, and so, it has to be two-tone. This implies (ii). □

Proof of Theorem 1.2. Let $L = \ell_1 \cup \ell_2$ be a 2-component link with $lk(\ell_1, \ell_2)$ is odd.
(i) Then L admits no two-tone D_n-colorings for any $n \geq 3$ by Theorem 1.1 (by the contraposition of (ii)⇒(i)).
(ii) By (i), if the link group G_L admits a surjective homomorphism to D_n for $n \geq 3$, then it is not induced from two-tone D_n-colorings. That is, the homomorphism must send Wirtinger

generators to either the trivial element and reflections in D_n or the trivial element and rotations in D_n. However, the latter cannot be surjective, and so, it is impossible. Therefore the homomorphism sends Wirtinger generators to either the trivial element and reflections in D_n. Such a homomorphism is induced from a Fox n-coloring on ℓ_1, ℓ_2 or L. □

Proof of Theorem 1.3. Let L be a link with at least 3 components. We show that G_L admits a surjective homomorphism to D_n.

Consider sub-links of 2 components in L. If some of them, say $L' = \ell'_1 \cup \ell'_2$, satisfies that $lk(\ell'_1, \ell'_2)$ is even, then by Theorem 1.2, $G_{L'}$ admits a surjective homomorphism to D_n and L' is two-tone D_n-colorable for n. It follows that G_L admits a surjective homomorphism to D_n via a surjection $G_L \to G_{L'}$ and L is two-tone D_n-colorable.

Suppose that for all the 2 component sub-links of L, the linking numbers of the two components are odd. Then, by Lemma 4.2, no such links have the determinant 0. Since L has at least 3 components, we can consider a sub-link of L with 3 components, say $L' = \ell_1 \cup \ell_2 \cup \ell_3$. For this link, $lk(\ell_1, \ell_2 \cup \ell_3)$ is even and $\det(\ell_2 \cup \ell_3) \neq 0$ holds. Then, by Proposition 3.1, $G_{L'}$ admits a surjective homomorphism to D_∞ and so a surjective homomorphism to D_n for every $n \geq 3$. This implies that G_L admits a surjective homomorphism to D_n for every $n \geq 3$. □

Proof of Corollary 1.4. Suppose that L is two-tone D_m-colorable for some odd $m \geq 3$. If L is a link with 2 components, then G_L admits a homomorphism to D_n for every $n \geq 3$ by Theorem 1.2 ((ii)⇒(iv)). If L has at least 3 components, then G_L admits a homomorphism to D_n for every $n \geq 3$ by Theorem 1.3.

Suppose that G_L admits a surjective homomorphism to D_n for $n \geq 3$. Then there is a D_n-coloring on a diagram of L. See Remark 2. If the coloring uses two-tone colors, then L contains a two-tone D_n-colorable sub-link. Otherwise, since the homomorphism is surjective, the coloring comes from a nontrivial Fox n-coloring on a diagram of a sub-link of L as in the proof of Theorem 1.2. □

REMARK 6. For the proof of Lemma 4.2, it is pointed out by the anonymous referee that the lemma is a direct consequence of the following two well-known formulas for Alexander polynomial Δ_L:

- $\Delta_L(1,1) = \pm lk(\ell_1, \ell_2)$ for a link $L = \ell_1 \cup \ell_2$ ([11])
- $\det(L) = 2|\Delta_L(-1,-1)|$ ([5, Theorem 1]).

(The second formula is a generalization of the Fox formula and a special case of the Mayberry-Murasugi formula [8], whose simple proof is given by Porti [9].) Moreover, the two formulas imply the stronger conclusion that $lk(\ell_1, \ell_2) \equiv 0 \pmod 2$ if and only if $\det(L) \equiv 0 \pmod 4$.

5. Finding two-tone colorings

Proof of Proposition 1.5. Suppose that there exists a trivial component ℓ_0 of a link L and, for every component $\ell \subset L - \ell_0$, $lk(\ell_0, \ell)$ is even. If a diagram of a link L admits a two-tone D_n-coloring for every odd $n \geq 3$ which assigns the arcs on ℓ_0 to a_i's and the other arcs to b_j's, then so does any diagram of L. Thus, to prove the proposition, it suffices to show that a particular diagram of L admits such a D_n-coloring.

Now we take a diagram D of L depicted in Figure 6. In the figure, D_0 is a sub-diagram corresponding to ℓ_0, which is a trivial knot diagram, and each box between D_0 and the remaining sub-diagram D_b contains a vertical full twist (see Figure 5 (right)).

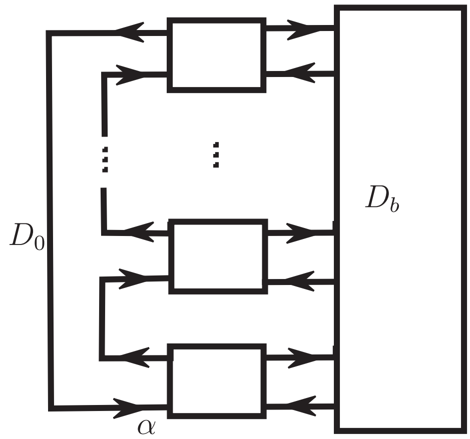

Fig. 6. The diagram D. Each box in the center contains a full twist

Consider the arc α in the figure, take an arc β_i from each component of $L - \ell_0$, and assign a_0 to α and b_1 to β_i's. Let us show that this assignment induces a two-tone D_n-coloring.

For the arc β_i, let ℓ be the component of $L - \ell_0$ containing β_i. Since $lk(\ell_0, \ell)$ is even for every component $\ell \subset L - \ell_0$, due to Lemma 2.2, the assigning β_i to b_1 induces a D_n-coloring on ℓ. In the same way, we can find a D_n-coloring on $L - \ell_0$.

Note that, on the sub-diagram corresponding to each component of $L - \ell_0$, an arc in the lower right of a box in the center is colored by b_1 or b_{n-1}. In particular, when the arc in the lower right is colored by b_1, then the arc in the upper right is colored in b_{n-1}, and vice versa.

Thus, by Lemma 2.2, for each component of $L - \ell_0$, the number of the boxes in the center with the arc in the lower right colored by b_1 is equal to the number of those with the arc colored in b_{n-1}.

Let m be the half of the linking number $lk(\ell_0, L - \ell_0)$. (Note that $lk(\ell_0, L - \ell_0)$ must be even, since $lk(\ell_0, \ell)$ is even for each component ℓ of $L - \ell_0$.) Then the number of the boxes in the center with the arc in the lower right colored by b_1 is m and the number of those with the arc colored in b_{n-1} is also m.

Again by Lemma 2.2, assigning α to a_0 induces assigning the arc in the upper left of the top box in center to $a_{0-2(m\cdot 1+m\cdot(-1))} = a_0$. This implies that the assignment induces a D_n-coloring on the whole diagram. By construction, the D_n-coloring is obviously two-tone.

Thus any diagram of L admits a two-tone D_n-coloring for every odd $n \geq 3$ which assigns the arcs on ℓ_0 to a_i's and the other arcs to b_j's. □

Appendix

The following was given by the anonymous referee for unifying and generalizing some of the arguments and results. The basic idea behind the approach is essentially identical with that of the proof of the key Proposition 3.1. However, it is quite different from the approach in the other parts.

Recognized as before, a D_n-coloring ($n \in \mathbb{Z}_{\geq 2} \cup \{\infty\}$) of a link diagram D representing a link L is nothing other than a homomorphism, γ, from the link group $G_L := \pi_1(S^3 - L)$ to the dihedral group

$$D_n = \langle a, b \mid a^2, b^n, (ab)^2 \rangle \cong \langle b \mid b_n \rangle \rtimes \langle a \mid a^2 \rangle,$$

that maps every meridian to a nontrivial element. Let $\nu : D_n \to \langle a \mid a^2 \rangle$ be the natural epimorphism. Then the coloring γ corresponds to a Fox coloring or a two-tone coloring according to whether (i) $\nu\gamma$ maps every meridian to the generator a or (ii) $\nu\gamma$ maps some meridian to a and some meridian to the trivial element.

Study of dihedral representations, more generally metableian representations, of link groups has a long history. In particular, a natural and useful viewpoint can be found in Hartley's article ([4]). The proof of the key Proposition 3.1 fits this viewpoint. On the other hand, for Lemma 4.1 and Proposition 1.5, which are intimately related with Proposition 3.1, the author give diagrammatic proofs, which have almost no relation with the proof of Proposition 3.1. Here, the following present unified proofs and generalizations of these results.

Homological proof of a generalization of Lemma 4.1 given in Remark 5. By the assumption of the lemma, G_L admits a two-tone D_n-representation $\gamma : G_L \to D_n$. Let L_a and L_b be the sublink of L consisting of the components whose meridians are mapped by $\nu\gamma$ to a or 1, respectively. Since γ maps the meridians of L_a to order 2 elements, it descends to a homomorphism, which we continue to denote by γ, from the quotient of G_L by the normal closure of the squares of meridians of L_a. The latter group is the orbifold fundamental group of the orbifold, $\mathcal{O}$, with underlying space $S^3 - L_b$ with singular set L_a of index 2. The double covering of $\mathcal{O}$ associated with the homomorphism $\nu\gamma : \pi_1^{orb}(\mathcal{O}) \to \langle a \mid a^2 \rangle$ is the manifold $M - \tilde{L}_b$ where M is the double branched covering of S^3 branched over L_a and $\tilde{L}_b$ is the inverse image of L_b in M. The fundamental group $\pi_1(M - \tilde{L}_b)$ is identified with the index 2 subgroup $\ker(\nu\gamma)$ of $\pi_1^{orb}(\mathcal{O})$, and the homomorphism $\gamma : \pi_1^{orb}(\mathcal{O}) \to D_n$ restricts to an abelian representation $\tilde{\gamma} : \pi_1(M - \tilde{L}_b) \to \langle b \mid b^n \rangle < D_n$.

Now suppose to the contrary that there is a component ℓ of L_b with $lk(\ell, L_a)$ odd. Then the inverse image $\tilde{\ell}$ of ℓ in $M - \tilde{L}_b$ is connected. Thus any two meridians of ℓ, regarded as elements of $\pi_1(M - \tilde{L}_b)$, are conjugate in $\pi_1(M - \tilde{L}_b)$, and so their images by γ, which are equal to the images by the abelian homomorphism $\tilde{\gamma}$, are identical in $\langle b \mid b^n \rangle < D_n$. However, this is impossible, because for a meridian μ_ℓ of ℓ and for a meridian μ_a of a component of L_a, we have $\gamma(\mu_a\mu_\ell\mu_a^{-1}) = \gamma(\mu_\ell)^{-1} \neq \gamma(\mu_\ell)$, though $\mu_a\mu_\ell\mu_a^{-1}$ is also a meridian of ℓ. (Here the inequality follows from the assumption that $n \geq 3$ is odd.) □

Though the above proof is lengthy, it ties up with the proof of Proposition 3.1 and it leads to a simple proof of the following generalization of Proposition 1.5.

Proposition A.1. *Let $L = L_0 \cup L_1$ be a link in S^3 satisfying the following conditions.*

(1) $\det(L_0) = 1$.

(2) *L_1 is non-empty, and every component of L_1 has an even linking number with L_0.*

Then there is a two-tone epimorphism from G_L to D_∞ for which $L_a = L_0$ and $L_b = L_1$, where L_a and L_b are the sublinks of L as in the "homological proof".

Proof. Let M be the double branched covering of S^3 branched along L_0 and $\tilde{L}_1$ the inverse image of L_1 in M. The assumptions imply that $H_1(M - \tilde{L}_1)$ is a free abelian group with basis $\{\mu_i, \mu'_i \mid 1 \leq i \leq r\}$ such that the homomorphism τ induced by the covering translation switches μ_i with μ'_i for each i. (Here r is the number of components of L_2, μ_i and μ'_i are meridians of the components of $\tilde{L}_1$ that are mapped to the i-th component of L_2.) Let Q be the semi-direct product of $H_1(M - \tilde{L}_1)$ with the order 2 cyclic group $\langle a \mid a^2 \rangle$, where the action of the latter group on the first group is given by τ. Then Q is a quotient of the link group G_L. (In fact it is a quotient of the orbifold fundamental group of the orbifold $\mathcal{O}$ with underlying space $S^3 - L_1$ with singular set L_0 of index 2, as defined in the homological proof of Lemma 4.1.) The proposition now follows from the fact that there is an epimorphism from Q to D_∞ defined by $a \mapsto a$, $\mu_i \mapsto b$ and $\mu'_i = \tau(\mu_i) = a\mu_i a^{-1} \mapsto b^{-1}$. □

The above proof and that of Lemma 4.1 work for links in a $\mathbb{Z}$-homology 3-sphere. Moreover, the same argument also imply the following further generalization.

Proposition A.2. *Let $L = L_0 \cup L_1$ be a link in a $\mathbb{Z}/2\mathbb{Z}$-homology 3-sphere S, and $n \geq 2$ an integer, satisfying the following conditions.*

(1) *The double branched covering M of S branched over L_0 is a $\mathbb{Z}/n\mathbb{Z}$-homology 3-sphere.*

(2) *L_1 is non-empty, and every component of L_1 has the trivial mod 2 linking number with L_0.*

Then there is a two-tone epimorphism from G_L to D_n for which $L_a = L_0$ and $L_b = L_1$, where L_a and L_b are the sublinks of L as in the "homological proof".

In fact, the assumption that L is a link in a $\mathbb{Z}/2\mathbb{Z}$-homology 3-sphere implies that there is a unique double branched covering branched along L, and the two conditions imply that $H_1(M - \tilde{L}_1; \mathbb{Z}/n\mathbb{Z})$ is the free $\mathbb{Z}/n\mathbb{Z}$-module that has a base consisting of meridians $\{\mu_i, \mu'_i \mid 1 \leq i \leq r\}$, such that the homomorphism τ induced by the covering translation switches μ_i with μ'_i for each i.

There are possible future problems (also given by the anonymous referee): It would be nice if one could give a unified diagrammatic proof to all of the results in the paper, including the key Proposition 3.1 and the results in the appendix. If successful, then it might bring our mathematical community a new deep insight into the link diagrams.

Also the results in this paper might give a hint to the following natural question:

Question. For $n = 1$ or 2, the "greatest common quotient" of the n-component link groups is the free abelian group $\mathbb{Z}_n$. For $n \geq 3$, is there a non-abelian group G bigger than $\mathbb{Z}_n$ (i.e., a non-commutative group with abelianization $\mathbb{Z}_n$), for which every n-component link group admits a (canonical) epimorphism onto G?

If such a group G exists, then by considering the G-coverings of link complements, one may be able to construct a link invariant stronger than the Alexander invariants, which are defined by using $\mathbb{Z}_n$-coverings.

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