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TORUS-EQUIVARIANTLY EMBEDDED TORIC MANIFOLDS ASSOCIATED TO AFFINE SUBSPACES

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Abstract
We study the closure of a complex subtorus in a toric manifold. If the closure of the complex subtorus is a smooth complex submanifold in the toric manifold, then the subtorus action on such submanifold is Hamiltonian. In this case, we may think of the embedding of the submanifold as torus-equivariant. We show that the image of the moment map for the Hamiltonian subtorus action on our submanifold coincides with the image of the Delzant polytope of the ambient toric manifold under the pullback of the inclusion of the tori. The submanifolds constructed in the present paper are called *torus-equivariantly embedded toric manifolds* with respect to the subtorus action.

1. Introduction

Delzant [5] established a one-to-one correspondence between compact symplectic toric manifolds and certain convex polytopes known as *Delzant polytopes*. Given a $2n$-dimensional compact symplectic toric manifold X, the image of a moment map for the Hamiltonian T^n-action on X is a Delzant polytope Δ in $(\mathfrak{t}^n)^* \cong \mathbb{R}^n$. Conversely, given a Delzant polytope Δ in $(\mathfrak{t}^n)^*$, we can construct a compact symplectic toric manifold X whose moment polytope is Δ. This construction is called the *Delzant construction*. From the Delzant construction, symplectic toric manifolds are canonically equipped with a Kähler structure [8, 3]. We can identify the complements of toric divisors in a symplectic toric manifold X with a complex torus $(\mathbb{C}^*)^n$, whose description allows us to consider complex coordinates in X.

1.1. Main Results. In this paper, we study complex submanifolds in compact toric manifolds X. From a k-dimensional affine subspace V in $\mathfrak{t}^n \cong \mathbb{R}^n$, we first construct a k-dimensional complex submanifold $C(V)$ in the toric divisor complements $\check{M} \cong (\mathbb{C}^*)^n$ of the toric manifold X. This construction is inspired by [11], and $C(V) \cong (\mathbb{C}^*)^k$ as Yamamoto noted there. In fact, $C(V) \cong (\mathbb{C}^*)^k$ is a complex subtorus of $\check{M} \cong (\mathbb{C}^*)^n$. We then consider the conditions of V when the (Zariski) closure $\overline{C(V)}$ is a k-dimensional complex submanifold in the toric manifold X (Section 4.1). While $C(V)$ is a complex submanifold in $\check{M} \cong (\mathbb{C}^*)^n$ for arbitrary affine subspace V as Yamamoto showed in [11, Lemma 6.1], $\overline{C(V)}$ may not be a complex submanifold in X (see Example 4.5 and Section 5).

Suppose that $\overline{C(V)}$ is a smooth complex submanifold in X. We then discuss the nature of the submanifolds $\overline{C(V)}$. Toric manifolds X are naturally equipped with a moment map

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$\mu : X \to (\mathfrak{t}^n)^*$ for the T^n-action on them. We can define the injective group homomorphism $i_V : T^k \to T^n$ by the data of V (see Equation 4.6). Because the T^n-action on X and $i_V : T^k \to T^n$ induce the T^k-action on $\overline{C(V)}$, we can determine the moment map $\overline{\mu} : \overline{C(V)} \to (\mathfrak{t}^k)^*$ by $\overline{\mu} = i_V^* \circ \mu \circ i$, where $i : \overline{C(V)} \to X$ is the embedding (see Section 4.3 for detail). We obtain the following diagram:

$$\begin{array}{ccc} X & \xrightarrow{\mu} & (\mathfrak{t}^n)^* \\ {\scriptstyle i}\uparrow & & \downarrow{\scriptstyle i_V^*} \\ \overline{C(V)} & \xrightarrow{\overline{\mu}} & (\mathfrak{t}^k)^*. \end{array}$$

We compare the image of the moment map $\overline{\mu} : \overline{C(V)} \to (\mathfrak{t}^k)^*$ for the T^k-action on our complex submanifold $\overline{C(V)}$ with the image of the moment map $\mu : X \to (\mathfrak{t}^n)^*$ for the T^n-action on the ambient toric manifold.

Theorem 1.1 (Theorem 4.20)**.** *Let $i_V : T^k \to T^n$ be an injective group homomorphism determined by a given affine subspace V in $\mathfrak{t}^n \cong \mathbb{R}^n$. Assume that $\overline{C(V)}$ is a complex submanifold in X. Then the image of $\overline{\mu}$ is equal to the image of $i_V^* \circ \mu$, i.e., $\overline{\mu}(\overline{C(V)}) = (i_V^* \circ \mu)(X)$.*

We call the complex submanifolds $\overline{C(V)}$ *torus-equivariantly embedded toric manifolds*.

1.2. Outline. This paper is organized as follows. In Section 2, we construct a system of complex coordinate charts on complex manifolds from matrices in $SL(n;\mathbb{Z})$. This construction helps us to consider a system of complex coordinate charts in toric manifolds. In Section 3, we review Delzant construction and construct a system of the inhomogeneous coordinate charts on compact toric manifolds using the construction established in Section 2. In Section 4, we give the conditions where the closure $\overline{C(V)}$ of a complex subtorus $C(V)$ is a complex submanifold in the ambient toric manifold. Moreover, we consider the moment maps for the subtorus action on our complex submanifolds and compare them with the moment maps for the torus action on ambient toric manifolds. In Section 5, we demonstrate some examples of torus-equivariantly embedded toric manifolds.

Throughout the paper, we express vectors as column vectors.

2. Construction of Coordinate Charts from Matrices in $SL(n;\mathbb{Z})$

This section is about construction of a system of complex coordinate charts from matrices in the special linear group $SL(n;\mathbb{Z})$. Our idea is similar to *the coordinate transformations* for compact toric manifolds by Duistermaat and Pelayo [6], but here we construct a system of complex coordinate charts in a general situation.

Let Λ be a set and $Q^\lambda \in SL(n;\mathbb{Z})$ a matrix corresponding to each $\lambda \in \Lambda$ and $\mathbb{C}^n_\lambda = \{z^\lambda = (z^\lambda_1, \ldots, z^\lambda_n) \in \mathbb{C}^n\} \cong \mathbb{C}^n$ for each $\lambda \in \Lambda$. We define a matrix $D^{\lambda\mu} = (Q^\lambda)^{-1}Q^\mu (= [d^{\lambda\mu}_{ij}])$ for any $\lambda, \mu \in \Lambda$ and a subset $U_{\lambda\mu} \subset \mathbb{C}^n_\lambda$ by

$$U_{\lambda\mu} = \{z^\lambda \in \mathbb{C}^n_\lambda \mid z^\lambda_j \neq 0 \text{ if } d^{\lambda\mu}_{jl} < 0 \text{ for some } l = 1, \ldots, n\}.$$

We introduce an equivalence relation on $\{U_{\lambda\mu}\}_{\lambda,\mu\in\Lambda}$.

DEFINITION 2.1. For $z^\lambda \in U_{\lambda\mu} \subset \mathbb{C}^n_\lambda$ and $z^\mu \in U_{\mu\lambda} \subset \mathbb{C}^n_\mu$, we define a binary relation $z^\lambda \sim z^\mu$ by

$$(z^\mu_1, \ldots, z^\mu_n) = \left(\prod_{j=1}^n \left(z^\lambda_j\right)^{d^{\lambda\mu}_{j1}}, \ldots, \prod_{j=1}^n \left(z^\lambda_j\right)^{d^{\lambda\mu}_{jn}} \right).$$

Proposition 2.2. *The binary relation* $\sim$ *defined in* Definition 2.1 *is an equivalence relation.*

Proof. We check that the binary relation $\sim$ satisfies the definition of equivalence relations.

Since we define $D^{\lambda\mu} = (Q^\lambda)^{-1}Q^\mu$ for $\lambda, \mu \in \Lambda$, we get $D^{\lambda\lambda} = (Q^\lambda)^{-1}Q^\lambda = E_n$, where E_n is the identity matrix. Hence, $z^\lambda = z^\lambda$, which means that $z^\lambda \sim z^\lambda$.

Since $D^{\lambda\mu} = (Q^\lambda)^{-1}Q^\mu$, we have $D^{\mu\lambda} = (Q^\mu)^{-1}Q^\lambda = (D^{\lambda\mu})^{-1}$. Suppose $z^\lambda \sim z^\mu$, then we see $z^\mu_i = \prod_{j=1}^n (z^\lambda_j)^{d^{\lambda\mu}_{ji}}$ for $i = 1, \ldots, n$. For $k = 1, \ldots, n$, we have

$$\prod_{i=1}^n \left(z^\mu_i\right)^{d^{\mu\lambda}_{ik}} = \prod_{i=1}^n \prod_{j=1}^n \left(z^\lambda_j\right)^{d^{\lambda\mu}_{ji} d^{\mu\lambda}_{ik}} = \prod_{j=1}^n \left(z^\lambda_j\right)^{\delta_{jk}} = z^\lambda_k.$$

Hence we obtain $z^\mu \sim z^\lambda$.

For $\lambda, \mu, \sigma \in \Lambda$, we have

$$D^{\lambda\sigma} = (Q^\lambda)^{-1}Q^\sigma = (Q^\lambda)^{-1}Q^\mu(Q^\mu)^{-1}Q^\sigma = D^{\lambda\mu}D^{\mu\sigma}.$$

Suppose $z^\lambda \sim z^\mu, z^\mu \sim z^\sigma$, then for $i = 1, \ldots, n$ we have

$$z^\sigma_i = \prod_{j=1}^n \left(z^\mu_j\right)^{d^{\mu\sigma}_{ji}} = \prod_{j=1}^n \left(\prod_{k=1}^n \left(z^\lambda_k\right)^{d^{\lambda\mu}_{kj}} \right)^{d^{\mu\sigma}_{ji}} = \prod_{k=1}^n \left(z^\lambda_k\right)^{d^{\lambda\sigma}_{ki}}.$$

Hence we obtain $z^\lambda \sim z^\sigma$. □

Proposition 2.3. *The quotient space* $X = \bigsqcup_{\lambda\in\Lambda} \mathbb{C}^n_\lambda/\sim$ *is a Hausdorff space.*

Proof. Define the projection $\mathrm{pr} : \bigsqcup_{\lambda\in\Lambda} \mathbb{C}^n_\lambda \to X$ to the quotient space. Take two distinct points $[x] \neq [y] \in X$. Let $U_{[x]}$ and $U_{[y]}$ be open subsets containing the points $[x]$ and $[y]$ respectively. Then we can write $\mathrm{pr}^{-1}(U_{[x]}), \mathrm{pr}^{-1}(U_{[y]})$ as follows:

$$\mathrm{pr}^{-1}(U_{[x]}) = \bigsqcup_{\lambda\in\Lambda} U^\lambda_{[x]},\ \mathrm{pr}^{-1}(U_{[y]}) = \bigsqcup_{\lambda\in\Lambda} U^\lambda_{[y]},$$

where $U^\lambda_{[x]}, U^\lambda_{[y]} \subset \mathbb{C}^n_\lambda \cong \mathbb{C}^n$ for $\lambda \in \Lambda$.

Let $B_\varepsilon(x)$ be an open ball of radius $\varepsilon > 0$. We define the map $\varphi_\lambda : \mathbb{C}^n_\lambda/\sim \to \mathbb{C}^n_\lambda$ by $\varphi_\lambda([z^\lambda]) = z^\lambda$. If $\mathbb{C}^n_\lambda$ contains the points x and y, then there exist $\varepsilon, \varepsilon' > 0$ such that $B_\varepsilon(x) \cap B_{\varepsilon'}(y) = \emptyset$. Thus we have $\mathrm{pr}^{-1}(\mathrm{pr}(B_\varepsilon(x))) \cap \mathrm{pr}^{-1}(\mathrm{pr}(B_{\varepsilon'}(y))) = \emptyset$.

If $x \in \mathbb{C}^n_\lambda, y \in \mathbb{C}^n_\mu$ $(\lambda \neq \mu)$, then there exists an element $\sigma \in \Lambda$ such that $\varphi_\sigma \circ \varphi_\lambda^{-1}(B_\varepsilon(x) \cap U_{\lambda\sigma}) \subset U^\sigma_{[x]}, \varphi_\sigma \circ \varphi_\mu^{-1}(B_{\varepsilon'}(y) \cap U_{\mu\sigma}) \subset U^\sigma_{[y]}$. Thus we can take sufficiently small $\varepsilon, \varepsilon' > 0$ such that $\mathrm{pr}^{-1}(\mathrm{pr}(\varphi_\sigma \circ \varphi_\lambda^{-1}(B_\varepsilon(x) \cap U_{\sigma\lambda}))) \cap \mathrm{pr}^{-1}(\mathrm{pr}(\varphi_\sigma \circ \varphi_\mu^{-1}(B_{\varepsilon'}(y) \cap U_{\sigma\mu}))) = \emptyset$.

Suppose that $x \in \mathbb{C}^n_\lambda \setminus U_{\lambda\mu}, y \in \mathbb{C}^n_\mu \setminus U_{\mu\lambda}$ $(\lambda \neq \mu)$. If there exist $\varepsilon, \varepsilon' > 0$ such that $\mathrm{pr}(B_\varepsilon(x)) \cap \mathrm{pr}(B_{\varepsilon'}(y)) \neq \emptyset$, then there exist $z_x \in B_\varepsilon(x) \cap U_{\lambda\mu}$ and $z_y \in B_{\varepsilon'}(y) \cap U_{\mu\lambda}$ such that $\mathrm{pr}(z_x) = \mathrm{pr}(z_y)$. Since $x \notin U_{\lambda\mu}$, we obtain

$$0 < |x - z_x| < \varepsilon.$$

We can retake ε smaller than $|x - z_x|$ so that $\mathrm{pr}(B_\varepsilon(x)) \cap \mathrm{pr}(B_{\varepsilon'}(y)) = \emptyset$. Thus we have $\mathrm{pr}^{-1}(\mathrm{pr}(B_\varepsilon(x))) \cap \mathrm{pr}^{-1}(\mathrm{pr}(B_{\varepsilon'}(y))) = \emptyset$.

Therefore, the quotient space X is a Hausdorff space. □

Let $U_\lambda = \{[z^\lambda] \in X \mid z^\lambda \in \mathbb{C}^n_\lambda\} \subset X$. Then we see $X = \bigcup_{\lambda\in\Lambda} U_\lambda$ from Proposition 2.2. We define a map $\varphi_\lambda : U_\lambda \to \mathbb{C}^n$ by $\varphi_\lambda([z^\lambda]) = z^\lambda$ for each $\lambda \in \Lambda$. The following lemma is obvious from the construction above.

Lemma 2.4. *For all $\lambda, \mu \in \Lambda$ such that $U_\lambda \cap U_\mu \neq \emptyset$, we have*

$$\varphi_\mu \circ \varphi_\lambda^{-1}(z^\lambda) = \left(\prod_{j=1}^{n} \left(z_j^\lambda\right)^{d_{j1}^{\lambda\mu}}, \ldots, \prod_{j=1}^{n} \left(z_j^\lambda\right)^{d_{jn}^{\lambda\mu}} \right). \tag{2.1}$$

Definition 2.5. The set $\{(U_\lambda, \varphi_\lambda)\}_{\lambda\in\Lambda}$ is a system of complex coordinate charts on X, whose coordinate transformation is given by Equation 2.1.

3. Toric Manifolds

In this section, we write the inhomogeneous coordinate charts on a toric manifold in terms of the coordinate charts given in Section 2. We also discuss the complements of toric divisors, which we call the *toric divisor complements*.

3.1. Convex Polytopes and Convex Cones. We review the definitions and some of the facts of *convex polytopes* and *convex cones* in $\mathbb{R}^n$, which are used later.

We first deal with convex polytopes, which are defined as follows:

Definition 3.1. Let $V = \{x_1, \ldots, x_s\} \neq \emptyset$ be a finite set of elements in $\mathbb{R}^n$. The convex hull $\Delta = \mathrm{conv}(V)$ of V is a convex polytope in $\mathbb{R}^n$. Concretely, Δ is written as

$$\Delta = \mathrm{conv}(V) = \left\{ \sum_{i=1}^{s} r_i x_i \;\middle|\; r_i \geq 0, \sum_{i=1}^{s} r_i = 1, x_i \in V \right\}.$$

If a convex polytope Δ can be written as Equation 3.1, then we say that Δ is generated by $V = \{x_1, \ldots, x_s\}$.

The next lemma is obvious.

Lemma 3.2. *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map. If $\Delta \subset \mathbb{R}^n$ is a convex polytope generated by $V = \{x_1, \ldots, x_s\}$, then $f(\Delta) \subset \mathbb{R}^m$ is also a convex polytope generated by $V_f = \{f(x_1), \ldots, f(x_s)\}$.*

We deal with convex cones, which are defined as follows:

Definition 3.3. A subset C in $\mathbb{R}^n$ is a (convex polyhedral) cone if there exist elements $v_1, \ldots, v_s \in C$ such that

$$C = \mathbb{R}_{\geq 0} v_1 + \cdots + \mathbb{R}_{\geq 0} v_s. \tag{3.1}$$

If a cone C can be written as Equation 3.1, then we say that C is generated by $\{v_1, \ldots, v_s\}$.

The next lemma is obvious.

Lemma 3.4. *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map. If $C \subset \mathbb{R}^n$ is a cone generated by $\{v_1, \dots, v_s\}$, then $f(C) \subset \mathbb{R}^m$ is also a cone generated by $\{f(v_1), \dots, f(v_s)\}$.*

Definition 3.5. Let $C \subset \mathbb{R}^n$ be a cone generated by $\{v_1, \dots, v_s\}$. The point $0 = (0, \dots, 0) \in C \subset \mathbb{R}^n$ is a *vertex* of C if C does not contain a nontrivial subspace.

Lemma 3.6. *Let $C \subset \mathbb{R}^n$ be a cone generated by $\{v_1, \dots, v_s\}$. The cone C does not contain a nontrivial subspace if and only if the following is satisfied:*

$$r_1 v_1 + \cdots + r_s v_s = 0,\ r_i \geq 0 \Rightarrow r_1 = \cdots = r_s = 0. \tag{3.2}$$

Proof. We first show that if C does not contain a nontrivial subspace, then Equation 3.2 holds. We give a proof by showing the contraposition.

Suppose that $r_1, \dots, r_s \in \mathbb{R}_{\geq 0}$ satisfy $\sum_{i=1}^{s} r_i v_i = 0$. Suppose further that there exists some $i_0 \in \{1, \dots, s\}$ such that $r_{i_0} > 0$. Then since we can calculate

$$v_{i_0} = -\frac{1}{r_{i_0}} \sum_{i \neq i_0} r_i v_i = -\sum_{i \neq i_0} \frac{r_i}{r_{i_0}} v_i,$$

$W := \{r v_{i_0} \mid r \in \mathbb{R}\} \subset C$ holds. Indeed, if $r \geq 0$, then $r v_{i_0} \in C$ by the definition of C; if otherwise, then since from the above calculation we see

$$r v_{i_0} = \sum_{i \neq i_0} (-r) \frac{r_i}{r_{i_0}} v_i$$

and $(-r)\frac{r_i}{r_{i_0}} \geq 0$ for any $i \neq i_0$, $r v_{i_0} \in C$. Since W is a nontrivial subspace in $\mathbb{R}^n$, we obtain the contraposition to the desired result.

We then show that if Equation 3.2 holds, then C does not contain a nontrivial subspace.

Let $W \neq \emptyset$ be a subspace contained in C. Since W is a linear space, if $w \in W$ then $-w \in W$ holds. Since $W \subset C$, there exsit $r_1, \dots, r_s, r'_1, \dots, r'_s \geq 0$ such that

$$w = \sum_{i=1}^{s} r_i v_i,\ -w = \sum_{i=1}^{s} r'_i v_i.$$

Since $w + (-w) = 0$, we see that

$$\sum_{i=1}^{s} (r_i + r'_i) v_i = 0.$$

Since we assume that Equation 3.2 holds, $r_i + r'_i = 0$ holds for any $i = 1, \dots, s$. Furthermore, since $r_1, \dots, r_s, r'_1, \dots, r'_s \geq 0$, $r_i = r'_i = 0$ holds for any $i = 1, \dots, s$. This implies that $w = 0$, i.e., $W = \{0\}$. □

From the above lemma, we can use the following definition of a *vertex* in a cone.

Definition 3.7. Let $C \subset \mathbb{R}^n$ be a cone generated by $\{v_1, \dots, v_s\}$. The point $0 = (0, \dots, 0) \in C \subset \mathbb{R}^n$ is a *vertex* of C if Equation 3.2 is satisfied.

3.2. An Alternative Construction of Toric Manifolds. We briefly review the Delzant construction [5] in order to construct inhomogeneous coordinate charts on a toric manifold. Delzant showed that there is a one-to-one correspondence between compact symplectic toric manifolds and Delzant polytopes, which are moment polytopes for the Hamiltonian torus action on toric manifolds (see [9, Chapter 1] for detailed explanations about the Delzant construction). Delzant polytopes are defined as follows:

DEFINITION 3.8. Delzant polytopes are convex polytopes Δ in $(\mathfrak{t}^n)^* \cong \mathbb{R}^n$ satisfying the following three conditions:

- simple; each vertex has n edges,
- rational; the direction vectors $v_1^\lambda, \ldots, v_n^\lambda$ from any vertex $\lambda \in \Lambda$ are integral vectors,
- smooth; the vectors $v_1^\lambda, \ldots, v_n^\lambda$ chosen as above form a basis of $\mathbb{Z}^n$,

where Λ is the set of the vertices in Δ.

We can define Delzant polytopes in terms of facets in Δ instead of edges (see [2, Theorem 4] for example).

DEFINITION 3.9. Delzant polytopes are convex polytopes Δ in $(\mathfrak{t}^n)^* \cong \mathbb{R}^n$ satisfying the following three conditions:

- simple; each vertex meets n facets,
- rational; the inward pointing normal vectors $u_1^\lambda, \ldots, u_n^\lambda$ for facets meeting a vertex $\lambda \in \Lambda$ are integral vectors,
- smooth; the vectors $u_1^\lambda, \ldots, u_n^\lambda$ chosen as above form a basis of $\mathbb{Z}^n$,

where Λ is the set of the vertices in Δ.

We can see that two ways to define Delzant polytopes are equivalent. Although we can find a similar result in [4, Proposition 2.2], we give a proof because we shall use the statement repeatedly.

Lemma 3.10. *Let $v_1^\lambda, \ldots, v_n^\lambda$ be the direction vectors and $u_1^\lambda, \ldots, u_n^\lambda$ the inward pointing normal vectors for $\lambda \in \Lambda$. Then,*

$$[v_1^\lambda \cdots v_n^\lambda] \begin{bmatrix} {}^t u_1^\lambda \\ \vdots \\ {}^t u_n^\lambda \end{bmatrix} = E_n,$$

where E_n denotes the identity matrix.

Proof. Let $e_1, \ldots, e_n$ be the standard basis in $\mathbb{R}^n$. Since the direction vectors $v_1^\lambda, \ldots, v_n^\lambda$ form a basis of $\mathbb{Z}^n$, there exists a square matrix B_n^λ such that

$$E_n = [e_1 \cdots e_n] = [v_1^\lambda \cdots v_n^\lambda] B_n^\lambda. \tag{3.3}$$

Since the matrix B_n^λ is the inverse matrix for the matrix $[v_1^\lambda \cdots v_n^\lambda]$, we see $B_n^\lambda [v_1^\lambda \cdots v_n^\lambda] = E_n$. Moreover, the matrix B_n^λ is in $GL(n; \mathbb{Z})$ from (3.3).

We define the vectors $u_1^\lambda, \ldots, u_n^\lambda \in \mathbb{Z}^n$ by

$$B_n^\lambda = \begin{bmatrix} {}^t u_1^\lambda \\ \vdots \\ {}^t u_n^\lambda \end{bmatrix}.$$

By calculating $B_n^\lambda[v_1^\lambda \cdots v_n^\lambda]$, we obtain

$$\begin{aligned} B_n^\lambda[v_1^\lambda \cdots v_n^\lambda] &= \begin{bmatrix} {}^t u_1^\lambda \\ \vdots \\ {}^t u_n^\lambda \end{bmatrix} [v_1^\lambda \cdots v_n^\lambda] \\ &= \begin{bmatrix} {}^t u_1^\lambda v_1^\lambda & \cdots & {}^t u_1^\lambda v_n^\lambda \\ \vdots & \ddots & \vdots \\ {}^t u_n^\lambda v_1^\lambda & \cdots & {}^t u_n^\lambda v_n^\lambda \end{bmatrix} \\ &= \begin{bmatrix} \langle u_1^\lambda, v_1^\lambda \rangle & \cdots & \langle u_1^\lambda, v_n^\lambda \rangle \\ \vdots & \ddots & \vdots \\ \langle u_n^\lambda, v_1^\lambda \rangle & \cdots & \langle u_1^\lambda, v_n^\lambda \rangle \end{bmatrix}. \end{aligned}$$

Since $B_n^\lambda[v_1^\lambda \cdots v_n^\lambda] = E_n$, we have $\langle u_i^\lambda, v_j^\lambda \rangle = \delta_{ij}$ (Kronecker's delta). We say that

$$u_1^\lambda \in \operatorname{span}\{v_2^\lambda, \ldots, v_n^\lambda\}^\perp, \ldots, u_n^\lambda \in \operatorname{span}\{v_1^\lambda, \ldots, v_{n-1}^\lambda\}^\perp,$$

which means that the vectors $u_1^\lambda, \ldots, u_n^\lambda$ are inward pointing normal vectors to facets meeting the vertex λ. □

For $\lambda \in \Lambda$, we define an $n \times n$ matrix $Q^\lambda = \left[v_1^\lambda, \ldots, v_n^\lambda\right] (= \left[Q_{ij}^\lambda\right])$. In general $\det Q^\lambda = \pm 1$ by the definition, but we assume $\det Q^\lambda = 1$ by changing the numbering of $v_1^\lambda, \ldots, v_n^\lambda$. We also define a matrix $D^{\lambda\mu}$ by $D^{\lambda\mu} = (Q^\lambda)^{-1} Q^\mu (= [d_{ij}^{\lambda\mu}])$ for each $\lambda, \mu \in \Lambda$ as we defined in Section 2.

From the construction in Section 2, we obtain a system of complex coordinate charts $\{(U_\lambda, \varphi_\lambda)\}_{\lambda \in \Lambda}$ on a toric manifold X associated with a Delzant polytope Δ.

REMARK 3.11. Azam, Cannizzo, and Lee explained the construction of symplectic toric manifolds with a system of the inhomogeneous coordinate charts from a data of Delzant polytopes [2]. In this case, the coordinate transformation of our system of complex coordinate charts $\{(U_\lambda, \varphi_\lambda)\}_{\lambda \in \Lambda}$ coincides with the one constructed in [2]. Hereafter, we call $\{(U_\lambda, \varphi_\lambda)\}_{\lambda \in \Lambda}$ a system of the inhomogeneous coordinate chart on a toric manifold.

From this section, we write X by a compact toric manifold of complex dimension n, Δ by the Delzant polytope of X, Λ by the set of the vertices in the polytope Δ.

REMARK 3.12. The coordinate transformation of the inhomogeneous coordinates also coincides with the one in algebraic geometry (see for example [7]). Note that we may have a fan of toric manifolds X by taking integral vectors inward pointing normal to each facets of Delzant polytopes of X.

3.3. Toric Divisor Complements. In this section, we construct a diffeomorphism between the complements of toric divisors in X and $(\mathbb{C}^*)^n$.

We define $\check{U}_\lambda = \{[z^\lambda] \in U_\lambda \mid z_1^\lambda z_2^\lambda \cdots z_n^\lambda \neq 0\} \subset X$ for each $\lambda \in \Lambda$, then we have $\check{M} = \bigcup_{\lambda\in\Lambda} \check{U}_\lambda$. We call $\check{M}$ be the *toric divisor complement*. Furthermore, we see $\check{U}_\lambda = \check{U}_\sigma = \check{M}$ for $\lambda, \sigma \in \Lambda$.

DEFINITION 3.13. We define a map $\phi_\lambda : \varphi_\lambda(\check{U}_\lambda) \to (\mathbb{C}^*)^n$ by

$$\phi_\lambda(z_1^\lambda, \ldots, z_n^\lambda) = \left(\prod_{j=1}^n (z_j^\lambda)^{\hat{Q}_{j1}^\lambda}, \ldots, \prod_{j=1}^n (z_j^\lambda)^{\hat{Q}_{jn}^\lambda}\right),$$

where $(Q^\lambda)^{-1} = [\hat{Q}_{ij}^\lambda]$.

Lemma 3.14. *For any $\lambda, \sigma \in \Lambda$, $\phi_\lambda \circ \varphi_\lambda = \phi_\sigma \circ \varphi_\sigma$.*

Proof. Since we define $D^{\sigma\lambda} = (Q^\sigma)^{-1}Q^\lambda$, we see $D^{\sigma\lambda}(Q^\lambda)^{-1} = (Q^\sigma)^{-1}$. For any $[z^\lambda] \in \check{U}_\lambda$, we have

$$\begin{aligned}
\phi_\lambda \circ \varphi_\lambda([z^\lambda]) =& \phi_\lambda \circ \varphi_\lambda\left(\left[\left(\prod_{j=1}^n (z_j^\sigma)^{d_{j1}^{\sigma\lambda}}, \ldots, \prod_{j=1}^n (z_j^\sigma)^{d_{jn}^{\sigma\lambda}}\right)\right]\right)\\
=& \phi_\lambda\left(\prod_{j=1}^n (z_j^\sigma)^{d_{j1}^{\sigma\lambda}}, \ldots, \prod_{j=1}^n (z_j^\sigma)^{d_{jn}^{\sigma\lambda}}\right)\\
=& \left(\prod_{i=1}^n \left(\prod_{j=1}^n (z_j^\sigma)^{d_{ji}^{\sigma\lambda}}\right)^{\hat{Q}_{i1}^\lambda}, \ldots, \prod_{i=1}^n \left(\prod_{j=1}^n (z_j^\sigma)^{d_{ji}^{\sigma\lambda}}\right)^{\hat{Q}_{in}^\lambda}\right)\\
=& \left(\prod_{j=1}^n (z_j^\sigma)^{\hat{Q}_{j1}^\sigma}, \ldots, \prod_{j=1}^n (z_j^\sigma)^{\hat{Q}_{jn}^\sigma}\right)\\
=& \phi_\sigma \circ \varphi_\sigma([z^\sigma]).
\end{aligned}$$

□

From Lemma 3.14, we can define the following map independent of the choice of $\lambda \in \Lambda$.

DEFINITION 3.15. We define a map $\phi : \check{M} \to (\mathbb{C}^*)^n = \{(z_1, \ldots, z_n) \mid z_1 z_2 \cdots z_n \neq 0\}$ by $\phi = \phi_\lambda \circ \varphi_\lambda$.

Next we construct the inverse map $\hat{\phi} : (\mathbb{C}^*)^n \to \check{M}$, which is actually similar to the construction of ϕ.

DEFINITION 3.16. We define a map $\hat{\phi}_\lambda : (\mathbb{C}^*)^n \to \varphi_\lambda(\check{U}_\lambda)$ by

$$\hat{\phi}_\lambda(z_1, \ldots, z_n) = \left(\prod_{j=1}^n (z_j)^{Q_{j1}^\lambda}, \ldots, \prod_{j=1}^n (z_j)^{Q_{jn}^\lambda}\right).$$

Lemma 3.17. *For any $\lambda, \sigma \in \Lambda$, $\varphi_\lambda^{-1} \circ \hat{\phi}_\lambda = \varphi_\sigma^{-1} \circ \hat{\phi}_\sigma$.*

Proof. Since we define $D^{\lambda\sigma} = (Q^\lambda)^{-1}Q^\sigma$, we see $Q^\lambda D^{\lambda\sigma} = Q^\sigma$. For any $z = (z_1, \ldots, z_n) \in (\mathbb{C}^*)^n$, we have

$$\varphi_\lambda^{-1} \circ \hat{\phi}_\lambda(z) = \varphi_\lambda^{-1}\left(\prod_{j=1}^n (z_j)^{Q_{j1}^\lambda}, \ldots, \prod_{j=1}^n (z_j)^{Q_{jn}^\lambda}\right)$$

$$
\begin{aligned}
&=\left[\left(\prod_{j=1}^{n}(z_j)^{Q^{\lambda}_{j1}},\dots,\prod_{j=1}^{n}(z_j)^{Q^{\lambda}_{jn}}\right)\right]\\
&=\left[\left(\prod_{i=1}^{n}\left(\prod_{j=1}^{n}(z_j)^{Q^{\lambda}_{ji}}\right)^{d^{\lambda\sigma}_{i1}},\dots,\prod_{i=1}^{n}\left(\prod_{j=1}^{n}(z_j)^{Q^{\lambda}_{ji}}\right)^{d^{\lambda\sigma}_{in}}\right)\right]\\
&=\left[\left(\prod_{j=1}^{n}(z_j)^{Q^{\sigma}_{j1}},\dots,\prod_{j=1}^{n}(z_j)^{Q^{\sigma}_{jn}}\right)\right]\\
&=\varphi_{\sigma}^{-1}\circ\hat{\phi}_{\sigma}(z). \qquad \square
\end{aligned}
$$

From Lemma 3.17, we can define the following map independent of the choice of $\lambda \in \Lambda$.

Definition 3.18. We define a map $\hat{\phi} : (\mathbb{C}^*)^n \to \check{M}$ by $\hat{\phi} = \varphi_{\lambda}^{-1} \circ \hat{\phi}_{\lambda}$.

We check that the map $\hat{\phi}$ defined in Definition 3.18 is the inverse map of ϕ defined in Definition 3.15.

Lemma 3.19. *$\hat{\phi} \circ \phi = id_{\check{M}}$, $\phi \circ \hat{\phi} = id_{(\mathbb{C}^*)^n}$.*

Proof. Since we define $\phi = \phi_{\lambda} \circ \varphi_{\lambda}$ and $\hat{\phi} = \varphi_{\lambda}^{-1} \circ \hat{\phi}_{\lambda}$, we obtain $\hat{\phi} \circ \phi = \varphi_{\lambda}^{-1} \circ \hat{\phi}_{\lambda} \circ \phi_{\lambda} \circ \varphi_{\lambda}$ and $\phi \circ \hat{\phi} = \phi_{\lambda} \circ \varphi_{\lambda} \circ \varphi_{\lambda}^{-1} \circ \hat{\phi}_{\lambda} = \phi_{\lambda} \circ \hat{\phi}_{\lambda}$. We say that it is sufficient to show that $\hat{\phi}_{\lambda} \circ \phi_{\lambda} = \mathrm{id}_{\varphi_{\lambda}(\check{U}_{\lambda})}$ and $\phi_{\lambda} \circ \hat{\phi}_{\lambda} = \mathrm{id}_{(\mathbb{C}^*)^n}$.

For $z^{\lambda} \in \varphi_{\lambda}(\check{U}_{\lambda})$, by similar calculation, we have

$$
\begin{aligned}
\hat{\phi}_{\lambda} \circ \phi_{\lambda}(z_1^{\lambda},\dots,z_n^{\lambda}) &=\hat{\phi}_{\lambda}\left(\prod_{j=1}^{n}(z_j^{\lambda})^{\hat{Q}^{\lambda}_{j1}},\dots,\prod_{j=1}^{n}(z_j^{\lambda})^{\hat{Q}^{\lambda}_{jn}}\right)\\
&=\left(\prod_{i=1}^{n}\left(\prod_{j=1}^{n}(z_j^{\lambda})^{\hat{Q}^{\lambda}_{ji}}\right)^{Q^{\lambda}_{i1}},\dots,\prod_{i=1}^{n}\left(\prod_{j=1}^{n}(z_j^{\lambda})^{\hat{Q}^{\lambda}_{ji}}\right)^{Q^{\lambda}_{in}}\right)\\
&=\left(\prod_{j=1}^{n}(z_j^{\lambda})^{\delta_{j1}},\dots,\prod_{j=1}^{n}(z_j^{\lambda})^{\delta_{jn}}\right)\\
&=(z_1^{\lambda},\dots,z_n^{\lambda}).
\end{aligned}
$$

Thus we obtain $\hat{\phi}_{\lambda} \circ \phi_{\lambda} = \mathrm{id}_{\varphi_{\lambda}(\check{U}_{\lambda})}$.

For $z \in (\mathbb{C}^*)^n$, by similar calculation, we have

$$
\begin{aligned}
\phi_{\lambda} \circ \hat{\phi}_{\lambda}(z_1,\dots,z_n) &=\phi_{\lambda}\left(\prod_{j=1}^{n}(z_j)^{Q^{\lambda}_{j1}},\dots,\prod_{j=1}^{n}(z_j)^{Q^{\lambda}_{jn}}\right)\\
&=\left(\prod_{i=1}^{n}\left(\prod_{j=1}^{n}(z_j)^{Q^{\lambda}_{ji}}\right)^{\hat{Q}^{\lambda}_{i1}},\dots,\prod_{i=1}^{n}\left(\prod_{j=1}^{n}(z_j)^{Q^{\lambda}_{ji}}\right)^{\hat{Q}^{\lambda}_{in}}\right)\\
&=(z_1,\dots,z_n).
\end{aligned}
$$

Thus we obtain $\phi_{\lambda} \circ \hat{\phi}_{\lambda} = \mathrm{id}_{(\mathbb{C}^*)^n}$. $\square$

4. Torus-equivariantly Embedded Toric Manifolds

We construct k-dimensional complex submanifolds $\overline{C(V)}$ in toric manifolds X associated to affine subspaces V in $\mathbb{R}^n \cong \mathfrak{t}^n$ and examine their fundamental properties.

In Section 4.1, we give the construction of $\overline{C(V)}$. In Section 4.2, we consider a Hamiltonian subtorus action on a toric manifold X. In Section 4.3, we consider the Hamiltonian torus action on $\overline{C(V)}$.

4.1. Construction of Torus-equivariantly Embedded Toric Manifolds. First, we will concentrate on Yamamoto's construction [11, Lemma 6.1] of complex submanifolds $C(V)$ in $(\mathbb{C}^*)^n$. Let $e_1, \dots, e_n$ be the standard basis in $\mathbb{R}^n$. We write $\langle , \rangle$ for the inner product of vectors. Fix $k = 1, \dots, n$. Let $p_1, \dots, p_k \in \mathbb{Z}^n$ be primitive vectors which are linearly independent, and $a \in \mathbb{R}^n$. Then we consider an affine subspace $V = \mathbb{R}p_1 + \cdots + \mathbb{R}p_k + a \cong \mathbb{R}^k$ in $\mathbb{R}^n$, which may have rational slope. Yamamoto constructed k-dimensional complex manifolds $C(V)$ in $(\mathbb{C}^*)^n$. Although we do not give the same statement as the original one, the statement is like as follows.

Proposition 4.1. *Given an affine subspace $V = \mathbb{R}p_1 + \cdots + \mathbb{R}p_k + a$ in $\mathbb{R}^n$, we can construct a complex submanifold $C(V) \cong (\mathbb{C}^*)^k$ in $\check{M} \cong (\mathbb{C}^*)^n$ by*

$$C(V) = \left\{ (e^{x_1+\sqrt{-1}y_1}, \dots, e^{x_n+\sqrt{-1}y_n}) \in (\mathbb{C}^*)^n \;\middle|\; \begin{array}{l} x_i = \displaystyle\sum_{l=1}^{k} \langle p_l, e_i \rangle u_l + \langle a, e_i \rangle, \\ y_i = \displaystyle\sum_{l=1}^{k} \langle p_l, e_i \rangle v_l \end{array} \right\},$$

where $(e^{u_1+\sqrt{-1}v_1}, \dots, e^{u_k+\sqrt{-1}v_k}) \in (\mathbb{C}^*)^k$.

Note that if $k = 0$, then $C(V)$ is a point $(e^{\langle a, e_1 \rangle}, \dots, e^{\langle a, e_n \rangle})$ in $(\mathbb{C}^*)^n$. We rewrite the expression of $C(V)$ in Proposition 4.1 as follows:

Proposition 4.2. *Let $C(V)$ be a complex submanifold in $\check{M}$ given in* Proposition 4.1. *There exists a primitive basis $q_{k+1}, \dots, q_n \in \mathbb{Z}^n$ of the orthogonal subspace to $V = \mathbb{R}p_1 + \cdots + \mathbb{R}p_k + a$ in $\mathbb{R}^n$ such that*

$$C(V) = \left\{ (e^{w_1}, \dots, e^{w_n}) \in (\mathbb{C}^*)^n \;\middle|\; {}^t[q_{k+1} \cdots q_n] \left(\begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix} - a \right) = 0 \right\},$$

where $w_1 = x_1 + \sqrt{-1}y_1, \dots, w_n = x_n + \sqrt{-1}y_n$.

Proof. Hereafter, we calculate angle coordinates $y = (y_1, \dots, y_n)$ and $v = (v_1, \dots, v_k)$ up to $2\pi\mathbb{Z}$.

From Proposition 4.1, we have

$$\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = [p_1 \cdots p_k] \begin{bmatrix} u_1 \\ \vdots \\ u_k \end{bmatrix} + a, \quad \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = [p_1 \cdots p_k] \begin{bmatrix} v_1 \\ \vdots \\ v_k \end{bmatrix}.$$

We obtain

$$\begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix} - a = \begin{bmatrix} x_1 + \sqrt{-1}y_1 \\ \vdots \\ x_n + \sqrt{-1}y_n \end{bmatrix} - a = [p_1 \cdots p_k] \begin{bmatrix} u_1 + \sqrt{-1}v_1 \\ \vdots \\ u_k + \sqrt{-1}v_k \end{bmatrix}.$$

We can take a basis of primitive vectors $q_{k+1}, \ldots, q_n \in \mathbb{Z}^n$ of the orthogonal subspace to V. Multiplying both sides by ${}^t[q_{k+1} \cdots q_n]$, we obtain

$$\begin{aligned} &{}^t[q_{k+1} \cdots q_n] \left(\begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix} - a \right) \\ =& \begin{bmatrix} {}^t q_{k+1} \\ \vdots \\ {}^t q_n \end{bmatrix} [p_1 \cdots p_k] \begin{bmatrix} u_1 + \sqrt{-1}v_1 \\ \vdots \\ u_k + \sqrt{-1}v_k \end{bmatrix} \\ =& \begin{bmatrix} \langle q_{k+1}, p_1 \rangle & \langle q_{k+1}, p_2 \rangle & \cdots & \langle q_{k+1}, p_k \rangle \\ \langle q_{k+2}, p_1 \rangle & \langle q_{k+2}, p_2 \rangle & \cdots & \langle q_{k+2}, p_k \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle q_n, p_1 \rangle & \langle q_n, p_2 \rangle & \cdots & \langle q_n, p_k \rangle \end{bmatrix} \begin{bmatrix} u_1 + \sqrt{-1}v_1 \\ \vdots \\ u_k + \sqrt{-1}v_k \end{bmatrix} \\ =&0. \end{aligned}$$

□

Note that this submanifold $C(V)$ can be regarded as a complex subtorus $(\mathbb{C}^*)^k$ in $(\mathbb{C}^*)^n$.

We will write $C(V)$ explicitly as a submanifold in $\check{M}$. Recall that $\check{M} = \bigcup_{\lambda \in \Lambda} \check{U}_\lambda$. Using the map $\hat{\phi}_\lambda : (\mathbb{C}^*)^n \to \varphi_\lambda(\check{U}_\lambda)$, we obtain

$$\hat{\phi}_\lambda(C(V)) = \{z^\lambda = e^{w^\lambda} \in \varphi_\lambda(\check{U}_\lambda) \mid {}^t[q_{k+1} \cdots q_n]((Q^\lambda)^{-1} w^\lambda - a) = 0\},$$

where $z^\lambda = e^{w^\lambda}$ means that $(z_1^\lambda, \ldots, z_n^\lambda) = (e^{w_1^\lambda}, \ldots, e^{w_n^\lambda})$.

Next we take the closure of $\hat{\phi}_\lambda(C(V))(\subset \varphi_\lambda(U_\lambda) \cong \mathbb{C}^n)$. Since $\hat{Q}_{il}^\lambda = \langle u_i^\lambda, e_l \rangle$, we have

$$\sum_{l=1}^{n} \hat{Q}_{il}^\lambda \langle q_j, e_l \rangle = \langle u_i^\lambda, q_j \rangle$$

for $i = 1, \ldots, n$ and $j = k+1, \ldots, n$. Define three subsets $\mathcal{I}_{\lambda,j}^+, \mathcal{I}_{\lambda,j}^-, \mathcal{I}_{\lambda,j}^0 \subset \{1, 2, \ldots, n\}$ by

$$\mathcal{I}_{\lambda,j}^+ = \left\{ i \in \{1, 2, \ldots, n\} \ \langle u_i^\lambda, q_j \rangle \geq 0 \right\}, \tag{4.1}$$

$$\mathcal{I}_{\lambda,j}^- = \left\{ i \in \{1, 2, \ldots, n\} \ \langle u_i^\lambda, q_j \rangle \leq 0 \right\}, \tag{4.2}$$

$$\mathcal{I}_{\lambda,j}^0 = \left\{ i \in \{1, 2, \ldots, n\} \ \langle u_i^\lambda, q_j \rangle = 0 \right\}, \tag{4.3}$$

for $\lambda \in \Lambda$ and $j = k+1, \ldots, n$. Note that $\mathcal{I}_{\lambda,j}^+ \cap \mathcal{I}_{\lambda,j}^- = \mathcal{I}_{\lambda,j}^0$ and $\mathcal{I}_{\lambda,j}^+ \cup \mathcal{I}_{\lambda,j}^- = \{1, 2, \ldots, n\}$ for any $\lambda \in \Lambda$ and $j = k+1, \ldots, n$.

From the expression of $\hat{\phi}_\lambda(C(V))$, direct calculation gives us

$$\begin{aligned} \langle q_l, (Q^\lambda)^{-1} w^\lambda - a \rangle &= \langle q_l, (Q^\lambda)^{-1} \log z^\lambda - a \rangle \\ &= \langle {}^t(Q^\lambda)^{-1} q_l, \log z^\lambda \rangle - \langle q_l, a \rangle \end{aligned}$$

$$= \left\langle \begin{bmatrix} \langle u_1^\lambda, q_l\rangle \\ \vdots \\ \langle u_n^\lambda, q_l\rangle \end{bmatrix}, \log z^\lambda \right\rangle - \langle a, q_l\rangle$$

$$= \log\left(\prod_{j=1}^{n} (z_j^\lambda)^{\langle u_j^\lambda, q_l\rangle}\right) - \langle a, q_l\rangle$$

for $l = k+1, \dots, n$. We can define $\overline{C(V)} = \bigcup_{\lambda\in\Lambda} \varphi_\lambda^{-1}(\overline{C_\lambda(V)}) \subset X$ by

$$\overline{C_\lambda(V)} = \left\{ z^\lambda \in \varphi_\lambda(U_\lambda) \;\; f_j^\lambda(z^\lambda) = 0 \text{ for } j = k+1, \dots, n \right\}, \tag{4.4}$$

where f_j^λ is defined by

$$f_j^\lambda(z^\lambda) = \prod_{i\in\mathcal{I}_{\lambda,j}^+} (z_i^\lambda)^{\langle u_i^\lambda, q_j\rangle} - e^{\langle a, q_j\rangle} \prod_{i\in\mathcal{I}_{\lambda,j}^-} (z_i^\lambda)^{-\langle u_i^\lambda, q_j\rangle} \tag{4.5}$$

for each $j = k+1, \dots, n$. Here, if $\mathcal{I}_{\lambda,j}^+ = \emptyset$, then $\prod_{i\in\mathcal{I}_{\lambda,j}^+} (z_i^\lambda)^{\langle u_i^\lambda, q_j\rangle} = 1$. Similarly, if $\mathcal{I}_{\lambda,j}^- = \emptyset$, then $\prod_{i\in\mathcal{I}_{\lambda,j}^-} (z_i^\lambda)^{-\langle u_i^\lambda, q_j\rangle} = 1$.

$\overline{C_\lambda(V)}$ is a zero locus of $f_{k+1}^\lambda, \dots, f_n^\lambda$. Note that if $k = 0$, then $\overline{C(V)}$ is a point in X.

REMARK 4.3. By the implicit function theorem, $\overline{C(V)}$ is a complex submanifold in X if the rank of the Jacobian matrix of $f_{k+1}^\lambda, \dots, f_n^\lambda$ is equal to $n-k$ for any points $p \in \overline{C(V)}$ and any $\lambda \in \Lambda$.

We demonstrate some examples for complex submanifolds $\overline{C(V)}$. Example 4.4 gives an example for a complex submanifold in $X = \mathbb{C}P^2$, while Example 4.5 deals with a subset in $X = \mathbb{C}P^2$ which does not become a complex submanifold in $\mathbb{C}P^2$. In the following two examples, define the points λ, μ, σ in $(\mathfrak{t}^2)^* \cong \mathbb{R}^2$ by

$$\lambda = (0,0),\ \mu = (2,0),\ \sigma = (0,2).$$

Let Δ be a polytope defined by the convex hull of the points λ, μ, σ. From the Delzant polytope Δ of $\mathbb{C}P^2$ we define the inward pointing normal vectors to the facets by

$$u_1^\lambda = \begin{bmatrix}1\\0\end{bmatrix}, u_2^\lambda = \begin{bmatrix}0\\1\end{bmatrix}, u_1^\mu = \begin{bmatrix}-1\\-1\end{bmatrix}, u_2^\mu = \begin{bmatrix}1\\0\end{bmatrix}, u_1^\sigma = \begin{bmatrix}0\\1\end{bmatrix}, u_2^\sigma = \begin{bmatrix}-1\\-1\end{bmatrix},$$

where $\Lambda = \{\lambda, \mu, \sigma\}$.

EXAMPLE 4.4. Let $X = \mathbb{C}P^2$, $k = 1$, V be spanned by $p = {}^t[1\ 1]$. We can choose a basis q of the orthogonal subspace to V as $q = {}^t[1\ -1]$. In this case, the subset $\overline{C(V)}$ is a complex submanifold in X. Indeed, we give $f^\lambda, f^\mu, f^\sigma$ by

$$f^\lambda = z_1^\lambda - z_2^\lambda,\ f^\mu = z_2^\mu - 1,\ f^\sigma = 1 - z_1^\sigma,$$

respectively. Since the Jacobian matrices are expressed as

$$Df^\lambda = [1\ -1],\ Df^\mu = [0\ 1],\ Df^\sigma = [1\ 0],$$

respectively, we see the rank of each matrix is one.

Example 4.5. Let $X = \mathbb{C}P^2$, $k = 1$, V be spanned by $p = {}^t[3\ 1]$. We can choose a basis q of the orthogonal subspace to V as $q = {}^t[1\ -3]$. In this case, the subset $\overline{C(V)}$ is not a complex submanifold in X. Indeed, we give $f^\lambda, f^\mu, f^\sigma$ by

$$f^\lambda = z_1^\lambda - (z_2^\lambda)^3,\ f^\mu = (z_1^\mu)^2 z_2^\mu - 1,\ f^\sigma = (z_1^\sigma)^3 - (z_2^\sigma)^2,$$

respectively. Note that $(0,0) \notin \overline{C_\mu(V)}$. Since the Jacobian matrices are expressed as

$$Df^\lambda = [1\ \ -3(z_2^\lambda)^2],\ Df^\mu = [2z_1^\mu z_2^\mu\ (z_1^\mu)^2],\ Df^\sigma = [3(z_1^\sigma)^2\ \ -2z_2^\sigma],$$

respectively, we see the rank of Df^λ and Df^μ is one. However, the rank of Df^σ becomes zero at the point $(z_1^\sigma, z_2^\sigma) = (0,0) \in \overline{C_\sigma(V)}$.

Notice that if we fix a toric manifold X, then we may classify examples of complex submanifolds $\overline{C(V)}$ in X in terms of the conditions of V. Other examples for $X = \mathbb{C}P^2$ are treated in Section 5.1.

Suppose that $\overline{C(V)}$ is a complex submanifold in X. Then, there exists a map $i : \overline{C(V)} \to X$ as an embedding. By the construction of $\overline{C(V)}$ in this section, if $\overline{C(V)}$ is a complex submanifold in X, then there exists an embedding $i_\lambda : \overline{C_\lambda(V)} \to \varphi_\lambda(U_\lambda)$ for each $\lambda \in \Lambda$.

4.2. Subtorus Actions on Toric Manifolds. In this section, we consider a subtorus action on toric manifolds in order to give a Hamiltonian torus action on a complex submanifold given in Section 4.1.

First we define a k-dimensional torus action on a complex n-dimensional toric manifold X. Given an affine subspace $V = \mathbb{R}p_1 + \cdots + \mathbb{R}p_k + a$ in $\mathbb{R}^n$, define a map $i_V : T^k \to T^n$ by

$$i_V(t_1, \ldots, t_k) = \left(\prod_{l=1}^k t_l^{\langle p_l, e_1 \rangle}, \ldots, \prod_{l=1}^k t_l^{\langle p_l, e_n \rangle} \right). \tag{4.6}$$

Recall that the n-dimensional torus T^n action on $\varphi_\lambda(U_\lambda) \cong \mathbb{C}^n$ is given by

$$\begin{array}{ccc} T^n \times \varphi_\lambda(U_\lambda) & \to & \varphi_\lambda(U_\lambda) \\ (t = (t_1, \ldots, t_n), z^\lambda = (z_1^\lambda, \ldots, z_n^\lambda)) & \mapsto & t \cdot z^\lambda \end{array}$$

for each $\lambda \in \Lambda$, where $t \cdot z^\lambda$ is defined by

$$t \cdot z^\lambda = \left(\prod_{j=1}^n t_j^{Q_{j1}^\lambda} z_1^\lambda, \ldots, \prod_{j=1}^n t_j^{Q_{jn}^\lambda} z_n^\lambda \right) = \left(\prod_{j=1}^n t_j^{\langle e_j, v_1^\lambda \rangle} z_1^\lambda, \ldots, \prod_{j=1}^n t_j^{\langle e_j, v_n^\lambda \rangle} z_n^\lambda \right).$$

This torus action is compatible with the torus action on $(\mathbb{C}^*)^n$, which is given by

$$\begin{array}{ccc} T^n \times (\mathbb{C}^*)^n & \to & (\mathbb{C}^*)^n \\ ((t_1, \ldots, t_n), (z_1, \ldots, z_n)) & \mapsto & (t_1 z_1, \ldots, t_n z_n). \end{array}$$

We can describe the k-dimensional torus T^k action on X by

$$\begin{array}{ccc} T^k \times \varphi_\lambda(U_\lambda) & \to & \varphi_\lambda(U_\lambda) \\ (t = (t_1, \ldots, t_k), z^\lambda) & \mapsto & i_V(t) \cdot z^\lambda, \end{array}$$

where

$$i_V(t) \cdot z^\lambda = \left(\prod_{l=1}^k t_l^{\langle p_l, v_1^\lambda \rangle} z_1^\lambda, \ldots, \prod_{l=1}^k t_l^{\langle p_l, v_n^\lambda \rangle} z_n^\lambda \right). \tag{4.7}$$

We define the subset $\mathcal{J}_\lambda = \{j_1, \ldots, j_m\} \subset \{1, \ldots, n\}$ by

$$\mathcal{J}_\lambda = \{j \in \{1,\dots,n\} \mid \langle p_1, v_j^\lambda\rangle = \cdots = \langle p_k, v_j^\lambda\rangle = 0\}$$

for $\lambda \in \Lambda$. If $\mathcal{J}_\lambda = \emptyset$, then we may interpret $\mathcal{J}_\lambda = \{j_1,\dots,j_m\}$ as $m = 0$. Note that since the vectors $p_1,\dots,p_k$ form a basis of the linear part of V, we see that

$$\max_{\lambda\in\Lambda} |\mathcal{J}_\lambda| \le n - k.$$

Since our complex manifold X coincides with the one constructed in [2] as compact toric manifolds, X is equipped with the torus invariant symplectic form ω given by Guillemin [8] (see also [9, Appendix 2]). Moreover, the T^n-action on X is Hamiltonian with respect to the symplectic form ω. We write the moment map for the Hamiltonian T^n-action on X is $\mu : X \to (\mathfrak{t}^n)^*$. The fundamental property of moment maps for Hamiltonian torus actions is the convexity theorem [1, 10].

REMARK 4.6. The T^k-action given in Equation 4.7 is also Hamiltonian with respect to ω and the moment map for the action is given by $i_V^* \circ \mu : X \to (\mathfrak{t}^k)^*$.

We study the fixed point set of the T^k-action on X.

Lemma 4.7. *Consider the T^k-action on X defined above. The fixed point set of the T^k-action on $\varphi_\lambda(U_\lambda)$ is $\{z^\lambda \in \varphi_\lambda(U_\lambda) \mid z_i^\lambda = 0, i \notin \mathcal{J}_\lambda\}$.*

Proof. For simplicity, we give the proof for the case when $\mathcal{J}_\lambda = \{i\}$.

If $\langle p_l, v_i^\lambda\rangle = 0$ for $l = 1,\dots,k$, then we have

$$i_V(t)\cdot(0,\dots,0,z_i^\lambda,0,0\dots,0) = \left(0,\dots,0,\prod_{l=1}^k t_l^{\langle p_l,v_i^\lambda\rangle} z_i^\lambda,0,\dots,0\right) = (0,\dots,0,z_i^\lambda,0,\dots,0).$$

Thus, the point $(0,\dots,0,z_i^\lambda,0,\dots,0)$ is a fixed point of the T^k-action on $\varphi_\lambda(U_\lambda)$.

Conversely, if $(0,\dots,0,z_i^\lambda,0,\dots,0)$ is a fixed point of the T^k-action on $\varphi_\lambda(U_\lambda)$, then we have

$$\left(0,\dots,0,\prod_{l=1}^k t_l^{\langle p_l,v_i^\lambda\rangle} z_i^\lambda,0,\dots,0\right) = (0,\dots,0,z_i^\lambda,0,\dots,0).$$

Since $(t_1,\dots,t_k) \in T^k$, we see that $\langle p_l, v_i^\lambda\rangle = 0$ for $l = 1,\dots,k$.

Note that if $\mathcal{J}_\lambda = \emptyset$, we see that the fixed point of the T^k-action on $\varphi_\lambda(U_\lambda)$ is $(0,\dots,0) \in \varphi_\lambda(U_\lambda)$. □

Note that the set $\{z^\lambda \in \varphi_\lambda(U_\lambda) \mid z_i^\lambda = 0, i \notin \mathcal{J}_\lambda\}$ corresponds to the m-face defined by the direction vectors $v_{j_1}^\lambda,\dots,v_{j_m}^\lambda$ in Δ for $j_1,\dots,j_m \in \mathcal{J}_\lambda$. If $\mathcal{J}_\lambda = \emptyset$, then the set $\{z^\lambda \in \varphi_\lambda(U_\lambda) \mid z_i^\lambda = 0, i \notin \mathcal{J}_\lambda\} = \{(0,\dots,0) \in \varphi_\lambda(U_\lambda)\}$ corresponds to the vertex λ, which is a 0-face in Δ. In particular, $\{z^\lambda \in \varphi_\lambda(U_\lambda) \mid z_i^\lambda = 0, i \notin \mathcal{J}_\lambda\} \ne \emptyset$ for any $\mathcal{J}_\lambda$.

4.3. Torus Actions on Torus-equivariantly Embedded Toric Manifolds. After giving torus actions on $\overline{C(V)}$, we consider the image of the moment map for the torus action.

Under the following diagram;

$$\begin{array}{ccccc} T^n & \times & \varphi_\lambda(U_\lambda) & \longrightarrow & \varphi_\lambda(U_\lambda) \\ \uparrow i_V & & \uparrow i_\lambda & & \uparrow i_\lambda \\ T^k & \times & \overline{C_\lambda(V)} & \longrightarrow & \overline{C_\lambda(V)}, \end{array}$$

a T^k-action on $\overline{C(V)}$ is defined by

$$\begin{array}{ccc} T^k \times \overline{C_\lambda(V)} & \to & \overline{C_\lambda(V)} \\ (t = (t_1, \ldots, t_k), z^\lambda) & \mapsto & i_V(t) \cdot z^\lambda, \end{array}$$

which makes the above diagram commutative.

From Lemma 4.7 and the definition of $\overline{C_\lambda(V)}$, the following lemma is obvious.

Lemma 4.8. *The fixed point set of the T^k-action on $\overline{C_\lambda(V)}$ is $\{z^\lambda \in \varphi_\lambda(U_\lambda) \mid z_i^\lambda = 0, i \notin \mathcal{J}_\lambda\} \cap \overline{C_\lambda(V)}$.*

It is clear that if $\{z^\lambda \in \varphi_\lambda(U_\lambda) \mid z_i^\lambda = 0, i \notin \mathcal{J}_\lambda\} \cap \overline{C_\lambda(V)} \neq \emptyset$, then there exists a fixed point of the T^k-action on $\overline{C_\lambda(V)}$.

Lemma 4.9. *Assume that $k \geq 1$. If there exists $j = k+1, \ldots, n$ such that $\langle u_i^\lambda, q_j\rangle > 0$ (or $\langle u_i^\lambda, q_j\rangle < 0$) for all $i = 1, \ldots, n$, then $\{z^\lambda \in \varphi_\lambda(U_\lambda) \mid z_i^\lambda = 0, i \notin \mathcal{J}_\lambda\} \cap \overline{C_\lambda(V)} = \emptyset$, i.e., there is no fixed point of the T^k-action on $\overline{C_\lambda(V)}$.*

Proof. For simplicity, we assume that there exists $j = k+1, \ldots, n$ such that $\langle u_i^\lambda, q_j\rangle > 0$ for all $i = 1, \ldots, n$. This implies that $\mathcal{I}_{\lambda,j}^+ = \{1, \ldots, n\}$ and $\mathcal{I}_{\lambda,j}^- = \emptyset$. As we noted in the definition of $f_j^\lambda(z^\lambda)$, we obtain

$$f_j^\lambda(z^\lambda) = \prod_{i\in\mathcal{I}_{\lambda,j}^+} (z_i^\lambda)^{\langle u_i^\lambda, q_j\rangle} - e^{\langle a, q_j\rangle} \prod_{i\in\mathcal{I}_{\lambda,j}^-} (z_i^\lambda)^{-\langle u_i^\lambda, q_j\rangle} = \prod_{i=1}^n (z_i^\lambda)^{\langle u_i^\lambda, q_j\rangle} - e^{\langle a, q_j\rangle}.$$

Since $e^{\langle a, q_j\rangle} \neq 0$, $f_j^\lambda(z^\lambda) = 0$ implies that $z_1^\lambda z_2^\lambda \cdots z_n^\lambda \neq 0$, i.e.,

$$\{f_j^\lambda(z^\lambda) = 0\} \subset \{z_1^\lambda z_2^\lambda \cdots z_n^\lambda \neq 0\}.$$

Recall that $|\mathcal{J}_\lambda| \leq n-k$ for any $\lambda \in \Lambda$. If $k \geq 1$, then there exists $i_0 \notin \mathcal{J}_\lambda$ such that

$$\{z^\lambda \in \varphi_\lambda(U_\lambda) \mid z_i^\lambda = 0, i \notin \mathcal{J}_\lambda\} \subset \{z_{i_0}^\lambda = 0\}.$$

It is clear that $\{z_1^\lambda z_2^\lambda \cdots z_n^\lambda \neq 0\} \cap \{z_{i_0}^\lambda = 0\} = \emptyset$, which implies that $\{z^\lambda \in \varphi_\lambda(U_\lambda) \mid z_i^\lambda = 0, i \notin \mathcal{J}_\lambda\} \cap \overline{C_\lambda(V)} = \emptyset$. Since $\overline{C_\lambda(V)} = \bigcap_{j=k+1}^n \{f_j^\lambda(z^\lambda) = 0\}$, we obtain the desired result. □

Note that if $\langle u_i^\lambda, q_j\rangle > 0$ (or $\langle u_i^\lambda, q_j\rangle < 0$) for all $i = 1, \ldots, n$ and some $j = k+1, \ldots, n$, then $\overline{C_\lambda(V)} \subset \varphi_\lambda(\check{U}_\lambda)$.

Remark 4.10. If $\overline{C(V)}$ is a complex submanifold in X, the T^k-action on $\overline{C(V)}$ is actually Hamiltonian with respect to the symplectic form $i^*\omega$ on $\overline{C(V)}$ for the inclusion $i : \overline{C(V)} \to X$. If $\mu : X \to (\mathfrak{t}^n)^*$ is the moment map for the T^n-action on X, then we can obtain the moment map for the T^k-action on $\overline{C(V)}$ by $\overline{\mu} = i_V^* \circ \mu \circ i : \overline{C(V)} \to (\mathfrak{t}^k)^*$.

We further examine the fixed points of the T^k-action on $\overline{C(V)}$.

Since the map $i_V : T^k \to T^n$ is defined by (4.6), we can write the pull back $i_V^* : (\mathfrak{t}^n)^* \to$

$(\mathfrak{t}^k)^*$ as

$$i_V^*(\xi) = (\langle p_1, \xi\rangle, \dots, \langle p_k, \xi\rangle)\,. \tag{4.8}$$

Note that this map i_V^* is a surjective linear map. Hence, we have

$$i_V^*(\mu(X)) = \{(\langle p_1, \xi\rangle, \dots, \langle p_k, \xi\rangle) \mid \xi \in \mu(X)\} \subset (\mathfrak{t}^k)^*.$$

Since X is a toric manifold, $\mu(X) = \Delta$ is a Delzant polytope. In this situation, we obtain the followings:

Corollary 4.11. *Let Δ be a Delzant polytope and $i_V^* : (\mathfrak{t}^n)^* \to (\mathfrak{t}^k)^*$ be the map defined in* Equation 4.8. *Then, $i_V^*(\Delta)$ is a convex polytope.*

Proof. As we noted, the map i_V^* is a linear map. By the definition of Delzant polytopes, Δ is a convex polytope. By Lemma 3.2, $i_V^*(\Delta)$ is a convex polytope. □

Lemma 4.12. *Suppose that $\mathcal{J}_\lambda = \{j_1, \dots, j_m\}$. Let $\mathcal{F}_\lambda$ be an m-face of Δ defined by the direction vectors $v_{j_1}^\lambda, \dots, v_{j_m}^\lambda$. Then, $i_V^*(\xi) = i_V^*(\lambda)$ holds for any $\xi \in \mathcal{F}_\lambda$.*

Proof. Since $\xi, \lambda \in \mathcal{F}_\lambda$, we have

$$\xi - \lambda = \sum_{l=1}^{m} \alpha_l v_{j_l}^\lambda$$

for some $\alpha_1, \dots, \alpha_m \in \mathbb{R}$. We calculate

$$\begin{aligned} i_V^*(\xi) - i_V^*(\lambda) &= i_V^*(\xi - \lambda) \\ &= \left(\langle p_1, \sum_{l=1}^{m} \alpha_l v_{j_l}^\lambda \rangle, \dots, \langle p_k, \sum_{l=1}^{m} \alpha_l v_{j_l}^\lambda \rangle \right) \\ &= \left(\sum_{l=1}^{m} \alpha_l \langle p_1, v_{j_l}^\lambda \rangle, \dots, \sum_{l=1}^{m} \alpha_l \langle p_k, v_{j_l}^\lambda \rangle \right) \\ &= (0, \dots, 0). \end{aligned}$$

Thus we obtain $i_V^*(\xi) = i_V^*(\lambda)$ for any $\xi \in \mathcal{F}_\lambda$. □

By the definition of Delzant polytopes, we can take the direction vectors $v_1^\lambda, \dots, v_n^\lambda \in \mathbb{Z}^n$ from the vertex λ of a Delzant polytope Δ and the vectors $v_1^\lambda, \dots, v_n^\lambda \in \mathbb{Z}^n$ can be chosen as a basis of $\mathbb{Z}^n$. We define the cone C_λ by

$$C_\lambda = \mathbb{R}_{\geq 0} v_1^\lambda + \dots + \mathbb{R}_{\geq 0} v_n^\lambda \subset (\mathfrak{t}^n)^* \cong \mathbb{R}^n.$$

In other words, C_λ is generated by $\{v_1^\lambda, \dots, v_n^\lambda\}$. Since the map i_V^* is linear, by Lemma 3.4, $i_V^*(C_\lambda)$ is the cone generated by $\{i_V^*(v_1^\lambda), \dots, i_V^*(v_n^\lambda)\}$. The cone $i_V^*(C_\lambda)$ can be written concretely by

$$i_V^*(C_\lambda) = \mathbb{R}_{\geq 0} i_V^*(v_1^\lambda) + \dots + \mathbb{R}_{\geq 0} i_V^*(v_n^\lambda) \subset (\mathfrak{t}^k)^* \cong \mathbb{R}^k.$$

DEFINITION 4.13. Let Δ be a Delzant polytope and λ be a vertex in Δ. The point $i_V^*(\lambda)$ is a vertex in the convex polytope $i_V^*(\Delta)$ if $0 = (0, \dots, 0) \in i_V^*(C_\lambda) \subset (\mathfrak{t}^k)^*$ is a vertex in the sense of Definition 3.7.

We have the relation between the vectors $p_1, \dots, p_k$, $q_{k+1}, \dots, q_n$, $v_1^\lambda, \dots, v_n^\lambda$, and $u_1^\lambda, \dots, u_n^\lambda$.

Lemma 4.14. *For any $l = 1, \dots, k$ and any $j = k+1, \dots, n$,*

$$\sum_{i=1}^{n} \langle p_l, v_i^\lambda \rangle \langle u_i^\lambda, q_j \rangle = 0 \tag{4.9}$$

holds.

Proof. Instead of Equation 4.9, we show the matrix equation

$$\begin{bmatrix} {}^t p_1 \\ \vdots \\ {}^t p_k \end{bmatrix} \begin{bmatrix} v_1^\lambda & \cdots & v_n^\lambda \end{bmatrix} \begin{bmatrix} {}^t u_1^\lambda \\ \vdots \\ {}^t u_n^\lambda \end{bmatrix} \begin{bmatrix} q_{k+1} & \cdots & q_n \end{bmatrix} = 0. \tag{4.10}$$

From Lemma 3.10, we have obtained

$$\begin{bmatrix} v_1^\lambda & \cdots & v_n^\lambda \end{bmatrix} \begin{bmatrix} {}^t u_1^\lambda \\ \vdots \\ {}^t u_n^\lambda \end{bmatrix} = E_n. \tag{4.11}$$

Since the vectors $q_{k+1}, \dots, q_n$ are taken to be an orthogonal basis of the orthogonal subspace to V, we have

$$\begin{bmatrix} {}^t p_1 \\ \vdots \\ {}^t p_k \end{bmatrix} \begin{bmatrix} q_{k+1} & \cdots & q_n \end{bmatrix} = 0. \tag{4.12}$$

From Equation 4.11 and Equation 4.12, we calculate

$$\begin{bmatrix} {}^t p_1 \\ \vdots \\ {}^t p_k \end{bmatrix} \begin{bmatrix} v_1^\lambda & \cdots & v_n^\lambda \end{bmatrix} \begin{bmatrix} {}^t u_1^\lambda \\ \vdots \\ {}^t u_n^\lambda \end{bmatrix} \begin{bmatrix} q_{k+1} & \cdots & q_n \end{bmatrix} = \begin{bmatrix} {}^t p_1 \\ \vdots \\ {}^t p_k \end{bmatrix} \begin{bmatrix} q_{k+1} & \cdots & q_n \end{bmatrix} = 0.$$

Hence, we obtain Equation 4.10. □

From Lemma 4.14, we see that

$$\begin{bmatrix} \sum_{i=1}^{n} \langle p_1, v_i^\lambda \rangle \langle u_i^\lambda, q_j \rangle \\ \vdots \\ \sum_{i=1}^{n} \langle p_k, v_i^\lambda \rangle \langle u_i^\lambda, q_j \rangle \end{bmatrix} = 0$$

holds for any $j = k+1, \dots, n$. This is equivalent to the equation:

$$\sum_{i=1}^{n} \langle u_i^\lambda, q_j \rangle \begin{bmatrix} \langle p_1, v_i^\lambda \rangle \\ \vdots \\ \langle p_k, v_i^\lambda \rangle \end{bmatrix} = 0.$$

From (4.8), the following equation

$$\sum_{i=1}^{n}\langle u_i^\lambda, q_j\rangle i_V^*(v_i^\lambda) = 0 \tag{4.13}$$

holds for any $j = k+1, \dots, n$. Since the set $\mathcal{J}_\lambda \subset \{1, \dots, n\}$ was defined by

$$\mathcal{J}_\lambda = \{i \in \{1, \dots, n\} \mid \langle p_1, v_i^\lambda\rangle = \cdots = \langle p_k, v_i^\lambda\rangle = 0\},$$

Equation 4.13 can be written as

$$\sum_{i\notin\mathcal{J}_\lambda}\langle u_i^\lambda, q_j\rangle i_V^*(v_i^\lambda) = 0. \tag{4.14}$$

Note that since $|\mathcal{J}_\lambda| \le n-k$, the number of the terms in the left hand side of Equation 4.14 should be greater than or equal to k.

Lemma 4.15. *Fix $j = k+1, \dots, n$. If the point $i_V^*(\lambda)$ is a vertex in the convex polytope $i_V^*(\Delta)$, then $\{\langle u_i^\lambda, q_j\rangle\}_{i\notin\mathcal{J}_\lambda}$ satisfies either of the following conditions:*

(1) $\langle u_i^\lambda, q_j\rangle = 0$ *holds for any* $i \notin \mathcal{J}_\lambda$,

(2) *there exist* $i_j \neq i'_j \notin \mathcal{J}_\lambda$ *such that* $\langle u_{i_j}^\lambda, q_j\rangle\langle u_{i'_j}^\lambda, q_j\rangle < 0$.

Proof. Regarding Equation 4.14 as a linear combination of the vectors $i_V^*(v_i^\lambda)$ $(i \notin \mathcal{J}_\lambda)$, the coefficients $\langle u_i^\lambda, q_j\rangle$ $(i \notin \mathcal{J}_\lambda)$ satisfy at least one of the following cases:

- $\langle u_i^\lambda, q_j\rangle \ge 0$ holds for any $i \notin \mathcal{J}_\lambda$,
- $\langle u_i^\lambda, q_j\rangle \le 0$ holds for any $i \notin \mathcal{J}_\lambda$,
- there exist $i_j \neq i'_j \notin \mathcal{J}_\lambda$ such that $\langle u_{i_j}^\lambda, q_j\rangle\langle u_{i'_j}^\lambda, q_j\rangle < 0$.

If the point $i_V^*(\lambda)$ is a vertex in the convex polytope $i_V^*(\Delta)$, then by Definition 3.7 and Equation 4.14 the first and second cases can be written as $\langle u_i^\lambda, q_j\rangle = 0$ holds for any $i \notin \mathcal{J}_\lambda$. □

To check whether the fixed point set of the T^k-action on $\overline{C_\lambda(V)}$ is empty or not, we can use Lemma 4.15.

Lemma 4.16. *Fix $j = k+1, \dots, n$. If the point $i_V^*(\lambda)$ is a vertex in the convex polytope $i_V^*(\Delta)$ and if there exist $i_j \neq i'_j$ such that $\langle u_{i_j}^\lambda, q_j\rangle\langle u_{i'_j}^\lambda, q_j\rangle < 0$, then*

$$\{z^\lambda \in \varphi_\lambda(U_\lambda) \mid z_{i_j}^\lambda = z_{i'_j}^\lambda = 0\} \subset \{z^\lambda \in \varphi_\lambda(U_\lambda) \mid f_j^\lambda(z^\lambda) = 0\} \tag{4.15}$$

holds.

Proof. For simplicity, we assume that $\langle u_{i_j}^\lambda, q_j\rangle > 0$, $\langle u_{i'_j}^\lambda, q_j\rangle < 0$. This implies that $i_j \in \mathcal{I}_{\lambda,j}^+ \setminus \mathcal{I}_{\lambda,j}^0$ and $i'_j \in \mathcal{I}_{\lambda,j}^- \setminus \mathcal{I}_{\lambda,j}^0$. From Equation 4.5, we obtain Equation 4.15. □

As a corollary to Lemma 4.16, if the point $i_V^*(\lambda)$ is a vertex in the convex polytope $i_V^*(\Delta)$ and if $\{\langle u_i^\lambda, q_j\rangle\}_{i\notin\mathcal{J}_\lambda}$ satisfies the condition (2) in Lemma 4.15 for a fixed j, then we obtain

$$\{z^\lambda \in \varphi_\lambda(U_\lambda) \mid z_{i_j}^\lambda = z_{i'_j}^\lambda = 0, i_j \neq i'_j \notin \mathcal{J}_\lambda\} \subset \{z^\lambda \in \varphi_\lambda(U_\lambda) \mid f_j^\lambda(z^\lambda) = 0\}$$

for $i_j \neq i'_j \notin \mathcal{J}_\lambda$ appearing in the statement of the condition (2) in Lemma 4.15.

When $\{\langle u_i^\lambda, q_j\rangle\}_{i\notin\mathcal{J}_\lambda}$ satisfies the condition (1) in Lemma 4.15, we obtain the following result:

Proposition 4.17. *Assume that the point $i_V^*(\lambda)$ is a vertex in the convex polytope $i_V^*(\Delta)$. We define a point $\tilde{z}^\lambda = (\tilde{z}_1^\lambda, \ldots, \tilde{z}_n^\lambda) \in \mathbb{C}^n$ by setting*

$$\tilde{z}_i^\lambda = \begin{cases} e^{\langle a, v_i^\lambda \rangle}, & i \in \mathcal{J}_\lambda, \\ 0, & i \notin \mathcal{J}_\lambda. \end{cases}$$

If there exists $j_0 = k+1, \ldots, n$ such that $\{\langle u_i^\lambda, q_{j_0} \rangle\}_{i \notin \mathcal{J}_\lambda}$ satisfies the condition (1) *in* Lemma 4.15*, then we obtain*

$$\tilde{z}^\lambda \in \{z^\lambda \in \varphi_\lambda(U_\lambda) \mid z_i^\lambda = 0, i \notin \mathcal{J}_\lambda\} \cap \overline{C_\lambda(V)},$$

i.e., the set $\{z^\lambda \in \varphi_\lambda(U_\lambda) \mid z_i^\lambda = 0, i \notin \mathcal{J}_\lambda\} \cap \overline{C_\lambda(V)}$ is not empty. In particular, there exists a fixed point of the T^k-action on $\overline{C_\lambda(V)}$.

Proof. It is clear that $\tilde{z}^\lambda \in \{z^\lambda \in \varphi_\lambda(U_\lambda) \mid z_i^\lambda = 0, i \notin \mathcal{J}_\lambda\}$. We show that $\tilde{z}^\lambda \in \overline{C_\lambda(V)} = \bigcap_{j=k+1}^n \{z^\lambda \in \varphi_\lambda(U_\lambda) \mid f_j^\lambda(z^\lambda) = 0\}$.

Since we assume that the point $i_V^*(\lambda)$ is a vertex in the convex polytope $i_V^*(\Delta)$, $\{\langle u_i^\lambda, q_j \rangle\}_{i \notin \mathcal{J}_\lambda}$ satisfies either of the condition (1) or the condition (2) in Lemma 4.15 for each fixed j. We use the result to check that $\tilde{z}^\lambda \in \{z^\lambda \in \varphi_\lambda(U_\lambda) \mid f_j^\lambda(z^\lambda) = 0\}$ for any j.

If $\{\langle u_i^\lambda, q_j \rangle\}_{i \notin \mathcal{J}_\lambda}$ satisfies the condition (1) in Lemma 4.15 for some j, then $i \notin \mathcal{J}_\lambda$ implies $\langle u_i^\lambda, q_j \rangle = 0$, i.e., $i \in \mathcal{I}_{\lambda,j}^0$ for such j. By considering the contraposition, $\mathcal{I}_{\lambda,j}^+ \cup \mathcal{I}_{\lambda,j}^- \setminus \mathcal{I}_{\lambda,j}^0 \subset \mathcal{J}_\lambda$. We calculate $\prod_{i \in \mathcal{I}_{\lambda,j}^+} (\tilde{z}_i^\lambda)^{\langle u_i^\lambda, q_j \rangle}$ and $\prod_{i \in \mathcal{I}_{\lambda,j}^-} (\tilde{z}_i^\lambda)^{-\langle u_i^\lambda, q_j \rangle}$ as

$$\prod_{i \in \mathcal{I}_{\lambda,j}^+} (\tilde{z}_i^\lambda)^{\langle u_i^\lambda, q_j \rangle} = \prod_{i \in \mathcal{I}_{\lambda,j}^+ \setminus \mathcal{I}_{\lambda,j}^0} (\tilde{z}_i^\lambda)^{\langle u_i^\lambda, q_j \rangle} = \prod_{i \in \mathcal{I}_{\lambda,j}^+ \setminus \mathcal{I}_{\lambda,j}^0} (e^{\langle a, v_i^\lambda \rangle})^{\langle u_i^\lambda, q_j \rangle},$$
$$\prod_{i \in \mathcal{I}_{\lambda,j}^-} (\tilde{z}_i^\lambda)^{-\langle u_i^\lambda, q_j \rangle} = \prod_{i \in \mathcal{I}_{\lambda,j}^- \setminus \mathcal{I}_{\lambda,j}^0} (\tilde{z}_i^\lambda)^{-\langle u_i^\lambda, q_j \rangle} = \prod_{i \in \mathcal{I}_{\lambda,j}^- \setminus \mathcal{I}_{\lambda,j}^0} (e^{\langle a, v_i^\lambda \rangle})^{-\langle u_i^\lambda, q_j \rangle} \neq 0.$$

Moreover, since $i \in \mathcal{I}_{\lambda,j}^0$ means that $\langle u_i^\lambda, q_j \rangle = 0$, we obtain

$$\prod_{i \in \mathcal{I}_{\lambda,j}^- \setminus \mathcal{I}_{\lambda,j}^0} (e^{\langle a, v_i^\lambda \rangle})^{-\langle u_i^\lambda, q_j \rangle} = \prod_{i \in \mathcal{I}_{\lambda,j}^-} (e^{\langle a, v_i^\lambda \rangle})^{-\langle u_i^\lambda, q_j \rangle},$$
$$\prod_{i \in \mathcal{I}_{\lambda,j}^+ \setminus \mathcal{I}_{\lambda,j}^0} (e^{\langle a, v_i^\lambda \rangle})^{\langle u_i^\lambda, q_j \rangle} = \prod_{i \in \mathcal{I}_{\lambda,j}^+} (e^{\langle a, v_i^\lambda \rangle})^{\langle u_i^\lambda, q_j \rangle}.$$

As we noted that $\mathcal{I}_{\lambda,j}^+ \cup \mathcal{I}_{\lambda,j}^- = \{1, \ldots, n\}$, we can calculate

$$\begin{aligned}
\frac{\prod_{i \in \mathcal{I}_{\lambda,j}^+} (\tilde{z}_i^\lambda)^{\langle u_i^\lambda, q_j \rangle}}{\prod_{i \in \mathcal{I}_{\lambda,j}^-} (\tilde{z}_i^\lambda)^{-\langle u_i^\lambda, q_j \rangle}} &= \frac{\prod_{i \in \mathcal{I}_{\lambda,j}^+ \setminus \mathcal{I}_{\lambda,j}^0} (e^{\langle a, v_i^\lambda \rangle})^{\langle u_i^\lambda, q_j \rangle}}{\prod_{i \in \mathcal{I}_{\lambda,j}^- \setminus \mathcal{I}_{\lambda,j}^0} (e^{\langle a, v_i^\lambda \rangle})^{-\langle u_i^\lambda, q_j \rangle}} \\
&= \frac{\prod_{i \in \mathcal{I}_{\lambda,j}^+} (e^{\langle a, v_i^\lambda \rangle})^{\langle u_i^\lambda, q_j \rangle}}{\prod_{i \in \mathcal{I}_{\lambda,j}^-} (e^{\langle a, v_i^\lambda \rangle})^{-\langle u_i^\lambda, q_j \rangle}} \\
&= \prod_{i \in \mathcal{I}_{\lambda,j}^+ \cup \mathcal{I}_{\lambda,j}^-} (e^{\langle a, v_i^\lambda \rangle})^{\langle u_i^\lambda, q_j \rangle} \\
&= \prod_{i=1}^n (e^{\langle a, v_i^\lambda \rangle})^{\langle u_i^\lambda, q_j \rangle}
\end{aligned}$$

$$= e^{\langle a, q_j \rangle}.$$

This calculation implies that $f_j^\lambda(\tilde{z}^\lambda) = 0$ for j such that $\{\langle u_i^\lambda, q_j\rangle\}_{i\notin\mathcal{J}_\lambda}$ satisfies the condition (1) in Lemma 4.15.

If $\{\langle u_i^\lambda, q_j\rangle\}_{i\notin\mathcal{J}_\lambda}$ does not satisfy the condition (1) in Lemma 4.15 for some j, i.e., if $\{\langle u_i^\lambda, q_j\rangle\}_{i\notin\mathcal{J}_\lambda}$ satisfies the condition (2) in Lemma 4.15 for some j, then by Lemma 4.16, there exist $i_j \neq i'_j \notin \mathcal{J}_\lambda$ such that Equation 4.15 holds for such j. By the definition of $\tilde{z}^\lambda$, $\tilde{z}^\lambda \in \{z^\lambda \in \varphi_\lambda(U_\lambda) \mid z_{i_j}^\lambda = z_{i'_j}^\lambda = 0\}$ holds for such j, which implies that $\tilde{z}^\lambda \in \{z^\lambda \in \varphi_\lambda(U_\lambda) \mid f_j^\lambda(z^\lambda) = 0\}$.

From the above discussion, $\tilde{z}^\lambda \in \{z^\lambda \in \varphi_\lambda(U_\lambda) \mid f_j^\lambda(z^\lambda) = 0\}$ holds for any $j = k+1, \ldots, n$. By Lemma 4.8, there exists a fixed point of the T^k-action on $\overline{C_\lambda(V)}$. □

We consider the case that $\{\langle u_i^\lambda, q_j\rangle\}_{i\notin\mathcal{J}_\lambda}$ does not satisfy the condition (1) in Lemma 4.15 for any $j = k+1, \ldots, n$, i.e., $\{\langle u_i^\lambda, q_j\rangle\}_{i\notin\mathcal{J}_\lambda}$ satisfies the condition (2) in Lemma 4.15 for any $j = k+1, \ldots, n$.

Proposition 4.18. *Assume that the point $i_V^*(\lambda)$ is a vertex in the convex polytope $i_V^*(\Delta)$. If $\{\langle u_i^\lambda, q_j\rangle\}_{i\notin\mathcal{J}_\lambda}$ satisfies the condition* (2) *in* Lemma 4.15 *for any $j = k+1, \ldots, n$, then there exists a fixed point of the T^k-action on $\overline{C_\lambda(V)}$.*

Proof. From Lemma 4.16, for any j, there exist $i_j \neq i'_j \notin \mathcal{J}_\lambda$ such that

$$\{z^\lambda \in \varphi_\lambda(U_\lambda) \mid z_{i_j}^\lambda = z_{i'_j}^\lambda = 0, i_j \neq i'_j \notin \mathcal{J}_\lambda\} \subset \{z^\lambda \in \varphi_\lambda(U_\lambda) \mid f_j^\lambda(z^\lambda) = 0\}$$

holds. Since for any j,

$$\{z^\lambda \in \varphi_\lambda(U_\lambda) \mid z_i^\lambda = 0, \text{ for any } i \notin \mathcal{J}_\lambda\}$$
$$\subset \{z^\lambda \in \varphi_\lambda(U_\lambda) \mid z_{i_j}^\lambda = z_{i'_j}^\lambda = 0,\ i_j \neq i'_j \notin \mathcal{J}_\lambda\}$$

holds for some $i_j \neq i'_j \notin \mathcal{J}_\lambda$, we obtain

$$\{z^\lambda \in \varphi_\lambda(U_\lambda) \mid z_i^\lambda = 0, i \notin \mathcal{J}_\lambda\} \subset \bigcap_{j=k+1}^{n} \{z^\lambda \in \varphi_\lambda(U_\lambda) \mid f_j^\lambda(z^\lambda) = 0\}.$$

Since the right hand side is equal to $\overline{C_\lambda(V)}$, we see that

$$\{z^\lambda \in \varphi_\lambda(U_\lambda) \mid z_i^\lambda = 0, i \notin \mathcal{J}_\lambda\} \subset \overline{C_\lambda(V)}.$$

In particular, as we noted that $\{z^\lambda \in \varphi_\lambda(U_\lambda) \mid z_i^\lambda = 0,\ i \notin \mathcal{J}_\lambda\} \neq \emptyset$, we obtain

$$\{z^\lambda \in \varphi_\lambda(U_\lambda) \mid z_i^\lambda = 0,\ i \notin \mathcal{J}_\lambda\} \cap \overline{C_\lambda(V)} \neq \emptyset.$$

By Lemma 4.8, there exists a fixed point of the T^k-action on $\overline{C_\lambda(V)}$. □

Proposition 4.19. *If the point $i_V^*(\lambda)$ is a vertex in the convex polytope $i_V^*(\Delta)$, then there exists a fixed point of the T^k-action on $\overline{C_\lambda(V)}$.*

Proof. By Lemma 4.15, if the point $i_V^*(\lambda)$ is a vertex in the convex polytope $i_V^*(\Delta)$, then $\{\langle u_i^\lambda, q_j\rangle\}_{i\notin\mathcal{J}_\lambda}$ satisfies either of the condition (1) or the condition (2) in Lemma 4.15 for each j.

If there exists j such that $\{\langle u_i^\lambda, q_j\rangle\}_{i\notin\mathcal{J}_\lambda}$ satisfies the condition (1) in Lemma 4.15, then by Proposition 4.17, there exists a fixed point of the T^k-action on $\overline{C_\lambda(V)}$. If otherwise, i.e., if $\{\langle u_i^\lambda, q_j\rangle\}_{i\notin\mathcal{J}_\lambda}$ satisfies the condition (2) in Lemma 4.15 for any j, then by Proposition 4.18, there exists a fixed point of the T^k-action on $\overline{C_\lambda(V)}$. □

By comparing the vertices in $\overline{\mu}(\overline{C(V)})$ with those in $i_V^*(\mu(X))$, we say more about the image of the moment map $\overline{\mu}$.

Theorem 4.20. *If $\overline{C(V)}$ is a complex submanifold in X, then we obtain $\overline{\mu}(\overline{C(V)}) = i_V^*(\mu(X))$ in* $(\mathfrak{t}^k)^*$.

Proof. Since the map $\overline{\mu}$ is the moment map, the image of $\overline{\mu}$ is the convex hull of the images of the fixed points of the T^k-action on $\overline{C(V)}$. We classified the fixed points of the T^k-action on X (Lemma 4.7) and those of the T^k-action on $\overline{C(V)}$ (Lemma 4.8).

Since $\overline{C(V)} \subset X$, we obtain $\overline{\mu}(\overline{C(V)}) \subset i_V^*(\mu(X)) = i_V^*(\Delta)$. In particular, by Lemma 4.12, if $z^\lambda \in \overline{C_\lambda(V)}$ is a fixed point of the T^k-action on $\overline{C_\lambda(V)}$, then $\overline{\mu}(z^\lambda) = i_V^*(\lambda) \in i_V^*(\Delta)$ for the vertex λ.

Since Proposition 4.19 shows that if $i_V^*(\lambda)$ is a vertex of $i_V^*(\Delta)$, then there exists a fixed point z^λ of the T^k-action on $\overline{C_\lambda(V)}$ such that $\overline{\mu}(z^\lambda) = i_V^*(\lambda)$.

Thus, the set of the vertices of $\overline{\mu}(\overline{C(V)})$ coincides with the set of the vertices of $i_V^*(\Delta)$. Since the map $\overline{\mu}$ is a moment map for the T^k-action on $\overline{C(V)}$, by the convexity theorem [1, 10], the image of $\overline{\mu}$ is the convex hull of the images of the fixed points of the T^k-action on $\overline{C(V)}$. Since $i_V^*(\Delta)$ is the convex hull of the images of the vertices of Δ by i_V^*, we obtain $\overline{\mu}(\overline{C(V)}) = i_V^*(\Delta)$. □

We say a submanifold $\overline{C(V)}$ to be a *torus-equivariantly embedded toric manifold* in a toric manifold X.

5. Examples of Torus-equivariantly Embedded Toric Manifolds

We demonstrate examples of $\overline{C(V)}$ and check whether they are torus-equivariantly embedded toric manifolds or not. When $\overline{C(V)}$ is smooth, we further draw figures of $\overline{D(V)} := \mu(\overline{C(V)})$ for each example.

5.1. Examples of Torus-equivariantly Embedded Toric Manifolds in $\mathbb{C}P^2$. We give examples for $\overline{C(V)}$ and check whether $\overline{C(V)}$ is a complex submanifold in $X = \mathbb{C}P^2$.

Delzant polytopes of $\mathbb{C}P^2$ are isosceles right triangles. As in Example 4.4 and Example 4.5, define the points λ, μ, σ in $(\mathfrak{t}^2)^* \cong \mathbb{R}^2$ by

$$\lambda = (0,0),\ \mu = (2,0),\ \sigma = (0,2).$$

Let Δ be a polytope defined by the convex hull of the points λ, μ, σ. We define the inward pointing normal vectors to the facets by

$$u_1^\lambda = \begin{bmatrix}1\\0\end{bmatrix}, u_2^\lambda = \begin{bmatrix}0\\1\end{bmatrix}, u_1^\mu = \begin{bmatrix}-1\\-1\end{bmatrix}, u_2^\mu = \begin{bmatrix}1\\0\end{bmatrix}, u_1^\sigma = \begin{bmatrix}0\\1\end{bmatrix}, u_2^\sigma = \begin{bmatrix}-1\\-1\end{bmatrix},$$

where $\Lambda = \{\lambda, \mu, \sigma\}$.

Example 5.1. Let $X = \mathbb{C}P^2$, $k = 1$, and $V = \mathbb{R}p + a$ ($a \in \mathbb{R}^2$) be an affine subspace spanned by $p = {}^t[1\ 0]$. Then, we can choose a basis q of the orthogonal subspace to the linear part of V as $q = {}^t[0\ 1]$. In this case, $\overline{C(V)}$ is a complex submanifold in X. Indeed, we give $f^\lambda, f^\mu, f^\sigma$ by

$$f^\lambda = z_2^\lambda - e^{\langle a,e_2\rangle},\ f^\mu = 1 - e^{\langle a,e_2\rangle} z_1^\mu,\ f^\sigma = z_1^\sigma - e^{\langle a,e_2\rangle} z_2^\sigma,$$

respectively. Since the Jacobian matrices are expressed as

$$Df^\lambda = [0 \quad 1],\ Df^\mu = [-e^{\langle a,e_2\rangle} \quad 0],\ Df^\sigma = [1 \quad -e^{\langle a,e_2\rangle}],$$

respectively, we see the rank of each matrix is one.

Fig. 5.1 describes $\overline{D(V)}$ when $a = (0,0)$. Fig. 5.2 describes $\overline{D(V)}$ when $a = (0, \log 2)$.

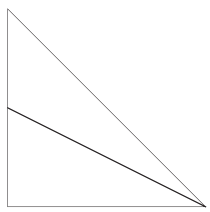

Fig. 5.1. $\overline{D(V)}$ in Example 5.1 when $a = (0,0)$

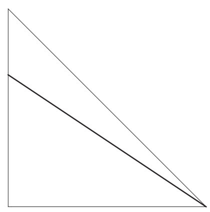

Fig. 5.2. $\overline{D(V)}$ in Example 5.1 when $a = (0, \log 2)$

Example 5.2. Let $X = \mathbb{C}P^2$, $k = 1$, and $V = \mathbb{R}p + a$ ($a \in \mathbb{R}^2$) be an affine subspace spanned by $p = {}^t[0\ 1]$. Then, we can choose a basis q of the orthogonal subspace to the linear part of V as $q = {}^t[1\ 0]$. In this case, $\overline{C(V)}$ is a complex submanifold in X. Indeed, we give $f^\lambda, f^\mu, f^\sigma$ by

$$f^\lambda = z_1^\lambda - e^{\langle a,e_1\rangle},\ f^\mu = z_2^\mu - e^{\langle a,e_1\rangle} z_1^\mu,\ f^\sigma = 1 - e^{\langle a,e_1\rangle} z_2^\sigma,$$

respectively. Since the Jacobian matrices are expressed as

$$Df^\lambda = [1 \quad 0],\ Df^\mu = [-e^{\langle a,e_1\rangle} \quad 1],\ Df^\sigma = [0 \quad -e^{\langle a,e_1\rangle}],$$

respectively, we see the rank of each matrix is one.

Fig. 5.3 describes $\overline{D(V)}$ when $a = (0,0)$. Fig. 5.4 describes $\overline{D(V)}$ when $a = (\log 2, 0)$.

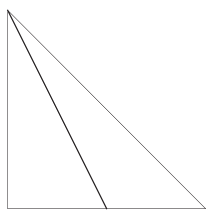

Fig. 5.3. $\overline{D(V)}$ in Example 5.2 when $a = (0,0)$

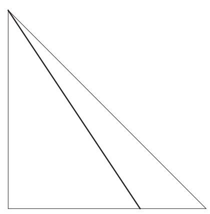

Fig. 5.4. $\overline{D(V)}$ in Example 5.2 when $a = (\log 2, 0)$

EXAMPLE 5.3. Let $X = \mathbb{C}P^2$, $k = 1$, and $V = \mathbb{R}p + a$ ($a \in \mathbb{R}^2$) be an affine subspace spanned by $p = {}^t[1\ 1]$. Then, we can choose a basis q of the orthogonal subspace to the linear part of V as $q = {}^t[1\ -1]$. In this case, we show in Example 4.4 that $\overline{C(V)}$ is a complex submanifold in X for $a = 0$. By similar calculation, we see that $\overline{C(V)}$ is a complex submanifold in X for arbitrary $a \in \mathbb{R}^2$.

Fig. 5.5 describes $\overline{D(V)}$ when $a = (0, 0)$. Fig. 5.6 describes $\overline{D(V)}$ when $a = (0, \log 2)$.

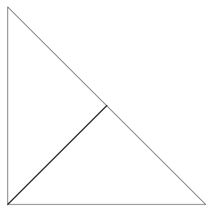

Fig. 5.5. $\overline{D(V)}$ in Example 5.3 when $a = (0, 0)$

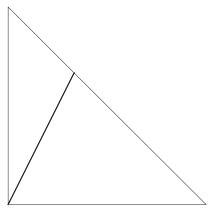

Fig. 5.6. $\overline{D(V)}$ in Example 5.3 when $a = (0, \log 2)$

EXAMPLE 5.4. Let $X = \mathbb{C}P^2$, $k = 1$, and $V = \mathbb{R}p + a$ ($a \in \mathbb{R}^2$) be an affine subspace spanned by $p = {}^t[1\ -1]$. Then, we can choose a basis q of the orthogonal subspace to the linear part of V as $q = {}^t[1\ 1]$. In this case, $\overline{C(V)}$ is a complex submanifold in X. Indeed, we give $f^\lambda, f^\mu, f^\sigma$ by

$$\begin{aligned} f^\lambda &= z_1^\lambda z_2^\lambda - e^{\langle a, e_1\rangle + \langle a, e_2\rangle}, \\ f^\mu &= z_2^\mu - e^{\langle a, e_1\rangle + \langle a, e_2\rangle}(z_1^\mu)^2, \\ f^\sigma &= e^{\langle a, e_1\rangle + \langle a, e_2\rangle}(z^\sigma - (z_2^\sigma)^2), \end{aligned}$$

respectively. Note that $(0, 0) \notin \overline{C_\lambda(V)}$. Since the Jacobian matrices are expressed as

$$\begin{aligned} Df^\lambda &= [z_2^\lambda \quad z_1^\lambda], \\ Df^\mu &= [-2e^{\langle a, e_1\rangle + \langle a, e_2\rangle} z_1^\mu \quad 1], \\ Df^\sigma &= [1 \quad -2e^{\langle a, e_1\rangle + \langle a, e_2\rangle} z_2^\sigma], \end{aligned}$$

respectively, we see the rank of each matrix is one.

Fig. 5.7 describes $\overline{D(V)}$ when $a = (0, 0)$. Fig. 5.8 describes $\overline{D(V)}$ when $a = (-\log 2, 0)$.

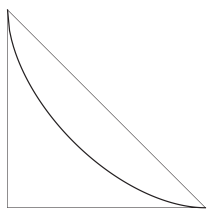

Fig. 5.7. $\overline{D(V)}$ in Example 5.4 when $a = (0, 0)$

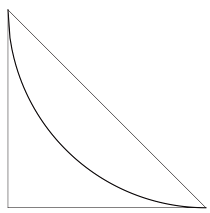

Fig. 5.8. $\overline{D(V)}$ in Example 5.4 when $a = (-\log 2, 0)$

Example 5.5. Let $X = \mathbb{C}P^2$, $k = 1$, and $V = \mathbb{R}p + a$ ($a \in \mathbb{R}^2$) be an affine subspace spanned by $p = {}^t[1\ 2]$. Then, we can choose a basis q of the orthogonal subspace to the linear part of V as $q = {}^t[2\ \ -1]$. In this case, $\overline{C(V)}$ is a complex submanifold in X. Indeed, we give $f^\lambda, f^\mu, f^\sigma$ by

$$\begin{aligned} f^\lambda &= (z_1^\lambda)^2 - e^{2\langle a,e_1\rangle - \langle a,e_2\rangle} z_2^\lambda, \\ f^\mu &= (z_2^\mu)^2 - e^{2\langle a,e_1\rangle - \langle a,e_2\rangle} z_1^\mu, \\ f^\sigma &= 1 - e^{2\langle a,e_1\rangle - \langle a,e_2\rangle} z_1^\sigma z_2^\sigma, \end{aligned}$$

respectively. Note that $(0,0) \notin \overline{C_\sigma(V)}$. Since the Jacobian matrices are expressed as

$$\begin{aligned} Df^\lambda &= [2z^\lambda \quad -e^{2\langle a,e_1\rangle - \langle a,e_2\rangle}], \\ Df^\mu &= [-e^{2\langle a,e_1\rangle - \langle a,e_2\rangle} \quad 2z_2^\mu], \\ Df^\sigma &= -e^{2\langle a,e_1\rangle - \langle a,e_2\rangle}[z_2^\sigma \quad z_1^\sigma], \end{aligned}$$

respectively, we see the rank of each matrix is one.

Fig. 5.9 describes $\overline{D(V)}$ when $a = (0,0)$. Fig. 5.10 describes $\overline{D(V)}$ when $a = (0, -\log 2)$.

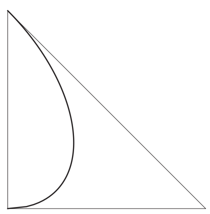

Fig. 5.9. $\overline{D(V)}$ in Example 5.5 when $a = (0,0)$

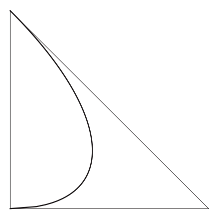

Fig. 5.10. $\overline{D(V)}$ in Example 5.5 when $a = (0, -\log 2)$

Example 5.6. Let $X = \mathbb{C}P^2$, $k = 1$, and $V = \mathbb{R}p + a$ ($a \in \mathbb{R}^2$) be an affine subspace spanned by $p = {}^t[2\ 1]$. Then, we can choose a basis q of the orthogonal subspace to the linear part of V as $q = {}^t[1\ \ -2]$. In this case, $\overline{C(V)}$ is a complex submanifold in X. Indeed, we give $f^\lambda, f^\mu, f^\sigma$ by

$$\begin{aligned} f^\lambda &= z_1^\lambda - e^{\langle a,e_1\rangle - 2\langle a,e_2\rangle} (z_2^\lambda)^2, \\ f^\mu &= z_1^\mu z_2^\mu - e^{\langle a,e_1\rangle - 2\langle a,e_2\rangle}, \\ f^\sigma &= z_2^\sigma - e^{\langle a,e_1\rangle - 2\langle a,e_2\rangle} (z_1^\sigma)^2, \end{aligned}$$

respectively. Note that $(0,0) \notin \overline{C_\mu(V)}$. Since the Jacobian matrices are expressed as

$$\begin{aligned} Df^\lambda &= [1 \quad -2e^{\langle a,e_1\rangle - 2\langle a,e_2\rangle} z_2^\lambda], \\ Df^\mu &= [z_2^\mu \quad z_1^\mu], \\ Df^\sigma &= [-2e^{\langle a,e_1\rangle - 2\langle a,e_2\rangle} z_1^\sigma \quad 1], \end{aligned}$$

respectively, we see the rank of each matrix is one.

Fig. 5.11 describes $\overline{D(V)}$ when $a = (0,0)$. Fig. 5.12 describes $\overline{D(V)}$ when $a = (-\log 2, 0)$.

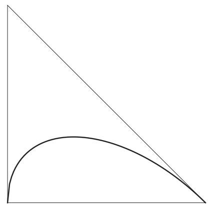

Fig. 5.11. $\overline{D(V)}$ in Example 5.6 when $a = (0, 0)$

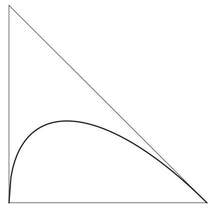

Fig. 5.12. $\overline{D(V)}$ in Example 5.6 when $a = (-\log 2, 0)$

In the following examples, we treat $\overline{C(V)}$ which does not become a complex submanifold in $\mathbb{C}P^2$.

EXAMPLE 5.7. Let $X = \mathbb{C}P^2$, $k = 1$, and $V = \mathbb{R}p + a$ ($a \in \mathbb{R}^2$) be an affine subspace spanned by $p = {}^t[1\ \alpha]$ for all integers α greater than or equal to three. Then, we can choose a basis q of the orthogonal subspace to V as $q = {}^t[\alpha\ -1]$. In this case, $\overline{C(V)}$ is not a complex submanifold in X. Indeed, we give $f^\lambda, f^\mu, f^\sigma$ by

$$\begin{aligned} f^\lambda &= (z_1^\lambda)^\alpha - e^{\alpha\langle a,e_1\rangle - \langle a,e_2\rangle} z_2^\lambda, \\ f^\mu &= (z_2^\mu)^\alpha - e^{\alpha\langle a,e_1\rangle - \langle a,e_2\rangle} (z_1^\mu)^{\alpha-1}, \\ f^\sigma &= 1 - e^{\alpha\langle a,e_1\rangle - \langle a,e_2\rangle} z_1^\sigma (z_2^\sigma)^{\alpha-1}, \end{aligned}$$

respectively. Note that $(0, 0) \notin \overline{C_\sigma(V)}$. Since the Jacobian matrices are expressed as

$$\begin{aligned} Df^\lambda &= [\alpha(z_1^\lambda)^{\alpha-1} \quad -e^{\alpha\langle a,e_1\rangle - \langle a,e_2\rangle}], \\ Df^\mu &= [-(\alpha-1)e^{\alpha\langle a,e_1\rangle - \langle a,e_2\rangle}(z_1^\mu)^{\alpha-2} \quad \alpha(z_2^\mu)^{\alpha-1}], \\ Df^\sigma &= -e^{\alpha\langle a,e_1\rangle - \langle a,e_2\rangle}[(z_2^\sigma)^{\alpha-1} \quad (\alpha-1)z_1^\sigma(z_2^\sigma)^{\alpha-2}], \end{aligned}$$

respectively, we see the rank of Df^λ and Df^σ is one. However, the rank of Df^μ becomes zero when $(z_1^\mu, z_2^\mu) = (0, 0) \in \overline{C_\mu(V)}$.

EXAMPLE 5.8. Let $X = \mathbb{C}P^2$, $k = 1$, and $V = \mathbb{R}p + a$ ($a \in \mathbb{R}^2$) be an affine subspace spanned by $p = {}^t[\alpha\ 1]$ for all integers α greater than or equal to three. Then, we can choose a basis q of the orthogonal subspace to V as $q = {}^t[1\ -\alpha]$. In this case, $\overline{C(V)}$ is not a complex submanifold in X. Indeed, we give $f^\lambda, f^\mu, f^\sigma$ by

$$\begin{aligned} f^\lambda &= z_1^\lambda - e^{\langle a,e_1\rangle - \alpha\langle a,e_2\rangle}(z_2^\lambda)^\alpha, \\ f^\mu &= (z_1^\mu)^{\alpha-1} z_2^\mu - e^{\langle a,e_1\rangle - \alpha\langle a,e_2\rangle}, \\ f^\sigma &= (z_2^\sigma)^{\alpha-1} - e^{\langle a,e_1\rangle - \alpha\langle a,e_2\rangle}(z_1^\sigma)^\alpha, \end{aligned}$$

respectively. Note that $(0, 0) \notin \overline{C_\mu(V)}$. Since the Jacobian matrices are expressed as

$$\begin{aligned} Df^\lambda &= [1 \quad -\alpha e^{\langle a,e_1\rangle - \alpha\langle a,e_2\rangle}(z_2^\lambda)^{\alpha-1}], \\ Df^\mu &= [(\alpha-1)(z_1^\mu)^{\alpha-2} z_2^\mu \quad (z_1^\mu)^{\alpha-1}], \\ Df^\sigma &= [-\alpha e^{\langle a,e_1\rangle - \alpha\langle a,e_2\rangle}(z_1^\sigma)^{\alpha-1} \quad (\alpha-1)(z_2^\sigma)^{\alpha-2}], \end{aligned}$$

respectively, we see the rank of Df^λ and Df^σ is one. However, the rank of Df^σ becomes zero at the point $(z_1^\sigma, z_2^\sigma) = (0,0) \in \overline{C_\sigma(V)}$.

Example 5.9. Let $X = \mathbb{C}P^2$, $k = 1$, and $V = \mathbb{R}p + a$ $(a \in \mathbb{R}^2)$ be an affine subspace spanned by $p = {}^t[1\ \ -\alpha]$ for all integers α greater than or equal to two. Then, we can choose a basis q of the orthogonal subspace to V as $q = {}^t[\alpha\ 1]$. In this case, $\overline{C(V)}$ is not a complex submanifold in X. Indeed, we give $f^\lambda, f^\mu, f^\sigma$ by

$$\begin{aligned}
f^\lambda &= (z_1^\lambda)^\alpha z_2^\lambda - e^{\alpha\langle a,e_1\rangle+\langle a,e_2\rangle},\\
f^\mu &= (z_2^\mu)^\alpha - e^{\alpha\langle a,e_1\rangle+\langle a,e_2\rangle}(z_1^\mu)^{\alpha+1},\\
f^\sigma &= z_1^\sigma - e^{\alpha\langle a,e_1\rangle+\langle a,e_2\rangle}(z_2^\sigma)^{\alpha+1},
\end{aligned}$$

respectively. Note that $(0,0) \notin \overline{C_\lambda(V)}$. Since the Jacobian matrices are expressed as

$$\begin{aligned}
Df^\lambda &= [\alpha(z_1^\lambda)^{\alpha-1}z_2^\lambda \quad (z_1^\lambda)^\alpha],\\
Df^\mu &= [-(\alpha+1)e^{\alpha\langle a,e_1\rangle+\langle a,e_2\rangle}(z_1^\mu)^\alpha \quad \alpha(z_2^\mu)^{\alpha-1}],\\
Df^\sigma &= [1 \quad -(\alpha+1)e^{\alpha\langle a,e_1\rangle+\langle a,e_2\rangle}(z_2^\sigma)^\alpha],
\end{aligned}$$

respectively, we see the rank of Df^λ and Df^σ is one. However, the rank of Df^μ becomes zero at the point $(z_1^\mu, z_2^\mu) = (0,0) \in \overline{C_\mu(V)}$.

Example 5.10. Let $X = \mathbb{C}P^2$, $k = 1$, and $V = \mathbb{R}p + a$ $(a \in \mathbb{R}^2)$ be an affine subspace spanned by $p = {}^t[\alpha\ \ -1]$ for all integers α greater than or equal to three. Then, we can choose a basis q of the orthogonal subspace to V as $q = {}^t[1\ \alpha]$. In this case, $\overline{C(V)}$ is not a complex submanifold in X. Indeed, we give $f^\lambda, f^\mu, f^\sigma$ by

$$\begin{aligned}
f^\lambda &= z_1^\lambda(z_2^\lambda)^\alpha - e^{\langle a,e_1\rangle+\alpha\langle a,e_2\rangle},\\
f^\mu &= z_2^\mu - e^{\langle a,e_1\rangle+\alpha\langle a,e_2\rangle}(z_1^\mu)^{\alpha+1},\\
f^\sigma &= (z_1^\sigma)^\alpha - e^{\langle a,e_1\rangle+\alpha\langle a,e_2\rangle}(z_2^\sigma)^{\alpha+1},
\end{aligned}$$

respectively. Note that $(0,0) \notin \overline{C_\lambda(V)}$. Since the Jacobian matrices are expressed as

$$\begin{aligned}
Df^\lambda &= [(z_2^\lambda)^\alpha \quad \alpha z_1^\lambda(z_2^\lambda)^{\alpha-1}],\\
Df^\mu &= [-(\alpha+1)e^{\langle a,e_1\rangle+\alpha\langle a,e_2\rangle}(z_1^\mu)^\alpha \quad 1],\\
Df^\sigma &= [\alpha(z_1^\sigma)^{\alpha-1} \quad -(\alpha+1)e^{\langle a,e_1\rangle+\alpha\langle a,e_2\rangle}(z_2^\sigma)^\alpha],
\end{aligned}$$

respectively, we see the rank of Df^λ and Df^σ is one. However, the rank of Df^σ becomes zero at the point $(z_1^\sigma, z_2^\sigma) = (0,0) \in \overline{C_\sigma(V)}$.

By similar calculation, we see that $\overline{C(V)}$ is not a complex submanifold in X if the slope of V is not the same as treated above.

Remark 5.11. We can classify all examples for complex submanifolds $\overline{C(V)}$ in X in terms of the conditions of V by direct calculation. In particular, when $X = \mathbb{C}P^2$, we can show that $\overline{C(V)}$ is a complex submanifold in $\mathbb{C}P^2$ if and only if the linear part of V is spanned by ${}^t[1\ 0]$, ${}^t[0\ 1]$, ${}^t[1\ 1]$, ${}^t[1\ 2]$, ${}^t[2\ 1]$, or ${}^t[1\ \ -1]$.

When $X = \mathbb{C}P^2$, we can determine the conditions that $\overline{C(V)}$ is a one-dimensional complex submanifold in X by the linear part of an affine subspace $V = \mathbb{R}p + a$ in $\mathbb{R}^2$.

5.2. Other Examples of Torus-equivariantly Embedded Toric Manifolds. We demonstrate other examples of torus-equivariantly embedded toric manifolds in toric manifolds other than $\mathbb{C}P^2$.

It is well-known that Delzant polytopes of $\mathbb{F}_1$ are shown in Fig. 5.13. Define the points λ, μ, σ, δ in $(\mathfrak{t}^2)^* \cong \mathbb{R}^2$ by

$$\lambda = (0,0),\ \mu = (2,0),\ \sigma = (1,1),\ \delta = (0,1).$$

Let Δ be a polytope defined by the convex hull of the points λ, μ, σ, δ. Let $\Lambda = \{\lambda, \mu, \sigma, \delta\}$ be a set of the vertices in the Delzant polytope of $\mathbb{F}_1$. We define the inward pointing normal vectors to the facets by

$$u_1^\lambda = \begin{bmatrix}1\\0\end{bmatrix}, u_2^\lambda = \begin{bmatrix}0\\1\end{bmatrix}, u_1^\mu = \begin{bmatrix}0\\1\end{bmatrix}, u_2^\mu = \begin{bmatrix}-1\\-1\end{bmatrix},$$
$$u_1^\sigma = \begin{bmatrix}-1\\-1\end{bmatrix}, u_2^\sigma = \begin{bmatrix}0\\-1\end{bmatrix}, u_1^\delta = \begin{bmatrix}0\\-1\end{bmatrix}, u_2^\delta = \begin{bmatrix}1\\0\end{bmatrix}.$$

EXAMPLE 5.12. Let X be a Hirzebruch surface $\mathbb{F}_1$ of degree one, $k = 1$, and $p = {}^t[1\ 0]$. Then, we can choose a basis of the orthogonal subspace to V as $q = {}^t[0\ 1]$. In this case, $\overline{C(V)}$ is a complex submanifold in X. Indeed, we give f^λ, f^μ, f^σ, f^δ by

$$f^\lambda = z_1^\lambda - 1,\ f^\mu = z_1^\mu - z_2^\mu,\ f^\sigma = 1 - z_1^\sigma z_2^\sigma,\ f^\delta = 1 - z_1^\delta,$$

respectively. Note that $(0,0) \notin \overline{C_\sigma(V)}$. Since the Jacobian matrices are expressed as

$$Df^\lambda = [0 \quad 1],\ Df^\mu = [1 \quad -1],\ Df^\sigma = [-z_2^\sigma \quad -z_1^\sigma],\ Df^\delta = [-1 \quad 0],$$

respectively, we see the rank of each matrix is one.

The image of $\mu\ |_{\overline{C(V)}}\colon \overline{C(V)} \to \mathbb{R}^2$ is given in Fig. 5.14.

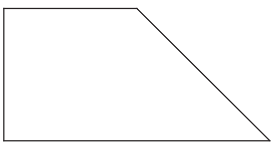

Fig. 5.13. a Delzant polytope of $\mathbb{F}_1$

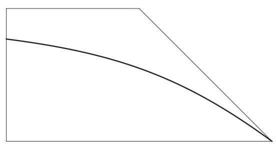

Fig. 5.14. $\mu(\overline{C(V)})$ in Example 5.12

Delzant polytopes of $\mathbb{C}P^3$ are the convex hull of the points:

$$\lambda = (0,0,0),\ \mu = (2,0,0),\ \sigma = (0,0,2),\ \delta = (0,2,0).$$

We can see a Delzant polytope of a blow up of $\mathbb{C}P^3$ at the point corresponding to δ as the convex hull of the points:

$$\lambda = (0,0,0),\ \mu = (2,0,0),\ \sigma = (0,0,2),$$
$$\delta_1 = (1,1,0),\ \delta_2 = (0,1,0),\ \delta_3 = (0,1,1).$$

We define the inward pointing normal vectors to the facets by

$$u_1^\lambda = \begin{bmatrix}1\\0\\0\end{bmatrix}, u_2^\lambda = \begin{bmatrix}0\\1\\0\end{bmatrix}, u_3^\lambda = \begin{bmatrix}0\\0\\1\end{bmatrix}, u_1^\mu = \begin{bmatrix}0\\1\\0\end{bmatrix}, u_2^\mu = \begin{bmatrix}-1\\-1\\-1\end{bmatrix}, u_3^\mu = \begin{bmatrix}0\\0\\1\end{bmatrix},$$
$$u_1^\sigma = \begin{bmatrix}0\\1\\0\end{bmatrix}, u_2^\sigma = \begin{bmatrix}1\\0\\0\end{bmatrix}, u_3^\sigma = \begin{bmatrix}-1\\-1\\-1\end{bmatrix}, u_1^{\delta_1} = \begin{bmatrix}0\\-1\\0\end{bmatrix}, u_2^{\delta_1} = \begin{bmatrix}0\\0\\1\end{bmatrix}, u_3^{\delta_1} = \begin{bmatrix}-1\\-1\\-1\end{bmatrix},$$
$$u_1^{\delta_2} = \begin{bmatrix}1\\0\\0\end{bmatrix}, u_2^{\delta_2} = \begin{bmatrix}0\\0\\1\end{bmatrix}, u_3^{\delta_2} = \begin{bmatrix}0\\-1\\0\end{bmatrix}, u_1^{\delta_3} = \begin{bmatrix}1\\0\\0\end{bmatrix}, u_2^{\delta_3} = \begin{bmatrix}0\\-1\\0\end{bmatrix}, u_3^{\delta_3} = \begin{bmatrix}-1\\-1\\-1\end{bmatrix}.$$

Example 5.13. Let X be a blow up of $\mathbb{C}P^3$ at the point corresponding to δ, $k = 2$, $p_1 = {}^t[1\ 0\ -1]$ and $p_2 = {}^t[0\ 1\ 0]$. Then, we can choose a basis of the orthogonal subspace to V as $q = {}^t[1\ 0\ 1]$. In this case, $\overline{C(V)}$ is a complex submanifold in X. Indeed, we give f^λ, f^μ, f^σ, f^{δ_1}, f^{δ_2}, f^{δ_3} by

$$f^\lambda = z_1^\lambda z_3^\lambda - 1, f^\mu = z_3^\mu - (z_2^\mu)^2, f^\sigma = z_2^\sigma - (z_3^\sigma)^2,$$
$$f^{\delta_1} = z_2^{\delta_1} - (z_3^{\delta_1})^2, f^{\delta_2} = z_1^{\delta_2} z_2^{\delta_2} - 1, f^{\delta_3} = z_1^{\delta_3} - (z_3^{\delta_3})^2,$$

respectively. Note that $(0,0,0) \notin \overline{C_\lambda(V)}$, and $(0,0,0) \notin \overline{C_{\delta_2}(V)}$. Since the Jacobian matrices are expressed as

$$Df^\lambda = [z_3^\lambda\ 0\ z_1^\lambda], Df^\mu = [0\ -2z_2^\mu\ 1], Df^\sigma = [0\ 1\ -2z_3^\sigma],$$
$$Df^{\delta_1} = [0\ 1\ -2z_3^{\delta_1}], Df^{\delta_2} = [z_2^{\delta_2}\ z_1^{\delta_2}\ 0], Df^{\delta_3} = [1\ 0\ -2z_3^{\delta_3}],$$

respectively, we see the rank of each matrix is one.

In Example 5.13, we have to suppose that X is a blow up of $\mathbb{C}P^3$ at the point corresponding to the vertex δ because $\overline{C(V)}$ is not a complex submanifold in $X = \mathbb{C}P^3$ when the linear part of an affine subspace V is spanned by $p_1 = {}^t[1\ 0\ -1]$ and $p_2 = {}^t[0\ 1\ 0]$.

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