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<th><strong>Title</strong></th>
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Osaka University
L∞ BOUNDEDNESS OF NONLINEAR EIGENFUNCTION UNDER SINGULAR VARIATION OF DOMAINS

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1. Introduction.

Let $M$ be a bounded domain in $\mathbb{R}^3$ with smooth boundary $\partial M$. Let $\tilde{w}$ be a point in $M$. We remove ball $B_\varepsilon$ of radius $\varepsilon$ with the center $\tilde{w}$ from $M$ and we get $M_\varepsilon = M \setminus \overline{B_\varepsilon}$.

Fix $p \in (1, 2)$. We fix $k > 0$. We put

$$\lambda(\varepsilon) = \inf_{u \in X} \left( \int_{M_\varepsilon} |\nabla u|^2 \, dx + k \int_{\partial B_\varepsilon} u^2 \, d\sigma \right),$$

where $X = \{ u \in H^1(M_\varepsilon), \ u = 0 \text{ on } \partial M \text{ and } u \geq 0 \text{ in } M_\varepsilon, \ |u|_{p+1, \varepsilon} = 1 \}$. Here $\|u\|_{L^p(M_\varepsilon)} = \|u\|_{q, \varepsilon}$. We see that there exists at least one solution $v_\varepsilon$ of the above problem which attains (1.1). We know that $v_\varepsilon$ satisfies $-\Delta v_\varepsilon(x) = \lambda(\varepsilon)v_\varepsilon(x)^p$ in $M_\varepsilon$, $v_\varepsilon(x) = 0$ on $\partial M$ and $kv_\varepsilon(x) + (\partial/\partial n)v_\varepsilon(x) = 0$ on $\partial B_\varepsilon$. Here $\partial/\partial n$ denotes the derivative along the exterior normal vector with respect to $M_\varepsilon$.

Let $S_\varepsilon$ denote the set of positive function $u_\varepsilon$ which attains the minimum of (1.1).

Main result of this paper is the following

**Theorem 1.** Fix $p \in (1, 2)$. Then, there exists a constant $C$ independent of $\varepsilon$ such that

$$\sup_{u_\varepsilon \in S_\varepsilon} \sup_{x \in M_\varepsilon} |v_\varepsilon(x)| < C.$$ 

Related topics are discussed in Lin [1], Ozawa [4].

2. Preliminary Lemmas.

We have the following Lemma 2.1.

**Lemma 2.1.** Consider the following equations.

$$\Delta v_\varepsilon(x) = 0 \quad x \in M \setminus \overline{B_\varepsilon}$$

Here $\omega \in S^2$. Then, the solution of (2.1), (2.2), (2.3) satisfies

$$|v_{\varepsilon}(x)| \leq C\varepsilon^2 r^{-1} \|\alpha\|_{C^{(\sigma/2)}(S^2)}$$

for any $\sigma > 1$. Here $C$ may depend on $\sigma$ but independent of $\varepsilon$. Here $r = |x - \hat{w}|$.

Proof. Let $\Delta_{S^2}$ denote the Laplace-Beltrami operator on $S^2$. It is well known that $-n(n + 1)$ is an eigenvalue of $\Delta_{S^2}$ whose multiplicity is $(2n + 1)$. We can write it explicitly by using the Legendre polynomial but we do not do it. We write complete orthonormal basis of $L^2(S^2)$ consisting of eigenfunction by $\{\varphi_j(\omega)\}_{j=1}^{\infty}$. If $\Delta \varphi_j(\omega) = -n(n + 1)\varphi_j(\omega)$, then we write $j$ as $j \in \mathbb{N}$. Thus, $\#\{j : j \in \mathbb{N}\} = 2n + 1.$

First we want to construct a solution of (2.1), (2.3).

We put

$$v^*(x) = \sum_{n=0}^{\infty} r^{-(n+1)} \sum_{j \in \mathbb{N}} c_{j,n} \varphi_j(\omega).$$

Then,

$$\Delta v^*(x) = 0 \quad \text{in} \quad \mathbb{R}^3 \setminus \bar{B}_\varepsilon.$$

We expand $\alpha$ by $\varphi_j(\omega)$ as

$$\alpha(\omega) = \sum_{n=0}^{\infty} \sum_{j \in \mathbb{N}} a_{j,n} \varphi_j(\omega).$$

By the equality (2.3), we have

$$(k c_{j,n} e^{-(n+1)} + (n+1)c_{j,n} e^{-(n+2)}) = a_{j,n}.$$ 

Then,

$$c_{j,n} = a_{j,n} e^{n+1}(k + (n+1)e^{-1})^{-1}.$$ 

Therefore,

$$|v^*(x)| \leq \varepsilon \sum_{n=0}^{\infty} \sum_{j \in \mathbb{N}} a_{j,n} (e/r)^{n+1}(e(k + (n+1))^{-1} \varphi_j(\omega).$$

We notice that we have

$$|\varphi_j(\omega)| \leq C(n + 1)^{1/2}$$

(2.5)
for $j \in (n)$ by the property of Legendre and associated Legendre polynomial. See Mizohata [2, p.312].

By the Schwarz inequality we have

$$|v_{x_0}^*| \leq C \epsilon^2 r^{-1} \left( \sum_{n=0}^{\infty} (n+1)^{\sigma} \sum_{j(e(n))} a_{j,n}^2 \right)^{1/2}$$

$$\times \left( \sum_{n=0}^{\infty} (n+1)^{-\sigma}(2n+1)n(n+1)^{-2} \right)^{1/2}.$$

If we take $\sigma > 1$, then

$$|v_{x_0}^*(x)| \leq C \epsilon^2 r^{-1} \left( \sum_{n=0}^{\infty} (n+1)^{\sigma} \sum_{j(e(n))} a_{j,n}^2 \right)^{1/2}$$

$$\leq C \epsilon^2 r^{-1} \|x\|_{H^{\sigma/2}(S^2)}.$$

Here we note that $\|x\|_{H^2(S^2)}$ is equivalent to $\left( \sum_{n=0}^{\infty} (n+1)^{4} \sum_{j(e(n))} a_{j,n}^2 \right)$, since the eigenvalue of $-\Delta_{S^2}$ is $n(n+1)$. We used representation of norm of fractional Sobolev space. Notice that $C^{\sigma/2}(S^2) \subset H^{\sigma/2}(S^2)$ for $\sigma < \sigma'$.

Now $v_{x_0}^*(x)$ does not satisfy $v_{x_0}^*(x) = 0$ on $\partial M$.

By the same procedure as in the repeated construction of the function $v_{x_0}^*(n)$ in Proposition 1 of Ozawa [3] we proved Lemma 2.1.

The following Lemm is very useful.

**Lemma 2.3.** There exists an extension operator $E: H^1(M) \ni u \rightarrow \tilde{u} \in H^1(M)$ satisfying the followings;

$$u(x) = \tilde{u}(x) \quad a.e. \ M$$

holds for any $u \in H^1(M)$,

$$\|\tilde{u}\|_{L^s(M)} \leq C\|u\|_{L^s,M} \quad (1 \leq s \leq \infty)$$

holds for any $u \in H^1(M) \cap L^s(M)$,

$$\|\tilde{u}\|_{H^1(M)} \leq C\|u\|_{H^1(M)} + \epsilon^{(s-2)/3(2s)} \|u\|_{L^s,M}$$

for any $u \in H^1(M) \cap L^s(M)$ with $2 \leq s < \infty$,

$$\|\tilde{u}\|_{H^1(M)} \leq C\|u\|_{H^1(M)} + C\epsilon^{(1/2)}\|u\|_{L^\infty,M}$$

holds for any $u \in H^1(M) \cap L^\infty(M)$. 
Proof. Without loss of generality, we may assume that \( \tilde{w} = 0 \). We take an arbitrary \( u \in H^1(M) \) and put

\[
\tilde{u}(x) = u(x) \quad x \in M_e
\]

\[
= u(\varepsilon^2 |x|^{-2}) \eta_\varepsilon(x) \quad x \in B_e,
\]

where \( \eta_\varepsilon \in C^\infty(R^3) \) satisfies \( 0 \leq \eta_\varepsilon \leq 1, \eta_\varepsilon = 1 \) on \( R^3 \setminus B_{\varepsilon/2} \), \( \eta_\varepsilon = 0 \) on \( B_{\varepsilon/4} \) and \( |\nabla \eta_\varepsilon| \leq 8\varepsilon^{-1} \). Notice that both \( \eta_\varepsilon(\varepsilon^2 |x|^{-2}) \) and \( (\nabla \eta_\varepsilon)(\varepsilon^2 |x|^{-2}) \) vanish on \( R^3 \setminus B_{4\varepsilon} \). Then, by using the Kelvin transformation of co-ordinates, \( y = \varepsilon^2 x \frac{\varepsilon}{x} \), we have

\[
\int_{B_e} |\tilde{u}(x)|^s dx \leq \int_{R^3 \setminus B_e} |u(y)|^s (\varepsilon^2 |y|^{-2})^s (\varepsilon |y|^{-1})^s dy
\]

\[
\leq \int_{M_e} |u(y)|^s dy \quad (1 \leq s < \infty),
\]

and

\[
\int_{B_e} |\nabla u(x)|^s dx \leq C \int_{B_e} |u(\varepsilon^2 |x|^{-2})|^2 (|\nabla \eta_\varepsilon(x)|^2 dx
\]

\[
+ C \int_{B_e} (\varepsilon |x|^{-1})^4 (|\nabla u(\varepsilon^2 |x|^{-2})|^2 ) \eta_\varepsilon(x)^2 dx
\]

\[
\leq C \varepsilon^4 \int_{M_e} |u(y)|^2 |y|^{-6} dy + C \int_{M_e} |\nabla u|^2 dy.
\]

By Hölder's inequality, we see that

\[
\int_{M_e} |u(y)|^s |y|^{-6} dy \leq \begin{cases} C\varepsilon^{-(1+(2/s))3} \|u\|^2_{L^s(M_e)} & (2 \leq s < \infty) \\ C\varepsilon^{-3} \|u\|^2_{L^\infty(M_e)} & (s = \infty). \end{cases}
\]

Thus, we get Lemma 2.3.

3. The Green function.

Let \( G(x,y) \) be the Green function of the Laplacian in \( M \) under the Dirichlet condition on \( \partial M \). We introduce the following kernel \( p_\varepsilon(x,y) \).

\[
p_\varepsilon(x,y) = G(x,y) + g(\varepsilon)G(x,\tilde{w})G(\tilde{w},y) + h(\varepsilon)\langle \nabla_w G(x,\tilde{w}), \nabla_w G(\tilde{w},y) \rangle,
\]

where

\[
\langle \nabla_w u(\tilde{w}), \nabla_w v(\tilde{w}) \rangle = \sum_{n=1}^{3} \frac{\partial u}{\partial \tilde{w}_i} \frac{\partial v}{\partial \tilde{w}_i} |w = \tilde{w}|
\]
when \( w = (w_1, w_2, w_3) \) is an orthonomal frame of \( \mathbb{R}^3 \).

Here we put

\[
g(e) = -(\gamma + (4\pi e)^{-1} + k^{-1}(4\pi)^{-1} e^{-2})^{-1},
\]

where

\[
\gamma = \lim_{x \to \tilde{w}} (G(x, \tilde{w}) - (4\pi)^{-1}|x - \tilde{w}|^{-1})
\]

and

\[
h(e) = k^{-1}/((4\pi)^{-1} e^{-2} + k^{-1}(2\pi)^{-1} e^{-3}).
\]

We put

\[
G(x, y) - (4\pi)^{-1}|x - y|^{-1} = S(x, y).
\]

Then, \( S(x, y) \in C^\infty(M \times M) \). We have the following:

\[
\langle \nabla_w G(x, \tilde{w}), \nabla_w G(\tilde{w}, y) \rangle
\]

\[
= (4\pi)^{-1} e^{-2} \frac{\partial}{\partial w_1} G(\tilde{w}, y) + \langle \nabla_w S(x, \tilde{w}), \nabla_w G(\tilde{w}, y) \rangle
\]

\[
\frac{\partial}{\partial x_1} \langle \nabla_w G(x, \tilde{w}), \nabla_w G(\tilde{w}, y) \rangle
\]

\[
= - (2\pi)^{-1} e^{-3} \frac{\partial}{\partial w_1} G(\tilde{w}, y) + \frac{\partial}{\partial x_1} \langle \nabla_w S(x, \tilde{w}), \nabla_w G(\tilde{w}, y) \rangle
\]

for \( x = (\epsilon, 0, 0) \).

Then,

\[
k p_s(x, y) + \frac{\partial}{\partial x_1} p_s(x, y)
\]

\[
=k G(x, y) + k g(e) G(x, \tilde{w}) G(\tilde{w}, y) + k h(e) \langle \nabla_w G(x, \tilde{w}), \nabla_w G(\tilde{w}, y) \rangle
\]

\[
- \frac{\partial}{\partial x_1} G(x, y) - g(e) \left( \frac{\partial}{\partial x_1} G(x, \tilde{w}) \right) G(\tilde{w}, y)
\]

\[
- h(e) \frac{\partial}{\partial x_1} \langle \nabla_w G(x, \tilde{w}), \nabla_w G(\tilde{w}, y) \rangle
\]

\[
=k G(x, y) - k G(\tilde{w}, y) - g(e) \frac{\partial}{\partial x_1} S(x, \tilde{w}) G(\tilde{w}, y)
\]
for \( x = (\varepsilon, 0, 0) \).

Let \( G_\varepsilon(x, y) \) denote the Green function of \( \Delta \) in \( M_\varepsilon \) satisfying \( G_\varepsilon(x, y) = 0 \) on \( \partial M \) and \( kG_\varepsilon(x, y) + \frac{\partial}{\partial y} G_\varepsilon(x, y) = 0 \) on \( \partial B_\varepsilon \).

Let \( G_\varepsilon \) and \( P_\varepsilon \) be the operator defined by

\[
G_\varepsilon f(x) = \int_{M_\varepsilon} G_\varepsilon(x, y) f(y) dy
\]

\[
P_\varepsilon f(x) = \int_{M_\varepsilon} p_\varepsilon(x, y) f(y) dy.
\]

We put \( Q_\varepsilon f(x) = P_\varepsilon f(x) \). Then, it satisfies

\[
\begin{align*}
\Delta Q_\varepsilon f(x) &= 0, & x &\in M_\varepsilon \\
Q_\varepsilon f(x) &= 0, & x &\in \partial M \\
kQ_\varepsilon f(x) + \frac{\partial}{\partial y} Q_\varepsilon f(x) &= kP_\varepsilon f(x) + \frac{\partial}{\partial y} P_\varepsilon f(x), & x &\in \partial B_\varepsilon.
\end{align*}
\]

We know that

\[
kP_\varepsilon f(x) + \frac{\partial}{\partial y} P_\varepsilon f(x)
= k(Gf)(x) - (Gf)(\bar{w}) - \frac{\partial}{\partial x_1} Gf(x) + \frac{\partial}{\partial w_1} Gf(\bar{w})
+ O(1)g(\varepsilon)Gf(\bar{w}) + O(h(\varepsilon))\frac{\partial}{\partial x_1} \langle \nabla_w S(x, \bar{w}), \nabla_w Gf(\bar{w}) \rangle
+ O(h(\varepsilon))\langle \nabla_w S(x, \bar{w}), \nabla_w Gf(\bar{w}) \rangle.
\]

4. Proof of Theorem 1.

We have the following decomposition of \( u_\varepsilon \).

\[
u_\varepsilon = - \lambda(\varepsilon)Q_\varepsilon u_\varepsilon^\ast + \lambda(\varepsilon)P_\varepsilon u_\varepsilon^\ast.
\]
Lemma 4.1. Fix $p \in (1, 2)$. Then,

\begin{equation}
\sup_{x \in M_\varepsilon} |P^*_x u^p_\varepsilon| \leq C(1 + \varepsilon \|u_\varepsilon\|_{\infty, e}) \tag{4.2}
\end{equation}

holds for any $\tau < 1$ which is close enough to 1.

Proof. Recall that

\[ P^*_x u^p_\varepsilon(x) = G\hat{u}^p_\varepsilon(x) + g(\varepsilon)G(x, \tilde{w})G\hat{u}^p_\varepsilon(\tilde{w}) + h(\varepsilon)\langle \nabla_w G(x, \tilde{w}), \nabla_w G\hat{u}^p_\varepsilon(\tilde{w}) \rangle. \]

Here $\hat{u}^p_\varepsilon(x)$ is the extension of $u^p_\varepsilon$ to $M$ which is zero outside $M_\varepsilon$.

We have

\[ |G\hat{u}^p_\varepsilon(\tilde{w})| \leq C\|u^p_\varepsilon\|_{\infty, e} \leq C\|u_\varepsilon\|_{p, e} \quad \text{for any } t > (3/2). \]

Notice that we can take $t > (3/2)$ as close as $3/2$ so that $\varepsilon u^p_\varepsilon$ is bounded.

We have

\[ \varepsilon |\nabla_w G\hat{u}^p_\varepsilon(\tilde{w})| \leq C\varepsilon \|u_\varepsilon\|_{\infty, e} \left( \int_{M_\varepsilon} u^p_\varepsilon^{-r}(y)|y - \tilde{w}|^{-2} dy \right) \]

\[ \leq C\varepsilon \|u_\varepsilon\|_{\infty, e} \|u_\varepsilon\|_{p + 1, e}^{(p + 1)} \left( \int_{M_\varepsilon} |y - w|^{-2r} dy \right)^{1/r} \]

where $r = (p + 1)/(1 + \tau)$. Thus, it does not exceed $\varepsilon \|u_\varepsilon\|_{\infty, e}$ since $2r < 3$, if we take $p \in (1, 2)$ and $\tau < 1$ as close as 1.

Lemma 4.2. Fix $p \in (1, 2)$. Then,

\begin{equation}
|Q^*_x u^p_\varepsilon| \leq C\varepsilon^{1 - \frac{p}{q}} \|u_\varepsilon\|_{\infty, e}^{(a/q)} \tag{4.3}
\end{equation}

holds for $q > 6, q > \mu$.

Proof. By Lemma 2.1 we see that

\begin{equation}
|Q_x f(x)| \leq C\varepsilon^{2r - 1} \|G\hat{f}\|_{C^{1 + (\sigma/2)}(M)} \tag{4.4}
\end{equation}

for $\sigma > 1$ by using Lemma 2.1 and (3.1). Here $\hat{f}$ is the extension of $f$ to $M$ which is zero outside $M_\varepsilon$.

We have

\[ \|G\hat{f}\|_{C^{1 + (\sigma/2)}(M)} \leq C\|f\|_{q, e} \]

for $q > 3/(1 - (\sigma/2))$. We take $\sigma > 1$ as close as 1.

Then, we have
(4.5)
\[ |Q_{e}f(x)| \leq C\|f\|_{q,e} \]
for \( q > 6, \ x \in M_{e} \).

We have
\[ \varepsilon\|u_{e}\|_{q,e}^{2} \leq \varepsilon\|u_{e}\|_{\infty}^{\mu(q)} \left( \int_{M_{e}}|u_{e}|^{p_{q}-\mu}dx \right)^{1/q} . \]

We take \( \mu < q \) as close as \( q \) and \( q > 6 \) as close as \( 6 \). Then, \( pq - \mu < 6 \) for \( p \in (1,2) \).

On the other hand we have
\[ \|u_{e}\|_{p_{q}-\mu,e} \leq \|\bar{u}_{e}\|_{L^{p_{q}-\mu}(M)} \leq C\|\bar{u}_{e}\|_{H^{1}(M)} \]
by the Sobolev embedding theorem using \( pq - \mu < 6 \). Here \( \bar{u}_{e} \) is an extension of \( u_{e} \) in Lemma 2.3 which is different from \( \hat{u}_{e} \). It should be noted, since \( \lambda(\varepsilon) \) is defined as an infimum of a functional so that it is easy to see that \( \lim_{\varepsilon \to 0} \sup \lambda(\varepsilon) < \infty \). Therefore,
\[ \|\bar{u}_{e}\|_{H^{1}(M)} \leq \|u_{e}\|_{H^{1}(M)} + C\varepsilon^{-1}\|u_{e}\|_{2,e} \]
by taking \( s = 2 \) in Lemma 2.3. Since \( u_{e} \) is a minimizer of (1.1) and
\[ \|u_{e}\|_{2,e} \leq C \]
we see that
\[ \|\bar{u}_{e}\|_{H^{1}(M)} \leq C\varepsilon^{-1} . \]

Therefore,
\[ \|u_{e}\|_{p_{q}-\mu,e} \leq C'' \varepsilon^{-1} . \]

Therefore, we get the desired result.

Proof of Theorem 1. We put \( \xi = \|u_{e}\|_{\infty,e} \).

By Lemma 4.1 and 4.2 we get
\[ \xi \leq C\left(1 + \varepsilon_{\nu}^{\tau} + \varepsilon^{1-p+(\mu/q)}\varepsilon^{\mu/q} \right) . \]

Since \( p \in (1,2) \), we can take \( \mu < q \) as close as \( q \) so that \( 1 - p + (\mu/q) > 0 \). We take \( \tau < 1 \). Therefore, \( \xi \leq C'' \). We get the desired result.

References


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