<table>
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<th>L^∞ boundedness of nonlinear eigenfunction under singular variation of domains</th>
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<tr>
<td>Author(s)</td>
<td>Ozawa, Shin</td>
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<tr>
<td>Citation</td>
<td>Osaka Journal of Mathematics. 32(2) P.363-P.371</td>
</tr>
<tr>
<td>Issue Date</td>
<td>1995</td>
</tr>
<tr>
<td>Text Version</td>
<td>publisher</td>
</tr>
<tr>
<td>URL</td>
<td><a href="https://doi.org/10.18910/10245">https://doi.org/10.18910/10245</a></td>
</tr>
<tr>
<td>DOI</td>
<td>10.18910/10245</td>
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https://ir.library.osaka-u.ac.jp/repo/ouka/all/

Osaka University
1. Introduction.

Let $M$ be a bounded domain in $\mathbb{R}^3$ with smooth boundary $\partial M$. Let $\tilde{w}$ be a point in $M$. We remove ball $B_\varepsilon$ of radius $\varepsilon$ with the center $\tilde{w}$ from $M$ and we get $M_\varepsilon = M \setminus B_\varepsilon$.

Fix $p \in (1, 2)$. We fix $k > 0$. We put

$$
A(\varepsilon) = \inf \left( \int_{M_\varepsilon} |\nabla u|^2 \, dx + k \int_{\partial B_\varepsilon} u^2 \, d\sigma_x \right),
$$

where $X = \{ u \in H^1(M_\varepsilon) \mid u = 0 \text{ on } \partial M \text{ and } u \geq 0 \text{ in } M_\varepsilon, \| u \|_{p+1, \varepsilon} = 1 \}$. Here $\| u \|_{L^q(M_\varepsilon)} = \| u \|_{q, \varepsilon}$. We see that there exists at least one solution $v_\varepsilon$ of the above problem which attains $(1.1)_\varepsilon$. We know that $v_\varepsilon$ satisfies $-\Delta v_\varepsilon(x) = \lambda(\varepsilon)v_\varepsilon(x)$ in $M_\varepsilon$, $v_\varepsilon(x) = 0$ on $\partial M$ and $kv_\varepsilon(x) + (\partial / \partial v_\varepsilon)v_\varepsilon(x) = 0$ on $\partial B_\varepsilon$. Here $\partial / \partial v_\varepsilon$ denotes the derivative along the exterior normal vector with respect to $M_\varepsilon$.

Let $S_\varepsilon$ denote the set of positive function $u_\varepsilon$ which attains the minimum of $(1.1)$. Main result of this paper is the following

**Theorem 1.** Fix $p \in (1, 2)$. Then, there exists a constant $C$ independent of $\varepsilon$ such that

$$
\sup_{u_\varepsilon \in S_\varepsilon} \sup_{x \in M_\varepsilon} |v_\varepsilon(x)| < C.
$$

Related topics are discussed in Lin [1], Ozawa [4].

2. Preliminary Lemmas.

We have the following Lemma 2.1.

**Lemma 2.1.** Consider the following equations.

$$
(2.1) \quad \Delta v_\varepsilon(x) = 0 \quad x \in M \setminus \overline{B_\varepsilon}
$$
where $\omega \in S^2$. Then, the solution of (2.1), (2.2), (2.3) satisfies

$$(2.4) \quad |v_\epsilon(x)| \leq C_\epsilon^2 r^{-1}\|\alpha\|_{C^0(\partial S^2)}$$

for any $\sigma > 1$. Here $C$ may depend on $\sigma$ but independent of $\epsilon$. Here $r = |x - \bar{w}|$.

Proof. Let $\Delta_{S^2}$ denote the Laplace-Beltrami operator on $S^2$. It is well known that $-n(n+1)$ is an eigenvalue of $\Delta_{S^2}$ whose multiplicity is $(2n+1)$. We can write it explicitly by using the Legendre polynomial but we do not do it. We write complete orthonormal basis of $L^2(S^2)$ consisting of eigenfunction by $\{\phi_j(\omega)\}_{j=1}^\infty$. If $\Delta\phi_j(\omega) = -n(n+1)\phi_j(\omega)$, then we write $j$ as $j \in (n)$. Thus, $\# \{j : j \in (n)\} = 2n+1$.

First we want to construct a solution of (2.1), (2.3).

We put

$$v^\ast(x) = \sum_{n=0}^\infty r^{-(n+1)} \sum_{j \in (n)} c_{j,n} \phi_j(\omega).$$

Then,

$$\Delta v^\ast(x) = 0 \quad \text{in} \quad R^3 \setminus \overline{B}_\epsilon.$$

We expand $\alpha$ by $\phi_j(\omega)$ as

$$\alpha(\omega) = \sum_{n=0}^\infty \sum_{j \in (n)} a_{j,n} \phi_j(\omega).$$

By the equality (2.3), we have

$$(k c_{j,n} e^{-(n+1)} + (n+1)c_{j,n} e^{-(n+2)}) = a_{j,n}.$$

Then,

$$c_{j,n} = a_{j,n} e^{n+1}(k + (n+1)e^{-1})^{-1}.$$ 

Therefore,

$$|v^\ast(x)| \leq \epsilon \sum_{n=0}^\infty \sum_{j \in (n)} a_{j,n} (e/r)^{n+1}(e k + (n+1))^{-1} \phi_j(\omega).$$

We notice that we have

$$(2.5) \quad |\phi_j(\omega)| \leq C(n+1)^{1/2}$$
for \( j \in (n) \) by the property of Legendre and associated Legendre polynomial. See Mizohata [2, p.312].

By the Schwarz inequality we have

\[
|v_e^*(x)| \leq C e^{2r-1} \left( \sum_{n=0}^{\infty} (n+1)^r \sum_{j_e(n)} a_{j,n}^2 \right)^{1/2} \\
\times \left( \sum_{n=0}^{\infty} (n+1)^{-\sigma(2n+1)} n(n+1)^{-2} \right)^{1/2}.
\]

If we take \( \sigma > 1 \), then

\[
(2.6) \quad |v^*(x)| \leq C e^{2r-1} \left( \sum_{n=0}^{\infty} (n+1)^r \sum_{j_e(n)} a_{j,n}^2 \right)^{1/2}
\]

\[
\leq C e^{2r-1} \|x\|_{H^{\sigma/2}(S^2)}.
\]

Here we note that \( \|x\|_{H^{\sigma}(S^2)} \) is equivalent to \( \left( \sum_{n=0}^{\infty} (n+1)^4 \sum_{j_e(n)} a_{j,n}^2 \right) \), since the eigenvalue of \( -\Delta_{S^2} \) is \( n(n+1) \). We used representation of norm of fractional Sobolev space. Notice that \( C^{\sigma/2}(S^2) \subset H^{\sigma/2}(S^2) \) for \( \sigma < \sigma' \).

Now \( v^*(x) \) does not satisfy \( v^*(x) = 0 \) on \( \partial M \).

By the same procedure as in the repeated construction of the function \( v_e^{(n)} \) in Proposition 1 of Ozawa [3] we proved Lemma 2.1.

The following Lemma is very useful.

**Lemma 2.3.** There exists an extension operator \( E : H^1(M) \ni u \to \bar{u} \in H^1(M) \) satisfying the followings;

\[
\bar{u}(x) = u(x) \quad a.e. \ M
\]

holds for any \( u \in H^1(M) \),

\[
(2.9) \quad \|\bar{u}\|_{L^s(M)} \leq C \|u\|_{s,t} \quad (1 \leq s \leq \infty)
\]

holds for any \( u \in H^1(M) \cap L^s(M) \),

\[
(2.10) \quad \|\bar{u}\|_{H^1(M)} \leq C \|u\|_{H^1(M)} + \varepsilon^{(s-2)(\frac{1}{2s}-1)} \|u\|_{s,t}
\]

for any \( u \in H^1(M) \cap L^s(M) \) with \( 2 \leq s < \infty \),

\[
(2.11) \quad \|\bar{u}\|_{H^1(M)} \leq C \|u\|_{H^1(M)} + C \varepsilon^{(1/2)} \|u\|_{\infty,t}
\]

holds for any \( u \in H^1(M) \cap L^\infty(M) \).
Proof. Without loss of generality, we may assume that \( \tilde{w} = 0 \). We take an arbitrary \( u \in H^1(M) \) and put

\[
\tilde{u}(x) = u(x) \quad x \in M,
\]

\[
= u(\varepsilon^2 x |x|^{-2}) \eta_{\varepsilon}(x) \quad x \in B_{\varepsilon},
\]

where \( \eta_{\varepsilon} \in C^\infty(R^3) \) satisfies \( 0 \leq \eta_{\varepsilon} \leq 1 \), \( \eta_{\varepsilon} = 1 \) on \( R^3 \setminus B_{\varepsilon/2} \), \( \eta_{\varepsilon} = 0 \) on \( B_{\varepsilon/4} \) and \( |\nabla \eta_{\varepsilon}| \leq 8\varepsilon^{-1} \). Notice that both \( \eta_{\varepsilon}(\varepsilon^2 x |x|^{-2}) \) and \( (\nabla \eta_{\varepsilon})(\varepsilon^2 x |x|^{-2}) \) vanish on \( R^3 \setminus B_{4\varepsilon} \).

Then, by using the Kelvin transformation of co-ordinates, \( y = \varepsilon^2 x |x|^{-2} \), we have

\[
\int_{B_{\varepsilon}} |\tilde{u}(x)|^s dx \leq \int_{R^3 \setminus B_{\varepsilon}} |u(y)|^s \eta_{\varepsilon}(\varepsilon^2 y |y|^{-2}) |(\varepsilon|y|^{-1})^6 dy
\]

\[
\leq \int_{M_{\varepsilon}} |u(y)|^s dy \quad (1 \leq s < \infty),
\]

and

\[
\int_{B_{\varepsilon}} |\nabla u(x)|^2 dx \leq C \int_{B_{\varepsilon}} |u(\varepsilon^2 x |x|^{-2})| |\nabla \eta_{\varepsilon}(x)|^2 dx
\]

\[
+ C \int_{B_{\varepsilon}} (\varepsilon|x|^{-1})^4 |(\nabla u)(\varepsilon^2 x |x|^{-2})|^2 \eta_{\varepsilon}(x)^2 dx
\]

\[
\leq C \varepsilon^4 \int_{M_{\varepsilon}} |u(y)|^2 |y|^{-6} dy + C \int_{M_{\varepsilon}} |\nabla u|^2 dy.
\]

By Hölder's inequality, we see that

\[
\int_{M_{\varepsilon}} |u(y)|^s |y|^{-6} dy \leq \begin{cases} C \varepsilon^{-1 + (2/s)/3} \|u\|^2_{L^1(M_{\varepsilon})} & (2 \leq s < \infty) \\ C \varepsilon^{-3} \|u\|^2_{L^\infty(M_{\varepsilon})} & (s = \infty). \end{cases}
\]

Thus, we get Lemma 2.3.

3. The Green function.

Let \( G(x,y) \) be the Green function of the Laplacian in \( M \) under the Dirichlet condition on \( \partial M \). We introduce the following kernel \( p_{\varepsilon}(x,y) \).

\[
p_{\varepsilon}(x,y) = G(x,y) + g(\varepsilon)G(x,\tilde{w})G(\tilde{w},y) + h(\varepsilon) \langle \nabla w G(x,\tilde{w}), \nabla w G(\tilde{w},y) \rangle,
\]

where

\[
\langle \nabla w \tilde{u}(\tilde{w}), \nabla w \tilde{v}(\tilde{w}) \rangle = \sum_{n=1}^3 \overline{\partial u} \overline{\partial v} \bigg|_{\tilde{w} = \tilde{w}},
\]
when $w=(w_1, w_2, w_3)$ is an orthonormal frame of $\mathbb{R}^3$.

Here we put

$$g(\epsilon) = -(y + (4\pi \epsilon)^{-1} + k^{-1}(4\pi)^{-1}\epsilon^{-2})^{-1},$$

where

$$y = \lim_{x \to \tilde{w}} (G(x, \tilde{w}) - (4\pi)^{-1}|x - \tilde{w}|^{-1})$$

and

$$h(\epsilon) = k^{-1}/((4\pi)^{-1}\epsilon^{-2} + k^{-1}(2\pi)^{-1}\epsilon^{-3}).$$

We put

$$G(x, y) - (4\pi)^{-1}|x - y|^{-1} = S(x, y).$$

Then, $S(x, y) \in C^\infty(M \times M)$. We have the following:

$$\langle \nabla_w G(x, \tilde{w}), \nabla_w G(\tilde{w}, y) \rangle$$

$$= (4\pi)^{-1}\epsilon^{-2} \frac{\partial}{\partial w_1} G(\tilde{w}, y) + \langle \nabla_w S(x, \tilde{w}), \nabla_w G(\tilde{w}, y) \rangle$$

$$\frac{\partial}{\partial x_1} \langle \nabla_w G(x, \tilde{w}), \nabla_w G(\tilde{w}, y) \rangle$$

$$= -2(2\pi)^{-1}\epsilon^{-3} \frac{\partial}{\partial w_1} G(\tilde{w}, y) + \frac{\partial}{\partial x_1} \langle \nabla_w S(x, \tilde{w}), \nabla_w G(\tilde{w}, y) \rangle$$

for $x=(\epsilon, 0, 0)$.

Then,

$$kp_s(x, y) + \frac{\partial}{\partial v_x} p_s(x, y)$$

$$= kG(x, y) + kg(\epsilon)G(x, \tilde{w})G(\tilde{w}, y) + kh(\epsilon)\langle \nabla_w G(x, \tilde{w}), \nabla_w G(\tilde{w}, y) \rangle$$

$$- \frac{\partial}{\partial x_1} G(x, y) - g(\epsilon)\left(\frac{\partial}{\partial x_1} G(x, \tilde{w})\right) G(\tilde{w}, y)$$

$$- h(\epsilon)\frac{\partial}{\partial x_1} \langle \nabla_w G(x, \tilde{w}), \nabla_w G(\tilde{w}, y) \rangle$$

$$= kG(x, y) - kG(\tilde{w}, y) - g(\epsilon)\frac{\partial}{\partial x_1} S(x, \tilde{w})G(\tilde{w}, y)$$
Let $G_{e}(x, y)$ denote the Green function of $\Delta$ in $M_{e}$ satisfying $G_{e}(x, y) = 0$ on $\partial M$ and $kG_{e}(x, y) + \frac{\partial}{\partial v_{x}}G_{e}(x, y) = 0$ on $\partial B_{r}$.

Let $G_{e}$ and $P_{e}$ be the operator defined by

$$G_{e}f(x) = \int_{M} G_{e}(x, y)f(y)dy$$

$$P_{e}f(x) = \int_{M} p_{e}(x, y)f(y)dy.$$ 

We put $Q_{e}f(x) = P_{e}f(x)$. Then, it satisfies

$$\Delta Q_{e}f(x) = 0 \quad x \in M_{e}$$

$$Q_{e}f(x) = 0 \quad x \in \partial M$$

$$kQ_{e}f(x) + \frac{\partial}{\partial v_{x}}Q_{e}f(x) = kP_{e}f(x) + \frac{\partial}{\partial v_{x}}P_{e}f(x) \quad x \in \partial B_{r}.$$ 

We know that

$$kP_{e}f(x) + \frac{\partial}{\partial v_{x}}P_{e}f(x)$$

$$= k(Gf)(x) - (Gf)(\bar{w}) - \frac{\partial}{\partial x_{1}}Gf(x) + \frac{\partial}{\partial w_{1}}Gf(\bar{w})$$

$$+ O(1)g(e)Gf(\bar{w}) + O(h(\varepsilon))\frac{\partial}{\partial x_{1}}\langle \nabla_{w}S(x, \bar{w}), \nabla_{w}Gf(\bar{w}) \rangle$$

$$+ O(h(\varepsilon))\langle \nabla_{w}S(x, \bar{w}), \nabla_{w}Gf(\bar{w}) \rangle.$$ 

4. **Proof of Theorem 1.**

We have the following decomposition of $u_{e}$.

$$u_{e} = -\lambda(e)Q_{e}u_{e}^{p} + \lambda(e)P_{e}u_{e}^{p}.$$
Lemma 4.1. Fix $p \in (1,2)$. Then,

$$\sup_{x \in M_{\delta}} |P_{\tau}u^\delta_p| \leq C(1 + \epsilon \|u_\epsilon\|_{\infty,\delta})$$

holds for any $\tau < 1$ which is close enough to 1.

Proof. Recall that

$$P_{\tau}u^\delta_p(x) = G\delta^\tau_p(x) + g(\epsilon)G(x, \bar{w})G\delta^\tau_p(\bar{w}) + h(\epsilon)\langle \nabla_{\epsilon}G(x, \bar{w}), \nabla_{\epsilon}G\delta^\tau_p(\bar{w}) \rangle.$$ 

Here $\delta^\tau_p(x)$ is the extension of $u^\delta_p$ to $M$ which is zero outside $M_{\delta}$.

We have $|G\delta^\tau_p(\bar{w})| \leq C\|u^\delta_p\|_{t,\delta} \leq C\|u_\epsilon\|_{p,\delta}$ for any $t > (3/2)$. Notice that we can take $t > (3/2)$ as close as 3/2 so that $tp < p + 1$ for $p \in (1,2)$. Therefore, $|G\delta^\tau_p(\bar{w})|$ is bounded.

We have

$$\epsilon|\nabla_{\epsilon}\delta^\tau_p(\bar{w})| \leq C\epsilon\|u_\epsilon\|_{\infty,\delta} \left(\int_{M_{\delta}} u^\delta_{p-1}(y)|y - \bar{w}|^{-2} dy \right)$$

$$\leq C\epsilon\|u_\epsilon\|_{\infty,\delta} \|u_\epsilon\|_{p,\delta}^{(p+1)} \left(\int_{M_{\delta}} |y - \bar{w}|^{-2\rho} dy \right)$$

where $\rho = (p + 1)/(1 + \tau)$. Thus, it does not exceed $\epsilon\|u_\epsilon\|_{\infty,\delta}$ since $2\rho < 3$, if we take $p \in (1,2)$ and $\tau < 1$ as close as 1.

Lemma 4.2. Fix $p \in (1,2)$. Then,

$$|Q_{\tau}u^\delta_p| \leq C\epsilon^{1-p+(\mu/q)}\|u_\epsilon\|_{(\mu/q),\delta}$$

holds for $q > 6$, $q > \mu$.

Proof. By Lemma 2.1 we see that

$$|Q_{\tau}f(x)| \leq C\epsilon^{1-p+(\mu/q)}\|G\hat{f}\|_{C^{1+\sigma/2}(M)}$$

for $\sigma > 1$ by using Lemma 2.1 and (3.1). Here $\hat{f}$ is the extension of $f$ to $M$ which is zero outside $M_{\delta}$.

We have

$$\|G\hat{f}\|_{C^{1+\sigma/2}(M)} \leq C\|f\|_{q,\delta}$$

for $q > 3/(1-(\sigma/2))$. We take $\sigma > 1$ as close as 1.

Then, we have
for $q > 6$, $x \in M_\varepsilon$.

We have

$$\varepsilon \| u_\varepsilon \|_{q, \varepsilon} \leq \| u_\varepsilon \|_{\frac{\mu(q)}{\omega_0}, \varepsilon} \left( \int_{M_\varepsilon} |u_\varepsilon|^{pq - \mu} \, dx \right)^{1/q}.$$ 

We take $\mu < q$ as close as $q$ and $q > 6$ as close as 6. Then, $pq - \mu < 6$ for $p \in (1, 2)$.

On the other hand, we have

$$\| u_\varepsilon \|_{pq - \mu, \varepsilon} \leq \| \tilde{u}_\varepsilon \|_{L^{pq - \mu}(M_\varepsilon)} \leq C \| \tilde{u}_\varepsilon \|_{H^1(M)}$$

by the Sobolev embedding theorem using $pq - \mu < 6$. Here $\tilde{u}_\varepsilon$ is an extension of $u_\varepsilon$ in Lemma 2.3 which is different from $\hat{u}_\varepsilon$. It should be noted, since $\lambda(\varepsilon)$ is defined as an infimum of a functional so that it is easy to see that $\lim_{\varepsilon \to 0} \sup \lambda(\varepsilon) < \infty$. Therefore,

$$\| \tilde{u}_\varepsilon \|_{H^1(M)} \leq \| u_\varepsilon \|_{H^1(M)} + C \varepsilon^{-1} \| u_\varepsilon \|_{2, \varepsilon}$$

by taking $s = 2$ in Lemma 2.3. Since $u_\varepsilon$ is a minimizer of (1.1) and

$$\| u_\varepsilon \|_{2, \varepsilon} \leq C$$

we see that

$$\| \tilde{u}_\varepsilon \|_{H^1(M)} \leq C \varepsilon^{-1}.$$ 

Therefore,

$$\| u_\varepsilon \|_{pq - \mu, \varepsilon} \leq C'' \varepsilon^{-1}.$$ 

Therefore, we get the desired result.

Proof of Theorem 1. We put $\xi = \| u_\varepsilon \|_{\infty, \varepsilon}$. By Lemma 4.1 and 4.2 we get

$$\xi \leq C(1 + \varepsilon^{p\tau} + \varepsilon^{1 - p + (\mu(q)/\tau)} \varepsilon^{\mu(q)}).$$

Since $p \in (1, 2)$, we can take $\mu < q$ as close as $q$ so that $1 - p + (\mu(q)/\tau) > 0$. We take $\tau < 1$. Therefore, $\xi \leq C''$. We get the desired result.

References


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