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L^∞ BOUNDEDNESS OF NONLINEAR EIGENFUNCTION UNDER SINGULAR VARIATION OF DOMAINS

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1. Introduction.

Let M be a bounded domain in $\boldsymbol{R}^3$ with smooth boundary ∂M. Let $\tilde{w}$ be a point in M. We remove ball B_ε of radius ε with the center $\tilde{w}$ from M and we get $M_\varepsilon = M \backslash \bar{B}_\varepsilon$.

Fix $p \in (1,2)$. We fix $k > 0$. We put

$$\lambda(\varepsilon) = \inf_{u \in X}\left(\int_{M_\varepsilon} |\nabla u|^2 dx + k\int_{\partial B_\varepsilon} u^2 d\sigma_x\right), \tag{1.1}$$

where $X = \{u \in H^1(M_\varepsilon),\ u = 0 \text{ on } \partial M \text{ and } u \geq 0 \text{ in } M_\varepsilon,\ \|u\|_{p+1,\varepsilon} = 1\}$. Here $\|u\|_{L^q(M_\varepsilon)} = \|u\|_{q,\varepsilon}$. We see that there exists at least one solution v_ε of the above problem which attains $(1.1)_\varepsilon$. We know that v_ε satisfies $-\Delta v_\varepsilon(x) = \lambda(\varepsilon) v_\varepsilon(x)^p$ in M_ε, $v_\varepsilon(x) = 0$ on ∂M and $k v_\varepsilon(x) + (\partial/\partial \nu_x) v_\varepsilon(x) = 0$ on ∂B_ε. Here $\partial/\partial \nu_x$ denotes the derivative along the exterior normal vector with respect to M_ε.

Let S_ε denote the set of positive function u_ε which attains the minimum of (1.1).

Main result of this paper is the following

Theorem 1. *Fix $p \in (1,2)$. Then, there exists a constant C independent of ε such that*

$$\sup_{u_\varepsilon \in S_\varepsilon} \sup_{x \in M_\varepsilon} |v_\varepsilon(x)| < C.$$

Related topics are discussed in Lin [1], Ozawa [4].

2. Preliminary Lemmas.

We have the following Lemma 2.1.

Lemma 2.1. *Consider the following equations.*

$$\Delta v_\varepsilon(x) = 0 \qquad x \in M \backslash \bar{B}_\varepsilon \tag{2.1}$$

(2.2) $$v_\varepsilon(x)=0 \qquad x\in\partial M$$

(2.3) $$k\, v_\varepsilon(x)+(\partial v_\varepsilon/\partial v_x)=\alpha(\omega) \qquad x=w+\varepsilon\omega\in\partial B_\varepsilon.$$

Here $\omega\in S^2$. *Then, the solution of* (2.1), (2.2), (2.3) *satisfies*

(2.4) $$|v_\varepsilon(x)|\le C\varepsilon^2 r^{-1}\|\alpha\|_{C^{(\sigma/2)}(S^2)}$$

for any $\sigma>1$. *Here C may depend on* σ *but independent of* ε. *Here* $r=|x-\tilde{w}|$.

Proof. Let Δ_{S^2} denote the Laplace-Beltrami oporator on S^2. It is well known that $-n(n+1)$ is an eigenvalue of Δ_{S^2} whose multiplicity is $(2n+1)$. We can write it explicitly by using the Legendre polynomial but we do not do it. We write complete orthonomal basis of $L^2(S^2)$ consisting of eigenfunction by $\{\varphi_j(\omega)\}_{j=1}^{\infty}$. If $\Delta\varphi_j(\omega)=-n(n+1)\varphi_j(\omega)$, then we write j as $j\in(n)$. Thus, $\#\{j;j\in(n)\}=2n+1$.

First we want to construct a solution of (2.1), (2.3).
We put

$$v^*(x)=\sum_{n=0}^{\infty} r^{-(n+1)}\sum_{j\in(n)} c_{j,n}\varphi_j(\omega).$$

Then,

$$\Delta v^*(x)=0 \quad \text{in} \quad \boldsymbol{R}^3\backslash\bar{B}_\varepsilon.$$

We expand α by $\varphi_j(\omega)$ as

$$\alpha(\omega)=\sum_{n=0}^{\infty}\sum_{j\in(n)} a_{j,n}\varphi_j(\omega).$$

By the equality (2.3), we have

$$(k\, c_{j,n}\varepsilon^{-(n+1)}+(n+1)c_{j,n}\varepsilon^{-(n+2)})=a_{j,n}.$$

Then,

$$c_{j,n}=a_{j,n}\varepsilon^{n+1}(k+(n+1)\varepsilon^{-1})^{-1}.$$

Therefore,

$$|v^*(x)|\le\varepsilon\sum_{n=0}^{\infty}\sum_{j\in(n)} a_{j,n}(\varepsilon/r)^{n+1}(\varepsilon k+(n+1))^{-1}\varphi_j(\omega).$$

We notice that we have

(2.5) $$|\varphi_j(\omega)|\le C(n+1)^{1/2}$$

for $j \in (n)$ by the property of Legendre and associated Legendre polynomial. See Mizohata [2, p.312].

By the Schwarz inequality we have

$$|v_\varepsilon^*| \le C\varepsilon^2 r^{-1}\left(\sum_{n=0}^{\infty}(n+1)^\sigma \sum_{j\in(n)} a_{j,n}^2\right)^{1/2} \times\left(\sum_{n=0}^{\infty}(n+1)^{-\sigma}(2n+1)n(n+1)^{-2}\right)^{1/2}.$$

If we take $\sigma > 1$, then

$$|v^*(x)| \le C\varepsilon^2 r^{-1}\left(\sum_{n=0}^{\infty}(n+1)^\sigma \sum_{j\in(n)} a_{j,n}^2\right)^{1/2} \le C\varepsilon^2 r^{-1}\|\alpha\|_{H^{\sigma/2}(S^2)}. \tag{2.6}$$

Here we note that $\|\alpha\|^2{}_{H^2(S^2)}$ is equivalent to $\left(\sum_{n=0}^{\infty}(n+1)^4 \sum_{j\in(n)} a_{j,n}^2\right)$, since the eigenvalue of $-\Delta_{S^2}$ is $n(n+1)$. We used representation of norm of fractional Sobolev space. Notice that $C^{\sigma'/2}(S^2) \subset H^{\sigma/2}(S^2)$ for $\sigma < \sigma'$.

Now $v^*(x)$ does not satisfy $v^*(x)=0$ on ∂M.
By the same procedure as in the repeated construction of the function $v_\varepsilon^{(n)}$ in Proposition 1 of Ozawa [3] we proved Lemma 2.1.

The following Lemm is very useful.

Lemma 2.3. *There exists an extension operator* $E: H^1(M_\varepsilon) \ni u \to Eu = \tilde{u} \in H^1(M)$ *satisfying the followings;*

$$\tilde{u}(x) \doteq u(x) \qquad a.e.\ M_\varepsilon \tag{2.9}$$

holds for any $u \in H^1(M_\varepsilon)$,

$$\|\tilde{u}\|_{L^s(M)} \le C\|u\|_{s,\varepsilon} \qquad (1 \le s \le \infty) \tag{2.10}$$

holds for any $u \in H^1(M_\varepsilon) \cap L^s(M_\varepsilon)$,

$$\|\tilde{u}\|_{H^1(M)} \le C\|u\|_{H^1(M_\varepsilon)} + \varepsilon^{((s-2)3/(2s))-1}\|u\|_{s,\varepsilon} \tag{2.11}$$

for any $u \in H^1(M_\varepsilon) \cap L^s(M_\varepsilon)$ *with* $2 \le s < \infty$,

$$\|\tilde{u}\|_{H^1(M)} \le C\|u\|_{H^1(M_\varepsilon)} + C\varepsilon^{(1/2)}\|u\|_{\infty,\varepsilon}$$

holds for any $u \in H^1(M_\varepsilon) \cap L^\infty(M_\varepsilon)$.

Proof. Without loss of generality, we may assume that $\tilde{w}=0$. We take an arbitrary $u\in H^1(M_\varepsilon)$ and put

$$\begin{aligned}\tilde{u}(x)&=u(x) \qquad x\in M_\varepsilon\\ &=u(\varepsilon^2x|x|^{-2})\eta_\varepsilon(x) \qquad x\in B_\varepsilon,\end{aligned}$$

where $\eta_\varepsilon\in C^\infty(\boldsymbol{R}^3)$ satisfies $0\le\eta_\varepsilon\le 1$, $\eta_\varepsilon=1$ on $\boldsymbol{R}^3\backslash\bar{B}_{\varepsilon/2}$, $\eta_\varepsilon=0$ on $B_{\varepsilon/4}$ and $|\nabla\eta_\varepsilon|\le 8\varepsilon^{-1}$. Notice that both $\eta_\varepsilon(\varepsilon^2x|x|^{-2})$ and $(\nabla\eta_\varepsilon)(\varepsilon^2x|x|^{-2})$ vanish on $\boldsymbol{R}^3\backslash B_{4\varepsilon}$. Then, by using the Kelvin transformation of co-ordinates, $y=\varepsilon^2x|x|^{-2}$, we have

$$\begin{aligned}\int_{B_\varepsilon}|\tilde{u}(x)|^s dx&\le\int_{\boldsymbol{R}^3\backslash\bar{B}_\varepsilon}|u(y)|^s\eta_\varepsilon(\varepsilon^2y|y|^{-2})^s(\varepsilon|y|^{-1})^6dy\\ &\le\int_{M_\varepsilon}|u(y)|^s dy \qquad (1\le s<\infty),\end{aligned}$$

and

$$\begin{aligned}\int_{B_\varepsilon}|\nabla u(x)|^2dx&\le C\int_{B_\varepsilon}|u(\varepsilon^2x|x|^{-2})|^2|(\nabla\eta_\varepsilon)(x)|^2dx\\ &\quad+C\int_{B_\varepsilon}(\varepsilon|x|^{-1})^4|(\nabla u)(\varepsilon^2x|x|^{-2})|^2\eta_\varepsilon(x)^2dx\\ &\le C\varepsilon^4\int_{M_\varepsilon}|u(y)|^2|y|^{-6}dy+C\int_{M_\varepsilon}|\nabla u|^2dy.\end{aligned}$$

By Hölder's inequality, we see that

$$\int_{M_\varepsilon}|u(y)|^2|y|^{-6}dy\le\begin{cases}C\varepsilon^{-(1+(2/s))3}\|u\|^2_{L^s(M_\varepsilon)} & (2\le s<\infty)\\ C\varepsilon^{-3}\|u\|^2_{L^\infty(M_\varepsilon)} & (s=\infty).\end{cases}$$

Thus, we get Lemma 2.3.

3. The Green function.

Let $G(x,y)$ be the Green function of the Laplacian in M under the Dirichlet condition on ∂M. We introduce the following kernel $p_\varepsilon(x,y)$.

$$p_\varepsilon(x,y)=G(x,y)+g(\varepsilon)G(x,\tilde{w})G(\tilde{w},y)+h(\varepsilon)\langle\nabla_wG(x,\tilde{w}),\nabla_wG(\tilde{w},y)\rangle,$$

where

$$\langle\nabla_wu(\tilde{w}),\nabla_wv(\tilde{w})\rangle=\sum_{n=1}^{3}\frac{\partial u}{\partial w_i}\frac{\partial v}{\partial w_i}\Big|_{w=\tilde{w}},$$

when $w=(w_1,w_2,w_3)$ is an orthonomal frame of $\boldsymbol{R}^3$.
Here we put

$$g(\varepsilon)=-(\gamma+(4\pi\varepsilon)^{-1}+k^{-1}(4\pi)^{-1}\varepsilon^{-2})^{-1},$$

where

$$\gamma=\lim_{x\to\tilde{w}}(G(x,\tilde{w})-(4\pi)^{-1}|x-\tilde{w}|^{-1})$$

and

$$h(\varepsilon)=k^{-1}/((4\pi)^{-1}\varepsilon^{-2}+k^{-1}(2\pi)^{-1}\varepsilon^{-3}).$$

We put

$$G(x,y)-(4\pi)^{-1}|x-y|^{-1}=S(x,y).$$

Then, $S(x,y)\in C^\infty(M\times M)$. We have the following:

$$\begin{aligned}&\langle\nabla_w G(x,\tilde{w}),\nabla_w G(\tilde{w},y)\rangle\\&=(4\pi)^{-1}\varepsilon^{-2}\frac{\partial}{\partial w_1}G(\tilde{w},y)+\langle\nabla_w S(x,\tilde{w}),\nabla_w G(\tilde{w},y)\rangle\\&\frac{\partial}{\partial x_1}\langle\nabla_w G(x,\tilde{w}),\nabla_w G(\tilde{w},y)\rangle\\&=-(2\pi)^{-1}\varepsilon^{-3}\frac{\partial}{\partial w_1}G(\tilde{w},y)+\frac{\partial}{\partial x_1}\langle\nabla_w S(x,\tilde{w}),\nabla_w G(\tilde{w},y)\rangle\end{aligned}$$

for $x=(\varepsilon,0,0)$.
Then,

$$\begin{aligned}&kp_\varepsilon(x,y)+\frac{\partial}{\partial v_x}p_\varepsilon(x,y)\\&=kG(x,y)+kg(\varepsilon)G(x,\tilde{w})G(\tilde{w},y)+kh(\varepsilon)\langle\nabla_w G(x,\tilde{w}),\nabla_w G(\tilde{w},y)\rangle\\&\quad-\frac{\partial}{\partial x_1}G(x,y)-g(\varepsilon)\left(\frac{\partial}{\partial x_1}G(x,\tilde{w})\right)G(\tilde{w},y)\\&\quad-h(\varepsilon)\frac{\partial}{\partial x_1}\langle\nabla_w G(x,\tilde{w}),\nabla_w G(\tilde{w},y)\rangle\\&=kG(x,y)-kG(\tilde{w},y)-g(\varepsilon)\frac{\partial}{\partial x_1}S(x,\tilde{w})G(\tilde{w},y)\end{aligned}$$

$$-\frac{\partial}{\partial x_1}G(x,y)+\frac{\partial}{\partial w_1}G(\tilde{w},y)+kh(\varepsilon)\langle\nabla_w S(x,\tilde{w}),\nabla_w G(\tilde{w},y)\rangle$$

$$-h(\varepsilon)\frac{\partial}{\partial x_1}\langle\nabla_w S(x,\tilde{w}),\nabla_w G(\tilde{w},y)\rangle+O(\varepsilon)g(\varepsilon)G(\tilde{w},y)$$

for $x=(\varepsilon,0,0)$.

Let $G_\varepsilon(x,y)$ denote the Green function of Δ in M_ε satisfying $G_\varepsilon(x,y)=0$ on ∂M and $kG_\varepsilon(x,y)+\frac{\partial}{\partial\nu_x}G_\varepsilon(x,y)=0$ on ∂B_ε.

Let $\boldsymbol{G}_\varepsilon$ and $\boldsymbol{P}_\varepsilon$ be the operator defined by

$$\boldsymbol{G}_\varepsilon f(x)=\int_{M_\varepsilon}G_\varepsilon(x,y)f(y)dy$$

$$\boldsymbol{P}_\varepsilon f(x)=\int_{M_\varepsilon}p_\varepsilon(x,y)f(y)dy.$$

We put $\boldsymbol{Q}_\varepsilon f(x)=\boldsymbol{P}_\varepsilon f(x)$. Then, it satisfies

$$\Delta\boldsymbol{Q}_\varepsilon f(x)=0 \qquad x\in M_\varepsilon$$

$$\boldsymbol{Q}_\varepsilon f(x)=0 \qquad x\in\partial M$$

$$k\boldsymbol{Q}_\varepsilon f(x)+\frac{\partial}{\partial\nu_x}\boldsymbol{Q}_\varepsilon f(x)=k\boldsymbol{P}_\varepsilon f(x)+\frac{\partial}{\partial\nu_x}\boldsymbol{P}_\varepsilon f(x) \qquad x\in\partial B_\varepsilon.$$

We know that

$$k\boldsymbol{P}_\varepsilon f(x)+\frac{\partial}{\partial\nu_x}\boldsymbol{P}_\varepsilon f(x) \tag{3.1}$$

$$=k((\boldsymbol{G}f)(x)-(\boldsymbol{G}f)(\tilde{w}))-\frac{\partial}{\partial x_1}\boldsymbol{G}f(x)+\frac{\partial}{\partial w_1}\boldsymbol{G}f(\tilde{w})$$

$$+O(1)g(\varepsilon)\boldsymbol{G}f(\tilde{w})+O(h(\varepsilon))\frac{\partial}{\partial x_1}\langle\nabla_w S(x,\tilde{w}),\nabla_w\boldsymbol{G}f(\tilde{w})\rangle$$

$$+O(h(\varepsilon))\langle\nabla_w S(x,\tilde{w}),\nabla_w\boldsymbol{G}f(\tilde{w})\rangle.$$

4. Proof of Theorem 1.

We have the following decomposition of u_ε.

$$u_\varepsilon=-\lambda(\varepsilon)\boldsymbol{Q}_\varepsilon u_\varepsilon^p+\lambda(\varepsilon)\boldsymbol{P}_\varepsilon u_\varepsilon^p. \tag{4.1}$$

Lemma 4.1. *Fix* $p\in(1,2)$. *Then,*

$$\sup_{x\in M_\varepsilon}|P_\varepsilon u_\varepsilon^p|\leq C(1+\varepsilon\|u_\varepsilon\|_{\infty,\varepsilon}^{\tau}) \tag{4.2}$$

holds for any $\tau<1$ *which is close enough to* 1.

Proof. Recall that

$$\begin{aligned}P_\varepsilon u_\varepsilon^p(x)=G\hat{u}_\varepsilon^p(x)+g(\varepsilon)G(x,\tilde{w})G\hat{u}_\varepsilon^p(\tilde{w})\\ +h(\varepsilon)\langle\nabla_w G(x,\tilde{w}),\nabla_w G\hat{u}_\varepsilon^p(\tilde{w})\rangle.\end{aligned}$$

Here $\hat{u}_\varepsilon^p(x)$ is the extension of u_ε^p to M which is zero outside M_ε.

We have $|Gu_\varepsilon^p(\tilde{w})|\leq C\|u_\varepsilon^p\|_{t,\varepsilon}\leq C'\|u_\varepsilon\|_{tp,\varepsilon}^p$ for any $t>(3/2)$. Notice that we can take $t>(3/2)$ as close as 3/2 so that $tp<p+1$ for $p\in(1,2)$. Therefore, $|G\hat{u}_\varepsilon^p(w)|$ is bounded.

We have

$$\begin{aligned}\varepsilon|\nabla_w Gu_\varepsilon^p(\tilde{w})|&\leq C\varepsilon\|u_\varepsilon\|_{\infty,\varepsilon}^{\tau}\left(\int_{M_\varepsilon}u_\varepsilon^{p-\tau}(y)|y-\tilde{w}|^{-2}dy\right)\\ &\leq C'\varepsilon\|u_\varepsilon\|_{\infty,\varepsilon}^{\tau}\|u_\varepsilon\|_{p+1,\varepsilon}^{(p-\tau)/(p+1)}\left(\int_{M_\varepsilon}|y-w|^{-2r}dy\right)^{1/r}\end{aligned}$$

where $r=(p+1)/(1+\tau)$. Thus, it does not exceed $\varepsilon\|u_\varepsilon\|_{\infty,\varepsilon}^{\tau}$, since $2r<3$, if we take $p\in(1,2)$ and $\tau<1$ as close as 1.

Lemma 4.2. *Fix* $p\in(1,2)$. *Then,*

$$|Q_\varepsilon u_\varepsilon^p|\leq C\varepsilon^{1-p+(\mu/q)}\|u_\varepsilon\|_{\infty,\varepsilon}^{(\mu/q)} \tag{4.3}$$

holds for $q>6$, $q>\mu$.

Proof. By Lemma 2.1 we see that

$$|Q_\varepsilon f(x)|\leq C\varepsilon^2 r^{-1}\|G\hat{f}\|_{C^{1+(\sigma/2)}(M)} \tag{4.4}$$

for $\sigma>1$ by using Lemma 2.1 and (3.1). Here $\hat{f}$ is the extension of f to M which is zero outside M_ε.

We have

$$\|G\hat{f}\|_{C^{1+(\sigma/2)}(M)}\leq C\|f\|_{q,\varepsilon}$$

for $q>3/(1-(\sigma/2))$. We take $\sigma>1$ as close as 1.

Then, we have

(4.5) $$|Q_\varepsilon f(x)| \leq C\varepsilon \|f\|_{q,\varepsilon}$$

for $q>6$, $x\in M_\varepsilon$.

We have

$$\varepsilon\|u_\varepsilon^p\|_{q,\varepsilon} \leq \varepsilon\|u_\varepsilon\|_{\infty,\varepsilon}^{(\mu/q)}\left(\int_{M_\varepsilon}|u_\varepsilon|^{pq-\mu}dx\right)^{1/q}.$$

We take $\mu<q$ as close as q and $q>6$ as close as 6. Then, $pq-\mu<6$ for $p\in(1,2)$.

On the other hand we have

$$\|u_\varepsilon\|_{pq-\mu,\varepsilon} \leq \|\tilde{u}_\varepsilon\|_{L^{pq-\mu}(M)} \leq C\|\tilde{u}_\varepsilon\|_{H^1(M)}$$

by the Sobolev embedding theorem using $pq-\mu<6$. Here $\tilde{u}_\varepsilon$ is an extension of u_ε in Lemma 2.3 which is different from $\hat{u}_\varepsilon$. It should be noted, since $\lambda(\varepsilon)$ is difined as an infimum of a functional so that it is easy to see that $\lim\sup_{\varepsilon\to 0}\lambda(\varepsilon)<\infty$. Therefore,

$$\|\tilde{u}_\varepsilon\|_{H^1(M)} \leq \|u_\varepsilon\|_{H^1(M_\varepsilon)} + C\varepsilon^{-1}\|u_\varepsilon\|_{2,\varepsilon}$$

by taking $s=2$ in Lemma 2.3. Since u_ε is a minimizer of (1.1) and

$$\|u_\varepsilon\|_{2,\varepsilon} \leq C$$

we see that

$$\|\tilde{u}_\varepsilon\|_{H^1(M)} \leq C\varepsilon^{-1}.$$

Therefore,

$$\|u_\varepsilon\|_{pq-\mu,\varepsilon} \leq C''\varepsilon^{-1}.$$

Therefore, we get the desired redult.

Proof of Theorem 1. We put $\xi=\|u_\varepsilon\|_{\infty,\varepsilon}$.
By Lemma 4.1 and 4.2 we get

$$\xi \leq C'(1+\varepsilon\xi^\tau+\varepsilon^{1-p+(\mu/q)}\xi^{\mu/q}).$$

Since $p\in(1,2)$, we can take $\mu<q$ as close as q so that $1-p+(\mu/q)>0$. We take $\tau<1$. Therefore, $\xi\leq C''$. We get the desired result.

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