

Title	On non-commutative quadratic extensions of a commutative ring
Author(s)	Kanzaki, Teruo
Citation	Osaka Journal of Mathematics. 10(3) p.597-p.605
Issue Date	1973
oaire:version	VoR
URL	https://doi.org/10.18910/10259
rights	
Note	

Osaka University Knowledge Archive : OUKA

<https://ir.library.osaka-u.ac.jp/>

Osaka University

ON NON-COMMUTATIVE QUADRATIC EXTENSIONS OF A COMMUTATIVE RING

TERUO KANZAKI

(Received January 30, 1973)

Let A be a ring with the identity element and B an extension ring of A with the common identity element. B is called a quadratic extension of A , if the residue module B/A is an invertible A - A -bimodule, i.e. $B/A \otimes_A \text{Hom}_A(B/A, A) \approx \text{Hom}_A(B/A, A) \otimes_A B/A \approx A$. In [4], [5] and [9], one has studied about commutative quadratic extensions. We like to extend these results to non-commutative quadratic extensions. But, in general, it is difficult. In this note, we shall study non-commutative quadratic extensions of a commutative ring. Let A be a commutative ring with the identity element. Let D be an A -algebra with the identity element such that D is a quadratic extension of a commutative subring B and B is a separable quadratic extension of A . In the section 1, we shall show that if A has no idempotents other than 0 and 1, then such an A -algebra D is either a commutative ring or a central A -algebra having B as a maximal commutative subring. We shall say that A -algebra D is a *quaternion A -algebra* with a maximal commutative and separable subalgebra B , if D is an A -algebra mentioned above and is a central separable A -algebra. In the section 2, we shall show that a quaternion A -algebra with a maximal commutative and separable subalgebra B is characterized by the separable quadratic extension B of A and a non-degenerate hermitian B -module (V, Φ) of rank one. Let (V, q) be a non-degenerate quadratic A -module such that V is a finitely generated projective A -module with a constant rank two. Then the Clifford algebra $C(V, q) = C_0(V, q) \oplus C_1(V, q)$ is a quaternion A -algebra with a maximal commutative and separable subalgebra $C_0(V, q)$. And, the quadratic A -module (V, q) is hyperbolic if and only if $[C_0(V, q)] = 1$ in $Q_S(A)$, where $Q_S(A)$ is the group of separable quadratic extensions of A (cf. [4], [5] and [9]).

1. Let A be a commutative ring with the identity element, and B a commutative and separable quadratic extension of A . Then B is characterized by an invertible A -module U , an A -linear map $f: U \rightarrow A$ and a quadratic form $q: U \rightarrow A$, as $B = A \oplus U$ and $x^2 = f(x)x + q(x)$ for $x \in U$ (cf. [4]). Let τ be an A -algebra automorphism of B defined by $\tau(a+x) = a+f(x)-x$ for $a \in A, x \in U$. Then we have $B^\tau = A$. Because, if x is in U and $\tau(x) = x$, then $f(x) = 0$, and $2x = 0$. From the fact that a bilinear form $D_{f,q}: U \times U \rightarrow A; (x, y) \mapsto f(x)f(y) + 2B_q(x, y)$

is non-degenerate (Theorem 1 in [4]), x is 0, consequently we have $B^r=A$. Therefore, B is a Galois extension of A with the Galois group $G(B/A)=\{I, \tau\}$. If A has no idempotents other than 0 and 1, then $G(B/A)=\{I, \tau\}$ is the group of all A -algebra automorphisms of B .

Let D be an A -algebra which is a quadratic extension of B . Then we have

(1.1) **Theorem.** *Let D and B be as above. If A has no idempotents other than 0 and 1, then D is either a commutative ring or a central A -algebra having the subalgebra B as a maximal commutative subring.*

Proof. Since the residue B - B -bimodule D/B is invertible, there exists an A -algebra automorphism σ of B such that $xb \equiv \sigma(b)x \pmod{B}$ for all $x \in D$ and $b \in B$. Then σ is either I or τ . If $\sigma=I$, then for each x in D , $d_x(b)=xb-bx$ is in B for all $b \in B$. The map $d_x: B \rightarrow B$ becomes a derivation of B over A . B is separable over A , hence every derivation of B over A is 0, and so $d_x=0$. Therefore, D is a B -algebra. Since D is a quadratic extension of B , D is a commutative ring. If $\sigma=\tau$, then for each $x \in D$, $d_x(b)=xb-\tau(b)x$ is in B for all $b \in B$, and the map $d_x: B \rightarrow B$ is a (τ, I) -derivation of B over A , i.e. $d_x(b_1 b_2) = d_x(b_1)b_2 + \tau(b_1)d_x(b_2)$ for b_1, b_2 in B , (cf. p. 170 in [6]). Since D/B is a projective left B -module, the exact sequence $0 \rightarrow B \rightarrow D \rightarrow D/B \rightarrow 0$ is split, i.e. there exists an invertible left B -submodule V of D such that $D=B \oplus V$. We consider the commutator ring $V_D(B) = \{x \in D; xb=bx \text{ for all } b \in B\}$, then $V_D(B) \supset B$. Now, we shall show $V_D(B) \cap V = 0$. If x is in $V_D(B) \cap V$, we have $d_x(b)=xb-\tau(b)x = bx-\tau(b)x \in B \cap V = 0$, and so $\tau(b)x=bx$ for all $b \in B$. Since $B \supset A$ is a Galois extension with the Galois group $G(B/A) = \{I, \tau\}$, there exist b_1, b_2, \dots, b_r and c_1, c_2, \dots, c_r in B such that $\sum_i c_i b_i = 1$ and $\sum_i c_i \tau(b_i) = 0$. Then $x = \sum c_i b_i x = \sum c_i \tau(b_i) x = 0$. Consequently, we get $V_D(B) = B$, i.e. B is a maximal commutative subring of D . Finally, we shall show that the center of D is A . Let c be an element of the center. c is contained in $B = V_D(B)$. For any $x \in V$, $cx = xc = d_x(c) + \tau(c)x$ in $B \oplus V = D$. Therefore, we have $cx = \tau(c)x$. Since V is faithful over B , $c = \tau(c)$, and c is contained in $B^{G(B/A)} = A$. Therefore, A is the center of D .

2. Let B be a commutative and separable quadratic extension of A , and D an A -algebra such that D is a quadratic extension of B . If D is central separable over A , then B is a maximal commutative subring of D . Because, when we regard D as $D \otimes_A B$ -left module by $d \otimes b \cdot x = dx$ for $d \otimes b \in D \otimes_A B$ and $x \in D$, D is a finitely generated projective $D \otimes_A B$ -module and $\text{Hom}_{D \otimes B}(D, D) \approx V_D(B) \supset B$. For every maximal ideal \mathfrak{m} of A , $\text{Hom}_{D \otimes B}(D, D) \otimes_A A_{\mathfrak{m}} \approx \text{Hom}_{D_{\mathfrak{m}} \otimes B_{\mathfrak{m}}}(D_{\mathfrak{m}}, D_{\mathfrak{m}}) \approx V_{D_{\mathfrak{m}}}(B_{\mathfrak{m}}) \supset B_{\mathfrak{m}}$. But, $A_{\mathfrak{m}}$ has no idempotents without 0 and 1, $V_{D_{\mathfrak{m}}}(B_{\mathfrak{m}}) = B_{\mathfrak{m}}$. Therefore, $V_D(B) = B$. B is a maximal commutative subring of D . We shall say that D is a *quaternion A -algebra* with a maximal commutative and separable subalgebra B , if D is an A -algebra defined above and is central separable over A . If A has no idempotents other than 0 and 1, and if D is non-commutative

and separable over A , then by (1.1), D is a quaternion A -algebra with a maximal commutative and separable subalgebra B .

(2.1) **Proposition.** *Let D be a quaternion A -algebra with a maximal commutative and separable subalgebra B . Then D is a generalized crossed product of B and $G(B/A)$ (defined in [3]). Therefore, there exists an invertible B - B -submodule of D such that $D=B\oplus V$ and $B=V\cdot V\approx V\otimes_B V$.*

Proof. D is a central separable A -algebra and contains a maximal commutative subalgebra B which is a Galois extension of A with the Galois group $G(B/A)=\{I, \tau\}$. By Proposition 3 in [3], D is a generalized crossed product of B and $G(B/A)$, and so D is written as $D=J_I\oplus J_\tau$, where $J_I=B$ and $J_\tau=\{x\in D; \tau(b)x=xb \text{ for all } b\in B\}$ are invertible B - B -bimodules. Furthermore, the map $f_{\tau,\tau}: J_\tau\otimes_B J_\tau\rightarrow J_I; x\otimes y\mapsto xy$ is a B - B -isomorphism. Put $V=J_\tau$, V is the required B - B -bimodule.

DEFINITION. Let $B\supset A$ be a commutative and separable quadratic extension which is a Galois extension with Galois group $G(B/A)=\{I, \tau\}$. For a left B -module M with an A -bilinear form $\Phi: M\times M\rightarrow B$, we shall call (M, Φ) a *hermitian B -module* if it satisfies

- 1) $\Phi(bx, y)=b\Phi(x, y)$,
- 2) $\Phi(x, y)=\tau(\Phi(y, x))$ for every $b\in B$ and $x, y\in M$.

We shall say that a hermitian B -module (M, Φ) is non-degenerate, if the A -linear map $M\rightarrow \text{Hom}_B(M, B); x\mapsto \Phi(-, x)$ is an isomorphism. Let (M_1, Φ_1) and (M_2, Φ_2) be hermitian B -modules. The product $(M_1, \Phi_1)\otimes(M_2, \Phi_2)$ is defined by $(M_1\otimes_B M_2, \Phi_1\otimes\Phi_2)$ where $\Phi_1\otimes\Phi_2: (M_1\otimes_B M_2)\times(M_1\otimes_B M_2)\rightarrow B; (x_1\otimes x_2, y_1\otimes y_2)\mapsto \Phi_1(x_1, y_1)\cdot\Phi_2(x_2, y_2)$. We denote by (B, I) a hermitian B -module defined by $I(b, b')=b\cdot\tau(b')$ for $b, b'\in B$.

If M_1 and M_2 are finitely generated projective B -modules, and if (M_1, Φ_1) and (M_2, Φ_2) are non-degenerate hermitian B -modules, then the product $(M_1, \Phi_1)\otimes(M_2, \Phi_2)$ is also non-degenerate.

(2.2) **Theorem.** *Let D be a quaternion A -algebra with a maximal commutative and separable subalgebra B . Then there exists a non-degenerate hermitian B -module (V, Φ) with an invertible B -bimodule V such that $D=B\oplus V, xb=\tau(b)x$ for $b\in B, x\in V$ and $xy=\Phi(x, y)$ for $x, y\in V$. Conversely, if (V, Φ) is any non-degenerate hermitian B -module with an invertible B -left module V , then an A -algebra $D=B\oplus V$ which is defined by $(b+x)\cdot(b'+x')=bb'+\Phi(x, x')+bx'+\tau(b')x$ for $b, b'\in B$ and $x, x'\in V$, is a quaternion A -algebra with a maximal commutative and separable subalgebra B .*

Proof. Let D be a quaternion A -algebra with a maximal commutative and separable subalgebra B . By (2.1), there exists an invertible B - B -bimodule V

such that $D=B\oplus V$ and $V\cdot V=B$. We define an A -bilinear map $\Phi : V \times V \rightarrow B$ by $\Phi(x, y)=xy$ for $x, y \in V$. We shall show that (V, Φ) is a non-degenerate hermitian B -module. Put $\Psi(x, y)=\Phi(x, y)-\tau(\Phi(y, x))$ for x, y in V . For any maximal ideal \mathfrak{m} of A , the localization $B_{\mathfrak{m}}$ is a semilocal ring, therefore $V_{\mathfrak{m}}$ is a free $B_{\mathfrak{m}}$ -module of rank 1. Let $V_{\mathfrak{m}}=B_{\mathfrak{m}}v$, $\Psi_{\mathfrak{m}}=\Psi \otimes I_{\mathfrak{m}}$, $\Phi_{\mathfrak{m}}=\Phi \otimes I_{\mathfrak{m}}$ and $\tau_{\mathfrak{m}}=\tau \otimes I_{\mathfrak{m}}$. Then we have $\Psi_{\mathfrak{m}}(bv, b'v)v=\Phi_{\mathfrak{m}}(bv, b'v)v-\tau_{\mathfrak{m}}(\Phi_{\mathfrak{m}}(b'v, bv))v=(vb'v)v-v(b'vbv)=b\tau_{\mathfrak{m}}(b')v^3-\tau_{\mathfrak{m}}(b')bv^3=0$ in $D_{\mathfrak{m}}$. Therefore, we have $\Psi_{\mathfrak{m}}=0$ for any maximal ideal \mathfrak{m} of A , and so $\Psi=0$, i.e. $\Phi(x, y)=\tau(\Phi(y, x))$ for every x, y in V . Since $V \otimes_B V \rightarrow B; x \otimes y \mapsto xy$ is B - B -isomorphism from (2.1), (V, Φ) is non-degenerate. Conversely, let (V, Φ) be any non-degenerate hermitian B -module with an invertible left B -module V . We can make a B - B -bimodule V by $xb=\tau(b)x$ for $b \in B, x \in V$. Since (V, Φ) is non-degenerate, the map $f_{\tau, \tau} : V \otimes_B V \rightarrow B; x \otimes y \mapsto \Phi(x, y)$ is a B - B -isomorphism as B - B -bimodules. By [3], we can construct a generalized crossed product $\Delta(f, B, \Psi, G)$ of B and $G=G(B/A)=\{I, \tau\}$ provided $\Psi; \Psi(I)=B, \Psi(\tau)=V$, and a factor set $f=\{I=f_{I, I}, f_{\tau, I}, f_{I, \tau}, f_{\tau, \tau}\}$, where $f_{I, \tau} : B \otimes_B V \rightarrow V; b \otimes x \mapsto bx, f_{\tau, I} : V \otimes_B B \rightarrow V; x \otimes b \mapsto xb$. To show the commutativity of the diagrams of the factor set, we need only to show the following commutative diagram:

$$\begin{array}{ccc}
 V \otimes_B V \otimes_B V & \xrightarrow{I \otimes f_{\tau, \tau}} & V \otimes_B B \\
 \downarrow f_{\tau, \tau} \otimes I & & \downarrow f_{\tau, I} \\
 B \otimes_B V & \xrightarrow{f_{I, \tau}} & V
 \end{array}$$

we shall show it by taking the localization with respect to a maximal ideal \mathfrak{m} of A . Then we have $f_{\tau, I} \circ (I \otimes f_{\tau, \tau})(av \otimes bv \otimes cv)=f_{\tau, I}(av \otimes f_{\tau, \tau}(bv \otimes cv))=av \cdot \Phi(bv, cv)=a\tau(b)c\tau(\Phi(v, v))v=a\tau(b)c\Phi(v, v)v=\Phi(av, bv)cv=f_{\tau, \tau}(av \otimes bv)cv=f_{I, \tau} \circ (f_{\tau, \tau} \otimes I)(av \otimes bv \otimes cv)$ for all av, bv, cv in $V_{\mathfrak{m}}=A_{\mathfrak{m}}v$. Therefore, the diagram is commutative. Thus, $D=B\oplus V=\Delta(f, B, \Psi, G)$ is an A -algebra defined the multiplication by $(b+x) \cdot (b'+x')=bb'+\Phi(x, x')+bx'+\tau(b')x$ for $b+x, b'+x'$ in $B\oplus V=D$. By Proposition 3 in [3], D is an Azumaya A -algebra, accordingly $D=B\oplus V$ is a quaternion A -algebra with a maximal commutative and separable subalgebra B .

We shall call (V, Φ) a non-degenerate hermitian B -module of rank 1 if (V, Φ) is a non-degenerate hermitian B -module and V is an invertible left B -module. For a non-degenerate hermitian B -module of rank 1, we denote by $D(B, V, \Phi)$ the quaternion A -algebra D with a maximal commutative and separable subalgebra B defined by (V, Φ) in (2.2)

(2.3) **Corollary.** *Let (V, Φ) and (V', Φ') be non-degenerate hermitian B -modules of rank 1. Then (V, Φ) and (V', Φ') are isometric if and only if there exists an A -algebra isomorphism of $D(B, V, \Phi)$ to $D(B, V', \Phi')$ which is identity map on B .*

Let (P, q) be a quadratic A -module with a quadratic form $q : P \rightarrow A$. We shall call that (P, q) is a non-degenerate quadratic A -module of rank n , if P is a finitely generated projective A -module with constant rank n , i.e. $[P_m : A_m] = n$ for every maximal ideal m of A , and $q : P \rightarrow A$ is non-degenerate.

(2.4) **Proposition.** *Let (V, q) be a non-degenerate A -module of rank 2. Then the Clifford algebra $C(V, q) = C_0(V, q) \oplus C_1(V, q)$ is a quaternion A -algebra with a maximal commutative and separable subalgebra $C_0(V, q)$, where $C_0(V, q)$ (resp. $C_1(V, q)$) is the subalgebra of $C(V, q)$ of homogeneous elements of degree 0 (resp. degree 1.)*

Proof. $C(V, q)$ is an Azumaya A -algebra, and $C_0(V, q)$ is a commutative and separable quadratic extension of A (Lemma 6 in [7]). Therefore, $V \approx C_1(V, q)$ is a finitely generated projective $C_0(V, q)$ -module. We shall show that $C_1(V, q)$ is an invertible $C_0(V, q)$ -module. It suffices to show that for the case where A is a local ring. Assume that A is a local ring. Then, $V = Au \oplus Av$ is a free A -module of rank 2. Since (V, q) is non-degenerate, we may assume that $q(u)$ is invertible in A . Then we have $C_0(V, q) = A \oplus Au$ and $C_1(V, q) \approx V = Au \oplus Av = C_1(V, q)u$. Since u is invertible in $C(V, q)$, $C_1(V, q)$ is a free $C_0(V, q)$ -module of rank 1.

(2.5) **Lemma.** *Let Λ be a Galois extension of a ring Γ with a Galois group G , and P a Λ -module. Then we have $\text{Hom}_\Gamma(P, \Gamma) = \text{Tr} \circ \text{Hom}_\Lambda(P, \Lambda)$, where $\text{Tr}(x) = \sum_{\sigma \in G} \sigma(x)$ for $x \in \Lambda$.*

Proof. Since $\Lambda \supset \Gamma$ is a Galois extension with a Galois group G , there exist x_1, x_2, \dots, x_n and y_1, y_2, \dots, y_n in Λ such that $\sum_i \sigma(x_i)y_i = \begin{cases} 1, & \sigma = I \\ 0, & \sigma \neq I \end{cases}$. Then, for f in $\text{Hom}_\Gamma(P, \Gamma)$, $F(-) = \sum_i x_i f(y_i -)$ is contained in $\text{Hom}_\Lambda(P, \Lambda)$, and $\text{Tr} \circ F(z) = \sum_i \text{Tr}(x_i f(y_i z)) = f(\sum_i \text{Tr}(x_i)y_i z) = f(z)$ for all $z \in P$. Therefore, f is in $\text{Tr} \circ \text{Hom}_\Lambda(P, \Lambda)$. The converse is clear.

(2.6) **Lemma.** *Let (P, Φ) be a non-degenerate hermitian B -dmodule. Then $(P, \text{Tr} \circ \Phi)$ is a non-degenerate bilinear A -module.*

Proof. $\text{Tr} \circ \Phi : P \times P \rightarrow A; (x, y) \mapsto \text{Tr}(\Phi(x, y)) = \Phi(x, y) + \tau(\Phi(x, y))$ is an A -bilinear form. We show that $P \rightarrow \text{Hom}_A(P, A); x \mapsto \text{Tr}(\Phi(-, x))$ is an A -isomorphism. If x is in P such that $\text{Tr}(\Phi(-, x)) = 0$, $\Phi(P, x)$ is an ideal of B and $\text{Tr}(\Phi(P, x)) = 0$. Let b_1, b_2, \dots, b_n and b'_1, b'_2, \dots, b'_n be elements in B such that $\sum_i b_i b'_i = 1$ and $\sum_i \tau(b_i)b'_i = 0$. Then, we have $b = \sum_i \text{Tr}(bb_i)b'_i = 0$ for every b in $\Phi(P, x)$, hence $\Phi(P, x) = 0$. Therefore, $x = 0$. From Lemma (2.5), $(P, \text{Tr} \circ \Phi)$ is non-degenerate.

(2.7) **Theorem.** *Let $D = D(B, V, \Phi)$ be quaternion A -algebra with a maximal*

commutative and separable subalgebra B . Then there exists an involution $\sigma: D \rightarrow D$ which is defined by $\sigma(b+v) = \tau(b) - v$ for $b \in B, v \in V$. We put $N(x) = x \cdot \sigma(x)$ and $T(x) = x + \sigma(x)$ for x in D . Then $N: D \rightarrow A$ is a quadratic form, (D, N) is a non-degenerate quadratic A -module of rank 4, and $D = B \perp V$. $T: D \rightarrow A$ is an A -linear map and $B_N(x, y) = T(x \cdot \sigma(y))$ for $x, y \in D$.

Proof. Let $x = b + v$ and $x' = b' + v'$ be elements in $D = B \oplus V$. Then we have $\sigma(xx') = \sigma(bb' + \Phi(v, v') + bv' + \tau(b')v) = \tau(bb' + \Phi(v, v')) - (bv' + \tau(b')v) = \tau(b) \cdot \tau(b') + \Phi(v, v') - v'\tau(b) - \tau(b')v = \sigma(x') \cdot \sigma(x)$, and $\sigma^2(x) = x$. Therefore, σ is an involution. Furthermore, $N(b+v) = b\sigma(b) - \Phi(v, v)$ and $T(b+v) = b + \tau(b)$ are contained in $B^c = A$, hence $N: D \rightarrow A$ is a quadratic form, and the bilinear form is $B_N(x, x') = N(x+x') - N(x) - N(x') = x\sigma(x') + x'\sigma(x) = T(x\sigma(x))$ for $x, x' \in D$. Therefore, we have $D = B \perp V$. To prove that (D, N) is non-degenerate, it suffices to show that $(B, N|B)$ and $(V, N|V)$ are non-degenerate. From Lemma (2.6), $\text{Tr} \circ \Phi$ and $\text{Tr} \circ I$ are non-degenerate, and $B_N(b, b') = T(b\tau(b')) = \text{Tr}(b\tau(b')) = \text{Tr} \circ I(b, b')$ for $b, b' \in B$ and $B_N(v, v') = T(v(-v')) = T(-\Phi(v, v')) = -\text{Tr} \circ \Phi(v, v')$ for $v, v' \in V$, hence $(B, N|B)$ and $(V, N|V)$ are non-degenerate.

In Theorem (2.7), we put $Q = -N|V$. Then (V, Q) is a non-degenerate quadratic A -module of rank 2.

(2.8) **Theorem.** *Let $D(B, V, \Phi)$ be a quaternion A -algebra with a maximal commutative and separable subalgebra B , and $N: D \rightarrow A$ and $Q = -N|V$ as defined before. Then $D(B, V, \Phi)$ is isomorphic to the Clifford algebra $C(V, Q)$ of the quadratic module (V, Q) as A -algebras.*

Proof. Since $Q(x)$ is equal to $x^2 = N(x)$ in $D(B, V, \Phi)$ for every $x \in V$, the inclusion map $V \rightarrow D(B, V, \Phi) = B \oplus V$ can be extended to an A -algebra homomorphism $\rho: C(V, Q) \rightarrow D(B, V, \Phi)$. From the fact that $C(V, Q)$ and $D(B, V, \Phi)$ are Azumaya algebras over A and are generated by V , we obtain that ρ is an A -isomorphism.

(2.9) **Lemma.** *Let V be any invertible B -module. Then for any f in $\text{Hom}_B(V, B)$ and x, y in V , we have $f(x)y = f(y)x$.*

Proof. Put $\Psi(x, y) = f(y)x - f(x)y$ for every $x, y \in V$, then $\Psi: V \times V \rightarrow V$ is a B -bilinear form. By taking the localization of V with respect to a maximal ideal \mathfrak{m} of A , we get easily $\Psi_{\mathfrak{m}} = 0$. Therefore, $\Psi = 0$.

(2.10) **Proposition.** *Let (V, Φ) be a non-degenerate hermitian B -module of rank 1. Then, the quaternion A -algebra $D(B, V \otimes_B V, \Phi \otimes \Phi)$ which is determined by $(V, \Phi) \otimes (V, \Phi) = (V \otimes_B V, \Phi \otimes \Phi)$, is A -algebra isomorphic to $\text{Hom}_A(V, V)$, and this isomorphism preserves the structure of B -modules.*

Proof. We can define a map $\theta: D(B, V \otimes_B V, \Phi \otimes \Phi) = B \oplus V \otimes_B V \rightarrow \text{Hom}_A(V, V)$ as follows: $\theta(b)(x) = bx$ for $b \in B, x \in V$, and $\theta(u \otimes v)(x) = \Phi(u, x)v$ for $u \otimes v \in V \otimes_B V, x \in V$. Then θ is an A -algebra homomorphism. Because, for $b \in B, u \otimes v \in V \otimes_B V$ and $x \in V$, we have $\theta(bu \otimes v)(x) = \Phi(bu, x)v = b\Phi(u, x)v = \theta(b) \circ \theta(u \otimes v)(x)$ and $\theta(u \otimes vb)(x) = \Phi(u, x)vb = \tau(b)\Phi(u, x)v = \Phi(u, bx)v = \theta(u \otimes v) \circ \theta(b)(x)$. And, for $u \otimes v, u' \otimes v' \in V \otimes_B V$ and $x \in V$, $\theta(u \otimes v) \circ \theta(u' \otimes v')(x) = \theta(u \otimes v)(\Phi(u', x')v') = \Phi(u, \Phi(u', x')v')v = \Phi(u, v')\Phi(x, u')v$. On the other hand, $\Phi(-, v')$ and $\Phi(-, u')$ are in $\text{Hom}_B(V, B)$, by Lemma (2.9) we get $\Phi(x, u')\Phi(u, v')v = \Phi(x, u')\Phi(v, v')u = \Phi(u, u')\Phi(v, v')x = \theta(\Phi(u, u')\Phi(v, v'))(x) = \theta((u \otimes v)(u' \otimes v'))(x)$. Thus, θ is an A -algebra homomorphism. Now we check that θ is an epimorphism. From Lemma (2.5), we have $\text{Hom}_A(V, V) \approx \text{Hom}_A(V, A) \otimes_A V \approx \text{Tr} \circ \text{Hom}_B(V, B) \otimes_A V \approx (\text{Tr} \circ \Phi(-, V)) \otimes_A V$. Therefore, any element f in $\text{Hom}_A(V, V)$ is written as $f = \sum_i \text{Tr} \circ \Phi(-, u_i)v_i = \sum_i (\Phi(-, u_i)v + \Phi(u_i, -)v_i)$ for some $u_i, v_i \in V$, and by Lemma (2.9), $f(x) = \sum_i \Phi(x, u_i)v_i + \sum_i \Phi(u_i, x)v_i = \sum_i \Phi(v_i, u_i)x + \theta(\sum_i u_i \otimes v_i)(x)$ for $x \in V$. Thus, we get $f = \theta(\sum_i \Phi(v_i, u_i) + \sum u_i \otimes v_i)$. Since $D(B, V \otimes_B V, \Phi \otimes \Phi)$ and $\text{Hom}_A(V, V)$ are Azumaya A -algebras, θ is an A -algebra isomorphism.

(2.11) **Corollary.** $D(B, B, I) \approx \text{Hom}_A(B, B)$ as A -algebras.

(2.12) **Corollary.** For any non-degenerate hermitian B -modules of rank 1 (V, Φ) and (V, Φ') , $(V \otimes_B V, \Phi \otimes \Phi)$ and $(V \otimes_B V, \Phi' \otimes \Phi')$ are isometric.

(2.13) **Theorem.** Let $D(B, V, \Phi)$ be a quaternion A -algebra with a maximal commutative and separable subalgebra B , and (V, Q) a non-degenerate quadratic A -module of rank 2 defined by $D(B, V, \Phi)$ in (2.8). Then, (V, Q) is hyperbolic if and only if $[B] = 1$ in $Q_S(A)$ (cf. [4]).

Proof. In (2.8), we obtained $D(B, V, \Phi) = C(V, Q) = B \oplus V, C_0(V, Q) = B$ and $C_1(V, Q) = V$. We assume that (V, Q) is hyperbolic. Then we may assume that $V = P \oplus P^*$ for some invertible A -module P and $P^* = \text{Hom}_A(P, A)$, and $Q(x+f) = f(x)$ for $x \in P, f \in P^*$. Since $P \cdot P = P^* \cdot P^* = 0$ in $C(V, Q)$, we get $C_0(V, Q) = A \oplus P \cdot P^* \approx A \oplus P \otimes_A P^*$. For any $\sum_i x_i f_i$ in $P \cdot P^*$, we have $(\sum_i x_i f_i)^2 = \sum_{i,j} x_i f_i x_j f_j = \sum_{i,j} x_i (f_i(x_j) - x_j f_i) f_j = \sum_{i,j} f_i(x_j) x_i f_j = \sum_{i,j} f_i(x_i) x_j f_j = (\sum_i f_i(x_i)) (\sum_i x_i f_i)$ using Lemma (2.9). We consider an A -isomorphism $\mu: P \cdot P^* (\approx P \otimes_B P^*) \rightarrow A$ defined by $\mu(\sum_i x_i f_i) = \sum_i f_i(x_i)$ for $\sum_i x_i f_i$ in $P \cdot P^*$. Then we have $(\sum_i x_i f_i)^2 = \mu(\sum_i x_i f_i) \sum_i x_i f_i$ for every $\sum_i x_i f_i$ in $P \cdot P^*$, hence $B = C_0(V, Q) \approx (P \otimes_A P^*, \mu, 0) \approx (A, 1, 0)$ as A -algebras (cf. [4]). Accordingly, $[B] = 1$ in $Q_S(A)$. Conversely, we assume $[B] = 1$ in $Q_S(A)$. Then the quadratic extension B of A has idempotents e_1 and e_2 such that $1 = e_1 + e_2, e_1 e_2 = 0$ and $B = Ae_1 \oplus Ae_2$. Furthermore, A -module V is written as a direct sum of A -submodules $e_1 V$ and $e_2 V$. Since the Galois group $G = G(B/A) = \{I, \tau\}$ is permutations of $\{e_1, e_2\}$, we have $Q(e_i, x) = \Phi(e_i, x, e_i x) = e_i \tau(e_i) \Phi(x, x) = e_1 e_2 \Phi(x, x) = 0$ for every

$x \in V$. Therefore, $e_1 V$ is totally isotropic. We have $(e_1 V)^\perp = e_1 V$, because, for $e_1 y + e_2 z \in (e_2 V)^\perp$, $0 = B_Q(e_1 y + e_2 z, e_1 x) = \Phi(e_1 y + e_2 z, e_1 x) + \Phi(e_1 x, e_1 y + e_2 z) = \Phi(e_2 z, e_1 x) + \Phi(e_1 x, e_2 z) = e_2 \Phi(z, x) + e_1 \Phi(x, z)$ in $e_2 A \oplus e_1 A = B$, hence $e_1 \Phi(x, z) = \Phi(x, e_2 z) = 0$ for all z in V . Therefore, we get $e_2 z = 0$. Accordingly, (V, Q) is hyperbolic (cf. [2]).

(2.14) **Corollary.** *Let (P, q) be any non-degenerate quadratic A -module of rank 2. Then (P, q) is hyperbolic if and only if $[C_0(P, q)] = 1$ in $Q_S(A)$.*

(2.15) **Corollary.** *If B is a quadratic extension of A such that $[B] = 1$ in $Q_S(A)$, then every quaternion A -algebra $D(B, V, \Phi)$ with a maximal commutative and separable subalgebra B is split, i.e. $[D(B, V, \Phi)] = 1$ in the Brauer group $B(A)$.*

(2.16) **Corollary.** *If A is commutative ring such that $Q_S(A) = 1$, then every non-degenerate quadratic A -module of rank 2 is hyperbolic.*

(2.17) **EXAMPLE.** *If A is the integers Z or the gaussian integers $Z[i]$, then every non-degenerate quadratic A -module of rank 2 is hyperbolic (cf. [5], [7]).*

(2.18) **REMARK.** Let K be a field, and (V, q) and (V', q') non-degenerate quadratic K -modules of rank 2. Then, (V, q) and (V', q') are isometric if and only if $[C(V, q)] = [C(V', q')]$ in the Brauer group $B(K)$ and $[C_0(V, q)] = [C_0(V', q')]$ in $Q_S(A)$.

Proof. For a field of characteristic $\neq 2$, this is obtained from Theorem 58:4 in [8], and for a field of characteristic 2, is obtained from Theorem 3 in [1].

OSAKA CITY UNIVERSITY

References

- [1] C. Arf: *Untersuchungen über quadratische Formen in Körpern der Charakteristik 2*, J. Reine Angew. Math. **183** (1941), 148–167.
- [2] H. Bass: *Lectures on Topics in Algebraic K-theory*, Tata Institute of Fundamental Research, Bombay, 1967.
- [3] T. Kanzaki: *On generalized crossed product and Brauer group*, Osaka J. Math. **5** (1968), 175–188.
- [4] ———: *On the quadratic extensions and the extended Witt ring of a commutative ring*, Nagoya Math. J. **49** (1973), 127–141.
- [5] K. Kitamura: *On free quadratic extensions over a commutative ring*, Osaka J. Math. **10** (1973), 15–20.
- [6] N. Jacobson: *Structure of Rings*, Providence, R.I., 1964.
- [7] A. Micali and O.E. Villamayor: *Sur les algèbres de Clifford*. II, J. Reine Angew. Math. **242** (1970), 61–90.

- [8] O.T. O'Meara: *Introduction to Quadratic Forms*, Berlin-Göttingen-Heidelberg: Springer, 1963.
- [9] C. Small: *The group of quadratic extensions*, J. Pure Appl. Algebra **2** (1972), 83–105.

