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TWO EXAMPLES OF STOCHASTIC FIELD THEORIES

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Abstract

We give some regularization in order to define rigorously a stochastic W.Z.N.W. model or a stochastic Chern-Simons theory. We show that the Markov property of the random field allows us to satisfy the glueing axiom of field theory (of Segal or Atiyah).

I. Introduction

This list of axioms of a d-dimensional field theory is strongly inspired of Segal's axiom of conformal field theory ([50]) and Atiyah's axiom of topological field theory ([5]).

Axioms. a d-dimensional field theory is given by the following data:
i) To any $d-1$ Riemannian manifold (Σ, g), we associate an Hilbert space $H(\Sigma, g)$ such that $H(\Sigma_1 \cup \Sigma_2, g_1 \cup g_2) = H(\Sigma_1, g_1) \otimes H(\Sigma_2, g_2)$ if $\Sigma_1 \cap \Sigma_2 = \emptyset$.
ii) If (V, g_V) is a bordism from (Σ_1, g_1) to (Σ_2, g_2) such that in a neighborhood of Σ_2, (V, g_V) is isometric to $([0, 1/2[\times \Sigma_2, dt \otimes g_1)$ and such that in a neighborhood of Σ_1, (U, g_U) is isometric to $(]1/2, 1] \times \Sigma_1, dt \otimes g_1)$, then (U, g_U) realizes a bounded linear map $H(V, g_V)$ from $H(\Sigma_1, g_1)$ into $H(\Sigma_2, g_2)$.
iii) These data have to satisfy the following requirement (Gluing property): let (V_1, g_{V_1}) a bordism from (Σ_1, g_1) into (Σ_2, g_2) and (V_2, g_{V_2}) be a bordism between (Σ_2, g_2) into (Σ_3, g_3). Let (W, g_W) the Riemannian manifold got by sewing V_1 and V_2 along their common boundary Σ_2. Then

$$H(W, g_W) = H(V_2, g_{V_2}) \circ H(V_1, g_{V_1}). \tag{1.1}$$

Let us remark that there are some difference with the traditional axioms of field theory:
-) The operator $H(V, g_V)$ is supposed only bounded and not Hilbert-Schmidt as it is traditional.
-) In the gluing axiom, we sew (V_1, g_{V_1}) and (V_2, g_{V_2}) along a piece of the output boundary of V_1 and a piece of the input boundary of V_2.

We are motivated in this work by a stochastic realization of these axioms, with in addition a technical modification.

The first example is involved with the stochastic Wess-Zumino-Novikov-Witten model (see [17], [18], [19] for a pedagogical introduction about the physicist model). This is a 2-dimensional field theory. This theory uses infinite dimensional processes over infinite dimensional manifolds. The construction of such processes was pioneered by Kuo ([30]) who has constructed the Brownian motion on infinite dimensional manifold. The Russian school ([6], [12], [7]) has a different way to construct processes on infinite dimensional manifolds. [2] and [13] have constructed the Ornstein-Uhlenbeck process on the loop space by using Dirichlet forms. [1] have constructed the heat-kernel measure over a loop group by using the Brownian motion on a loop group. Our construction is related to this work and to the construction of diffusion processes on $M-2$ Banach manifolds of Brzezniak-Elworthy ([8]). Related works are the papers of Brzezniak-Léandre ([9], [10]) and Léandre ([31], [32], [33], [34], [35], [36], [37], [38], [39], [40], [41]). As in [37] and in [38], our random fields are C^k. This leads to a simplification with the treatment of [33] for instance: namely no construction of stochastic integrals is required in the treatment of the Wess-Zumino term.

The second example is the 3-dimensional stochastic Chern-Simons theory. Unlike the traditional Chern-Simons theory, our stochastic Chern-Simons theory is not a topological field theory, because we average the connections under a Gaussian measure, instead of the Lebesgue measure as it is classical in Chern-Simons theory ([5], [3], [53]).

In these two examples, we deduce from the gluing property a relation with operads by chosing bordism between always the same connected (Σ, g) and as exit boundary a finite disjoint union of the same (Σ, g). This relation was pioneered for conformal field theory by Huang-Lepowsky ([23]) and Kimura-Stasheff-Voronov ([26]).

The geometrical data of this paper are taken from the work of Freed ([16]).

II. Stochastic Wess-Zumino-Novikov-Witten model

Let us consider the case of a two dimensional field theory. V is a Riemannian surface with exit and input boudary loops endowed with the canonical metric on S^1 on each on the connected components of the boundary.

Let us consider $\overline{V}$ got from V by sewing disk along the boudaries. $\overline{V}$ has a canonical metric, inherited from V. Let $\Delta_{\overline{V}}$ be the Laplace Beltrami operator on $\overline{V}$. Let $H_{\overline{V}}$ the Hilbert space of maps f from $\overline{V}$ into R such that:

$$\int_{\overline{V}} \big(\Delta_{\overline{V}}^k + 1\big) f(z) \big(\Delta_{\overline{V}}^k + 1\big) f(z)\, dm_{\overline{V}}(z) < \infty \tag{2.1}$$

where $dm_{\overline{V}}(z)$ denotes the Riemannian measure on $\overline{V}$.

Let $B_{\overline{V},t}$ be the Brownian motion with values in $H_{\overline{V}}$. It has reproducing Hilbert space:

$$\int_{[0,1]}\int_{\overline{V}}\frac{\partial}{\partial t}\left(\Delta_{\overline{V}}^{k}+1\right)f(t,z)\frac{\partial}{\partial t}\left(\Delta_{\overline{V}}^{k}+1\right)f(t,z)\,dt\,dm_{\overline{V}}(z)<\infty \tag{2.2}$$

with initial condition $f(0,z) = 0$. If k is big enough independent from r, $(t,z) \to B_{\overline{V},t}(z)$ is continous in $t \in [0,1]$ and C^r in $z \in \overline{V}$ (see [37]).

Let $\overline{S}_1 = [0,1] \times S_1$ where we sew disk along the boundary. $\overline{S}_1$ inherites a canonical Riemannian structure. Let $H_{\overline{S}_1}$ be the Hilbert space of maps from $\overline{S}_1$ into R such that

$$\int_{\overline{S}_1}\left(\Delta_{\overline{S}_1}^{k}+I\right)f(z)\left(\Delta_{\overline{S}_1}^{k}+1\right)f(z)\,dm_{\overline{S}_1}(z)<\infty. \tag{2.3}$$

Let $B_{\overline{S}_1,t}$ be the Brownian motion with values in $H_{\overline{S}_1}$. It has as reproducing Hilbert space the set of maps f from $[0,1] \times \overline{S}_1$ into R such that:

$$\int_{[0,1]}\int_{\overline{S}_1}\frac{\partial}{\partial t}\left(\Delta_{\overline{S}_1}^{k}+1\right)f(t,z)\frac{\partial}{\partial t}\left(\Delta_{\overline{S}_1}^{k}+1\right)f(t,z)\,dt\,dm_{\overline{S}_1}(z)<\infty \tag{2.4}$$

with initial condition $f(0,z) = 0$ ($\Delta_{\overline{S}_1}$ denotes the Laplace-Beltrami operator on $\overline{S}_1$).

Let $g_V(z)$ be a map from V into [0,1] equal to 1 on V where we have removed the output collars $[0,1/2[\times\Sigma_2$ and where we have removed the input collars $]1/2;1]\times \Sigma_1$. We suppose that g_V is equal to zero on a neighborhood of the boundaries of V.

Let g^{out} be a smooth map from $[0,1/2]$ into $[0,1]$ equal to 0 only in 0 and equal to 1 in a neighborhood of $1/2$. Let g^{in} be a smooth map from $[1/2,1]$ equal to 0 only in 1 and equal to 1 in a neighborhood of $1/2$.

We consider the Gaussian random field parametrized by $U \times [0,1]$:

$$B_{V,.}(.) = g_V(.)B_{\overline{V},.}(.) + \sum_{in} g^{in} B^{in}_{\overline{S}_1,.}(.) + \sum_{out} g^{out} B^{out}_{\overline{S}_1,.}(.) \tag{2.5}$$

where we take independent Brownian motion on $H_{\overline{S}_1}$ which are independent of the Brownian motion $B_{\overline{V}}$. We have a body process and some boundary processes which are independent themselves and of the body process.

An object $V_{tot,k} = (V_1 \cup V_2 .. \cup V_k)$ is constructed inductively as follows: V_1 is a Riemann surface constructed as before. $V_{tot,k+1}$ is constructed from $V_{tot,k}$ where we sew some exit boundaries of $V_{tot,k}$ along some input boundaries of V_{k+1}. Let us remark that in the present theory, we don't consider $V_{tot,k}$ as a Riemannian manifold, but as the sequence $(V_1, .., V_k)$ and the way we sew V_{k+1} to $V_{tot,k}$ inductively. Namely, if we consider the random fields parametrized by $V_{tot,k} \times [0,1]$ considered as a global Riemannian manifold done by (2.5), it is different from the random field $B_{V_{tot,k}}$ constructed as below. In particular, the sewing collars in $V_{tot,k}$ are independent in the construction below, and are not independent in the construction (2.5).

We can construct inductively $B_{V_{tot,k+1}}$ as follows: if $k = 1$, it is B_V. $B_{V_{k+1}}$ is constructed from Brownian motion independent of those which have constructed $B_{V_{tot,k}}$, except for the Brownian motions in the input boundaries of $B_{V_{k+1}}$ which coincide with the Brownian motion in the output boundaries of $V_{tot,k}$ which are sewed to the corresponding input boundaries of V_{k+1}. By this procedure, if $z \in V_{tot}$, we get a process $(t, z) \to B_{V_{tot},t}(z)$ which is continuous in t and C^r in $z \in V_{tot}$.

Let G be a compact simply connected Lie group. We consider Airault-Malliavin equation ([1])

$$d_t g_{V_{tot},t}(z) = g_{V_{tot},t}(z) \sum e_i d_t B^i_{V_{tot}}(z) \tag{2.6}$$

starting from e. $B^i_{V_{tot}}$ are independent copies of $B_{V_{tot}}$ and e_i an orthogonal basis of the Lie algebra of G.

Let us remark that we can construct the formal action driving the non-linear Random field $g_{V_{tot}}$ by using large deviation theory ([19], [31]). $t \to B_{V_{tot},t}(.)$ is a Brownian motion on a Hilbert space whose reproducing kernel $\|.\|$ is deduced from (2.1), (2.3) and (2.5). It has formally the Gaussian law:

$$d\mu = \frac{1}{Z} \exp\left[-\int_0^1 \frac{1}{2}\frac{\partial}{\partial t}\|h_t(.)\|^2\right] dD(h) \tag{2.7}$$

where $dD(h)$ is the formal Lebesgue measure on fields parametrized by $V_{tot} \times [0, 1]$ into $Lie(G)$. Let us consider the equation

$$d_t g_{V_{tot},t,\epsilon}(z) = \epsilon g_{V_{tot},t,\epsilon}(z) \sum e_i d_t B^i_{V_{tot},t}(z). \tag{2.8}$$

The following large deviation estimate holds: let us consider a borelian subset O on the space of maps from $V_{tot} \times [0, 1]$ into G for the uniform topology. $int\, O$ denotes its interior for the uniform topology and $clos\ 0$ its adherence for the uniform topology. We have when $\epsilon \to 0$:

$$-\inf_{g_{V_{tot}}(h)\in int\ O}\left(\int_0^1 \frac{\partial}{\partial t}\|h_t(.)\|^2\, dt\right) \le \liminf 2\epsilon^2 \log P\{g_{V_{tot},.,\epsilon}(.) \in O\}, \tag{2.9}$$

$$\limsup 2\epsilon^2 \log P\{g_{V_{tot},.,\epsilon}(.) \in O\} \le -\inf_{g_{V_{tot}}(h)\in clos\ O}\left(\int_0^1 \frac{\partial}{\partial t}\|h_t(.)\, dt\|^2\right). \tag{2.10}$$

In order to define $g_{V_{tot},t}(z)(h)$, we replace formally in (2.6) $d_t B^i_{V_{tot}}(z)$ by $d_t h^i_{V_{tot}}(z)$.

By proceeding as in [37] we get:

Theorem II.1. *If k is big enough, the random field parametrized by V_{tot} $z \to g_{V_{tot},1}(z)$ is C^r. Moreover the restriction to this random field to the connected components of the boundary of V_{tot} are independents and have the same law.*

Let us recall some geometrical background about the Wess-Zumino-Novikov-Witten model ([16]). Let V be an oriented surface with boundaries. Let g be a C^r map from V into G conveniently extended into a map $g_t(z)$ from $[0,1] \times V$ into G such that $g_0(z) = e$. We define the Wess-Zumino term:

$$W_V(g) = -\frac{1}{6}\int_{[0,1]\times V} \langle g^{-1}dg \wedge [g^{-1}dg \wedge g^{-1}dg]\rangle \tag{2.11}$$

where $\langle , \rangle$ is the canonical normalized Killing form on the Lie algebra of G. We suppose that the 3-form which is integrated in (2.11) represents an element of $H^3(G; Z)$ (see [16] for this hypothesis). $\exp[2\pi\sqrt{-1}\, W_V(g)]$ can be identified canonically to an element of $K_{\partial V, \partial g}$ where K is an Hermitian line bundle over the set of C^r maps from ∂V into G. Let ∂V_i be the oriented connected components of ∂V. We have a canonical inclusion map π_i from ∂V_i in ∂V. We deduce from it a map $\overline{\pi}_i$ from the set of maps from ∂V in G into the set of maps from ∂V_i into G. Let Λ_i be the hermitian bundle on the set of maps from ∂V_i into G constructed in [16]. $K = \otimes \overline{\pi}_i^* \Lambda_i$ endowed with its natural metric inherited from each Λ_i. We denote it $\otimes_{exit}\Lambda \otimes_{in} \Lambda$. Moreover, we can realize this expression as a map from the tensor products of Hermitian line bundle Λ over the exit loop groups defined by restricting the field over each exit boundary to the tensor product of Hermitian line bundles Λ over the input loop groups defined by restricting the field over each connected component of the input boundary. Therefore $\exp[2\pi\sqrt{-1}\, W_V(g)]$ can be realized as an application from $\otimes_{exit}\Lambda$ into $\otimes_{in}\Lambda$ of modulus 1. This application is consistent with the operation of sewing surface.

Let V_{tot} and the restriction of $g_{V_{tot},1}(.)$ to one connected component of the boundary of V_{tot}. Let Ξ' be the Hilbert space of section of Λ over the C^r loop group $L^r(G)$ endowed with the law arising from restricting the field to one boundary loops. Let V_i be such a boundary loop. The laws of $g_{V_{tot,1}}(.)$ restricted to each V_i are the same. Let $\Psi_i(g_{V_{tot}},1(.)|_{V_i})$ a section of Λ on the set of loops defined by V_i. $|\Psi_i(g_{V_{tot}},1(.)|_{V_i})|$ denotes a random variable which is $g_{V_{tot}},1(.)|_{V_i}$ measurable, where $g_{V_{tot}},1(.)|_{V_i}$ denotes the restriction to the random field to V_i. We put

$$\|\Psi_i\|^2_{\Xi'_i} = E[|\Psi_i(g_{V_{tot}},1(.)|_{V_i})|^2]. \tag{2.12}$$

But the previous Hilbert norm don't depend of the chosen boundary loop V_i, and we get the definition of the Hilbert space Ξ'_i. Let $L^2([0,1] \times V_i)$ be the Hilbert space of L^2 functionals with respect of $g_{V_{tot,.}}(.)$ restricted to $[0,1] \times V_i$. We put $\Xi_i = \Xi'_i \otimes L^2([0,1] \times V_i)$. We get always the same Hilbert space Ξ independent of the choice of V_{tot}. If $g_{V_{tot}},1(z)$ is the random map from V_{tot} into G, we deduce the random map from $[0,1] \times V_{tot}$ into G, $(t,z) \to g_{V_{tot}},1(z)$ with the the boudary condition $g_{V_{tot}},0(z) = e$. We deduce from this the Wess-Zumino term $\exp[2\pi\sqrt{-1}\, W_{V_{tot}}(g_{V_{tot}},1)]$. Let us recall that the cinetic term in the W.Z.N.W. model is equal to $\exp[-I(g)]$ where $I(g)$ is the energy from the map g from V_{tot} into G. Compare with (2.9) and (2.10). In this work, we will consider the same topological term and we will consider another cinetic term given by the law of $g_{V_{tot},1}(.)$.

DEFINITION II.2. $H(V_{tot}, g_{V_{tot}})$ is the operator from $\otimes_{out}\Xi$ into $\otimes_{in}\Xi$ where we put the tensor product along respectively the connected components of the exit boundary of V_{tot} and of the input boundaries of V_{tot} defined as follows: let Ψ_i a section of Λ at the i^{th} connected component of the exit boundary:

$$H(V_{tot}, g_{V_{tot}}) \otimes_{out} \Psi_i = E[\exp[2\pi\sqrt{-1}\, W_{V_{tot}}(g_{V_{tot},1})] \otimes_{out} \Psi_i | B'([0,1]\times\Sigma_1)] \tag{2.13}$$

where $B'([0,1]\times\Sigma_1)$ is the σ-algebra spanned by the random field $g_{V_{tot},.}(.)$ restricted to the input data $[0,1]\times\Sigma_1$.

Let $(V^1_{tot}, g^1_{V_{tot}})$ and (V^2_{tot}, g^2_{tot}) and $(W_{tot}, g_{W_{tot}})$ got by sewing V^1_{tot} along some exit boundaries coinciding with some input boundaries of V^2_{tot}. We call the sewing boundary $\tilde\Sigma$ in W_{tot}. We call $B([0,1]\times\tilde\Sigma)$ the sigma algebra defined by (4.2) for the random field $(t,z)\to g_{W_{tot},t}(z)$ parametrized by $[0,1]\times W_{tot}$. From Theorem IV.2, it satisfies (4.4). We deduce:

Theorem II.3. *We have*:

$$H(W_{tot}, g_{W_{tot}}) = H(V^1_{tot}, g_{V^1_{tot}}) \circ H^2(V^2_{tot}, g_{V^2_{tot}}) \tag{2.14}$$

where the composition goes for the Hilbert spaces which arises from the sewing boundaries.

Proof. Let Σ_s be the sewing boundary in W_{tot}. We get almost surely:

$$\begin{aligned}&\exp[2\pi\sqrt{-1}\, W_{W_{tot}}(g_{W_{tot},1})]\\ &\quad = \exp[2\pi\sqrt{-1}\, W_{V^1_{tot}}(g_{V^1_{tot},1})] \circ \exp[2\pi\sqrt{-1}\, W_{V^2_{tot}}(g_{V^2_{tot},1})].\end{aligned} \tag{2.15}$$

By Markov property, the two term in the right hand side of (2.15) are conditionnally independent to $B'([0,1]\times\Sigma_s)$. We have:

$$\begin{aligned}&H(W_{tot}, g_{W_{tot}}) \otimes_{out} \Psi_i\\ &= E[E[\exp[2\pi\sqrt{-1}\, W_{W_{tot}}(g_{W_{tot},1})] \otimes_{out} \Psi_i | B'(\Sigma_1\cup\Sigma_s)] | B'([0,1]\times\Sigma_1)].\end{aligned} \tag{2.16}$$

But we have:

$$\begin{aligned}&E[\exp[2\pi\sqrt{-1}\, W_{W_{tot}}(g_{W_{tot},1})] | B'(\Sigma_1\cup\Sigma_s)] = E[\exp[2\pi\sqrt{-1}\, W_{V^1_{tot}}(g_{V^1_{tot},1})]\\ &\quad \circ \exp[2\pi\sqrt{-1}\, W_{V^2_{tot}}(g_{V^2_{tot},1})] \otimes_{out} \Psi_i | B'([0,1]\times(\Sigma_1\cup\Sigma_s)].\end{aligned} \tag{2.17}$$

By (4.4), where we choose as O the interior of V^1_{tot} in W_{tot}, the right hand-side in (2.17) is equal to

$$E[\exp[2\pi\sqrt{-1}\,W_{V^1_{tot}}(g_{V^1_{tot},1})](E[\exp[2\pi\sqrt{-1}\,W_{V^2_{tot}}(g_{V^2_{tot},1})] \otimes_{out} \Psi_i]|B'([0,1]\times\Sigma_s)]|B'([0,1]\times\Sigma_1)]. \tag{2.18}$$

In (4.4), this decomposition formula is true for functionals, but we can come back to this case in (2.17) by introducing an orthonormal basis of Ξ. □

If $(V_{tot}, g_{V_{tot}})$ have only one connected component in the input boundary and n connected component in the output boudary, we say that $(V_{tot}, g_{V_{tot}})$ belongs to $E(n)$. An element of $E(n)$ realizes an element of $Hom(\Xi^{\otimes n}, \Xi)$.

In particular, we will consider as $E(n)$ the case of $1+n$-punctured sphere $S_{tot}(1,n)$ with one input loop and n output loops, by taking care of the history where we glue some subspheres in $S_{tot}(1,n)$. $E(n)$ is very similar to the space of trees with one root and n exit vertices. Trees are an archetype of an operad: if $A(n)$ denotes the space of trees with n-exit vertices, we get an operation of $A(n)\times A(r_1)\times\cdots\times A(r_n)$ by grafting trees. This action is compatible with the action of symmetric group got by relabelling the exit vertices. $E(n)$ should correspond to the parameter set of a branching process on the loop space, time of branching being included: the branching mechanism is got when a loop splitts in two loops (and not by creating two loops as it is classical in branching process theory). $E(n)$ inherits an action of the symmetric group by labelling the connected components of the exit boudary. The action of sewing punctured spheres realizes a map from $E(n)\times E(r_1)\times\cdots\times E(r_n)$ into $E(\sum r_i)$ which is compatible with the action of the symmetric group. We say that $E(n)$ is an operad. On the other hand, $Hom(\Xi^n, \Xi)$ realizes clearly an operad, by composition of the homomorphisms. We get from Theorem II.3:

Theorem II.4. *If $(W_{tot}, g_{W_{tot}})$ belongs to $E(n)$, $H(W_{tot}, g_{W_{tot}})$ realizes a map from the operad $E(n)$ into the operad $Hom(\Xi^{\otimes n}, \Xi)$.*

If we consider the case of the punctured sphere, this corresponds to a kind of Branching process on the loop space.

III. Stochastic Chern-Simons theory

We consider now as (V, g_V) the case of an oriented 3-dimensional manifold V with boundaries having connected components some oriented Riemannian surfaces (Σ_i, g_{Σ_i}). The input boundaries are called Σ_i^{in} and the output boundaries are called Σ_i^{out}. This means that V realizes a bordism from $\bigcup\Sigma_i^{in}$ into $\bigcup\Sigma_i^{out}$. We can find a 3-dimensional manifold whose boundary is Σ_i. Let us consider $\overline{V}$ got from V by sewing these 3-dimensional manifolds to each Σ_i. $\overline{V}$ has a Riemannian metric inherited from Σ_i. Let $\Delta_{\overline{V}}$ be the Hodge Laplacian operating on 1-forms on $\overline{V}$ with values in the Lie algebra of a compact simply connected Lie group G, endowed with the natural Killing metric. We introduce the Sobolev space $H_{\overline{V}}$ of 1-form ω with values in $Lie(G)$ such that:

$$\int_{\overline{V}} \left\langle \left(\Delta^k_{\overline{V}} + 1\right)\omega, \left(\Delta^k_{\overline{V}} + 1\right)\omega \right\rangle dm_{\overline{V}} < \infty. \tag{3.1}$$

We denote by $\omega_{\overline{V}}$ the centered Gaussian measure in $H_{\overline{V}}$. If k is big enough, $\omega_{\overline{V}}(z)$ is almost surely a 1-form which is C^r.

Let $\overline{\Sigma}$ got from $[0,1] \times \Sigma$ by sewing these 3-dimensional manifolds along the boundary. $\overline{\Sigma}$ inherites a canonical metric from the metric on Σ. Let $\Delta_{\overline{\Sigma}}$ be the Laplacian operating on 1-form on $\overline{\Sigma}$ with values in $Lie(G)$. Let $H_{\overline{\Sigma}}$ be the Hilbert Sobolev space of 1-forms ω on $\overline{\Sigma}$ with values in $Lie(G)$ such that:

$$\int_{\overline{\Sigma}} \left\langle \left(\Delta^k_{\overline{\Sigma}} + 1\right)\omega, \left(\Delta^k_{\overline{\Sigma}} + 1\right)\omega \right\rangle dm_{\overline{\Sigma}} < \infty. \tag{3.2}$$

We consider the centered Gaussian measure on $H_{\overline{\Sigma}}$. This gives a random 1-form $\omega_{\tilde{\Sigma}}$ which is C^r if k is big enough.

Let $g_V(z)$ be a map from V into [0,1] equal to 1 on V where we have removed the output collars $[0, 1/2[\times \Sigma_i^{out}$ and where we have removed the input collars $]1/2; 1] \times \Sigma_i^{in}$. We suppose that g_V is equal to zero on a neighborhood of the boudaries of V.

Let g^{out} be a smooth map from $[0, 1/2]$ into $[0, 1]$ equal to 0 only in 0 and equal to 1 in a neighborhood of 1/2. Let g^{in} be a smooth map from $[1/2, 1]$ equal to 0 only in 1 and equal to 1 in a neighborhood of 1/2.

Let V be constructed as above. We consider the Gaussian random field:

$$\omega_V = g_V \omega_{\overline{V}} + \sum_{in} g^{in} \omega_{\overline{\Sigma}_i^{in}} + \sum_{out} g^{out} \omega_{\overline{\Sigma}_i^{out}} \tag{3.3}$$

where we take independent $\omega_{\overline{V}}$, $\omega_{\overline{\Sigma}_i^{in}}$ and $\omega_{\overline{\Sigma}_i^{out}}$. ω_V is a random C^r 1-form on V with values in $Lie(G)$.

Let us consider the trivial bundle $V \times G$ on V. By this trivialization, ω_V realizes a random C^r connection on this bundle.

An object $V_{tot,k} = (V_1 \cup V_2 .. \cup V_k)$ is constructed inductively as follows: V_1 is a 3-dimensional oriented Riemannian manifold constructed as before. $V_{tot,k+1}$ is constructed from $V_{tot,k}$ where we sew some exit boudaries of $V_{tot,k}$ along some input boundaries of V_{k+1}.

We can construct inductively $\omega_{V_{tot,k+1}}$ as follows: if $k = 1$, it is ω_V. $\omega_{V_{k+1}}$ is constructed from Gaussian fields independent of those which have constructed $\omega_{V_{tot,k}}$, except for the Gaussian fields in the input boudaries of $\omega_{V_{tot,k}}$ which coincide with the Gaussian fields in the output boudaries of $V_{tot,k}$ which are sewed to the corresponding input boudaries of V_{k+1}.

Theorem III.1. *If k is big enough, $\omega_{V_{tot}}$ is almost surely C^r.*

Let us recall some background about the Chern-Simons functional (see [16]). If Σ_i is connected, we can construct an Hermitian line bundle $\Lambda(\Sigma_i)$ over the set of C^r connection over Σ_i of the trivial bundle $\Sigma_i \times G$ on Σ_i. Let us do the following hypothesis: let σ be the invariant 3-form on G which is equal to $\sigma(X, Y, Z) = \langle X, [Y, Z]\rangle$ at the level of the Lie algebra. Let us suppose that $1/6\sigma$ represents an element of $H^3(G; Z)$.

Under this hypothesis, it is possible to define as it was used for instance in [16] the Chern-Simons functional $\exp[2\pi\sqrt{-1}\, C_{C.S}(\omega_V)]$ where ω_V is a connection on V as a linear application of modulus one from $\otimes_{out}\Lambda(\Sigma_i^{out})(\omega_{\Sigma_i^{out}})$ into $\otimes_{in}\Lambda(\Sigma_i^{in})(\omega_{\Sigma_i^{in}})$ where we restrict the connection (ω_V) to the input and output boundaries Σ_i of V. We call ω_{Σ_i} these restrictions. These operations are consistent with the operation of sewing 3-dimensional manifolds.

Let us recall, if V has no boundary, that the Chern-Simons action is equal to

$$\frac{k}{2\pi}\int_V Tr\left[\omega_V \wedge d\omega_V + \frac{2}{3}\omega_V \wedge \omega_V \wedge \omega_V\right] \tag{3.4}$$

wherec Tr is got by imbedding the Lie group G into $SO(n)$ for some big convenient n.

Let $H(\Sigma, g_\Sigma)$ the Hilbert space of sections of $\Lambda(\Sigma)$ for the measure got by restricting $\omega_{\overline{\Sigma}}$ to Σ.

DEFINITION III.2. $H(V_{tot}, g_{V_{tot}})$ is the operator from $\otimes_{out}H(\Sigma_i^{out}, g_{\Sigma_i^{out}})$ into the Hilbert space $\otimes_{in}H(\Sigma_i^{in}, g_{\Sigma_i^{in}})$ defined as follows: let Ψ_i^{out} belonging to $H(\Sigma_i^{out}, g_{\Sigma_i^{out}})$:

$$H(V_{tot}, g_{V_{tot}}) \otimes_{out} \Psi_I^{out} = E\left[\exp\left[2\pi\sqrt{-1}\, S_{CS}(\omega_{V_{tot}})\right] \otimes \Psi_i^{out}(\omega_{\Sigma_i^{out}}) | B'\left(\bigcup \Sigma_i^{in}\right)\right] \tag{3.5}$$

where $B'(\bigcup \Sigma_i^{in})$ is the σ-algebra spanned by the restriction ω_V to the union of input boudaries Σ_i^{in}.

Let $(V_{tot}^1, g_{V_{tot}^1})$ and $(V_{tot}^2, g_{V_{tot}^2})$ and $(W_{tot}, g_{W_{tot}})$ got by sewing V_{tot}^1 and V_{tot}^2 along some exit boundaries from V_{tot}^1 and some input boundaries of V_{tot}^2. Since the stochastic Chern-Simons functional $\exp[2\pi\sqrt{-1}\, S_{C.S}(\omega_{V_{tot}^i})]$ is measurable for the σ-algebra spanned by the fields $\omega_{V_{tot}^i}$, we deduce from Theorem IV.2:

Theorem III.3. *We have*:

$$H(W_{tot}, g_{W_{tot}}) = H(V_{tot}^1, g_{V_{tot}^1}) \circ H^2(V_{tot}^2, g_{V_{tot}^2}) \tag{3.6}$$

where the composition goes from the Hilbert spaces which arise from the sewing boundary.

If $(V_{tot}, g_{V_{tot}})$ has only one connected component in the input boundary (Σ, g_Σ) and n-connected component in the output boundary constituted of the same (Σ, g_Σ), we say that we have an element of $E_n(\Sigma, g_\Sigma)$. The collection of $E_n(\Sigma, g_\Sigma)$ constitutes an operad when (Σ, g_Σ) is fixed. We put $\Xi = H(\Sigma, g_\Sigma)$. An element of $E_n(\Sigma, g_\Sigma)$ realizes an element of $Hom(\Xi^{\otimes n}, \Xi)$.

Theorem III.4. *If $(W_{tot}, g_{W_{tot}})$ belongs to $E_n(\Sigma, g_\Sigma)$, $H(W_{tot}, g_{W_{tot}})$ realizes a map from the operad $E_n(\Sigma, g_\Sigma)$ into the operad $Hom(\Xi^{\otimes n}, \Xi)$.*

IV. Appendix

This appendix constitutes a brief review concerning the Markov property for Gaussian random fields. We refer to [29] and references therein for more details.

(Ω, F, P) be a probability space, and $X(z)$ a Gaussian continuous centered random field with parameter space a finite manifold T endowed with a Riemannian distance d.

If O is an open subset of T, we define

$$B(O) = \sigma(X(z); z \in O) \tag{4.1}$$

anf for an closed subset D, we define

$$B(D) = \bigcap_{\epsilon>0} B(D_\epsilon) \tag{4.2}$$

where $D_\epsilon = \{z \in T : \inf_{z' \in D} d(z, z') < \epsilon\}$.

DEFINITION IV.1. A random field has the Markov property with respect to an open set O if for any $B(\overline{O})$-measurable functional ψ:

$$E[\psi | B(O^c)] = E[\psi | B(\partial O)]. \tag{4.3}$$

A random field is G-markov if it has the Markov property with respect to all open sets O.

Markov property with respect to O is equivalent to the following statement: for any event A_1 $B(\overline{O})$-measurable and for any event A_2 $B(O^c)$-measurable:

$$P(A_1 \cap A_2 | B(\partial O)] = P(A_1 | B(\partial O)] P[A_2 | B(\partial O)]. \tag{4.4}$$

Let us recall that the reproducing Hilbert space H of the continuous Gaussian random field is given as follows: if X is a linear random variable of the Gaussian random field, we put:

$$f_X(z) = E[XX(z)] \tag{4.5}$$

and

$$\langle f_X, f_Y\rangle = E[XY]. \tag{4.6}$$

If $e_z(z')$ is the covariance of the Gaussian random field,

$$E[X(z)X(z')] = e_z(z') \tag{4.7}$$

we have

$$f(z) = \langle f, e_z(.)\rangle \tag{4.8}$$

Let us recall ([29] Theorem 5.1):

Theorem IV.2. *A random continuous Gaussian field X with reproducing Hilbert space H is a Markov field if and only if the two following conditions are checked*:
i) *For all f_1, $f_2 \in H$ with support disjoint, $\langle f_1, f_2\rangle = 0$.*
ii) *if $f \in H$ is such that $f = f_1 + f_2$ with disjoint supports, then f_1 and f_2 belong to H.*

We have a natural generalization of Theorem IV.2 to the case where the random field takes its values in R^d.

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