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## TWO EXAMPLES OF STOCHASTIC FIELD THEORIES

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### Abstract

We give some regularization in order to define rigorously a stochastic W.Z.N.W. model or a stochastic Chern-Simons theory. We show that the Markov property of the random field allows us to satisfy the glueing axiom of field theory (of Segal or Atiyah).

### I. Introduction

This list of axioms of a  $d$ -dimensional field theory is strongly inspired of Segal's axiom of conformal field theory ([50]) and Atiyah's axiom of topological field theory ([5]).

**Axioms.** a  $d$ -dimensional field theory is given by the following data:

- i) To any  $d - 1$  Riemannian manifold  $(\Sigma, g)$ , we associate an Hilbert space  $H(\Sigma, g)$  such that  $H(\Sigma_1 \cup \Sigma_2, g_1 \cup g_2) = H(\Sigma_1, g_1) \otimes H(\Sigma_2, g_2)$  if  $\Sigma_1 \cap \Sigma_2 = \emptyset$ .
- ii) If  $(V, g_V)$  is a bordism from  $(\Sigma_1, g_1)$  to  $(\Sigma_2, g_2)$  such that in a neighborhood of  $\Sigma_2$ ,  $(V, g_V)$  is isometric to  $([0, 1/2[ \times \Sigma_2, dt \otimes g_1)$  and such that in a neighborhood of  $\Sigma_1$ ,  $(U, g_U)$  is isometric to  $(]1/2, 1] \times \Sigma_1, dt \otimes g_1)$ , then  $(U, g_U)$  realizes a bounded linear map  $H(V, g_V)$  from  $H(\Sigma_1, g_1)$  into  $H(\Sigma_2, g_2)$ .
- iii) These data have to satisfy the following requirement (Gluing property): let  $(V_1, g_{V_1})$  a bordism from  $(\Sigma_1, g_1)$  into  $(\Sigma_2, g_2)$  and  $(V_2, g_{V_2})$  be a bordism between  $(\Sigma_2, g_2)$  into  $(\Sigma_3, g_3)$ . Let  $(W, g_W)$  the Riemannian manifold got by sewing  $V_1$  and  $V_2$  along their common boundary  $\Sigma_2$ . Then

$$(1.1) \quad H(W, g_W) = H(V_2, g_{V_2}) \circ H(V_1, g_{V_1}).$$

Let us remark that there are some difference with the traditional axioms of field theory:

- ) The operator  $H(V, g_V)$  is supposed only bounded and not Hilbert-Schmidt as it is traditional.
- ) In the gluing axiom, we sew  $(V_1, g_{V_1})$  and  $(V_2, g_{V_2})$  along a piece of the output boundary of  $V_1$  and a piece of the input boundary of  $V_2$ .

We are motivated in this work by a stochastic realization of these axioms, with in addition a technical modification.

The first example is involved with the stochastic Wess-Zumino-Novikov-Witten model (see [17], [18], [19] for a pedagogical introduction about the physicist model). This is a 2-dimensional field theory. This theory uses infinite dimensional processes over infinite dimensional manifolds. The construction of such processes was pioneered by Kuo ([30]) who has constructed the Brownian motion on infinite dimensional manifold. The Russian school ([6], [12], [7]) has a different way to construct processes on infinite dimensional manifolds. [2] and [13] have constructed the Ornstein-Uhlenbeck process on the loop space by using Dirichlet forms. [1] have constructed the heat-kernel measure over a loop group by using the Brownian motion on a loop group. Our construction is related to this work and to the construction of diffusion processes on  $M - 2$  Banach manifolds of Brzezniak-Elworthy ([8]). Related works are the papers of Brzezniak-Léandre ([9], [10]) and Léandre ([31], [32], [33], [34], [35], [36], [37], [38], [39], [40], [41]). As in [37] and in [38], our random fields are  $C^k$ . This leads to a simplification with the treatment of [33] for instance: namely no construction of stochastic integrals is required in the treatment of the Wess-Zumino term.

The second example is the 3-dimensional stochastic Chern-Simons theory. Unlike the traditional Chern-Simons theory, our stochastic Chern-Simons theory is not a topological field theory, because we average the connections under a Gaussian measure, instead of the Lebesgue measure as it is classical in Chern-Simons theory ([5], [3], [53]).

In these two examples, we deduce from the gluing property a relation with operads by choosing bordism between always the same connected  $(\Sigma, g)$  and as exit boundary a finite disjoint union of the same  $(\Sigma, g)$ . This relation was pioneered for conformal field theory by Huang-Lepowsky ([23]) and Kimura-Stasheff-Voronov ([26]).

The geometrical data of this paper are taken from the work of Freed ([16]).

## II. Stochastic Wess-Zumino-Novikov-Witten model

Let us consider the case of a two dimensional field theory.  $V$  is a Riemannian surface with exit and input boundary loops endowed with the canonical metric on  $S^1$  on each on the connected components of the boundary.

Let us consider  $\bar{V}$  got from  $V$  by sewing disk along the boundaries.  $\bar{V}$  has a canonical metric, inherited from  $V$ . Let  $\Delta_{\bar{V}}$  be the Laplace Beltrami operator on  $\bar{V}$ . Let  $H_{\bar{V}}$  the Hilbert space of maps  $f$  from  $\bar{V}$  into  $R$  such that:

$$(2.1) \quad \int_{\bar{V}} (\Delta_{\bar{V}}^k + 1) f(z) (\Delta_{\bar{V}}^k + 1) f(z) dm_{\bar{V}}(z) < \infty$$

where  $dm_{\bar{V}}(z)$  denotes the Riemannian measure on  $\bar{V}$ .

Let  $B_{\bar{V},t}$  be the Brownian motion with values in  $H_{\bar{V}}$ . It has reproducing Hilbert

space:

$$(2.2) \quad \int_{[0,1]} \int_{\overline{V}} \frac{\partial}{\partial t} (\Delta_{\overline{V}}^k + 1) f(t, z) \frac{\partial}{\partial t} (\Delta_{\overline{V}}^k + 1) f(t, z) dt dm_{\overline{V}}(z) < \infty$$

with initial condition  $f(0, z) = 0$ . If  $k$  is big enough independent from  $r$ ,  $(t, z) \rightarrow B_{\overline{V},t}(z)$  is continuous in  $t \in [0, 1]$  and  $C^r$  in  $z \in \overline{V}$  (see [37]).

Let  $\overline{S}_1 = [0, 1] \times S_1$  where we sew disk along the boundary.  $\overline{S}_1$  inherits a canonical Riemannian structure. Let  $H_{\overline{S}_1}$  be the Hilbert space of maps from  $\overline{S}_1$  into  $R$  such that

$$(2.3) \quad \int_{\overline{S}_1} (\Delta_{\overline{S}_1}^k + I) f(z) (\Delta_{\overline{S}_1}^k + 1) f(z) dm_{\overline{S}_1}(z) < \infty.$$

Let  $B_{\overline{S}_1,t}$  be the Brownian motion with values in  $H_{\overline{S}_1}$ . It has as reproducing Hilbert space the set of maps  $f$  from  $[0, 1] \times \overline{S}_1$  into  $R$  such that:

$$(2.4) \quad \int_{[0,1]} \int_{\overline{S}_1} \frac{\partial}{\partial t} (\Delta_{\overline{S}_1}^k + 1) f(t, z) \frac{\partial}{\partial t} (\Delta_{\overline{S}_1}^k + 1) f(t, z) dt dm_{\overline{S}_1}(z) < \infty$$

with initial condition  $f(0, z) = 0$  ( $\Delta_{\overline{S}_1}$  denotes the Laplace-Beltrami operator on  $\overline{S}_1$ ).

Let  $g_V(z)$  be a map from  $V$  into  $[0,1]$  equal to 1 on  $V$  where we have removed the output collars  $[0, 1/2] \times \Sigma_2$  and where we have removed the input collars  $]1/2; 1] \times \Sigma_1$ . We suppose that  $g_V$  is equal to zero on a neighborhood of the boundaries of  $V$ .

Let  $g^{out}$  be a smooth map from  $[0, 1/2]$  into  $[0, 1]$  equal to 0 only in 0 and equal to 1 in a neighborhood of  $1/2$ . Let  $g^{in}$  be a smooth map from  $]1/2, 1]$  equal to 0 only in 1 and equal to 1 in a neighborhood of  $1/2$ .

We consider the Gaussian random field parametrized by  $U \times [0, 1]$ :

$$(2.5) \quad B_{V, \cdot}(\cdot) = g_V(\cdot) B_{\overline{V}, \cdot}(\cdot) + \sum_{in} g^{in} B_{\overline{S}_1, \cdot}^{in}(\cdot) + \sum_{out} g^{out} B_{\overline{S}_1, \cdot}^{out}(\cdot)$$

where we take independent Brownian motion on  $H_{\overline{S}_1}$  which are independent of the Brownian motion  $B_{\overline{V}}$ . We have a body process and some boundary processes which are independent themselves and of the body process.

An object  $V_{tot,k} = (V_1 \cup V_2 \dots \cup V_k)$  is constructed inductively as follows:  $V_1$  is a Riemann surface constructed as before.  $V_{tot,k+1}$  is constructed from  $V_{tot,k}$  where we sew some exit boundaries of  $V_{tot,k}$  along some input boundaries of  $V_{k+1}$ . Let us remark that in the present theory, we don't consider  $V_{tot,k}$  as a Riemannian manifold, but as the sequence  $(V_1, \dots, V_k)$  and the way we sew  $V_{k+1}$  to  $V_{tot,k}$  inductively. Namely, if we consider the random fields parametrized by  $V_{tot,k} \times [0, 1]$  considered as a global Riemannian manifold done by (2.5), it is different from the random field  $B_{V_{tot,k}}$  constructed as below. In particular, the sewing collars in  $V_{tot,k}$  are independent in the construction below, and are not independent in the construction (2.5).

We can construct inductively  $B_{V_{tot,k+1}}$  as follows: if  $k = 1$ , it is  $B_V$ .  $B_{V_{k+1}}$  is constructed from Brownian motion independent of those which have constructed  $B_{V_{tot,k}}$ , except for the Brownian motions in the input boundaries of  $B_{V_{k+1}}$  which coincide with the Brownian motion in the output boundaries of  $V_{tot,k}$  which are sewed to the corresponding input boundaries of  $V_{k+1}$ . By this procedure, if  $z \in V_{tot}$ , we get a process  $(t, z) \rightarrow B_{V_{tot},t}(z)$  which is continuous in  $t$  and  $C^r$  in  $z \in V_{tot}$ .

Let  $G$  be a compact simply connected Lie group. We consider Airault-Malliavin equation ([1])

$$(2.6) \quad d_t g_{V_{tot},t}(z) = g_{V_{tot},t}(z) \sum e_i d_t B_{V_{tot}}^i(z)$$

starting from  $e$ .  $B_{V_{tot}}^i$  are independent copies of  $B_{V_{tot}}$  and  $e_i$  an orthogonal basis of the Lie algebra of  $G$ .

Let us remark that we can construct the formal action driving the non-linear Random field  $g_{V_{tot}}$  by using large deviation theory ([19], [31]).  $t \rightarrow B_{V_{tot},t}(\cdot)$  is a Brownian motion on a Hilbert space whose reproducing kernel  $\|\cdot\|$  is deduced from (2.1), (2.3) and (2.5). It has formally the Gaussian law:

$$(2.7) \quad d\mu = \frac{1}{Z} \exp \left[ - \int_0^1 \frac{1}{2} \frac{\partial}{\partial t} \|h_t(\cdot)\|^2 \right] dD(h)$$

where  $dD(h)$  is the formal Lebesgue measure on fields parametrized by  $V_{tot} \times [0, 1]$  into  $Lie(G)$ . Let us consider the equation

$$(2.8) \quad d_t g_{V_{tot},t,\epsilon}(z) = \epsilon g_{V_{tot},t,\epsilon}(z) \sum e_i d_t B_{V_{tot},t}^i(z).$$

The following large deviation estimate holds: let us consider a borelian subset  $O$  on the space of maps from  $V_{tot} \times [0, 1]$  into  $G$  for the uniform topology.  $int O$  denotes its interior for the uniform topology and  $clos O$  its adherence for the uniform topology. We have when  $\epsilon \rightarrow 0$ :

$$(2.9) \quad - \inf_{g_{V_{tot}}(h) \in int O} \left( \int_0^1 \frac{\partial}{\partial t} \|h_t(\cdot)\|^2 dt \right) \leq \liminf 2\epsilon^2 \log P\{g_{V_{tot},\epsilon}(\cdot) \in O\},$$

$$(2.10) \quad \limsup 2\epsilon^2 \log P\{g_{V_{tot},\epsilon}(\cdot) \in O\} \leq - \inf_{g_{V_{tot}}(h) \in clos O} \left( \int_0^1 \frac{\partial}{\partial t} \|h_t(\cdot)\|^2 dt \right).$$

In order to define  $g_{V_{tot},t}(z)(h)$ , we replace formally in (2.6)  $d_t B_{V_{tot}}^i(z)$  by  $d_t h_{V_{tot}}^i(z)$ .

By proceeding as in [37] we get:

**Theorem II.1.** *If  $k$  is big enough, the random field parametrized by  $V_{tot}$   $z \rightarrow g_{V_{tot},1}(z)$  is  $C^r$ . Moreover the restriction to this random field to the connected components of the boundary of  $V_{tot}$  are independents and have the same law.*

Let us recall some geometrical background about the Wess-Zumino-Novikov-Witten model ([16]). Let  $V$  be an oriented surface with boundaries. Let  $g$  be a  $C^r$  map from  $V$  into  $G$  conveniently extended into a map  $g_t(z)$  from  $[0, 1] \times V$  into  $G$  such that  $g_0(z) = e$ . We define the Wess-Zumino term:

$$(2.11) \quad W_V(g) = -\frac{1}{6} \int_{[0,1] \times V} \langle g^{-1} dg \wedge [g^{-1} dg \wedge g^{-1} dg] \rangle$$

where  $\langle , \rangle$  is the canonical normalized Killing form on the Lie algebra of  $G$ . We suppose that the 3-form which is integrated in (2.11) represents an element of  $H^3(G; \mathbb{Z})$  (see [16] for this hypothesis).  $\exp[2\pi\sqrt{-1} W_V(g)]$  can be identified canonically to an element of  $K_{\partial V, \partial G}$  where  $K$  is an Hermitian line bundle over the set of  $C^r$  maps from  $\partial V$  into  $G$ . Let  $\partial V_i$  be the oriented connected components of  $\partial V$ . We have a canonical inclusion map  $\pi_i$  from  $\partial V_i$  in  $\partial V$ . We deduce from it a map  $\bar{\pi}_i$  from the set of maps from  $\partial V$  in  $G$  into the set of maps from  $\partial V_i$  into  $G$ . Let  $\Lambda_i$  be the hermitian bundle on the set of maps from  $\partial V_i$  into  $G$  constructed in [16].  $K = \otimes \bar{\pi}_i^* \Lambda_i$  endowed with its natural metric inherited from each  $\Lambda_i$ . We denote it  $\otimes_{exit} \Lambda \otimes_{in} \Lambda$ . Moreover, we can realize this expression as a map from the tensor products of Hermitian line bundle  $\Lambda$  over the exit loop groups defined by restricting the field over each exit boundary to the tensor product of Hermitian line bundles  $\Lambda$  over the input loop groups defined by restricting the field over each connected component of the input boundary. Therefore  $\exp[2\pi\sqrt{-1} W_V(g)]$  can be realized as an application from  $\otimes_{exit} \Lambda$  into  $\otimes_{in} \Lambda$  of modulus 1. This application is consistent with the operation of sewing surface.

Let  $V_{tot}$  and the restriction of  $g_{V_{tot},1}(\cdot)$  to one connected component of the boundary of  $V_{tot}$ . Let  $\Xi'$  be the Hilbert space of section of  $\Lambda$  over the  $C^r$  loop group  $L^r(G)$  endowed with the law arising from restricting the field to one boundary loops. Let  $V_i$  be such a boundary loop. The laws of  $g_{V_{tot},1}(\cdot)$  restricted to each  $V_i$  are the same. Let  $\Psi_i(g_{V_{tot},1}(\cdot)|_{V_i})$  a section of  $\Lambda$  on the set of loops defined by  $V_i$ .  $|\Psi_i(g_{V_{tot},1}(\cdot)|_{V_i})|$  denotes a random variable which is  $g_{V_{tot},1}(\cdot)|_{V_i}$  measurable, where  $g_{V_{tot},1}(\cdot)|_{V_i}$  denotes the restriction to the random field to  $V_i$ . We put

$$(2.12) \quad \|\Psi_i\|_{\Xi'_i}^2 = E[|\Psi_i(g_{V_{tot},1}(\cdot)|_{V_i})|^2].$$

But the previous Hilbert norm don't depend of the chosen boundary loop  $V_i$ , and we get the definition of the Hilbert space  $\Xi'_i$ . Let  $L^2([0, 1] \times V_i)$  be the Hilbert space of  $L^2$  functionals with respect of  $g_{V_{tot},1}(\cdot)$  restricted to  $[0, 1] \times V_i$ . We put  $\Xi_i = \Xi'_i \otimes L^2([0, 1] \times V_i)$ . We get always the same Hilbert space  $\Xi$  independent of the choice of  $V_{tot}$ . If  $g_{V_{tot},1}(z)$  is the random map from  $V_{tot}$  into  $G$ , we deduce the random map from  $[0, 1] \times V_{tot}$  into  $G$ ,  $(t, z) \rightarrow g_{V_{tot},1}(z)$  with the the boudary condition  $g_{V_{tot},0}(z) = e$ . We deduce from this the Wess-Zumino term  $\exp[2\pi\sqrt{-1} W_{V_{tot}}(g_{V_{tot},1})]$ . Let us recall that the cinetic term in the W.Z.N.W. model is equal to  $\exp[-I(g)]$  where  $I(g)$  is the energy from the map  $g$  from  $V_{tot}$  into  $G$ . Compare with (2.9) and (2.10). In this work,

we will consider the same topological term and we will consider another cinetic term given by the law of  $g_{V_{tot},1}(\cdot)$ .

DEFINITION II.2.  $H(V_{tot}, g_{V_{tot}})$  is the operator from  $\otimes_{out} \Xi$  into  $\otimes_{in} \Xi$  where we put the tensor product along respectively the connected components of the exit boundary of  $V_{tot}$  and of the input boundaries of  $V_{tot}$  defined as follows: let  $\Psi_i$  a section of  $\Lambda$  at the  $i^{th}$  connected component of the exit boundary:

$$(2.13) \quad H(V_{tot}, g_{V_{tot}}) \otimes_{out} \Psi_i = E[\exp[2\pi\sqrt{-1} W_{V_{tot}}(g_{V_{tot},1})] \otimes_{out} \Psi_i | B'([0, 1] \times \Sigma_1)]$$

where  $B'([0, 1] \times \Sigma_1)$  is the  $\sigma$ -algebra spanned by the random field  $g_{V_{tot},\cdot}(\cdot)$  restricted to the input data  $[0, 1] \times \Sigma_1$ .

Let  $(V_{tot}^1, g_{V_{tot}^1}^1)$  and  $(V_{tot}^2, g_{V_{tot}^2}^2)$  and  $(W_{tot}, g_{W_{tot}})$  got by sewing  $V_{tot}^1$  along some exit boundaries coinciding with some input boundaries of  $V_{tot}^2$ . We call the sewing boundary  $\tilde{\Sigma}$  in  $W_{tot}$ . We call  $B([0, 1] \times \tilde{\Sigma})$  the sigma algebra defined by (4.2) for the random field  $(t, z) \rightarrow g_{W_{tot},t}(z)$  parametrized by  $[0, 1] \times W_{tot}$ . From Theorem IV.2, it satisfies (4.4). We deduce:

**Theorem II.3.** *We have:*

$$(2.14) \quad H(W_{tot}, g_{W_{tot}}) = H(V_{tot}^1, g_{V_{tot}^1}^1) \circ H^2(V_{tot}^2, g_{V_{tot}^2}^2)$$

where the composition goes for the Hilbert spaces which arises from the sewing boundaries.

Proof. Let  $\Sigma_s$  be the sewing boundary in  $W_{tot}$ . We get almost surely:

$$(2.15) \quad \begin{aligned} & \exp[2\pi\sqrt{-1} W_{W_{tot}}(g_{W_{tot},1})] \\ & = \exp[2\pi\sqrt{-1} W_{V_{tot}^1}(g_{V_{tot}^1,1})] \circ \exp[2\pi\sqrt{-1} W_{V_{tot}^2}(g_{V_{tot}^2,1})]. \end{aligned}$$

By Markov property, the two term in the right hand side of (2.15) are conditionnally independent to  $B'([0, 1] \times \Sigma_s)$ . We have:

$$(2.16) \quad \begin{aligned} & H(W_{tot}, g_{W_{tot}}) \otimes_{out} \Psi_i \\ & = E[E[\exp[2\pi\sqrt{-1} W_{W_{tot}}(g_{W_{tot},1})] \otimes_{out} \Psi_i | B'(\Sigma_1 \cup \Sigma_s)] | B'([0, 1] \times \Sigma_1)]. \end{aligned}$$

But we have:

$$(2.17) \quad \begin{aligned} & E[\exp[2\pi\sqrt{-1} W_{W_{tot}}(g_{W_{tot},1})] | B'(\Sigma_1 \cup \Sigma_s)] = E[\exp[2\pi\sqrt{-1} W_{V_{tot}^1}(g_{V_{tot}^1,1})] \\ & \circ \exp[2\pi\sqrt{-1} W_{V_{tot}^2}(g_{V_{tot}^2,1})] \otimes_{out} \Psi_i | B'([0, 1] \times (\Sigma_1 \cup \Sigma_s))]. \end{aligned}$$

By (4.4), where we choose as  $O$  the interior of  $V_{tot}^1$  in  $W_{tot}$ , the right hand-side in (2.17) is equal to

$$(2.18) \quad E[\exp[2\pi\sqrt{-1} W_{V_{tot}^1}(g_{V_{tot}^1,1})]](E[\exp[2\pi\sqrt{-1} W_{V_{tot}^2}(g_{V_{tot}^2,1})]] \otimes_{out} \Psi_i] | B'([0, 1] \times \Sigma_S) | B'([0, 1] \times \Sigma_1).$$

In (4.4), this decomposition formula is true for functionals, but we can come back to this case in (2.17) by introducing an orthonormal basis of  $\Xi$ . □

If  $(V_{tot}, g_{V_{tot}})$  have only one connected component in the input boundary and  $n$  connected component in the output boudary, we say that  $(V_{tot}, g_{V_{tot}})$  belongs to  $E(n)$ . An element of  $E(n)$  realizes an element of  $Hom(\Xi^{\otimes n}, \Xi)$ .

In particular, we will consider as  $E(n)$  the case of  $1+n$ -punctured sphere  $S_{tot}(1, n)$  with one input loop and  $n$  output loops, by taking care of the history where we glue some subspheres in  $S_{tot}(1, n)$ .  $E(n)$  is very similar to the space of trees with one root and  $n$  exit vertices. Trees are an archetype of an operad: if  $A(n)$  denotes the space of trees with  $n$ -exit vertices, we get an operation of  $A(n) \times A(r_1) \times \dots \times A(r_n)$  by grafting trees. This action is compatible with the action of symmetric group got by relabelling the exit vertices.  $E(n)$  should correspond to the parameter set of a branching process on the loop space, time of branching being included: the branching mechanism is got when a loop splitts in two loops (and not by creating two loops as it is classical in branching process theory).  $E(n)$  inherits an action of the symmetric group by labelling the connected components of the exit boudary. The action of sewing punctured spheres realizes a map from  $E(n) \times E(r_1) \times \dots \times E(r_n)$  into  $E(\sum r_i)$  which is compatible with the action of the symmetric group. We say that  $E(n)$  is an operad. On the other hand,  $Hom(\Xi^n, \Xi)$  realizes clearly an operad, by composition of the homomorphisms. We get from Theorem II.3:

**Theorem II.4.** *If  $(W_{tot}, g_{W_{tot}})$  belongs to  $E(n)$ ,  $H(W_{tot}, g_{W_{tot}})$  realizes a map from the operad  $E(n)$  into the operad  $Hom(\Xi^{\otimes n}, \Xi)$ .*

If we consider the case of the punctured sphere, this corresponds to a kind of Branching process on the loop space.

### III. Stochastic Chern-Simons theory

We consider now as  $(V, g_V)$  the case of an oriented 3-dimensional manifold  $V$  with boundaries having connected components some oriented Riemannian surfaces  $(\Sigma_i, g_{\Sigma_i})$ . The input boundaries are called  $\Sigma_i^{in}$  and the output boundaries are called  $\Sigma_i^{out}$ . This means that  $V$  realizes a bordism from  $\bigcup \Sigma_i^{in}$  into  $\bigcup \Sigma_i^{out}$ . We can find a 3-dimensional manifold whose boundary is  $\Sigma_i$ . Let us consider  $\bar{V}$  got from  $V$  by sewing these 3-dimensional manifolds to each  $\Sigma_i$ .  $\bar{V}$  has a Riemannian metric inher-



ited from  $\Sigma_i$ . Let  $\Delta_{\overline{V}}$  be the Hodge Laplacian operating on 1-forms on  $\overline{V}$  with values in the Lie algebra of a compact simply connected Lie group  $G$ , endowed with the natural Killing metric. We introduce the Sobolev space  $H_{\overline{V}}$  of 1-form  $\omega$  with values in  $Lie(G)$  such that:

$$(3.1) \quad \int_{\overline{V}} \langle (\Delta_{\overline{V}}^k + 1)\omega, (\Delta_{\overline{V}}^k + 1)\omega \rangle dm_{\overline{V}} < \infty.$$

We denote by  $\omega_{\overline{V}}$  the centered Gaussian measure in  $H_{\overline{V}}$ . If  $k$  is big enough,  $\omega_{\overline{V}}(z)$  is almost surely a 1-form which is  $C^r$ .

Let  $\overline{\Sigma}$  got from  $[0, 1] \times \Sigma$  by sewing these 3-dimensional manifolds along the boundary.  $\overline{\Sigma}$  inherits a canonical metric from the metric on  $\Sigma$ . Let  $\Delta_{\overline{\Sigma}}$  be the Laplacian operating on 1-form on  $\overline{\Sigma}$  with values in  $Lie(G)$ . Let  $H_{\overline{\Sigma}}$  be the Hilbert Sobolev space of 1-forms  $\omega$  on  $\overline{\Sigma}$  with values in  $Lie(G)$  such that:

$$(3.2) \quad \int_{\overline{\Sigma}} \langle (\Delta_{\overline{\Sigma}}^k + 1)\omega, (\Delta_{\overline{\Sigma}}^k + 1)\omega \rangle dm_{\overline{\Sigma}} < \infty.$$

We consider the centered Gaussian measure on  $H_{\overline{\Sigma}}$ . This gives a random 1-form  $\omega_{\overline{\Sigma}}$  which is  $C^r$  if  $k$  is big enough.

Let  $g_V(z)$  be a map from  $V$  into  $[0,1]$  equal to 1 on  $V$  where we have removed the output collars  $[0, 1/2[ \times \Sigma_i^{out}$  and where we have removed the input collars  $]1/2; 1] \times \Sigma_i^{in}$ . We suppose that  $g_V$  is equal to zero on a neighborhood of the boudaries of  $V$ .

Let  $g^{out}$  be a smooth map from  $[0, 1/2]$  into  $[0, 1]$  equal to 0 only in 0 and equal to 1 in a neighborhood of  $1/2$ . Let  $g^{in}$  be a smooth map from  $[1/2, 1]$  equal to 0 only in 1 and equal to 1 in a neighborhood of  $1/2$ .

Let  $V$  be constructed as above. We consider the Gaussian random field:

$$(3.3) \quad \omega_V = g_V \omega_{\overline{V}} + \sum_{in} g^{in} \omega_{\overline{\Sigma}_i^{in}} + \sum_{out} g^{out} \omega_{\overline{\Sigma}_i^{out}}$$

where we take independent  $\omega_{\overline{V}}$ ,  $\omega_{\overline{\Sigma}_i^{in}}$  and  $\omega_{\overline{\Sigma}_i^{out}}$ .  $\omega_V$  is a random  $C^r$  1-form on  $V$  with values in  $Lie(G)$ .

Let us consider the trivial bundle  $V \times G$  on  $V$ . By this trivialization,  $\omega_V$  realizes a random  $C^r$  connection on this bundle.

An object  $V_{tot,k} = (V_1 \cup V_2 \dots \cup V_k)$  is constructed inductively as follows:  $V_1$  is a 3-dimensional oriented Riemannian manifold constructed as before.  $V_{tot,k+1}$  is constructed from  $V_{tot,k}$  where we sew some exit boudaries of  $V_{tot,k}$  along some input boundaries of  $V_{k+1}$ .

We can construct inductively  $\omega_{V_{tot,k+1}}$  as follows: if  $k = 1$ , it is  $\omega_V$ .  $\omega_{V_{k+1}}$  is constructed from Gaussian fields independent of those which have constructed  $\omega_{V_{tot,k}}$ , except for the Gaussian fields in the input boundaries of  $\omega_{V_{tot,k}}$  which coincide with the Gaussian fields in the output boudaries of  $V_{tot,k}$  which are sewed to the corresponding

input boudaries of  $V_{k+1}$ .

**Theorem III.1.** *If  $k$  is big enough,  $\omega_{V_{tot}}$  is almost surely  $C^r$ .*

Let us recall some background about the Chern-Simons functional (see [16]). If  $\Sigma_i$  is connected, we can construct an Hermitian line bundle  $\Lambda(\Sigma_i)$  over the set of  $C^r$  connection over  $\Sigma_i$  of the trivial bundle  $\Sigma_i \times G$  on  $\Sigma_i$ . Let us do the following hypothesis: let  $\sigma$  be the invariant 3-form on  $G$  which is equal to  $\sigma(X, Y, Z) = \langle X, [Y, Z] \rangle$  at the level of the Lie algebra. Let us suppose that  $1/6\sigma$  represents an element of  $H^3(G; Z)$ .

Under this hypothesis, it is possible to define as it was used for instance in [16] the Chern-Simons functional  $\exp[2\pi\sqrt{-1} C_{C,S}(\omega_V)]$  where  $\omega_V$  is a connection on  $V$  as a linear application of modulus one from  $\otimes_{out}\Lambda(\Sigma_i^{out})(\omega_{\Sigma_i^{out}})$  into  $\otimes_{in}\Lambda(\Sigma_i^{in})(\omega_{\Sigma_i^{in}})$  where we restrict the connection  $(\omega_V)$  to the input and output boundaries  $\Sigma_i$  of  $V$ . We call  $\omega_{\Sigma_i}$  these restrictions. These operations are consistent with the operation of sewing 3-dimensional manifolds.

Let us recall, if  $V$  has no boundary, that the Chern-Simons action is equal to

$$(3.4) \quad \frac{k}{2\pi} \int_V Tr \left[ \omega_V \wedge d\omega_V + \frac{2}{3} \omega_V \wedge \omega_V \wedge \omega_V \right]$$

wherc  $Tr$  is got by imbedding the Lie group  $G$  into  $SO(n)$  for some big convenient  $n$ .

Let  $H(\Sigma, g_\Sigma)$  the Hilbert space of sections of  $\Lambda(\Sigma)$  for the measure got by restricting  $\omega_\Sigma$  to  $\Sigma$ .

DEFINITION III.2.  $H(V_{tot}, g_{V_{tot}})$  is the operator from  $\otimes_{out}H(\Sigma_i^{out}, g_{\Sigma_i^{out}})$  into the Hilbert space  $\otimes_{in}H(\Sigma_i^{in}, g_{\Sigma_i^{in}})$  defined as follows: let  $\Psi_i^{out}$  belonging to  $H(\Sigma_i^{out}, g_{\Sigma_i^{out}})$ :

$$(3.5) \quad H(V_{tot}, g_{V_{tot}}) \otimes_{out} \Psi_i^{out} = E \left[ \exp[2\pi\sqrt{-1} S_{C,S}(\omega_{V_{tot}})] \otimes \Psi_i^{out}(\omega_{\Sigma_i^{out}}) | B'(\cup \Sigma_i^{in}) \right]$$

where  $B'(\cup \Sigma_i^{in})$  is the  $\sigma$ -algebra spanned by the restriction  $\omega_V$  to the union of input boudaries  $\Sigma_i^{in}$ .

Let  $(V_{tot}^1, g_{V_{tot}^1})$  and  $(V_{tot}^2, g_{V_{tot}^2})$  and  $(W_{tot}, g_{W_{tot}})$  got by sewing  $V_{tot}^1$  and  $V_{tot}^2$  along some exit boundaries from  $V_{tot}^1$  and some input boundaries of  $V_{tot}^2$ . Since the stochastic Chern-Simons functional  $\exp[2\pi\sqrt{-1} S_{C,S}(\omega_{V_{tot}^i})]$  is measurable for the  $\sigma$ -algebra spanned by the fields  $\omega_{V_{tot}^i}$ , we deduce from Theorem IV.2:

**Theorem III.3.** *We have:*

$$(3.6) \quad H(W_{tot}, g_{W_{tot}}) = H(V_{tot}^1, g_{V_{tot}^1}) \circ H^2(V_{tot}^2, g_{V_{tot}^2})$$

where the composition goes from the Hilbert spaces which arise from the sewing boundary.

If  $(V_{tot}, g_{V_{tot}})$  has only one connected component in the input boundary  $(\Sigma, g_\Sigma)$  and  $n$ -connected component in the output boundary constituted of the same  $(\Sigma, g_\Sigma)$ , we say that we have an element of  $E_n(\Sigma, g_\Sigma)$ . The collection of  $E_n(\Sigma, g_\Sigma)$  constitutes an operad when  $(\Sigma, g_\Sigma)$  is fixed. We put  $\Xi = H(\Sigma, g_\Sigma)$ . An element of  $E_n(\Sigma, g_\Sigma)$  realizes an element of  $Hom(\Xi^{\otimes n}, \Xi)$ .

**Theorem III.4.** *If  $(W_{tot}, g_{W_{tot}})$  belongs to  $E_n(\Sigma, g_\Sigma)$ ,  $H(W_{tot}, g_{W_{tot}})$  realizes a map from the operad  $E_n(\Sigma, g_\Sigma)$  into the operad  $Hom(\Xi^{\otimes n}, \Xi)$ .*

**IV. Appendix**

This appendix constitutes a brief review concerning the Markov property for Gaussian random fields. We refer to [29] and references therein for more details.

$(\Omega, F, P)$  be a probability space, and  $X(z)$  a Gaussian continuous centered random field with parameter space a finite manifold  $T$  endowed with a Riemannian distance  $d$ .

If  $O$  is an open subset of  $T$ , we define

$$(4.1) \quad B(O) = \sigma(X(z); z \in O)$$

anf for an closed subset  $D$ , we define

$$(4.2) \quad B(D) = \bigcap_{\epsilon > 0} B(D_\epsilon)$$

where  $D_\epsilon = \{z \in T : \inf_{z' \in D} d(z, z') < \epsilon\}$ .

DEFINITION IV.1. A random field has the Markov property with respect to an open set  $O$  if for any  $B(\overline{O})$ -measurable functional  $\psi$ :

$$(4.3) \quad E[\psi | B(O^c)] = E[\psi | B(\partial O)].$$

A random field is  $G$ -markov if it has the Markov property with respect to all open sets  $O$ .

Markov property with respect to  $O$  is equivalent to the following statement: for any event  $A_1$   $B(\overline{O})$ -measurable and for any event  $A_2$   $B(O^c)$ -measurable:

$$(4.4) \quad P(A_1 \cap A_2 | B(\partial O)) = P(A_1 | B(\partial O))P[A_2 | B(\partial O)].$$

Let us recall that the reproducing Hilbert space  $H$  of the continuous Gaussian random field is given as follows: if  $X$  is a linear random variable of the Gaussian random

field, we put:

$$(4.5) \quad f_X(z) = E[XX(z)]$$

and

$$(4.6) \quad \langle f_X, f_Y \rangle = E[XY].$$

If  $e_z(z')$  is the covariance of the Gaussian random field,

$$(4.7) \quad E[X(z)X(z')] = e_z(z')$$

we have

$$(4.8) \quad f(z) = \langle f, e_z(\cdot) \rangle$$

Let us recall ([29] Theorem 5.1):

**Theorem IV.2.** *A random continuous Gaussian field  $X$  with reproducing Hilbert space  $H$  is a Markov field if and only if the two following conditions are checked:*

- i) *For all  $f_1, f_2 \in H$  with support disjoint,  $\langle f_1, f_2 \rangle = 0$ .*
- ii) *if  $f \in H$  is such that  $f = f_1 + f_2$  with disjoint supports, then  $f_1$  and  $f_2$  belong to  $H$ .*

We have a natural generalization of Theorem IV.2 to the case where the random field takes its values in  $R^d$ .

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