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ON ISOMORPHIC POWER SERIES RINGS

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Introduction

Let $A$ and $B$ be commutative rings with an identity. In this paper we investigate the following question raised by M.J. O'Malley [4]. Can there be an isomorphism of $A$ onto $B$ whenever the formal power series rings $A[[X_1, \ldots, X_n]]$ and $B[[Y_1, \ldots, Y_n]]$ are isomorphic? We shall say that $A$ is $n$-power invariant if whenever $C$ is a ring and $A[[X_1, \ldots, X_n]] \cong C[[Y_1, \ldots, Y_n]]$, then we have $A \cong C$. A ring $A$ will be said to be strongly $n$-power invariant if whenever $C$ is a ring and $\phi$ is an isomorphism of $A[[X_1, \ldots, X_n]]$ onto $C[[Y_1, \ldots, Y_n]]$, then there exists a $C$-automorphism $\psi$ of $C[[Y_1, \ldots, Y_n]]$ such that $\phi(X_i) = \psi(Y_i)$ for $1 \leq i \leq n$. The present paper consists of three parts. In the first part we shall give a characterization of $A$-automorphisms of $A[[X_1, \ldots, X_n]]$. The second part will deal with higher derivations on a complete local ring and we shall determine a necessary and sufficient condition in order that a complete local ring $A$ is isomorphic to a formal power series ring $A_0[[X]]$. M.J. O'Malley has proved that semisimple rings (the Jacobson radical = (0)) are strongly 1-power invariant [4]. In the last part we shall show that semisimple rings are strongly $n$-power invariant for any positive integer $n$. In particular an affine domain over a field is strongly $n$-power invariant for any $n$. Next we shall prove that if $A$ and $B$ are local rings which may not be noetherian (see [2], p. 13) and $A[[X_1, \ldots, X_n]] \cong B[[Y_1, \ldots, Y_n]]$ under $\phi$, then there is either a $B$-automorphism $\psi$ of $B[[Y_1, \ldots, Y_n]]$ satisfying $\phi(X_i) = \psi(Y_i)$ for $1 \leq i \leq n$ or $A$ (resp. $B$) is isomorphic to a formal power series ring $A_0[[X]]$ (resp. $B_0[[Y]]$). From this we shall easily conclude that a local ring $A$ which may not be noetherian is either strongly $n$-power invariant for any $n$, or $A$ is isomorphic to a formal power series ring $A_0[[X]]$. Furthermore we shall show that any noetherian local ring is $n$-power invariant for any $n$.

Throughout this paper all rings are assumed to be commutative and contain an identity.

1. $A$-automorphisms of $A[[X_1, \ldots, X_n]]$

We denote the Jacobson radical of a ring $A$ by $\mathfrak{J}(A)$. In this section let
us suppose that a ring $A$ satisfies the condition $\bigcap_{m=1}^{\infty} \mathfrak{P}(A)^m = (0)$. As is well-known we have $\bigcap_{m=1}^{\infty} \mathfrak{P}(A)^m = (0)$ when $A$ is noetherian.

**Proposition 1.** Let $B$ be a ring and let $\varphi$ be an isomorphism of $A[[X_1, \ldots, X_n]]$ onto $B[[Y_1, \ldots, Y_n]]$. Let $\varphi(X_i) = b_i + b_i Y_1 + \cdots + b_n Y_n + \cdots$ for $1 \leq i \leq n$, where $b_i, b_{ij} \in B$. We set $B = (b_1, \ldots, b_n)$, the ideal of $B$ generated by $b_1, \ldots, b_n$. Then we have

1. $\bigcap_{m=1}^{\infty} \mathfrak{P}(B)^m = (0)$ and $\mathfrak{B} \subseteq \mathfrak{P}(B),$

2. $B$ is complete in the $B$-adic topology,

3. for any power series $\sum a_{i_1, \ldots, i_n} X_1^{i_1} \cdots X_n^{i_n} \in A[[X_1, \ldots, X_n]]$, $\sum \varphi(a_{i_1, \ldots, i_n}) X_1^{i_1} \cdots \varphi(X_n)^{i_n}$ is a well defined power series in $B[[Y_1, \ldots, Y_n]]$ and we have

4. $\varphi(\sum a_{i_1, \ldots, i_n} X_1^{i_1} \cdots X_n^{i_n}) = \sum \varphi(a_{i_1, \ldots, i_n}) \varphi(X_1)^{i_1} \cdots \varphi(X_n)^{i_n}$.

**Proof.** (1) Since $\mathfrak{P}(A[[X_1, \ldots, X_n]]) = \mathfrak{P}(A[[X_1, \ldots, X_n]]) + (X_1, \ldots, X_n)$ and $\bigcap_{m=1}^{\infty} \mathfrak{P}(A)^m = (0)$, we get $\bigcap_{m=1}^{\infty} \mathfrak{P}(A[[X_1, \ldots, X_n]]) = (0)$. On the other hand $\varphi(\mathfrak{P}(A[[X_1, \ldots, X_n]])) = (0)$ and hence $\bigcap_{m=1}^{\infty} \mathfrak{P}(B[[Y_1, \ldots, Y_n]]) = (0)$. Then it is easy to see that $\bigcap_{m=1}^{\infty} \mathfrak{P}(B)^m = (0)$. In order to show $\mathfrak{B} \subseteq \mathfrak{P}(B)$, we have only to prove that $b_i \in \mathfrak{P}(B)$ for $1 \leq i \leq n$. For each $b \in B$, $1 + \varphi^{-1}(b) X_i$ is a unit of $A[[X_1, \ldots, X_n]]$ and hence $\varphi(1 + \varphi^{-1}(b) X_i) = 1 + b_i + b_{i1} Y_1 + \cdots + b_{in} Y_n + \cdots$ is a unit of $B[[Y_1, \ldots, Y_n]]$. Therefore $1 + b_{i1}$ is a unit of $B$ for each $b \in B$ and so $b_i \in \mathfrak{P}(B)$ as asserted. If $B$ is $\mathfrak{B}$-adic complete, $B[[Y_1, \ldots, Y_n]]$ is complete in the $B[[Y_1, \ldots, Y_n]] + (Y_1, \ldots, Y_n)$-adic topology. Then the assertion (3) is obvious. Thus it is sufficient to prove (2). (2) We set $B = (b_1, \ldots, b_n)$, the ideal of $B$ generated by $b_1, \ldots, b_n$. The sequence of ideals $\{B_k\}$ defines a topology on $B$ which is equivalent to the $B$-adic topology on $B$. Let $\{e_k\}$ be a Cauchy sequence of $B$ in the $B$-adic topology. Then $\{e_k\}$ is a Cauchy sequence with respect to the topology defined by $\{B_k\}$. It is therefore immediate to see that there exists a subsequence $\{d_k\}$ of $\{e_k\}$ such that $d_k = \sum_{i=0}^{\infty} r_{ij} b_i^j + \cdots + r_{ik} b_k^i$ for each $k$, where $r_{ij} \in B$. Let $f_{ij} = \varphi^{-1}(r_{ij}) \in A[[X_1, \ldots, X_n]]$ and we set $f = \sum_{i=0}^{\infty} (f_{i1} X_1^i + \cdots + f_{in} X_n^i)$ which is a well defined power series in $A[[X_1, \ldots, X_n]]$. If $B^*$ is the $\mathfrak{B}$-adic completion of $B$, then we have the canonical injection $\iota: B[[Y_1, \ldots, Y_n]] \to B^*[[Y_1, \ldots, Y_n]]$. We shall identify $B[[Y_1, \ldots, Y_n]]$ with the subring $B[[Y_1, \ldots, Y_n]]$ of $B^*[[Y_1, \ldots, Y_n]]$ and for $h \in B[[Y_1, \ldots, Y_n]]$ we shall denote $\iota(h)$ by $h$. The sequence $\{\sum_{i=0}^{\infty} (r_{ij} \varphi(X_i)^i + \cdots + r_{ik} \varphi(X_k)^i)\}$ is obviously a Cauchy sequence of $B[[Y_1, \ldots, Y_n]]$ under the $B[[Y_1, \ldots, Y_n]] + (Y_1, \ldots, Y_n)$-adic topology. Hence $\sum_{i=0}^{\infty} (r_{ij} \varphi(X_i)^i + \cdots + r_{ik} \varphi(X_k)^i)$ is a well defined power series in $B^*[[Y_1, \ldots, Y_n]]$. On the other hand we have
\[ \varphi(f) = \sum_{i=0}^{n}(r_i \varphi(X_i)i + \cdots + r_{in} \varphi(X_n)i) \]
\[ = \varphi(f) - \varphi(\sum_{i=0}^{n}(f_i \cdot X_i + \cdots + f_{in}X_i)) \]
\[ = \varphi(\sum_{i=0}^{n-k+1}(f_i \cdot X_i + \cdots + f_{in}X_i)) \]
\[ = \varphi(X_i)^{k+1} + \varphi(\sum_{i=0}^{n-k+1}f_i \cdot X_i^{n-k+1}) + \cdots + \varphi(X_n)^k \cdot \varphi(\sum_{i=0}^{n-k+1}f_iX_n^{n-k+1}) \]
in \( B[[Y_1, \cdots, Y_n]] \). Hence we get
\[ \psi(f) = \sum_{i=0}^{n}(\varphi(Y_i)i + \cdots + \varphi(Y_n)i) \]
\[ = \sum_{i=0}^{n}(r_i b_i + \cdots + r_{in}b_n) + g \]
in \( B^*[[Y_1, \cdots, Y_n]] \), where \( g \in B^*[[Y_1, \cdots, Y_n]] \) and \( g \) has no constant term. Hence we see that \( \{d_\lambda \} \to c \), the constant term of \( \varphi(f) \). Since \( \varphi(f) \in B[[Y_1, \cdots, Y_n]] \) we have \( c \in B \). Thus \( \{\tilde{c}_\lambda \} \to c \) and it follows that \( B \) is complete in its \( \mathfrak{A} \)-adic topology.

**Theorem 2.** Let \( Y_i = a_i + a_{i1}X_1 + \cdots + a_{in}X_n + \cdots \in A[[X_1, \cdots, X_n]] \) for \( 1 \leq i \leq n \). We set \( \mathfrak{A} = (a_1, \cdots, a_n) \), the ideal of \( A \) generated by \( a_1, \cdots, a_n \). Then there exists an \( A \)-automorphism \( \varphi \) of \( A[[X_1, \cdots, X_n]] \) such that \( \varphi(X_i) = Y_i \) for \( 1 \leq i \leq n \) if and only if the following conditions hold:

1. \( \mathfrak{A} \subset \mathfrak{F}(A) \) and \( A \) is complete in the \( \mathfrak{A} \)-adic topology,
2. the matrix

\[
\begin{pmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{m1} & a_{m2} & \cdots & a_{mn}
\end{pmatrix}
\]
is invertible.

**Proof.** We assume that there exists an \( A \)-automorphism \( \varphi \) of \( A[[X_1, \cdots, X_n]] \) satisfying \( \varphi(X_i) = Y_i \) for \( 1 \leq i \leq n \). Then it follows from Proposition 1 that \( \mathfrak{A} \subset \mathfrak{F}(A) \) and \( A \) is complete in the \( \mathfrak{A} \)-adic topology. Let \( \varphi^{-1}(X_i) = b_i + b_{i1}X_1 + \cdots + b_{in}X_n + \cdots \) for \( 1 \leq i \leq n \). Then we get
\[ X_i = \varphi^{-1}(\varphi(X_i)) \]
\[ = a_i + a_{i1} \varphi^{-1}(X_1) + \cdots + a_{in} \varphi^{-1}(X_n) + \cdots \]
by Proposition 1 applied to an isomorphism \( \varphi^{-1} \). Comparing the coefficients of \( X \)'s we have
\[ \sum_{i=1}^{n}a_{ik}b_{kj} \equiv \delta_{ij} \pmod{\mathfrak{F}(A)} \]
where \( \delta_{ij} \) denotes the Kronecker's symbol, because the coefficients of \( X \)'s in
\( \varphi^{-1}(X_1)^{i_1} \cdots \varphi^{-1}(X_n)^{i_n}(i_1 + \cdots + i_n \geq 2) \) belong to the ideal \((b_1, \ldots, b_n) \subseteq \mathfrak{A}(A)\).

Then \( \det(a_{ij}) \det(b_{ij}) \equiv 1 \pmod{3^f(-4)} \) and hence \( \det(a_{ij}) \) is a unit of \( A \) as asserted. Conversely we assume that the conditions (1) and (2) are satisfied. Since \( A \) is complete in the \( \mathfrak{A} \)-adic topology, \( A[[X_1, \ldots, X_n]] \) is complete in its \( \mathfrak{A}[[X_1, \ldots, X_n]] \)-adic topology and hence \( \sum a_{i_1 \ldots i_n} Y_1^{i_1} \cdots Y_n^{i_n} \) is a well defined power series in \( A[[X_1, \ldots, X_n]] \). If we set \( \varphi(\sum a_{i_1 \ldots i_n} X_1^{i_1} \cdots X_n^{i_n}) = \sum a_{i_1 \ldots i_n} Y_1^{i_1} \cdots Y_n^{i_n} \), then we have \( \varphi \) is an \( A \)-endomorphism of \( A[[X_1, \ldots, X_n]] \) satisfying \( \varphi(X_i) = Y_i \) for \( 1 \leq i \leq n \). In fact we shall show that \( \varphi \) is an automorphism. Let us consider an \( A \)-endomorphism \( \tau \) of \( A[[X_1, \ldots, X_n]] \) defined by \( \tau(X_i) = X_i - a_i \) for \( 1 \leq i \leq n \). It is immediate to see that \( \tau \) is an automorphism and hence we have only to show that \( \varphi \tau \) is an automorphism in order to complete our proof. Since \( \varphi \tau(X_i) = a_{i_1} X_1 + \cdots + a_{i_n} X_n + \cdots \) for \( 1 \leq i \leq n \), it is sufficient to prove assertion under the additional assumption: 

\[ a_{i_1} = 0 \text{ for } 1 \leq i \leq n. \]

The matrix \( (a_{ij}) \) being invertible, we can resolve

\[ X_i = b_{i_1} Y_1 + \cdots + b_{i_n} Y_n + f_i(X_1, \ldots, X_n) \]

for \( 1 \leq i \leq n \) conversely, where the non-zero terms of \( f_i(X_1, \ldots, X_n) \) are of degree \( \geq 2 \) in \( X_1, \ldots, X_n \). Now we have \( f_i(X_1, \ldots, X_n) = f_i(b_{i_1} Y_1 + \cdots + b_{i_n} Y_n + f_i(X_1, \ldots, X_n), \ldots, b_{j_1} Y_1 + \cdots + b_{j_n} Y_n + f_j(X_1, \ldots, X_n), \ldots, b_{n_1} Y_1 + \cdots + b_{n_n} Y_n + f_n(X_1, \ldots, X_n)) = \sum c_{j_1 \ldots j_n} Y_1 + \cdots + c_{n_1 \ldots n_n} Y_n \). Here the non-zero terms of \( g_i(X_1, \ldots, X_n) \) are of degree \( \geq 3 \) in \( X_1, \ldots, X_n \). We repeat this procedure and eventually we can write

\[ X_i = \sum b_{i_1 \ldots i_n} Y_1^{i_1} \cdots Y_n^{i_n}. \]

Since \( a_{i_1} = 0 \) for \( 1 \leq i \leq n \), we must have \( b_{i_1 \ldots i_n} = 0 \). Then it is easy to see that \( \varphi \) is a surjection. Next we shall prove that \( \varphi \) is an injection. To the contrary, let us suppose that there is a non-zero power series \( f(X_1, \ldots, X_n) \in A[[X_1, \ldots, X_n]] \) satisfying

\[ \varphi(f(X_1, \ldots, X_n)) = f(Y_1, \ldots, Y_n) = 0. \]

Let \( k \) be the degree of first non-zero terms in \( f(X_1, \ldots, X_n) \). Since \( a_{i_1} = 0 \) for \( 1 \leq i \leq n \), we have \( f(0, \ldots, 0) = 0 \) and hence \( k > 0 \). As is \( f(Y_1, \ldots, Y_n) = 0 \), we get

\[ \sum \sum a_{i_1 \ldots i_n} a_{i_1 \ldots i_n} X_1 + \cdots + a_{i_1 \ldots i_n} X_n)^{i_1} \cdots (a_{i_1 \ldots i_n} X_1 + \cdots + a_{i_1 \ldots i_n} X_n)^{i_n} = 0, \]

with some \( a_{i_1 \ldots i_n} \neq 0 \). Now the matrix \( (a_{ij}) \) is invertible by our assumption and therefore we have

\[ A[[X_1, \ldots, X_n]] = A[a_{i_1} X_1 + \cdots + a_{i_n} X_n, \ldots, a_{n_1} X_1 + \cdots + a_{n_n} X_n]. \]

This implies that \( a_{i_1} X_1 + \cdots + a_{i_n} X_n, \ldots, a_{n_1} X_1 + \cdots + a_{n_n} X_n \) are algebraically independent over \( A \) by the proof of (1.1) in [1]. Thus we obtain a contradiction and our proof is complete.

2. A condition that a complete local ring is isomorphic to a formal power series ring

Let \( A \) be a ring. A higher derivation on \( A \) is an infinite sequence of endomorphisms \( D = \{ \delta_0, \delta_1, \delta_2, \ldots \} \) of the underlying additive group of \( A \) satisfying the conditions: (1) \( \delta_0 = \) the identity mapping of \( A \) and (2) \( \delta_n(ab) = \sum \delta_i(a) \delta_j(b) \) for any \( a, b \in A \) and \( n \).

**Lemma 1.** Let \( A \) be a ring and let \( D = \{ \delta_0, \delta_1, \delta_2, \ldots \} \) be an infinite se-
The following conditions are equivalent:

1. \( D \) is a higher derivation on \( A \).
2. The mapping \( \phi: a \mapsto \delta_0(a) + \delta_1(a)t + \delta_2(a)t^2 + \cdots \) is a ring homomorphism of \( A \) into \( A[[t]] \) such that \( \pi\phi(a) = a \) for every \( a \in A \) where \( \pi \) is the homomorphism: \( \sum a_i t^i \mapsto a_0 \).

Proof. The equivalence between (1) and (2) is nothing but a reformulation of the definition.

**Lemma 2.** Let \( A \) be a ring and let \( \mathfrak{A} \) be an ideal of \( A \) such that \( \bigcap_{m=1}^{\infty} \mathfrak{A}^m = (0) \). Suppose that \( A \) is complete in the \( \mathfrak{A} \)-adic topology and let \( D = \{ \delta_0, \delta_1, \delta_2, \ldots \} \) be a higher derivation on \( A \). We assume that there exists an element \( u \in \mathfrak{A} \) such that \( \delta_i(u) = 1 \) and \( \delta_i(u) = 0 \) for \( i \geq 2 \). Then \( A \) contains a subring \( A_0 \) having the following properties: (1) \( u \) is analytically independent over \( A_0 \) and (2) \( A \) is the power series ring \( A_0[[u]] \).

Proof. The mapping \( \sigma: A \to A \), given by \( \sigma(a) = \sum_{i=0}^{\infty} (-1)^i \delta_i(a) u^i \) is a ring homomorphism. We put \( \text{Im}(\sigma) = A_0 \). \( A_0 \) is a subring of \( A \). From the definition of \( \sigma \) it follows that \( a = \sigma(a) + \delta_0(a) u - \delta_1(a) u^2 + \cdots \) for \( a \in A \). Similarly we see \( \delta_0(a) = \sigma(\delta_0(a)) + \delta_1(a) u - \delta_2(a) u^2 + \cdots \) and therefore we can write \( a = \sigma(a) + \delta(\delta_0(a)) u + (\delta_1(a) - \delta_2(a)) u^2 + (-\delta_3(a) + \delta_4(a)) u^3 + \cdots \). Proceeding in this way we have \( a = \sum_{i=0}^{\infty} a_i u^i \) with \( a_i \in A_0 \). Next we shall prove that \( u \) is analytically independent over \( A_0 \). Since \( \delta_i(u) = 1 \) and \( \delta_i(u) = 0 \) for \( i \geq 2 \), we get \( u \in \ker(\sigma) \). For \( a \in A_0 \) there exists \( b \in A \) such that \( a = \sigma(b) = b - \delta_0(b) u + \delta_2(b) u^2 - \cdots \). Thus it follows that \( a = b - ur \) for some \( c \in A \). If \( a \in \ker(\sigma) \cap A_0 \), we obtain \( b = a + uc \in \ker(\sigma) \) and hence \( a = \sigma(b) = 0 \). Let us suppose that \( \sum_{i=0}^{\infty} a_i u^i = 0 \) with \( a_i \in A_0 \). Since \( a_0 = - (\sum_{i=0}^{\infty} a_i u^i) u \) and \( u \in \ker(\sigma) \), we have \( a_0 \in \ker(\sigma) \cap A_0 = (0) \). By induction it will be shown that all \( a_i = 0 \). If we assume \( a_i = 0 \) for \( 0 \leq i \leq n \), we get \( 0 = a_{n+1} u^{n+1} + a_{n+2} u^{n+2} + \cdots \). Then we have \( 0 = \delta_{n+1}(a_{n+1} u^{n+1} + a_{n+2} u^{n+2} + \cdots) = a_{n+1} + ub \) for some \( b \in A \) and therefore \( a_{n+1} \in \ker(\sigma) \cap A_0 = (0) \) as desired. Hence \( A \) is the power series ring \( A_0[[u]] \).

An ideal \( \mathfrak{N} \) of a ring \( A \) is said to be differential if we have \( \delta_i(\mathfrak{N}) \subseteq \mathfrak{N} \) for every higher derivation \( \{ \delta_0, \delta_1, \delta_2, \ldots \} \) on \( A \).

**Theorem 3.** A complete local ring \( A \) is isomorphic to a formal power series ring \( A_0[[X]] \) if and only if the maximal ideal \( \mathfrak{M} \) of \( A \) is not differential.

Proof. We assume that \( A \) is isomorphic to a formal power series ring \( A_0[[X]] \). Then \( A_0 \) is a complete local ring. Let \( \mathfrak{M}_0 \) be the maximal ideal of \( A_0 \). It is well-known that the maximal ideal of \( A_0[[X]] \) is \( \mathfrak{M}_0[[X]] + (X) \). We consider a mapping \( \delta_n \) of \( A_0[[X]] \) into itself defined by \( \delta_n(\sum_{i=0}^{\infty} a_i X^i) = \sum_{i=0}^{\infty} (\delta_i a_i) X^{i-n} \) where \( (\delta_i) = 0 \) for \( i < n \). It is easy to see that \( \{ \delta_0, \delta_1, \delta_2, \ldots \} \) is a
higher derivation on $A_0[[X]]$. Since $\delta(X)=1$, the ideal $\mathfrak{M}_0[[X]]+(X)$ is not differential and hence $\mathfrak{M}$ is so. Conversely we assume that the maximal ideal $\mathfrak{M}$ of $A$ is not differential. Then exists a higher derivation $\{\delta_0, \delta_1, \delta_2, \cdots\}$ on $A$ such that $\delta_i(u)$ is a unit of $A$ for some $u \in \mathfrak{M}$. By Lemma 1 the mapping $\phi: a \to \sum_{i=0}^\infty \delta_i(a)t^i$ is a ring homomorphism of $A$ into the power series ring $A[[t]]$. We shall set $s=\delta_1(u)t+\delta_2(u)t^2+\cdots$. Since $\delta_i(u)$ is a unit of $A$, we can resolve $t=u_s+u_s^2+\cdots (u_i \in A)$ conversely, where $u_i=\delta_i(u)^{-1}$ is a unit of $A$. Obviously $s$ is analytically independent over $A$ and we have $A[[t]]=A[[s]]$. For $a \in A$ we shall define $d_n(a) \in A$ by the following identity:

$$a+\delta_1(a)t+\delta_2(a)t^2+\cdots+\delta_n(a)t^n+\cdots = a+\delta_1(a)(u_s+u_s^2+\cdots)+\delta_2(a)(u_s+u_s^2+\cdots)^2+\cdots = a+d_1(a)s+d_2(a)s^2+\cdots+d_n(a)s^n+\cdots.$$  

Then the mapping $\psi: a \to a+d_1(a)s+d_2(a)s^2+\cdots$ is a ring homomorphism of $A$ into $A[[s]]$. It follows from Lemma 1 that $\{d_0=1, d_1, d_2, \cdots\}$ is a higher derivation on $A$. Since $u+\delta_1(u)t+\delta_2(u)t^2+\cdots=u+s$, we have $d_i(u)=1$ and $d_i(u)=0$ for $i \geq 2$. Hence by Lemma 2 we see that $A$ is isomorphic to a formal power series ring $A_0[[X]]$.

3. Power invariant rings and strongly power invariant rings

Let $A$ be a ring. We say that $A$ is $n$-power invariant if whenever $B$ is a ring and $A[[X_1, \cdots, X_n]]\approx B[[Y_1, \cdots, Y_n]]$, then we have $A \approx B$. $A$ is said to be strongly $n$-power invariant if whenever $B$ is a ring and $A[[X_1, \cdots, X_n]] \approx B[[Y_1, \cdots, Y_n]]$ under $\phi$, then there exists a $B$-automorphism $\psi$ of $B[[Y_1, \cdots, Y_n]]$ such that $\phi(X_i)=\psi(Y_i)$ for $1 \leq i \leq n$. We first observe that if $A$ is strongly $n$-power invariant and $A[[X_1, \cdots, X_n]] \approx B[[Y_1, \cdots, Y_n]]$ under $\phi$, there is a $B$-automorphism $\psi$ of $B[[Y_1, \cdots, Y_n]]$ such that $\phi(X_i)=\psi(Y_i)$ for $1 \leq i \leq n$ and hence $\psi^{-1}\phi$ is an isomorphism of $A[[X_1, \cdots, X_n]]$ onto $B[[Y_1, \cdots, Y_n]]$ satisfying $\psi^{-1}\phi(X_i)=Y_i$ for $1 \leq i \leq n$. Hence we have

$$A \approx A[[X_1, \cdots, X_n]]/(X_1, \cdots, X_n) \approx B[[Y_1, \cdots, Y_n]]/(Y_1, \cdots, Y_n) \approx B.$$  

Thus a strongly $n$-power invariant ring $A$ is $n$-power invariant.

**Theorem 4.** A semisimple ring $A$ (the Jacobson radical of $A=(0)$) is strongly $n$-power invariant for any $n$.

**Proof.** Let $B$ be a ring and let $\phi$ be an isomorphism of $A[[X_1, \cdots, X_n]]$ onto $B[[Y_1, \cdots, Y_n]]$. By Proposition 1 we have

* This result is essentially due to M.J. O'Malley [4].
\( \varphi(X_i) = b_i + b_{i1}Y_1 + \cdots + b_{in}Y_n + \cdots (1 \leq i \leq n), \)
\( \varphi^{-1}(Y_i) = a_iX_1 + \cdots + a_{in}X_n + \cdots (1 \leq i \leq n) \)

where \( b_i \in \mathfrak{I}(B), b_{ij} \in B, a_{ij} \in A \) and \( B \) is \((b_1, \ldots, b_n)\)-adic complete. Let \( \varphi(a_{ij}) = b_{ij} + b_{ij1}Y_1 + \cdots + b_{ijn}Y_n + \cdots \) for \( 1 \leq i, j \leq n \), where \( b_{ij}', b_{ijk} \in B \). Then by Proposition 1

\[
Y_i = \varphi(\varphi^{-1}(Y_i)) = \varphi(a_{i1})\varphi(X_1) + \cdots + \varphi(a_{in})\varphi(X_n) + \cdots
= \sum_{j=1}^{n} (b_{ij}' + \sum_{k=1}^{n} b_{ijk}Y_k + \cdots)(b_j + \sum_{k=1}^{n} b_{jk}Y_k + \cdots) + \cdots.
\]

Equating the coefficients of \( Y_i \)'s we have

\[
\sum_{j=1}^{n} b_{ij}'b_{jk} = \delta_{ik} (\text{mod. } \mathfrak{I}(B))
\]

because the coefficients of \( Y_i \)'s in \( \varphi(a)\varphi(X_1)^{i1}\cdots\varphi(X_n)^{in} (a \in A, i_1 + \cdots + i_n \geq 2) \) belong to \( \mathfrak{I}(B) \). Then it is immediate to see that the matrix \((b_{ij}')\) is invertible. Thus it follows from Theorem 2 that there exists a \( B \)-automorphism \( \psi \) of \( B[[Y_1, \ldots, Y_n]] \) satisfying \( \varphi(X_i) = \psi(Y_i) \) for \( 1 \leq i \leq n \).

**Corollary.** An affine domain \( A \) over a field is strongly \( n \)-power invariant for any \( n \).

Proof. By Hilbert's Nullstellensatz we see that \( \mathfrak{I}(A) = (0) \). Now our assertion follows from Theorem 4.

From now on we exclusively consider local rings which may not be noetherian (see [2], p. 13) and for such a ring \( A \) we denote the unique maximal ideal by \( \mathfrak{M}(A) \).

**Theorem 5.** Let \( A \) be a local ring which may not be noetherian and let \( \varphi \) be an isomorphism of \( A[[X_1, \ldots, X_n]] \) onto \( B[[Y_1, \ldots, Y_n]] \). Then we have the following facts:

1. \( B \) is a local ring which may not be noetherian.
2. There is either a \( B \)-automorphism \( \psi \) of \( B[[Y_1, \ldots, Y_n]] \) satisfying \( \varphi(X_i) = \psi(Y_i) \) for \( 1 \leq i \leq n \), or \( A \) (resp. \( B \)) contains a local ring \( A_0 \) (resp. \( B_0 \)) which may not be noetherian and an element \( a \in \mathfrak{M}(A) \) (resp. \( b \in \mathfrak{M}(B) \)) such that a (resp. \( b \)) is analytically independent over \( A_0 \) (resp. \( B_0 \)) and \( A = A_0[[a]] \) (resp. \( B = B_0[[b]] \)).

Proof. (1) It is obvious by Proposition 1.

(2) By Proposition 1 we can express

\[
\varphi(X_i) = b_i + b_{i1}Y_1 + \cdots + b_{in}Y_n + \cdots (1 \leq i \leq n),
\]
\[
\varphi^{-1}(Y_i) = a_i + a_{i1}X_1 + \cdots + a_{in}X_n + \cdots (1 \leq i \leq n)
\]

where \( a_i \in \mathfrak{M}(A) \) and \( b_i \in \mathfrak{M}(B) \) for \( 1 \leq i \leq n \). Here \( A \) is \((a_1, \ldots, a_n)\)-adic complete...
and $B$ is $(b_1, \ldots, b_n)$-adic complete. Let

$$\varphi(a_i) = b'_i + b'_i Y_1 + \cdots + b'_{in} Y_n + \cdots (1 \leq i \leq n),$$

$$\varphi(a_{ij}) = b''_{ij} + b'_{ij} Y_1 + \cdots + b'_{ijn} Y_n + \cdots (1 \leq i, j \leq n).$$

We see that $b'_i$ is in $\mathbb{M}(B)$, as is $a_i \in \mathbb{M}(A)$. If the matrix

$$\begin{pmatrix}
    b_{11} & b_{12} & \cdots & b_{1m} \\
    b_{21} & b_{22} & \cdots & b_{2m} \\
    \vdots & \vdots & \ddots & \vdots \\
    b_{n1} & b_{n2} & \cdots & b_{nm}
\end{pmatrix}
$$

is invertible, then it follows from Theorem 2 that there exists a $B$-automorphism $\psi$ of $B[[Y_1, \ldots, Y_n]]$ satisfying $\varphi(X_i) = \psi(Y_i)$ for $1 \leq i \leq n$. To the contrary we assume that matrix $(\#)$ is not invertible. From Proposition 1 we have

$$Y_i = \varphi(\varphi^{-1}(Y_i))$$

$$= \varphi(a_i) + \varphi(a_{i1}) \varphi(X_1) + \cdots + \varphi(a_{in}) \varphi(X_n) + \cdots$$

$$= (b'_i + \sum_{k=1}^{n} b'_{ik} Y_k + \cdots) + \sum_{k=1}^{n} (b''_{ij} + \sum_{h=1}^{n} b'_{ijk} Y_h + \cdots)$$

$$= (b'_i + \sum_{k=1}^{n} b'_{ik} Y_k + \cdots) + \cdots.$$

Comparing the coefficients of $Y_i$ we get

$$\sum_{k=1}^{n} b''_{ij} b'_{ik} = \delta_{ik} (\text{mod. } \mathbb{M}(B))$$

because the coefficients of $Y_i$ in $\varphi(a_i) \varphi(X_1) \cdots \varphi(X_n)^n (a \in A, i_1 + \cdots + i_n \geq 2)$ belong to $\mathbb{M}(B)$. Thus we have

$$\begin{pmatrix}
    b_{11}'' & b_{12}'' & \cdots & b_{1m}'' \\
    b_{11}'' & b_{12}'' & \cdots & b_{1m}'' \\
    \vdots & \vdots & \ddots & \vdots \\
    b_{n1}'' & b_{n2}'' & \cdots & b_{nm}''
\end{pmatrix}
= \begin{pmatrix}
    1 - b'_{11} & b'_{11} & \cdots & b'_{1m} \\
    -b'_{21} & 1 - b'_{22} & \cdots & b'_{2m} \\
    \vdots & \vdots & \ddots & \vdots \\
    -b'_{n1} & -b'_{n2} & \cdots & 1 - b'_{nm}
\end{pmatrix}
$$

(mod. $\mathbb{M}(B)$). By our assumption the matrix $(\#)$ is not invertible and so $\det (b_{ij}) \in \mathbb{M}(B)$. Since $\det (\delta_{ij} - b_{ij}) \equiv \det (b_{ij}'') \det (b_{ij})$ (mod. $\mathbb{M}(B)$), we must have $\det (\delta_{ij} - b_{ij}) \in \mathbb{M}(B)$ where $\delta_{ij}$ is the Kronecker's symbol. Hence there exists a $b_{ij} \in \mathbb{M}(B)$. Then it is easy to see that the matrix

$$\begin{pmatrix}
    1 \\
    \vdots \\
    1 \\
    b'_{i1} \cdots b'_{ij} \cdots b'_{in} \\
    1
\end{pmatrix}$$

is invertible.
is invertible. Now we shall show that \( B \) is \((b_i')\)-adic complete. Let \( \{c_k\} \) be a Cauchy sequence in \( B \) under the \((b_i')\)-adic topology. Then there is a subsequence \( \{d_k\} \) of \( \{c_k\} \) such that \( d_k = \sum_{j=0}^{r_j} b_i'^j \) for each \( k \), where \( r_j \in B \). Let \( f_j = \varphi^{-1}(r_j) \) and we set \( f = \sum_{j=0}^{r_j} a_i'^j f_j \) which is a well defined power series in \( A[[X_1, \ldots, X_n]] \), because \( A \) is \((a_1, \ldots, a_n)\)-adic complete. Then \( \varphi(f) = \sum_{j=0}^{r_j} \varphi(a_i') f_j = \sum_{j=0}^{r_j} b_i'^j + g \) in \( B^*[[Y_1, \ldots, Y_n]] \), where \( B^* \) denotes the \((b_i')\)-adic completion of \( B \) and \( g \) has no constant term. Since \( \varphi(f) \equiv B[[Y_1, \ldots, Y_n]] \), we see that \( \sum_{j=0}^{r_j} b_i'^j \in B \), that is, \( \{d_k\} \) converges in \( B \) and hence \( \{c_k\} \) converges in \( B \). Together with \( b_i' \in \mathfrak{m}(B) \), it follows from Theorem 2 that there exists a \( B \)-automorphism \( \sigma \) of \( B[[Y_1, \ldots, Y_n]] \) such that \( \sigma(Y_j) = \varphi(a_i) \) and \( \sigma(Y_k) = Y_k \) for \( k \neq j \), that is, \( \varphi(a_i) \) is analytically independent over \( B[[Y_1, \ldots, Y_{j-1}, Y_{j+1}, \ldots, Y_n]] \) and \( B[[Y_1, \ldots, Y_n]] = B[[Y_1, \ldots, Y_{j-1}, \varphi(a_i), Y_{j+1}, \ldots, Y_n]] \). We consider the following sequence of ring homomorphisms:

\[
A \xrightarrow{i} A[X_1, \ldots, X_n] \xrightarrow{\varphi} B[[Y_1, \ldots, Y_n]] = B[[Y_1, \ldots, Y_{j-1}, Y_{j+1}, \ldots, Y_n]]
\]

\[
\varphi(a_i), Y_{j+1}, \ldots, Y_n] \xrightarrow{\tau} B[[Y_1, \ldots, Y_{j-1}, \varphi(a_i), Y_{j+1}, \ldots, Y_n]]
\]

where \( \varphi(a) = a \) for \( a \in A \). \( \varphi \) is the given isomorphism, \( \tau(\varphi(a_i)) = \varphi(a_i) + t, \tau(Y_k) = Y_k + t \) for \( k \neq j, \varphi^{-1} \) is the isomorphism induced by \( \varphi^{-1} \), and \( \nu(X_i) = 0 \) for \( 1 \leq i \leq n \). We set \( \rho \) the composite of these homomorphisms. Then \( \rho \) is a ring homomorphism of \( A \) into \( A[[t]] \) such that \( \pi \rho(a) = a \) where \( \pi \) is the homomorphism: \( \sum_i a_i t^i \to a_i \). Thus we can express \( \rho(a) = a + \delta_1(a)t + \delta_2(a)t^2 + \cdots \).

Thence \( \{1, \delta_1, \delta_2, \ldots\} \) is a higher derivation on \( A \) by Lemma 1. Since \( \rho(a_i) = a_i + t \), we have \( \delta_1(a_i) = 1, \delta_j(a_i) = 0 \) for \( j \geq 2 \) and by Lemma 2 we see that \( A \) contains a subring \( A_0 \) satisfying the properties: \( a_i \) is analytically independent over \( A_0 \) and \( A = A_0[[a_i]] \). It is obvious that \( A_0 \) is a local ring which may not be noetherian. On the other hand

\[
X_l = \varphi^{-1}(\varphi(X_l)) = \varphi^{-1}(b_i) + \varphi^{-1}(b_{i_1}) + \cdots + \varphi^{-1}(b_{i_m}) \varphi^{-1}(Y_l) + \cdots
\]

We set

\[
\varphi^{-1}(b_i) = a_i' + a_{i_1}' X_1 + \cdots + a_{i_m}' X_n + \cdots (1 \leq l \leq n),
\]

\[
\varphi^{-1}(b_{i_m}) = a_{i_m}' + a_{i_1m} X_1 + \cdots + a_{i_m} X_n + \cdots (1 \leq l, m \leq n).
\]

Here \( a_i' \) is in \( \mathfrak{m}(A) \), as is \( b_i \in \mathfrak{m}(B) \). Thus

\[
X_l = (a_i' + \sum_{k=1}^n a_{i_k} X_k + \cdots) + \sum_{m=1}^n (a_{i_1m}' + \sum_{k=1}^n a_{i_1k} X_k + \cdots) + \sum_{m=1}^n (a_{i_2m} X_1 + \cdots + a_{i_m} X_n + \cdots) + \cdots.
\]
Comparing the coefficients of $X$'s we get
\[ \sum_{m=1}^n a_{im}'' a_{mh} + a_{ih} = \delta_{ih} \pmod{\mathcal{M}(A)}. \]
In the matrix notation
\[ (a_{ij}')(a_{ij}) \equiv (\delta_{ij} - a_{ij}) \pmod{\mathcal{M}(A)}. \]
Now we have $\det(\varphi^{-1}(b_{im})) \equiv \det(a_{im}) \pmod{\{X_1, \ldots, X_n\}}$. Since $\det(\varphi^{-1}(b_{im})) = \varphi^{-1}(\det(b_{im}))$ and $\det(b_{im}) \in \mathcal{M}(B)$ by our assumption, it is immediate to see that $\det(a_{im}) \in \mathcal{M}(A)$. Thus the same argument as above implies that some $a_{im} \in \mathcal{M}(A)$ and we have $A[[X_1, \ldots, X_n]] = A[[X_1, \ldots, X_{m-1}, \varphi^{-1}(b_i), X_{m+1}, \ldots, X_n]]$. Then we see that $B$ contains a subring $B_0$ satisfying the properties: $b_i$ is analytically independent over $B_0$ and $B = B_0[[b_i]]$. Obviously $B_0$ is a local ring which may not be noetherian and our proof is now complete.

**Theorem 6.** Let $A$ be a local ring which may not be noetherian. Then we have only one of the followings:

1. $A$ is strongly $n$-power invariant for any $n$.
2. $A$ is isomorphic to a formal power series ring $A_0[[X]]$.

**Proof.** We assume that $A$ is not strongly $n$-power invariant for some $n$. Then we have a ring $B$ and an isomorphism $\varphi: A[[X_1, \ldots, X_n]] \cong B[[Y_1, \ldots, Y_n]]$ such that there is never a $B$-automorphism $\psi$ of $B[[Y_1, \ldots, Y_n]]$ satisfying $\varphi(X_i) = \psi(Y_i)$ for $1 \leq i \leq n$. Now Theorem 5 implies that $A$ must be isomorphic to a power series ring $A_0[[X]]$. Conversely it is easy to see that a power series ring $A_0[[X]]$ is not strongly $n$-power invariant for any $n$.

Thus a local ring which may not be noetherian can simply be called to be strongly power invariant without reference to the number $n$ of variables.

**Corollary 1.** An artinian local ring is strongly power invariant.

**Proof.** An artinian local ring $A$ is not isomorphic to a power series ring $A_0[[X]]$ and hence $A$ is strongly power invariant.

**Corollary 2.** Let $P$ be a point on an irreducible affine algebraic curve over an algebraically closed field $k$ and let $A$ be the local ring of $P$. Then the following conditions are equivalent:

1. $P$ is a singular point.
2. The completion $\hat{A}$ is strongly power invariant.

**Proof.** Let us suppose that $P$ is non-singular. Then it is obvious that $\hat{A}$ is isomorphic to the power series ring $k[[X]]$ and hence by Theorem 6 $\hat{A}$ is not strongly power invariant. Conversely we assume that $\hat{A}$ is not strongly power invariant. Then it follows from Theorem 6 that $\hat{A}$ is isomorphic to a
formal power series ring $A_0[[X]]$. Since $\hat{A}$ is reduced and $\dim \hat{A}=1$, $A_0$ is reduced and $\dim A_0=0$. Now it is immediate to show that $A_0 \approx k$ and therefore $\hat{A} \approx k[[X]]$. Hence $P$ is non-singular.

**Corollary 3.** Let $V$ be an irreducible affine variety over a field of characteristic zero and let $A$ be the local ring of a component of the singular locus of $V$. Then the completion $\hat{A}$ is strongly power invariant.

Proof. If $\hat{A}$ is not strongly power invariant, $\hat{A}$ is isomorphic to a formal power series ring $A_0[[X]]$. Then we can obtain a contradiction by the same argument as that of Theorem 5 in [5].

**Theorem 7.** Let $A$ be a complete local ring. Then $A$ is strongly power invariant if and only if the maximal ideal $\mathfrak{m}(A)$ of $A$ is differential.

Proof. The assertion follows from Theorem 3 and Theorem 6 immediately.

**Theorem 8***) A noetherian local ring is $n$-power invariant for any $n$.

Proof. Let $A$ be a noetherian local ring. We shall prove our assertion by induction on Krull dimension of $A$. If $\dim A=0$, then $A$ is strongly power invariant by Corollary 1 of Theorem 6 and hence $A$ is $n$-power invariant for any $n$ according to the remark preceding to Theorem 4. Let us suppose $\dim A>0$. Let $B$ be a ring and let $A[[X_1, \ldots, X_n]] \approx B[[Y_1, \ldots, Y_n]]$ under $\phi$. If there exists a $B$-automorphism $\psi$ of $B[[Y_1, \ldots, Y_n]]$ such that $\psi(X_i)=\phi(Y_i)$ for $1 \leq i \leq n$, then $A\approx B$ by the remark preceding to Theorem 4. Unless such an automorphism exists, it follows from Theorem 5 that $A$(resp. $B$) is a power series ring $A_0[[a]]$ (resp. $B_0[[b]]$). Here $A_0$ and $B_0$ are local rings. Thus we have an isomorphism $A_0[[a, X_1, \ldots, X_n]] \approx B_0[[b, Y_1, \ldots, Y_n]]$. Since $\dim A_0 < \dim A$, our induction hypothesis means that $A_0$ is $n$-power invariant for any $n$. Hence we have $A_0 \approx B_0$ and $A=A_0[[a]] \approx B_0[[b]]=B$, as desired.

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References


** After this paper is completed, the author has observed that E. Hamann obtained the result: a quasi-local ring is $n$-power invariant for any $n$, in her paper “On Power Invarianee”, to appear.