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Osaka University
ON ISOMORPHIC POWER SERIES RINGS

YASUNORI ISHIBASHI

(Received June 5, 1975)

Introduction

Let $A$ and $B$ be commutative rings with an identity. In this paper we investigate the following question raised by M.J. O'Malley [4]. Can there be an isomorphism of $A$ onto $B$ whenever the formal power series rings $A[[X_1, \ldots, X_n]]$ and $B[[Y_1, \ldots, Y_n]]$ are isomorphic? We shall say that $A$ is $n$-power invariant if whenever $C$ is a ring and $A[[X_1, \ldots, X_n]] \cong C[[Y_1, \ldots, Y_n]]$, then we have $A \cong C$. A ring $A$ will be said to be strongly $n$-power invariant if whenever $C$ is a ring and $\varphi$ is an isomorphism of $A[[X_1, \ldots, X_n]]$ onto $C[[Y_1, \ldots, Y_n]]$, then there exists a $C$-automorphism $\psi$ of $C[[Y_1, \ldots, Y_n]]$ such that $\varphi(X_i) = \psi(Y_i)$ for $1 \leq i \leq n$. The present paper consists of three parts. In the first part we shall give a characterization of $A$-automorphisms of $A[[X_1, \ldots, X_n]]$. The second part will deal with higher derivations on a complete local ring and we shall determine a necessary and sufficient condition in order that a complete local ring $A$ is isomorphic to a formal power series ring $A[[X]]$. M.J. O'Malley has proved that semisimple rings (the Jacobson radical $= (0)$) are strongly 1-power invariant [4]. In the last part we shall show that semisimple rings are strongly $n$-power invariant for any positive integer $n$. In particular an affine domain over a field is strongly $n$-power invariant for any $n$. Next we shall prove that if $A$ and $B$ are local rings which may not be noetherian (see [2], p. 13) and $A[[X_1, \ldots, X_n]] \cong B[[Y_1, \ldots, Y_n]]$ under $\varphi$, then there is either a $B$-automorphism $\psi$ of $B[[Y_1, \ldots, Y_n]]$ satisfying $\varphi(X_i) = \psi(Y_i)$ for $1 \leq i \leq n$ or $A$ (resp. $B$) is isomorphic to a formal power series ring $A_0[[X]]$ (resp. $B_0[[Y]]$). From this we shall easily conclude that a local ring $A$ which may not be noetherian is either strongly $n$-power invariant for any $n$, or $A$ is isomorphic to a formal power series ring $A_0[[X]]$. Furthermore we shall show that any noetherian local ring is $n$-power invariant for any $n$.

Throughout this paper all rings are assumed to be commutative and contain an identity.

1. $A$-automorphisms of $A[[X_1, \ldots, X_n]]$

We denote the Jacobson radical of a ring $A$ by $\mathfrak{J}(A)$. In this section let
Proposition 1. Let $B$ be a ring and let $\varphi$ be an isomorphism of $A[[X_1, \cdots, X_n]]$ onto $B[[Y_1, \cdots, Y_n]]$. Let $\varphi(X_i) = b_i + b_{i1}Y_1 + \cdots + b_{in}Y_n$ for $1 \leq i \leq n$, where $b_i, b_{ij} \in B$. We set $\mathfrak{B} = (b_1, \cdots, b_n)$, the ideal of $B$ generated by $b_1, \cdots, b_n$. Then we have

1. $\bigcap_{m=1}^{\infty} \mathfrak{I}(B)^m = (0)$ and $\mathfrak{B} \subset \mathfrak{I}(B)$,
2. $B$ is complete in the $\mathfrak{B}$-adic topology,
3. for any power series $\sum a_{i1} \cdots a_{in} X_1^{i1} \cdots X_n^{in} \in A[[X_1, \cdots, X_n]]$, $\sum \varphi(a_{i1} \cdots a_{in}) \varphi(X_1)^{i1} \cdots \varphi(X_n)^{in}$ is a well defined power series in $B[[Y_1, \cdots, Y_n]]$ and we have $\varphi(\sum a_{i1} \cdots a_{in} X_1^{i1} \cdots X_n^{in}) = \sum \varphi(a_{i1} \cdots a_{in}) \varphi(X_1)^{i1} \cdots \varphi(X_n)^{in}$.

Proof. (1) Since $\mathfrak{I}(A[[X_1, \cdots, X_n]]) = \mathfrak{I}(A)[[X_1, \cdots, X_n]] + (X_1, \cdots, X_n)$ and $\bigcap_{m=1}^{\infty} \mathfrak{I}(A)^m = (0)$, we get $\bigcap_{m=1}^{\infty} \mathfrak{I}(A[[X_1, \cdots, X_n]])^m = (0)$. On the other hand $\varphi(\mathfrak{I}(A[[X_1, \cdots, X_n]])) = \mathfrak{I}(B[[Y_1, \cdots, Y_n]])$ and hence $\bigcap_{m=1}^{\infty} \mathfrak{I}(B[[Y_1, \cdots, Y_n]])^m = (0)$. Then it is easy to see that $\bigcap_{m=1}^{\infty} \mathfrak{I}(B)^m = (0)$. In order to show $\mathfrak{B} \subset \mathfrak{I}(B)$, we have only to prove that $b_i \in \mathfrak{I}(B)$ for $1 \leq i \leq n$. For each $b_i \in B$, $1 + \varphi^{-1}(b_i)X_i$ is a unit of $A[[X_1, \cdots, X_n]]$ and hence $\varphi(1 + \varphi^{-1}(b_i)X_i) = 1 + bb_i + b_{i1}Y_1 + \cdots + b_{in}Y_n + \cdots$ is a unit of $B[[Y_1, \cdots, Y_n]]$. Therefore $1 + bb_i$ is a unit of $B$ for each $b_i \in B$ and so $b_i \in \mathfrak{I}(B)$ as asserted. If $B$ is $\mathfrak{B}$-adic complete, $B[[Y_1, \cdots, Y_n]]$ is complete in the $(\mathfrak{B}[[Y_1, \cdots, Y_n]] + (Y_1, \cdots, Y_n))$-adic topology. Then the assertion (3) is obvious. Thus it is sufficient to prove (2). (2) We set $\mathfrak{B}_k = (b_1^k, \cdots, b_n^k)$, the ideal of $B$ generated by $b_1^k, \cdots, b_n^k$. The sequence of ideals $\{\mathfrak{B}_k\}$ defines a topology on $B$ which is equivalent to the $\mathfrak{B}$-adic topology on $B$. Let $\{e_k\}$ be a Cauchy sequence of $B$ in the $\mathfrak{B}$-adic topology. Then $\{e_k\}$ is a Cauchy sequence with respect to the topology defined by $\{\mathfrak{B}_k\}$. It is therefore immediate to see that there exists a subsequence $\{d_k\}$ of $\{e_k\}$ such that $d_k = \sum_{i=0}^{\infty} r_{i1}b_1^i + \cdots + r_{ik}b_k^i$ for each $k$, where $r_{ij} \in B$. Let $f_{ij} = \varphi^{-1}(r_{ij}) \in A[[X_1, \cdots, X_n]]$ and we set $f = \sum_{i=0}^{\infty} (f_{i1}X_1^i + \cdots + f_{in}X_n^i)$ which is a well defined power series in $A[[X_1, \cdots, X_n]]$. If $B^*$ is the $\mathfrak{B}$-adic completion of $B$, then we have the canonical injection $\iota: B[[Y_1, \cdots, Y_n]] \to B^*[[Y_1, \cdots, Y_n]]$. We shall identify $B[[Y_1, \cdots, Y_n]]$ with the subring $\iota(B[[Y_1, \cdots, Y_n]])$ of $B^*[[Y_1, \cdots, Y_n]]$ and for $h \in B[[Y_1, \cdots, Y_n]]$ we shall denote $\iota(h)$ by $h$. The sequence $\{\sum_{i=0}^{\infty} (f_{i1}\varphi(X_1)^i + \cdots + f_{in}\varphi(X_n)^i)\}_{k}$ is obviously a Cauchy sequence of $B[[Y_1, \cdots, Y_n]]$ under the $(\mathfrak{B}[[Y_1, \cdots, Y_n]] + (Y_1, \cdots, Y_n))$-adic topology. Hence $\sum_{i=0}^{\infty} (f_{i1}\varphi(X_1)^i + \cdots + f_{in}\varphi(X_n)^i)$ is a well defined power series in $B^*[[Y_1, \cdots, Y_n]]$. On the other hand we have
in $B[[Y_1, \ldots, Y_n]]$. Hence we get
\[ \psi(f) = \sum_{i=0}^{k} (r_i X_i^i + \cdots + r_n X_n^i) \]
where $r_i, g \in B[[Y_1, \ldots, Y_n]]$ and $g$ has no constant term. Hence we see that $\{d_k\} \to c$, the constant term of $\varphi(f)$. Since $\varphi(f) \in B[[Y_1, \ldots, Y_n]]$ we have $c \in B$. Thus $\{e_k\} \to c$ and it follows that $B$ is complete in its $\mathfrak{B}$-adic topology.

**Theorem 2.** Let $Y_i = a_{i1} + a_{i2} X_1 + \cdots + a_{in} X_n + \cdots \in A[[X_1, \ldots, X_n]]$ for $1 \leq i \leq n$. We set $\mathfrak{A} = (a_1, \ldots, a_n)$, the ideal of $A$ generated by $a_1, \ldots, a_n$. Then there exists an $A$-automorphism $\varphi$ of $A[[X_1, \ldots, X_n]]$ such that $\varphi(X_i) = Y_i$ for $1 \leq i \leq n$ if and only if the following conditions hold:

1. $\mathfrak{A} \subseteq \mathfrak{B}(A)$ and $A$ is complete in the $\mathfrak{B}$-adic topology,
2. the matrix

\[
\begin{pmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{m1} & a_{m2} & \cdots & a_{mn}
\end{pmatrix}
\]
is invertible.

**Proof.** We assume that there exists an $A$-automorphism $\varphi$ of $A[[X_1, \ldots, X_n]]$ satisfying $\varphi(X_i) = Y_i$ for $1 \leq i \leq n$. Then it follows from Proposition 1 that $\mathfrak{A} \subseteq \mathfrak{B}(A)$ and $A$ is complete in the $\mathfrak{B}$-adic topology. Let $\varphi^{-1}(X_i) = b_{i1} X_1 + \cdots + b_{in} X_n + \cdots$ for $1 \leq i \leq n$. Then we get
\[
X_i = \varphi^{-1}(\varphi(X_i)) = a_i + a_{i1} \varphi^{-1}(X_1) + \cdots + a_{in} \varphi^{-1}(X_n) + \cdots
\]
by Proposition 1 applied to an isomorphism $\varphi^{-1}$. Comparing the coefficients of $X$'s we have
\[
\sum_{i=1}^{n} a_{ik} b_{kj} \equiv \delta_{ij} \pmod{\mathfrak{B}(A)}
\]
where $\delta_{ij}$ denotes the Kronecker's symbol, because the coefficients of $X$'s in
$\varphi^{-1}(X_i)\cdots\varphi^{-1}(X_n)^{i_1+\cdots+i_n}(i_1+\cdots+i_n \geq 2)$ belong to the ideal $(b_1, \ldots, b_n) \subseteq \mathfrak{Y}(A)$. Then $\det(a_{ij})\det(b_{ij}) \equiv 1 \pmod{\mathfrak{Y}(A)}$ and hence $\det(a_{ij})$ is a unit of $A$ as asserted. Conversely we assume that the conditions (1) and (2) are satisfied. Since $A$ is complete in the $\mathfrak{Y}$-adic topology, $A[[X_1, \ldots, X_n]]$ is complete in its $(\mathfrak{Y}[[X_1, \ldots, X_n]]+(X_1, \ldots, X_n))$-adic topology and hence $\sum a_{i_1\ldots i_n}Y_1^{i_1}\cdots Y_n^{i_n}$ is a well defined power series in $A[[X_1, \ldots, X_n]]$. If we set $\varphi(\sum a_{i_1\ldots i_n}X_1^{i_1}\cdots X_n^{i_n}) = \sum a_{i_1\ldots i_n}Y_1^{i_1}\cdots Y_n^{i_n}$, then we see that $\varphi$ is an $A$-endomorphism of $A[[X_1, \ldots, X_n]]$ satisfying $\varphi(X_i) = Y_i$ for $1 \leq i \leq n$. In fact we shall show that $\varphi$ is an automorphism. Let us consider an $A$-endomorphism $\tau$ of $A[[X_1, \ldots, X_n]]$ defined by $\tau(X_i) = X_i - a_i$ for $1 \leq i \leq n$. It is immediate to see that $\tau$ is an automorphism and hence we have only to show that $\varphi \tau$ is an automorphism in order to complete our proof. Since $\varphi \tau(X_i) = a_1X_1 + \cdots + a_nX_n + \cdots$ for $1 \leq i \leq n$, it is sufficient to prove assertion under the additional assumption: $a_i = 0$ for $1 \leq i \leq n$. The matrix $(a_{ij})$ being invertible, we can resolve $X_i = b_{i_1}Y_1 + \cdots + b_{in}Y_n + f_i(X_1, \ldots, X_n)$ for $1 \leq i \leq n$ conversely, where the non-zero terms of $f_i(X_1, \ldots, X_n)$ are of degree $\geq 2$ in $X_1, \ldots, X_n$. Now we have $f_i(X_1, \ldots, X_n) = f_i(b_{i_1}Y_1 + \cdots + b_{in}Y_n + f_i(X_1, \ldots, X_n), \ldots, b_{i_1}Y_1 + \cdots + b_{in}Y_n + f_i(X_1, \ldots, X_n), \ldots, b_{i_1}Y_1 + \cdots + b_{in}Y_n + f_i(X_1, \ldots, X_n)) = \sum f_{i+j}(Y_1 + \cdots + Y_n + g_i(X_1, \ldots, X_n))$. Here the non-zero terms of $g_i(X_1, \ldots, X_n)$ are of degree $\geq 3$ in $X_1, \ldots, X_n$. We repeat this procedure and eventually we can write $X_i = \sum b_{i_1\ldots i_n}Y_1^{i_1}\cdots Y_n^{i_n}$. Since $a_i = 0$ for $1 \leq i \leq n$, we must have $b_{i_1\ldots i_n} = 0$. Then it is easy to see that $\varphi$ is a surjection. Next we shall prove that $\varphi$ is an injection. To the contrary, let us suppose that there is a non-zero power series $f(X_1, \ldots, X_n) \in A[[X_1, \ldots, X_n]]$ satisfying $\varphi(f(X_1, \ldots, X_n)) = f(Y_1, \ldots, Y_n) = 0$. Let $k$ be the degree of first non-zero terms in $f(X_1, \ldots, X_n)$. Since $\varphi \tau(X_i) = 0$ for $1 \leq i \leq n$, we have $f(0, \ldots, 0) = 0$ and hence $k > 0$. As is $f(Y_1, \ldots, Y_n) = 0$, we get $\sum a_{i_1\ldots i_n}f_i(a_1X_1 + \cdots + a_nX_n)^{i_1+\cdots+i_n}(a_1X_1 + \cdots + a_nX_n)^{i_1+\cdots+i_n}(a_1X_1 + \cdots + a_nX_n)^{i_1+\cdots+i_n} = 0$, with some $a_{i_1\ldots i_n} \neq 0$. Now the matrix $(a_{ij})$ is invertible by our assumption and therefore we have $A[X_1, \ldots, X_n] = A[a_1X_1 + \cdots + a_nX_n, \ldots, a_1X_1 + \cdots + a_nX_n]$. This implies that $a_1X_1 + \cdots + a_nX_n$ are algebraically independent over $A$ by the proof of (1.1) in [1]. Thus we obtain a contradiction and our proof is complete.

2. A condition that a complete local ring is isomorphic to a formal power series ring

Let $A$ be a ring. A higher derivation on $A$ is an infinite sequence of endomorphisms $D = \{\delta_0, \delta_1, \delta_2, \cdots\}$ of the underlying additive group of $A$ satisfying the conditions: (1) $\delta_0 =$ the identity mapping of $A$ and (2) $\delta_n(ab) = \sum_{i+j=n} \delta_i(a) \delta_j(b)$ for any $a, b \in A$ and $n$.

Lemma 1. Let $A$ be a ring and let $D = \{\delta_0, \delta_1, \delta_2, \cdots\}$ be an infinite se-
quence of mappings of $A$ into itself. Then the following conditions are equivalent:

1. $D$ is a higher derivation on $A$.
2. The mapping $\varphi: a \mapsto \delta_0(a) + \delta_1(a)t + \delta_2(a)t^2 + \cdots$ is a ring homomorphism of $A$ into $A[[t]]$ such that $\pi \varphi(a) = a$ for every $a \in A$ where $\pi$ is the homomorphism: $\sum a_i t^i \mapsto a_0$.

Proof. The equivalence between (1) and (2) is nothing but a reformulation of the definition.

Lemma 2. Let $A$ be a ring and let $\mathfrak{A}$ be an ideal of $A$ such that $\bigcap_{m=1}^{\infty} \mathfrak{A}^m = (0)$. Suppose that $A$ is complete in the $\mathfrak{A}$-adic topology and let $D = \{\delta_0, \delta_1, \delta_2, \ldots\}$ be a higher derivation on $A$. We assume that there exists an element $w \in \mathfrak{A}$ such that $\delta_i(w) = 1$ and $\delta_i(w) = 0$ for $i \geq 2$. Then $A$ contains a subring $A_0$ having the following properties: (1) $u$ is analytically independent over $A_0$ and (2) $A$ is the power series ring $A_0[[u]]$.

Proof. The mapping $\sigma: A \mapsto A$, given by $\sigma(a) = \sum_{i=0}^{\infty} (-1)^i \delta_i(a) u^i$ is a ring homomorphism. We put $\text{Im} (\sigma) = A_0$. $A_0$ is a subring of $A$. From the definition of $\sigma$ it follows that $a = \sigma(a) + \delta_i(a) u - \delta_0(a) u^2 + \cdots$ for $a \in A$. Similarly we see $\delta_i(a) = \sigma(\delta_i(a)) + \delta_i^2(a) u - \delta_i(\delta_i(a)) u^2 + \cdots$ and therefore we can write $a = \sigma(a) + \sigma(\delta_i(a)) u + (\delta_i^2(a) - \delta_i(\delta_i(a)) u^2 + (-\delta_i(\delta_i(a)) + \delta_i(a)) u^3 + \cdots$. Proceeding in this way we have $a = \sum_{i=0}^{\infty} a_i u^i$ with $a_i \in A_0$. Next we shall prove that $u$ is analytically independent over $A_0$. Since $\delta_i(u) = 1$ and $\delta_i(w) = 0$ for $i \geq 2$, we get $u \in \text{Ker}(\sigma)$. For $a \in A_0$ there exists $b \in A$ such that $a = \sigma(b) = b - \delta_i(b) u + \delta_i(b) u^2 - \cdots$. Thus it follows that $a - b - uc \in \text{Ker}(\sigma)$ for some $c \in A$. If $a \in \text{Ker}(\sigma) \cap A_0$, we obtain $b = a + uc \in \text{Ker}(\sigma)$ and hence $a = \sigma(b) = 0$. Let us suppose that $\sum_{i=0}^{n} a_i u^i = 0$ with $a_i \in A_0$. Since $a_0 = \sum_{i=0}^{n} a_i u^{i-1}$ and $u \in \text{Ker}(\sigma)$, we have $a_0 \in \text{Ker}(\sigma) \cap A_0 = (0)$. By induction it will be shown that all $a_i = 0$. If we assume $a_n = 0$ for $0 \leq i \leq n$, we get $0 = a_{n+1} u^{n+1} + a_{n+2} u^{n+2} + \cdots$. Then we have $0 = \delta_{n+1}(a_{n+1} u^{n+1} + a_{n+2} u^{n+2} + \cdots) = a_{n+1} + ub$ for some $b \in A$ and therefore $a_{n+1} \in \text{Ker}(\sigma) \cap A_0 = (0)$ as desired. Hence $A$ is the power series ring $A_0[[u]]$.

An ideal $\mathfrak{A}$ of a ring $A$ is said to be differential if we have $\delta_i(\mathfrak{A}) \subset \mathfrak{A}$ for every higher derivation $\{\delta_0, \delta_1, \delta_2, \ldots\}$ on $A$.

Theorem 3. A complete local ring $A$ is isomorphic to a formal power series ring $A_0[[X]]$ if and only if the maximal ideal $\mathfrak{M}$ of $A$ is not differential.

Proof. We assume that $A$ is isomorphic to a formal power series ring $A_0[[X]]$. Then $A_0$ is a complete local ring. Let $\mathfrak{M}_0$ be the maximal ideal of $A_0$. It is well-known that the maximal ideal of $A_0[[X]]$ is $\mathfrak{M}_0[[X]] + (X)$. We consider a mapping $\delta_n$ of $A_0[[X]]$ into itself defined by $\delta_n(\sum_{i=0}^{\infty} a_i X^i) = \sum_{i=0}^{n} (\delta_i) a_i X^{i-n}$ where $(\delta_i) = 0$ for $i < n$. It is easy to see that $\{\delta_0, \delta_1, \delta_2, \ldots\}$ is a
higher derivation on \( A[[X]] \). Since \( \delta_i(X)=1 \), the ideal \( \mathfrak{M}[[X]]+(X) \) is not differential and hence \( \mathfrak{M} \) is so. Conversely, we assume that the maximal ideal \( \mathfrak{M} \) of \( A \) is not differential. Then exists a higher derivation \( \{\delta_o, \delta_1, \delta_2, \ldots\} \) on \( A \) such that \( \delta_i(u) \) is a unit of \( A \) for some \( u \in \mathfrak{M} \). By Lemma 1 the mapping \( \varphi: a \to \sum_{i=0}^\infty \delta_i(a)t^i \) is a ring homomorphism of \( A \) into the power series ring \( A[[t]] \). We shall set \( s=\delta_i(u)t+\delta_2(u)t^2+\cdots \). Since \( \delta_i(u) \) is a unit of \( A \), we can resolve \( t=u_1s+u_2s^2+\cdots \) (\( u_i \in A \)) conversely, where \( u_i=\delta_i(u)^{-1} \) is a unit of \( A \). Obviously \( s \) is analytically independent over \( A \) and we have \( A[[t]]=A[[s]] \). For \( a \in A \) we shall define \( d_n(a) \in A \) by the following identity:

\[
\begin{align*}
a&+\delta_1(a)t+\delta_2(a)t^2+\cdots+\delta_n(a)t^n+\cdots \\
&=a+\delta_1(a)(u_1s+u_2s^2+\cdots)+\delta_2(a)(u_1s+u_2s^2+\cdots)^2+\cdots \\
&=a+d_1(a)s+d_2(a)s^2+\cdots+d_n(a)s^n+\cdots.
\end{align*}
\]

Then the mapping \( \varphi: a \to a+d_1(a)s+d_2(a)s^2+\cdots \) is a ring homomorphism of \( A \) into \( A[[s]] \). It follows from Lemma 1 that \( \{d_0, d_1, d_2, \ldots\} \) is a higher derivation on \( A \). Since \( u+\delta_i(u)t+\delta_2(u)t^2+\cdots=u+s \), we have \( d_i(u)=1 \) and \( d_i(u)=0 \) for \( i \geq 2 \). Hence by Lemma 2 we see that \( A \) is isomorphic to a formal power series ring \( A[[X]] \).

### 3. Power invariant rings and strongly power invariant rings

Let \( A \) be a ring. We say that \( A \) is \( n \)-power invariant if whenever \( B \) is a ring and \( A[[X_1, \ldots, X_n]] \cong B[[Y_1, \ldots, Y_n]] \), then we have \( A \cong B \). \( A \) is said to be strongly \( n \)-power invariant if whenever \( B \) is a ring and \( A[[X_1, \ldots, X_n]] \cong B[[Y_1, \ldots, Y_n]] \) under \( \varphi \), then there exists a \( B \)-automorphism \( \psi \) of \( B[[Y_1, \ldots, Y_n]] \) such that \( \varphi(X_i)=\psi(Y_i) \) for \( 1 \leq i \leq n \). We first observe that if \( A \) is strongly \( n \)-power invariant and \( A[[X_1, \ldots, X_n]] \cong B[[Y_1, \ldots, Y_n]] \) under \( \varphi \), there is a \( B \)-automorphism \( \psi \) of \( B[[Y_1, \ldots, Y_n]] \) such that \( \varphi(X_i)=\psi(Y_i) \) for \( 1 \leq i \leq n \) and hence \( \psi^{-1}\varphi \) is an isomorphism of \( A[[X_1, \ldots, X_n]] \) onto \( B[[Y_1, \ldots, Y_n]] \) satisfying \( \psi^{-1}\varphi(X_i)=Y_i \) for \( 1 \leq i \leq n \). Hence we have

\[
A \cong A[[X_1, \ldots, X_n]]/(X_1, \ldots, X_n) \cong B[[Y_1, \ldots, Y_n]]/(Y_1, \ldots, Y_n)
\]

\( \cong B \).

Thus a strongly \( n \)-power invariant ring \( A \) is \( n \)-power invariant.

**Theorem 4.**

A semisimple ring \( A \) (the Jacobson radical of \( A=(0) \)) is strongly \( n \)-power invariant for any \( n \).

**Proof.** Let \( B \) be a ring and let \( \varphi \) be an isomorphism of \( A[[X_1, \ldots, X_n]] \) onto \( B[[Y_1, \ldots, Y_n]] \). By Proposition 1 we have

\* This result is essentially due to M.J. O'Malley [4].
\[ \varphi(X_i) = b_i + b_{i1} Y_1 + \cdots + b_{in} Y_n + \cdots (1 \leq i \leq n), \]
\[ \varphi^{-1}(Y_i) = a_{i1} X_1 + \cdots + a_{in} X_n + \cdots (1 \leq i \leq n) \]

where \( b_i \in \mathfrak{P}(B) \), \( b_{ij} \in B \), \( a_{ij} \in A \) and \( B \) is \((b_1, \ldots, b_n)\)-adic complete. Let \( \varphi(a_{ij}) = b_{ij}' + b_{ij1} Y_1 + \cdots + b_{ijn} Y_n + \cdots \) for \( 1 \leq i, j \leq n \), where \( b_{ij}', b_{ijk} \in B \). Then by Proposition 1

\[ Y_i = \varphi(\varphi^{-1}(Y_i)) \]
\[ = \varphi(a_{i1})\varphi(X_1) + \cdots + \varphi(a_{in})\varphi(X_n) + \cdots \]
\[ = \sum_{i=1}^{n}(b_{ij} + \sum_{k=1}^{n}b_{ijk} Y_k + \cdots )(b_j + \sum_{k=1}^{n}b_{jk} Y_k + \cdots ) + \cdots . \]

Equating the coefficients of \( Y_i \)'s we have

\[ \sum_{j=1}^{n}b_{ij}'/b_{jk} = \delta_{ik} (\text{mod. } \mathfrak{P}(B)) \]

because the coefficients of \( Y_i \)'s in \( \varphi(a)\varphi(X_1)^{i1}\cdots \varphi(X_n)^{in}(a \in A, i_1 + \cdots + i_n \geq 2) \) belong to \( \mathfrak{P}(B) \). Then it is immediate to see that the matrix \((b_{ij})\) is invertible. Thus it follows from Theorem 2 that there exists a \( B \)-automorphism \( \psi \) of \( B[[Y_1, \ldots, Y_n]] \) satisfying \( \varphi(X_i) = \psi(Y_i) \) for \( 1 \leq i \leq n \).

**Corollary.** An affine domain \( A \) over a field is strongly \( n \)-power invariant for any \( n \).

Proof. By Hilbert's Nullstellensatz we see that \( \mathfrak{P}(A) = (0) \). Now our assertion follows from Theorem 4.

From now on we exclusively consider local rings which may not be noetherian (see [2], p. 13) and for such a ring \( A \) we denote the unique maximal ideal by \( \mathfrak{M}(A) \).

**Theorem 5.** Let \( A \) be a local ring which may not be noetherian and let \( \varphi \) be an isomorphism of \( A[[X_1, \ldots, X_n]] \) onto \( B[[Y_1, \ldots, Y_n]] \). Then we have the following facts:

1. \( B \) is a local ring which may not be noetherian.
2. There is either a \( B \)-automorphism \( \psi \) of \( B[[Y_1, \ldots, Y_n]] \) satisfying \( \varphi(X_i) = \psi(Y_i) \) for \( 1 \leq i \leq n \), or \( A \) (resp. \( B \)) contains a local ring \( A_0 \) (resp. \( B_0 \)) which may not be noetherian and an element \( a \in \mathfrak{M}(A) \) (resp. \( b \in \mathfrak{M}(B) \)) such that a (resp. \( b \)) is analytically independent over \( A_0 \) (resp. \( B_0 \)) and \( A = A_0[[a]] \) (resp. \( B = B_0[[b]] \)).

Proof. (1) It is obvious by Proposition 1.

(2) By Proposition 1 we can express

\[ \varphi(X_i) = b_i + b_{i1} Y_1 + \cdots + b_{in} Y_n + \cdots (1 \leq i \leq n), \]
\[ \varphi^{-1}(Y_i) = a_{i1} X_1 + \cdots + a_{in} X_n + \cdots (1 \leq i \leq n) \]

where \( a_i \in \mathfrak{M}(A) \) and \( b_i \in \mathfrak{M}(B) \) for \( 1 \leq i \leq n \). Here \( A \) is \((a_1, \ldots, a_n)\)-adic complete.
and $B$ is $(b_1, \ldots, b_n)$-adic complete. Let

$$
\varphi(a_i) = b_{i1}' + b_{i2}'Y_1 + \cdots + b_{im}'Y_m + \cdots (1 \leq i \leq n),
$$

$$
\varphi(a_{ij}) = b_{ij}'' + b_{ij3}Y_1 + \cdots + b_{ijn}Y_n + \cdots (1 \leq i, j \leq n).
$$

We see that $b_i'$ is in $\mathfrak{M}(B)$, as is $a_i \in \mathfrak{M}(A)$. If the matrix

$$
(\#)
$$

is invertible, then it follows from Theorem 2 that there exists a $B$-automorphism $\psi$ of $B[[Y_1, \ldots, Y_n]]$ satisfying $\varphi(X_i) = \psi(Y_i)$ for $1 \leq i \leq n$. To the contrary we assume that matrix $(\#)$ is not invertible. From Proposition 1 we have

$$
Y_i = \varphi(\varphi^{-1}(Y_i))
$$

$$
= \varphi(a_i) + \varphi(a_{i1})\varphi(X_1) + \cdots + \varphi(a_{in})\varphi(X_n) + \cdots
$$

$$
= (b_{i1}' + \sum_{k=1}^{m} b_{ik}'Y_k + \cdots) + \sum_{j=1}^{n} (b_{ij}'' + \sum_{k=1}^{m} b_{ijk}Y_k + \cdots)
$$

$$
(b_{ij} + \sum_{k=1}^{n} b_{jk}Y_k + \cdots) + \cdots.
$$

Comparing the coefficients of $Y$'s we get

$$
\sum_{j=1}^{n} b_{ij}''b_{jk} + b_{ik}' \equiv \delta_{ib} \pmod{\mathfrak{M}(B)}
$$

because the coefficients of $Y$'s in $\varphi(a)\varphi(X_1)^{i_1}\cdots \varphi(X_n)^{i_n}(a \in A, i_1 + \cdots + i_n \geq 2)$ belong to $\mathfrak{M}(B)$. Thus we have

$$
\begin{pmatrix}
  b_{i1}'' & b_{i2}'' & \cdots & b_{im}'' \\
  b_{i1}'' & b_{i2}'' & \cdots & b_{im}'' \\
  \vdots & \vdots & \ddots & \vdots \\
  b_{i1}'' & b_{i2}'' & \cdots & b_{im}''
\end{pmatrix}
\begin{pmatrix}
  b_{11} & b_{12} & \cdots & b_{1m} \\
  b_{21} & b_{22} & \cdots & b_{2m} \\
  \vdots & \vdots & \ddots & \vdots \\
  b_{n1} & b_{n2} & \cdots & b_{nm}
\end{pmatrix}
\equiv
\begin{pmatrix}
  1 & -b_{11}' & -b_{12}' & \cdots & -b_{1m}' \\
  -b_{21}' & 1 & -b_{22}' & \cdots & -b_{2m}' \\
  \vdots & \vdots & \ddots & \vdots \\
  -b_{n1}' & -b_{n2}' & \cdots & 1 - b_{nm}'
\end{pmatrix}
\pmod{\mathfrak{M}(B)}
$$

By our assumption the matrix $(\#)$ is not invertible and so $\det (b_{ij}) \in \mathfrak{M}(B)$. Since $\det(\delta_{ij} - b_{ij}) \equiv \det(b_{ij})\det(b_{ij}) \pmod{\mathfrak{M}(B)}$, we must have $\det(\delta_{ij} - b_{ij}) \in \mathfrak{M}(B)$ where $\delta_{ij}$ is the Kronecker's symbol. Hence there exists an $b_{ij} \in \mathfrak{M}(B)$. Then it is easy to see that the matrix

$$
\begin{pmatrix}
  1 & \cdots & 1 \\
  \vdots & \ddots & \vdots \\
  b_{i1}' & \cdots & b_{in}' \\
  1 & \cdots & 1
\end{pmatrix}
$$
is invertible. Now we shall show that $B$ is $(b_i')$-adic complete. Let $\{c_k\}$ be a Cauchy sequence in $B$ under the $(b_i')$-adic topology. Then there is a subsequence $\{d_k\}$ of $\{c_k\}$ such that $d_k = \sum_{j=0}^{r_j} b_i'^j$ for each $k$, where $r_j \in B$. Let $f = \varphi^{-1}(r_j)$ and we set $f = \sum_{j=0}^{r_j} b_i'^j f_j$ which is a well defined power series in $A[[X_1, \ldots, X_n]]$, because $A$ is $(a_1, \ldots, a_n)$-adic complete. Then $\varphi(f) = \sum_{j=0}^{r_j} \varphi(a_i)' r_j = \sum_{j=0}^{r_j} b_i'^j f_j + g$ in $B^*[[Y_1, \ldots, Y_n]]$, where $B^*$ denotes the $(b_i')$-adic completion of $B$ and $g$ has no constant term. Since $\varphi(f) \in B[[Y_1, \ldots, Y_n]]$, we see that $\sum_{j=0}^{r_j} b_i'^j f_j \in B$, that is, $\{d_k\}$ converges in $B$ and hence $\{c_k\}$ converges in $B$. Together with $b_i' \in \mathfrak{m}(B)$, it follows from Theorem 2 that there exists a $B$-automorphism $\sigma$ of $B[[Y_1, \ldots, Y_n]]$ such that $\sigma(Y_j) = \varphi(a_i)$ and $\sigma(Y_k) = Y_k$ for $k \neq j$, that is, $\varphi(a_i)$ is analytically independent over $B[[Y_1, \ldots, Y_{j-1}, Y_{j+1}, \ldots, Y_n]]$ and $B[[Y_1, \ldots, Y_n]] = B[[Y_1, \ldots, Y_{j-1}, \varphi(a_i), Y_{j+1}, \ldots, Y_n]]$. We consider the following sequence of ring homomorphisms:

$$
A \to A[[X_1, \ldots, X_n]] \overset{\varphi}{\to} B[[Y_1, \ldots, Y_n]] = B[[Y_1, \ldots, Y_{j-1}, Y_{j+1}, \ldots, Y_n]]
$$

$$
\varphi^{-1}([f]) \cong A[[X_1, \ldots, X_n]] [([f]) \to A[[f]]
$$

where $\iota(a) = a$ for $a \in A$, $\varphi$ is the given isomorphism, $\tau(\varphi(a_i)) = \varphi(a_i) + t$, $\tau(Y_k) = Y_k + t$ for $k \neq j$, $\varphi^{-1}$ is the isomorphism induced by $\varphi^{-1}$, and $\varphi(X_1) = 0$ for $1 \leq i \leq n$. We set $\rho$ the composite of these homomorphisms. Then $\rho$ is a ring homomorphism of $A$ into $A[[f]]$ such that $\pi \rho(a) = a$ where $\pi$ is the homomorphism: $\sum_i a_i t^i \to a_0$. Thus we can express $\rho(a) = a_0 + \delta_1(a) t + \delta_2(a) t^2 + \cdots$.

Thence $\{1, \delta_1, \delta_2, \cdots\}$ is a higher derivation on $A$ by Lemma 1. Since $\rho(a_i) = a_i + t$, we have $\delta_1(a_i) = 1$, $\delta_j(a_i) = 0$ for $j \geq 2$ and by Lemma 2 we see that $A$ contains a subring $A_0$ satisfying the properties: $a_i$ is analytically independent over $A_0$ and $A = A_0[[a_i]]$. It is obvious that $A_0$ is a local ring which may not be noetherian. On the other hand

$$
X_l = \varphi^{-1}(\varphi(X_l))
$$

$$
= \varphi^{-1}(b_1) + \varphi^{-1}(b_1) \varphi^{-1}(X_1) + \cdots + \varphi^{-1}(b_{1n}) \varphi^{-1}(Y_n) + \cdots.
$$

We set

$$
\varphi^{-1}(b_1) = a_1' + a_1' X_1 + \cdots + a_{1n}' X_n + \cdots (1 \leq l \leq n),
$$

$$
\varphi^{-1}(b_{1m}) = a_{1m}' + a_{1m} X_1 + \cdots + a_{1mn} X_n + \cdots (1 \leq i, m \leq n).
$$

Here $a_i'$ is in $\mathfrak{m}(A)$, as is $b_i \in \mathfrak{m}(B)$. Thus

$$
X_l = (a_1' + a_{1n}' X_n + \cdots) + \sum_{m=1}^{n} (a_{1m}' + a_{1mn} X_n + \cdots) + (a_m + a_{mn} k X_n + \cdots) + \cdots.
$$
Comparing the coefficients of \( X \)'s we get
\[
\sum_{m+n} a_{mm}'' a_{mh} + a_{kh} = \delta_{ik} \pmod{\mathcal{M}(A)}.
\]

In the matrix notation
\[
(a_{ij}'')(a_{ij}) \equiv (\delta_{ij} - a_{ij}) \pmod{\mathcal{M}(A)}.
\]

Now we have
\[
\det(\phi^{-1}(b_{lm})) \equiv \det(a_{lm}'') \pmod{(X_1, \ldots, X_n)}.
\]

Since \( \det(\phi^{-1}(b_{lm})) = \phi^{-1}(\det(b_{lm})) \) and \( \det(b_{lm}) \in \mathcal{M}(B) \) by our assumption, it is immediate to see that \( \det(a_{lm}'') \in \mathcal{M}(A) \). Thus the same argument as above implies that some \( a_{lm}' \in \mathcal{M}(A) \) and we have
\[
A[[X_1, \ldots, X_n]] = A[[X_1, \ldots, X_{m-1}, \phi^{-1}(b_l), X_{m+1}, \ldots, X_n]].
\]

Then we see that \( B \) contains a subring \( B_0 \) satisfying the properties: \( b_l \) is analytically independent over \( B_0 \) and \( B = B_0[[b_l]] \). Obviously \( B_0 \) is a local ring which may not be noetherian and our proof is now complete.

**Theorem 6.** Let \( A \) be a local ring which may not be noetherian. Then we have only one of the followings:

1. \( A \) is strongly \( n \)-power invariant for any \( n \).
2. \( A \) is isomorphic to a formal power series ring \( A_0[[X]] \).

Proof. We assume that \( A \) is not strongly \( n \)-power invariant for some \( n \). Then we have a ring \( B \) and an isomorphism \( \phi : A[[X_1, \ldots, X_n]] \cong B[[Y_1, \ldots, Y_n]] \) such that there is never a \( B \)-automorphism \( \psi \) of \( B[[Y_1, \ldots, Y_n]] \) satisfying \( \phi(X_i) = \psi(Y_i) \) for \( 1 \leq i \leq n \). Now Theorem 5 implies that \( A \) must be isomorphic to a power series ring \( A_0[[X]] \). Conversely it is easy to see that a power series ring \( A_0[[X]] \) is not strongly \( n \)-power invariant for any \( n \).

Thus a local ring which may not be noetherian can simply be called to be strongly power invariant without reference to the number \( n \) of variables.

**Corollary 1.** An artinian local ring is strongly power invariant.

Proof. An artinian local ring \( A \) is not isomorphic to a power series ring \( A_0[[X]] \) and hence \( A \) is strongly power invariant.

**Corollary 2.** Let \( P \) be a point on an irreducible affine algebraic curve over an algebraically closed field \( k \) and let \( A \) be the local ring of \( P \). Then the following conditions are equivalent:

1. \( P \) is a singular point.
2. The completion \( \hat{A} \) is strongly power invariant.

Proof. Let us suppose that \( P \) is non-singular. Then it is obvious that \( \hat{A} \) is isomorphic to the power series ring \( k[[X]] \) and hence by Theorem 6 \( \hat{A} \) is not strongly power invariant. Conversely we assume that \( \hat{A} \) is not strongly power invariant. Then it follows from Theorem 6 that \( \hat{A} \) is isomorphic to a
formal power series ring $A[[X]]$. Since $\hat{A}$ is reduced and $\dim \hat{A}=1$, $A_0$ is reduced and $\dim A_0=0$. Now it is immediate to show that $A_0 \cong k$ and therefore $\hat{A} \cong k[[X]]$. Hence $P$ is non-singular.

**Corollary 3.** Let $V$ be an irreducible affine variety over a field of characteristic zero and let $A$ be the local ring of a component of the singular locus of $V$. Then the completion $\hat{A}$ is strongly power invariant.

Proof. If $\hat{A}$ is not strongly power invariant, $\hat{A}$ is isomorphic to a formal power series ring $A[[X]]$. Then we can obtain a contradiction by the same argument as that of Theorem 5 in [5].

**Theorem 7.** Let $A$ be a complete local ring. Then $A$ is strongly power invariant if and only if the maximal ideal $\mathfrak{m}(A)$ of $A$ is differential.

Proof. The assertion follows from Theorem 3 and Theorem 6 immediately.

**Theorem 8***) A noetherian local ring is $n$-power invariant for any $n$.

Proof. Let $A$ be a noetherian local ring. We shall prove our assertion by induction on Krull dimension of $A$. If $\dim A=0$, then $A$ is strongly power invariant by Corollary 1 of Theorem 6 and hence $A$ is $n$-power invariant for any $n$ according to the remark preceding to Theorem 4. Let us suppose $\dim A>0$. Let $B$ be a ring and let $A[[X_1, \ldots, X_n]] \cong B[[Y_1, \ldots, Y_n]]$ under $\varphi$. If there exists a $B$-automorphism $\psi$ of $B[[Y_1, \ldots, Y_n]]$ such that $\varphi(X_i)=\psi(Y_i)$ for $1 \leq i \leq n$, then $A \cong B$ by the remark preceding to Theorem 4. Unless such an automorphism exists, it follows from Theorem 5 that $A$ (resp. $B$) is a power series ring $A_0[[a]]$ (resp. $B_0[[b]]$). Here $A_0$ and $B_0$ are local rings. Thus we have an isomorphism $A_0[[a, X_1, \ldots, X_n]] \cong B_0[[b, Y_1, \ldots, Y_n]]$. Since $\dim A_0 < \dim A$, our induction hypothesis means that $A_0$ is $n$-power invariant for any $n$. Hence we have $A_0 \cong B_0$ and $A=A_0[[a]] \cong B_0[[b]]=B$, as desired.

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**References**


** After this paper is completed, the author has observed that E. Hamann obtained the result: a quasi-local ring is $n$-power invariant for any $n$, in her paper “On Power Invariance”, to appear.}