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RELATIVE Ext GROUPS, RESOLUTIONS, AND SCHANUEL CLASSES

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Abstract

Given a precovering (also called contravariantly finite) class F there are three natural approaches to a homological dimension with respect to F: One based on Ext functors relative to F, one based on F-resolutions, and one based on Schanuel classes relative to F. In general these approaches do not give the same result. In this paper we study relations between the three approaches above, and we give necessary and sufficient conditions for them to agree.

1. Introduction

The fact that the category of modules over any ring R has enough projectives is a cornerstone in classical homological algebra. The existence of enough projective modules has three important consequences:

- For every module A, and $n \geqslant 0$ one can define the Ext *functor*,

$$\mathrm{Ext}_R^n(-, A),$$

with well-known properties, see [4, Chapter V].

- Every module M admits a *projective resolution*, cf. [4, Chapter V]:

$$\cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0.$$

- Every module M represents a projective equivalence class $[M]$, and to this one can associate its *Schanuel class*,

$$\mathcal{S}([M]) = [\mathrm{Ker}\,\pi],$$

where $\pi\colon P \to M$ is any epimorphism and P is projective. One can also consider the iterated Schanuel maps $\mathcal{S}^n(-)$ for $n \geqslant 0$, see Schanuel's lemma [14, Chapter 4, Theorem A].

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The three fundamental types of objects described above—Ext functors, projective resolutions, and Schanuel classes—are linked together as nicely as one could hope for, in the sense of the following well-known result (see [4, Chapter V, Proposition 2.1]):

Theorem A. *For any R-module M, and any integer $n \geqslant 0$ the following conditions are equivalent*:
$(E_{M,n})$ $\mathrm{Ext}_R^{n+1}(M, A) = 0$ *for all R–modules A.*
$(R_{M,n})$ *There exists a projective resolution for M of length n,*

$$0 \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_0 \longrightarrow M \longrightarrow 0.$$

$(S_{M,n})$ $\mathcal{S}^n([M]) = [0]$.

The equivalent conditions of this theorem define what it means for M to have projective dimension $\leqslant n$. Note how the conditions above are labelled according to the mnemonic rules: "E" for Ext, "R" for Resolution, and "S" for Schanuel.

In relative homological algebra, one substitutes the class of projective modules by any other *precovering* class F, see 2.2. The fact that F is precovering allows for well-defined constructions (see [8, Chapter 8] and [9, Lemma 2.2]) of:

- Ext functors $\underline{\mathrm{Ext}}_{\mathsf{F}}^n(-, A)$ relative to F;
- F-resolutions, $\cdots \to F_2 \to F_1 \to F_0 \to 0$; and
- Schanuel maps $\mathcal{S}_{\mathsf{F}}^n(-)$ relative to F.

The study of relative homological algebra goes back to [10, 11], and there is much literature on the subject. Just to mention a few examples: Special relative Ext functors (or cohomology theories) have been studied in e.g. [2, 7, 13, 15], and special relative resolutions and precovers have been investigated in e.g. [3, 5, 6, 12]. Relative Schanuel classes appear in e.g. [9, 16].

One could hope that there might exist an "F-version" of Theorem A, indeed, one would need such a theorem to have a rich and flexible notion of an F-dimension. Unfortunately, Theorem A fails for a general precovering class F! The aim of this paper is to understand, for a given precovering class F, the different kind of obstructions which keep the F-version of Theorem A from being true. Our results can be summarized in the following diagram:

$$\begin{tikzcd}
& (E) \arrow[dl, Rightarrow, shift right, "\text{Lemma 5.3, Corollary 5.5, Example 5.2}"'] \arrow[dr, Rightarrow, shift left, "\text{Lemma 3.3, Theorem 3.9, Proposition 3.8}"] & \\
(S) \arrow[ur, Rightarrow, shift right, "\text{Proposition 5.1}"'] \arrow[rr, Rightarrow, shift left, "\text{Theorem 4.4, Example 4.2}"] & & (R). \arrow[ul, Rightarrow, shift left, "\text{Proposition 3.1}"] \arrow[ll, Rightarrow, shift left, "\text{Theorem 4.8, Example 4.7}"]
\end{tikzcd}$$

For example, this diagram tells that for a general precovering class F, information about the implication:

$$(*) \qquad (R_{M,n}) \implies (S_{M,n}) \quad \text{for all modules } M \text{ and all integers } n \geqslant 0$$

can be found in Theorem 4.8 which, in fact, asserts that $(*)$ is equivalent to the property that every mono F-precover is an isomorphism. Furthermore, 4.7 gives examples of classes F for which $(*)$ fails.

The paper is organized as follows: Section 2 is preliminary and recalls the definitions of Ext functors, resolutions, and Schanuel classes with respect to F. In Section 3 we investigate the relationship between (E) and (R); in Section 4 the one between (R) and (S), and in Section 5 the one between (S) and (E), as illustrated above.

2. Preliminaries

SETUP 2.1. Throughout, R will be a ring, and all modules will be left R-modules. We write $\mathsf{Mod}\, R$ for the category of (left) R-modules, and Ab for the category of abelian groups. F will be any precovering class of modules, cf. 2.2 below, which contains 0 and is closed under isomorphism and finite direct sums.

PRECOVERING CLASSES 2.2. For definitions and results on precovering classes we generally follow [8, Chapter 5 and 8]. We mention here just a few notions which will be important for this paper.

Let F be a class of modules. An F-*precover* of a module M is a homomorphism $F \to M$ with $F \in \mathsf{F}$, such that given any other homomorphism $F' \to M$ with $F' \in \mathsf{F}$ there exists a factorization,

$$\begin{tikzcd} & F' \arrow[dl, dashed] \arrow[d] \\ F \arrow[r] & M. \end{tikzcd}$$

If every module admits an F-precover then F is called *precovering*. An (augmented) F-*resolution* of a module M is a complex (which is not necessarily exact),

$$\cdots \longrightarrow F_2 \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0 \xrightarrow{\partial_0} M \longrightarrow 0,$$

with $F_0, F_1, F_2, \ldots \in \mathsf{F}$, such that

$$\cdots \longrightarrow (F, F_2) \xrightarrow{(F,\partial_2)} (F, F_1) \xrightarrow{(F,\partial_1)} (F, F_0) \xrightarrow{(F,\partial_0)} (F, M) \longrightarrow 0$$

is exact for all $F \in \mathsf{F}$. When F is precovering, and $T\colon \mathsf{Mod}\, R \to \mathsf{Ab}$ is a contravariant additive functor, then one can well-define the n-th right derived functor of T relative to F,

$$\mathrm{R}^n_{\mathsf{F}}T\colon \ \mathsf{Mod}\ R \to \mathsf{Ab}.$$

One computes $\mathrm{R}^n_{\mathsf{F}}T(M)$ by taking a non-augmented F-resolution of M, applying T to it, and then taking the n-th cohomology group of the resulting complex. For a module A we write:

$$\underline{\mathrm{Ext}}^n_{\mathsf{F}}(-, A) = \mathrm{R}^n_{\mathsf{F}}\,\mathrm{Hom}_R(-, A).$$

Note that we underline the Ext for good reasons: There is also a notion of a *preenveloping* class. If G is preenveloping then one can right derive the Hom functor in the covariant variable with respect to G. Thus for each R-module B there are functors $\overline{\mathrm{Ext}}^n_{\mathsf{G}}(B, -)$. However, in general,

$$\underline{\mathrm{Ext}}^n_{\mathsf{F}}(B, A) \ncong \overline{\mathrm{Ext}}^n_{\mathsf{G}}(B, A)$$

even if $\mathsf{F} = \mathsf{G}$ is both precovering and preenveloping. For example, over the ring $R = \mathbb{Z}$, the class $\mathsf{F} = \mathsf{G} = \mathsf{Inj}\,\mathbb{Z}$ of injective (i.e. divisible) $\mathbb{Z}$-modules is both precovering, cf. [8, Theorem 5.4.1], and preenveloping. As $\mathbb{Q}/\mathbb{Z}$ is injective as $\mathbb{Z}$-module, it is trivial that

$$\underline{\mathrm{Ext}}^1_{\mathsf{Inj}\,\mathbb{Z}}(\mathbb{Q}/\mathbb{Z}, \mathbb{Z}) = 0,$$

however, for the classical Ext we have

$$\overline{\mathrm{Ext}}^1_{\mathsf{Inj}\,\mathbb{Z}}(\mathbb{Q}/\mathbb{Z}, \mathbb{Z}) \cong \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z}, \mathbb{Q}/\mathbb{Z}) \neq 0.$$

F-EQUIVALENCE 2.3. Two modules K and K' are called F-*equivalent*, and we write $K \equiv_{\mathsf{F}} K'$, if there exist $F, F' \in \mathsf{F}$ with $K \oplus F' \cong K' \oplus F$. We use $[K] = [K]_{\mathsf{F}}$ to denote the F-equivalence class containing K.

Now let M be any module. By the version of Schanuel's lemma found in [9, Lemma 2.2], the kernels of any two F-precovers of M are F-equivalent. Thus the class $[\mathrm{Ker}\,\varphi]$, where $\varphi\colon F \to M$ is any F-precover of M, is a well-defined object depending only on M. We write

$$\mathcal{S}_{\mathsf{F}}(M) = [\mathrm{Ker}\,\varphi].$$

As F is closed under finite direct sums; cf. Setup 2.1, it is not hard to see that $\mathcal{S}_{\mathsf{F}}(M)$ only depends on the F-equivalence class of M, and hence we get the induced Schanuel map:

$$\mathsf{Mod}\ R/{\equiv_{\mathsf{F}}} \xrightarrow{\mathcal{S}_{\mathsf{F}}} \mathsf{Mod}\ R/{\equiv_{\mathsf{F}}}.$$

For $n > 0$ we write $\mathcal{S}^n_{\mathsf{F}}$ for the n-fold composition of $\mathcal{S}_{\mathsf{F}}$ with itself, and we set $\mathcal{S}^0_{\mathsf{F}} = \mathrm{id}$.

This paper is all about studying relations between the conditions from the following definition.

DEFINITION 2.4. For any module M and any integer $n \geqslant 0$ we consider the conditions:
($E_{M,n}$) $\underline{\mathrm{Ext}}_{\mathsf{F}}^{n+1}(M, A) = 0$ for all modules A.
($R_{M,n}$) There exists an (augmented) F-resolution of the form

$$0 \longrightarrow F_n \longrightarrow \cdots \longrightarrow F_0 \longrightarrow M \longrightarrow 0.$$

($S_{M,n}$) $\mathcal{S}_{\mathsf{F}}^{n}([M]) = [0]$.

The conditions in Definition 2.4 are labelled according to the mnemonic rules: “E” for Ext, “R” for Resolution, and “S” for Schanuel.

3. Relative Ext functors and resolutions

In this section we study how the Ext condition and the resolution condition of Definition 2.4 are related. It is straightforward, cf. Proposition 3.1 below, that the resolution condition implies the Ext condition. The converse is, in general, not true, but in Theorem 3.9 we give a sufficient condition on F for this to happen.

Proposition 3.1. *For any precovering class* F *we have*:

$$(R_{M,n}) \implies (E_{M,n}) \quad \text{for all modules } M \text{ and all integers } n \geqslant 0.$$

EXAMPLE 3.2. There exist precovering classes which are not closed under direct summands: Let R be a left Noetherian ring which is not Quasi-Frobenius, and set $D = R \oplus E$ where $E \neq 0$ is any injective R-module. Let F be the class of all modules which are isomorphic to $D^{(\Lambda)}$ for some index set Λ (here $D^{\varnothing} = 0$). Note that F is precovering as for example an F-precover of a module M is given by the natural map

$$D^{(\mathrm{Hom}_R(D,M))} \to M.$$

To see that F is not closed under direct summands we note that E is a direct summand of $D \in \mathsf{F}$. However, there exists no set Λ for which $E \cong D^{(\Lambda)}$.

The example above makes the following lemma relevant:

Lemma 3.3. *A necessary condition for* F *to satisfy the implication*:

$$(E_{M,0}) \implies (R_{M,0}) \quad \text{for all modules} \quad M,$$

is that F *is closed under direct summands.*

Proof. Assume that F is not closed under direct summands. Then there exists an $F \in \mathsf{F}$ and a direct summand M of F with $M \notin \mathsf{F}$. We claim that $(E_{M,0})$ holds but that $(R_{M,0})$ does not:

As M is a direct summand of F, and as F is closed under finite direct sums, cf. Setup 2.1, the abelian group $\underline{\mathrm{Ext}}^1_{\mathsf{F}}(M, A)$ is a direct summand of $\underline{\mathrm{Ext}}^1_{\mathsf{F}}(F, A)$ for every module A. The latter is zero as $F \in \mathsf{F}$, and hence also $\underline{\mathrm{Ext}}^1_{\mathsf{F}}(M, A) = 0$. Now suppose for contradiction that there do exist an F-resolution of M of length zero:

$$0 \longrightarrow F_0 \xrightarrow{\partial_0} M \longrightarrow 0.$$

We claim that ∂_0 must be an isomorphism (contradicting the fact that $M \notin \mathsf{F}$). As M is a direct summand of F there is an embedding $\iota\colon M \to F$ and a projection $\pi\colon F \to M$ with $\pi\iota = \mathrm{id}_M$. As ∂_0 is an F-precover of M, we get a factorization:

$$\begin{tikzcd} & F \arrow[dl, dashed, "\varphi"'] \arrow[d, "\pi"] \\ F_0 \arrow[r, "\partial_0"'] & M \end{tikzcd}$$

It follows that $\partial_0(\varphi\iota) = \pi\iota = \mathrm{id}_M$, so ∂_0 is epi and the sequence

$$(\dagger) \qquad 0 \longrightarrow \mathrm{Ker}\,\partial_0 \longrightarrow F_0 \underset{\varphi\iota}{\overset{\partial_0}{\rightleftarrows}} M \longrightarrow 0$$

splits. By assumption, $\mathrm{Hom}_R(G, \partial_0)$ is mono for all $G \in \mathsf{F}$, so by $(\dagger)$ it follows that $\mathrm{Hom}_R(G, \mathrm{Ker}\,\partial_0) = 0$ for all $G \in \mathsf{F}$. In particular,

$$\mathrm{Hom}_R(F_0, \mathrm{Ker}\,\partial_0) = 0,$$

and therefore $\mathrm{Ker}\,\partial_0 = 0$ since $\mathrm{Ker}\,\partial_0$ is a direct summand of F_0. Consequently, ∂_0 is an isomorphism. □

Lemma 3.4. *For a homomorphism $\varphi\colon F \to M$ the following two conditions are equivalent*:

(a) *Every endomorphism $g\colon M \to M$ with $g\varphi = \varphi$ is an automorphism.*
(b) *Every endomorphism $g\colon M \to M$ with $g\varphi = \varphi$ admits a left inverse.*

Proof. We only need to show that (b) implies (a): If $g\varphi = \varphi$ then (b) gives a homomorphism $v\colon M \to M$ with $vg = \mathrm{id}_M$. Now

$$v\varphi = vg\varphi = \mathrm{id}_M\varphi = \varphi,$$

so another application of (b) gives that also v has a left inverse. As v has g as a right inverse, v must be an automorphisms with $v^{-1} = g$. □

DEFINITION 3.5. A homomorphism $\varphi\colon F \to M$ satisfying the equivalent conditions of Lemma 3.4 is called *almost epi* (Auslander referred to these as *left minimal*, see [1, Chapter 1.2]). The precovering class F is called *precovering by almost epimorphisms* if every module has an F-precover which is almost epi.

EXAMPLE 3.6. Clearly, every epimorphism is almost epi but the converse is, in general, not true as for example

$$\mathbb{Z} \xrightarrow{2\cdot} \mathbb{Z}$$

is an almost epimorphism of abelian groups. It follows from Lemma 3.7 below that if a precovering class contains all free modules, then it is precovering by almost epimorphisms.

Lemma 3.7. *If there exists an almost epi homomorphism $\varphi\colon F \to M$ with $F \in \mathsf{F}$ then every F-precover of M is almost epi.*

Proof. If $\tilde{\varphi}\colon \tilde{F} \to M$ is any F-precover of M then there exists a factorization,

$$\begin{tikzcd} & F \arrow[dl, dashed, "\psi"'] \arrow[d, "\varphi"] \\ \tilde{F} \arrow[r, "\tilde{\varphi}"'] & M. \end{tikzcd}$$

For any endomorphism $g\colon M \to M$ with $g\tilde{\varphi} = \tilde{\varphi}$ it follows that

$$g\varphi = g\tilde{\varphi}\psi = \tilde{\varphi}\psi = \varphi,$$

and hence g must be an automorphism since φ is almost epi. $\square$

The next proposition gives much more information than 3.6, namely that there do indeed exist module classes F which are precovering by almost epimorphisms, without every F-precover being epi. We postpone the proof of Proposition 3.8 to the end of this section.

Proposition 3.8. *Consider the local ring $R = \mathbb{Z}/4\mathbb{Z}$. We denote the generator $2+4\mathbb{Z}$ of the maximal ideal by ξ, and the residue class field $R/(\xi) \cong \mathbb{F}_2$ by k. Furthermore, if $\mathsf{F} = \mathsf{Add}\,k$ is the class of all direct summands of set-indexed coproducts of copies of k, then*:

(a) F *is precovering by almost epimorphisms, cf.* Definition 3.5.

(b) *R does not admit an epi F-precover.*

(c) *The exists mono F-precovers which are not isomorphisms.*

The reason we are interested in classes which are precovering by almost epimorphisms is because of the next result:

Theorem 3.9. *Assume that* F *is closed under direct summands and is precovering by almost epimorphisms. Then*

$$(E_{M,n}) \Longrightarrow (R_{M,n}) \quad \textit{for all modules} \quad M \quad \textit{and all integers} \quad n \geqslant 0.$$

Proof. First we deal with the case $n = 0$: Thus let M be any module, and assume that $\underline{\mathrm{Ext}}^1_{\mathsf{F}}(M, A) = 0$ for all modules A. We must prove the existence of an F-resolution of M of length zero,

$$0 \longrightarrow G_0 \longrightarrow M \longrightarrow 0.$$

By assumption on F we can build an F-resolution of M by successively taking almost epi F-precovers $\varphi_0, \varphi_1, \varphi_2, \ldots$:

$$\begin{tikzcd}
 & 0 \arrow[dr] & & 0 & & & & 0 & \\
 & & K_1 \arrow[ur, dashed] \arrow[dr, "i_1"] & & & & M \arrow[ur, dashed] \arrow[dr, equal] & & \\
\cdots \arrow[r] & F_2 \arrow[ur, "\varphi_2"] \arrow[rr, "\partial_2"] & & F_1 \arrow[rr, "\partial_1"] \arrow[dr, "\varphi_1"'] & & F_0 \arrow[ur, "\varphi_0"] \arrow[rr, "\partial_0"] & & M \arrow[r] & 0 \\
 & & & & K_0 \arrow[ur, "i_0"'] \arrow[dr, dashed] & & & & \\
 & & & 0 \arrow[ur] & & 0 & & &
\end{tikzcd}$$

We keep in mind that the F-precovers φ_n are not necessarily epi, and this is the reason why some of the arrows in the diagram above have been dotted. Applying $\mathrm{Hom}_R(-, A)$, for any module A, to the $\mathrm{Hom}_R(\mathsf{F}, -)$ exact complex,

$$0 \longrightarrow K_0 \longrightarrow F_0 \longrightarrow M \longrightarrow 0,$$

induces by [8, Theorem 8.2.3 (2)] an exact sequence of relative Ext groups,

$$(*) \qquad \underline{\mathrm{Ext}}^0_{\mathsf{F}}(F_0, A) \xrightarrow{q} \underline{\mathrm{Ext}}^0_{\mathsf{F}}(K_0, A) \longrightarrow \underline{\mathrm{Ext}}^1_{\mathsf{F}}(M, A) = 0.$$

As $F_0 \in \mathsf{F}$ we have $\underline{\mathrm{Ext}}^0_{\mathsf{F}}(F_0, A) = \mathrm{Hom}_R(F_0, A)$. Furthermore,

$$\underline{\mathrm{Ext}}^0_{\mathsf{F}}(K_0, A) = \mathrm{Ker}\,\mathrm{Hom}_R(\partial_2, A) = \{ f \in \mathrm{Hom}_R(F_1, A) \mid f\partial_2 = 0\},$$

and q is given by $g \mapsto g\partial_1$ for $g \in \mathrm{Hom}_R(F_0, A)$. Applying these considerations to $A = K_0$ and $\varphi_1 \in \underline{\mathrm{Ext}}^0_{\mathsf{F}}(K_0, K_0)$, exactness of $(*)$ implies the existence of a $g \in \mathrm{Hom}_R(F_0, K_0)$

with $g\partial_1 = \varphi_1$, that is, $gi_0\varphi_1 = \varphi_1$. As φ_1 is almost epi, $gi_0 \colon K_0 \to K_0$ must be an automorphism, and hence the sequence

$$\begin{tikzcd} 0 \arrow[r] & K_0 \arrow[r, shift left, dashed, "i_0"] & F_0 \arrow[l, shift left, "(gi_0)^{-1}g"] \arrow[r, "\pi_0"] & F_0/K_0 \arrow[r] & 0 \end{tikzcd}$$

is split exact. In particular, $F_0/K_0 \in \mathsf{F}$ as F is closed under direct summands. It follows easily that the by $\varphi_0 \colon F_0 \to M$ induced homomorphism $\bar{\varphi}_0 \colon F_0/K_0 \to M$ is a mono F-precover of M, and thus

$$0 \longrightarrow F_0/K_0 \xrightarrow{\bar{\varphi}_0} M \longrightarrow 0$$

is an F-resolution of M of length zero.

For $n > 0$ we proceed by induction: If $\underline{\mathrm{Ext}}_{\mathsf{F}}^{n+1}(M, -) = 0$ then we take an F-precover $\partial_0 \colon F_0 \to M$ of M. By [8, Theorem 8.2.3 (2)] the complex

$$(\dagger) \qquad 0 \longrightarrow \mathrm{Ker}\,\partial_0 \longrightarrow F_0 \xrightarrow{\partial_0} M \longrightarrow 0$$

induces a long exact sequence of relative Ext groups:

$$0 = \underline{\mathrm{Ext}}_{\mathsf{F}}^{n}(F_0, -) \longrightarrow \underline{\mathrm{Ext}}_{\mathsf{F}}^{n}(\mathrm{Ker}\,\partial_0, -) \longrightarrow \underline{\mathrm{Ext}}_{\mathsf{F}}^{n+1}(M, -) = 0.$$

It follows that $\underline{\mathrm{Ext}}_{\mathsf{F}}^{n}(\mathrm{Ker}\,\partial_0, -) = 0$, so the induction hypothesis implies that $\mathrm{Ker}\,\partial_0$ admits an F-resolution of length $n-1$, say,

$$(\ddagger) \qquad 0 \longrightarrow F_n \longrightarrow \cdots \longrightarrow F_1 \longrightarrow \mathrm{Ker}\,\partial_0 \longrightarrow 0.$$

Gluing $(\ddagger)$ onto $(\dagger)$ we get an F-resolution of M of length n. $\square$

Proof of Proposition 3.8. Note that $R = \mathbb{Z}/4\mathbb{Z}$ is a two-dimensional k-vector space with basis $\{1, \xi\}$, so every element of R has a unique representation of the form $a + b\xi$ where $a, b \in k \cong \mathbb{F}_2$.

Just as in Example 3.2 it follows that $\mathsf{F} = \mathsf{Add}\, k$ is precovering, but shortly we shall prove this more directly. It is useful to observe that a homomorphism $F \to M$ with $F \in \mathsf{F}$ is an F-precover of M if and only if every homomorphism $k \to M$ admits a factorization:

$$(\natural) \qquad \begin{tikzcd} & k \arrow[dl, dashed] \arrow[d] \\ F \arrow[r] & M. \end{tikzcd}$$

One important consequence of this is that if $F_j \to M_j$ is a family of F-precovers then the coproduct $\coprod_j F_j \to \coprod_j M_j$ is again an F-precover. For every $c \in k$ there is an R-linear map

$$\varphi_c \colon k \to R, \quad a \mapsto ac\xi,$$

and it is not hard to see that every R-linear map $k \to R$ has the form φ_c for some $c \in k$. Combining this with the commutative diagram

$$\begin{tikzcd} & k \arrow[dl, "c\cdot"'] \arrow[d, "\varphi_c"] \\ k \arrow[r, "\varphi_1"'] & R, \end{tikzcd}$$

the observation $(\natural)$ implies that $\varphi_1 \colon k \to R$ is an F-precover of R. Clearly, φ_1 is mono, and since it is not epi, R cannot be the homomorphic image of any module from F. This proves parts (b) and (c) of the proposition.

It remains to prove part (a), namely that every R-module admits an almost epi F-precover. It is well-known that every module over $R = \mathbb{Z}/4\mathbb{Z}$ is isomorphic to one of the form $k^{(I)} \oplus R^{(J)}$ for suitable index sets I and J. Hence we only need to show that the module $k^{(I)} \oplus R^{(J)}$ has an almost epi F-precover. By the observation $(\natural)$ it follows that

$$\begin{matrix} k^{(I)} \\ \oplus \\ k^{(J)} \end{matrix} \xrightarrow{\varphi = \begin{pmatrix} \mathrm{id}_{k^{(I)}} & 0 \\ 0 & \varphi_1^{(J)} \end{pmatrix}} \begin{matrix} k^{(I)} \\ \oplus \\ R^{(J)} \end{matrix}$$

is an F-precover. To argue that φ is almost epi we let

$$\begin{matrix} k^{(I)} \\ \oplus \\ R^{(J)} \end{matrix} \xrightarrow{g = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}} \begin{matrix} k^{(I)} \\ \oplus \\ R^{(J)} \end{matrix}$$

be any endomorphism with $\varphi = g\varphi$. We must prove that g is an automorphism. By assumption,

$$(*) \qquad \begin{pmatrix} \mathrm{id}_{k^{(I)}} & 0 \\ 0 & \varphi_1^{(J)} \end{pmatrix} = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \begin{pmatrix} \mathrm{id}_{k^{(I)}} & 0 \\ 0 & \varphi_1^{(J)} \end{pmatrix} = \begin{pmatrix} g_{11} & g_{12}\varphi_1^{(J)} \\ g_{21} & g_{22}\varphi_1^{(J)} \end{pmatrix}.$$

In particular it follows that $g_{11} = \mathrm{id}_{k^{(I)}}$ and $g_{21} = 0$, so g takes the form

$$g = \begin{pmatrix} \mathrm{id}_{k^{(I)}} & g_{12} \\ 0 & g_{22} \end{pmatrix}.$$

If we can prove that $g_{22}\colon R^{(J)} \to R^{(J)}$ is an automorphism, then g must be an automorphism as well with inverse

$$g^{-1} = \begin{pmatrix} \mathrm{id}_{k^{(I)}} & -g_{12}\, g_{22}^{-1} \\ 0 & g_{22}^{-1} \end{pmatrix}.$$

To see that g_{22} is an automorphism we use another relation from $(*)$, namely that $\varphi_1^{(J)} = g_{22}\varphi_1^{(J)}$, or equivalently, $(g_{22} - \mathrm{id}_{R^{(J)}})\varphi_1^{(J)} = 0$. Since $\operatorname{Im}\varphi_1 = k\xi \subseteq R$ it is not hard to see that

$$g_{22} - \mathrm{id}_{R^{(J)}} = \xi f$$

for some homomorphism $f\colon R^{(J)} \to R^{(J)}$. Now, using that R has characteristic two and that $\xi^2 = 0$ it follows that:

$$g_{22}^2 = (\mathrm{id}_{R^{(J)}} + \xi f)^2 = \mathrm{id}_{R^{(J)}}^2 + 2\xi f + \xi^2 f^2 = \mathrm{id}_{R^{(J)}},$$

and thus g_{22} is an automorphism which is its own inverse. □

4. Relative resolutions and Schanuel maps

In this section we study how the resolution condition and the Schanuel condition of Definition 2.4 are related. In general, neither of these two conditions imply the other, however, in Theorems 4.4 and 4.8 we give necessary and sufficient conditions for this phenomenon to happen.

DEFINITION 4.1. We say that F is *weakly closed under direct summands* if for any $F \in \mathsf{F}$ and any direct summand M in F with $F/M \in \mathsf{F}$, the module M belongs to F.

EXAMPLE 4.2. There exist precovering classes which are not closed under set-indexed coproducts: A trivial example can be constructed over a field $R = k$ by letting F be the class of k-vector spaces of dimension, say, $\neq \aleph_0$. In fact, it is easy to see that F is not weakly closed under direct summands either.

A little more natural is the precovering class F from Example 3.2, which is not closed under direct summands. As F *is* closed under set-indexed coproducts, it follows from Proposition 4.3 below that F is not even weakly closed under direct summands.

Proposition 4.3. *A precovering class F is closed under direct summands if and only if F is weakly closed under direct summands and closed under set-indexed* (*respectively, countable*) *coproducts in* $\mathsf{Mod}\, R$.

Proof. "If": Let M be a direct summand of $F \in \mathsf{F}$, that is, there exists some module M' with $F = M \oplus M'$. Using Eilenberg's swindle we consider $F^{(\mathbb{N})}$ and note that

$$(*) \qquad M \oplus F^{(\mathbb{N})} \cong F^{(\mathbb{N})}.$$

As F is closed under countable coproducts, $F^{(\mathbb{N})} \in \mathsf{F}$, and then $(*)$ implies that $M \in \mathsf{F}$ since F is weakly closed under direct summands.

"Only if": If F is closed under direct summands then obviously F is also weakly closed under direct summands. Since F is precovering and closed under direct summands, the argument in [8, proof of Theorem 5.4.1, (2) $\Rightarrow$ (1)] shows that F is closed under set-indexed coproducts. □

The reason we are interested in classes which are weakly closed under direct summands is because of the next result.

Theorem 4.4. *A precovering class* F *satisfies*:

$$(\natural) \qquad (S_{M,n}) \Longrightarrow (R_{M,n}) \quad \text{for all modules} \quad M \quad \text{and all integers} \quad n \geqslant 0$$

if and only if F *is weakly closed under direct summands*.

Proof. "Only if": Let M be a direct summand of a module F where $F, F/M \in \mathsf{F}$. As $M \oplus F/M \cong 0 \oplus F$ we see that M is F-equivalent to 0, that is, $\mathcal{S}^0_{\mathsf{F}}([M]) = [M] = [0]$. Now the assumption $(\natural)$ implies the existence of an F-resolution of M of length zero,

$$(*) \qquad 0 \longrightarrow F_0 \xrightarrow{\partial_0} M \longrightarrow 0.$$

As in the end of the proof of Lemma 3.3 we see that ∂_0 is an isomorphism, and hence $M \cong F_0 \in \mathsf{F}$ as desired.

"If": We prove $(\natural)$ by induction on n: Suppose that $n = 0$ and that $\mathcal{S}^0_{\mathsf{F}}([M]) = [M] = [0]$. By definition there exist F', $F \in \mathsf{F}$ with $M \oplus F' \cong 0 \oplus F \cong F$, and since F is weakly closed under direct summands it follows that $F_0 := M \in \mathsf{F}$. Thus $0 \to F_0 \xrightarrow{=} M \to 0$ is an F-resolution of M of length zero. The induction step is straightforward: If $n > 0$ and $\mathcal{S}^n_{\mathsf{F}}([M]) = [0]$ then we take an F-precover

$$(\dagger) \qquad 0 \longrightarrow \operatorname{Ker} \partial \longrightarrow F_0 \xrightarrow{\partial} M \longrightarrow 0.$$

It follows that $\mathcal{S}^{n-1}_{\mathsf{F}}([\operatorname{Ker} \partial]) = [0]$, so the induction hypothesis implies the existence of an F-resolution of $\operatorname{Ker} \partial$ of length $n - 1$. Pasting this resolution together with $(\dagger)$ gives an F-resolution of M of length n. □

DEFINITION 4.5. A (precovering) class F is said to be *separating* if for every module $M \neq 0$ there exists a non-zero homomorphism $F \to M$ with $F \in \mathsf{F}$.

Lemma 4.6. *For a precovering class* F *the following hold*:
(a) *If every mono* F*-precover is an isomorphism then* F *is separating.*
(b) *If* F *is separating and* $\partial : A \to B$ *is a homomorphism such that* $\mathrm{Hom}_R(F, \partial)$ *is mono for all* $F \in \mathsf{F}$*, then* ∂ *is mono.*

Proof. "(a)": If M is a module with $\mathrm{Hom}_R(F, M) = 0$ for all $F \in \mathsf{F}$ then the map $0 \to M$ is a mono F-precover. Thus $0 \to M$ is an isomorphism by assumption, that is, $M = 0$.

"(b)": Applying, for any $F \in \mathsf{F}$, the left exact functor $\mathrm{Hom}_R(F, -)$ to

$$0 \longrightarrow \mathrm{Ker}\,\partial \longrightarrow A \xrightarrow{\partial} B$$

and using that $\mathrm{Hom}_R(F, \partial)$ is mono, we get that $\mathrm{Hom}_R(F, \mathrm{Ker}\,\partial) = 0$. As F is separating it follows that $\mathrm{Ker}\,\partial = 0$, that is, ∂ is mono. □

EXAMPLE 4.7. In Proposition 3.8 we saw an example of a precovering class F for which there exist mono F-precovers which are not isomorphisms. We now give two additional (more natural) examples:
(a) Let R be a commutative Noetherian ring which is not Artinian. As R is Noetherian the class $\mathsf{F} = \mathsf{Inj}\, R$ of injective R-modules is precovering by [8, Theorem 5.4.1]. However, as R is not Artinian, F is not separating by [16, Corollary 2.4.11], and hence Lemma 4.6 (a) implies that there must exist mono F-precovers which are not isomorphisms.
(b) Let R be a commutative integral domain, and consider for any module M its *torsion submodule*,

$$M_T = \{x \in M \mid rx = 0 \text{ for some } r \in R \setminus \{0\}\}.$$

A module M is called *torsion* if $M_T = M$, and of course the torsion submodule of any module is torsion. The torsion modules constitutes a precovering class, in fact, given a module M it is not hard to see that the inclusion $M_T \to M$ is a torsion precover of M. In particular, $0 = R_T \to R$ is a mono torsion precover of R which is not an isomorphism.

The following result shows why we are interested in precovering classes for which every mono precover is an isomorphism.

Theorem 4.8. *A precovering class* F *satisfies*:

$$(\flat) \qquad (R_{M,n}) \Longrightarrow (S_{M,n}) \quad \textit{for all modules} \quad M \quad \textit{and all integers} \quad n \geqslant 0$$

if and only if every mono F*-precover is an isomorphism.*

Proof. "Only if": Any mono F-precover $\varphi\colon F_0 \to M$ gives an F-resolution of M of length zero, $0 \to F_0 \xrightarrow{\varphi} M \to 0$. Thus our assumption implies that $\mathcal{S}^0_{\mathsf{F}}([M]) = [M] = [0]$. In particular, M is a homomorphic image of some $F \in \mathsf{F}$, and hence the F-precover φ must be epi. Consequently, φ is an isomorphism.

"If": We prove ($\flat$) by induction on n, beginning with the case $n = 0$: Thus, assume that M admits an F-resolution of length zero, say,

$$(*) \qquad 0 \longrightarrow F_0 \xrightarrow{\ \partial\ } M \longrightarrow 0.$$

We must argue that $\mathcal{S}^0_{\mathsf{F}}([M]) = [0]$. Actually, we prove something even stronger, namely that $M \in \mathsf{F}$. Since $(*)$ is an F-resolution, $\mathrm{Hom}_R(F, \partial)$ is an isomorphism for all $F \in \mathsf{F}$. Hence our assumption and Lemma 4.6 (a) and (b) gives that $\partial\colon F_0 \to M$ is a mono F-precover. Another application of our assumption then gives that ∂ is an isomorphism, and thus $M \cong F_0 \in \mathsf{F}$.

The induction step is easy: Suppose that M admits an F-resolution of length $n > 0$, say,

$$(\dagger) \qquad 0 \longrightarrow F_n \longrightarrow \cdots \longrightarrow F_1 \xrightarrow{\ \partial_1\ } F_0 \xrightarrow{\ \partial_0\ } M \longrightarrow 0.$$

We break up $(\dagger)$ into two complexes,

$$(1) \qquad 0 \longrightarrow F_n \longrightarrow \cdots \longrightarrow F_1 \xrightarrow{\ \hat{\partial}_1\ } \mathrm{Ker}\,\partial_0 \longrightarrow 0,$$

$$(2) \qquad 0 \longrightarrow \mathrm{Ker}\,\partial_0 \longrightarrow F_0 \xrightarrow{\ \partial_0\ } M \longrightarrow 0,$$

where $\hat{\partial}_1$ is the co-restriction of ∂_1 to $\mathrm{Ker}\,\partial_0$. It is not hard to see that (1) is an F-resolution of $\mathrm{Ker}\,\partial_0$, and hence the induction hypothesis gives that $\mathcal{S}^{n-1}_{\mathsf{F}}([\mathrm{Ker}\,\partial_0]) = [0]$. By (2), $\mathcal{S}_{\mathsf{F}}([M]) = [\mathrm{Ker}\,\partial_0]$, and it follows that $\mathcal{S}^n_{\mathsf{F}}([M]) = \mathcal{S}^{n-1}_{\mathsf{F}}\mathcal{S}_{\mathsf{F}}([M]) = [0]$, as desired. $\square$

5. Relative Schanuel maps and Ext functors

In this final section we compare the Schanuel condition and the Ext condition of Definition 2.4. While it is true that the Schanuel condition implies the Ext condition, cf. Proposition 5.1, the converse is, in general, not true by Lemma 5.3. However, in Corollary 5.5 we give a sufficient condition for this to happen.

Proposition 5.1. *For any precovering class* F *we have*:

$$(S_{M,n}) \implies (E_{M,n}) \quad \textit{for all modules} \quad M \quad \textit{and all integers} \quad n \geqslant 0.$$

Proof. We use induction on n. If $\mathcal{S}^0_{\mathsf{F}}([M]) = [0]$ then, in particular, M is a direct summand of some $F \in \mathsf{F}$. As $\underline{\mathrm{Ext}}^1_{\mathsf{F}}(F, -) = 0$ it follows that $\underline{\mathrm{Ext}}^1_{\mathsf{F}}(M, -) = 0$.

Now let $n > 0$ and assume that $\mathcal{S}^n_{\mathsf{F}}([M]) = [0]$. If $\varphi\colon F \to M$ is an F-precover then $\mathcal{S}^{n-1}_{\mathsf{F}}([\operatorname{Ker}\varphi]) = [0]$, and the induction hypothesis implies that $\underline{\operatorname{Ext}}^n_{\mathsf{F}}(\operatorname{Ker}\varphi, -) = 0$. By [8, Theorem 8.3.2 (2)] there is an induced long exact sequence:

$$0 = \underline{\operatorname{Ext}}^n_{\mathsf{F}}(\operatorname{Ker}\varphi, -) \longrightarrow \underline{\operatorname{Ext}}^{n+1}_{\mathsf{F}}(M, -) \longrightarrow \underline{\operatorname{Ext}}^{n+1}_{\mathsf{F}}(F, -) = 0$$

from which the desired conclusion follows. □

EXAMPLE 5.2. We have already seen examples of classes where mono precovers are not necessarily isomorphisms, cf. Proposition 3.8 and Example 4.7.

Lemma 5.3. *A necessary condition for* F *to satisfy the implication*:

$$(E_{M,0}) \Longrightarrow (S_{M,0}) \quad \textit{for all modules} \quad M,$$

is that every mono F*-precover is an isomorphism.*

Proof. Let $F \to M$ be a mono F-precover. The $\operatorname{Hom}_R(\mathsf{F}, -)$ exact complex $0 \to 0 \to F \to M \to 0$ gives a long exact sequence:

$$0 = \underline{\operatorname{Ext}}^0_{\mathsf{F}}(0, -) \longrightarrow \underline{\operatorname{Ext}}^1_{\mathsf{F}}(M, -) \longrightarrow \underline{\operatorname{Ext}}^1_{\mathsf{F}}(F, -) = 0,$$

from which it follows that $\underline{\operatorname{Ext}}^1_{\mathsf{F}}(M, -) = 0$. By our assumptions we then get $\mathcal{S}_{\mathsf{F}}([M]) = [M] = [0]$, in particular, M is a homomorphic image of some module in F. Therefore the mono F-precover $F \to M$ must be surjective as well, and hence an isomorphism. □

REMARK 5.4. In particular, the class F from Proposition 3.8 satisfies the implication "$(E) \Rightarrow (R)$" but not "$(E) \Rightarrow (S)$", cf. Theorem 3.9 and Lemma 5.3.

Assuming the necessary condition from Lemma 5.3, the following result is an immediate corollary of Theorems 3.9 and 4.8.

Corollary 5.5. *Assume that every mono* F*-precover is an isomorphism, that* F *is closed under direct summands, and that* F *is precovering by almost epimorphisms. Then*

$$(E_{M,n}) \Longrightarrow (S_{M,n}) \quad \textit{for all modules} \quad M \quad \textit{and all integers} \quad n \geqslant 0.$$

REMARK 5.6. The dual notion of a precover is a *preenvelope*, see [8, Chapter 6]. For a *preenveloping* class G, the reader can imagine how to construct Ext functors, resolutions, and Schanuel maps relative to G, see also [8, Chapter 8].

Not surprisingly, every result in this this paper has an analogue in this “preenveloping context”. We leave it as an exercise for the interested reader to verify this claim.

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References

[1] M. Auslander, I. Reiten and S.O. Smalø: Representation Theory of Artin Algebras, Cambridge Studies in Advanced Mathematics **36**, Cambridge Univ. Press, Cambridge, 1995.
[2] L.L. Avramov and A. Martsinkovsky: *Absolute, relative, and Tate cohomology of modules of finite Gorenstein dimension*, Proc. London Math. Soc. (3) **85** (2002), no. 2, 393–440.
[3] L. Bican, R. El Bashir and E. Enochs: *All modules have flat covers*, Bull. London Math. Soc. **33** (2001), no. 4, 385–390.
[4] H. Cartan and S. Eilenberg: Homological Algebra, Princeton Univ. Press, Princeton, N.J., 1956.
[5] E.E. Enochs: *Torsion free covering modules*, Proc. Amer. Math. Soc. **14** (1963), 884–889.
[6] E.E. Enochs: *Injective and flat covers, envelopes and resolvents*, Israel J. Math. **39** (1981), no. 3, 189–209.
[7] E.E. Enochs and O.M.G. Jenda: *Balanced functors applied to modules*, J. Algebra **92** (1985), no. 2, 303–310.
[8] E.E. Enochs and O.M.G. Jenda: Relative Homological Algebra, de Gruyter Expositions in Mathematics **30**, Walter de Gruyter & Co. Berlin, 2000.
[9] E.E. Enochs, O.M.G. Jenda and L. Oyonarte: *λ and μ-dimensions of modules*, Rend. Sem. Mat. Univ. Padova **105** (2001), 111–123.
[10] A. Heller: *Homological algebra in abelian categories*, Ann. of Math. (2) **68** (1958), 484–525.
[11] G. Hochschild: *Relative homological algebra*, Trans. Amer. Math. Soc. **82** (1956), 246–269.
[12] O.M.G. Jenda: *Injective resolvents and preenvelopes*, Classical and categorical algebra (Durban, 1985), Quaestiones Math. **9** (1986), no. 1–4, 301–309.
[13] P. Jørgensen: *Existence of Gorenstein projective resolutions and Tate cohomology*, J. Eur. Math. Soc. (JEMS) **9** (2007), 59–76.
[14] I. Kaplansky: Commutative Rings, revised edition, The University of Chicago Press, Chicago, Ill.-London, Ill., 1974.
[15] D. Simson: *On pure global dimension of locally finitely presented Grothendieck categories*, Fund. Math. **96** (1977), no. 2, 91–116.
[16] J. Xu: Flat Covers of Modules, Lecture Notes in Math. **1634**, Springer, Berlin, 1996.

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