

Title	On a construction of recurrent Markov chains
Author(s)	Kondô, Ryôji
Citation	Osaka Journal of Mathematics. 1969, 6(1), p. 13-28
Version Type	VoR
URL	https://doi.org/10.18910/10267
rights	
Note	

Osaka University Knowledge Archive : OUKA

<https://ir.library.osaka-u.ac.jp/>

Osaka University

ON A CONSTRUCTION OF RECURRENT MARKOV CHAINS

RYŌJI KONDO

(Received August 31, 1968)

Let S be a denumerable (possibly finite) set and \mathbf{B} the space of all real valued and bounded functions defined on S . For a given measure μ , strictly positive at each point of S , we shall denote by $N(\mu)$ the collection of functions f such that the support of f is finite and $\langle \mu, f \rangle = \sum_{x \in S} \mu(x)f(x) = 0$. A linear operator R from $N(\mu)$ to \mathbf{B} is said to satisfy the *semi-complete maximum principle* if it has the following property:

(S.C.M) For any $f \in N(\mu)$, if $Rf \leq m$ on the set $\{f > 0\}$, then $Rf \leq m$ everywhere, where m is a real constant.

We know that if R is a weak potential operator for a recurrent semi-group $(P_t)_{t \geq 0}$ with an invariant measure μ , it satisfies this maximum principle [7, p. 337]. In this work we shall consider the converse problem: Given a measure μ and a linear operator R satisfying (S.C.M), can we find a recurrent semi-group $(P_t)_{t \geq 0}$ which has μ as an invariant measure and R as a weak potential operator?

If μ is bounded, this problem has an affirmative answer, which will be stated in section 2. However, if μ is unbounded, there are several cases, for example, some operators are weak potential operators for transient semi-groups with invariant measure μ and others are never weak potential operators for any Markov semi-group with invariant measure μ . We shall give such examples in section 3. The appropriate conditions under which the problem is solved are not known yet. In section 1 we shall study, for later use, another type of maximum principle which is satisfied by weak potential operators (weak inverses in Orey [10]) for recurrent Markov chains with discrete parameters.

1. Potential operators satisfying the reinforced semi-complete maximum principle

Throughout this work notations and terminology are mainly taken from [7]. We shall denote the collection of all non-empty finite subsets of S by \mathcal{K} . Further, for each $E \in \mathcal{K}$, we shall use the following notations:

- f_E The function restricted to E .
 ν_E The measure restricted to E .
 \mathbf{B}_E The space of all functions f_E .
 \mathbf{N}^E The space of functions of $\mathbf{N}(\mu)$ with supports in E .

For any function f on S , $f^+ = \sup(f, 0)$ and $f^- = \sup(-f, 0)$. The indicator function of a set E will be denoted by χ_E .

A linear operator G from $\mathbf{N}(\mu)$ to \mathbf{B} is said to satisfy the *reinforced semi-complete maximum principle* if it has the following property:

(R.S.C.M) For any function $f \in \mathbf{N}(\mu)$, if $Gf \leq m$ on the set $\{f > 0\}$, then $Gf \leq m - f^-$ everywhere, where m is a real constant.

Let G be a linear operator from $\mathbf{N}(\mu)$ to \mathbf{B} satisfying (R.S.C.M).

Lemma 1. *G is non-singular in the sense: If f is a non-zero element of $\mathbf{N}(\mu)$, then Gf is never equal to a constant on the support of f . So that $Gf=0$ implies $f=0$.*

Proof. Let f be a non-zero element of $\mathbf{N}(\mu)$ and $Gf = m$ on the support of f , where m is a constant. From (R.S.C.M) it follows that $Gf \leq m - f^-$ everywhere and hence, $m = Gf \leq m - f^-$ on the set $\{f < 0\}$. Therefore $f^- = 0$. Similarly we have $f^+ = 0$, for, $-m = G(-f) \leq -m - (-f)^- = -m - f^+$ on the set $\{f > 0\}$. Thus $f = 0$, which is a contradiction.

Lemma 2. *There is a family of (signed) measures $(\lambda^E)_{E \in \mathcal{K}}$ on S such that; (i) the support of each λ^E is contained in E , (ii) $\langle \lambda^E, 1 \rangle = 1$ and (iii) $\langle \lambda^E, Gf \rangle = 0$ for all $f \in \mathbf{N}^E$. Such a family is unique.*

Proof. Let $E \in \mathcal{K}$ and the number of elements of E be n . Then the linear dimensions of \mathbf{B}_E and \mathbf{N}^E are equal to n and $n-1$ respectively. Let us define a linear operator G^E from \mathbf{N}^E to \mathbf{B}_E by

$$(1.1) \quad G^E f = (Gf)_E \quad \text{for } f \in \mathbf{N}^E.$$

From Lemma 1 it follows that if $G^E f = 0$, then $f = 0$ and that 1_E , the restriction of the function 1 to E , does not belong to the range $G^E(\mathbf{N}^E)$. Therefore, since $\dim G^E(\mathbf{N}^E) = \dim \mathbf{N}^E = n-1$ and $1_E \notin G^E(\mathbf{N}^E)$, we can find exactly one linear functional l_E on \mathbf{B}_E such that $l_E(g_E) = 0$ if and only if $g_E \in G^E(\mathbf{N}^E)$ and $l_E(1_E) = 1$. Thus if we define the measure λ^E by $\lambda^E(y) = l_E((\chi_{\{y\}})_E)$ for $y \in E$ and $\lambda^E(y) = 0$ for $y \in S \setminus E$, the family $(\lambda^E)_{E \in \mathcal{K}}$ is the desired one. The uniqueness of $(\lambda^E)_{E \in \mathcal{K}}$ is obvious from the above proof.

Let $g \in \mathbf{B}$ and $E \in \mathcal{K}$. If we put $h_E = (g - \langle \lambda^E, g \rangle)_E$, then $l_E(h_E) = \langle \lambda^E, g \rangle - \langle \lambda^E, g \rangle = 0$, so that we can find unique $f^E \in \mathbf{N}^E$ such that $h_E = G^E f^E$. Now

let us define the mappings H^E and Π^E from \mathbf{B} to \mathbf{B} by

$$(1.2) \quad H^E g = Gf^E + \langle \lambda^E, g \rangle$$

and

$$(1.3) \quad \Pi^E g = Gf^E + \langle \lambda^E, g \rangle - f^E = H^E g - f^E$$

respectively. Obviously, H^E and Π^E are linear and $H^E g = \Pi^E g$ on $S \setminus E$.

Lemma 3. (i) If $g \geq 0$ on E , then $H^E g \geq 0$ and $\Pi^E g \geq 0$ everywhere. (ii) $H^E 1 = 1$ and $\Pi^E 1 = 1$. (iii) If $E, F \in \mathcal{K}$ and $E \subseteq F$, then $H^F H^E g = H^E g$ and $\Pi^F H^E g = \Pi^E g$.

Proof. Let $g \geq 0$ on E and $H^E g = Gf^E + \langle \lambda^E, g \rangle$ where $f^E \in N^E$. Since $Gf^E + \langle \lambda^E, g \rangle = g$ on E , $Gf^E \geq -\langle \lambda^E, g \rangle$ on the support of f^E . Therefore, using (R.S.C.M), we have

$$Gf^E \geq -\langle \lambda^E, g \rangle + (f^E)^+$$

everywhere, so that

$$H^E g = Gf^E + \langle \lambda^E, g \rangle \geq (f^E)^+ \geq 0$$

and

$$\Pi^E g = Gf^E + \langle \lambda^E, g \rangle - f^E \geq Gf^E + \langle \lambda^E, g \rangle - (f^E)^+ \geq 0$$

everywhere. Thus, the assertion (i) is true. Next, if $H^E 1 = Gf^E + \langle \lambda^E, 1 \rangle$, then $f^E = 0$ by Lemma 1. Therefore $H^E 1 = \Pi^E 1 = 1$, which implies (ii). Finally, let $E \subseteq F$ and let

$$\begin{aligned} h &= H^E g = Gf^E + \langle \lambda^E, g \rangle & (f^E \in N^E) \\ H^F h &= Gf^F + \langle \lambda^F, h \rangle & (f^F \in N^F). \end{aligned}$$

Since $H^F h = h$ on F , we have

$$Gf^F + \langle \lambda^F, h \rangle = Gf^E + \langle \lambda^E, g \rangle$$

on F . Therefore

$$G(f^F - f^E) = \langle \lambda^E, g \rangle - \langle \lambda^F, h \rangle = \text{const.}$$

on the support of $f^F - f^E$. Using Lemma 1, we have $f^F = f^E$ and $\langle \lambda^E, g \rangle = \langle \lambda^F, h \rangle$, which implies $H^F H^E g = H^E g$ and that

$$\Pi^F H^E g = H^F h - f^F = h - f^E = \Pi^E g.$$

Thus the assertion (iii) was proved.

From this lemma we can see that H^E and Π^E are Markov kernels on S and that for each $x \in S$ the supports of measures $H^E(x, \cdot)$ and $\Pi^E(x, \cdot)$ are contained in E .

Corollary. *If $E, F \in \mathcal{K}$, $E \subseteq F$ and g is a non-negative function on S with support in E , then $\Pi^E g \geq \Pi^F g$ everywhere.*

$$\begin{aligned}
 \text{For,} \quad \Pi^E g(x) &= \Pi^F H^E g(x) \\
 &= \sum_{y \in E} \Pi^F(x, y) g(y) + \sum_{y \in S \setminus E} \Pi^E(x, y) H^E g(y) \\
 &\geq \sum_{y \in E} \Pi^F(x, y) g(y) \\
 &= \Pi^F g(x)
 \end{aligned}$$

for all $x \in S$.

Theorem 1. *Let μ be a bounded measure which is strictly positive everywhere and G a linear operator from $N(\mu)$ to \mathbf{B} satisfying the reinforced semi-complete maximum principle. Then there is a kernel P on S such that*

$$(1.4) \quad P \geq 0 \quad \text{and} \quad P1 = 1,$$

$$(1.5) \quad \mu P = \mu,$$

$$(1.6) \quad (I - P)Gf = f \quad \text{for all } f \in N(\mu).$$

Such a kernel is unique.¹⁾

Further, P is irreducible recurrent in the sense:

$$(1.7) \quad \sum_{n=0}^{\infty} P^n(x, y) = \infty \quad \text{for all } (x, y) \in S \times S.$$

Proof. Let $(E_n)_{n \geq 1}$ be an increasing sequence of \mathcal{K} with the union S and $x, y \in S$. Then, there is some n such that $y \in E_k$ for all $k \geq n$. So that, by Corollary of Lemma 2, we have

$$\Pi^{E_n}(x, y) \geq \Pi^{E_{n+1}}(x, y) \geq \dots \geq 0.$$

Therefore the limit;

$$(1.8) \quad P(x, y) = \lim_{n \rightarrow \infty} \Pi^{E_n}(x, y),$$

exists for any $(x, y) \in S \times S$. We shall prove the kernel P defined by (1.8) has

1) Precisely speaking, a Markov kernel satisfying (1.6), if it exists, is unique, even if μ is unbounded. We can see this in the proof of the theorem. Similar circumstance occurs in Lemma 5 and Theorem 2 in the next section.

all the properties stated in the theorem. Since Π^{E_n} are Markov kernels, P is obviously sub-Markov kernel, that is, $P \geq 0$ and $P1 \leq 1$, by Fatou's inequality. From the definition of the kernel H^E , we can find $f^{E_n} \in N^{E_n}$ such that

$$H^{E_n}(x, y) = Gf^{E_n}(x) + \lambda^{E_n}(y).$$

Since,

$$\Pi^{E_n}(x, y) = H^{E_n}(x, y) - f^{E_n}(x),$$

we have,

$$\begin{aligned} \sum_{x \in E_n} \mu(x) \Pi^{E_n}(x, y) &= \sum_{x \in E_n} \mu(x) H^{E_n}(x, y) - \sum_{x \in E_n} \mu(x) f^{E_n}(x) \\ &= \mu(y), \end{aligned}$$

whenever $y \in E_n$. On the other hand, since $0 \leq \chi_{E_n}(x) \Pi^{E_n}(x, y) \leq 1$, $\lim_n \chi_{E_n}(x) \Pi^{E_n}(x, y) = P(x, y)$ and μ is a bounded measure, we have

$$\begin{aligned} \mu P(y) &= \sum_{x \in S} \mu(x) (\lim_{n \rightarrow \infty} \chi_{E_n}(x) \Pi^{E_n}(x, y)) \\ &= \lim_{n \rightarrow \infty} \sum_{x \in S} \mu(x) \chi_{E_n}(x) \Pi^{E_n}(x, y) \\ &= \mu(y) \end{aligned}$$

for all $y \in S$. Thus (1.5) was proved. From (1.5) it follows that $\langle \mu, P1 \rangle = \langle \mu P, 1 \rangle = \langle \mu, 1 \rangle$. Since $0 \leq P1 \leq 1$, we have $P1 = 1$ almost everywhere with respect to μ . However, since μ is strictly positive everywhere, we have $P1 = 1$. That is, (1.4) is true. Let $f \in N(\mu)$ and $g = Gf + \|Gf\|$, where $\| \cdot \|$ denotes the uniform norm in B . If n is so large that the support of f is contained in E_n , we have

$$\begin{aligned} \Pi^{E_n} g(x) &= \Pi^{E_n} Gf(x) + \|Gf\| \\ &= Gf(x) - f(x) + \|Gf\| \end{aligned}$$

for all $x \in S$ and hence, noting that $g \geq 0$, we have

$$Pg(x) \leq \liminf_{n \rightarrow \infty} \Pi^{E_n} g(x) = Gf(x) - f(x) + \|Gf\|,$$

which implies

$$(1.9) \quad PGf \leq Gf - f.$$

Similarly, by replacing f to $-f$ in (1.9), we have $PGf \geq Gf - f$, so that $PGf = Gf - f$ which proves (1.6). If \tilde{P} is any kernel satisfying (1.4) and (1.6), then for any $g \in B$

$$\begin{aligned} \tilde{P}g &= \lim_{n \rightarrow \infty} \tilde{P}H^{E_n}g = \lim_{n \rightarrow \infty} \tilde{P}(Gf^{E_n} + \langle \lambda^{E_n}, g \rangle) \\ &= \lim_{n \rightarrow \infty} (Gf^{E_n} - f^{E_n} + \langle \lambda^{E_n}, g \rangle) = \lim_{n \rightarrow \infty} PH^{E_n}g = Pg, \end{aligned}$$

where $H^E_n g = Gf^E_n + \langle \lambda^E_n, g \rangle$ and $f^E_n \in N^E_n$. Thus the uniqueness of P is proved. Finally we shall prove (1.7). If there is some $y \in S$ such that

$$\sum_{n=0}^{\infty} P^n(y, y) < \infty,$$

then

$$\sum_{n=0}^{\infty} P^n(x, y) \leq \sum_{n=0}^{\infty} P^n(y, y) < \infty$$

for all $x \in S$. Consequently $\lim_n P^n(x, y) = 0$ for all $x \in S$.

Therefore, using (1.5), we have

$$\mu(y) = \sum_{x \in S} \mu(x) (\lim_n P^n(x, y)) = 0,$$

which contradicts the assumption that μ is strictly positive everywhere. Thus (1.7) is true when $x=y$. To show (1.7) in the case $x \neq y$, it is sufficient that we prove there is some n such that $P^n(x, y) > 0$. Let us introduce the function e_y in $N(\mu)$ by

$$e_y(z) = \begin{cases} 1 & z = x \\ -\mu(x)/\mu(y) & z = y \\ 0 & \text{otherwise.} \end{cases}$$

If $P^n(x, y) = 0$ for all $n \geq 0$, we have

$$\begin{aligned} \sum_{k=0}^n P^k(x, x) &= \sum_{k=0}^n P^k e_y(x) \\ &= G e_y(x) - P^{n+1} G e_y(x) \\ &= [G e_y(x) - G e_y(y)] - P^{n+1} [G e_y - G e_y(y)](x) \\ &\leq G e_y(x) - G e_y(y), \end{aligned}$$

because $G e_y \geq G e_y(y)$ everywhere. Consequently we have

$$\sum_{k=0}^{\infty} P^k(x, x) \leq G e_y(x) - G e_y(y) < \infty$$

which is a contradiction. Thus the theorem was proved.

In the proof of this theorem, we have used essentially the boundedness of the measure μ . Examples of operators G for unbounded measures will be given and discussed in section 3.

2. The potential operators satisfying the semi-complete maximum principle.

Let μ be a measure on S , strictly positive everywhere, and R a linear operator from $N(\mu)$ to B which satisfies the semi-complete maximum principle.

In this section we shall assume always that μ is bounded. For each positive number α , we put $G_\alpha = I + \alpha R$, where I is the identity operator. Evidently G_α is a linear operator from $N(\mu)$ to B .

Lemma 4. *G_α satisfies the reinforced semi-complete maximum principle.*

Proof. Let $G_\alpha f \leq m$ on the set $\{f > 0\}$, where m is a real constant. Then $\alpha Rf \leq G_\alpha f \leq m$ on the set $\{f > 0\}$, so that $\alpha Rf \leq m$ everywhere by (S. C. M). Therefore $-f^- + \alpha Rf \leq m - f^-$ everywhere. Hence we have $G_\alpha f = -f^- + \alpha Rf \leq m - f^-$ on the set $\{f \leq 0\}$, which implies $G_\alpha f \leq m - f^-$ everywhere.

Since G_α satisfies (R. S. C. M), we can apply Theorem 1 to G_α , so that there is a kernel Q_α on S which has all the properties in Theorem 1. Put $R_\alpha = Q_\alpha / \alpha$, then

Lemma 5. *The family of kernels $(R_\alpha)_{\alpha > 0}$ satisfies the following conditions:*

$$(2.1) \quad \alpha R_\alpha \geq 0 \text{ and } \alpha R_\alpha 1 = 1 ,$$

$$(2.2) \quad \alpha \mu R_\alpha = \mu ,$$

$$(2.3) \quad R_\alpha - R_\beta + (\alpha - \beta) R_\alpha R_\beta = 0 ,$$

$$(2.4) \quad (I - \alpha R_\alpha) Rf = R_\alpha f \quad \text{for all } f \in N(\mu) .$$

Such a family is unique.

Further

$$(2.5) \quad \lim_{\alpha \rightarrow 0} R_\alpha(x, y) = \infty \quad \text{for all } (x, y) \in S \times S .$$

Proof. (2.1), (2.2) and (2.4) are the same as (1.4), (1.5) and (1.6) of Theorem 1 respectively and the uniqueness of such a family is a consequence of Theorem 1, too. So we have only to prove (2.3) and (2.5). Let us denote by $(\lambda_\alpha^E)_{E \in \mathcal{K}}$ the family of measures satisfying (i), (ii) and (iii) of Lemma 2 for G_α and by H_α^E the kernel defined by (1.2) with respect to G_α and λ_α^E . If $g \in B$ and $H_\beta^E g = G_\beta f^E + \langle \lambda_\beta^E, g \rangle$, where $f^E \in N^E$, then, noting the relation

$$H_\beta^E g = G_\alpha f^E + (\beta - \alpha) Rf^E + \langle \lambda_\beta^E, g \rangle ,$$

we have

$$R_\alpha H_\beta^E g = Rf^E + (\beta - \alpha) R_\alpha Rf^E + \langle \lambda_\beta^E, g \rangle / \alpha .$$

Since

$$R_\beta H_\beta^E g = Rf^E + \langle \lambda_\beta^E, g \rangle / \beta ,$$

we have

$$\begin{aligned} R_\alpha H_\beta^E g - R_\beta H_\beta^E g \\ = (\beta - \alpha) [R_\alpha Rf^E + \langle \lambda_\beta^E, g \rangle / \alpha \beta] . \end{aligned}$$

We can easily verify that the last term is equal to $(\beta - \alpha) R_\alpha R_\beta H_\beta^E g$, so that

$$(2.6) \quad R_\alpha H_\beta^E g - R_\beta H_\beta^E g = (\beta - \alpha) R_\alpha R_\beta H_\beta^E g$$

for all $g \in B$, $E \in \mathcal{K}$ and $\alpha, \beta > 0$. Let $(E_n)_{n \geq 1}$ be an increasing sequence of sets in \mathcal{K} with the union S . Since $\|H_\beta^{E_n} g\| \leq \|g\|$ and $\lim_n H_\beta^{E_n} g(x) = g(x)$ for all $x \in S$, we have

$$\begin{aligned} R_\alpha g - R_\beta g &= \lim_{n \rightarrow \infty} [R_\alpha H_\beta^{E_n} g - R_\beta H_\beta^{E_n} g] \\ &= (\beta - \alpha) \lim_n R_\alpha R_\beta H_\beta^{E_n} g \\ &= (\beta - \alpha) R_\alpha R_\beta g, \end{aligned}$$

which proves (2.4). Finally we shall prove (2.5). First we prove the inequality

$$(2.7) \quad R_\alpha(x, y) \leq R_\alpha(y, y).$$

Since $\beta R_{\alpha+\beta}$ is a sub-Markov kernel on S and $I + \beta R_\alpha = \sum_{n=0}^{\infty} (\beta R_{\alpha+\beta})^n$, we have

$$(2.8) \quad I(x, y) + \beta R_\alpha(x, y) \leq I(y, y) + \beta R_\alpha(y, y)$$

for all $(x, y) \in S \times S$. Hence, dividing both side of (2.8) by β , and letting $\beta \rightarrow \infty$, we obtain (2.7). If there is some $y \in S$ such that $\lim_{\alpha \rightarrow 0} R_\alpha(y, y) < \infty$, then $\lim_{\alpha \rightarrow 0} \alpha R_\alpha(x, y) = 0$ for all $x \in S$ by (2.7). Therefore

$$\mu(y) = \lim_{\alpha \rightarrow 0} \alpha \mu R_\alpha(y) = \mu(\lim_{\alpha \rightarrow 0} \alpha R_\alpha)(y) = 0,$$

which is a contradiction. Thus (2.5) is true when $x = y$. Let $r_\beta(x) = R_\beta(x, y) / R_\beta(y, y)$ and $r(x) = \lim_{\beta \rightarrow 0} r_\beta(x)$. From (2.7) it follows that $0 \leq r_\beta(x) \leq 1$ for all $x \in S$. Since the resolvent equation (2.3) implies

$$\alpha R_\alpha r_\beta(x) = \beta R_\alpha r_\beta(x) + r_\beta(x) - R_\alpha(x, y) / R_\beta(y, y)$$

and since

$$\begin{aligned} 0 &\leq R_\alpha(x, y) / R_\beta(y, y) \leq 1 / \alpha R_\beta(y, y), \\ 0 &\leq \beta R_\alpha r_\beta(x) \leq \beta / \alpha, \end{aligned}$$

we have

$$\alpha R_\alpha r(x) \leq \liminf_{\beta \rightarrow 0} \alpha R_\alpha r_\beta(x) \leq \liminf_{\beta \rightarrow 0} r_\beta(x) = r(x)$$

for all $x \in S$, which implies the function r is excessive with respect to the kernel $Q_\alpha = \alpha R_\alpha$. By Theorem 1, Q_α is irreducible recurrent, so that r should be a constant function, which is proved in [5, p. 226]. Since $r(y) = 1$, we have

$$(2.9) \quad r(x) = \lim_{\alpha \rightarrow 0} R_\alpha(x, y) / R_\alpha(y, y) = 1$$

for all $x \in S$, which implies $\lim_{\alpha \rightarrow 0} R_\alpha(x, y) = \infty$ for all $(x, y) \in S \times S$. Thus the theorem was proved.

Using (2.9), we can obtain easily the following corollaries:

Corollary 1. $\lim_{\alpha \rightarrow 0} \alpha R_\alpha(x, y) = \mu(y) / \langle \mu, 1 \rangle$ for all $(x, y) \in S \times S$.

Corollary 2. For each $f \in N(\mu)$ there exists the limit $R_0 f = \lim_{\alpha \rightarrow 0} R_\alpha f$ and

$$R_0 f = Rf - \langle \mu, Rf \rangle / \langle \mu, 1 \rangle \quad \text{for all } f \in N(\mu)$$

and hence, the linear operator R_0 satisfies (S.C.M), too.

Let $a \in S$ and define the function f_y by

$$f_y(x) = \begin{cases} 1 & x = y \\ -\mu(y)/\mu(a) & x = a \\ 0 & \text{otherwise.} \end{cases}$$

If we put ${}^a R(x, y) = Rf_y(x) - Rf_y(a)$, then ${}^a R$ is a non-negative kernel on S with ${}^a R(a, y) = {}^a R(x, a) = 0$ for all $x, y \in S$.

Corollary 3. Put

$${}^a R_\alpha(x, y) = R_\alpha(x, y) - R_\alpha(x, a) R_\alpha(a, y) / R_\alpha(a, a)$$

then $({}^a R_\alpha)_{\alpha > 0}$ is a sub-Markov resolvent with $\lim_{\alpha \rightarrow 0} {}^a R_\alpha = {}^a R$.

The meaning of these corollaries will be made clear later.

Theorem 2. Let μ be a bounded measure on S , strictly positive everywhere, and R a linear operator from $N(\mu)$ to \mathbf{B} which satisfies the semi-complete maximum principle. Then there exists a family of kernels $(P_t)_{t > 0}$ such that :

$$(2.9) \quad P_t \geq 0 \quad \text{and} \quad P_t 1 = 1 \quad \text{for all } t > 0.$$

$$(2.10) \quad P_t P_s = P_{t+s} \quad \text{for all } s, t > 0.$$

$$(2.11) \quad \mu P_t = \mu \quad \text{for all } t > 0.$$

(2.12) The functions $t \rightarrow P_t(x, y)$ are continuous in the open interval $(0, \infty)$ for all $(x, y) \in S \times S$.

$$(2.13) \quad (I - P_t) Rf(x) = \int_0^t P_s f(x) ds \quad \text{for all } f \in N(\mu), x \in S \text{ and } t > 0.^{2)}$$

Such a family is unique.

2) If a linear operator R from $N(\mu)$ to \mathbf{B} satisfies (2.13) for a Markov semi-group $(P_t)_{t > 0}$, it will be called a *weak potential operator* for $(P_t)_{t > 0}$.

Further $(P_t)_{t>0}$ is irreducible recurrent in the sense :

$$(2.14) \quad \int_0^\infty P_t(x, y) dt = \infty \quad \text{for all } (x, y) \in S \times S.$$

Proof. Let $(R_\alpha)_{\alpha>0}$ be the family constructed in Lemma 6. Since it satisfies (2.1) and (2.3), using the result of Reuter [12], we can find $(P_t)_{t>0}$ which satisfies (2.9), (2.10), (2.11) and

$$(2.15) \quad R_\alpha(x, y) = \int_0^\infty e^{-\alpha t} P_t(x, y) dt \quad \text{for all } (x, y) \in S \times S.$$

Since the functions $t \rightarrow \mu P_t(y)$ are continuous in $(0, \infty)$ and

$$\int_0^\infty e^{-\alpha t} \mu P_t(y) dt = \mu R_\alpha(y) / \alpha = \int_0^\infty e^{-\alpha t} \mu(y) dt,$$

we have (2.11) by the uniqueness of the inverse Laplace transform. We remark here that, for any $f \in \mathbf{B}$ and $x \in S$, the function $t \rightarrow P_t f(x)$ is continuous in $(0, \infty)$. In fact, if $0 \leq f \leq 1$, the functions $t \rightarrow P_t f(x)$ and $t \rightarrow P_t(1-f)(x) = 1 - P_t f(x)$ are lower-semi-continuous in $(0, \infty)$ and hence, the function $t \rightarrow P_t f(x)$ is continuous in $(0, \infty)$. The general case is reduced to this case by the usual procedure. From this remark we know that the both sides of (2.13) are continuous with respect to t in $(0, \infty)$. Since the Laplace transform of (2.13) is equal to (2.4), (2.13) is true by the property of the Laplace transform. Similarly the uniqueness of $(P_t)_{t>0}$ is followed from Lemma 6 and the uniqueness of the inverse Laplace transform. Relation (2.14) is evident by

$$\int_0^\infty P_t(x, y) dt = \lim_{\alpha \rightarrow 0} R_\alpha(x, y) = \infty.$$

Thus the theorem was proved.

Corollary 1 of Lemma 6 implies the *ergodic property* of $(P_t)_{t>0}$; $\lim_{t \rightarrow \infty} P_t(x, y) = \mu(y) / \langle \mu, 1 \rangle$, and Corollary 2 implies the *normality* of $(P_t)_{t>0}$; for any $f \in N(\mu)$ and $x \in S$, there exists the limit; $R_0 f(x) = \lim_{t \rightarrow \infty} \int_0^t P_s f(x) ds$, and which satisfies the equation (2.13), too.

Now we discuss the continuity of $(P_t)_{t>0}$ at $t=0$.

Theorem 3. *Under the same conditions of Theorem 1, the relation*

$$(2.16) \quad \lim_{t \rightarrow 0} P_t(x, y) = I(x, y) \quad \text{for all } (x, y) \in S \times S$$

holds if and only if R is non-singular.

Proof. First let us assume that $(P_t)_{t>0}$ satisfies (2.16). Let f be a non-

zero element of $N(\mu)$ and $Rf=m$ on the support of f , where m is a constant. Since R satisfies (S.C.M), $Rf=m$ everywhere, so that $\int_0^t P_s f(x) ds = 0$ for all $x \in S$. Therefore, from (2.15) it follows that

$$f(x) = \lim_{t \rightarrow 0} [\int_0^t P_s f(x) ds] / t = 0$$

for all $x \in S$, which is a contradiction. Therefore if f is a non-zero element of $N(\mu)$, Rf is never equal to a constant on the support of f , which is the meaning of that R is non-singular. Conversely we assume that R is non-singular. In this case we can define a family of measures $(\lambda^E)_{E \in \mathcal{K}}$ and a family of Markov kernels $(H^E)_{E \in \mathcal{K}}$ corresponding to R in the same way as stated in Lemma 2 and Lemma 3 of section 1 respectively. Let $(E_n)_{n \geq 1}$ be an increasing sequence of \mathcal{K} with the union S and further let $g = \chi_{\{y\}}$ and

$$H^{E_n} g = Rf^{E_n} + \langle \lambda^{E_n}, g \rangle,$$

where $f^{E_n} \in N^{E_n}$. Then, using (2.9) and (2.13), we have

$$\begin{aligned} (2.17) \quad P_t H^{E_n} g &= P_t Rf^{E_n} + \langle \lambda^{E_n}, g \rangle \\ &= Rf^{E_n} - \int_0^t P_s f^{E_n} ds + \langle \lambda^{E_n}, g \rangle \\ &= H^{E_n} g - \int_0^t P_s f^{E_n} ds \end{aligned}$$

for each n and $t > 0$. On the other hand, we know that, for each $(x, y) \in S \times S$, there exists the limit

$$(2.18) \quad W(x, y) = \lim_{t \rightarrow 0} P_t(x, y)$$

and the kernel W is a sub-Markov kernel with $W^2 = W$ [1, p. 118]. Therefore, using Fatou's inequality, we have

$$\begin{aligned} (2.19) \quad WH^{E_n} g(x) &\leq \liminf_{t \rightarrow 0} [H^{E_n} g(x) - \int_0^t P_s f^{E_n}(x) ds] \\ &= H^{E_n} g(x) \end{aligned}$$

for each n and $x \in S$. Noting that $0 \leq H^{E_n} g \leq 1$ and $\lim_n H^{E_n} g(x) = \chi_{\{y\}}(x) = I(x, y)$ for all $x \in S$, we have from (2.19)

$$(2.20) \quad W(x, y) \leq I(x, y) \quad \text{for all } (x, y) \in S \times S.$$

Thus $W(x, y) = w(x)I(x, y)$, where w is a function on S which takes only two values 0 or 1, for $W^2 = W$. However, since

$$\mu(y)w(y) = \mu W(y) = \lim_{t \rightarrow 0} \mu P_t(y) = \mu(y)$$

for all $y \in S$ and since μ is strictly positive everywhere, we have $w=1$ on S . Therefore.

$$I = W = \lim_{t \rightarrow 0} P_t.$$

Thus the theorem was proved.

Now the meaning of Corollary 3 of Lemma 5 is the following. Assume that R is non-singular, then the corresponding semi-group $(P_t)_{t>0}$ in Theorem 3 is continuous at $t=0$. In this case we can find a Markov process $X=(\Omega, \mathcal{M}, (X_t)_{t \geq 0}, (\theta_t)_{t \geq 0}, (P_x)_{x \in S})$ with an enlarged state space \bar{S} such that

$$P_x(X_t = y) = P_t(x, y) \quad \text{for all } (x, y) \in S \times S \text{ and } t > 0$$

(for precise definitions, see [7]). For any $a \in S$, if we define the family of kernels $({}^aP_t)_{t>0}$ by

$${}^aP_t(x, y) = P_x(X_t = y, t < T^a) \quad \text{for } (x, y) \in S \times S,$$

where T^a denotes the first hitting time of the set $\{a\}$, then $({}^aP_t)_{t>0}$ is a sub-Markov semi-group which is continuous at $t=0$. Corollary 3 shows that $({}^aP_t)_{t>0}$ is transient and its potential kernel is aR .

3. Examples

In this section we shall give examples of operators satisfying (R.S.C.M) with unbounded measures. Since (R.S.C.M) implies (S.C.M), these are also examples of non-singular operators satisfying (S.C.M).

EXAMPLE 1. Let S be the set of all integers and $\mu(x)=1$ for all $x \in S$. Define a linear operator G by

$$(3.1) \quad Gf(x) = -\sum_{y \in S} |y-x| f(y) \quad \text{for } f \in N(\mu).$$

Then, by simple calculations, we have the following formulae;

$$(3.2) \quad Gf(x) = Gf(x-1) + 2 \sum_{y \geq x} f(y),$$

$$(3.3) \quad Gf(x) = Gf(x+1) + 2 \sum_{x \geq y} f(y),$$

$$(3.4) \quad Gf(x) = \frac{1}{2} [Gf(x-1) + Gf(x+1)] + f(x)$$

for all $x \in S$. If the support of f is contained in $\{a, a+1, \dots, b\}$, by (3.3) and (3.2), $Gf(x)=Gf(a)$ for $x < a$ and $Gf(x)=Gf(b)$ for $x > b$, respectively. Therefore Gf is bounded on S , that is, G maps $N(\mu)$ into B . To show that G satisfies

(R.S.C.M) we assume $Gf \leq m$ on the set $\{f > 0\} = \{a_1, a_2, \dots, a_p\}$, where $a_1 < a_2 < \dots < a_p$. For each $x < a_1$, using (3.3), we have $Gf(x) \leq Gf(x+1) + f(x)$ and $Gf(x+1) \leq Gf(a_1)$, so that $Gf(x) \leq Gf(a_1) + f(x) \leq m - f^-(x)$. Similarly, for each $x > a_p$, using (3.2), we have $Gf(x) \leq m - f^-(x)$. For $a_k < x < a_{k+1}$, using (3.4), we have $Gf(x) \leq \sup (Gf(a_k), Gf(a_{k+1})) + f(x) \leq m - f^-(x)$, $k=1, 2, \dots, p-1$. Therefore $Gf \leq m - f^-$ everywhere, so that G satisfies (R.S.C.M). Let us introduce a Markov kernel P on S by

$$P(x, y) = \begin{cases} 1/2 & y = x \pm 1 \\ 0 & \text{otherwise,} \end{cases}$$

then μ is an invariant measure for P and the relation (1.6) of Theorem 1 holds for all $f \in \mathbf{N}(\mu)$, for, (1.6) is equivalent to (3.4) in this case. Further, since P is the transition function of (simple) symmetric random walk of dimension one, it is irreducible recurrent. Thus, Theorem 1 is valid for G , though μ is unbounded. If we define the Markov semi-group $(P_t)_{t>0}$ by $P_t = e^{t(P-I)}$, that is,

$$P_t(x, y) = e^{-t} \sum_{n=0}^{\infty} \frac{1}{n!} (tP)^n(x, y) \quad \text{for } (x, y) \in S \times S,$$

it has an invariant measure μ and a weak potential operator G . Obviously $(P_t)_{t>0}$ is irreducible recurrent, so that Theorem 2 is valid for G , too.

EXAMPLE 2. Let S and μ be the same in Example 1. Define a linear operator G from $\mathbf{N}(\mu)$ to \mathbf{B} by

$$(3.5) \quad Gf(x) = \sum_{y \geq x} f(y) \quad \text{for all } f \in \mathbf{N}(\mu).$$

To show that G satisfies (R.S.C.M) we assume that $Gf \leq m$ on the set $\{f > 0\} = \{a_1, a_2, \dots, a_p\}$, where $a_1 < a_2 < \dots < a_p$. Since $0 \leq Gf(a_1) \leq m$, m should be non-negative. If $a_{k-1} < x < a_k$,

$$\begin{aligned} Gf(x) &= Gf(x+1) + f(x) \leq Gf(a_k) + f(x) \\ &\leq m - f^-(x), \end{aligned}$$

$k=1, 2, \dots, p$ (we regard a_0 as $-\infty$). If $x > a_p$,

$$\begin{aligned} Gf(x) &= Gf(x+1) + f(x) \leq \sup (Gf(a_p), 0) + f(x) \\ &\leq m - f^-(x). \end{aligned}$$

Consequently $Gf \leq m - f^-$ everywhere, which shows that G satisfies (R.S.C.M).

Let us now define a Markov kernel P on S by

$$P(x, y) = \begin{cases} 1 & y = x+1 \\ 0 & \text{otherwise.} \end{cases}$$

Obviously, P has μ as an invariant measure and satisfies the relation (1.6) of Theorem 1. However, since $\sum_{n=0}^{\infty} P^n(x, y) = 0$ or $=1$ according as $x > y$ or $x \leq y$, P is not irreducible recurrent. If we define a Markov semi-group $(P_t)_{t \geq 0}$ by $P_t = e^{t(P-I)}$, it has μ as an invariant measure and G as a weak potential operator. But it is transient in the sense:

$$\int_0^{\infty} P_t(x, y) dt < \infty \quad \text{for all } (x, y) \in S \times S.$$

EXAMPLE 3. Let $S = \{0, 1, \dots\}$ and $\mu(x) = 1$ for all $x \in S$. Define a linear operator G from $N(\mu)$ to B by

$$(3.6) \quad Gf(x) = \sum_{y \geq x} f(y) \quad \text{for all } f \in N(\mu).$$

That G satisfies (R.S.C.M) is proved in the same way as stated in Example 2. Let us introduce a Markov kernel P on S by

$$P(x, y) = \begin{cases} 1 & y = x+1 \\ 0 & \text{otherwise.} \end{cases}$$

Then P satisfies (1.6) of Theorem 1. Since a Markov kernel satisfying (1.6) is unique, P is only such a kernel. However, the relation; $1 = \mu(0) > \mu P(0) = 0$, shows that μ is not an invariant measure for P . If a Markov semi-group $(P_t)_{t \geq 0}$ with a weak potential operator G exists, it should be equal to that defined by $P_t = e^{t(P-I)}$. Since μ is not an invariant measure for $(P_t)_{t \geq 0}$, there is never Markov semi-group which has μ as an invariant measure and G as a weak potential operator.

Finally we notice some remarks on our problem. We shall assume again that S is any denumerable set and μ is any measure on S , strictly positive everywhere. Let R be a non-singular operator from $N(\mu)$ to B satisfying (R. C. M.), for example, an operator satisfying (R. S. C. M.). Take a function g on S which is strictly positive everywhere and $\langle \mu, g \rangle < \infty$. Define a measure $\tilde{\mu}$ on S by $\tilde{\mu}(x) = g(x)\mu(x)$ for all $x \in S$. Then, $f \in N(\tilde{\mu})$ if and only if $gf \in N(\mu)$, so that we may define a linear operator \tilde{R} from $N(\tilde{\mu})$ to B by $\tilde{R}f = R(gf)$. We can easily verify that \tilde{R} is also a non-singular operator satisfying (R. C. M.). Since $\tilde{\mu}$ is bounded, by Theorem 2 and 3, we can find a Markov semi-group $(\tilde{P}_t)_{t \geq 0}$ which is continuous at $t=0$ and has $\tilde{\mu}$ and \tilde{R} as its own invariant measure and weak potential operator, respectively. Let $\tilde{X} = (\Omega, \mathcal{M}, (\tilde{X}_t)_{t \geq 0}, (\tilde{\theta}_t)_{t \geq 0}, (P_x)_{x \in S})$ be a Markov process with a state space \bar{S} , some metric completion of S , such that

$$\tilde{P}_t(x, y) = P_x(\tilde{X}_t = y) \quad \text{for all } (x, y) \in S \times S.$$

Let us introduce an additive functional $(A_t)_{t \geq 0}$ for \tilde{X} by

$$A_t = \begin{cases} \int_0^t [1/g(X_s)] ds & \text{for } t < T \\ \infty & \text{for } t \geq T, \end{cases}$$

where $T = \sup \{t: \int_0^t [1/g(X_s)] ds < \infty\}$. Further we put $C_t = s$ if and only if $A_s = t$ for $s \in [0, T)$. If we denote $X_t = \tilde{X}_{C_t}$ and $\theta_t = \tilde{\theta}_{C_t}$, $X = (\Omega, \mathcal{M}, (X_t)_{t \geq 0}, (\theta_t)_{t \geq 0}, (P_x)_{x \in S})$ is a Markov process with a state space \tilde{S} , too. Using properties of \tilde{X} , we can prove that a family of kernels $(P_t)_{t > 0}$ on S defined by; $P_t(x, y) = P_x(X_t = y)$ for all $(x, y) \in S \times S$, is a sub-Markov semi-group on S , continuous at $t=0$. If the condition;

$$(3.7) \quad P_x(T = \infty) = 1 \quad \text{for all } x \in S,$$

is satisfied, we can prove that $(P_t)_{t > 0}$ is an irreducible recurrent Markov semi-group with an invariant measure μ and a weak potential operator R . In Example 1, condition (3.7) is true, however, in Example 2 and 3, (3.7) is not true. Unwillingly, we could not express these facts as analytic conditions on R .

SHIZUOKA UNIVERSITY

Bibliography

- [1] K.L. Chung: Markov Chains with Stationary Transition Probabilities, Springer-Verlag, 1960.
- [2] E.B. Dynkin: Markov Processes I, II, Springer-Verlag, 1965.
- [3] W. Feller: *On boundaries and lateral conditions for the Kolmogorov differential equations*, Ann. of Math. **65** (1957), 527-570.
- [4] G.A. Hunt: *Markov processes and potentials* II, Illinois J. Math. **1** (1957), 316-396.
- [5] J.G. Kemeny and J.L. Snell: *Potentials for denumerable Markov chains*, J. Math. Anal. Appl. **3** (1961), 196-260.
- [6] J.G. Kemeny and J.L. Snell: *Boundary theory for recurrent Markov chains*, Trans. Amer. Math. Soc. **106** (1963), 495-520.
- [7] R. Kondō: *On weak potential operators for recurrent Markov chains with continuous parameters*, Osaka J. Math. **4** (1967), 327-344.
- [8] H. Kunita and T. Watanabe: *Some theorems concerning resolvents over locally compact spaces*, Fifth Berkeley Symp. Math. Statist. Probability, Berkeley, II, Part II (1967), 131-164.
- [9] P.A. Meyer: Probability and Potentials, Blaisdell Publishing Company, 1966.
- [10] S. Orey: *Potential kernels for recurrent Markov chains*, J. Math. Anal. Appl. **8** (1964), 104-132.

- [11] D. Ray: *Resolvents, transition functions and strongly Markovian processes*, Ann. of Math. **70** (1959), 43–78.
- [12] G.E.H. Reuter: *Note on resolvents of denumerable submarkovian processes*, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **9** (1967), 16–19.