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RESTRICTING THE TRANSFORMATION GROUP IN EQUIVARIANT CW COMPLEXES

Dedicated to Professor Shôrô Araki on his sixtieth birthday

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In the theory of equivariant CW complexes (for non-discrete transformation groups) most of the basic properties and elementary results, such as the equivariant skeletal approximation theorem and the equivariant Whitehead theorem, work in a natural and expected way; see for example Matumoto [13] and Illman [6, Section 2]. However, there is one important and very basic operation that causes some problems. This is the operation of restricting the action of the transformation group to a closed subgroup. The problem here is that when in a G-CW complex X (where say G is a compact Lie group) one restricts the given action of G to a closed subgroup H of G, the H-space X *does not* in general inherit an induced structure of an H-CW complex, at least not in any natural way. An example which demonstrates this problem is given in Section 2.

By Waner [18, Proposition 3.8] we know that if X is a G-CW complex and H is a closed subgroup of the compact Lie group G, then there exists an H-CW complex ΓX of the same H-homotopy type as the H-space X. However, the proof by Waner is such that the H-CW complex ΓX is not finite, or finite-dimensional, even if the given G-CW complex has this property. (See the proof of Lemma 4.7 in Section 5 of [18].) For many possible applications such an H-CW complex ΓX is inadequate.

A good construction for an H-CW complex $\check{X}$, with the same H-homotopy type as the underlying H-space of a given G-CW complex X, ought to have the following properties. When G is a compact Lie group and X is a finite G-CW complex the H-CW complex $\check{X}$ should be finite. The H-CW complex $\check{X}$ should be finite-dimensional whenever X is a finite-dimensional G-CW complex. In order to be really useful the construction should in fact preserve the topological dimensions of the spaces and all the appropriate fixed point sets. The construction should also preserve the H-isotropy types, without introducing any new ones.

Theorem A gives a construction with all the desirable properties mentioned above. Observe that in Theorem A the transformation group G *is an arbitrary*

Lie group and H is a compact subgroup of G. The property that $\check{X}$ is a finite H-CW complex when X is a finite G-CW complex can of course hold in general only when G is compact; see Corollary B.

Theorem A. *Suppose that G is a Lie group and H is a compact subgroup of G. Let X be a G-CW complex. Then there exist an H-CW complex $\check{X}$ and an H-homotopy equivalence $\eta\colon X\to\check{X}$, such that the following properties hold.*

1) *For every $K<H$, $\check{X}^K$ is compact if and only if X^K is compact.*
2) $\dim \check{X}^K = \dim X^K$, *for every $K<H$.*
3) *The H-isotropy types occurring in $\check{X}$ and in X are exactly the same.*

Here dim denotes the topological dimension, more precisely the covering dimension, of a space. For equivariant CW complexes and fixed point sets in equivariant CW complexes, as in Theorem A, the covering dimension agrees with the "geometric" dimension of such a space, see Lemma 1.2. The notation $K<H$ means that K is a closed subgroup of H. By (2) we in particular have that $\check{X}$ is a finite-dimensional H-CW complex if X is a finite-dimensional G-CW complex.

Corollary B. *Suppose that G is a compact Lie group and that $H<G$. If X is a finite G-CW complex then $\check{X}$ is a finite H-CW complex, and conversely.*

It is a well-known fact that, when G is a compact group a G-CW complex X is finite if and only if X is compact. Hence Corollary B follows directly from (1) of Theorem A, by taking $K=\{e\}$.

DEFINITION C. Let G and H be as in Theorem A. An H-reduction of a G-CW complex X consists of an H-CW complex $\check{X}$ and an H-homotopy equivalence $\eta\colon X\to\check{X}$ such that conditions (1)–(3) in Theorem A are satisfied.

If (X, X_0) is a G-CW pair and $\eta\colon X\to\check{X}$ and $\eta_0\colon X_0\to\check{X}_0$ are H-reductions of X and X_0, respectively, we say that η extends η_0 if $\check{X}_0$ is an H-subcomplex of $\check{X}$ and $\eta|X_0=\eta_0$. The proof of Theorem A is such that it immediately also gives a relative version of Theorem A. Using the terminology introduced above, the relative version of Theorem A can be stated as follows.

Theorem A (rel). *Suppose that G and H are as in Theorem A. Let (X, X_0) be a G-CW pair. Then there exist an H-CW pair $(\check{X}, \check{X}_0)$ and an H-map $\eta\colon (X, X_0)\to(\check{X}, \check{X}_0)$ such that $\eta\colon X\to\check{X}$ and $\eta|\colon X_0\to\check{X}_0$ are H-reductions of X and X_0, respectively. In fact, any H-reduction $\eta_0\colon X_0\to\check{X}_0$ of X_0 extends to an H-reduction $\eta\colon X\to\check{X}$ of X.*

In the case of a finite filtration $X_0\subset X_1\subset\cdots\subset X_m=X$ of a G-CW complex X one directly obtains a filtered version of Theorem A by repeated use of the relative version of Theorem A. But in some situations it is important to be able to deal with infinite filtrations and also in this case Theorem A generalizes to give the following filtered version. (This filtered version of Theorem A is used in [11].)

Theorem A (filt). *Suppose that G and H are as in Theorem A. Let $X_0\subset X_1\subset X_2\subset\cdots$ be a filtration of a G-CW complex X by G-subcomplexes, and let $\eta_0\colon X_0\to\check{X}_0$ be an H-reduction of X_0. Then there exist an H-reduction $\eta\colon X\to\check{X}$ of X and a filtration $\check{X}_0\subset\check{X}_1\subset\check{X}_2\subset\cdots$ of $\check{X}$ by H-subcomplexes, beginning with the given H-CW complex $\check{X}_0$, such that η induces an H-reduction $\eta_i=\eta|\colon X_i\to\check{X}_i$ of X_i, for each $i\geq 1$, and η extends η_0.*

Theorem A, as well as its relative and filtered versions, are proved in Section 5. The contents of the other sections of this paper are as follows. Section 1 gives some general preliminaries and Section 2 contains the example referred to at the very beginning of this introduction. In Section 3 we present some results concerning the equivariant homotopy type of adjunction spaces. These results are crucial for the construction given in the proof of Theorem A. The aim of Section 4 is to establish the result given in Corollary 4.4, and we use the telescope construction to achieve this. We need Corollary 4.4 for the proof of Theorem A, in the case of an infinite-dimensional G-CW complex.

As a typical example of the use of the results of this paper we now give an application of Theorem A to the theory of homotopy representations for compact Lie groups. For the definition of a homotopy representation we refer to tom Dieck [3, Definition 10.1]. Let us here only recall that a finite homotopy representation for a compact Lie group G is a finite G-CW complex X, such that for each $K<G$ the fixed point set X^K is homotopy equivalent to the sphere $S^{n(K)-1}$, where $\dim X^K=n(K)-1$. In addition two further technical conditions must be satisfied, see conditions (v) and (vi) in tom Dieck [3, Definition 10.1]. Now suppose that X and Y are finite homotopy representations for a compact Lie group G. In order to define the product of the homotopy representations X and Y one considers the join $X*Y$ with the induced diagonal G-action. Since X and Y are finite G-CW complexes, the join $X*Y$ is a finite $(G\times G)$-CW complex, but we need to consider $X*Y$ as a G-space via the diagonal G-action; i.e., we consider $X*Y$ with the induced action of the diagonal subgroup G of $G\times G$. In this situation Theorem A and Corollary B give us a finite G-CW complex $(X*Y)^\vee$ with the same G-homotopy type as the G-space $X*Y$, such that

$$\dim((X*Y)^\vee)^K=\dim(X*Y)^K \tag{1}$$

for every $K<G$. Furthermore the G-isotropy types occurring in $(X*Y)^\vee$ and in $X*Y$ are exactly the same. Since $(X*Y)^\vee$ is G-homotopy equivalent to the G-space $X*Y$, it follows that $((X*Y)^\vee)^K$ is homotopy equivalent to $(X*Y)^K=$

$X^K * Y^K$ for each $K<G$. Thus

$$((X*Y)^{\vee})^K \simeq S^{n(K)-1} * S^{n'(K)-1} = S^{n(K)+n'(K)-1},$$

where $\dim X^K = n(K)-1$ and $\dim Y^K = n'(K)-1$. Furthermore, we have dim $(X*Y)^K = \dim X^K + \dim Y^K + 1$. (The covering dimension need not always be multiplicative, but this equality follows easily from Lemma 1.2 since $X*Y$ is a $(G\times G)$-CW complex.) Hence the basic dimension equality (1) gives us

$$\dim((X*Y)^{\vee})^K = n(K)+n'(K)-1 . \tag{2}$$

The fact that the G-isotropy types occurring in $(X*Y)^{\vee}$ are exactly the same as the G-isotropy types occurring in $X*Y$ implies that $(X*Y)^{\vee}$ also satisfies the technical conditions (v) and (vi) in tom Dieck [3, Definition 10.1]. Thus we have shown that $(X*Y)^{\vee}$ is a finite homotopy representation for G.

By Dim X we denote the dimension function of a homotopy representation X. It is an integer-valued function defined on the set of all conjugacy classes of closed subgroups of G. We have that Dim X is defined by

$$\mathrm{Dim}\, X(K) = n(K)$$

for each conjugacy class (K) of a closed subgroup. Here $X^K \simeq S^{n(K)-1}$ and dim $X^K = n(K)-1$, as before. It follows from (2) that the dimension function Dim $(X*Y)^{\vee}$ of the finite homotopy representation $(X*Y)^{\vee}$ for G satisfies the equality

$$\mathrm{Dim}\,(X*Y)^{\vee} = \mathrm{Dim}\, X + \mathrm{Dim}\, Y ,$$

as it should. Compare with the discussion on page 169 in tom Dieck [3].

Another typical application of Theorem A is given in [11]. The proofs of the main results in [11] use the filtered version of Theorem A and rely in an absolutely essential way on the dimension equality given by part (2) of Theorem A.

Some further developments related to this paper are as follows. When G is a compact Lie group and X is a finite G-CW complex the finite H-CW complex $\check{X}$ is unique up to simple H-homotopy type, see [10]. Furthermore, we show in [10] that there exists a well-defined restriction homomorphism between equivariant Whitehead groups Res_H^G: $\mathrm{Wh}_G(X)\rightarrow \mathrm{Wh}_H(X)$, for every $H<G$. Additional properties as well as applications of the restriction homomorphisms Res_H^G will be given in future papers. For a brief outline and an announcement of some of our main results in this direction we refer to [9].

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1. Preliminaries

By a G-space X, where G is a locally compact group, we mean a topological space X on which G acts on the left. By a compactly generated space we mean a Hausdorff space X with the property that a subset A of X is closed in X if (and only if) A intersects every compact subset of X in a closed set (cf. Steenrod [17]). In our use of terminology a normal space is always assumed also to be a Hausdorff space. We shall use some elementary facts about the covering dimension of a normal space, which can be found in standard books on dimension theory. We use Nagami [16] as our reference on a couple of occasions.

For the basic facts about equivariant CW complexes (in the case of non-discrete transformation groups), such as the equivariant skeletal approximation theorem and the equivariant Whitehead theorem, we refer to the original papers by Matumoto [12], [13] and Illman [6], [7], and also to the exposition given in tom Dieck [3]. The elementary fact given in Lemma 1.1 below is perhaps not explicitly stated anywhere in the existing literature, so we present it here.

Lemma 1.1. *Let X be a G-CW complex, where G is a Lie group. Then the topological space X is compactly generated and normal.*

Proof. The 0-skeleton X^0 of X is a disjoint union of homogeneous spaces of the form G/P, where $P<G$. Hence X^0 is compactly generated and normal. Now let $n\geq 1$ and assume inductively that X^{n-1} is compactly generated and normal. The n-skeleton X^n of X is obtained from X^{n-1} as an adjunction space $X^n=X^{n-1}\cup_\varphi(\dot{\bigcup}_{j\in J} D^n\times G/P_j)$, where $P_j<G$ for each $j\in J$, and $\varphi\colon \dot{\bigcup}_{j\in J} S^{n-1}\times G/P_j \to X^{n-1}$ is a G-map. Clearly each $D^n\times G/P_j$ is compactly generated and normal.

Since X^n is a quotient of the disjoint union $X^{n-1}\dot{\cup}(\dot{\bigcup}_{j\in J} D^n\times G/P_j)$, which is compactly generated, and X^n is Hausdorff we have by [17, 2.6] that X^n is compactly generated. It is also a well-known fact that the normality property is preserved in forming adjunction spaces, see for example [2, Theorem 8.B.4]. Hence X^n is normal. This completes the inductive step.

Now, since the topology of X is coherent with respect to the family of all n-skeletons X^n, $n\geq 0$, and each X^n is compactly generated and normal we have by [17, Lemma 9.2] that X is compactly generated and by [2, Theorem 7.D.2] that X is normal. $\square$

Lemma 1.2. *Let X be a G-CW complex, where G is a Lie group. For any compact subgroup K of G we have*

$$\dim X^K = \sup_{j\in J} \dim(\mathring{c}_j)^K$$

where the supremum is over the set of all G-cells of X. In particular $\dim X = \sup_{j\in J} \dim \mathring{c}_j$.

Proof. By Lemma 1.1 X is normal and since the fixed point set X^K is closed in X it follows that X^K is normal. We have $X^K=\bigcup_{n\geq 0}(X^n)^K$, where X^n denotes the equivariant n-skeleton of X. By the countable sum theorem for the covering dimension (see e.g. [16, Theorem 9.10]) and the fact that the covering dimension is monotonic on closed subsets we therefore obtain that $\dim X^K = \sup_{n\geq 0} \dim (X^n)^K$.

We claim that

$$(*) \qquad \dim(X^n)^K = \max\{\dim(X^{n-1})^K, \dim(X^n - X^{n-1})^K\} .$$

This follows by [16, Theorem 9.11] and the fact that dim is monotonic on closed subsets, using the following observations. The difference $X^n - X^{n-1}$ is a disjoint union of open G-cells $\mathring{D}^n \times G/P_i$, $i \in J_n$, where J_n denotes the set of all G-n-cells of X. Each $\mathring{D}^n \times G/P_i$, $i \in J_n$, is a smooth G-manifold. Hence $(X^n - X^{n-1})^K$ is a disjoint union of smooth manifolds $\mathring{D}^n \times (G/P_i)^K$, $i \in J_n$, and in particular $(X^n - X^{n-1})^K$ is normal. Thus we see that $\dim(X^n - X^{n-1})^K$ equals the maximal dimension occurring among the manifolds $\mathring{D}^n \times (G/P_i)^K$, all $i \in J_n$; i.e.,

$$d = \dim(X^n - X^{n-1})^K = \max_{i \in J_n} \{\dim(\mathring{D}^n \times (G/P_i)^K)\}$$

(We have $d \leq n + \dim G$.) For every closed subset C of $(X^n)^K$ such that $C \subset (X^n - X^{n-1})^K$ we clearly have $\dim C \leq d$ and there also exists such a set C with $\dim C = d$, for example $C = \bigcup_{i \in J_n} D^n_{1/2} \times (G/P_i)^K$. Hence [16, Theorem 9.11] and the monotonicity of dim on closed subsets imply that $(*)$ holds. Now Lemma 1.2 follows by induction in n. $\square$

We will use the important general fact that every smooth G-manifold, where G is a compact Lie group, can be given the structure of a G-CW complex. The precise form, of this general result, that we shall use (in fact in this paper only in a special case) is the following.

Theorem 1.3. *Suppose that G is a compact Lie group and that M is a smooth G-manifold with boundary ∂M, (which may be empty). Then there exist a G-CW pair (A, B) and a G-homeomorphism $\alpha\colon (A, B) \to (M, \partial M)$ of G-pairs.*

A proof of this theorem is given in Illman [8], where one also finds references to the original sources Matumoto [12] and Illman [6], [7] for this result. In the present paper we do not need the uniqueness result for G-CW structures on smooth G-manifolds proved by Matumoto and Shiota [14]. But this uniqueness result plays a crucial role in our paper [10].

2. An example

Let S^1 be the circle group; i.e., $S^1 = \{\zeta \in \boldsymbol{C} \mid |\zeta| = 1\}$. We now construct a finite S^1-CW complex X in the following way. Let $L = D^2 = \{z \in \boldsymbol{C} \mid |z| \leq 1\}$, and let S^1 act on D^2 in the standard way; i.e., the action $S^1 \times L \to L$ is given by

$(\zeta, z) \mapsto \zeta z$, for every $\zeta \in S^1$ and every $z \in L$. Then L is a 1-dimensional S^1-CW complex with three equivariant cells. The S^1-equivariant 0-cells of L are the origin and the boundary $\partial D^2 = S^1$. There is one S^1-equivariant 1-cell of the form $I \times S^1$, which has been attached to the 0-skeleton L^0 in such a way that $\{0\} \times S^1$ is attached to the origin by the constant map and $\{1\} \times S^1$ is attached to $\partial D^2 = S^1$ by the identity map. The S^1-CW complex X, that we wish to consider, is obtained by adjoining an S^1-equivariant 2-cell of the form $D^2 \times S^1$ to L via some strange attaching S^1-map $\psi: \partial D^2 \times S^1 \to L$.

Let $\omega: \partial D^2 \to L$ be a space filling curve such that the image of ω is the whole disk $D^2 = L$; i.e.,

$$\omega(\partial D^2) = L .$$

For the existence of such maps, see for example [2, Section 9.B]. Now define an S^1-map

$$\psi: \partial D^2 \times S^1 \to L$$

by setting $\psi(z, \zeta) = \zeta\omega(z)$, for every $z \in \partial D^2$ and every $\zeta \in S^1$. Since ψ is an S^1-map from the boundary $\partial D^2 \times S^1$ of the S^1-equivariant 2-cell $D^2 \times S^1$ into the (S^1-equivariant) 1-skeleton $L^1 = L$, the adjunction space

$$X = L \cup_\psi (D^2 \times S^1)$$

is (by definition) an S^1-CW complex.

Suppose now that D^2 and S^1 have been given arbitrary CW structures and that $D^2 \times S^1$ has the product CW structure, or any subdivision of the product structure. We claim that the space X cannot have a CW structure, which is induced from the CW structures on $L = D^2$ and $D^2 \times S^1$. (By an induced CW structure on X is meant a CW structure on X such that the family of all open cells of X consists of all open cells of $L = D^2$ and all open cells of $D^2 \times S^1$.) Assume on the contrary that there exists an induced CW structure on the adjunction space X. Let v be any vertex of S^1; we may assume that $v = 1 \in S^1$. Then $D^2 = D^2 \times \{1\}$ is a subcomplex of $D^2 \times S^1$ and hence $Y = L \cup_\omega (D^2 \times \{1\})$ is a subcomplex of X. But it is a well-known fact that the space Y cannot have a CW structure, see [20, p. 51]. Therefore Y cannot be a subcomplex of X. This contradiction proves our claim.

3. Adjunction spaces and equivariant homotopy type

By G we denote, as before, an arbitrary Lie group. (In this section and in Section 4 G could as well be any locally compact group.) Let X be a G-space and (A, B) a G-pair, where B is a closed G-subset of A. Given a G-map φ: $B \to X$ we can form the adjunction space $X \cup_\varphi A$. Since G is locally compact and the natural projection $p: X \dot\cup A \to X \cup_\varphi A$ is an identification map it follows that $\mathrm{id}\times p\colon G\times(X\dot{\cup} A)\to G\times(X\cup_\varphi A)$ is an identification map (see e.g. [4, Theorem XII 4.1]). Hence the induced action of G on $X\cup_\varphi A$ is continuous, and therefore the adjunction space $X\cup_\varphi A$ is a well-defined G-space.

Now assume that the G-pair (A, B), where B is closed in A, has the G-homotopy extension property. This assumption holds if and only if $A\times\{0\}\cup B\times I$ is a G-retract of $A\times I$. Furthermore, $A\times\{0\}\cup B\times I$ is a G-retract of $A\times I$ if and only if $A\times\{0\}\cup B\times I$ is a strong G-deformation retract of $A\times I$. This last fact follows by the same proof as the one in R. Brown [1, Lemma 7.2.3] for the corresponding non-equivariant case. (R. Brown attributes this Lemma 7.2.3 to D. Puppe.) Thus our assumption that the G-pair (A, B), where B is closed in A, has the G-homotopy extension property is equivalent to the assumption:

$(*)$ $\quad A\times\{0\}\cup B\times I$ is a strong G-deformation retract of $A\times I$.

(In Section 5, where we use the results of this section, (A, B) is a G-CW pair and by a well-known property of G-CW pairs condition $(*)$ holds in this case.)

Suppose that the G-pair (A, B), where B is closed in A, satisfies condition $(*)$ and that $\varphi_0, \varphi_1\colon B\to X$ are two G-maps that are G-homotopic. We shall show that the adjunction spaces $X\cup_{\varphi_0}A$ and $X\cup_{\varphi_1}A$ have the same G-homotopy type. Let $\Phi\colon B\times I\to X$ be a G-homotopy from φ_0 to φ_1. We form the adjunction space $X\cup_\Phi(A\times I)$. It follows directly from $(*)$ that $X\cup_\Phi(A\times\{0\}\cup B\times I)$ is a strong G-deformation retract of $X\cup_\Phi(A\times I)$. Similarly $X\cup_\Phi(A\times\{1\}\cup B\times I)$ is a strong G-deformation retract of $X\cup_\Phi(A\times I)$. Let $\hat{i}_0\colon X\cup_\Phi(A\times\{0\}\cup B\times I)\to X\cup_\Phi(A\times I)$ denote the obvious inclusion. Choose a G-retraction $r_1\colon A\times I\to A\times\{1\}\cup B\times I$ and let $\hat{r}_1\colon X\cup_\Phi(A\times I)\to X\cup_\Phi(A\times\{1\}\cup B\times I)$ denote the G-retraction induced by r_1. Define

$$k(\Phi, r_1)\colon X\cup_{\varphi_0}A\to X\cup_{\varphi_1}A$$

to be the composite map

$$X\cup_{\varphi_0}A\xrightarrow{\cong}X\cup_\Phi(A\times\{0\}\cup B\times I)\xrightarrow{\hat{i}_0}X\cup_\Phi(A\times I)\xrightarrow{\hat{r}_1}$$
$$X\cup_\Phi(A\times\{1\}\cup B\times I)\xrightarrow{\cong}X\cup_{\varphi_1}A\,,$$

where the first and the last map are natural G-homeomorphisms, which we shall use as identifications. Since both $\hat{i}_0$ and $\hat{r}_1$ are G-homotopy equivalences, it follows that $k(\Phi, r_1)$ is a G-homotopy equivalence.

In the following we shall leave out from our notation the natural G-homeomorphisms appearing in the above composite map as the first and last map, and simply write

$$k(\Phi, r_1)\colon X\cup_{\varphi_0}A\xrightarrow{\hat{i}_0}X\cup_\Phi(A\times I)\xrightarrow{\hat{r}_1}X\cup_{\varphi_1}A\,.$$

Here i_0 now denotes the inclusion $A \to A\times I$, given by $a \mapsto (a, 0)$, but one must keep in mind that r_1 is a G-retraction from $A\times I$ onto $A\times\{1\}\cup B\times I$ so that $\hat{r}_1$ is well-defined.

Any two G-retractions from $A\times I$ onto $A\times\{1\}\cup B\times I$ are G-homotopic rel $(A\times\{1\}\cup B\times I)$ and thus in particular rel $B\times I$. Hence the choice of another G-retraction r': $A\times I\to A\times\{1\}\cup B\times I$ gives rise to a G-homotopy equivalence $k(\Phi, r_1')$: $X\cup_{\varphi_0} A\to X\cup_{\varphi_1} A$, which is G-homotopic to $k(\Phi, r_1)$ rel X. Thus we see that the G-homotopy class rel X determined by $k(\Phi, r_1)$ is independent of the choise of the retraction r_1. We shall in fact, by a slight abuse of notation and terminology, often use

$$k(\Phi)\colon X\cup_{\varphi_0} A\to X\cup_{\varphi_1} A$$

to denote any G-homotopy equivalence of the form $k(\Phi, r_1)$ and call $k(\Phi)$ the G-homotopy equivalence determined by Φ. Observe that we have $k(\Phi)\,|\,X=\mathrm{id}_X$. If we by Φ^{-1} denote the inverse homotopy of Φ we have that $k(\Phi^{-1})$ is a G-homotopy inverse of $k(\Phi)$ rel X.

In 3.1–3.3 below X and Y denote arbitrary G-spaces and (A, B) denotes a G-pair which has the G-homotopy extension property and where B is closed in A. Furthermore, φ: $B\to X$ is a G-map. Our discussion above already proved the following result.

Proposition 3.1. *Suppose that the G-maps φ_0, φ_1: $B\to X$ are G-homotopic and that Φ: $B\times I\to X$ is a G-homotopy from φ_0 to φ_1. Then*

$$k(\Phi)\colon X\cup_{\varphi_0} A\to X\cup_{\varphi_1} A$$

is a G-homotopy equivalence and $k(\Phi)\,|\,X=\mathrm{id}_X$. Furthermore, $k(\Phi^{-1})$ is a G-homotopy inverse of $k(\Phi)$ rel X. $\square$

If f: $X\to Y$ is a G-map we define

$$\bar{f}\colon X\cup_{\varphi} A\to Y\cup_{f\varphi} A$$

to be the G-map induced by f and the identity map on A. We call $\bar{f}$ the canonical extension of f.

Lemma 3.2. *Suppose that the G-maps f_0, f_1: $X\to Y$ are G-homotopic and that F: $X\times I\to Y$ is a G-homotopy from f_0 to f_1. Then the diagram*

$$\begin{tikzcd} X\cup_{\varphi} A \arrow[r, "\bar{f}_0"] \arrow[dr, "\bar{f}_1"'] & Y\cup_{f_0\varphi} A \arrow[d, "k(\theta)", "\simeq"'] \\ & Y\cup_{f_1\varphi} A \end{tikzcd}$$

is G-homotopy commutative. Here $\theta = F\circ(\varphi\times\mathrm{id})\colon B\times I\to Y$ *and* $k(\theta)$ *denotes the corresponding G-homotopy equivalence as given by Proposition* 3.1.

Proof. The composite map

$$(X\cup_\varphi A)\times I=(X\times I)\cup_{\varphi\times\mathrm{id}}(A\times I)\xrightarrow{\bar F} Y\cup_\theta(A\times I)\xrightarrow{\hat r_1} Y\cup_{f_1\varphi}A\,,$$

is a G-homotopy from $k(\theta)\circ\bar f_0$ to $\bar f_1$. □

Proposition 3.3. *If* $f\colon X\to Y$ *is a G-homotopy equivalence then so is its canonical extension* $\bar f\colon X\cup_\varphi A\to Y\cup_{f\varphi}A$.

Proof. Let $h\colon Y\to X$ be a G-homotopy inverse of f. Since $h\circ f$ is G-homotopic to id_X we have by Proposition 3.1 and Lemma 3.2 that there is a G-homotopy equivalence $k\colon X\cup_{hf\varphi}A\to X\cup_\varphi A$ such that the composite map

$$X\cup_\varphi A\xrightarrow{\overline{h\circ f}} X\cup_{hf\varphi}A\xrightarrow{k} X\cup_\varphi A$$

is G-homotopic to the identity; i.e., $k\circ\bar h\circ\bar f\simeq\mathrm{id}$. Thus $k\circ\bar h$ is a left G-homotopy inverse of $\bar f$. In the same way one shows that $\bar h$ has a left G-homotopy inverse. Since k is a G-homotopy equivalence it is now an easy formal consequence of the above facts that $\bar f$ is a G-homotopy equivalence, with $k\circ\bar h$ as a G-homotopy inverse (cf. [15, p. 22–23]). □

The result in Proposition 3.1 is a generalization of the classical result by J.H.C. Whitehead [19, Lemma 5] on the homotopy type of a space with a cell attached, and Proposition 3.3 gives a similar extension of an old result (cf. Milnor [15, Lemma 3.7.] and Hilton [5, Proposition 6.8]). Both Proposition 3.1 and 3.3 are well known facts, the presence of a transformation group does not in this case add anything essentially new.

4. The telescope construction

We shall use a straightforward equivariant generalization of the ordinary telescope construction. (We refer to the Appendix of Milnor [15] for a description of the ordinary telescope construction and its basic properties.) As in Section 3, the group G may in this section be an arbitrary locally compact group but we will use the results of this section only in the case when G is a Lie group.

Let X be a G-space and let $X_0\subset X_1\subset X_2\subset\cdots$ be an expanding sequence of closed G-subsets of X such that $X=\bigcup_{n\geq 0}X_n$. Then we define the corresponding telescope to be the space

$$X_\Sigma = X_0\times[0,1]\cup X_1\times[1,2]\cup X_2\times[2,3]\cup\cdots$$

with the induced action of G. The topology of X is the relative topology from $X\times\boldsymbol{R}$. We say that X is the *G-homotopy direct limit* of the sequence $\{X_n\}_{n\geq 0}$

if the natural projection $p: X_\Sigma \to X$, given by $p(x, t)=x$, is a G-homotopy equivalence. A completely straightforward equivariant version of the proof of Theorem A in the Appendix of [15] proves the following result.

Proposition 4.1. *Suppose that X is the G-homotopy direct limit of $\{X_n\}_{n\geq 0}$ and Y is the G-homotopy direct limit of $\{Y_n\}_{n\geq 0}$. Let $f: X \to Y$ be a G-map which induces G-homotopy equivalences $f_n=f|: X_n \to Y_n$ for all $n \geq 0$. Then f is a G-homotopy equivalence.* $\square$

For any $n \geq 0$ we denote

$$X_{\Sigma,n} = X_0 \times [0, 1] \cup X_1 \times [1, 2] \cup \cdots \cup X_n \times [n, n+1] .$$

First we need the following fact.

Lemma 4.2. *Assume that X is compactly generated and that the topology of X is coherent with $\{X_n\}_{n\geq 0}$. Then the topology of X_Σ is coherent with $\{X_{\Sigma,n}\}_{n\geq 0}$.*

Proof. Suppose $A \subset X_\Sigma$ is such that $A \cap X_{\Sigma,n}$ is closed in $X_{\Sigma,n}$ for all $n \geq 0$. We shall show that A is closed in X_Σ. Since X is compactly generated and $\boldsymbol{R}$ is locally compact it follows that $X \times \boldsymbol{R}$ is compactly generated (see Theorem 4.3 in [17]). Hence X_Σ is compactly generated since it is a closed subset of $X \times \boldsymbol{R}$. Thus it is enough to prove that $C \cap A$ is closed for every compact subset C of X_Σ. That this fact holds is seen as follows.

Let $C \subset X_\Sigma$ be compact. Then there exists an $m \geq 0$ such that $p_2(C) \subset [0, m+1]$, where $p_2: X \times \boldsymbol{R} \to \boldsymbol{R}$ denotes the natural projection. Then $C \subset X_{\Sigma,m}$ and thus $C \cap A = C \cap A \cap X_{\Sigma,m}$. Since $A \cap X_{\Sigma,m}$ is closed, by assumption, it now follows that $C \cap A$ is closed. $\square$

Proposition 4.3. *Let X be a G-CW complex and let $X_0 \subset X_1 \subset \cdots$ be an expanding sequence of G-subcomplexes of X such that $\bigcup_{n\geq 0} X_n = X$. Then the corresponding telescope X_Σ is a G-CW complex, with each $X_{\Sigma,n}$ as a G-subcomplex. Furthermore, X is the G-homotopy direct limit of $\{X_n\}_{n\geq 0}$.*

Proof. Clearly each $X_{\Sigma,n}$ is a G-CW complex with $X_{\Sigma,n-1}$ as a G-subcomplex. Thus, in order to prove that X_Σ is a G-CW complex we only need to show that the topology of X_Σ is coherent with the family $\{X_{\Sigma,n}\}_{n\geq 0}$. This is seen as follows.

We know by Lemma 1.1 that X is compactly generated. Since the topology of the G-CW complex X is coherent with the family of all closed G-cells it also follows that the topology of X is coherent with the family $\{X_n\}_{n\geq 0}$. Thus Lemma 4.2 implies that the topology of X_Σ is coherent with $\{X_{\Sigma,n}\}_{n\geq 0}$.

In order to establish the last claim in Proposition 4.3 we must prove that $p: X_\Sigma \to X$ is a G-homotopy equivalence. Since X_Σ and X are G-CW complexes, it is, by the equivariant Whitehead theorem, enough to show the following: For each $K<G$ the map $p^K: X^K_{\Sigma}\to X^K$ induces isomorphisms

$$(1) \qquad p^K_*: \pi_q(X^K_{\Sigma}, x_0) \to \pi_q(X^K, p(x_0))$$

for every $q\geq 0$ and any $x_0\in X^K_{\Sigma}$.

Clearly $p_n=p|: X_{\Sigma,n}\to X_n$ is a G-homotopy equivalence for all $n\geq 0$. Hence we have for each $K<G$ that $p^K_n: X^K_{\Sigma,n}\to X^K_n$ is an ordinary homotopy equivalence and therefore we have for all $n\geq 0$ that the maps

$$(2) \qquad (p^K_n)_*: \pi_q(X^K_{\Sigma,n}, x_0) \to \pi_q(X^K_n, p(x_0))$$

are isomorphisms for every $q\geq 0$ and any $x_0\in X^K_{\Sigma,n}$. Since X and X_{Σ} are G-CW complexes it follows that a compact subset of X or of X_{Σ} is contained in some X_m or in some $X_{\Sigma,k}$, respectively, where $m, k\in N$. This fact together with the fact that (2) holds for all $n\geq 0$ implies that (1) holds. □

Corollary 4.4. *Let $X_0\subset X_1\subset\cdots$ be a filtration of a G-CW complex X by G-subcomplexes, and $Y_0\subset Y_1\subset\cdots$ a filtration of an H-CW complex Y by H-subcomplexes, where H is a closed subgroup of G. Suppose $f: X\to Y$ is an H-map which induces H-homotopy equivalences $f_n=f|: X_n\to Y_n$ for all $n\geq 0$. Then $f: X\to Y$ is an H-homotopy equivalence.*

Proof. We know by Proposition 4.3 that the G-CW complex X is the G-homotopy direct limit of $\{X_n\}_{n\geq 0}$. It follows that the H-space X is the H-homotopy direct limit of the H-spaces $\{X_n\}_{n\geq 0}$. Since we also know by Proposition 4.3 that the H-CW complex Y is the H-homotopy direct limit of $\{Y_n\}_{n\geq 0}$, the result follows by Proposition 4.1. □

5. Proof of Theorem A

In this section G denotes an arbitrary Lie group and H is a fixed *compact* subgroup of G. Recall that the notation $P<G$ means that P is a closed subgroup of G.

Proof of Theorem A. Let X be an arbitrary G-CW complex and let $X^0\subset X^1\subset X^2\subset\cdots$ be the filtration of X by skeletons. The 0-skeleton X^0 is a disjoint union of various G-orbits of the form G/P, where $P<G$. When we consider X^0 with the induced action of H we have that H acts smoothly on each homogeneous space G/P by multiplication on the left. Thus, by Theorem 1.3 there exist an H-CW complex $(X^0)^\vee$ and an H-homeomorphism $\eta_0: X^0\xrightarrow{\cong}(X^0)^\vee$. Then $(X_0)^\vee$ of course satisfies properties (1)–(3) in Theorem A, with respect to X^0, and in particular $\eta_0: X^0\to(X^0)^\vee$ is an H-reduction of X^0 in the sense of Definition C.

Now let $n\geq 1$, and assume inductively that we have constructed an H-reduction $\eta=\eta_{n-1}\colon X^{n-1}\to(X^{n-1})^{\vee}$ of X^{n-1}. The n-skeleton X^n of X is given as an adjunction space

$$X^n = X^{n-1}\cup_{\psi}(\dot{\bigcup_{j\in J}} D^n\times G/P_j)$$

where $P_j<G$, for each $j\in J$, and $\psi\colon \dot{\bigcup_{j\in J}}(S^{n-1}\times G/P_j)\to X^{n-1}$ is a G-map. When we consider the induced action of H, each $D^n\times G/P_j$ is a smooth H-manifold with boundary equal to $S^{n-1}\times G/P_j$. Hence there exist by Theorem 1.3 an *H-CW* pair (A, B) and an H-homeomorphism

$$\alpha\colon (A, B)\overset{\cong}{\to}(\dot{\bigcup_{j\in J}} D^n\times G/P_j,\ \dot{\bigcup_{j\in J}} S^{n-1}\times G/P_j)\,.$$

Thus we obtain an induced H-homeomorphism

$$\hat{\alpha}\colon X^{n-1}\cup_{\varphi} A\overset{\cong}{\to} X^{n-1}\cup_{\psi}(\dot{\bigcup_{j\in J}} D^n\times G/P_j) = X^n$$

where $\varphi=\psi\circ(\alpha\,|)\colon B\to X^{n-1}$. Furthermore $\hat{\alpha}\,|\,X^{n-1}=\mathrm{id}$.

Since $\eta\colon X^{n-1}\to(X^{n-1})^{\vee}$ is an H-homotopy equivalence its canonical extension

$$\bar{\eta}\colon X^{n-1}\cup_{\varphi} A\to(X^{n-1})^{\vee}\cup_{\eta\phi} A$$

is also an H-homotopy equivalence by Proposition 3.3. By the equivariant skeletal approximation theorem (see Matumoto [13, Theorem 4.4] or Illman [6, Proposition 2.3]) the H-map $\eta\circ\varphi\colon B\to(X^{n-1})^{\vee}$, between two *H-CW* complexes, is H-homotopic to a skeletal H-map $\mu\colon B\to(X^{n-1})^{\vee}$. We now define $(X^n)^{\vee}$ to be the adjunction space

$$(X^n)^{\vee} = (X^{n-1})^{\vee}\cup_{\mu} A\,.$$

Since $\mu\colon B\to(X^{n-1})^{\vee}$ is a skeletal H-map it follows that $(X^n)^{\vee}$ is an *H-CW* complex which contains $(X^{n-1})^{\vee}$ as an H-subcomplex. Because the H-maps $\eta\circ\varphi$ and $\mu\colon B\to(X^{n-1})^{\vee}$ are H-homotopic there exists by Proposition 3.1 an H-homotopy equivalence

$$k\colon (X^{n-1})^{\vee}\cup_{\eta\varphi} A\overset{\cong}{\to}(X^{n-1})^{\vee}\cup_{\mu} A = (X^n)^{\vee}\,,$$

such that $k\,|\,(X^{n-1})^{\vee}=\mathrm{id}$. Thus,

$$\tilde{\eta} = k\circ\bar{\eta}\circ(\hat{\alpha})^{-1}\colon X^n\to(X^n)^{\vee}$$

is an H-homotopy equivalence, and $\tilde{\eta}\,|\,X^{n-1}=\eta\colon X^{n-1}\to(X^{n-1})^{\vee}$.

We shall now prove that $(X^n)^{\vee}$ satisfies properties (1)–(3) of Theorem A, with respect to X^n. This will then show that $\eta_n=\tilde{\eta}\colon X^n\to(X^n)^{\vee}$ is an H-reduction of X^n, which extends $\eta_{n-1}=\eta: X^{n-1}\to(X^{n-1})^\vee$.

Let us first verify that $(X^n)^\vee$ satisfies condition (1) of Theorem A, with respect to X^n. Assume that $K<H$ is such that $(X^n)^K$ is compact. Then also $(X^{n-1})^K$ is compact and hence by the inductive assumption $((X^{n-1})^\vee)^K$ is compact. We now make the following simple observations. The disjoint union $\dot{\bigcup}_{j\in J}(D^n_{1/2}\times G/P_j)$, where $D^n_{1/2}=\{x\in D^n \mid ||x||\leq 1/2\}$, is a closed G-subset of X^n. Furthermore, $\dot{\bigcup}_{j\in J}(D^n_{1/2}\times G/P_j)$ is, in the obvious way, G-homeomorphic and hence also H-homeomorphic to the space $\dot{\bigcup}_{j\in J}(D^n\times G/P_j)$, which in turn is H-homeomorphic to A. Since $(X^n)^K$ is compact it now follows from the above observations that also A^K is compact. Hence we now have that $((X^n)^\vee)^K=((X^{n-1})^\vee)^K\cup_\mu A^K$ is compact. A completely similar argument shows that if $((X^n)^\vee)^K$ is compact, then so is $(X^n)^K$.

Next we prove that $(X^n)^\vee$ satisfies condition (2) of Theorem A. Let K be a closed subgroup of H. It follows by Lemma 1.1 that X^n and $(X^n)^\vee$ are normal spaces, and hence also the fixed point sets $(X^n)^K$ and $((X^n)^\vee)^K$ are normal. By the inductive assumption we have that

$$\dim(X^{n-1})^K = \dim((X^n)^\vee)^K = d' .$$

Both X^n-X^{n-1} and $(X^n)^\vee-(X^{n-1})^\vee$ are H-homeomorphic to the disjoint union $\dot{\bigcup}_{j\in J}\mathring{D}^n\times G/P_j$. Hence the fixed point sets $(X^n-X^{n-1})^K$ and $((X^n)^\vee-(X^{n-1})^\vee)^K$ are normal and homeomorphic to each other. In particular they have the same dimension; i.e.,

$$\dim(X^n-X^{n-1})^K = \dim((X^n)^\vee-(X^{n-1})^\vee)^K = d'' .$$

For any closed subset C of $(X^n)^K$ such that $C\subset(X^n-X^{n-1})^K$ we have that dim $C\leq d''$, and clearly there exists such a closed subset C with dim $C=d''$. It now follows by [16, Theorem 9.11], and the fact that the covering dimension is monotonic on closed subsets, that

$$\dim(X^n)^K = \max\{d', d''\} .$$

In exactly the same way we see that dim $((X^n)^\vee)^K=\max\{d', d''\}$, and hence dim $(X^n)^K=\dim((X^n)^\vee)^K$. This shows that $(X^n)^\vee$ satisfies property (2) of Theorem A.

In order to see that the H-isotropy types occurring in X^n and in $(X^n)^\vee$ agree one only needs to observe the following. By the inductive assumption the H-isotropy types occurring in X^{n-1} are the same ones as in $(X^{n-1})^\vee$. Also the H-isotropy types occurring in $X^n-X^{n-1}=\dot{\bigcup}_{j\in J}(\mathring{D}^n\times G/P_j)$ and in $(X^n)^\vee-(X)^{n-1}=A-B$ agree since these two H-spaces are H-homeomorphic.

We have now completed the inductive step and proved that $\eta_n=\tilde{\eta}: X^n\to$

$(X^n)^\vee$ is an H-reduction of X^n, which extends $\eta_{n-1}=\eta\colon X^{n-1}\to(X^{n-1})^\vee$.

Now define $\check{X}=\bigcup_{n\geq 0}(X^n)^\vee$, where $\check{X}$ is given the topology coherent with the family $\{(X^n)^\vee\}_{n\geq 0}$, and let $\eta\colon X\to\check{X}$ be the H-map that extends every $\eta_n\colon X^n\to(X^n)^\vee$, $n\geq 0$. Then $\check{X}$ is an H-CW complex and η is an H-homotopy equivalence by Corollary 4.4. Suppose that $K<H$ is such that X^K is compact. Then there exists an $m\geq 0$ such that $X^K\subset X^m$, and hence $X^K=(X^m)^K$. Thus we have that $(X^n-X^{n-1})^K=\emptyset$ for all $n\geq m+1$. Since X^n-X^{n-1} is H-homeomorphic to $(X^n)^\vee-(X^{n-1})^\vee$, for each $n\geq 0$, it follows that also $((X^n)^\vee-(X^{n-1})^\vee)^K=\emptyset$ for all $n\geq m+1$. Therefore $(\check{X})^K\subset(X^m)^\vee$, and hence $(\check{X})^K=((X^m)^\vee)^K$. But by what we already proved above, the fact that $(X^m)^K$ is compact implies that $((X^m)^\vee)^K$ is compact. This shows that if X^K is compact then also $\check{X}^K$ is compact. A completely similar argument shows that if $\check{X}^K$ is compact then also X^K is compact.

Now let $K<H$. It follows by Lemma 1.1 that both X and $\check{X}$ are normal spaces, and hence also the fixed point sets X^K and $\check{X}^K$ are normal. We have that $X^K=\bigcup_{n\geq 0}(X^n)^K$ and $(\check{X})^K=\bigcup_{n\geq 0}((X^n)^\vee)^K$. Since we already proved that $\dim(X^n)^K=\dim((X^n)^\vee)^K$ for all $n\geq 0$, it now follows by the countable sum theorem for the covering dimension (see e.g. [16, Theorem 9.10]) and the fact that dim is monotonic on closed subsets, that $\dim X^K=\dim\check{X}^K$. The fact that for each $n\geq 0$ the H-isotropy types occurring in X^n are the same as the ones occurring in $(X^n)^\vee$ implies that the H-isotropy types occurring in $\check{X}$ and in X are the same. This completes the proof of Theorem A. □

Proof of Theorem A (rel). Given a G-CW-pair (X, X_0) and an H-reduction $\eta_0\colon X_0\to\check{X}_0$ of X_0 one constructs an H-reduction $\eta\colon X\to\check{X}$ of X which extends η_0 exactly in the same way as in the above proof of Theorem A. This time the induction is over the filtration of X by the skeletons of (X, X_0); i.e., over the filtration $X_0\subset(X, X_0)^0\subset(X, X_0)^1\subset(X, X_0)^2\subset\cdots$, where $(X, X_0)^n=X_0\cup X^n$ for each $n\geq 0$. □

Proof of Theorem A (filt). Repeated use of Theorem A (rel) gives us an H-reduction $\eta_i\colon X_i\to\check{X}_i$, such that η_i extends η_{i-1}, for every $i\geq 0$. This gives us an H-CW complex $\check{X}=\bigcup_{i\geq 0}X_i$ and an H-map $\eta\colon X\to\check{X}$ such that $\eta\,|\,X_i=\eta_i$ for all $i\geq 0$. It follows by Corollary 4.4 that η is an H-homotopy equivalence. The fact that $\eta\colon X\to\check{X}$ is an H-reduction of X; i.e., the fact that $\check{X}$ satisfies conditions (1)–(3) in Theorem A, is seen exactly in the same way as the corresponding fact is proved in the proof of Theorem A. □

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