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Research Paper

Shuffle product of desingularized multiple zeta functions at integer points

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ABSTRACT

In this paper, we investigate the "shuffle-type" formula for special values of desingularized multiple zeta functions at integer points. It is proved by giving an iterated integral/differential expression for the desingularized multiple zeta functions at integer points.

Contents

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0. Introduction

We begin with the *desingularized multiple zeta function* (desingularized MZF for short) which was introduced by Furusho, Komori, Matsumoto, and Tsumura ([6]):

$$\zeta_r^{\mathrm{des}}(s_1,\dots,s_r) = \lim_{\substack{c\to 1\\ c\neq 1}} \frac{1}{(1-c)^r}\prod_{k=1}^{r}\frac{1}{(e^{2\pi i s_k}-1)\Gamma(s_k)}\int_{\mathcal{C}_\epsilon^r}\tilde{\mathfrak{H}}_r(t_1,\dots,t_r;c)\prod_{k=1}^{r} t_k^{s_k-1}dt_k \tag{0.1}$$

for complex variables $s_1,\dots,s_r$, where $\mathcal{C}_\epsilon$ is the *Hankel contour* (see Definition 1.1), and for $c\in\mathbb{R}$, we put

$$\tilde{\mathfrak{H}}_r(t_1,\dots,t_r;c) := \prod_{j=1}^{r}\left(\frac{1}{\exp(\sum_{k=j}^{r} t_k)-1} - \frac{c}{\exp(c\sum_{k=j}^{r} t_k)-1}\right) \in \mathbb{C}[[t_1,\dots,t_r]].$$

The desingularized MZF was introduced to resolve the infinitely many singularities of the *multiple zeta function* (MZF for short), under the motivation of finding a suitable meaning of the special values of MZF $\zeta_r(s_1,\dots,s_r)$ at non-positive integer points $(s_1,\dots,s_r)\in\mathbb{Z}_{\le 0}^r$. Refer to §1.1 for more information on MZFs and desingularized ones. Here, we will describe some properties of the desingularized MZFs that are shown in [6].

(i) $\zeta_r^{\mathrm{des}}(s_1,\dots,s_r)$ is entire on whole space $\mathbb{C}^r$.
(ii) $\zeta_r^{\mathrm{des}}(s_1,\dots,s_r)$ is expressed as a finite "linear" combination of MZFs.
(iii) Special values of $\zeta_r^{\mathrm{des}}(s_1,\dots,s_r)$ at non-positive integer points are calculated explicitly by using Seki-Bernoulli numbers.

Furthermore, the first-named author showed the following:

Proposition 1.7. *(cf. [10, Theorem 2.7]) For* $s_1, \dots, s_p \in \mathbb{Z}$, $l_1, \dots, l_q \in \mathbb{Z}_{\geq 0}$, *we have*

$$\begin{aligned}&\zeta_p^{\mathrm{des}}(s_1, \dots, s_p)\zeta_q^{\mathrm{des}}(-l_1, \dots, -l_q)\\ &= \sum_{\substack{i_b + j_b = l_b \\ i_b, j_b \geq 0 \\ 1 \leq b \leq q}} \prod_{a=1}^{q} (-1)^{i_a} \binom{l_a}{i_a} \zeta_{p+q}^{\mathrm{des}}(s_1, \dots, s_{p-1}, s_p - i_1 - \cdots - i_q, -j_1, \dots, -j_q).\end{aligned}$$

In this paper, we examine the so-called shuffle product formula for special values of desingularized MZFs at "integer" points. As a generalization of Proposition 1.7 and the main theorem of this paper, we show the following:

Theorem 4.8. *Let* $\mathcal{H}$ *be a Hopf algebra which is defined in* (2.3)*. We define the* $\mathbb{Q}$*-linear map* $\zeta_{\sqcup\!\sqcup}^{\mathrm{des}} : \mathcal{H} \to \mathbb{R}$ *by* $\zeta_{\sqcup\!\sqcup}^{\mathrm{des}}(1) := 1$ *and*

$$\zeta_{\sqcup\!\sqcup}^{\mathrm{des}}(j^{k_r}y \cdots j^{k_1}y) := \lim_{t \to 1-0} \mathrm{Li}^{\mathrm{des}}(k_1, \dots, k_r)(t),$$

for $k_1, \dots, k_r \in \mathbb{Z}$*. Then, this map* $\zeta_{\sqcup\!\sqcup}^{\mathrm{des}}$ *forms a* $\mathbb{Q}$*-algebra homomorphism. For* $\mathrm{Li}^{\mathrm{des}}(k_1, \dots, k_r)(t)$*, see Definition 2.10.*

As a consequence of this theorem, the following holds

$$\zeta_{\sqcup\!\sqcup}^{\mathrm{des}}(j^{k_r}y \cdots j^{k_1}y \sqcup\!\sqcup_0 j^{l_s}y \cdots j^{l_1}y) = \zeta_{\sqcup\!\sqcup}^{\mathrm{des}}(j^{k_r}y \cdots j^{k_1}y)\zeta_{\sqcup\!\sqcup}^{\mathrm{des}}(j^{l_s}y \cdots j^{l_1}y) \tag{0.2}$$

for $r, s \in \mathbb{N}$, $k_1, \dots, k_r, l_1, \dots, l_s \in \mathbb{Z}$. We call this product formula *the shuffle-type formula*. For $k_1, \dots, k_r, l_1, \dots, l_s \in \mathbb{N}$, one can see that (0.2) matches the same shuffle product of MZVs.

The following proposition is key to proving Theorem 4.8.

Theorem 4.6. *For* $\mathbf{k} \in \mathbb{Z}^r$*, we have*

$$\mathrm{Li}^{\mathrm{des}}(\mathbf{k})(t) = \widehat{\zeta}_r^{\mathrm{des}}(\mathbf{k})(t).$$

Here, $\mathrm{Li}^{\mathrm{des}}(\mathbf{k})(t)$ is defined by a certain iterated integral (Definition 2.10) and $\widehat{\zeta}_r^{\mathrm{des}}(\mathbf{k})(t)$ is a special value of a certain function $\widehat{\zeta}_r^{\mathrm{des}}(\mathbf{s})(t)$ on $\mathbb{C}^r$ at an integer point $\mathbf{k}$ defined by the Hankel contour integral (Definition 4.1). By definition, we know that $\mathrm{Li}^{\mathrm{des}}(\mathbf{k})(t)$ satisfies the shuffle product formula, and we show that, in the limit as $t \to 1$, the function $\widehat{\zeta}_r^{\mathrm{des}}(\mathbf{s})(t)$ coincides with the desingularized MZF $\zeta_r^{\mathrm{des}}(\mathbf{s})$ (Proposition 4.2). Therefore, we attain our main theorem.

In order to prove Theorem 4.6, we show that both $\mathrm{Li}^{\mathrm{des}}(\mathbf{k})(t)$ and $\widehat{\zeta}_r^{\mathrm{des}}(\mathbf{k})(t)$ share the same representation. More precisely, we show the following two statements:

Theorem 3.16. *For* $\mathbf{k} \in \mathbb{Z}^r$*, we have*

$$\mathrm{Li}^{\mathrm{des}}(\mathbf{k})(t) = \sum_{q \geq 0} (-1)^q \frac{(\log t)^q}{q!} \sum_{\substack{\mathbf{l}=(l_j) \in \mathbb{Z}_{\geq 0}^r \\ \mathbf{m}=(m_j) \in \mathbb{Z}^r \\ |\mathbf{m}|=-q}} a^r_{\mathbf{l},\mathbf{m}}(q) \left(\prod_{j=1}^r (k_j)_{l_j} \right) \mathrm{Li}_{\mathbf{k}+\mathbf{m}}(t). \tag{0.3}$$

Theorem 4.5. *For* $\mathbf{k} \in \mathbb{Z}^r$*, we have*

$$\widehat{\zeta}_r^{\mathrm{des}}(\mathbf{k})(t) = \sum_{q \geq 0} (-1)^q \frac{(\log t)^q}{q!} \sum_{\substack{\mathbf{l}=(l_j) \in \mathbb{Z}_{\geq 0}^r \\ \mathbf{m}=(m_j) \in \mathbb{Z}^r \\ |\mathbf{m}|=-q}} a^r_{\mathbf{l},\mathbf{m}}(q) \left(\prod_{j=1}^r (k_j)_{l_j} \right) \mathrm{Li}_{\mathbf{k}+\mathbf{m}}(t). \tag{0.4}$$

See Proposition 1.3 for definition of the Pochhammer symbol $(k_j)_{l_j}$, and see (A.1) for the symbol $a^r_{\mathbf{l},\mathbf{m}}(q)$.

The construction of this paper goes as follows: In §1, we recall the MZF and its desingularization. And then, we observe that the shuffle product formula for MZVs holds for special values of the desingularized MZFs with depth 1 at positive integer points (Proposition 1.10). In §2, we review a certain Hopf algebra derived from the differential relation of multiple polylogarithms. We show that special values of $\zeta_r^{\mathrm{des}}(s_1, \ldots, s_r)$ at all-negative integer points $(s_1, \ldots, s_r) \in \mathbb{Z}_{<0}^r$ have an "*iterated differential expression*". After that, we introduce $\mathrm{Li}^{\mathrm{des}}(\mathbf{k})(t)$ by an iterated integral expression. In §3, we define a function $Z(\mathbf{k}; t)$ to express $\mathrm{Li}^{\mathrm{des}}(\mathbf{k})(t)$ as a finite linear combination of multiple polylogarithms. At the end of §3, we show that $Z(\mathbf{k}; t)$ coincides with $\mathrm{Li}^{\mathrm{des}}(\mathbf{k})(t)$. In §4, we introduce an $(r+1)$-fold analytic function $\widehat{\zeta}_r^{\mathrm{des}}(s_1, \ldots, s_r)(t)$ defined by a Hankel contour integral. By showing $\widehat{\zeta}_r^{\mathrm{des}}(\mathbf{k})(t)$ and $Z(\mathbf{k}; t)$ are equal, we prove our main theorem (Theorem 4.8). In Appendix A, we give an explicit expression of $Z(\mathbf{k}; t)$ in terms of $\mathrm{Li}_{\mathbf{k}}(t)$.

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1. The MZFs and desingularized MZFs

In this section, we review the multiple zeta functions in §1.1. In §1.2, we recall the definition of the desingularized MZFs which is introduced by Furusho, Komori, Matsumoto, and Tsumura in [6], and explain some properties of them. In §1.3, we observe that special values of the desingularized MZFs in one variable case at all positive integers satisfy the shuffle product formula.

1.1. The MZFs

In this subsection, we recall the multiple zeta functions.

Let $r \in \mathbb{N}$. The Euler-Zagier *multiple zeta function* (MZF for short) is the r-fold complex analytic function defined by

$$\zeta_r(s_1, s_2, \dots, s_r) = \sum_{0<n_1<n_2<\cdots<n_r} \frac{1}{n_1^{s_1} n_2^{s_2} \cdots n_r^{s_r}}.$$

It converges absolutely in the region

$$\mathcal{D}_r := \{(s_1, \dots, s_r) \in \mathbb{C}^r \mid \Re(s_{r-j+1} + \cdots + s_r) > j \text{ for } 1 \le j \le r\}.$$

Here, $\Re(s)$ means the real part of $s \in \mathbb{C}$. In the early 2000s, Zhao ([12]) and Akiyama, Egami, and Tanigawa ([1]) independently showed that the MZF can be meromorphically continued to $\mathbb{C}^r$. In particular, in [1], the set of singularities of the MZF is determined as all $(s_1, \dots, s_r) \in \mathbb{C}^r$ satisfying either of the following conditions;

$$\begin{aligned} s_r &= 1, \\ s_{r-1} + s_r &= 2, 1, 0, -2, -4, -6, \dots, \\ \sum_{i=1}^{k} s_{r-i+1} &\in \mathbb{Z}_{\le k}, \quad (k = 3, 4, \dots, r). \end{aligned}$$

This shows that almost all non-positive integer points are located in the set of singularities. In addition, they are known to be points of indeterminacy. Only the special values $\zeta(-k)$ ($k \in \mathbb{Z}_{\ge 0}$) and $\zeta_2(-k_1, -k_2)$ ($k_1, k_2 \in \mathbb{Z}_{\ge 0}$ with $k_1 + k_2$ odd) are well-defined. It is one of the fundamental problems to give a nice meaning of "$\zeta_r(-k_1, \dots, -k_r)$" for $k_1, \dots, k_r \in \mathbb{Z}_{\ge 0}$. Regarding this, desingularization method (§1.2) and the renormalization method (§2.1, 2.2) are known.

1.2. Desingularized MZFs

In this subsection, we recall the definition of the desingularized MZFs and their remarkable properties.

We start with the generating function[1] $\tilde{\mathfrak{H}}_r(t_1, \dots, t_r; c) \in \mathbb{C}[[t_1, \dots, t_r]]$ (cf. [6, Definition 1.9]); for $c \in \mathbb{R}$, we put

$$\begin{aligned} \tilde{\mathfrak{H}}_r(t_1, \dots, t_r; c) &:= \prod_{j=1}^{r} \left(\frac{1}{\exp(\sum_{k=j}^{r} t_k) - 1} - \frac{c}{\exp(c \sum_{k=j}^{r} t_k) - 1} \right) \\ &= \prod_{j=1}^{r} \left(\sum_{m \ge 1} (1 - c^m) B_m \frac{(\sum_{k=j}^{r} t_k)^{m-1}}{m!} \right). \end{aligned}$$

[1] It is denoted by $\tilde{\mathfrak{H}}_r((t_j); (1); c)$ in [6].

Here, B_m ($m \in \mathbb{Z}_{\geq 0}$) is the m-th Seki-Bernoulli number which is defined by

$$\frac{x}{e^x - 1} := \sum_{m \geq 0} \frac{B_m}{m!} x^m.$$

We note that $B_0 = 1$, $B_1 = -\frac{1}{2}$ and $B_2 = \frac{1}{6}$.

Definition 1.1 *([6, Definition 3.1]).* For $s_1, \ldots, s_r \in \mathbb{C} \setminus \mathbb{Z}$, we define

$$\zeta_r^{\mathrm{des}}(s_1, \ldots, s_r) := \lim_{\substack{c \to 1 \\ c \neq 1}} \frac{1}{(1-c)^r} C(\mathbf{s}) \int_{\mathcal{C}_\epsilon^r} \tilde{\mathfrak{H}}_r(t_1, \ldots, t_r; c) \prod_{k=1}^{r} t_k^{s_k - 1} dt_k, \tag{1.1}$$

where $\mathcal{C}_\epsilon$ is the Hankel contour, that is, the path consisting of the positive axis (top side), a counterclockwise circle around the origin of radius ϵ (sufficiently small), and the positive real axis (bottom side). We put

$$C(\mathbf{s}) := C(s_1, \ldots, s_k) := \prod_{k=1}^{r} \frac{1}{(e^{2\pi i s_k} - 1)\Gamma(s_k)}. \tag{1.2}$$

We call $\zeta_r^{\mathrm{des}}(s_1, \ldots, s_r)$ the *desingularized MZF.*

The next proposition guarantees the convergence of the right-hand side of (1.1).

Proposition 1.2 *([6, Theorem 3.4]). The function $\zeta_r^{\mathrm{des}}(s_1, \ldots, s_r)$ can be analytically continued to $\mathbb{C}^r$ as an entire function in $(s_1, \ldots, s_r) \in \mathbb{C}^r$ by the following integral expression;*

$$\begin{aligned} &\zeta_r^{\mathrm{des}}(s_1, \ldots, s_r) \\ &= C(\mathbf{s}) \cdot \int_{\mathcal{C}_\epsilon^r} \prod_{j=1}^{r} \left(\frac{1}{\exp(\sum_{k=j}^{r} t_k) - 1} - \frac{\sum_{k=j}^{r} t_k \exp(\sum_{k=j}^{r} t_k)}{(\exp(\sum_{k=j}^{r} t_k) - 1)^2} \right) \prod_{k=1}^{r} t_k^{s_k - 1} dt_k. \end{aligned}$$

We review useful notations to show another wonderful property of $\zeta_r^{\mathrm{des}}(s_1, \ldots, s_r)$. For indeterminates u_j and v_j ($1 \leq j \leq r$), we set

$$\mathcal{G}_r := \mathcal{G}_r(u_1, \ldots, u_r; v_1, \ldots, v_r) := \prod_{j=1}^{r} \{1 - (u_j v_j + \cdots + u_r v_r)(v_j^{-1} - v_{j-1}^{-1})\} \tag{1.3}$$

with the convention $v_0^{-1} := 0$, and we define the set of integers $\{a_{\mathbf{l},\mathbf{m}}^r\}$ by

$$\mathcal{G}_r(u_1, \ldots, u_r; v_1, \ldots, v_r) = \sum_{\substack{\mathbf{l}=(l_j) \in \mathbb{Z}_{\geq 0}^r \\ \mathbf{m}=(m_j) \in \mathbb{Z}^r \\ |\mathbf{m}|=0}} a_{\mathbf{l},\mathbf{m}}^r \prod_{j=1}^{r} u_j^{l_j} v_j^{m_j}. \tag{1.4}$$

Here, we put $|\mathbf{m}| := m_1 + \cdots + m_r$.

One of the remarkable properties of the desingularized MZFs is that it can be represented as a finite "linear" combination of MZFs.

Proposition 1.3 *([6, Theorem 3.8]). For $s_1, \ldots, s_r \in \mathbb{C}$, we have the following equality between meromorphic functions of the complex variables $(s_1, \ldots, s_r)$;*

$$\zeta_r^{\mathrm{des}}(s_1, \ldots, s_r) = \sum_{\substack{\mathbf{l}=(l_j)\in\mathbb{Z}_{\geq 0}^r \\ \mathbf{m}=(m_j)\in\mathbb{Z}^r \\ |\mathbf{m}|=0}} a_{\mathbf{l},\mathbf{m}}^r \left(\prod_{j=1}^r (s_j)_{l_j} \right) \zeta_r(s_1 + m_1, \ldots, s_r + m_r),$$

where $(s)_k$ is the Pochhammer symbol, that is, $(s)_0 := 1$ and $(s)_k := \prod_{j=1}^k (s + j - 1)$ for $k \in \mathbb{N}$ and $s \in \mathbb{C}$.

Let us see some examples of Proposition 1.3.

Examples 1.4. ([6, §4]). In the case $r = 1$, we have

$$\zeta_1^{\mathrm{des}}(s) = (1 - s)\zeta(s).$$

We also see that $\zeta_1^{\mathrm{des}}(1) = -1$.

In the case $r = 2$, we have

$$\begin{aligned} \zeta_2^{\mathrm{des}}(s_1, s_2) =& (s_1 - 1)(s_2 - 1)\zeta_2(s_1, s_2) + s_2(s_2 + 1 - s_1)\zeta_2(s_1 - 1, s_2 + 1) \\ & - s_2(s_2 + 1)\zeta_2(s_1 - 2, s_2 + 2). \end{aligned} \tag{1.5}$$

By [7, Proposition 4.9], we have

$$\zeta_2^{\mathrm{des}}(1, 1) = \frac{1}{2}. \tag{1.6}$$

We consider the following generating function $Z(t_1, \ldots, t_r)$ of the special value $\zeta_r^{\mathrm{des}}(-k_1, \ldots, -k_r)$ $(k_1, \ldots, k_r \in \mathbb{Z}_{\geq 0})$ which will be employed in the later section:

$$Z(t_1, \ldots, t_r) := \sum_{k_1, \ldots, k_r \geq 0} \frac{(-t_1)^{k_1} \cdots (-t_r)^{k_r}}{k_1! \cdots k_r!} \zeta_r^{\mathrm{des}}(-k_1, \ldots, -k_r). \tag{1.7}$$

This is explicitly calculated as follows.

Proposition 1.5 *([6, Theorem 3.7]). We have*

$$Z(t_1, \ldots, t_r) = \prod_{i=1}^r \frac{(1 - t_i - \cdots - t_r)e^{t_i + \cdots + t_r} - 1}{(e^{t_i + \cdots + t_r} - 1)^2}.$$

In terms of $\zeta_r^{\mathrm{des}}(-k_1,\dots,-k_r)$ *for* $k_1,\dots,k_r \in \mathbb{Z}_{\geq 0}$*, the above equation is reformulated to*

$$\zeta_r^{\mathrm{des}}(-k_1,\dots,-k_r) = (-1)^{k_1+\cdots+k_r} \sum_{\substack{\nu_{1i}+\cdots+\nu_{ii}=k_i \\ 1\leq i\leq r \\ \nu_{ij}\in\mathbb{Z}_{\geq 0}}} \prod_{i=1}^{r} \frac{k_i!}{\prod_{j=i}^{r}\nu_{ij}!} B_{\nu_{ii}+\cdots+\nu_{ir}+1}.$$

We recall that the first-named author showed the following formula for special values of the desingularized MZFs.

Proposition 1.6 *([9, Proposition 4.8]). For* $s_1,\dots,s_{r-1}\in\mathbb{C}$ *and* $k\in\mathbb{Z}_{\geq 0}$*, we have*

$$\zeta_r^{\mathrm{des}}(s_1,\dots,s_{r-1},-k) = \sum_{i=0}^{k} \binom{k}{i} \zeta_{r-1}^{\mathrm{des}}(s_1,\dots,s_{r-1}-k+i)\zeta_1^{\mathrm{des}}(-i).$$

Proposition 1.7 *([10, Theorem 2.7]). For* $s_1,\dots,s_p\in\mathbb{C}$*,* $l_1,\dots,l_q\in\mathbb{Z}_{\geq 0}$*, we have*

$$\begin{aligned}&\zeta_p^{\mathrm{des}}(s_1,\dots,s_p)\zeta_q^{\mathrm{des}}(-l_1,\dots,-l_q)\\ &= \sum_{\substack{i_b+j_b=l_b \\ i_b,j_b\geq 0 \\ 1\leq b\leq q}} \prod_{a=1}^{q} (-1)^{i_a}\binom{l_a}{i_a} \zeta_{p+q}^{\mathrm{des}}(s_1,\dots,s_{p-1},s_p-i_1-\cdots-i_q,-j_1,\dots,-j_q).\end{aligned}$$

Our motivation of this paper is based on the question of whether we can naturally extend Proposition 1.7 for all special values at integer points of the desingularized MZFs.

1.3. Observation on the shuffle product of $\zeta_1^{\mathrm{des}}(k)$

In this subsection, we present a simple, but important observation.

At first, we remark on some special values of the double zeta function.

Lemma 1.8 *([2, Proposition 4]). For* $n\in\mathbb{N}_{\geq 2}$ *we have*

$$\lim_{s\to 1}(s-1)\zeta_2(n,s) = \zeta(n). \tag{1.8}$$

Proof. It is a direct consequence of the expansion

$$\zeta_2(n,s) = \frac{\zeta(n)}{s-1} + O(1),$$

where O is Landau's symbol. □

The followings are known for the cases $n\leq 0$.

Remark 1.9 *([7, Example 4.2]).* The following two hold:

$$\zeta_2(0,s) = \zeta(s-1) - \zeta(s),$$
$$\zeta_2(-1,s) = \frac{1}{2}\left\{\zeta(s-2) - \zeta(s-1)\right\}$$

for $s \in \mathbb{C}$ except for singularities. Hence, we have

$$\zeta_2(-1,s) = \frac{1}{2}\zeta_2(0,s-1) \tag{1.9}$$

for s and $s-1$ neither of which is a singularity.

By an elementary calculation, we find the following.

Proposition 1.10. *For n, $m \in \mathbb{N}$, we have*

$$\zeta_1^{\mathrm{des}}(n)\zeta_1^{\mathrm{des}}(m) = \sum_{j=1}^{n+m-1} \left\{ \binom{j-1}{n-1} + \binom{j-1}{m-1} \right\} \zeta_2^{\mathrm{des}}(n+m-j,j), \tag{1.10}$$

where we use the usual convention $\binom{k}{i} = 0$ for $k < i$.

Proof. If $n = m = 1$, by (1.6), we have

$$\zeta_1^{\mathrm{des}}(1)^2 = 1 = 2\zeta_2^{\mathrm{des}}(1,1).$$

We prove (1.10) for $m, n \geq 2$. To save space, we set

$$b_{j,n} := \binom{j-1}{n-1} \qquad (j,\ n \in \mathbb{N}).$$

Since n and m are symmetric, we compute as follows:

$$\begin{aligned}
&\sum_{j=1}^{n+m-1} \binom{j-1}{n-1} \zeta_2^{\mathrm{des}}(n+m-j,j) \\
&= \sum_{j=1}^{n+m-1} b_{j,n} \Big\{ (n+m-j-1)(j-1)\zeta_2(n+m-j,j) \\
&\quad + j(2j+1-n-m)\zeta_2(n+m-j-1,j+1) - j(j+1)\zeta_2(n+m-j-2,j+2) \Big\}.
\end{aligned}$$

Notice that the equality holds by (1.5). By rearranging each term, we have

$$\begin{aligned}
&\sum_{j=1}^{n+m-1}\binom{j-1}{n-1}\zeta_2^{\mathrm{des}}(n+m-j,j)\\
&=\sum_{j=3}^{n+m-1}\Big\{b_{j,n}(n+m-j-1)(j-1)+b_{j-1,n}(j-1)(2j-1-n-m)\\
&\qquad -b_{j-2,n}(j-2)(j-1)\Big\}\zeta_2(n+m-j,j)+R^1_{n,m}+R^2_{n,m},
\end{aligned}$$

where

$$\begin{aligned}
R^1_{n,m}:=\;& b_{1,n}(n+m-2)\zeta(n+m-1)-b_{2,n}(n+m-3)\zeta_2(n+m-2,2)\\
&-b_{1,n}(3-n-m)\zeta_2(n+m-2,2),\\
R^2_{n,m}:=\;& b_{n+m-1,n}(n+m-1)^2\zeta_2(0,n+m)\\
&-b_{n+m-2,n}(n+m-2)(n+m-1)\zeta_2(0,n+m)\\
&-b_{n+m-1,n}(n+m-1)(n+m)\zeta_2(-1,n+m+1).
\end{aligned}$$

Note that by applying (1.8) to $\zeta_2^{\mathrm{des}}(n+m-1,1)$, the term $\zeta(n+m-1)$ appears in the definition of $R^1_{n,m}$. By using $\binom{n-1}{r}=\binom{n}{r}\frac{n-r}{n}$ (i.e. $b_{j-1,n}=b_{j,n}\frac{j-n}{j-1}$), we have

$$\begin{aligned}
&b_{j,n}(n+m-j-1)(j-1)+b_{j-1,n}(j-1)(2j-1-n-m)-b_{j-2,n}(j-2)(j-1)\\
&=\Big\{(n+m-j-1)(j-1)+(j-n)(2j-1-n-m)-(j-n-1)(j-n)\Big\}b_{j,n}\\
&=(1-n)(1-m)b_{j,n}. \qquad (1.11)
\end{aligned}$$

We next compute $R^1_{n,m}$ and $R^2_{n,m}$. We can easily see that the following holds for $n,m\in\mathbb{N}_{\geq 2}$:

$$R^1_{n,m}=-b_{2,n}(m-1)\zeta_2(n+m-2,2). \qquad (1.12)$$

By using $b_{j-1,n}=b_{j,n}\frac{j-n}{j-1}$, we have

$$\begin{aligned}
R^2_{n,m}=\;& b_{n+m-1,n}(n+m-1)^2\zeta_2(0,n+m)\\
&-b_{n+m-1,n}(m-1)(n+m-1)\zeta_2(0,n+m)\\
&-b_{n+m-1,n}(n+m-1)(n+m)\zeta_2(-1,n+m+1). \qquad (1.13)
\end{aligned}$$

By using (1.9), we obtain

$$R^2_{n,m}=b_{n+m-1,n}(n-m)(n+m-1)\zeta_2(-1,n+m+1). \qquad (1.14)$$

By (1.11), (1.12), and (1.14), we have

$$\sum_{j=1}^{n+m-1}\binom{j-1}{n-1}\zeta_2^{\mathrm{des}}(n+m-j,j)$$
$$=(1-n)(1-m)\sum_{j=3}^{n+m-1}b_{j,n}\zeta_2(n+m-j,j)-b_{2,n}(m-1)\zeta_2(n+m-2,2)$$
$$+b_{n+m-1,n}(n-m)(n+m-1)\zeta_2(-1,n+m+1).$$

Because $b_{1,n}=0$ for $n\geq 2$, we get

$$\sum_{j=1}^{n+m-1}\binom{j-1}{n-1}\zeta_2^{\mathrm{des}}(n+m-j,j)=(1-n)(1-m)\sum_{j=1}^{n+m-1}b_{j,n}\zeta_2(n+m-j,j)$$
$$+b_{n+m-1,n}(n-m)(n+m-1)\zeta_2(-1,n+m+1).$$

Since $\binom{p+q}{p}=\binom{p+q}{q}$ for p, $q\in\mathbb{N}$, we have $b_{n+m-1,n}=b_{n+m-1,m}$. Thus, we have

$$\sum_{j=1}^{n+m-1}\left\{\binom{j-1}{n-1}+\binom{j-1}{m-1}\right\}\zeta_2^{\mathrm{des}}(n+m-j,j)$$
$$=(1-n)(1-m)\sum_{j=1}^{n+m-1}(b_{j,n}+b_{j,m})\zeta_2(n+m-j,j).$$

Because $m,n\geq 2$, by the shuffle product formula of MZVs, we calculate as

$$\sum_{j=1}^{n+m-1}\left\{\binom{j-1}{n-1}+\binom{j-1}{m-1}\right\}\zeta_2^{\mathrm{des}}(n+m-j,j)$$
$$=(1-n)(1-m)\zeta(n)\zeta(m)=\zeta_1^{\mathrm{des}}(n)\zeta_1^{\mathrm{des}}(m).$$

Hence, the equation (1.10) holds for $m,n\geq 2$. In the same way, we can prove (1.10) for $n=1$ and $m\geq 2$ (or $n\geq 2$ and $m=1$). □

In Theorem 4.8 in §4, we will generalize this proposition to higher depth.

2. Construction of $\mathrm{Li}^{\mathrm{des}}(\mathbf{k})(t)$

In this section, we review the Hopf algebra $(\mathcal{H},ш_0,\Delta_0)$ and its Hopf subalgebra $(\mathcal{H}_-,ш_0,\Delta_0)$ which are introduced in [4], and we prove some algebraic relations on $\mathcal{H}_-$ (Proposition 2.2 and Proposition 2.3). By using these relations, in §2.2, we give a reinterpretation of special values of the desingularized MZFs at non-positive integer points (Corollary 2.8). In §2.3, we give a quick review of multiple polylogarithms and introduce functions $\mathrm{Li}^{\mathrm{des}}(\mathbf{k})(t)$. And then, we mention the relationship between these functions $\mathrm{Li}^{\mathrm{des}}(\mathbf{k})(t)$ for $\mathbf{k}\in\mathbb{Z}_{\leq 0}^r$ and special values $\zeta_r^{\mathrm{des}}(\mathbf{k})$.

2.1. A certain Hopf algebra

We recall the definition of the Hopf algebra $(\mathcal{H}_-, ш_0, \Delta_0)$ introduced in [4] and its properties. Put $\mathbb{Q}\langle d,j,y\rangle$ to be the non-commutative polynomial ring generated by three letters d, j, y with $dj = jd = 1$. We define the product $ш_0 : \mathbb{Q}\langle d,j,y\rangle^{\otimes 2} \to \mathbb{Q}\langle d,j,y\rangle$ by $w ш_0 1 := 1 ш_0 w := w$ and

$$\begin{aligned} yu ш_0 v := u ш_0 yv &:= y(u ш_0 v), \\ ju ш_0 jv &:= j(u ш_0 jv) + j(ju ш_0 v), \\ du ш_0 dv &:= d(u ш_0 dv) - u ш_0 d^2v, \\ du ш_0 jv &:= d(u ш_0 jv) - u ш_0 v, \\ ju ш_0 dv &:= d(ju ш_0 v) - u ш_0 v, \end{aligned} \tag{2.1}$$

for any words u, v, w in $\mathbb{Q}\langle d,j,y\rangle$. Then the pair $(\mathbb{Q}\langle d,j,y\rangle, ш_0)$ forms a non-commutative, non-associative $\mathbb{Q}$-algebra. We define $\mathcal{T}$ to be the linear subspace over $\mathbb{Q}$ of $\mathbb{Q}\langle d,j,y\rangle$ generated by

$$\left\{ j^{k_1}y \cdots j^{k_{r-1}}yj^{k_r} \in \mathbb{Q}\langle d,j,y\rangle \mid r \in \mathbb{N}, k_1, \dots, k_r \in \mathbb{Z}, k_r \neq 0 \right\}.$$

Note that the symbol j^{-k} means d^k for $k \in \mathbb{N}$. And we define $\mathcal{L}$ to be the two-sided ideal of $(\mathbb{Q}\langle d,j,y\rangle, ш_0)$ generated by

$$\left\{ j^k\{d(u ш_0 v) - du ш_0 v - u ш_0 dv\} \mid k \in \mathbb{Z}, u, v\text{:words of } \mathbb{Q}\langle d,j,y\rangle \text{ ending in } y \right\}. \tag{2.2}$$

It is proved that the subspace $\mathcal{T}$ is a two-sided ideal of $(\mathbb{Q}\langle d,j,y\rangle, ш_0)$ in [4, Lemma 3.4]. We define the quotient algebra

$$\mathcal{H} := \mathbb{Q}\langle d,j,y\rangle/(\mathcal{T} + \mathcal{L}). \tag{2.3}$$

Then the pair $(\mathcal{H}, ш_0)$ is a commutative, associative $\mathbb{Q}$-algebra ([4, Proposition 3.5]).

We consider the non-commutative polynomial ring $\mathbb{Q}\langle d,y\rangle$ generated by two letters d and y. The pair $(\mathbb{Q}\langle d,y\rangle, ш_0)$ forms a subalgebra of $(\mathbb{Q}\langle d,j,y\rangle, ш_0)$. We put

$$\mathcal{T}_- := \mathbb{Q}\langle d,y\rangle \cap \mathcal{T}, \qquad \mathcal{L}_- := \mathbb{Q}\langle d,y\rangle \cap \mathcal{L},$$

and define

$$\mathcal{H}_- := \mathbb{Q}\langle d,y\rangle/(\mathcal{T}_- + \mathcal{L}_-).$$

Then there exists a coproduct Δ_0 such that the triple $(\mathcal{H}_-, ш_0, \Delta_0)$ forms a commutative Hopf algebra over $\mathbb{Q}$ (see [4, §3.3.2–3.3.6] for definition of Δ_0). We define the reduced coproduct $\widetilde{\Delta}_0(w) := \Delta_0(w) - 1 \otimes w - w \otimes 1$ for any words w in $\mathcal{H}_-$. In [8, Proposition 2.5], the reduced coproduct $\widetilde{\Delta}_0$ is explicitly given by

$$\begin{aligned} &\widetilde{\Delta}_0(d^{k_1}y\cdots d^{k_r}y) = \sum_{i+j=k_1}\binom{k_1}{i}d^iy\otimes_{\mathrm{sym}} d^jd^{k_2}y\cdots d^{k_r}y \\ &+\sum_{p=2}^{r-1}\sum_{\substack{i_l+j_l=k_l\\1\le l\le p}}\prod_{a=1}^{p}\binom{k_a}{i_a}\sum_{\substack{\{u_q,v_q\}=\{d^{i_q},d^{j_q}y\}\\1\le q\le p-1}} u_1\cdots u_{p-1}d^{i_p}y\otimes_{\mathrm{sym}} v_1\cdots v_{p-1}d^{j_p}d^{k_{p+1}}y\cdots d^{k_r}y, \end{aligned} \tag{2.4}$$

for $r \geq 2$ and $k_1, \ldots, k_r \in \mathbb{Z}_{\geq 0}$. Here, $w_1 \otimes_{\mathrm{sym}} w_2 := w_1 \otimes w_2 + w_2 \otimes w_1$ for $w_1, w_2 \in \mathcal{H}_-$.

Proposition 2.1 *([4, Corollary 3.24 for $\lambda = 0$]). For any word $w(\neq 1)$ in $\mathcal{H}_-$, we have*

$$(2^{\mathrm{dep}(w)} - 2)w = \sqcup\!\sqcup_0 \circ \widetilde{\Delta}_0(w).$$

Here, the symbol dep(w) *means the number of y appearing in the word w.*

We prove the following by using the above proposition.

Proposition 2.2. *Let $r \geq 3$ and $k_1, \ldots, k_r \in \mathbb{Z}_{\geq 0}$. We have*

$$\begin{aligned} &\sum_{\substack{i_l+j_l=k_l\\1\le l\le p}}\prod_{a=1}^{p}\binom{k_a}{i_a}u_1\cdots u_{p-1}d^{i_p}y \sqcup\!\sqcup_0 v_1\cdots v_{p-1}d^{j_p}d^{k_{p+1}}y\cdots d^{k_r}y \\ &\qquad = \sum_{i+j=k_1}\binom{k_1}{i}d^iy \sqcup\!\sqcup_0 d^jd^{k_2}y\cdots d^{k_r}y, \end{aligned}$$

for $p \in \{2, \ldots, r-1\}$ and $(u_q, v_q) \in \{(d^{i_q}, d^{j_q}y), (d^{j_q}y, d^{i_q})\}$ $(1 \leq q \leq p-1)$.

Proof. For $r \geq 1$, we consider the following generating functions

$$\mathcal{G}(t_1,\ldots,t_r) := \sum_{k_1,\ldots,k_r\geq 0}\frac{t_1^{k_1}\cdots t_r^{k_r}}{k_1!\cdots k_r!}d^{k_1}y\cdots d^{k_r}y \ \ (\in \mathcal{H}_-[[t_1,\ldots,t_r]])\,.$$

When $r = 2$, by using the equation (2.4) and Proposition 2.1, we have

$$d^{k_1}yd^{k_2}y = \sum_{i+j=k_1}\binom{k_1}{i}d^iy \sqcup\!\sqcup_0 d^{j+k_2}y,$$

for $k_1, k_2 \in \mathbb{Z}_{\geq 0}$. So we get

$$\mathcal{G}(t_1,t_2) = \sum_{k_1,k_2\geq 0}\frac{t_1^{k_1}t_2^{k_2}}{k_1!k_2!}\sum_{i+j=k_1}\binom{k_1}{i}d^iy \sqcup\!\sqcup_0 d^{j+k_2}y = \sum_{i,j,k_2\geq 0}\frac{t_1^{i+j}t_2^{k_2}}{i!j!k_2!}d^iy \sqcup\!\sqcup_0 d^{j+k_2}y$$

$$= \left\{ \sum_{i \geq 0} \frac{t_1^i}{i!} d^i y \right\} ш_0 \left\{ \sum_{k \geq 0} \frac{1}{k!} d^k y \sum_{j+k_2=k} \binom{k}{j} t_1^j t_2^{k_2} \right\}$$
$$= \mathcal{G}(t_1) ш_0 \left\{ \sum_{k \geq 0} \frac{(t_1+t_2)^k}{k!} d^k y \right\} = \mathcal{G}(t_1) ш_0 \mathcal{G}(t_1+t_2).$$

Therefore, we get

$$\mathcal{G}(t_1, t_2) = \mathcal{G}(t_1) ш_0 \mathcal{G}(t_1+t_2). \tag{2.5}$$

When $r \geq 3$, by direct calculation, we know

$$\sum_{k_1,\ldots,k_r \geq 0} \frac{t_1^{k_1} \cdots t_r^{k_r}}{k_1! \cdots k_r!} \left\{ \sum_{\substack{i_l+j_l=k_l \\ 1 \leq l \leq p}} \prod_{a=1}^{p} \binom{k_a}{i_a} u_1 \cdots u_{p-1} d^{i_p} y ш_0 v_1 \cdots v_{p-1} d^{j_p} d^{k_{p+1}} y \cdots d^{k_r} y \right\}$$
$$= \mathcal{G}(t_1 \circ_1 t_2 \circ_1 \cdots \circ_{p-1} t_p) ш_0 \mathcal{G}(t_1 \diamond_1 t_2 \diamond_2 \cdots \diamond_{p-1} t_p + t_{p+1}, t_{p+2}, \ldots, t_r).$$

Here, the symbols $\circ_q$ and $\diamond_q$ are defined by

$$(\circ_q, \diamond_q) := \begin{cases} (+, \; \textbf{,} \;) & \left((u_q, v_q) = (d^{i_q}, d^{j_q} y)\right), \\ (\; \textbf{,} \; , +) & \left((u_q, v_q) = (d^{j_q} y, d^{i_q})\right), \end{cases}$$

for $1 \leq q \leq p-1$. By using (2.5) and by induction on $r \geq 3$, we get

$$\begin{aligned} &\mathcal{G}(t_1 \circ_1 t_2 \circ_1 \cdots \circ_{p-1} t_p) ш_0 \mathcal{G}(t_1 \diamond_1 t_2 \diamond_2 \cdots \diamond_{p-1} t_p + t_{p+1}, t_{p+2}, \ldots, t_r) \\ &= \mathcal{G}(t_1) ш_0 \mathcal{G}(t_1+t_2) ш_0 \cdots ш_0 \mathcal{G}(t_1+\cdots+t_r) \\ &= \mathcal{G}(t_1) ш_0 \mathcal{G}(t_1+t_2, t_3, \ldots, t_r). \end{aligned}$$

Because we have

$$\begin{aligned} &\mathcal{G}(t_1) ш_0 \mathcal{G}(t_1+t_2, t_3, \ldots, t_r) \\ &= \sum_{k_1,\ldots,k_r \geq 0} \frac{t_1^{k_1} \cdots t_r^{k_r}}{k_1! \cdots k_r!} \left\{ \sum_{i+j=k_1} \binom{k_1}{i} d^i y ш_0 d^j d^{k_2} y \cdots d^{k_r} y \right\}, \end{aligned}$$

hence we obtain the claim. □

Proposition 2.3. *For $r \geq 2$ and $k_1, \ldots, k_r \in \mathbb{Z}_{\geq 0}$, we have*

$$d^{k_1} y \cdots d^{k_r} y = \sum_{i+j=k_1} \binom{k_1}{i} d^i y ш_0 d^j d^{k_2} y \cdots d^{k_r} y.$$

Proof. When $r = 2$, we obtain the claim by the equation (2.4). Set $r \geq 3$. By using the equation (2.4) and Proposition 2.1, we have

$$\begin{aligned} d^{k_1}y\cdots d^{k_r}y =& \frac{1}{2^{r-1}-1}\left\{\sum_{i+j=k_1}\binom{k_1}{i}d^iy \sqcup\!\sqcup_0 d^jd^{k_2}y\cdots d^{k_r}y\right.\\ &+\sum_{p=2}^{r-1}\sum_{\substack{i_l+j_l=k_l\\1\leq l\leq p}}\prod_{a=1}^{p}\binom{k_a}{i_a}\\ &\left.\times\sum_{\substack{\{u_q,v_q\}=\{d^{i_q},d^{j_q}y\}\\1\leq q\leq p-1}} u_1\cdots u_{p-1}d^{i_p}y \sqcup\!\sqcup_0 v_1\cdots v_{p-1}d^{j_p}d^{k_{p+1}}y\cdots d^{k_r}y\right\}. \end{aligned}$$

Note that $\sqcup\!\sqcup_0 (w_1 \otimes_{\mathrm{sym}} w_2) = 2(w_1 \sqcup\!\sqcup_0 w_2)$ for words w_1, w_2 in $\mathcal{H}_-$. By using Proposition 2.2, we get

$$\begin{aligned} d^{k_1}y\cdots d^{k_r}y =& \frac{1}{2^{r-1}-1}\left\{\sum_{i+j=k_1}\binom{k_1}{i}d^iy \sqcup\!\sqcup_0 d^jd^{k_2}y\cdots d^{k_r}y\right.\\ &\left.+(2^{r-1}-2)\sum_{i+j=k_1}\binom{k_1}{i}d^iy \sqcup\!\sqcup_0 d^jd^{k_2}y\cdots d^{k_r}y\right\}\\ =& \sum_{i+j=k_1}\binom{k_1}{i}d^iy \sqcup\!\sqcup_0 d^jd^{k_2}y\cdots d^{k_r}y. \end{aligned}$$

Hence, we obtain the claim. □

2.2. Generalization of the shuffle type renormalization

In this subsection, we first recall the algebraic Birkhoff decomposition which is a fundamental tool in the work of Connes and Kreimer ([3]) on their Hopf algebraic approach to the renormalization of the perturbative quantum field theory.

Let $\mathbb{Q}[[z, z^{-1}]$ be the $\mathbb{Q}$-algebra consisting of all formal Laurent series. Put $G = G(\mathcal{H}_-, \mathbb{Q}[[z, z^{-1}])$ to be the set of all $\mathbb{Q}$-linear map $\varphi : \mathcal{H}_- \to \mathbb{Q}[[z, z^{-1}]$ with $\varphi(1) = 1$. For $\varphi, \psi \in G$, we define the *convolution* $\varphi * \psi \in G$ by

$$\varphi * \psi := m \circ (\varphi \otimes \psi) \circ \Delta_0.$$

Here, m is the natural product of $\mathbb{Q}[[z, z^{-1}]$. Then the pair $(G, *)$ forms a group, whose unit is given by a map $e = u \circ \varepsilon_{\mathcal{H}_-}$. Here, the map u is the natural unit of $\mathbb{Q}[[z, z^{-1}]$ and the map $\varepsilon_{\mathcal{H}_-}$ is the counit of $\mathcal{H}_-$ (see [11] for detail).

Theorem 2.4 *([3], [4]: **algebraic Birkhoff decomposition**). For $\varphi \in G$, there exist unique $\mathbb{Q}$-linear maps $\varphi_+ : \mathcal{H}_- \to \mathbb{Q}[[z]]$ and $\varphi_- : \mathcal{H}_- \to \mathbb{Q}[z^{-1}]$ with $\varphi_-(1) = 1 \in \mathbb{Q}$ such that*

$$\varphi = \varphi_-^{*(-1)} * \varphi_+.$$

Here, the symbol $\varphi_-^{*(-1)}$ means the inverse element of φ_- in $(G(\mathcal{H}_-, \mathbb{Q}[[z, z^{-1}]), *)$. Moreover, the maps φ_+ and φ_- form algebra homomorphisms if the map φ is an algebra homomorphism.

Remark 2.5. We consider the projection $\pi : \mathbb{Q}[[z, z^{-1}] \to \mathbb{Q}[z^{-1}]$ defined by

$$\pi\left(\sum_{n \geq -N} a_n z^n\right) := \sum_{n=-N}^{-1} a_n z^n$$

for $a_n \in \mathbb{Q}$ and $N \in \mathbb{Z}_{\geq 0}$. By the above theorem, we can inductively calculate φ_+ and φ_- by

$$\varphi_-(w) = -\pi\left(\varphi(w) + \sum_{(w)} \varphi_-(w')\varphi(w'')\right),$$

$$\varphi_+(w) = (\mathrm{Id} - \pi)\left(\varphi(w) + \sum_{(w)} \varphi_-(w')\varphi(w'')\right).$$

Here, we use Sweedler's notation of the reduced coproduct $\widetilde{\Delta}_0$ by

$$\widetilde{\Delta}_0(w) = \sum_{(w)} w' \otimes w''.$$

Note that, when $\varphi(\mathcal{H}_-) \subset \mathbb{Q}[[z]]$, we have

$$\varphi_-(w) = 0, \qquad \varphi_+(w) = \varphi(w) \tag{2.6}$$

for any word $w \in \mathcal{H}_-$.

Let $f(z)$ be a Laurent series in $\mathbb{Q}[[z, z^{-1}]$. We define the $\mathbb{Q}$-linear map $\phi_f : \mathcal{H}_- \to \mathbb{Q}[[z, z^{-1}]$ by $\phi_f(1) := 1$ and

$$\phi_f(d^{k_1}y \cdots d^{k_r}y) := \partial_z^{k_1}\left[f(z)\partial_z^{k_2}\left[f(z)\cdots\partial_z^{k_r}[f(z)]\cdots\right]\right], \tag{2.7}$$

for $k_1, \ldots, k_r \in \mathbb{Z}_{\geq 0}$. Here, the symbol ∂_z is the derivative by z. By the Leibniz rule of the derivative ∂_z, we know that the map ϕ_f forms an algebra homomorphism. By applying the above theorem to this map ϕ_f, we get the algebra homomorphism $\phi_{f,+} : \mathcal{H}_- \to \mathbb{Q}[[z]]$. We define $F_f(k_1, \dots, k_r) \in \mathbb{Q}$ by

$$F_f(k_1, \dots, k_r) := \lim_{z\to 0} \phi_{f,+}(d^{k_r}y \cdots d^{k_1}y)$$

for $r \in \mathbb{N}$ and $k_1, \dots, k_r \in \mathbb{Z}_{\geq 0}$. We set the generating function of $F_f(k_1, \dots, k_r)$ by

$$\mathcal{F}_f(z_1, \dots, z_r) := \sum_{k_1,\dots,k_r \geq 0} \frac{z_1^{k_1} \cdots z_r^{k_r}}{k_1! \cdots k_r!} F_f(k_1, \dots, k_r).$$

The following explicit formulae of $\mathcal{F}_f(z_1, \dots, z_r)$ holds.

Theorem 2.6. *For $r \in \mathbb{N}$, we have*

$$\mathcal{F}_f(z_1, \dots, z_r) = f_{\geq 0}(z_r) f_{\geq 0}(z_{r-1} + z_r) \cdots f_{\geq 0}(z_1 + \cdots + z_r).$$

Here, $f_{\geq 0}(z)$ is defined by

$$f_{\geq 0}(z) := \sum_{n \geq 0} \frac{a_n}{n!} z^n$$

for $f(z) = \sum_{n \geq -N} \frac{a_n}{n!} z^n$ $(N \in \mathbb{Z}_{\geq 0})$ with $a_n \in \mathbb{Q}$.

Proof. When $r = 1$, we have $\widetilde{\Delta}_0(d^k y) = 0$, so by Remark 2.5, we calculate as

$$F_f(k_1) = \lim_{z\to 0} \phi_+(d^{k_1}y) = \lim_{z\to 0} (\mathrm{Id} - \pi)\left(\phi(d^{k_1}y)\right) = \lim_{z\to 0} (\mathrm{Id} - \pi)\left(\partial_z^{k_1}[f(z)]\right) = a_{k_1}.$$

Therefore, we get

$$\mathcal{F}_f(z_1) = \sum_{k_1 \geq 0} \frac{z_1^{k_1}}{k_1!} F_f(k_1) = \sum_{k_1 \geq 0} \frac{z_1^{k_1}}{k_1!} a_{k_1} = f_{\geq 0}(z_1). \tag{2.8}$$

Let $r \geq 2$. By Proposition 2.3, we have

$$\begin{aligned}
&F_f(k_1, \dots, k_r) \\
&= \lim_{z\to 0} \phi_{f,+}(d^{k_r}y \cdots d^{k_1}y) = \lim_{z\to 0} \phi_{f,+}\left(\sum_{i+j=k_r} \binom{k_r}{i} d^i y \sqcup_0 d^j d^{k_{r-1}}y \cdots d^{k_1}y \right) \\
&= \lim_{z\to 0} \sum_{i+j=k_r} \binom{k_r}{i} \phi_{f,+}\left(d^i y\right) \phi_{f,+}\left(d^j d^{k_{r-1}}y \cdots d^{k_1}y\right) \\
&= \sum_{i+j=k_r} \binom{k_r}{i} F_f(i) F_f(k_1, \dots, k_{r-2}, k_{r-1} + j).
\end{aligned}$$

So we calculate as

$$\begin{aligned}
\mathcal{F}_f(z_1,\dots,z_r) &= \sum_{k_1,\dots,k_r\geq 0} \frac{z_1^{k_1}\cdots z_r^{k_r}}{k_1!\cdots k_r!} F_f(k_1,\dots,k_r) \\
&= \sum_{k_1,\dots,k_r\geq 0} \frac{z_1^{k_1}\cdots z_r^{k_r}}{k_1!\cdots k_r!} \sum_{i+j=k_r} \binom{k_r}{i} F_f(i) F_f(k_1,\dots,k_{r-2},k_{r-1}+j) \\
&= \left\{ \sum_{i\geq 0} \frac{z_r^i}{i!} F_f(i) \right\} \\
&\quad \times \left\{ \sum_{k_1,\dots,k_{r-2},k\geq 0} \frac{z_1^{k_1}\cdots z_{r-2}^{k_{r-2}}}{k_1!\cdots k_{r-2}!k!} F_f(k_1,\dots,k_{r-2},k) \sum_{k_{r-1}+j=k} \binom{k}{j} z_{r-1}^{k_{r-1}} z_r^j \right\} \\
&= \left\{ \sum_{i\geq 0} \frac{z_r^i}{i!} F_f(i) \right\} \left\{ \sum_{k_1,\dots,k_{r-2},k\geq 0} \frac{z_1^{k_1}\cdots z_{r-2}^{k_{r-2}}(z_{r-1}+z_r)^k}{k_1!\cdots k_{r-2}!k!} F_f(k_1,\dots,k_{r-2},k) \right\} \\
&= \mathcal{F}_f(z_r)\mathcal{F}_f(z_1,\dots,z_{r-2},z_{r-1}+z_r).
\end{aligned}$$

By the equation (2.8), we get

$$\mathcal{F}_f(z_1,\dots,z_r) = f_{\geq 0}(z_r)\mathcal{F}_f(z_1,\dots,z_{r-2},z_{r-1}+z_r).$$

Hence, by using this equation repeatedly, we obtain the claim. □

By Theorem 2.6, we know that the special values $F_f(k_1,\dots,k_r)$ are independent of the principal part of $f(z)$. When $f(z)\in\mathbb{Q}[[z]]$, the following theorem holds.

Theorem 2.7. *Let $G(k_1,\dots,k_r)\in\mathbb{Q}$ and put $f(z) := \sum_{k\geq 0}\frac{z^k}{k!}a(k)$ with $a(k)\in\mathbb{Q}$. Then the following two statements are equivalent:*

(1) *For $r\geq 1$, we have*

$$\sum_{k_1,\dots,k_r\geq 0} \frac{z_1^{k_1}\cdots z_r^{k_r}}{k_1!\cdots k_r!} G(k_1,\dots,k_r) = f(z_r)f(z_{r-1}+z_r)\cdots f(z_1+\cdots+z_r).$$

(2) *For $r\geq 1$ and $k_1,\dots,k_r\in\mathbb{Z}_{\geq 0}$, we have*

$$G(k_1,\dots,k_r) = \lim_{z\to 0}\partial_z^{k_r}\Big[f(z)\partial_z^{k_{r-1}}\big[f(z)\cdots\partial_z^{k_1}[f(z)]\cdots\big]\Big].$$

Proof. At first, we prove (1) from (2). We consider the map ϕ_f in (2.7) for $f(z) = \sum_{k\geq 0}\frac{z^k}{k!}a(k)$. Then, by the assumption (2) and (2.6), we have

$$G(k_1,\dots,k_r)=\lim_{z\to 0}\phi_f(d^{k_r}y\cdots d^{k_1}y)=\lim_{z\to 0}\phi_{f,+}(d^{k_r}y\cdots d^{k_1}y).$$

So by Theorem 2.6, we obtain (1).

Next, we prove (2) from (1) by induction on r. When $r=1$, it is obvious from the equation of (1). Assume that (2) holds for the case $(1\le)r<r_0$. When $r=r_0$, by (1), we have

$$\begin{aligned}&\sum_{k_1,\dots,k_r\ge 0}\frac{z_1^{k_1}\cdots z_r^{k_r}}{k_1!\cdots k_r!}G(k_1,\dots,k_r)\\&=f(z_r)\left\{\sum_{k_1,\dots,k_{r-1}\ge 0}\frac{z_1^{k_1}\cdots z_{r-2}^{k_{r-2}}(z_{r-1}+z_r)^{k_{r-1}}}{k_1!\cdots k_{r-2}!k_{r-1}!}G(k_1,\dots,k_{r-1})\right\}.\end{aligned}$$

By comparing coefficients of the term $z_1^{k_1}\cdots z_r^{k_r}$ of both sides, we get

$$G(k_1,\dots,k_r)=\sum_{i+j=k_r}\binom{k_r}{i}G(i)G(k_1,\dots,k_{r-2},k_{r-1}+j).$$

By the induction hypothesis and the Leibniz rule of ∂_z, we have

$$\begin{aligned}G(k_1,\dots,k_r)&=\sum_{i+j=k_r}\binom{k_r}{i}\left\{\lim_{z\to 0}\partial_z^i\left[f(z)\right]\right\}\left\{\lim_{z\to 0}\partial_z^{j+k_{r-1}}\left[f(z)\cdots\partial_z^{k_1}\left[f(z)\right]\cdots\right]\right\}\\&=\lim_{z\to 0}\partial_z^{k_r}\Big[f(z)\partial_z^{k_{r-1}}\left[f(z)\cdots\partial_z^{k_1}\left[f(z)\right]\cdots\right]\Big].\end{aligned}$$

Hence, we finish the proof. □

We put

$$g(z):=\frac{e^z\{(1+z)-e^z\}}{(e^z-1)^2}\left(=\frac{ze^z}{e^z-1}\cdot\frac{1}{z}\left(\frac{z}{e^z-1}-1\right)\right)\in\mathbb{Q}[[z]].\tag{2.9}$$

We note that this formal series $g(z)$ corresponds with $Z(t_1)|_{t_1=-z}$ in Proposition 1.5 for $r=1$ in §1.2.

Corollary 2.8. *For any $r\ge 1$ and $k_1,\dots,k_r\in\mathbb{Z}_{\ge 0}$, we have*

$$\zeta_r^{\mathrm{des}}(-k_1,\dots,-k_r)=\lim_{z\to 0}\partial_z^{k_r}\Big[g(z)\partial_z^{k_{r-1}}\left[g(z)\cdots\partial_z^{k_1}\left[g(z)\right]\cdots\right]\Big].$$

Proof. Note that the generating function $Z(t_1,\dots,t_r)$ of $\zeta_r^{\mathrm{des}}(-k_1,\dots,-k_r)$ (see (1.7) for detail) coincides with the generating function $\mathcal{F}_g(-t_1,\dots,-t_r)$ for $g(z)$ in (2.9) by Proposition 1.5. Hence, by Theorem 2.7, we get the claim. □

2.3. Definition of $\mathrm{Li}^{\mathrm{des}}(\mathbf{k})(t)$

In this subsection, we review the multiple polylogarithms. And then, we introduce $\mathrm{Li}^{\mathrm{des}}(\mathbf{k})(t)$ which is defined by a certain iterated integration. At the end of this section, we show the equality between the limit value $\lim_{t\to 1-0}\mathrm{Li}^{\mathrm{des}}(\mathbf{k})(t)$ and the special value $\zeta_r^{\mathrm{des}}(\mathbf{k})$ when $(k_1,\dots,k_r)\in\mathbb{Z}^r_{\le 0}$ (Proposition 2.13).

The *multiple polylogarithms* (MPLs for short) $\mathrm{Li}_{k_1,\dots,k_r}(t)$ with $(k_1,\dots,k_r)\in\mathbb{Z}^r$, is the complex analytic function defined by the following series:

$$\mathrm{Li}_{k_1,\dots,k_r}(t) := \sum_{0<n_1<\cdots<n_r}\frac{t^{n_r}}{n_1^{k_1}\cdots n_r^{k_r}}$$

which converges for $t\in\mathbb{C}$ with $|t|<1$. The MPL for the case $r=1$, $k=1$ is given by $\mathrm{Li}_1(t) = -\log(1-t)$. Consult [13] for many topics related to MPLs. We will describe some properties of MPLs. The first to mention is the following *iterated integral* expression:

$$\mathrm{Li}_{k_1,\dots,k_r}(t) = \int_0^t \underbrace{\frac{dt}{t}\circ\cdots\circ\frac{dt}{t}}_{k_r-1}\circ\frac{dt}{1-t}\circ\cdots\circ\underbrace{\frac{dt}{t}\circ\cdots\circ\frac{dt}{t}}_{k_1-1}\circ\frac{dt}{1-t}, \tag{2.10}$$

where $(k_1,\dots,k_r)\in\mathbb{N}^r$. This yields analytic continuation to a bigger region. More precisely, MPLs can be analytically continued to the universal unramified covering of $\mathbb{P}^1(\mathbb{C})\setminus\{0,1,\infty\}$.

It is known that $\mathrm{Li}_{-k_1,\dots,-k_r}(t)$ is a rational function of t for $(k_1,\dots,k_r)\in\mathbb{Z}^r_{\ge 0}$, for instance,

$$\mathrm{Li}_0(t) = \frac{t}{1-t},\qquad \mathrm{Li}_{-1}(t) = \frac{t}{(1-t)^2},\qquad \mathrm{Li}_{-2}(t) = \frac{t(t+1)}{(1-t)^3}. \tag{2.11}$$

The following differential equation holds for MPLs. By definition, one can easily see that

$$\frac{d}{dt}\mathrm{Li}_{k_1,\dots,k_r}(t) = \begin{cases}\frac{1}{t}\mathrm{Li}_{k_1,\dots,k_r-1}(t), & (k_r\neq 1)\\ \frac{1}{1-t}\mathrm{Li}_{k_1,\dots,k_{r-1}}(t), & (k_r=1).\end{cases}$$

We note that $\frac{1}{1-t}\mathrm{Li}_{k_1,\dots,k_{r-1}}(t) = \frac{1}{t}\mathrm{Li}_{k_1,\dots,k_{r-1},0}(t)$ for $k_r=1$. By the above differential equation, we see that possible singularities of MPLs for any indices $(k_1,\dots,k_r)\in\mathbb{Z}^r$ are $t=0,\ 1$.

Put $\mathbb{R}[[t]]$ to be the algebra of formal power series. We consider the subalgebra $\mathcal{P}_{(0,1)}$ of $t\mathbb{R}[[t]]$ defined by

$$\mathcal{P}_{(0,1)} := \{f(t)\in t\mathbb{R}[[t]] \mid f(t) \text{ converges for } t\in(0,1)\}. \tag{2.12}$$

We also set

$$\mathcal{A} := \mathcal{P}_{(0,1)}[\log t] = \left\{ \sum_{\substack{1\le i \\ 0\le j\le N}} a_{ij} t^i (\log t)^j \,\middle|\, N \in \mathbb{Z}_{\geq 0}, \forall j \in \mathbb{Z}_{\geq 0}, \sum_{1\le i} a_{ij} t^i \in \mathcal{P}_{(0,1)} \right\}.$$

Here, $\log t$ means just a formal function denoting $\int_0^t \frac{dz}{z}$.

Lemma 2.9. *For $f \in \mathcal{A}$, we define*[2] *formal integration and derivation:*

$$J[f] := \int_0^t f(z)\frac{dz}{z}, \qquad D[f] := t\frac{df}{dt}.$$

Then J and D form operators on $\mathcal{A}$, and $J \circ D = D \circ J = \mathrm{Id}$.

Proof. It is easy to show $J \circ D = D \circ J = \mathrm{Id}$, so we prove $J[f], D[f] \in \mathcal{A}$ for any $f \in \mathcal{A}$. It is enough to prove $D\left[g(t)(\log t)^l\right], J\left[g(t)(\log t)^l\right] \in \mathcal{A}$ for $g(t) \in \mathcal{P}_{(0,1)}$ and $l \geq 1$. We have

$$D\left[g(t)(\log t)^l\right] = D\left[g(t)\right](\log t)^l + g(t)D\left[(\log t)^l\right] = D\left[g(t)\right](\log t)^l + g(t)l(\log t)^{l-1}.$$

Because we have $D\left[g(t)\right] \in \mathcal{P}_{(0,1)}$ by $g(t) \in \mathcal{P}_{(0,1)}$, we get $D\left[g(t)(\log t)^l\right] \in \mathcal{A}$.

On the other hand, by using integration by parts, we have

$$J\left[f_1 f_2\right] = J\left[f_1\right] f_2 - J\left[J[f_1]D[f_2]\right],$$

for $f_1, f_2 \in \mathcal{A}$. By using this for $f_1 = g(t)$ and $f_2 = (\log t)^l$, we calculate as

$$\begin{aligned} J\left[g(t)(\log t)^l\right] &= \left[J\left[g(t)\right](\log t)^l\right]_0^t - J\left[J[g(t)]D[(\log t)^l]\right] \\ &= J\left[g(t)\right](\log t)^l - \lim_{t\to+0} J\left[g(t)\right](\log t)^l - J\left[J[g(t)]l(\log t)^{l-1}\right]. \end{aligned}$$

We have $J\left[g(t)\right] \in \mathcal{P}_{(0,1)}$ by $g(t) \in \mathcal{P}_{(0,1)}$ and have $\lim_{t\to+0} t^k(\log t)^l = 0$ for $k, l \geq 1$, so we get

$$\lim_{t\to+0} J\left[g(t)\right](\log t)^l = 0.$$

Therefore, we get

$$J\left[g(t)(\log t)^l\right] = J\left[g(t)\right](\log t)^l - J\left[J[g(t)]l(\log t)^{l-1}\right].$$

[2] These maps J, D are also considered in [4, §3.1] as operators on a certain $\mathbb{C}$-algebra of power series.

Because we have $J\left[h(t)(\log t)^0\right] \in \mathcal{A}$ for $h(t) \in \mathcal{P}_{(0,1)}$, we inductively obtain $J\left[g(t)(\log t)^l\right] \in \mathcal{A}$ for $l \geq 1$. Hence, we finish the proof. □

By the above lemma, we denote $J^{-1} = D$. By using these operators J, D on $\mathcal{A}$, we introduce the following elements in $\mathcal{A}$.

Definition 2.10. Put $\mathbf{k} := (k_1, \dots, k_r) \in \mathbb{Z}^r$ for $r \in \mathbb{N}$. We define

$$\mathrm{Li}^{\mathrm{des}}(0)(t) := \mathrm{Li}_0(t) + \log t \cdot \mathrm{Li}_{-1}(t) = \frac{t}{1-t} + \frac{t \log t}{(1-t)^2},$$

and

$$\mathrm{Li}^{\mathrm{des}}(\mathbf{k})(t) := J^{k_r}\left[\mathrm{Li}^{\mathrm{des}}(0)(t) J^{k_{r-1}}\left[\mathrm{Li}^{\mathrm{des}}(0)(t) \cdots J^{k_1}[\mathrm{Li}^{\mathrm{des}}(0)(t)] \cdots\right]\right].$$

Because $\mathrm{Li}_0(t)$ and $\mathrm{Li}_{-1}(t)$ are in $\mathcal{P}_{(0,1)}$, the element $\mathrm{Li}^{\mathrm{des}}(0)(t)$ is in $\mathcal{A}$, and by Lemma 2.9, we get $\mathrm{Li}^{\mathrm{des}}(\mathbf{k})(t) \in \mathcal{A}$.

For $\mathbf{k} = (k_1, \dots, k_r) \in \mathbb{Z}^r$, we set

$$\begin{cases} \mathbf{k}' := (k_1, \dots, k_{r-1}, k_r - 1) \in \mathbb{Z}^r, \\ \mathbf{k}^{(j)} := (k_1, \dots, k_{r-1}, k_r - j) \in \mathbb{Z}^r, \end{cases} \tag{2.13}$$

for $j \in \mathbb{Z}$. Note that $\mathbf{k}^{(1)} = \mathbf{k}'$.

Remark 2.11. By the above definition, we have

$$D[\mathrm{Li}^{\mathrm{des}}(\mathbf{k})(t)] = \mathrm{Li}^{\mathrm{des}}(\mathbf{k}')(t), \qquad \mathrm{Li}^{\mathrm{des}}(\mathbf{k}, 0)(t) = \mathrm{Li}^{\mathrm{des}}(0)(t)\mathrm{Li}^{\mathrm{des}}(\mathbf{k})(t),$$

for $\mathbf{k} \in \mathbb{Z}^r$.

Recall the quotient algebra $\mathcal{H}$ defined in (2.3).

Lemma 2.12. *We define the $\mathbb{Q}$-linear map $\psi : \mathcal{H} \to \mathcal{A}$ by $\psi(1) := 1$ and*

$$\psi(j^{k_1} y \cdots j^{k_r} y) := \mathrm{Li}^{\mathrm{des}}(k_r, \dots, k_1)(t),$$

for $k_1, \dots, k_r \in \mathbb{Z}$. Then this map ψ forms an algebra homomorphism.[3]

Proof. By Definition 2.10, we have

$$\psi(j^{k_1} y \cdots j^{k_r} y) = J^{k_1}\left[\mathrm{Li}^{\mathrm{des}}(0)(t) J^{k_2}\left[\mathrm{Li}^{\mathrm{des}}(0)(t) \cdots J^{k_r}[\mathrm{Li}^{\mathrm{des}}(0)(t)] \cdots\right]\right].$$

[3] This lemma is an analogue of [4, Lemma 3.6] for ordinary MPLs.

By the Leibniz rule, the integration by parts and the definition (2.1), it is clear that the map ψ is an algebra homomorphism. □

The element $\mathrm{Li}^{\mathrm{des}}(-k_1,\dots,-k_r)(t) \in \mathcal{A}$ defined in Definition 2.10 converges to $\zeta_r^{\mathrm{des}}(-k_1,\dots,-k_r)$ when $(k_1,\dots,k_r) \in \mathbb{Z}_{\geq 0}^r$ under the limit $t \to 1-0$.

Proposition 2.13. *For $r \in \mathbb{N}$ and $(k_1,\dots,k_r) \in \mathbb{Z}_{\geq 0}^r$, we have*

$$\lim_{t\to 1-0} \mathrm{Li}^{\mathrm{des}}(-k_1,\dots,-k_r)(t) = \zeta_r^{\mathrm{des}}(-k_1,\dots,-k_r).$$

Proof. Because we have $J^{-1} = D$, we get $J^{-k} = D^k$ for $k \geq 0$. Therefore, for $k_1,\dots,k_r \in \mathbb{Z}_{\geq 0}$, we have

$$\mathrm{Li}^{\mathrm{des}}(-k_1,\dots,-k_r)(t) = D^{k_r}\left[\mathrm{Li}^{\mathrm{des}}(0)(t) D^{k_{r-1}}\left[\mathrm{Li}^{\mathrm{des}}(0)(t)\cdots D^{k_1}[\mathrm{Li}^{\mathrm{des}}(0)(t)]\cdots\right]\right].$$

Consider changing variables $t = e^z$. Then we have

$$D = t\frac{d}{dt} = e^z \frac{dz}{dt}\cdot\frac{d}{dz} = \frac{d}{dz} = \partial_z,$$

and, by (2.11), we get

$$\mathrm{Li}^{\mathrm{des}}(0)(t) = \frac{t}{1-t} + \log t \frac{t}{(1-t)^2} = \frac{e^z}{1-e^z} + \frac{ze^z}{(1-e^z)^2} = g(z).$$

Here, $g(z)$ is given in (2.9). So we get

$$\mathrm{Li}^{\mathrm{des}}(-k_1,\dots,-k_r)(t) = \partial_z^{k_r}\left[g(z)\partial_z^{k_{r-1}}\left[g(z)\cdots\partial_z^{k_1}[g(z)]\cdots\right]\right].$$

When $t \to 1$, we have $z \to 0$, so by Corollary 2.8, we obtain

$$\lim_{t\to 1} \mathrm{Li}^{\mathrm{des}}(-k_1,\dots,-k_r)(t) = \zeta_r^{\mathrm{des}}(-k_1,\dots,-k_r).$$

Hence, we finish the proof. □

In Theorem 4.6 in §4, we will generalize this proposition.

3. Linear combinations of MPLs

In this section, we consider certain finite linear combinations $Z(\mathbf{k};t)$ of MPLs to express $\mathrm{Li}^{\mathrm{des}}(\mathbf{k})(t)$ as a finite linear combination of MPLs (Definition 3.11). In §3.1, we introduce the function $\zeta_r^{\mathrm{des}}(s_1,\dots,s_r)(t)$ and explain its properties. By using these, we show some properties of $Z(\mathbf{k};t)$. In §3.2, we show that $Z(\mathbf{k};t)$ coincides with $\mathrm{Li}^{\mathrm{des}}(\mathbf{k})(t)$ (Theorem 3.16).

3.1. Definition of $Z(\mathbf{k})$ and its properties

In this subsection, we consider a certain finite linear combination of MPLs. To this end, we first consider certain multiple zeta functions and their properties which will be employed in later sections.

Definition 3.1 *(cf. [5]).* For $0 < t < 1$ and any $\mathbf{s} = (s_1, \ldots, s_r) \in \mathbb{C}^r$, we set

$$\zeta_r^{ш}(\mathbf{s}; t) = \zeta_r^{ш}(s_1, \ldots, s_r; t) := \sum_{0<n_1<\cdots<n_r} \frac{t^{n_r}}{n_1^{s_1} \cdots n_r^{s_r}}.$$

In [5, Corollary 2.2], it is shown that the function $\zeta_r^{ш}(\mathbf{s}; t)$ can be analytically continued to $\mathbb{C}^r$ as an entire function. We also denote $\zeta_r^{ш}(\mathbf{s}; t)$ as $\zeta_r^{ш}((s_j); t)$.

Remark 3.2. By definition, we have

$$\zeta_r^{ш}(k_1, \ldots, k_r; t) = \mathrm{Li}_{k_1,\ldots,k_r}(t),$$

for $k_1, \ldots, k_r \in \mathbb{Z}$. It is clear that for $(s_1, \ldots, s_r) \in \mathcal{D}_r$,

$$\lim_{t \to 1-0} \zeta_r^{ш}(s_1, \ldots, s_r; t) = \zeta_r(s_1, \ldots, s_r).$$

Here, the region $\mathcal{D}_r$ is defined as follows:

$$\mathcal{D}_r := \{(s_1, \ldots, s_r) \in \mathbb{C}^r \mid \Re(s_{r-j+1} + \cdots + s_r) > j \text{ for } 1 \leq j \leq r\}.$$

However, we note that $\lim_{t \to 1-0} \zeta_r^{ш}(s_1, \ldots, s_r; t)$ diverges for $(s_1, \ldots, s_r) \in \mathbb{C}^r \setminus \mathcal{D}_r$.

We consider the following generating function. For $c \in \mathbb{R}$ and $0 < t < 1$, we put

$$\tilde{\mathfrak{H}}_r(t_1, \ldots, t_r; c; t) := \prod_{j=1}^{r} \left(\frac{1}{\frac{1}{t}\exp(\sum_{k=j}^{r} t_k) - 1} - \frac{c}{\frac{1}{t}\exp(c\sum_{k=j}^{r} t_k) - 1} \right).$$

Definition 3.3 *(cf. [6, Definition 3.1]).* For $0 < t < 1$, $s_1, \ldots, s_r \in \mathbb{C} \setminus \mathbb{Z}$, we define

$$\zeta_r^{\mathrm{des}}(s_1, \ldots, s_r)(t) := \lim_{\substack{c \to 1 \\ c \neq 1}} \frac{1}{(1-c)^r} C(\mathbf{s}) \int_{\mathcal{C}_{\epsilon_t}^r} \tilde{\mathfrak{H}}_r(t_1, \ldots, t_r; c; t) \prod_{k=1}^{r} t_k^{s_k - 1} dt_k, \tag{3.1}$$

where $\mathcal{C}_{\epsilon_t}$ is the Hankel contour, that is, the path consisting of the positive axis (top side), a circle around the origin of radius ϵ_t (with $0 < \epsilon_t < |\log t|$), and the positive real axis (bottom side). The symbol $C(\mathbf{s})$ is defined in (1.2).

Remark 3.4. We note that the limit $\lim_{t\to 1-0}\zeta_r^{\mathrm{des}}(s_1,\dots,s_r)(t)$ does not always exist. For instance, consider the case $r=1$: we have

$$\zeta_1^{\mathrm{des}}(s)(t)=(1-s)\zeta^{\sqcup\!\sqcup}(s;t).$$

Thus, $\lim_{t\to 1-0}\zeta_r^{\mathrm{des}}(s)(t)$ diverges for $s\in\mathbb{C}\setminus\mathcal{D}_1$.

The following two propositions are analogues of Proposition 1.3 and 1.6, and can be proved exactly in the same way.

Proposition 3.5. *For $s_1,\dots,s_r\in\mathbb{C}$, and $0<t<1$, we have the following equality between meromorphic functions of the complex variables $(s_1,\dots,s_r)$;*

$$\zeta_r^{\mathrm{des}}(s_1,\dots,s_r)(t)=\sum_{\substack{\mathbf{l}=(l_j)\in\mathbb{Z}_{\geq 0}^r\\ \mathbf{m}(m_j)\in\mathbb{Z}^r\\ |\mathbf{m}|=0}} a^r_{\mathbf{l},\mathbf{m}}\left(\prod_{j=1}^r (s_j)_{l_j}\right)\zeta_r^{\sqcup\!\sqcup}(s_1+m_1,\dots,s_r+m_r;t),$$

where the coefficient $a^r_{\mathbf{l},\mathbf{m}}$ is defined in (1.4), and the symbol $(s)_k$ is the Pochhammer symbol (see Proposition 1.3 for the definition).

Proposition 3.6. *For $s_1,\dots,s_{r-1}\in\mathbb{C}$, $k\in\mathbb{Z}_{\geq 0}$ and $0<t<1$, we have*

$$\zeta_r^{\mathrm{des}}(s_1,\dots,s_{r-1},-k)(t)=\sum_{i=0}^k\binom{k}{i}\zeta_{r-1}^{\mathrm{des}}(s_1,\dots,s_{r-1}-k+i)(t)\zeta_1^{\mathrm{des}}(-i)(t).$$

Definition 3.7. We define a family $\{Z_q(\mathbf{k})\}_{q\geq 0,r\geq 1,\mathbf{k}\in\mathbb{Z}^r}$ in $\mathcal{P}_{(0,1)}$ (defined in (2.12)) by

$$Z_0(\mathbf{k}):=Z_0(\mathbf{k};t):=\zeta_r^{\mathrm{des}}(\mathbf{k})(t),$$

and for $q\geq 1$,

$$Z_q(\mathbf{k}):=Z_q(\mathbf{k};t):=\sum_{\substack{i+j=q\\ i,j\geq 0}}(-1)^j\binom{q}{i}D^i\left[Z_0(\mathbf{k}^{(j)})\right]. \tag{3.2}$$

By Remark 3.2 and Proposition 3.5, we have

$$Z_0(\mathbf{k})=Z_0(\mathbf{k};t)=\sum_{\substack{\mathbf{l}=(l_j)\in\mathbb{Z}_{\geq 0}^r\\ \mathbf{m}=(m_j)\in\mathbb{Z}^r\\ |\mathbf{m}|=0}} a^r_{\mathbf{l},\mathbf{m}}\left(\prod_{j=1}^r (k_j)_{l_j}\right)\mathrm{Li}_{\mathbf{k}+\mathbf{m}}(t), \tag{3.3}$$

so the above definition is well-defined. In Appendix A, we give explicit formulae of $Z_q(\mathbf{k})$ in terms of $\mathrm{Li}_{\mathbf{k}+\mathbf{m}}(t)$ for $\mathbf{m}\in\mathbb{Z}^r$.

The following recurrence formula holds for the above element $Z_q(\mathbf{k})$.

Proposition 3.8. *For $q \geq 1$, $r \geq 1$ and $\mathbf{k} \in \mathbb{Z}^r$, we have*

$$Z_q(\mathbf{k}) = D\left[Z_{q-1}(\mathbf{k})\right] - Z_{q-1}(\mathbf{k}').$$

Proof. By using the following Lemma 3.9 for $f(i,j) = (-1)^j D^i\left[Z_0(\mathbf{k}^{(j)})\right]$, we have

$$\begin{aligned}
&Z_q(\mathbf{k}) \\
&= \sum_{i+j=q-1} (-1)^j \binom{q-1}{i} D^{i+1}\left[Z_0(\mathbf{k}^{(j)})\right] - \sum_{i+j=q-1} (-1)^j \binom{q-1}{i} D^{i}\left[Z_0(\mathbf{k}^{(j+1)})\right] \\
&= D\left[\sum_{i+j=q-1} (-1)^j \binom{q-1}{i} D^{i}\left[Z_0(\mathbf{k}^{(j)})\right]\right] - \sum_{i+j=q-1} (-1)^j \binom{q-1}{i} D^{i}\left[Z_0\left((\mathbf{k}')^{(j)}\right)\right] \\
&= D\left[Z_{q-1}(\mathbf{k})\right] - Z_{q-1}(\mathbf{k}').
\end{aligned}$$

Hence, we obtain the claim. □

Lemma 3.9. *Let f be a map on $\mathbb{Z}_{\geq 0}^2$. For $q \geq 1$, we have*

$$\sum_{i+j=q} \binom{q}{i} f(i,j) = \sum_{i+j=q-1} \binom{q-1}{i} f(i+1,j) + \sum_{i+j=q-1} \binom{q-1}{i} f(i,j+1).$$

Proof. Because we have the recurrence relation of binomial coefficients

$$\binom{q}{i} = \binom{q-1}{i-1} + \binom{q-1}{i}$$

for $1 \leq i \leq q-1$ and we have

$$\binom{q}{0} = \binom{q}{q} = 1,$$

we obtain the claim. □

For simplicity, we sometimes denote (3.3) as

$$Z_0(\mathbf{k}) = \sum_{\mathbf{m} \in \mathbb{Z}^r} p_{\mathbf{m}}(\mathbf{k}) \mathrm{Li}_{\mathbf{k}+\mathbf{m}}(t). \tag{3.4}$$

Here, $p_{\mathbf{m}}(\mathbf{k})$ means $p_{\mathbf{m}}(x_1, \ldots, x_r)|_{x_i = k_i}$ where a family $\{p_{\mathbf{m}}(x_1, \ldots, x_r)\}_{\mathbf{m} \in \mathbb{Z}^r}$ in $\mathbb{Q}[x_1, \ldots, x_r]$ with $p_{\mathbf{m}}(x_1, \ldots, x_r) = 0$ except for a finite number of $\mathbf{m} \in \mathbb{Z}^r$. We denote $\deg_{x_r}(p_{\mathbf{m}}(x_1, \ldots, x_r))$ by the degree of the polynomial $p_{\mathbf{m}}(x_1, \ldots, x_r)$ in x_r, and for $r \geq 1$, we put

$$d_r := \max\{\deg_{x_r}(p_{\mathbf{m}}(x_1,\dots,x_r)) \mid \mathbf{m}\in\mathbb{Z}^r\}.$$

Proposition 3.10. *For $q \geq d_r + 1$ and for $\mathbf{k}\in\mathbb{Z}^r$, we have*

$$Z_q(\mathbf{k}) = 0.$$

Proof. If we have $Z_q(\mathbf{k}) = 0$ for $q = d_r + 1$, we inductively get $Z_q(\mathbf{k}) = 0$ for $q > d_r + 1$ by Proposition 3.8. So it is sufficient to prove $Z_q(\mathbf{k}) = 0$ for $q = d_r+1$. By the expression (3.4) and the definition (3.2), we have the following representation:

$$Z_q(\mathbf{k}) = \sum_{\mathbf{m}\in\mathbb{Z}^r} c_q(\mathbf{k};\mathbf{m})\mathrm{Li}_{\mathbf{k}^{(q)}+\mathbf{m}}(t),$$

where the symbol $c_q(\mathbf{k};\mathbf{m})$ means $c_q(x_1,\dots x_r;\mathbf{m})|_{x_i=k_i}$ with

$$c_q(x_1,\dots x_r;\mathbf{m}) := \sum_{i+j=q}(-1)^j\binom{q}{i}p_{\mathbf{m}}(x_1,\dots,x_{r-1},x_r-j)\ \ (\in\mathbb{Q}[x_1,\dots,x_r]).$$

We note that

$$\max\{\deg_{x_r}(c_0(x_1,\dots x_r;\mathbf{m})) \mid \mathbf{m}\in\mathbb{Z}^r\} = d_r. \tag{3.5}$$

By Proposition 3.8, we get

$$c_q(\mathbf{k};\mathbf{m}) = c_{q-1}(\mathbf{k};\mathbf{m}) - c_{q-1}(\mathbf{k}';\mathbf{m})$$

for $q\geq 1$ and for any $\mathbf{k}\in\mathbb{Z}^r$. Therefore, we have

$$\begin{aligned}&\deg_{x_r}(c_q(x_1,\dots x_r;\mathbf{m})) \\ &= \begin{cases}\deg_{x_r}(c_{q-1}(x_1,\dots x_r;\mathbf{m}))-1 & (\deg_{x_r}(c_{q-1}(x_1,\dots x_r;\mathbf{m}))\geq 1),\\ 0 & (\deg_{x_r}(c_{q-1}(x_1,\dots x_r;\mathbf{m}))=0).\end{cases}\end{aligned} \tag{3.6}$$

By using (3.5) and (3.6), we get

$$\deg_{x_r}(c_{d_r}(x_1,\dots x_r;\mathbf{m})) = 0$$

for any $\mathbf{m}\in\mathbb{Z}^r$, that is, we have $c_{d_r}(x_1,\dots x_r;\mathbf{m})\in\mathbb{Q}[x_1,\dots,x_{r-1}]$. Hence, we obtain

$$c_{d_r+1}(x_1,\dots x_r;\mathbf{m}) = 0$$

for any $\mathbf{m}\in\mathbb{Z}^r$, that is, $c_{d_r+1}(\mathbf{k};\mathbf{m}) = 0$ for $\mathbf{k},\mathbf{m}\in\mathbb{Z}^r$, so we get $Z_{d_r+1}(\mathbf{k}) = 0$. □

Definition 3.11. For $\mathbf{k} \in \mathbb{Z}^r$, we define an element $Z(\mathbf{k}) \in \mathcal{A}(= \mathcal{P}_{(0,1)}[\log t])$ by

$$Z(\mathbf{k}) := Z(\mathbf{k}; t) := \sum_{q \geq 0} \frac{(-\log t)^q}{q!} Z_q(\mathbf{k}). \tag{3.7}$$

Remark 3.12. By Proposition 3.10, the right-hand side of the equation (3.7) is a finite sum, so the above definition is well-defined.

Examples 3.13. We consider the case $r = 1$. By the definition (3.3), $Z_0(k)$ is presented as follows:

$$Z_0(k) = (1-k)\mathrm{Li}_k(t).$$

Then $Z_1(k)$ is calculated to be

$$Z_1(k) = -(\log t)\,\mathrm{Li}_{k-1}(t).$$

By Proposition 3.10, we know $Z_q(k) = 0$ for $q \geq 2$. Thus, we obtain

$$Z(k) = Z_0(k) - Z_1(k) = (1-k)\mathrm{Li}_k(t) + (\log t)\mathrm{Li}_{k-1}(t) \tag{3.8}$$

for $k \in \mathbb{Z}$.

We next show that $Z(\mathbf{k})$ satisfies the differential equation which holds for MPLs and $\mathrm{Li}^{\mathrm{des}}(\mathbf{k})(t)$'s in Definition 2.10.

Theorem 3.14. *For* $\mathbf{k} \in \mathbb{Z}^r$, *we have*

$$D\left[Z(\mathbf{k})\right] = Z(\mathbf{k}').$$

Proof. We have

$$D\left[Z(\mathbf{k})\right] - Z(\mathbf{k}') = D\left[\sum_{q \geq 0} \frac{(-\log t)^q}{q!} Z_q(\mathbf{k})\right] - \sum_{q \geq 0} \frac{(-\log t)^q}{q!} Z_q(\mathbf{k}')$$

$$= D\left[Z_0(\mathbf{k})\right] - Z_0(\mathbf{k}') + D\left[\sum_{q \geq 1} \frac{(-\log t)^q}{q!} Z_q(\mathbf{k})\right] - \sum_{q \geq 1} \frac{(-\log t)^q}{q!} Z_q(\mathbf{k}').$$

By Definition 3.7, we get

$$D\left[Z(\mathbf{k})\right] - Z(\mathbf{k}')$$

$$= D\left[Z_0(\mathbf{k})\right] - Z_0(\mathbf{k}') + D\left[\sum_{q \geq 1} \frac{(-\log t)^q}{q!} \sum_{i+j=q} (-1)^j \binom{q}{i} D^i\left[Z_0(\mathbf{k}^{(j)})\right]\right]$$

$$-\sum_{q\geq 1}\frac{(-\log t)^q}{q!}\sum_{i+j=q}(-1)^j\binom{q}{i}D^i\left[Z_0(\mathbf{k}^{(j+1)})\right].$$

By applying the Leibniz rule to the third term, we calculate as

$$\begin{aligned}
&D\left[Z(\mathbf{k})\right]-Z(\mathbf{k}')\\
&=D\left[Z_0(\mathbf{k})\right]-Z_0(\mathbf{k}')-\sum_{q\geq 1}\frac{(-\log t)^{q-1}}{(q-1)!}\sum_{i+j=q}(-1)^j\binom{q}{i}D^i\left[Z_0(\mathbf{k}^{(j)})\right]\\
&\quad+\sum_{q\geq 1}\frac{(-\log t)^q}{q!}\sum_{i+j=q}(-1)^j\binom{q}{i}D^{i+1}\left[Z_0(\mathbf{k}^{(j)})\right]\\
&\quad+\sum_{q\geq 1}\frac{(-\log t)^q}{q!}\sum_{i+j=q}(-1)^{j+1}\binom{q}{i}D^i\left[Z_0(\mathbf{k}^{(j+1)})\right].
\end{aligned}$$

By applying Lemma 3.9 for $f(i,j)=(-1)^jD^i\left[Z_0(\mathbf{k}^{(j)})\right]$ to the fourth and the fifth terms, we get

$$\begin{aligned}
&D\left[Z(\mathbf{k})\right]-Z(\mathbf{k}')\\
&=D\left[Z_0(\mathbf{k})\right]-Z_0(\mathbf{k}')-\sum_{q\geq 1}\frac{(-1)^{q-1}}{(q-1)!}(\log t)^{q-1}\sum_{i+j=q}(-1)^j\binom{q}{i}D^i\left[Z_0(\mathbf{k}^{(j)})\right]\\
&\quad+\sum_{q\geq 1}\frac{(-1)^q}{q!}(\log t)^q\sum_{i+j=q+1}(-1)^j\binom{q+1}{i}D^i\left[Z_0(\mathbf{k}^{(j)})\right].
\end{aligned}$$

By rearranging the first, second, and fourth terms, and by changing variables of summation of the third term from $q\geq 1$ to $q\geq 0$, we obtain

$$\begin{aligned}
D\left[Z(\mathbf{k})\right]-Z(\mathbf{k}')=&-\sum_{q\geq 0}\frac{(-1)^q}{q!}(\log t)^q\sum_{i+j=q+1}(-1)^j\binom{q+1}{i}D^i\left[Z_0(\mathbf{k}^{(j)})\right]\\
&+\sum_{q\geq 0}\frac{(-1)^q}{q!}(\log t)^q\sum_{i+j=q+1}(-1)^j\binom{q+1}{i}D^i\left[Z_0(\mathbf{k}^{(j)})\right]=0.
\end{aligned}$$

Hence, we finish the proof. □

3.2. Coincidence of $Z(\mathbf{k};t)$ *with* $\mathrm{Li}^{\mathrm{des}}(\mathbf{k})(t)$

In this subsection, we show that $\mathrm{Li}^{\mathrm{des}}(\mathbf{k})(t)$ can be expressed as a certain “linear” combination of MPLs (Theorem 3.16).

By definition of $Z_0(\mathbf{k})$ and Proposition 3.6, we have the following equation:

$$Z_0(\mathbf{k},-k_{r+1})=\sum_{i+j=k_{r+1}}\binom{k_{r+1}}{i}Z_0(-i)Z_0(\mathbf{k}^{(j)}).\tag{3.9}$$

Here, $r \in \mathbb{N}$, $\mathbf{k} \in \mathbb{Z}^r$ and $k_{r+1} \in \mathbb{Z}_{\geq 0}$.

Proposition 3.15. *For $r \in \mathbb{N}$ and $\mathbf{k} \in \mathbb{Z}^r$, we have*

$$Z(\mathbf{k}, 0) = Z(0)Z(\mathbf{k}).$$

Proof. Let $q \geq 0$. By the definition (3.2) of $Z_q(\mathbf{k})$ and the equality (3.9), we have

$$\begin{aligned} Z_q(\mathbf{k}, 0) &= \sum_{i+j=q} (-1)^j \binom{q}{i} D^i \left[Z_0(\mathbf{k}, -j) \right] \\ &= \sum_{i+j=q} (-1)^j \binom{q}{i} D^i \left[\sum_{b+d=j} \binom{j}{b} Z_0(-b) Z_0(\mathbf{k}^{(d)}) \right]. \end{aligned}$$

By the Leibniz rule, we calculate as

$$\begin{aligned} Z_q(\mathbf{k}, 0) &= \sum_{i+j=q} (-1)^j \binom{q}{i} \sum_{b+d=j} \binom{j}{b} \left\{ \sum_{a+c=i} \binom{i}{a} D^a \left[Z_0(-b) \right] D^c \left[Z_0(\mathbf{k}^{(d)}) \right] \right\} \\ &= \sum_{a+c+b+d=q} (-1)^{b+d} \frac{q!}{b!d!a!c!} D^a \left[Z_0(-b) \right] D^c \left[Z_0(\mathbf{k}^{(d)}) \right] \\ &= \sum_{i+j=q} \binom{q}{i} \left\{ \sum_{a+b=i} (-1)^b \binom{i}{a} D^a \left[Z_0(-b) \right] \right\} \left\{ \sum_{c+d=j} (-1)^d \binom{j}{c} D^c \left[Z_0(\mathbf{k}^{(d)}) \right] \right\}. \end{aligned}$$

By using the definition (3.2) of $Z_q(\mathbf{k})$ again, we get

$$Z_q(\mathbf{k}, 0) = \sum_{i+j=q} \binom{q}{i} Z_i(0) Z_j(\mathbf{k}).$$

Therefore, we obtain

$$\begin{aligned} Z(\mathbf{k}, 0) &= \sum_{q \geq 0} \frac{(-\log t)^q}{q!} Z_q(\mathbf{k}, 0) = \sum_{q \geq 0} \frac{(-\log t)^q}{q!} \sum_{i+j=q} \binom{q}{i} Z_i(0) Z_j(\mathbf{k}) \\ &= \left\{ \sum_{i \geq 0} \frac{(-\log t)^i}{i!} Z_i(0) \right\} \left\{ \sum_{j \geq 0} \frac{(-\log t)^j}{j!} Z_j(\mathbf{k}) \right\} = Z(0)Z(\mathbf{k}). \end{aligned}$$

Hence, we finish the proof. □

By using the above proposition, we obtain the following.

Theorem 3.16. *For $r \in \mathbb{N}$ and $\mathbf{k} \in \mathbb{Z}^r$, we have*

$$\mathrm{Li}^{\mathrm{des}}(\mathbf{k})(t) = Z(\mathbf{k}).$$

Proof. We prove this claim by induction on r. When $r = 1$, by Definition 2.10 and (3.8), we have

$$\mathrm{Li}^{\mathrm{des}}(0)(t) = \mathrm{Li}_0(t) + \log t \cdot \mathrm{Li}_{-1}(t) = Z(0). \tag{3.10}$$

So by Remark 2.11 and Theorem 3.14, we have

$$\mathrm{Li}^{\mathrm{des}}(k_1)(t) = J^{k_1}\left[\mathrm{Li}^{\mathrm{des}}(0)(t)\right] = J^{k_1}\left[Z(0)\right] = Z(k_1).$$

Hence the claim holds for $r = 1$. Assume that the claim holds for $r_0 \in \mathbb{N}$. Put $\mathbf{k} = (k_1, \ldots, k_{r_0+1})$ with $k_1, \ldots, k_{r_0+1} \in \mathbb{Z}$. Similarly to the case of $r = 1$, we have

$$\begin{aligned}\mathrm{Li}^{\mathrm{des}}(\mathbf{k})(t) &= J^{k_{r_0+1}}\left[\mathrm{Li}^{\mathrm{des}}(k_1, \ldots, k_{r_0}, 0)(t)\right] \\ &= J^{k_{r_0+1}}\left[\mathrm{Li}^{\mathrm{des}}(0)(t)\mathrm{Li}^{\mathrm{des}}(k_1, \ldots, k_{r_0})(t)\right].\end{aligned}$$

On the other hand, by using Theorem 3.14 and Proposition 3.15, we get

$$Z(\mathbf{k}) = J^{k_{r_0+1}}\left[Z(k_1, \ldots, k_{r_0}, 0)\right] = J^{k_{r_0+1}}\left[Z(0)Z(k_1, \ldots, k_{r_0})\right].$$

Hence, by (3.10) and the induction hypothesis, we obtain the claim. □

4. Main results

In this section, we introduce one-parameterized desingularized MZFs (Definition 4.1). After that, we prove the shuffle-type formula for special values of desingularized MZFs at integer points (Theorem 4.8). We also give some examples.

4.1. Definition of $\widehat{\zeta}_r^{\mathrm{des}}(\mathbf{s})(t)$

We introduce a new function $\widehat{\zeta}_r^{\mathrm{des}}(\mathbf{s})(t)$, and we investigate several properties of this function. More precisely, we show that $\widehat{\zeta}_r^{\mathrm{des}}(\mathbf{s})(t)$ converges to $\zeta_r^{\mathrm{des}}(\mathbf{s})$ under the limit $t \to 1-0$.

Definition 4.1. For $s_1, \ldots, s_r \in \mathbb{C} \setminus \mathbb{Z}$, and $0 < t < 1$, we define

$$\widehat{\zeta}_r^{\mathrm{des}}(s_1, \ldots, s_r)(t) := \prod_{k=1}^{r} \frac{1}{(e^{2\pi i s_k} - 1)\Gamma(s_k)} \tag{4.1}$$

$$\cdot \int_{\mathcal{C}_\epsilon^r} \prod_{j=1}^{r} \left(\frac{1}{\frac{1}{t}\exp(\sum_{k=j}^{r} x_k) - 1} + \frac{(\log t - \sum_{k=j}^{r} x_k)\frac{1}{t}\exp(\sum_{k=j}^{r} x_k)}{\{\frac{1}{t}\exp(\sum_{k=j}^{r} x_k) - 1\}^2} \right) \prod_{k=1}^{r} x_k^{s_k - 1} dx_k.$$

We note that the convergence of the right-hand side of (4.1) can be justified by the following proposition.

Proposition 4.2. *For any $t \in (0,1)$, the function $\widehat{\zeta}_r^{\mathrm{des}}(s_1,\ldots,s_r)(t)$ can be analytically continued to $\mathbb{C}^r$ as an entire function in $(s_1,\ldots,s_r) \in \mathbb{C}^r$. Moreover, for any $(s_1,\ldots,s_r) \in \mathbb{C}^r$, we have*

$$\lim_{t \to 1-0} \widehat{\zeta}_r^{\mathrm{des}}(s_1,\ldots,s_r)(t) = \zeta_r^{\mathrm{des}}(s_1,\ldots,s_r).$$

For the proof of Proposition 4.2, we prepare two lemmas. Let $\mathcal{N}(\epsilon) := \{z \in \mathbb{C} \mid |z| < \epsilon\}$ and $\mathcal{S}(\theta) := \{z \in \mathbb{C} \mid |\arg z| \le \theta\}$. Then one can easily obtain:

Lemma 4.3 *([6, Lemma 3.5]). There exist $\epsilon > 0$ and $0 < \theta < \pi/2$ such that*

$$\sum_{k=j}^{r} x_k \in \mathcal{N}(1) \cup \mathcal{S}(\theta)$$

for any $x_j,\ldots,x_r \in \mathcal{C}_\epsilon$ $(1 \le j \le r)$, where $\mathcal{C}_\epsilon$ is the Hankel contour involving a circle around the origin of radius ϵ.

Lemma 4.4 *(cf. [6, Lemma 3.6]). For any $t \in (0,1)$ and $y \in \mathcal{N}(1) \cup \mathcal{S}(\theta)$, there exists a constant $A > 0$ independent of t such that*

$$\left| \frac{1}{\frac{1}{t}e^y - 1} + \frac{(\log t - y)\frac{1}{t}e^y}{(\frac{1}{t}e^y - 1)^2} \right| < A e^{-\Re y/2}.$$

Proof. We set

$$F(t;y) := \frac{1}{\frac{1}{t}e^y - 1} + \frac{(\log t - y)\frac{1}{t}e^y}{(\frac{1}{t}e^y - 1)^2}.$$

We first note that

$$\frac{1}{t}e^y - 1 = y - \log t + \frac{1}{2}(y - \log t)^2 + O\big((y - \log t)^3\big), \quad (y \to \log t).$$

Thus, we get

$$F(t;y) = \frac{\frac{1}{t}e^y - 1 + (\log t - y)\frac{1}{t}e^y}{(\frac{1}{t}e^y - 1)^2} = \frac{-\frac{1}{2}(y - \log t)^2 + O\big((y - \log t)^3\big)}{(y - \log t)^2 + O\big((y - \log t)^3\big)}.$$

If $\log t < -1$, that is, $\log t \notin \mathcal{N}(1) \cup \mathcal{S}(\theta)$, then $F(t;y)$ is holomorphic for all $y \in \mathcal{N}(1) \cup \mathcal{S}(\theta)$. One can also see that $F(t;y)$ has the limit value when $y \to \log t$ for $-1 \le \log t < 0$. Hence, there exists $C > 0$ such that for any $y \in \mathcal{N}(1)$,

$$|F(t;y)| < C.$$

We next consider the case $y \in \mathcal{S}(\theta) \setminus \mathcal{N}(1)$. Note that there does not exist $t \in (0,1)$ such that $\log t = y$ in this case. Thus, there exists $A_1,\ A_2 > 0$ such that

$$\left|\frac{1}{\frac{1}{t}e^y - 1}\right| < A_1 e^{-\Re y/2}, \quad \left|\frac{(\log t - y)\frac{1}{t}e^y}{(\frac{1}{t}e^y - 1)^2}\right| < A_2 e^{-\Re y/2}.$$

Therefore, we obtain the claim by putting $A := A_1 + A_2$. □

Proof of Proposition 4.2. We use the notation used in Lemmas 4.3 and 4.4. We put

$$G(x_1,\ldots,x_r) := A^r \prod_{j=1}^{r} \exp\left(-\Re\left(\sum_{k=j}^{r} t_k/2\right)\right) = A^r \prod_{j=1}^{r} \exp\left(-\Re\left(t_k \frac{k(k+1)}{4}\right)\right).$$

Then, one can easily show that

$$\left|\prod_{j=1}^{r} F\left(t; \sum_{k=j}^{r} x_k\right)\right| < G(x_1,\ldots,x_r), \quad (x_1,\ldots,x_r \in \mathcal{C}), \tag{4.2}$$

$$\int\limits_{\mathcal{C}_\epsilon^r} G(x_1,\ldots,x_r) \prod_{k=1}^{r} |t_k^{s_k-1} dt_k| < \infty. \tag{4.3}$$

Since the integral on the right-hand side of (4.1) is holomorphic for all $s_1,\ldots,s_r \in \mathbb{C}$, we see that $\widehat{\zeta}_r^{\mathrm{des}}(s_1,\ldots,s_r)(t)$ can be meromorphically continued to $\mathbb{C}^r$. We also see that its possible singularities are located on hyperplanes $s_k = l_k \in \mathbb{N}$ $(k = 1,2,\ldots,r)$. For $s_k = l_k \in \mathbb{N}$, one can show that the integration of (4.1) is zero by using the residue theorem. Thus, $\widehat{\zeta}_r^{\mathrm{des}}(s_1,\ldots,s_r)(t)$ has no singularity on all the hyperplanes $s_k = l_k$ $(k = 1,2,\ldots,r)$. In other words, $\widehat{\zeta}_r^{\mathrm{des}}(s_1,\ldots,s_r)(t)$ is entire on $\mathbb{C}^r$. The latter part of the claim is proved by (4.2), (4.3) and Lebesgue's convergence theorem. □

4.2. Shuffle product of desingularized MZFs at integer points

In this subsection, we show that the shuffle product formula holds for products of special values at integer points of desingularized MZFs. At the end of this section, explicit formulas for the product of depth 1 and depth 1, and that of depth 1 and depth 2, are given.

Theorem 4.5. *For any $r \geq 1$, $\mathbf{k} = (k_1, \ldots, k_r) \in \mathbb{Z}^r$ and for $t \in (0,1)$, we have*

$$\widehat{\zeta}_r^{\mathrm{des}}(k_1, \ldots, k_r)(t) = Z(\mathbf{k}).$$

Proof. We set $[r] := \{1, 2, \ldots, r\}$ for $r \in \mathbb{N}$. We first calculate the integrand of the right-hand side in the equation (4.1):

$$\begin{aligned}
&\prod_{j=1}^{r}\left(\frac{1}{\frac{1}{t}\exp(\sum_{k=j}^{r} x_k) - 1} + \frac{(\log t - \sum_{k=j}^{r} x_k)\frac{1}{t}\exp(\sum_{k=j}^{r} x_k)}{\{\frac{1}{t}\exp(\sum_{k=j}^{r} x_k) - 1\}^2}\right) \\
&= \prod_{j=1}^{r}\left(\frac{1}{\frac{1}{t}\exp(\sum_{k=j}^{r} x_k) - 1} - \frac{\frac{1}{t}\sum_{k=j}^{r} x_k \exp(\sum_{k=j}^{r} x_k)}{\{\frac{1}{t}\exp(\sum_{k=j}^{r} x_k) - 1\}^2}\right. \\
&\quad \left. + \frac{\frac{1}{t}\exp(\sum_{k=j}^{r} x_k)}{\{\frac{1}{t}\exp(\sum_{k=j}^{r} x_k) - 1\}^2}\log t\right) \\
&= \sum_{J \subset [r]} \prod_{j \in [r] \setminus J}\left(\frac{1}{\frac{1}{t}\exp(\sum_{k=j}^{r} x_k) - 1} - \frac{\frac{1}{t}\sum_{k=j}^{r} x_k \exp(\sum_{k=j}^{r} x_k)}{\{\frac{1}{t}\exp(\sum_{k=j}^{r} x_k) - 1\}^2}\right) \\
&\quad \cdot \prod_{j \in J}\left(\frac{\frac{1}{t}\exp(\sum_{k=j}^{r} x_k)}{\{\frac{1}{t}\exp(\sum_{k=j}^{r} x_k) - 1\}^2}\right)(\log t)^{\# J} \\
&= \sum_{q \geq 0}(\log t)^q \left\{\sum_{\substack{J \subset [r] \\ \# J = q}} \prod_{j \in [r] \setminus J}\left(\frac{1}{\frac{1}{t}\exp(\sum_{k=j}^{r} x_k) - 1} - \frac{\frac{1}{t}\sum_{k=j}^{r} x_k \exp(\sum_{k=j}^{r} x_k)}{\{\frac{1}{t}\exp(\sum_{k=j}^{r} x_k) - 1\}^2}\right)\right. \\
&\quad \left. \cdot \prod_{j \in J}\left(\frac{\frac{1}{t}\exp(\sum_{k=j}^{r} x_k)}{\{\frac{1}{t}\exp(\sum_{k=j}^{r} x_k) - 1\}^2}\right)\right\}.
\end{aligned}$$

We note that the empty summation is interpreted as 0.

$$\begin{aligned}
&\prod_{j=1}^{r}\left(\frac{1}{\frac{1}{t}\exp(\sum_{k=j}^{r} x_k) - 1} + \frac{(\log t - \sum_{k=j}^{r} x_k)\frac{1}{t}\exp(\sum_{k=j}^{r} x_k)}{\{\frac{1}{t}\exp(\sum_{k=j}^{r} x_k) - 1\}^2}\right) \\
&= \sum_{q \geq 0}(\log t)^q \left\{\sum_{\substack{J \subset [r] \\ \# J = q}} \sum_{K \subset [r] \setminus J} (-1)^{\# K} \prod_{j \in [r] \setminus (J \cup K)}\left(\frac{1}{\frac{1}{t}\exp(\sum_{k=j}^{r} x_k) - 1}\right)\right. \\
&\quad \left. \cdot \prod_{j \in K}\left(\frac{\frac{1}{t}\sum_{k=j}^{r} x_k \exp(\sum_{k=j}^{r} x_k)}{\{\frac{1}{t}\exp(\sum_{k=j}^{r} x_k) - 1\}^2}\right) \prod_{j \in J}\left(\frac{\frac{1}{t}\exp(\sum_{k=j}^{r} x_k)}{\{\frac{1}{t}\exp(\sum_{k=j}^{r} x_k) - 1\}^2}\right)\right\}
\end{aligned}$$

$$= \sum_{q \geq 0} (\log t)^q \left\{ \sum_{\substack{J \subset [r] \\ \#J = q}} \sum_{K \subset [r] \setminus J} (-1)^{\#K} \prod_{j \in [r] \setminus (J \cup K)} \left(\frac{1}{\frac{1}{t} \exp(\sum_{k=j}^r x_k) - 1} \right) \right.$$

$$\left. \cdot \prod_{j \in K} \left(\sum_{k=j}^r x_k \right) \prod_{j \in J \cup K} \left(\frac{\frac{1}{t} \exp(\sum_{k=j}^r x_k)}{\{\frac{1}{t} \exp(\sum_{k=j}^r x_k) - 1\}^2} \right) \right\}.$$

Using two relations as formal power series

$$\frac{1}{\frac{1}{t} e^x - 1} = \sum_{n \geq 1} (t e^{-x})^n, \qquad \frac{\frac{1}{t} e^x}{(\frac{1}{t} e^x - 1)^2} = \sum_{n \geq 1} n (t e^{-x})^n,$$

we have

$$\prod_{j=1}^r \left(\frac{1}{\frac{1}{t} \exp(\sum_{k=j}^r x_k) - 1} + \frac{(\log t - \sum_{k=j}^r x_k) \frac{1}{t} \exp(\sum_{k=j}^r x_k)}{\{\frac{1}{t} \exp(\sum_{k=j}^r x_k) - 1\}^2} \right)$$

$$= \sum_{q \geq 0} (\log t)^q \left\{ \sum_{\substack{J \subset [r] \\ \#J = q}} \sum_{K \subset [r] \setminus J} (-1)^{\#K} \prod_{j \in [r] \setminus (J \cup K)} \left(\sum_{n_j \geq 1} t^{n_j} \exp\left(-n_j \sum_{k=j}^r x_k \right) \right) \right.$$

$$\left. \cdot \prod_{j \in K} \left(\sum_{k=j}^r x_k \right) \prod_{j \in J \cup K} \left(\sum_{n_j \geq 1} n_j t^{n_j} \exp\left(-n_j \sum_{k=j}^r x_k \right) \right) \right\}$$

$$= \sum_{q \geq 0} (\log t)^q \left\{ \sum_{\substack{J \subset [r] \\ \#J = q}} \sum_{K \subset [r] \setminus J} (-1)^{\#K} \sum_{\substack{n_j \geq 1 \\ j \in [r] \setminus (J \cup K)}} \prod_{j \in [r] \setminus (J \cup K)} t^{n_j} \exp\left(-n_j \sum_{k=j}^r x_k \right) \right.$$

$$\left. \cdot \prod_{j \in K} \left(\sum_{k=j}^r x_k \right) \sum_{\substack{n_j \geq 1 \\ j \in J \cup K}} \prod_{j \in J \cup K} n_j t^{n_j} \exp\left(-n_j \sum_{k=j}^r x_k \right) \right\}$$

$$= \sum_{q \geq 0} (\log t)^q \left\{ \sum_{\substack{J \subset [r] \\ \#J = q}} \sum_{K \subset [r] \setminus J} (-1)^{\#K} \sum_{n_1, \ldots, n_r \geq 1} \prod_{j=1}^r t^{n_j} \exp\left(-n_j \sum_{k=j}^r x_k \right) \right.$$

$$\left. \cdot \prod_{j \in K} \left(\sum_{k=j}^r x_k \right) \left(\prod_{j \in J \cup K} n_j \right) \right\}$$

$$= \sum_{q \geq 0} (\log t)^q \left\{ \sum_{\substack{J \subset [r] \\ \#J = q}} \sum_{K \subset [r] \setminus J} (-1)^{\#K} \sum_{n_1, \ldots, n_r \geq 1} t^{n_1 + \cdots + n_r} \prod_{j=1}^r \exp\left(-x_j \sum_{k=1}^j n_k \right) \right.$$

$$\cdot \prod_{j\in K}\left(\sum_{k=j}^{r} x_k\right)\left(\prod_{j\in J\cup K} n_j\right)\Bigg\}.$$

Similarly to [6, (3.18)], by putting

$$\prod_{j\in K}\left(\sum_{k=j}^{r} x_k\right) = \sum_{\mathbf{l}\in\mathbb{Z}_{\geq 0}^{r}} b_{K,\mathbf{l}} \prod_{j=1}^{r} x_j^{l_j}, \tag{4.4}$$

we get

$$\begin{aligned}
&\prod_{j=1}^{r}\left(\frac{1}{\frac{1}{t}\exp(\sum_{k=j}^{r} x_k)-1} + \frac{(\log t - \sum_{k=j}^{r} x_k)\frac{1}{t}\exp(\sum_{k=j}^{r} x_k)}{\{\frac{1}{t}\exp(\sum_{k=j}^{r} x_k)-1\}^2}\right) \qquad (4.5)\\
&= \sum_{q\geq 0}(\log t)^q \left\{\sum_{\substack{J\subset[r]\\ \#J=q}}\sum_{K\subset[r]\setminus J}(-1)^{\#K}\sum_{n_1,\dots,n_r\geq 1} t^{n_1+\cdots+n_r}\left(\prod_{j\in J\cup K} n_j\right)\right.\\
&\quad\left.\cdot \sum_{\mathbf{l}\in\mathbb{Z}_{\geq 0}^{r}} b_{K,\mathbf{l}} \prod_{j=1}^{r} x_j^{l_j} \prod_{j=1}^{r}\exp\left(-x_j\sum_{k=1}^{j} n_k\right)\right\}.
\end{aligned}$$

Assume that $\Re s_j$ is sufficiently large for $1 \leq j \leq r$. By using (4.5), we have

$$\begin{aligned}
&\widehat{\zeta}_r^{\mathrm{des}}(s_1,\dots,s_r)(t)\\
&= \prod_{k=1}^{r}\frac{1}{\Gamma(s_k)}\int_{[0,\infty)^r}\prod_{k=1}^{r} x_k^{s_k-1}dx_k \sum_{q\geq 0}(\log t)^q\left\{\sum_{\substack{J\subset[r]\\ \#J=q}}\sum_{K\subset[r]\setminus J}(-1)^{\#K}\right.\\
&\quad\left.\cdot\sum_{n_1,\dots,n_r\geq 1} t^{n_1+\cdots+n_r}\left(\prod_{j\in J\cup K} n_j\right)\sum_{\mathbf{l}\in\mathbb{Z}_{\geq 0}^{r}} b_{K,\mathbf{l}}\prod_{j=1}^{r} x_j^{l_j}\prod_{j=1}^{r}\exp\left(-x_j\sum_{k=1}^{j} n_k\right)\right\}\\
&= \prod_{k=1}^{r}\frac{1}{\Gamma(s_k)}\sum_{q\geq 0}(\log t)^q\left\{\sum_{\substack{J\subset[r]\\ \#J=q}}\sum_{K\subset[r]\setminus J}(-1)^{\#K}\sum_{n_1,\dots,n_r\geq 1} t^{n_1+\cdots+n_r}\left(\prod_{j\in J\cup K} n_j\right)\right.\\
&\quad\left.\cdot\sum_{\mathbf{l}\in\mathbb{Z}_{\geq 0}^{r}} b_{K,\mathbf{l}}\prod_{j=1}^{r}\int_{[0,\infty)}\exp\left(-x_j\sum_{k=1}^{j} n_k\right)x_j^{s_j+l_j-1}dx_j\right\}.
\end{aligned}$$

Because we have $\int_{[0,\infty)} e^{-nx}x^{s-1}dx = n^{-s}\Gamma(s)$, we calculate as

$$\begin{aligned}&\widehat{\zeta}_r^{\mathrm{des}}(s_1,\ldots,s_r)(t)\\&=\prod_{k=1}^{r}\frac{1}{\Gamma(s_k)}\sum_{q\geq 0}(\log t)^q\left\{\sum_{\substack{J\subset[r]\\ \#J=q}}\sum_{K\subset[r]\setminus J}(-1)^{\#K}\sum_{n_1,\ldots,n_r\geq 1}t^{n_1+\cdots+n_r}\left(\prod_{j\in J\cup K}n_j\right)\right.\\&\quad\left.\cdot\sum_{\mathbf{l}\in\mathbb{Z}_{\geq 0}^r}b_{K,\mathbf{l}}\prod_{j=1}^{r}\frac{\Gamma(s_j+l_j)}{(n_1+\cdots+n_j)^{s_j+l_j}}\right\}.\end{aligned}$$

Here, we have

$$\begin{aligned}\prod_{j\in J\cup K}n_j&=\prod_{j\in J\cup K}\left(\sum_{k=1}^{j}n_k-\sum_{k=1}^{j-1}n_k\right)\\&=\sum_{I\subset(J\cup K)\setminus\{1\}}\prod_{j\in(J\cup K)\setminus I}\left(\sum_{k=1}^{j}n_k\right)\prod_{j\in I}\left(-\sum_{k=1}^{j-1}n_k\right)\\&=\sum_{I\subset(J\cup K)\setminus\{1\}}(-1)^{\#I}\prod_{j\in(J\cup K)\setminus I}\left(\sum_{k=1}^{j}n_k\right)\prod_{j+1\in I}\left(\sum_{k=1}^{j}n_k\right),\end{aligned}$$

so by using this, we get

$$\begin{aligned}&\widehat{\zeta}_r^{\mathrm{des}}(s_1,\ldots,s_r)(t)\\&=\prod_{k=1}^{r}\frac{1}{\Gamma(s_k)}\sum_{q\geq 0}(\log t)^q\left\{\sum_{\substack{J\subset[r]\\ \#J=q}}\sum_{K\subset[r]\setminus J}(-1)^{\#K}\sum_{I\subset(J\cup K)\setminus\{1\}}(-1)^{\#I}\right.\\&\quad\left.\cdot\sum_{\mathbf{l}\in\mathbb{Z}_{\geq 0}^r}b_{K,\mathbf{l}}\prod_{j=1}^{r}\Gamma(s_j+l_j)\sum_{n_1,\ldots,n_r\geq 1}\frac{t^{n_1+\cdots+n_r}}{\prod_{j=1}^{r}(n_1+\cdots+n_j)^{s_j+l_j-\delta_{j\in(J\cup K)\setminus I}-\delta_{j+1\in I}}}\right\},\end{aligned}$$

where we use the symbol

$$\delta_{i\in I}:=\begin{cases}1 & (i\in I),\\ 0 & (i\notin I),\end{cases}$$

for $I\subset J\cup K$. Therefore, we have

$$\widehat{\zeta}_r^{\mathrm{des}}(s_1,\ldots,s_r)(t)=\sum_{q\geq 0}(\log t)^q\left\{\sum_{\substack{J\subset[r]\\ \#J=q}}\sum_{K\subset[r]\setminus J}\sum_{I\subset(J\cup K)\setminus\{1\}}(-1)^{\#K+\#I}\right. \tag{4.6}$$

$$\left.\cdot \sum_{\mathbf{l}\in\mathbb{Z}_{\geq 0}^r} b_{K,\mathbf{l}} \prod_{j=1}^r \frac{\Gamma(s_j+l_j)}{\Gamma(s_j)} \zeta_r^{\sqcup\!\sqcup}((s_j+l_j-\delta_{j\in(J\cup K)\setminus I}-\delta_{j+1\in I});t)\right\}$$

$$= \sum_{q\geq 0} (\log t)^q \left\{ \sum_{\substack{J\subset[r]\\ \#J=q}} \sum_{K\subset[r]\setminus J} \sum_{I\subset(J\cup K)\setminus\{1\}} (-1)^{\#K+\#I} \right.$$

$$\left.\cdot \sum_{\mathbf{l}\in\mathbb{Z}_{\geq 0}^r} b_{K,\mathbf{l}} \left(\prod_{j=1}^r (s_j)_{l_j}\right) \zeta_r^{\sqcup\!\sqcup}((s_j+l_j-\delta_{j\in(J\cup K)\setminus I}-\delta_{j+1\in I});t)\right\}.$$

We next put

$$\mathcal{H}((u_j);(v_j)) := \sum_{\substack{J\subset[r]\\ \#J=q}} \sum_{K\subset[r]\setminus J} \sum_{I\subset(J\cup K)\setminus\{1\}} (-1)^{\#K+\#I} \sum_{\mathbf{l}\in\mathbb{Z}_{\geq 0}^r} b_{K,\mathbf{l}} \prod_{j=1}^r u_j^{l_j} v_j^{l_j-\delta_{j\in(J\cup K)\setminus I}-\delta_{j+1\in I}}.$$

Then, by using (4.4), we have

$$\begin{aligned}
&\mathcal{H}((u_j);(v_j)) \\
&= \sum_{\substack{J\subset[r]\\ \#J=q}} \sum_{K\subset[r]\setminus J} \sum_{I\subset(J\cup K)\setminus\{1\}} (-1)^{\#K+\#I} \prod_{j\in K}\left(\sum_{k=j}^r u_k v_k\right) \prod_{j=1}^r v_j^{-\delta_{j\in(J\cup K)\setminus I}-\delta_{j+1\in I}} \\
&= \sum_{\substack{J\subset[r]\\ \#J=q}} \sum_{K\subset[r]\setminus J} \sum_{I\subset(J\cup K)\setminus\{1\}} (-1)^{\#K+\#I} \prod_{j\in K}\left(\sum_{k=j}^r u_k v_k\right) \left(\prod_{j\in(J\cup K)\setminus I} v_j^{-1}\right)\left(\prod_{j\in I} v_{j-1}^{-1}\right).
\end{aligned}$$

Because $(-1)^{\#((J\cup K)\setminus I)} = (-1)^{\#J+\#K-\#I} = (-1)^{\#J}(-1)^{\#K+\#I}$, we calculate as

$$\begin{aligned}
&\mathcal{H}((u_j);(v_j)) \\
&= \sum_{\substack{J\subset[r]\\ \#J=q}} (-1)^{\#J} \sum_{K\subset[r]\setminus J} \prod_{j\in K}\left(\sum_{k=j}^r u_k v_k\right) \left\{ \sum_{I\subset(J\cup K)\setminus\{1\}} \left(-\prod_{j\in(J\cup K)\setminus I} v_j^{-1}\right)\left(\prod_{j\in I} v_{j-1}^{-1}\right)\right\} \\
&= \sum_{\substack{J\subset[r]\\ \#J=q}} (-1)^{\#J} \sum_{K\subset[r]\setminus J} \prod_{j\in K}\left(\sum_{k=j}^r u_k v_k\right) \prod_{j\in J\cup K} (-v_j^{-1}+v_{j-1}^{-1}) \\
&= \sum_{\substack{J\subset[r]\\ \#J=q}} (-1)^{\#J} \prod_{j\in J} (-v_j^{-1}+v_{j-1}^{-1}) \sum_{K\subset[r]\setminus J} \prod_{j\in K} \left\{\left(\sum_{k=j}^r u_k v_k\right)(-v_j^{-1}+v_{j-1}^{-1})\right\}.
\end{aligned}$$

Here, for any set $I_1 \subset I_2$ and for indeterminates a_j ($j \in I_2$), we have

$$\sum_{I_1 \subset I_2} \prod_{j \in I_1} a_j = \prod_{j \in I_2} (a_j + 1),$$

where we set the summation of the left-hand side as 1 for $I_1 = \emptyset$. This relation with $I_1 = K$, $I_2 = [r] \setminus J$, and $a_j = \left(\sum_{k=j}^{r} u_k v_k\right)\left(-v_j^{-1} + v_{j-1}^{-1}\right)$ yields

$$\begin{aligned}
\mathcal{H}((u_j);(v_j)) &= \sum_{\substack{J \subset [r] \\ \#J = q}} \prod_{j \in J} \left(v_j^{-1} - v_{j-1}^{-1}\right) \prod_{j \in [r] \setminus J} \left\{ \left(\sum_{k=j}^{r} u_k v_k \right) \left(-v_j^{-1} + v_{j-1}^{-1}\right) + 1 \right\} \\
&= \sum_{\substack{J \subset [r] \\ \#J = q}} \prod_{j \in J} \left(v_j^{-1} - v_{j-1}^{-1}\right) \prod_{j \in [r] \setminus J} \left\{ 1 - (u_j v_j + \cdots + u_r v_r)\left(v_j^{-1} - v_{j-1}^{-1}\right) \right\} \\
&= \prod_{j=1}^{r} \left\{ 1 - (u_j v_j + \cdots + u_r v_r)\left(v_j^{-1} - v_{j-1}^{-1}\right) \right\} \\
&\quad \cdot \sum_{\substack{J \subset [r] \\ \#J = q}} \prod_{j \in J} \left(v_j^{-1} - v_{j-1}^{-1}\right) \prod_{j \in J} \left\{ 1 - (u_j v_j + \cdots + u_r v_r)\left(v_j^{-1} - v_{j-1}^{-1}\right) \right\}^{-1} \\
&= \prod_{j=1}^{r} \left\{ 1 - (u_j v_j + \cdots + u_r v_r)\left(v_j^{-1} - v_{j-1}^{-1}\right) \right\} \\
&\quad \cdot \sum_{\substack{J \subset [r] \\ \#J = q}} \prod_{j \in J} \frac{v_j^{-1} - v_{j-1}^{-1}}{1 - (u_j v_j + \cdots + u_r v_r)\left(v_j^{-1} - v_{j-1}^{-1}\right)}.
\end{aligned}$$

By the definition (1.3) of $\mathcal{G}_r$, the definition (A.2) of $G_{r,k}$ and Lemma A.1.(2), we have

$$\mathcal{H}((u_j);(v_j)) = \mathcal{G}_r \sum_{\substack{J \subset [r] \\ \#J = q}} \prod_{j \in J} G_{r,j} = \frac{(-1)^q}{q!} \left(v_r^{-1} \frac{\partial}{\partial u_r} \right)^q [\mathcal{G}_r],$$

that is, we obtain

$$\mathcal{H}((u_j);(v_j)) = \frac{(-1)^q}{q!} \left(v_r^{-1} \frac{\partial}{\partial u_r} \right)^q [\mathcal{G}_r].$$

Therefore, by using (A.1) and (4.6), we get

$$\widehat{\zeta}_r^{\mathrm{des}}(s_1, \ldots, s_r)(t) = \sum_{q \geq 0} \frac{(-\log t)^q}{q!} \left\{ \sum_{\substack{\mathbf{l} = (l_j) \in \mathbb{Z}_{\geq 0}^r \\ \mathbf{m} = (m_j) \in \mathbb{Z}^r \\ |\mathbf{m}| = -q}} a_{\mathbf{l},\mathbf{m}}^r(q) \left(\prod_{j=1}^{r} (s_j)_{l_j} \right) \zeta_r^{\sqcup\!\sqcup}(\mathbf{s} + \mathbf{m}; t) \right\}.$$

Because $\zeta_r^{\sqcup\!\sqcup}(\mathbf{k};t) = \mathrm{Li}_{\mathbf{k}}(t)$ for $\mathbf{k} \in \mathbb{Z}^r$, by Proposition A.2, we obtain

$$\widehat{\zeta}_r^{\mathrm{des}}(k_1,\dots,k_r)(t) = \sum_{q\geq 0} \frac{(-\log t)^q}{q!} \left\{ \sum_{\substack{\mathbf{l}=(l_j)\in\mathbb{Z}_{\geq 0}^r \\ \mathbf{m}=(m_j)\in\mathbb{Z}^r \\ |\mathbf{m}|=-q}} a_{\mathbf{l},\mathbf{m}}^r(q) \left(\prod_{j=1}^r (k_j)_{l_j}\right) \mathrm{Li}_{\mathbf{k}+\mathbf{m}}(t) \right\}$$

$$= \sum_{q\geq 0} \frac{(-\log t)^q}{q!} Z_q(\mathbf{k}) = Z(\mathbf{k}).$$

Hence, we finish the proof. □

Theorem 4.6. *For any $k_1,\dots,k_r \in \mathbb{Z}$ and for any $t \in (0,1)$, we have*

$$\widehat{\zeta}_r^{\mathrm{des}}(k_1,\dots,k_r)(t) = \mathrm{Li}^{\mathrm{des}}(k_1,\dots,k_r)(t).$$

Especially, we have

$$\lim_{t\to 1-0} \mathrm{Li}^{\mathrm{des}}(k_1,\dots,k_r)(t) = \zeta_r^{\mathrm{des}}(k_1,\dots,k_r). \tag{4.7}$$

Proof. By Theorem 3.16 and Theorem 4.5, we have

$$\widehat{\zeta}_r^{\mathrm{des}}(k_1,\dots,k_r)(t) = Z(\mathbf{k}) = \mathrm{Li}^{\mathrm{des}}(k_1,\dots,k_r)(t).$$

By using this equation, we get (4.7):

$$\lim_{t\to 1-0} \mathrm{Li}^{\mathrm{des}}(\mathbf{k})(t) = \lim_{t\to 1-0} \widehat{\zeta}_r^{\mathrm{des}}(\mathbf{k})(t) = \zeta_r^{\mathrm{des}}(\mathbf{k}).$$

Here, the second equality holds by Proposition 4.2. □

Remark 4.7. Proposition 4.6 is a generalization of Proposition 2.13.

We are now ready to present our main theorem.

Theorem 4.8. *We define the $\mathbb{Q}$-linear map $\zeta_{\sqcup\!\sqcup}^{\mathrm{des}} : \mathcal{H} \to \mathbb{R}$ by $\zeta_{\sqcup\!\sqcup}^{\mathrm{des}}(1) := 1$ and*

$$\zeta_{\sqcup\!\sqcup}^{\mathrm{des}}(j^{k_r}y\cdots j^{k_1}y) := \zeta_r^{\mathrm{des}}(k_1,\dots,k_r),$$

for $k_1,\dots,k_r \in \mathbb{Z}$. Then, this map $\zeta_{\sqcup\!\sqcup}^{\mathrm{des}}$ forms a $\mathbb{Q}$-algebra homomorphism, i.e., the "shuffle-type" formula holds for special values at any integer points of desingularized MZFs.

Proof. By Lemma 2.12 and (4.7), this map $\zeta^{\rm des}_{\sqcup\!\sqcup}$ forms a $\mathbb{Q}$-algebra homomorphism. Hence, we obtain the claim. □

Remark 4.9. This theorem is a generalization of Proposition 1.7 and Proposition 1.10.

We show some examples of the product of two special values of desingularized MZFs.

Examples 4.10. We calculate as the product $\zeta_1^{\rm des}(-k)\zeta_1^{\rm des}(l)$ for $k, l \in \mathbb{N}$. By definition of $\sqcup\!\sqcup_0$ (2.1), one can calculate the following: For k, $l \in \mathbb{N}$, we have

$$d^k y \sqcup\!\sqcup_0 j^l y = \sum_{i=0}^{\min\{k,l-1\}} (-1)^i \binom{k}{i} d^{k-i} y j^{l-i} y + (-1)^l \sum_{i=0}^{k-l} \binom{k-1-i}{l-1} d^{k-l-i} y d^i y.$$

We note that the empty summation is interpreted as 0. Thus we have

$$\begin{aligned}\zeta_1^{\rm des}(-k)\zeta_1^{\rm des}(l) &= \sum_{i=0}^{\min\{k,l-1\}} (-1)^i \binom{k}{i} \zeta_2^{\rm des}(l-i, -k+i)\\ &\quad + (-1)^l \sum_{i=0}^{k-l} \binom{k-1-i}{l-1} \zeta_2^{\rm des}(-i, -k+l+i).\end{aligned}$$

Examples 4.11. We calculate as the product $\zeta_2^{\rm des}(l,-k)\zeta_1^{\rm des}(m)$ for $k, l, m \in \mathbb{N}$. The following holds for k, l, $m \in \mathbb{N}$:

$$\begin{aligned}&d^k y j^l y \sqcup\!\sqcup_0 j^m y\\ &= \sum_{i=0}^{\min\{k,l-1\}} \sum_{p=1}^{l+m-i-1} (-1)^i \binom{k}{i} \left\{ \binom{p-1}{l-1} + \binom{p-1}{m-i-1} \right\} d^{k-i} y j^p y j^{l+m-i-p} y\\ &\quad + (-1)^m \sum_{i=0}^{k-m} \binom{m-1+i}{m-1} d^i y d^{k-m-i} y j^l y.\end{aligned}$$

Thus we have

$$\begin{aligned}&\zeta_2^{\rm des}(l,-k)\zeta_1^{\rm des}(m)\\ &= \sum_{i=0}^{\min\{k,l-1\}} \sum_{p=1}^{l+m-i-1} (-1)^i \binom{k}{i} \left\{ \binom{p-1}{l-1} + \binom{p-1}{m-i-1} \right\} \zeta_3^{\rm des}(l+m-i-p, p, -k+i)\\ &\quad + (-1)^m \sum_{i=0}^{k-m} \binom{m-1+i}{m-1} \zeta_3^{\rm des}(l, -k+m+i, -i).\end{aligned}$$

Based on some numerical experiments with depth 1, we believe that we will be able to prove Proposition 1.7 for $s_1, \dots, s_p \in \mathbb{Z}$ from (2.1) and (2.2).

Appendix A. Explicit formulae of $Z_q(\mathbf{k})$

In this appendix, we give explicit formulae of $Z_q(\mathbf{k})$ in Definition 3.7 in terms of $\mathrm{Li}_{\mathbf{k}+\mathbf{m}}(t)$ for $\mathbf{m} \in \mathbb{Z}^r$. As a consequence, we find an explicit expression of $Z(\mathbf{k})$ which is required to prove Theorem 4.5.

Recall the definitions of $\mathcal{G}_r$ and $\{a^r_{\mathbf{l},\mathbf{m}}\}$ (see (1.3) and (1.4) for detail). For $r \geq q \geq 0$, we define the set of integers $\{a^r_{\mathbf{l},\mathbf{m}}(q)\}$ by

$$\left(v_r^{-1}\frac{\partial}{\partial u_r}\right)^q [\mathcal{G}_r] = \sum_{\substack{\mathbf{l}=(l_j)\in\mathbb{Z}^r_{\geq 0}\\ \mathbf{m}=(m_j)\in\mathbb{Z}^r\\ |\mathbf{m}|=-q}} a^r_{\mathbf{l},\mathbf{m}}(q) \prod_{j=1}^r u_j^{l_j} v_j^{m_j}. \tag{A.1}$$

It is clear that $a^r_{\mathbf{l},\mathbf{m}}(0) = a^r_{\mathbf{l},\mathbf{m}}$. For $r \geq k \geq 1$, we put

$$G_{r,k} := G_{r,k}((u_j);(v_j)) := \frac{v_k^{-1} - v_{k-1}^{-1}}{1 - (u_k v_k + \cdots + u_r v_r)(v_k^{-1} - v_{k-1}^{-1})}. \tag{A.2}$$

We note that we have

$$v_r^{-1}\frac{\partial}{\partial u_r}[G_{r,k}] = G_{r,k}^2. \tag{A.3}$$

Recall the notations $(x)_q$ and $\mathbf{m}^{(q)}$ ($x \in \mathbb{C}$, $\mathbf{m} \in \mathbb{Z}^r$ and $q \in \mathbb{Z}$) for the following lemma (see Proposition 1.3 and (2.13) for detail).

Lemma A.1. *The following two statements hold.*

(1) *For $q \in [r]$, $\mathbf{l} = (l_j) \in \mathbb{Z}^r_{\geq 0}$ and $\mathbf{m} = (m_j) \in \mathbb{Z}^r$ with $|\mathbf{m}| = -q$, we have*

$$a^r_{\mathbf{l},\mathbf{m}}(q) = (l_r + 1)_q \cdot a^r_{\mathbf{l}^{(-q)},\mathbf{m}^{(-q)}}.$$

(2) *For $q \in [r]$, we have*

$$\left(v_r^{-1}\frac{\partial}{\partial u_r}\right)^q [\mathcal{G}_r] = (-1)^q q! \cdot \mathcal{G}_r \sum_{\substack{J\subset[r]\\ \#J=q}} \left(\prod_{j\in J} G_{r,j}\right). \tag{A.4}$$

Proof. We first prove the claim (1). By definition of $\mathcal{G}_r$, we calculate as

$$\left(v_r^{-1}\frac{\partial}{\partial u_r}\right)^q [\mathcal{G}_r] = \sum_{\substack{\mathbf{l}=(l_j)\in\mathbb{Z}^r_{\geq 0}\\ \mathbf{m}=(m_j)\in\mathbb{Z}^r\\ |\mathbf{m}|=0}} a^r_{\mathbf{l},\mathbf{m}} \left(\prod_{j=1}^{r-1} u_j^{l_j} v_j^{m_j}\right) (l_r - q + 1)_q u_r^{l_r - q} v_r^{m_r - q}.$$

By replacing $l_r - q$ to l_r and $m_r - q$ to m_r, we have

$$\left(v_r^{-1}\frac{\partial}{\partial u_r}\right)^q [\mathcal{G}_r] = \sum_{\substack{\mathbf{l}=(l_j)\in\mathbb{Z}_{\geq 0}^r \\ \mathbf{m}=(m_j)\in\mathbb{Z}^r \\ |\mathbf{m}|=-q}} a^r_{\mathbf{l}^{(-q)},\mathbf{m}^{(-q)}} \left(\prod_{j=1}^{r-1} u_j^{l_j} v_j^{m_j}\right) (l_r+1)_q u_r^{l_r} v_r^{m_r}$$

$$= \sum_{\substack{\mathbf{l}=(l_j)\in\mathbb{Z}_{\geq 0}^r \\ \mathbf{m}=(m_j)\in\mathbb{Z}^r \\ |\mathbf{m}|=-q}} (l_r+1)_q \cdot a^r_{\mathbf{l}^{(-q)},\mathbf{m}^{(-q)}} \prod_{j=1}^{r} u_j^{l_j} v_j^{m_j}.$$

Therefore, by comparing this and (A.1), we obtain the claim (1).

We next prove the claim (2) by induction on q. We note that, for any rational functions $f_1, \ldots, f_r$ in u_r, we have

$$\frac{\partial}{\partial u_r}\left[\prod_{j=1}^{r} f_j\right] = \left(\prod_{j=1}^{r} f_j\right) \sum_{k=1}^{r} \frac{1}{f_k}\frac{\partial}{\partial u_r}[f_k]. \tag{A.5}$$

When $q = 1$, by using (A.5) and the definition of $\mathcal{G}_r$, we directly have

$$\left(v_r^{-1}\frac{\partial}{\partial u_r}\right)[\mathcal{G}_r] = v_r^{-1}\mathcal{G}_r \sum_{k=1}^{r} \frac{-v_r(v_k^{-1} - v_{k-1}^{-1})}{1 - (u_k v_k + \cdots + u_r v_r)(v_k^{-1} - v_{k-1}^{-1})} \tag{A.6}$$

$$= -\mathcal{G}_r \sum_{k=1}^{r} G_{r,k} = (-1)^1 1! \cdot \mathcal{G}_r \sum_{\substack{J\subset[r] \\ \#J=1}} \left(\prod_{j\in J} G_{r,j}\right).$$

So the equation (A.4) holds for $q = 1$. Assume the equation (A.4) holds for $q\,(\geq 1)$ or less. Then, by the induction hypothesis, we have

$$\left(v_r^{-1}\frac{\partial}{\partial u_r}\right)^{q+1} [\mathcal{G}_r] = \left(v_r^{-1}\frac{\partial}{\partial u_r}\right)\left[(-1)^q q! \cdot \mathcal{G}_r \sum_{\substack{J\subset[r] \\ \#J=q}} \left(\prod_{j\in J} G_{r,j}\right)\right]$$

$$= (-1)^q q! \left(v_r^{-1}\frac{\partial}{\partial u_r}\right)[\mathcal{G}_r] \sum_{\substack{J\subset[r] \\ \#J=q}} \left(\prod_{j\in J} G_{r,j}\right)$$

$$+ (-1)^q q! \cdot \mathcal{G}_r \sum_{\substack{J\subset[r] \\ \#J=q}} \left(v_r^{-1}\frac{\partial}{\partial u_r}\right)\left[\prod_{j\in J} G_{r,j}\right].$$

By applying (A.6) to the first term and by applying (A.5) to the second term, we calculate as

$$\begin{aligned}&\left(v_r^{-1}\frac{\partial}{\partial u_r}\right)^{q+1}[\mathcal{G}_r]\\&=(-1)^{q+1}q!\cdot\mathcal{G}_r\left\{\left(\sum_{k=1}^{r}G_{r,k}\right)\sum_{\substack{J\subset[r]\\ \#J=q}}\left(\prod_{j\in J}G_{r,j}\right)\right.\\&\qquad\left.-\sum_{\substack{J\subset[r]\\ \#J=q}}\left(\prod_{j\in J}G_{r,j}\right)\sum_{k\in J}\frac{v_r^{-1}}{G_{r,k}}\frac{\partial}{\partial u_r}[G_{r,k}]\right\}.\end{aligned}$$

By using (A.3) in the second term, we get

$$\begin{aligned}&\left(v_r^{-1}\frac{\partial}{\partial u_r}\right)^{q+1}[\mathcal{G}_r]\\&=(-1)^{q+1}q!\cdot\mathcal{G}_r\left\{\left(\sum_{k=1}^{r}G_{r,k}\right)\sum_{\substack{J\subset[r]\\ \#J=q}}\left(\prod_{j\in J}G_{r,j}\right)-\sum_{\substack{J\subset[r]\\ \#J=q}}\left(\prod_{j\in J}G_{r,j}\right)\sum_{k\in J}G_{r,k}\right\}\\&=(-1)^{q+1}q!\cdot\mathcal{G}_r\sum_{\substack{J\subset[r]\\ \#J=q}}\left(\prod_{j\in J}G_{r,j}\right)\left\{\sum_{k=1}^{r}G_{r,k}-\sum_{k\in J}G_{r,k}\right\}\\&=(-1)^{q+1}q!\cdot\mathcal{G}_r\sum_{\substack{J\subset[r]\\ \#J=q}}\left\{\left(\prod_{j\in J}G_{r,j}\right)\sum_{k\in[r]\setminus J}G_{r,k}\right\}\\&=(-1)^{q+1}q!\cdot\mathcal{G}_r\sum_{\substack{J\subset[r]\\ \#J=q}}\sum_{k\in[r]\setminus J}\left\{\left(\prod_{j\in J}G_{r,j}\right)G_{r,k}\right\}.\end{aligned}$$

By putting $I=J\cup\{k\}$, we have $\#I=\#J+1=q+1$. So we calculate as

$$\begin{aligned}&\left(v_r^{-1}\frac{\partial}{\partial u_r}\right)^{q+1}[\mathcal{G}_r]\\&=(-1)^{q+1}q!\cdot\mathcal{G}_r\sum_{\substack{I\subset[r]\\ \#I=q+1}}\sum_{k\in I}\left\{\left(\prod_{j\in I\setminus\{k\}}G_{r,j}\right)G_{r,k}\right\}\end{aligned}$$

$$= (-1)^{q+1} q! \cdot \mathcal{G}_r \sum_{\substack{I \subset [r] \\ \#I = q+1}} \sum_{k \in I} \left(\prod_{j \in I} G_{r,j} \right)$$

$$= (-1)^{q+1} q! \cdot \mathcal{G}_r \sum_{\substack{I \subset [r] \\ \#I = q+1}} (q+1) \left(\prod_{j \in I} G_{r,j} \right) = (-1)^{q+1} (q+1)! \cdot \mathcal{G}_r \sum_{\substack{I \subset [r] \\ \#I = q+1}} \left(\prod_{j \in I} G_{r,j} \right).$$

Therefore, the equation (A.4) holds for $q+1$. Hence, we obtain the claim (2). □

By the above lemma, we get an explicit expression of $Z_q(\mathbf{k})$.

Proposition A.2. *For any $q \in [r]$, we have*

$$Z_q(\mathbf{k}) = \sum_{\substack{\mathbf{l}=(l_j) \in \mathbb{Z}_{\geq 0}^r \\ \mathbf{m}=(m_j) \in \mathbb{Z}^r \\ |\mathbf{m}| = -q}} a^r_{\mathbf{l},\mathbf{m}}(q) \left(\prod_{j=1}^{r} (k_j)_{l_j} \right) \mathrm{Li}_{\mathbf{k}+\mathbf{m}}(t). \tag{A.7}$$

Proof. By Definition 3.7, we have

$$Z_q(\mathbf{k}) = \sum_{\substack{i+j=q \\ i,j \geq 0}} (-1)^j \binom{q}{i} D^i \left[Z_0(\mathbf{k}^{(j)}) \right].$$

By Proposition 3.5 and by $\zeta_r^{ш}(\mathbf{k}; t) = \mathrm{Li}_{\mathbf{k}}(t)$, we calculate as

$$Z_q(\mathbf{k}) = \sum_{\substack{i+j=q \\ i,j \geq 0}} (-1)^j \binom{q}{i} D^i \left[\sum_{\substack{\mathbf{l}=(l_j) \in \mathbb{Z}_{\geq 0}^r \\ \mathbf{m}=(m_j) \in \mathbb{Z}^r \\ |\mathbf{m}| = 0}} a^r_{\mathbf{l},\mathbf{m}} \left(\prod_{a=1}^{r-1} (k_a)_{l_a} \right) (k_r - j)_{l_r} \mathrm{Li}_{\mathbf{k}+\mathbf{m}^{(j)}}(t) \right]$$

$$= \sum_{\substack{i+j=q \\ i,j \geq 0}} (-1)^j \binom{q}{i} \sum_{\substack{\mathbf{l}=(l_j) \in \mathbb{Z}_{\geq 0}^r \\ \mathbf{m}=(m_j) \in \mathbb{Z}^r \\ |\mathbf{m}| = 0}} a^r_{\mathbf{l},\mathbf{m}} \left(\prod_{a=1}^{r-1} (k_a)_{l_a} \right) (k_r - j)_{l_r} \mathrm{Li}_{\mathbf{k}+\mathbf{m}^{(q)}}(t).$$

By replacing $l_r - q$ to l_r and $m_r - q$ to m_r, we get

$$Z_q(\mathbf{k}) = \sum_{\substack{i+j=q \\ i,j \geq 0}} (-1)^j \binom{q}{i} \sum_{\substack{\mathbf{l}=(l_j) \in \mathbb{Z}_{\geq 0}^r \\ \mathbf{m}=(m_j) \in \mathbb{Z}^r \\ |\mathbf{m}| = -q}} a^r_{\mathbf{l}^{(-q)},\mathbf{m}^{(-q)}} \left(\prod_{a=1}^{r-1} (k_a)_{l_a} \right) (k_r - j)_{l_r+q} \mathrm{Li}_{\mathbf{k}+\mathbf{m}}(t)$$

$$= \sum_{\substack{\mathbf{l}=(l_j)\in\mathbb{Z}^r_{\geq 0}\\ \mathbf{m}=(m_j)\in\mathbb{Z}^r\\ |\mathbf{m}|=-q}} a^r_{\mathbf{l}^{(-q)},\mathbf{m}^{(-q)}} \left(\prod_{a=1}^{r-1}(k_a)_{l_a}\right)\left\{\sum_{\substack{i+j=q\\ i,j\geq 0}}(-1)^j\binom{q}{i}(k_r-j)_{l_r+q}\right\}\mathrm{Li}_{\mathbf{k}+\mathbf{m}}(t).$$

By using the following Lemma A.3 for $l = l_r$ and $s = k_r$, we have

$$Z_q(\mathbf{k}) = \sum_{\substack{\mathbf{l}=(l_j)\in\mathbb{Z}^r_{\geq 0}\\ \mathbf{m}=(m_j)\in\mathbb{Z}^r\\ |\mathbf{m}|=-q}} a^r_{\mathbf{l}^{(-q)},\mathbf{m}^{(-q)}} \left(\prod_{a=1}^{r-1}(k_a)_{l_a}\right)(l_r+1)_q\cdot(k_r)_{l_r}\mathrm{Li}_{\mathbf{k}+\mathbf{m}}(t)$$

$$= \sum_{\substack{\mathbf{l}=(l_j)\in\mathbb{Z}^r_{\geq 0}\\ \mathbf{m}=(m_j)\in\mathbb{Z}^r\\ |\mathbf{m}|=-q}} \left\{(l_r+1)_q\cdot a^r_{\mathbf{l}^{(-q)},\mathbf{m}^{(-q)}}\right\}\left(\prod_{a=1}^{r}(k_a)_{l_a}\right)\mathrm{Li}_{\mathbf{k}+\mathbf{m}}(t).$$

By Lemma A.1.(1), we obtain (A.7), hence we finish the proof. □

The following lemma is used in the above proof.

Lemma A.3. *For $l, q \geq 0$, and $s \in \mathbb{C}$, we have*

$$\sum_{\substack{i+j=q\\ i,j\geq 0}}(-1)^j\binom{q}{i}(s-j)_{l+q} = (l+1)_q\cdot(s)_l. \tag{A.8}$$

Proof. We prove this by induction on $q \geq 0$. It is clear that the claim holds for $q = 0$. When $q = 1$, we calculate as the left hand side of (A.8) as

$$(s)_{l+1}-(s-1)_{l+1} = (s)_l\cdot(s+l)-(s)_l\cdot(s-1) = (l+1)\cdot(s)_l. \tag{A.9}$$

Assume the equation (A.8) holds for $q-1\,(\geq 0)$ or less. Then, by using Lemma 3.9 for $f(i,j) = (s-j)_{l+q}$, we have

$$\begin{aligned}
&\sum_{\substack{i+j=q\\ i,j\geq 0}}(-1)^j\binom{q}{i}(s-j)_{l+q}\\
&= \sum_{\substack{i+j=q-1\\ i,j\geq 0}}(-1)^j\binom{q-1}{i}(s-j)_{l+q} + \sum_{\substack{i+j=q-1\\ i,j\geq 0}}(-1)^{j+1}\binom{q-1}{i}(s-j-1)_{l+q}\\
&= (l+2)_{q-1}\cdot(s)_{l+1}-(l+2)_{q-1}\cdot(s-1)_{l+1}\\
&= (l+2)_{q-1}(l+1)\cdot(s)_l\\
&= (l+1)_q\cdot(s)_l.
\end{aligned}$$

We used the induction hypothesis in the second equality and the equation (A.9) in the third equality. Hence, we finish the proof. □

Data availability

No data was used for the research described in the article.

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