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REMARKS ON HYPERBOLIC SYSTEMS OF FIRST ORDER WITH CONSTANT COEFFICIENT CHARACTERISTIC POLYNOMIALS

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Abstract

In this paper we shall deal with hyperbolic systems of first order with constant coefficient characteristic polynomials and give a necessary and sufficient condition for the Cauchy problem to be C^∞ well-posed under the maximal rank condition (see the condition (R) below).

1. Introduction

The Cauchy problem for hyperbolic operators has been studied by many authors, and necessary and sufficient conditions for C^∞ well-posedness have been obtained in restricted situations (see, e.g., [19], [6], [4], [22], [12], [26], [20], [10], [11], [17] and [21]). We think that it is meaningfully to obtain necessary and sufficient conditions for C^∞ well-posedness in various restricted situations as a step forward in the study of C^∞ well-posedness.

In [26] we proved that the Cauchy problem for a single higher order operator $P(x, D)$ with constant coefficient hyperbolic principal part is C^∞ well-posed (in $\mathbb{R}^n$) and has the finite propagation property if and only if $P(x, \xi)$ is hyperbolic in the sense of Gårding for each $x \in \mathbb{R}^n$. We should note that the sufficiency was proved by Dunn [5] and that the condition of the finite propagation property can be removed by applying the arguments in this paper. We shall attempt to extend the result to hyperbolic systems of first order whose characteristic polynomials have constant coefficients, and give a necessary and sufficient condition for C^∞ well-posedness under the maximal rank condition in this paper.

Concerning the necessity of C^∞ well-posedness for hyperbolic systems of first order, Benvenuti, Bernardi and Bove defined invariantly "determinants" of the systems under the maximal rank condition in [1] and gave necessary conditins of Ivrii-Petkov type by means of "determinants." The assumption on the rank was removed by Bove and Nishitani [3]. The necessary conditions obtained in [1] is not sufficient ones for the hyperbolic systems treated here. For hyperbolic systems with constant multiplicities a necessary and sufficient condition is obtained under the maximal rank condition (see

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[22] and [12]). The maximal rank condition considerably simplifies the problems. One of our aim is to illustrate how to prove the sufficiency of C^∞ well-posedness of the Cauchy problem for general hyperbolic systems satisfying the maximal rank condition by applying the results in [14]. Another aim is to show how to prove the necessity by transforming microlocally the systems. A part of the results here was announced in [30] without proof.

Let $A_1(x), \dots, A_n(x), L_0(x) \in C^\infty(\mathbb{R}^n; M_m(\mathbb{C}))$, and put

$$L_1(x,\xi) = \sum_{j=1}^{n} \xi_j A_j(x), \quad L(x,\xi) = L_1(x,\xi) + L_0(x),$$

where m is an integer ≥ 2, $M_{m,m'}(R)$ denotes the collection of all $m \times m'$ matrices with their entries in R, $M_m(R) = M_{m,m}(R)$, $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n$ and $C^\infty(\mathbb{R}^n; V)$ denotes the collection of C^∞ functions defined on $\mathbb{R}^n$ with their values in V. For $t \in \mathbb{R}$ we consider the Cauchy problem

$$(\mathrm{CP})_t \qquad \begin{cases} L(x,D)u(x) = f(x) & \text{in} \quad \mathbb{R}^n, \\ \operatorname{supp} u \subset \{x \in \mathbb{R}^n;\, x_1 \geq t\} \end{cases}$$

in the C^∞ (or $\mathscr{D}'$) category, where $D = (D_1, \dots, D_n) = -i(\partial/\partial x_1, \dots, \partial/\partial x_n)$ and $f = {}^t(f_1, \dots, f_m)$ satisfies $\operatorname{supp} f \subset \{x \in \mathbb{R}^n;\, x_1 \geq t\}$. Assume that
(H) $A_1(x) = I_m$, $\det L_1(x,\xi)$ does not depend on x and $p(\xi) \equiv \det L_1(x,\xi)$ is hyperbolic with respect to $\vartheta = (1, 0, \dots, 0) \in \mathbb{R}^n$, i.e., $p(\xi - i\vartheta) \neq 0$ for any $\xi \in \mathbb{R}^n$, where I_m denotes the identity matrix of order m.

We also assume that $L(x,D)$ satisfies the maximal rank condition, i.e.,
(R) $\operatorname{rank} L_1(x,\xi) = m-1$ for any $(x,\xi) \in \mathbb{R}^n \times S^{n-1}$ with $dp(\xi) = 0$, where $S^{n-1} = \{\xi \in \mathbb{R}^n;\, |\xi| = 1\}$.

Under the assumption (R) we shall reduce the Cauchy problem $(\mathrm{CP})_t$ to that of single higher order operators in a microlocal sense. Let $x^0 \in \mathbb{R}^n$ and $\xi^{0\prime} = (\xi_2^0, \dots, \xi_n^0) \in S^{n-2}$. By a linear coordinate transformation of $x' = (x_2, \dots, x_n)$ we may assume that $\xi^{0\prime} = (0, \dots, 0, 1) \in \mathbb{R}^{n-1}$. Write

$$p(\xi^0 + \lambda\vartheta) = \prod_{j=1}^{r} (\lambda + \lambda_j)^{m_j}, \quad \lambda_1 < \lambda_2 < \cdots < \lambda_r,$$

where $\xi^0 = (0, \dots, 0, 1) \in \mathbb{R}^n$. Then there are an open neighborhood V of x^0, an open conic neighborhood $\mathscr{C}'$ of $\xi^{0\prime}$ in $\mathbb{R}^{n-1} \setminus \{0\}$, $S(x,\xi') \in C^\infty(\mathbb{R}^n \times \mathscr{C}'; M_m(\mathbb{C}))$ and $A^j(x,\xi') \in C^\infty(V \times \mathscr{C}'; M_{m_j}(\mathbb{C}))$ $(1 \leq j \leq r)$ such that $\det S(x,\xi') \neq 0$ for $(x,\xi') \in V \times \mathscr{C}'$, the entries of $S(x,\xi')$ are (positively) homogeneous of degree 0, $(A^j(x,\xi^{0\prime}) - \lambda_j I_{m_j})^{m_j} = 0$ and

$$S(x,\xi')^{-1} A(x,\xi') S(x,\xi') = \operatorname{diag}(A^1(x,\xi'), \dots, A^r(x,\xi')) \quad \text{in} \quad V \times \mathscr{C}',$$

where $\xi' = (\xi_2, \dots, \xi_n) \in \mathbb{R}^{n-1}$, $A(x, \xi') = \sum_{j=2}^n \xi_j A_j(x)$ and

$$\mathrm{diag}(A^1, \dots, A^r) = \begin{pmatrix} A^1 & & 0 \\ & \ddots & \\ 0 & & A^r \end{pmatrix}$$

(see Lemma 2.2 below). We can assume without loss of generality that $A_2(x), \dots, A_n(x), L_0(x) \in \mathscr{B}(\mathbb{R}^n; M_m(\mathbb{C}))$, i.e., $\sup_{x\in\mathbb{R}^n}\left(\sum_{j=2}^n |D^\alpha A_j(x)| + |D^\alpha L_0(x)|\right) \le C_\alpha$ for every $\alpha \in (\mathbb{Z}_+)^n$, where $|A|$ denotes the matrix norm of A and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$ $(= \{0, 1, 2, \dots\})$. We say that a symbol $a(x,\xi)$ belongs to $S^\kappa_{1,0}$ if $a(x,\xi) \in C^\infty(T^*\mathbb{R}^n)$ and $\left|a^{(\alpha)}_{(\beta)}(x, \xi)\right| \le C_{\alpha,\beta}\langle\xi\rangle^{\kappa-|\alpha|}$ for any $(x, \xi) \in T^*\mathbb{R}^n$ $(\simeq \mathbb{R}^n \times \mathbb{R}^n)$ and $\alpha, \beta \in (\mathbb{Z}_+)^n$, where $\kappa \in \mathbb{R}$, $\langle\xi\rangle = (1 + |\xi|^2)^{1/2}$ and $a^{(\alpha)}_{(\beta)}(x, \xi) = \partial^\alpha_\xi D^\beta_x a(x, \xi)$. Similarly, we say that $a(x, \xi') \in S^\kappa_{1,0}(\mathbb{R} \times T^*\mathbb{R}^{n-1})$ if $a(x, \xi') \in C^\infty(\mathbb{R} \times T^*\mathbb{R}^{n-1})$ and $\left|a^{(\alpha')}_{(\beta)}(x, \xi')\right| \le C_{\alpha',\beta}\langle\xi'\rangle^{\kappa-|\alpha'|}$ for any $(x, \xi') \in \mathbb{R} \times T^*\mathbb{R}^{n-1}$ $(\simeq \mathbb{R}^n \times \mathbb{R}^{n-1})$ and $\alpha' = (\alpha_2, \dots, \alpha_n) \in (\mathbb{Z}_+)^{n-1}$ and $\beta \in (\mathbb{Z}_+)^n$, where $a^{(\alpha')}_{(\beta)}(x, \xi') = \partial^{\alpha'}_{\xi'} D^\beta_x a(x, \xi')$. Moreover, we say that a symbol $a(x,\xi)$ belongs to $\mathscr{S}^{k,l}_{1,0}$ if $a(x,\xi) = \sum_{j=0}^{[k]} a_j(x,\xi')\xi_1^j$ and the $a_j(x,\xi')$ are classical symbols and $a_j(x,\xi') \in S^{k+l-j}_{1,0}(\mathbb{R} \times T^*\mathbb{R}^{n-1})$, where $k,l \in \mathbb{R}$, $[k]$ denotes the largest integer $\le k$ and $\mathscr{S}^{k,l}_{1,0} = \{0\}$ if $k < 0$. We write $\mathscr{S}^k_{1,0} = \mathscr{S}^{k,0}_{1,0}$, $\mathscr{S}^{k,-\infty}_{1,0} = \bigcap_{l\in\mathbb{R}} \mathscr{S}^{k,l}_{1,0}$ and $\mathscr{S}^\infty_{1,0} = \bigcup_{k\ge0} \mathscr{S}^k_{1,0}$. By $\mathscr{L}^{k,l}_{1,0}$ we denote the set of pseudodifferential operators whose symbols belong to $\mathscr{S}^{k,l}_{1,0}$. We also write $\mathscr{L}^k_{1,0} = \mathscr{L}^{k,0}_{1,0}$. We need the following block-diagonalization (see, e.g., §3.3 of [18] and, also, Lemma 2.3 below): There are classical symbols $S_\pm(x,\xi') \in M_m\left(\mathscr{S}^0_{1,0}\right)$, $\widetilde{A}^\mu(x,\xi') \in M_{m_\mu}\left(\mathscr{S}^{0,1}_{1,0}\right)$ and $C^\mu(x,\xi') \in M_{m_\mu}\left(\mathscr{S}^0_{1,0}\right)$ $(1 \le \mu \le r)$ and $Q(x, \xi') \in M_m\left(\mathscr{S}^{0,-1}_{1,0}\right)$ such that

$$\begin{aligned}
&S_+(x, \xi') = S(x, \xi'), \quad \widetilde{A}^\mu(x, \xi') = A^\mu(x, \xi') \\
&\qquad\qquad \text{for} \quad (x, \xi') \in V \times \mathscr{C}' \quad \text{with} \quad |\xi'| \ge 1 \quad \text{and} \quad 1 \le \mu \le r, \\
&S_-(x, D')S_+(x, D') \equiv I \bmod \mathscr{L}^{0,-\infty}_{1,0} \quad \text{in} \quad V \times \mathscr{C}', \\
&S_-(x, D')L(x, D)S_+(x, D')(I + Q(x, D')) \\
&\equiv (I + Q(x, D'))\left\{D_1 I + \mathrm{diag}\left(\widetilde{A}^1(x, D') + C^1(x, D'), \dots, \widetilde{A}^r(x, D') + C^r(x, D')\right)\right\} \\
&\bmod \mathscr{L}^{1,-\infty}_{1,0} \qquad\qquad \text{in} \quad V \times \mathscr{C}',
\end{aligned}$$

with modifications of V and $\mathscr{C}'$ if necessary, where I denotes the identity operator and $D' = (D_2, \dots, D_n)$. Here $A(x, D) \equiv B(x, D) \bmod \mathscr{L}^{k,-\infty}_{1,0}$ in $V \times \mathscr{C}'$ means that $(A(x,\xi) - B(x,\xi))\psi(x,\xi') \in M_m\left(\mathscr{S}^{k,-\infty}_{1,0}\right)$ for any $\psi(x,\xi') \in \mathscr{S}^0_{1,0}$ with $\mathrm{supp}\,\psi \subset V \times \mathscr{C}'$. We fix $\mu \in \{1, 2, \dots, r\}$. Then it follows from the assumption (R) that there is $T_{\mu,1} \in$

$M_{m_\mu}(\mathbb{C})$ such that $\det T_{\mu,1} \neq 0$ and

$$T_{\mu,1}^{-1} A^\mu(x^0, \xi^{0\prime}) T_{\mu,1} = \begin{pmatrix} \lambda_\mu & 1 & & \mathbf{0} \\ & \ddots & \ddots & \\ & & & 1 \\ \mathbf{0} & & & \lambda_\mu \end{pmatrix}.$$

Putting

$$T_{\mu,2} = \begin{pmatrix} 0 & & & \mathbf{0} & 1 \\ 1 & 0 & & & 0 \\ & \ddots & \ddots & & \vdots \\ & & & 0 & 0 \\ \mathbf{0} & & & 1 & 0 \end{pmatrix} \in M_{m_\mu}(\mathbb{C}),$$

we have

$$L^\mu(x,\xi) \equiv T_{\mu,1}^{-1}\big(\xi_1 I_{m_\mu} + \widetilde{A}^\mu(x,\xi') + C^\mu(x,\xi')\big) T_{\mu,1} T_{\mu,2}$$

$$= \begin{pmatrix} \xi_n & 0 & & \mathbf{0} & \xi_1 + \lambda_\mu \xi_n \\ \xi_1 + \lambda_\mu \xi_n & \xi_n & & & 0 \\ & \ddots & \ddots & & \vdots \\ & & & \xi_n & 0 \\ \mathbf{0} & & & \xi_1 + \lambda_\mu \xi_n & 0 \end{pmatrix} + \widetilde{L}^\mu(x,\xi'), \tag{1.1}$$

$$\widetilde{L}^\mu(x,\xi') = \widetilde{L}_1^\mu(x,\xi') + \widetilde{L}_0^\mu(x,\xi'), \quad \widetilde{L}_1^\mu(x^0,\xi^{0\prime}) = 0,$$

where $\widetilde{L}_1^\mu(x,\xi') \in \mathscr{S}_{1,0}^{0,1}$, $\widetilde{L}_0^\mu(x,\xi') \in \mathscr{S}_{1,0}^0$ and $\widetilde{L}_1^\mu(x,\xi')$ is (positively) homogeneous of degree 1 for $|\xi'| \geq 1$. We may assume that $\xi_n + \widetilde{L}_{1,j,j}^\mu(x,\xi') \neq 0$ for $1 \leq j \leq m_\mu$, $x \in V$ and $\xi' \in \mathscr{C}'$ with $|\xi'| \geq 1$, modifying V and $\mathscr{C}'$ if necessary, where $\widetilde{L}_1^\mu(x,\xi') = \big(\widetilde{L}_{1,j,k}^\mu(x,\xi')\big)$. Now we perform the following elementary transformations on $L^\mu(x,D)$ in turn from $j = 1$ to $j = m_\mu - 1$: (i) We multiply the j-th column on the right by a pseudodifferential operator in $\mathscr{L}_{1,0}^0$ and add it to the k-th column to annihilate the (j,k)-th entry modulo $\mathscr{L}_{1,0}^{0,-\infty}$ in $V \times \mathscr{C}'$ $(j < k \leq m_\mu - 1)$. (ii) We multiply the j-th column on the right by an operator in $\mathscr{L}_{1,0}^{j,-j}$ and add it to the m_μ-th column to annihilate the (j, m_μ)-th entry modulo $\mathscr{L}_{1,0}^{j,-\infty}$ in $V \times \mathscr{C}'$. (iii) We multiply the k-th row on the left by an operator in $\mathscr{L}_{1,0}^0$ and add it to the $(j+1)$-th row to eliminate terms containing D_1 in the $(j+1, k-1)$-th entry $(j+2 \leq k \leq m_\mu)$. As a result we have a matrix-valued operator $\widetilde{\mathscr{L}}^{\mu}(x, D)$ satisfying the following:

$$\widetilde{\mathscr{L}}^{\mu}(x,D) \equiv \big(\widetilde{\mathscr{L}}^{\mu}_{j,k}(x,D)\big) = \begin{pmatrix} D_n + \tilde{l}^{\mu}_1(x,D') & & & 0 \\ & \ddots & & \\ & & D_n + \tilde{l}^{\mu}_{m_\mu-1}(x,D') & \\ * & & & \tilde{l}^{\mu}(x,D) \end{pmatrix} + \widetilde{\mathscr{R}}^{\mu}(x,D) \tag{1.2}$$

in $V \times \mathscr{C}'$, $\tilde{l}^{\mu}_j(x,\xi') \in \mathscr{S}^{0,1}_{1,0}$, $\tilde{l}^{\mu,0}_j(x^0,\xi^{0\prime}) = 0$ and $\widetilde{\mathscr{L}}^{\mu}_{j+1,j}(x,\xi) \in \mathscr{S}^{1}_{1,0}$ $(1 \le j \le m_\mu - 1)$, $\widetilde{\mathscr{L}}^{\mu}_{j,k}(x,\xi) \equiv \widetilde{\mathscr{L}}^{\mu}_{j,k}(x,\xi') \in \mathscr{S}^{0,1}_{1,0}$ $(3 \le j \le m_\mu,\ 1 \le k \le j-2)$, $\tilde{l}^{\mu}(x,\xi) \in \mathscr{S}^{m_\mu,-m_\mu+1}_{1,0}$, $\widetilde{\mathscr{R}}^{\mu}_{j,k}(x,\xi) \in \mathscr{S}^{0,-\infty}_{1,0}$ if $k \le m_\mu - 1$, and $\widetilde{\mathscr{R}}^{\mu}_{j,m_\mu}(x,\xi) \in \mathscr{S}^{m_\mu,-\infty}_{1,0}$, where $\tilde{l}^{\mu,0}_j(x,\xi')$ denotes the principal symbol of $\tilde{l}^{\mu}_j(x,\xi')$ and $\widetilde{\mathscr{R}}^{\mu}(x,\xi) = \big(\widetilde{\mathscr{R}}^{\mu}_{j,k}(x,\xi)\big)$. Here $A(x,D) = B(x,D)$ in $V \times \mathscr{C}'$ means that $A(x,\xi) = B(x,\xi)$ for $(x,\xi') \in V \times \mathscr{C}'$. Next we multiply $\widetilde{\mathscr{L}}^{\mu}(x,D)$ on the right by $\mathrm{diag}\big(k^{\mu}_1(x,D'),\dots,k^{\mu}_{m_\mu-1}(x,D'),k^{\mu}(x,D')\big)$, where $k^{\mu}_j(x,D')$ is a parametrix of $D_n + \tilde{l}^{\mu}_j(x,D')$ in $V \times \mathscr{C}'$ $(1 \le j \le m_\mu - 1)$ and $k^{\mu}(x,D') = (-1)^{m_\mu-1}\big(D_n + \tilde{l}^{\mu}_1(x,D')\big)\cdots\big(D_n + \tilde{l}^{\mu}_{m_\mu-1}(x,D')\big)$. Finally we perform elementary transformations for the rows to annihilate the off-diagonal entries in $V \times \mathscr{C}'$, and we have a matrix-valued operator $\mathscr{L}^{\mu}(x,D) \in M_{m_\mu}\big(\mathscr{L}^{m_\mu}_{1,0}\big)$ satisfying the following:

$$\begin{aligned} &\mathscr{L}^{\mu}(x,\xi) = \mathrm{diag}(1,\dots,1,l^{\mu}(x,\xi)) + \mathscr{R}^{\mu}(x,\xi) \quad \text{in} \quad V \times \mathscr{C}', \\ &l^{\mu}(x,\xi) = (\xi_1 + \lambda_\mu \xi_n)^{m_\mu} + \sum_{k=1}^{m_\mu} l^{\mu}_k(x,\xi')(\xi_1 + \lambda_\mu \xi_n)^{m_\mu-k} \in \mathscr{S}^{m_\mu}_{1,0}, \\ &l^{\mu,0}(x^0,\xi_1,0,\dots,0,\xi_n) = (\xi_1 + \lambda_\mu \xi_n)^{m_\mu}, \end{aligned} \tag{1.3}$$

$\mathscr{R}^{\mu}_{j,k}(x,\xi) \in \mathscr{S}^{j-1,-\infty}_{1,0}$ if $k \le m_\mu - 1$, and $\mathscr{R}^{\mu}_{j,m_\mu}(x,\xi) \in \mathscr{S}^{m_\mu+j-1,-\infty}_{1,0}$, where $\mathscr{R}^{\mu}(x,\xi) = \big(\mathscr{R}^{\mu}_{j,k}(x,\xi)\big)$ and $l^{\mu,0}(x,\xi)$ denotes the principal symbol of $l^{\mu}(x,\xi)$ (see Lemma 2.4 below). We write $V = V(x^0,\xi^{0\prime})$ and $\mathscr{C}' = \mathscr{C}'(x^0,\xi^{0\prime})$ since V and $\mathscr{C}'$ depend on x^0 and $\xi^{0\prime}$. Note that $l^{\mu,0}(x,\xi) \equiv l^{\mu,0}(\xi)$ and $p(\xi) = l^{1,0}(\xi)\cdots l^{r,0}(\xi)$ in $V \times \mathscr{C}'$. We define

$$Q(x,\xi;x^0,\xi^{0\prime}) = l^1(x,\xi)\cdots l^r(x,\xi) - p(\xi)$$

for $(x^0,\xi^{0\prime}) \in \mathbb{R}^n \times S^{n-2}$ and $(x,\xi) \in V(x^0,\xi^{0\prime}) \times \mathbb{R} \times \mathscr{C}'(x^0,\xi^{0\prime})$, and impose the following condition (L) for sufficiency of C^∞ well-posedness:
(L) For each $(x^0,\xi^{0\prime}) \in \mathbb{R}^n \times S^{n-2}$ there is $C > 0$ such that

$$|Q(x,\xi;x^0,\xi^{0\prime})| \le C|p(\xi - i\vartheta)| \tag{1.4}$$

for $x \in V(x^0,\xi^{0\prime})$ with $x_1 \ge 0$ and $\xi \in \mathbb{R} \times \mathscr{C}'(x^0,\xi^{0\prime})$.

The condition (L) is equivalent to the following condition (L)$'$:
(L)$'$ For each $(x^0, \xi^{0\prime}) \in \mathbb{R}^n \times S^{n-2}$ and $x \in V(x^0, \xi^{0\prime})$ with $x_1 \geq 0$ there is $C > 0$ such that

$$|Q(x,\xi;x^0,\xi^{0\prime})| \leq C|p(\xi - i\vartheta)| \quad \text{for} \quad \xi \in \mathbb{R} \times \mathscr{C}'(x^0,\xi^{0\prime}).$$

Indeed, it is obvious that the condition (L) implies the condition (L)$'$. By Lemma 4.3 below the condition (L)$'$ implies the condition (L), with modifications of $V(x^0,\xi^{0\prime})$ and $\mathscr{C}'(x^0,\xi^{0\prime})$ if necessary. Note that (1.4) is always satisfied if $(dp)(\xi_1,\xi^{0\prime}) \neq 0$ for any $\xi_1 \in \mathbb{R}$. We say that the Cauchy problem (CP)$_t$ is C^∞ well-posed if the following two conditions are satisfied:
(E) For any $f \in C^\infty(\mathbb{R}^n;\mathbb{C}^m)$ with $\operatorname{supp} f \subset \{x \in \mathbb{R}^n; x_1 \geq t\}$ there is $u \in C^\infty(\mathbb{R}^n;\mathbb{C}^m)$ satisfying (CP)$_t$.
(U) If $s > t$, $u \in C^\infty(\mathbb{R}^n;\mathbb{C}^m)$, $\operatorname{supp} u \subset \{x \in \mathbb{R}^n; x_1 \geq t\}$ and $\operatorname{supp} L(x,D)u \subset \{x \in \mathbb{R}^n; x_1 \geq s\}$, then $\operatorname{supp} u \subset \{x \in \mathbb{R}^n; x_1 \geq s\}$.

Theorem 1.1. *Assume that the conditions* (H) *and* (R) *are satisfied.*
(i) *We assume that the condition* (L) *is satisfied. Let $t > 0$, and let $f \in \mathscr{D}'$ satisfy $\operatorname{supp} f \subset \{x \in \mathbb{R}^n; x_1 \geq t\}$. Then the Cauchy problem* (CP)$_t$ *has a solution $u \in \mathscr{D}'$. If $x^0 \in \mathbb{R}^n$, $u \in \mathscr{D}'$ satisfies* (CP)$_t$ *and $f = 0$ near $\{x^0\} - \Gamma(p,\vartheta)^*$, then $x^0 \notin \operatorname{supp} u$, where $\Gamma(p,\vartheta)$ denotes the connected component of the set $\{\xi \in \mathbb{R}^n; p(\xi) \neq 0\}$ which contains ϑ, and $\Gamma^* = \left\{x \in \mathbb{R}^n; x \cdot \xi \ (\equiv \sum_{j=1}^n x_j\xi_j) \geq 0 \text{ for any } \xi \in \Gamma\right\}$. Moreover, if $u \in \mathscr{D}'$ satisfies* (CP)$_t$ *and $f \in C^\infty(\mathbb{R}^n;\mathbb{C}^m)$, then $u \in C^\infty(\mathbb{R}^n;\mathbb{C}^m)$, i.e., the Cauchy problem* (CP)$_t$ *is C^∞ well-posed.*
(ii) *If the Cauchy problem* (CP)$_t$ *is C^∞ well-posed for any $t > 0$, then the condition* (L) *is satisfied.*

REMARK. (i) It is possible to reduce the operator $L(x,D)$ to the operator of the form (2.6) below in different ways from the proof of Lemma 2.4. Then Theorem 1.1 is still valid.
(ii) Under the conditions (H) and (R) the condition (L) is a necessary and sufficient condition for (CP)$_t$ to be C^∞ well-posed for any $t > 0$. Therefore, the condition (L) does not depend on the reduction procedure.
(iii) The Cauchy problem for a single higher order operator $P(x,D)$ with constant coefficient hyperbolic principal part is C^∞ well-posed (in $\mathbb{R}^n$) if and only if $P(x,\xi)$ is hyperbolic with respect to ϑ for each $x \in \mathbb{R}^n$ (see [5] and [26]), which is equivalent to the condition that for every $x \in \mathbb{R}^n$ there is $C > 0$ satisfying $|P(x,\xi)|/|p(\xi - i\vartheta)| \leq C$ for $\xi \in \mathbb{R}^n$, where $p(\xi)$ is the principal part of $P(x,\xi)$ (see Svensson [23]).

The remainder of this paper is organized as follows. In §2 we shall reduce $L(x,D)$ to an operator $\operatorname{diag}(1,\ldots,1,l^1(x,D),\ldots,l^r(x,D))$ in a microlocal sense, by multiplying $L(x,D)$ on the both sides by invertible matrix-valued operators in $M_m\left(\mathscr{L}_{1,0}^{m-1}\right)$. In the proof of the assertion (ii) of Theorem 1.1 we shall use the Tarski-Seidenberg theorem. So we have to prove that the Tarski-Seidenberg theorem can be applicable to the $l^\mu(x,\xi)$. In §3 we shall prove the assertion (i) of Theorem 1.1, applying the results in [14]. In doing so, some simple modifications are necessary. The assertion (ii) of Theorem 1.1 will be proved in §4, applying the arguments in Ivrii-Petkov [10] and [26]. We shall modify the arguments in [9] to remove the assumption on the finite propagation property (see, also, [16]). Some examples and remarks will be given in §5.

2. Reduction to a simple form

We begin with several definitions.

DEFINITION 2.1. (i) The subset $\mathscr{A}$ of $\mathbb{R}^N$ is said to be a semi-algebraic set if $\mathscr{A}$ is a finite union of finite intersections of sets defined by a real polynomial equation or inequality.

(ii) Let V be an open subset of $\mathbb{R}^n$ and $\mathscr{C}'$ be an open conic semi-algebraic subset of $\mathbb{R}^{n-1}\setminus\{0\}$, and let $f(x,\xi)$ be a symbol in $C^\infty(V\times\mathbb{R}\times(\mathscr{C}'\cup(-\mathscr{C}')))$ such that $f(x,\lambda\xi)=\lambda^l f(x,\xi)$ for $(x,\xi)\in V\times\mathbb{R}\times(\mathscr{C}'\cup(-\mathscr{C}'))$ and $\lambda\in\mathbb{R}\setminus\{0\}$, where $l\in\mathbb{Z}$ and $-\mathscr{C}'=\{-\xi'\in\mathbb{R}^{n-1};\xi'\in\mathscr{C}'\}$. We say that $f(x,\xi)$ is an *SA*-symbol in $V\times\mathscr{C}'$ if there are $N\in\mathbb{N}$, a semi-algebraic set $\mathscr{A}$ in $\mathbb{R}^{N+n-1}$ and a polynomial $F(x,\xi,\lambda)$ of (ξ,λ) $(\equiv(\xi,\lambda_1,\dots,\lambda_N))$ and a polynomial $G(x,\xi',\lambda)$ of (ξ',λ), whose coefficients belong to $C^\infty(V)$, satisfying the following:

(1) The projection $\pi\equiv\pi_{N,n-1}\colon\mathscr{A}\ni(\xi',\lambda)\mapsto\xi'\in\mathbb{R}^{n-1}$ is injective, $\pi(\mathscr{A})=\mathscr{C}'\cup(-\mathscr{C}')$, and the $\Lambda_j(\xi')$ are real analytic and homogeneous in $\mathscr{C}'\cup(-\mathscr{C}')$, where $\pi^{-1}(\xi')=(\xi',\Lambda_1(\xi'),\dots,\Lambda_N(\xi'))$.

(2) $F(x,\xi_1,\pi^{-1}(\xi'))$ and $G(x,\pi^{-1}(\xi'))$ are homogeneous in $\xi\in\mathbb{R}\times(\mathscr{C}'\cup(-\mathscr{C}'))$ and $\xi'\in\mathscr{C}'\cup(-\mathscr{C}')$, respectively.

(3) $G(x,\pi^{-1}(\xi'))\neq 0$ and $f(x,\xi)=G(x,\pi^{-1}(\xi'))^{-1}F(x,\xi_1,\pi^{-1}(\xi'))$ for $x\in V$ and $\xi\in\mathbb{R}\times(\mathscr{C}'\cup(-\mathscr{C}'))$.

(iii) Let V be an open subset of $\mathbb{R}^n$ and $\mathscr{C}'$ be an open conic semi-algebraic subset of $\mathbb{R}^{n-1}\setminus\{0\}$, and let $f(x,\xi)$ be a symbol in $\mathscr{S}^{k,l}_{1,0}$ which has an asymptotic expansion of the form $f(x,\xi)\sim\sum_{j=0}^\infty f^j(x,\xi)$, where $k\in\mathbb{Z}_+$, $l\in\mathbb{Z}$ and $f^j(x,\xi)$ is positively homogeneous of degree $k+l-j$ $(j\in\mathbb{Z}_+)$. We say that $f(x,\xi)$ is an S-symbol in $V\times\mathscr{C}'$ if for each $\alpha,\beta\in(\mathbb{Z}_+)^n$ and $j\in\mathbb{Z}_+$ $f^{j(\alpha)}_{(\beta)}(x,\xi)$ is an *SA*-symbol in $V\times\mathscr{C}'$.

Let $x^0\in\mathbb{R}^n$ and $\xi^{0\prime}=(\xi^0_2,\dots,\xi^0_n)\in S^{n-2}$. By a linear coordinate transformation of x' we may assume that $\xi^{0\prime}=(0,\dots,0,1)\in\mathbb{R}^{n-1}$. Write

$$\begin{aligned}&p(\xi^0+\lambda\vartheta)\ (\equiv p(\lambda,\xi^{0\prime}))\\&=\prod_{j=1}^r(\lambda+\lambda_j)^{m_j},\quad \lambda_1<\lambda_2<\cdots<\lambda_r\end{aligned}$$

as in §1, and

$$p(\lambda,\xi')=\prod_{j=1}^{m}(\lambda+\tau_j(\xi')),\quad \tau_1(\xi)\le\tau_2(\xi')\le\cdots\le\tau_m(\xi').$$

We choose $\delta_j>0$ $(1\le j\le r)$ and an open neighborhood U' of $\xi^{0\prime}$ so that $\{\lambda\in\mathbb{C};\ |\lambda+\lambda_j|\le\delta_j\}$ $(1\le j\le r)$ are mutually disjoint and $\tau_k(\xi')\in\{\lambda\in\mathbb{R};\ |\lambda+\lambda_j|\le\delta_j/2\}$ for $1\le j\le r$, $m_1+\cdots+m_{j-1}+1\le k\le m_1+\cdots+m_j$ and $\xi'\in U'$. Put

$$P_j(x,\xi')=(2\pi i)^{-1}\oint_{|\lambda+\lambda_j|=\delta_j}L_1(x,\lambda,\xi')^{-1}\,d\lambda$$

for $(x,\xi')\in\mathbb{R}^n\times U'$ and $1\le j\le r$. Then we have rank $P_j(\xi')=m_j$ and we can choose an open neighborhood V of x^0 and m_j column vectors $P_j^1(x,\xi'),\ldots,P_j^{m_j}(x,\xi')$ of $P_j(x,\xi')$ so that $P_j^k(x,\xi')$ $(1\le k\le m_j)$ are linearly independent for each $(x,\xi')\in V\times U'$, modifying U' if necessary. We put

$$S(x,\xi')=\left(P_1^1(x,\xi'),\ldots,P_1^{m_1}(x,\xi'),P_2^1(x,\xi'),\ldots,P_r^{m_r}(x,\xi')\right).$$

Then $S(x,\xi')$ can be defined for $(x,\xi')\in\mathbb{R}^n\times(\mathscr{C}'\cup(-\mathscr{C}'))$ by homogeneity, i.e., $S(x,\xi')=S(x,\pm\xi'/|\xi'|)$ if $(x,\xi')\in\mathbb{R}^n\times(\pm\mathscr{C}')$, where $\mathscr{C}'=\{\lambda\xi'\in\mathbb{R}^{n-1};\xi'\in U',\ |\xi'|=1$ and $\lambda>0\}$. We may assume that $\mathscr{C}'$ is semi-algebraic. Since

$$R_{j,k,l}(\xi')\equiv(2\pi i)^{-1}\oint_{|\lambda+\lambda_j|=\delta_j}\frac{\lambda^k}{p(\lambda,\xi')^l}\,d\lambda$$

is a rational function of fundamental symmetric functions of $\{\tau_{m_1+\cdots+m_{\mu-1}+\nu}(\xi')\}_{\nu=1,\ldots,m_\mu}$ $(1\le\mu\le r)$ for $1\le j\le r$, $k\in\mathbb{Z}_+$ and $l\in\mathbb{N}$, the entries of $S^{(\alpha)}_{(\beta)}(x,\xi')$ are SA-symbols in $V\times\mathscr{C}'$ for $\alpha,\beta\in(\mathbb{Z}_+)^n$. Indeed, for $\xi'\in\mathscr{C}'\cup(-\mathscr{C}')$ the $R_{j,k,l}(\xi')$ are analytic and expressed as rational functions of $(\zeta_1,\ldots,\zeta_m)$, where $(\zeta_1,\ldots,\zeta_m)$ is defined by the following system of real polynomial equations and inequalities for $\xi'\in\mathscr{C}'\cup(-\mathscr{C}')$:

$$\tau_1\le\tau_2\le\cdots\le\tau_m,\quad p(\xi)=\prod_{j=1}^{m}(\xi_1+\tau_j),$$

$$\prod_{j=1}^{m}(\xi_1+\tau_{m_1+\cdots+m_{\mu-1}+j})=\xi_1^{m_\mu}+\sum_{j=1}^{m_\mu}\zeta_{m_1+\cdots+m_{\mu-1}+j}\xi_1^{m_\mu-j}\quad(1\le\mu\le r).$$

Therefore, we have the following

Lemma 2.2. *There are an open neighborhood V of x^0 in $\mathbb{R}^n$, an open conic semi-algebaraic subset $\mathscr{C}'$ of $\mathbb{R}^{n-1}\setminus\{0\}$, $S(x,\xi')\in C^\infty(\mathbb{R}^n\times(\mathscr{C}'\cup(-\mathscr{C}'));M_m(\mathbb{C}))$ and*

$A^j(x,\xi') \in C^\infty(V \times \mathscr{C}'; M_{m_j}(\mathbb{C}))$ $(1 \le j \le r)$ *such that* $S(x,\xi')$ *and* $A^j(x,\xi')$ *are homogeneous of degree* 0 *and* 1, *respectively,* $\det S(x,\xi') \neq 0$ *for* $(x,\xi') \in V \times \mathscr{C}'$, *the entries of* $S(x,\xi')$ *and the* $A^j(x,\xi')$ *are SA-symbols in* $V \times \mathscr{C}'$, $(A^j(x,\xi^{0\prime}) - \lambda_j I_{m_j})^{m_j} = 0$ *and*

$$S(x,\xi')^{-1}A(x,\xi')S(x,\xi') = \operatorname{diag}(A^1(x,\xi'), \dots, A^r(x,\xi'))$$

for $(x,\xi') \in V \times (\mathscr{C}' \cup (-\mathscr{C}'))$.

Modifying V and $\mathscr{C}'$, if necessary, we can construct $S_\pm(x,\xi') \in M_m\big(\mathscr{S}^0_{1,0}\big)$ so that $S_+(x,\xi') = S(x,\xi')$ for $(x,\xi') \in V \times (\mathscr{C}' \cup (-\mathscr{C}'))$ with $|\xi'| \ge 1$ and $S_-(x,D') \times S_+(x,D') \equiv I \bmod \mathscr{L}^{0,-\infty}_{1,0}$ in $V \times (\mathscr{C}' \cup (-\mathscr{C}'))$. It is easy to see that the entries of $S_\pm(x,\xi')$ are S-symbols in $V \times \mathscr{C}'$. We can assume without loss of generality that $A_2(x), \dots,$ $A_n(x), L_0(x) \in \mathscr{B}(\mathbb{R}^n; M_m(\mathbb{C}))$. Although the following lemma is well-known, for completeness we give the proof (see, e.g., [18]).

Lemma 2.3. *There are symbols* $Q(x,\xi') \in M_m\big(\mathscr{S}^{0,-1}_{1,0}\big)$ *and* $\widetilde{A}^\mu(x,\xi') \in M_{m_\mu}\big(\mathscr{S}^{0,1}_{1,0}\big)$ *and* $C^\mu(x,\xi') \in M_{m_\mu}\big(\mathscr{S}^0_{1,0}\big)$ $(1 \le \mu \le r)$ *such that the entries of* $Q(x,\xi')$, *the* $\widetilde{A}^\mu(x,\xi')$ *and the* $C^\mu(x,\xi')$ *are S-symbols in* $V \times \mathscr{C}'$ *and*

(2.1)
$$\widetilde{A}^\mu(x,\xi') = A^\mu(x,\xi') \quad \textit{for} \quad (x,\xi') \in V \times \mathscr{C}' \quad \textit{with} \quad |\xi'| \ge 1 \quad \textit{and} \quad 1 \le \mu \le r,$$
$$S_-(x,D')L(x,D)S_+(x,D')(I + Q(x,D'))$$
$$\equiv (I + Q(x,D')) \times \big\{D_1 I + \operatorname{diag}\big(\widetilde{A}^1(x,D') + C^1(x,D'), \dots, \widetilde{A}^r(x,D') + C^r(x,D')\big)\big\}$$
$$\bmod \mathscr{L}^{1,-\infty}_{1,0} \qquad \textit{in} \quad V \times (\mathscr{C}' \cup (-\mathscr{C}')).$$

Proof. Write

$$Q(x,\xi') \sim \sum_{j=1}^\infty Q^j(x,\xi'), \quad C^\mu(x,\xi') \sim \sum_{j=0}^\infty C^{\mu,j}(x,\xi') \tag{2.2}$$

(asymptotic expansions), where $Q^j(x,\xi')$ $(\in C^\infty(\mathbb{R} \times (T^*\mathbb{R}^{n-1} \setminus 0); M_m(\mathbb{C})))$ and $C^{\mu,j}(x,\xi')$ $(\in C^\infty(\mathbb{R} \times (T^*\mathbb{R}^{n-1} \setminus 0); M_{m_\mu}(\mathbb{C})))$ are positively homogeneous of degree $-j$ and $T^*\mathbb{R}^{n-1} \setminus 0 \simeq \mathbb{R}^{n-1} \times (\mathbb{R}^{n-1} \setminus \{0\})$. We also write

$$\widetilde{A}(x,D') = S_-(x,D')A(x,D')S_+(x,D') \in \mathscr{L}^{0,1}_{1,0},$$
$$\widetilde{A}(x,\xi') \sim \sum_{j=-1}^\infty \widetilde{A}_j(x,\xi') \quad \text{(asymptotic expansion)},$$

where $\widetilde{A}_j(x,\xi')$ $(\in C^\infty(\mathbb{R} \times (T^*\mathbb{R}^{n-1} \setminus 0); M_m(\mathbb{C})))$ is positively homogeneous of degree $-j$. By Lemma 2.2 we have

$$\widetilde{A}_{-1}(x,\xi') = \operatorname{diag}(A^1(x,\xi'), \dots, A^r(x,\xi'))$$

for $(x,\xi')\in V\times(\mathscr{C}'\cup(-\mathscr{C}'))$. Choose $\widetilde{A}^\mu(x,\xi')\in M_{m_\mu}\big(\mathscr{S}^{0,1}_{1,0}\big)$ $(1\le\mu\le r)$ so that the $\widetilde{A}^\mu(x,\xi')$ are positively homogeneous for $|\xi'|\ge 1$ and

$$\operatorname{diag}\big(\widetilde{A}^1(x,\xi'),\dots,\widetilde{A}^r(x,\xi')\big)=\widetilde{A}_{-1}(x,\xi')$$

for $(x,\xi')\in V\times(\mathscr{C}'\cup(-\mathscr{C}'))$ with $|\xi'|\ge 1$. We put

$$\begin{aligned}B(x,D')&=S_-(x,D')S_{+(e_1)}(x,D')+\widetilde{A}(x,D')\\&\quad-\operatorname{diag}\big(\widetilde{A}^1(x,D'),\dots,\widetilde{A}^r(x,D')\big)+S_-(x,D')L_0(x)S_+(x,D').\end{aligned}$$

and write

$$B(x,\xi')\sim\sum_{j=-1}^{\infty}B^j(x,\xi'),$$

where $e_1=(1,0,\dots,0)\in(\mathbb{Z}_+)^n$ and $B^j(x,\xi')$ is positively homogeneous of degree $-j$ $(j\in\mathbb{Z}_+)$. It is easy to see that $B^{-1}(x,\xi')=0$ in $V\times(\mathscr{C}'\cup(-\mathscr{C}'))$ and that the entries of $B(x,\xi')$ are S-symbols in $V\times\mathscr{C}'$. If (2.1) holds, then $\{Q^j(x,\xi')\}$ and $\{C^{\mu,j}(x,\xi')\}$ satisfy

$$\begin{aligned}&B^l(x,\xi')+Q^l_{(e_1)}(x,\xi')\\&+\sum_{\substack{j+|\alpha'|=l+1\\ j\ge 1}}\frac{1}{\alpha'!}\operatorname{diag}(A^{1(\alpha')}(x,\xi'),\dots,A^{r(\alpha')}(x,\xi'))Q^j_{(\alpha')}(x,\xi')\\&+\sum_{\substack{j+k+|\alpha'|=l\\ j\ge 1,\,k\ge 0}}\frac{1}{\alpha'!}B^{k(\alpha')}(x,\xi')Q^j_{(\alpha')}(x,\xi')\\&=\operatorname{diag}(C^{1,l}(x,\xi'),\dots,C^{r,l}(x,\xi'))\\&\quad+\sum_{\substack{j+|\alpha'|=l+1\\ j\ge 1}}\frac{1}{\alpha'!}Q^{j(\alpha')}(x,\xi')\operatorname{diag}\big(A^1_{(\alpha')}(x,\xi'),\dots,A^r_{(\alpha')}(x,\xi')\big)\\&\quad+\sum_{\substack{j+k+|\alpha'|=l\\ j\ge 1,\,k\ge 0}}\frac{1}{\alpha'!}Q^{j(\alpha')}(x,\xi')\operatorname{diag}\big(C^{1,k}_{(\alpha')}(x,\xi'),\dots,C^{r,k}_{(\alpha')}(x,\xi')\big)\end{aligned}\tag{2.3}$$

in $V\times(\mathscr{C}'\cup(-\mathscr{C}'))$ $(l=0,1,2,\dots)$, where $Q^0(x,\xi')\equiv 0$. If $A_{\mu,\nu}$ is an $m_\mu\times m_\nu$ matrix $(1\le\mu,\nu\le r)$ and $A=(A_{\mu,\nu})_{\substack{\mu\downarrow 1,\dots,r\\ \nu\to 1,\dots,r}}$, we define $\mathbb{B}_{\mu,\nu}(A)=A_{\mu,\nu}$. We choose $Q(x,\xi')$

so that $\mathbb{B}_{\mu,\mu}(Q(x,\xi'))=0$ for $1\le\mu\le r$. Then it follows from (2.3) that

$$C^{\mu,l}(x,\xi')=\mathbb{B}_{\mu,\mu}(B^l(x,\xi')) + \sum_{\substack{j+k+|\alpha'|=l\\ j\ge1,\,k\ge0}}\sum_{\gamma=1}^{r}\frac{1}{\alpha'!}\mathbb{B}_{\mu,\gamma}(B^{k(\alpha')}(x,\xi'))\mathbb{B}_{\gamma,\mu}\big(Q^j_{(\alpha')}(x,\xi')\big), \tag{2.4}$$

$$\begin{aligned}&\mathbb{B}_{\mu,\nu}(Q^{l+1}(x,\xi'))A^\nu(x,\xi')-A^\mu(x,\xi')\mathbb{B}_{\mu,\nu}(Q^{l+1}(x,\xi'))\\ &=\mathbb{B}_{\mu,\nu}(B^l(x,\xi'))+\mathbb{B}_{\mu,\nu}\big(Q^l_{(e_1)}(x,\xi')\big)\\ &\quad+\sum_{\substack{j+|\alpha'|=l+1\\1\le j\le l}}\frac{1}{\alpha'!}A^{\mu(\alpha')}(x,\xi')\mathbb{B}_{\mu,\nu}\big(Q^j_{(\alpha')}(x,\xi')\big)\\ &\quad+\sum_{\substack{j+k+|\alpha'|=l\\ j\ge1,\,k\ge0}}\sum_{\gamma=1}^{r}\frac{1}{\alpha'!}\mathbb{B}_{\mu,\gamma}(B^{k(\alpha')}(x,\xi'))\mathbb{B}_{\gamma,\nu}\big(Q^j_{(\alpha')}(x,\xi')\big)\\ &\quad-\sum_{\substack{j+|\alpha'|=l+1\\1\le j\le l}}\frac{1}{\alpha'!}\mathbb{B}_{\mu,\nu}(Q^{j(\alpha')}(x,\xi'))A^\nu_{(\alpha')}(x,\xi')\\ &\quad-\sum_{\substack{j+k+|\alpha'|=l\\ j\ge1,\,k\ge0}}\frac{1}{\alpha'!}\mathbb{B}_{\mu,\nu}(Q^{j(\alpha')}(x,\xi'))C^{\nu,k}_{(\alpha')}(x,\xi')\end{aligned} \tag{2.5}$$

in $V\times(\mathscr{C}'\cup(-\mathscr{C}'))$ for $l=0,1,2,\dots$ and $1\le\mu,\nu\le r$ with $\nu\ne\mu$. Let $A\equiv(a_{i,k})\in M_{n_1,n_2}(\mathbb{C})$ and $B\equiv(b_{j,l})\in M_{l_1,l_2}(\mathbb{C})$. Then the Kronecker product $A\otimes B\equiv(c_{\lambda,\mu})\in M_{n_1l_1,n_2l_2}(\mathbb{C})$ is defined by $c_{\lambda,\mu}=a_{i,k}b_{j,l}$ for $\lambda=(i,j)$ and $\mu=(k,l)$, where $\lambda\in\{(i,j)\in\mathbb{N}^2;\,1\le i\le n_1$ and $1\le j\le l_1\}$ and $\mu\in\{(k,l)\in\mathbb{N}^2;\,1\le k\le n_2$ and $1\le l\le l_2\}$ are properly arranged. Assume that $n_1=n_2$, $l_1=l_2$ and the repeated eigenvalues of A and B are $\{\lambda_j\}_{1\le j\le n_1}$ and $\{\tau_k\}_{1\le k\le l_1}$, respectively. Then the eigenvalues of $I_{l_1}\otimes {}^tA-B\otimes I_{n_1}\equiv(h_{\lambda,\mu})$ are $\{\lambda_j-\tau_k\}_{1\le j\le n_1,\,1\le k\le l_1}$. For $X=(x_{i,j})$, $F=(f_{i,j})\in M_{l_1,n_1}(\mathbb{C})$ the equation $XA-BX=F$ is equivalent to $\sum_\mu h_{\lambda,\mu}x_\mu=f_\lambda$ for $\lambda=(i,j)$ with $1\le i\le l_1$ and $1\le j\le n_1$. In particular, the equation is uniquely solvable in X if $\lambda_j\ne\tau_k$ for $1\le j\le n_1$ and $1\le k\le l_1$. The eigenvalues of $A^\nu(x,\xi')$ are different from those of $A^\mu(x,\xi')$ for $\mu\ne\nu$ and $(x,\xi')\in V\times\mathscr{C}'$. Therefore, by (2.4) and (2.5) we can determine $\{Q^j(x,\xi')\}$ and $\{C^{\mu,j}(x,\xi')\}$ in $V\times\mathscr{C}'$, inductively. Modifying the symbols outside $\{(x,\xi')\in V\times(\mathscr{C}'\cup(-\mathscr{C}'));\,|\xi'|\ge1\}$, we can choose $Q(x,\xi')$ and $C^\mu(x,\xi')$ $(1\le\mu\le r)$ so that (2.1) and (2.2) are valid. It is easy to see that the entries of $Q(x,\xi')$ and the $C^\mu(x,xi')$ are S-symbols in $V\times\mathscr{C}'$. □

As stated in §1, using elementary transformations we have the following

Lemma 2.4. *With modifications of V and $\mathscr{C}'$ if necessary, there are symbols $N_1(x,\xi)\in M_m\big(\mathscr{S}^{m-1,-m+1}_{1,0}\big)$, $N_2(x,\xi)\equiv(N_{2,j,k}(x,\xi))\in M_m\big(\mathscr{S}^{m-1}_{1,0}\big)$ and $l^j(x,\xi)\in\mathscr{S}^{m_j}_{1,0}$*

$(1 \le j \le r)$ *satisfying the following*:
(i) $N_{2,j,k}(x,\xi) \equiv N_{2,j,k}(x,\xi') \in \mathscr{S}^{0,-1}_{1,0}$ *for* $1 \le k \le m-r$, $N_{2,j,j}(x,\xi) \equiv N_{2,j,j}(x,\xi')$ *for* $m-r < j \le m$, $N_{2,j,m-r+k}(x,\xi) \in \mathscr{S}^{m_k-1}_{1,0}$ *for* $1 \le k \le r$, *and* $N_1(x,D)$ *and* $N_2(x,D)$ *have parametrices in* $M_m\big(\mathscr{L}^{m-1}_{1,0}\big)$.
(ii) $l^j(x,\xi)$ *can be written in the form*

$$l^j(x,\xi) = (\xi_1 + \lambda_j \xi_n)^{m_j} + \sum_{k=1}^{m_j} l^j_k(x,\xi')(\xi_1 + \lambda_j\xi_n)^{m_j-k},$$

$l^j_k(x,\xi') \in \mathscr{S}^{0,k}_{1,0}$ $(1 \le j \le r,\ 1 \le k \le m_j)$, *The* $l^j_k(x,\xi')$ *are* S*-symbols in* $V \times \mathscr{C}'$, $l^{j,0}_k(x,\xi') \equiv l^{j,0}_k(\xi')$ *and* $l^{j,0}_k(\xi^{0\prime}) = 0$, *where* $l^{j,0}_k(x,\xi')$ *denotes the principal symbol of* $l^j_k(x,\xi')$.
(iii)

$$N_1(x,D)L(x,D)N_2(x,D) = \operatorname{diag}(1,\dots,1,l^1(x,D),\dots,l^r(x,D)) + R(x,D) \tag{2.6}$$

in $V \times (\mathscr{C}' \cup (-\mathscr{C}'))$, *where* $R(x,\xi) \in M_m\big(\mathscr{S}^{2m-1,-\infty}_{1,0}\big)$.

Proof. We fix $\mu \in \{1,2,\dots,r\}$. As stated in §1, there are symbols $\widetilde{N}_\nu(x,\xi) \equiv \big(\widetilde{N}_{\nu,j,k}(x,\xi)\big) \in M_{m_\mu}\big(\mathscr{S}^{m_\mu-1}_{1,0}\big)$ $(\nu = 1,2)$ satisfying the following: (i) $\widetilde{N}_{\nu,j,j}(x,\xi) = 1$, $\widetilde{N}_{\nu,j,k}(x,\xi) = 0$ $(j > k)$, $\widetilde{N}_{1,j,k}(x,\xi) \equiv \widetilde{N}_{1,j,k}(x,\xi') \in \mathscr{S}^0_{1,0}$ $(j < k \le m_\mu)$, $\widetilde{N}_{2,j,k}(x,\xi) \equiv \widetilde{N}_{2,j,k}(x,\xi') \in \mathscr{S}^0_{1,0}$ for $j < k < m_\mu$ and $\widetilde{N}_{2,j,m_\mu}(x,\xi) \in \mathscr{S}^{m_\mu-1,-m_\mu+1}_{1,0}$. (ii) $\widetilde{N}_1(x,D)$ $(= \widetilde{N}_1(x,D'))$ and $\widetilde{N}_2(x,D)$ have the inverses in $M_{m_\mu}\big(\mathscr{L}^0_{1,0}\big)$ and $M_{m_\mu}\big(\mathscr{L}^{m_\mu-1}_{1,0}\big)$, respectively. (iii)

$$\widetilde{N}_1(x,D)L^\mu(x,D)\widetilde{N}_2(x,D) = \widetilde{\mathscr{L}}^\mu(x,D) \quad \text{in} \quad V \times (\mathscr{C}' \cup (-\mathscr{C}')),$$

where $L^\mu(x,\xi)$ is as in (1.1) and $\widetilde{\mathscr{L}}^\mu(x,\xi) \equiv \big(\widetilde{\mathscr{L}}^\mu_{j,k}(x,\xi)\big)$ has the same form as in (1.2) which is valid in $V \times (\mathscr{C}' \cup (-\mathscr{C}'))$. We note that $f(x,\xi) + g(x,\xi)$ and $f(x,\xi)g(x,\xi)$ are S-symbols in $V \times \mathscr{C}'$ if $f(x,\xi)$ and $g(x,\xi)$ are S-symbols in $V \times \mathscr{C}'$. Let $h(x,\xi') \in \mathscr{S}^{0,1}_{1,0}$ be an S-symbol in $V \times \mathscr{C}'$ such that $h^0(x^0,\xi^{0\prime}) = 0$, where $h^0(x,\xi')$ denotes the principal symbol of $h(x,\xi')$. Then there is a symbol $k(x,\xi') \in \mathscr{S}^{0,-1}_{1,0}$ such that $k(x,\xi')$ is an S-symbol in $V \times \mathscr{C}'$ and $k(x,D')$ is a parametrix of $(D_n + h(x,D'))$ in $V \times \mathscr{C}'$, with modifications of V and $\mathscr{C}'$ if necessary. Therefore, $\widetilde{N}_{\nu,j,k}(x,\xi)$ and $\widetilde{\mathscr{L}}^\mu_{j,k}(x,\xi)$ $(\nu = 1,2,$ $1 \le j,k \le m_\mu)$ are S-symbols in $V \times \mathscr{C}'$. It follows from the reduction procedure that

$$\tilde{l}^\mu(x,D) - (-1)^{m_\mu-1}k^\mu_{m_\mu-1}(x,D')\cdots k^\mu_2(x,D')k^\mu_1(x,D')D_1^{m_\mu} \in \mathscr{L}^{m_\mu-1,1}_{1,0}$$

in $V \times (\mathscr{C}' \cup (-\mathscr{C}'))$, i.e., $\big(\tilde{l}^\mu(x,D) - (-1)^{m_\mu-1}k^\mu_{m_\mu-1}(x,D')\cdots k^\mu_1(x,D')D_1^{m_\mu}\big) \times \psi(x,D') \in \mathscr{L}^{m_\mu-1,1}_{1,0}$ for any $\psi(x,\xi') \in \mathscr{S}^0_{1,0}$ with supp $\psi \subset V \times (\mathscr{C}' \cup (-\mathscr{C}'))$, where $k^\mu_j(x,D')$ is a parametrix of $D_n + \tilde{l}^\mu_j(x, D')$ in $V \times (\mathscr{C}' \cup (-\mathscr{C}'))$ $(1 \le j \le m_\mu - 1)$. Now we multiply $\widetilde{\mathscr{L}}^\mu(x, D)$ on the right by $\operatorname{diag}\bigl(k^\mu_1(x, D'), \dots, k^\mu_{m_\mu-1}(x, D'), k^\mu(x, D')\bigr)$ and, then, on the left by operators corresponding to elementary transformations for the rows to annihilate the off-diagonal entries in $V \times (\mathscr{C}' \cup (-\mathscr{C}'))$, where $k^\mu(x, D') = (-1)^{m_\mu-1}\bigl(D_n + \tilde{l}^\mu_1(x, D')\bigr) \cdots \bigl(D_n + \tilde{l}^\mu_{m_\mu-1}(x, D')\bigr)$. Finally we obtain $\mathscr{L}^\mu(x, D)$ of the form (1.3) in $V \times (\mathscr{C}' \cup (-\mathscr{C}'))$, i.e., there are $N^\mu_\nu(x,\xi) \equiv \bigl(N^\mu_{\nu,j,k}(x,\xi)\bigr) \in M_m\bigl(\mathscr{S}^{m_\mu-1}_{1,0}\bigr)$ $(\nu = 1, 2)$ satisfying the following: (i) $N^\mu_{2,j,j}(x, \xi) = k^\mu_j(x, \xi')$ $(1 \le j \le m_\mu - 1)$, $N^\mu_{2,m_\mu,m_\mu}(x, \xi) = k^\mu(x, \xi')$, $N^\mu_{2,j,k}(x, \xi) = 0$ $(j > k)$, $N^\mu_{1,j,k}(x, \xi) \in \mathscr{S}^{j-1,-j+1}_{1,0}$, $N^\mu_{2,j,k}(x, \xi) \equiv N^\mu_{2,j,k}(x, \xi') \in \mathscr{S}^{0,-1}_{1,0}$ $(k < m_\mu)$, $N^\mu_{2,j,m_\mu}(x, \xi) \in \mathscr{S}^{m_\mu-1}_{1,0}$, and the $N^\mu_{\nu,j,k}(x, \xi)$ are S-symbols in $V \times \mathscr{C}'$. (ii) $N^\mu_1(x, D)$ has the inverse in $M_{m_\mu}\bigl(\mathscr{L}^{m_\mu-1}_{1,0}\bigr)$ and $N^\mu_2(x, D)$ has a parametrix in $M_{m_\mu}\bigl(\mathscr{L}^{m_\mu-1}_{1,0}\bigr)$. (iii)

$$N^\mu_1(x, D)L^\mu(x, D)N^\mu_2(x, D) = \mathscr{L}^\mu(x, D) \quad \text{in} \quad V \times (\mathscr{C}' \cup (-\mathscr{C}')).$$

This proves the lemma. □

Let $q(x, \xi)$ be an SA-symbol in $V \times \mathscr{C}'$, and assume that

$$\sup_{\xi \in \mathbb{R} \times \mathscr{C}',\, |\xi'| \ge 1} \left| \frac{q(x, \xi)}{p(\xi - i\vartheta)} \right| < \infty \quad \text{for any} \quad x \in V. \tag{2.7}$$

By definition there are $N \in \mathbb{N}$, a semi-algebraic subset $\mathscr{A}$ of $\mathbb{R}^{N+n-1}$ and a polynomial $F(x, \xi, \lambda)$ of (ξ, λ) $(\equiv (\xi, \lambda_1, \dots, \lambda_N))$ and a polynomial $G(x, \xi', \lambda)$ of (ξ', λ), whose coefficients belong to $C^\infty(V)$, satisfying the conditions (1)–(3) of Definition 2.1 (ii) with $f(x, \xi)$ replaced by $q(x, \xi)$. Write

$$F(x, \xi, \lambda) = \sum_{|\alpha|+|\hat\beta| \le M} a_{\alpha,\hat\beta}(x)\xi^\alpha \lambda^{\hat\beta}, \quad a_{\alpha,\hat\beta}(x) \in C^\infty(V),$$

where $M \in \mathbb{Z}_+$, $\hat\beta = (\beta_1, \dots, \beta_N) \in (\mathbb{Z}_+)^N$ and $\lambda^{\hat\beta} = \lambda_1^{\beta_1} \cdots \lambda_N^{\beta_N}$. We put

$$\mathscr{B} = \Bigl\{ \boldsymbol{b} = (b_{\alpha,\hat\beta};\, |\alpha| + |\hat\beta| \le M) \in \mathbb{C}^{N'};$$
$$\sup_{\xi \in \mathbb{R} \times \mathscr{C}',\, |\xi'| \ge 1} |\xi'|^{-\kappa} \left| \sum_{|\alpha|+|\hat\beta| \le M} \frac{b_{\alpha,\hat\beta}\xi^\alpha \Lambda(\xi')^{\hat\beta}}{p(\xi - i\vartheta)} \right| < \infty \Bigr\},$$

where $N' = \binom{M+N-n}{M}$, $\Lambda(\xi') = (\Lambda_1(\xi'), \dots, \Lambda_N(\xi'))$ is as in the condition (1) of Definition 2.1 (ii) and $G(x, \xi', \Lambda(\xi'))$ is homogeneous of degree κ in $\xi' \in \mathscr{C}' \cup (-\mathscr{C}')$.

Then $\mathscr{B}$ is a subspace of $\mathbb{C}^{N'}$. Let $\{\boldsymbol{b}^j\}_{j=1,\dots,l}$ be a basis of $\mathscr{B}$. From (2.7) we have $\boldsymbol{a}(x) \equiv \bigl(a_{\alpha,\hat{\beta}}(x); |\alpha| + |\hat{\beta}| \le M\bigr) \in \mathscr{B}$. Therefore, there are $a_j(x) \in C^\infty(V)$ $(1 \le j \le l)$ such that $\boldsymbol{a}(x) = \sum_{j=1}^l a_j(x)\boldsymbol{b}^j$. This gives the following

Lemma 2.5. *Under the above assumptions there are $l \in \mathbb{N}$, SA-symbols $q_j(\xi)$ $(1 \le j \le l)$ in $V \times \mathscr{C}'$* (or $\mathbb{R}^n \times \mathscr{C}'$), *$a_j(x) \in C^\infty(V)$ $(1 \le j \le l)$, $\kappa \in \mathbb{Z}$ and $e(x,\xi') \in C^\infty(V \times (\mathscr{C}' \cup (-\mathscr{C}')))$ such that $e(x,\xi')$ is homogeneous of degree κ in $\xi' \in \mathscr{C}' \cup (-\mathscr{C}')$, $e(x,\xi') \neq 0$ for $(x,\xi') \in V \times \mathscr{C}'$ and*

$$\sup_{\xi \in \mathbb{R}\times\mathscr{C}',\, |\xi'| \ge 1} \left| \frac{q_j(\xi)}{p(\xi - i\vartheta)} \right| |\xi'|^{-\kappa} < \infty \quad (1 \le j \le l),$$

$$q(x,\xi) = \sum_{j=1}^{l} a_j(x) e(x,\xi')^{-1} q_j(\xi).$$

3. Proof of the assertion (i) of Theorem 1.1

In this section we shall prove the assertion (i) of Theorem 1.1, applying Theorem 1.4 of [14]. We use the notations in [14]. The conditions on $\mathscr{N}^1(x,D;z^{0\prime})$, $\mathscr{N}^2(x,D;z^{0\prime})$ and $R(x,D;z^{0\prime})$ in (1.3) of [14] are slightly different from the conditions satisfied by $N_1(x,D)$, $N_2(x,D)$ and $R(x,D)$ in (2.6). Therefore, we must show that Theorem 1.4 of [14] is still valid if, instead of the conditions there, $\mathscr{N}^1(x,D;z^{0\prime}) \equiv \bigl(\mathscr{N}^1_{j,k}(x,D;z^{0\prime})\bigr)$, $\mathscr{N}^2(x,D;z^{0\prime}) \equiv \bigl(\mathscr{N}^2_{j,k}(x,D;z^{0\prime})\bigr)$ and $R(x,D;z^{0\prime}) \equiv (R_{j,k}(x,D;z^{0\prime}))$ satisfy the following:

(N-1) There is $m', m'', \bar{m} \in \mathbb{Z}_+$ and $\mathscr{M}(x,\xi;z^{0\prime}) \equiv (\mathscr{M}_{j,k}(x,\xi;z^{0\prime}))$ such that $\mathscr{N}^1_{j,k}(x,\xi;z^{0\prime}) \in \mathscr{S}^{l_{0,j}-m_k+m',-m'}_{1,0}$, $\mathscr{M}_{j,k}(x,\xi;z^{0\prime}) \in \mathscr{S}^{m_j-l_{0,k}+m'',-m''}_{1,0}$ and $\mathscr{M}(x,D;z^{0\prime})$ is a microlocal parametrix of $\mathscr{N}^1(x,D;z^{0\prime})$ (mod $\mathscr{L}^{\bar{m},-\infty}_{1,0}$) in a conic neighborhood $\mathscr{C}'(\mathrm{z}^{0\prime})$ of $\mathrm{z}^{0\prime}$.

(N-2) $\mathscr{N}^2_{j,k}(x,\xi;z^{0\prime}) \in \mathscr{S}^{-n_j-l_{\nu,k}+m',-m'}_{1,0}$ and $R(x,\xi;z^{0\prime}) \in \mathscr{S}^{l_{0,j}-l_{\nu,k}+m',-\infty}_{1,0}$.

In particular, (N-1) implies that the parametrix $\mathscr{M}(x,D;z^{0\prime})$ of $\mathscr{N}^1(x,D;z^{0\prime})$ is a differential operator of x_1. We assume that the above conditions (N-1) and (N-2) are satisfied, and we shall modify the proof of Lemmas 4.11, 4.13 and 4.14 in [14]. By (N-2) we have

$$\left|\mathscr{R}^{(\alpha)}_{j,k(\beta)}(x,\xi;z^{0\prime};\gamma)\right| \le C_{M,\alpha,\beta} \langle\xi\rangle_\gamma^{l_{0,k}-l_{\nu,j}+m'} \langle\xi'\rangle_\gamma^{-M}$$

for $M \in \mathbb{Z}_+$. Therefore, Lemma 4.11 of [14] is valid. We put $\widetilde{\mathscr{N}}^\mu(x,\xi;z^{0\prime};\gamma) = {}^t\mathscr{N}^\mu(x,\xi;z^{0\prime})(1-\Theta_{\gamma/8}(\xi'))$ $(\mu = 1,2)$ and $\widetilde{\mathscr{M}}(x,\xi;z^{0\prime};\gamma) = {}^t\mathscr{M}(x,\xi;z^{0\prime})(1-\Theta_{\gamma/8}(\xi'))$, where ${}^t\mathscr{N}^\mu(x,D;z^{0\prime})$ and ${}^t\mathscr{N}^\mu(x,\xi;z^{0\prime})$ denote the transposed operator of $\mathscr{N}^\mu(x,D;z^{0\prime})$ and the symbol of ${}^t\mathscr{N}^\mu(x,D;z^{0\prime})$, respectively. It follows from (N-1) that for any $\Psi(x,\xi') \in S^0_{1,0}(\mathbb{R} \times T^*\mathbb{R}^{n-1})$ with $\operatorname{supp}\Psi(x,\xi') \subset -\mathscr{C}'(\mathrm{z}^{0\prime})$ there is $r_\psi(x,\xi;\gamma) \equiv$

$(r_{\Psi,j,k}(x,\xi;\gamma)) \in M_m\big(\mathscr{S}^{\bar{m},-\infty}_{1,0}\big)$ satisfying

$$\widetilde{\mathscr{N}}^1(x,D;z^{0\prime};\gamma)\widetilde{\mathscr{M}}(x,D;z^{0\prime};\gamma)\Psi_\gamma(x,D') = \Psi_\gamma(x,D')I_m + r_\psi(x,D;\gamma),$$
$$\big|r^{(\alpha)}_{\Psi,j,k(\beta)}(x,\xi;\gamma)\big| \le C_{\alpha,\beta,M}\langle\xi\rangle_\gamma^{\bar{m}-\alpha_1}\langle\xi'\rangle_\gamma^{-M} \quad (M\in\mathbb{Z}_+),$$

where $\Psi_\gamma(x,\xi') = (1-\Theta_{\gamma/2}(\xi'))\Psi(x,\xi')$. Instead of ${}^t\widetilde{\mathscr{N}}^1(x,D;z^{0\prime};b;\gamma)^{-1}$ we simply use $\widetilde{\mathscr{M}}(x,D+i\gamma\vartheta;z^{0\prime};\gamma) \equiv \big(\widetilde{\mathscr{M}}_{j,k}(x,D+i\gamma\vartheta;z^{0\prime};\gamma)\big)$. Note that

$$\big|\widetilde{\mathscr{M}}^{(\alpha)}_{j,k(\beta)}(x,\xi+i\gamma\vartheta;z^{0\prime};\gamma)\big| \le C_{\alpha,\beta}\langle\xi\rangle_\gamma^{m_k-l_{0,j}+m''-\alpha_1}\langle\xi'\rangle_\gamma^{-m''-|\alpha'|}.$$

The estimates in Lemma 4.13 of [14] are replaced by

$$\big\|\widetilde{\mathscr{M}}(x,D+i\gamma\vartheta;z^{0\prime};\gamma)v\big\|_{+,\gamma,\{l_{0,j}\},(l,s)} \le C\|v\|_{+,\gamma,\{m_j\},(l+m'',s-m'')},$$
$$\big\|\Psi_1(x,D')\widetilde{\mathscr{M}}(x,D+i\gamma\vartheta;z^{0\prime};\gamma)\Psi_2(x,D')v\big\|_{+,\gamma,\{l_{0,j}\},(l,s)}$$
$$\le C_M\|v\|_{+,\gamma,\{m_j\},(l+m'',-M)} \quad (M\in\mathbb{Z}_+)$$

if $\Psi_j(x,\xi') \in S^0_{1,0}(\mathbb{R}\times T^*\mathbb{R}^{n-1})$ $(j=1,2)$ satisfy $\operatorname{supp}\Psi_1 \cap \operatorname{supp}\Psi_2 = \emptyset$ and $\big|\Psi^{(\alpha')}_{j(\beta)}(x,\xi')\big| \le C_{\alpha',\beta}\langle\xi'\rangle_\gamma^{-|\alpha'|}$. The proof of the above estimates is obvious while the proof of Lemma 4.13 of [14] is not so simple. Instead of (4.70) and (4.71) of [14], we have

$$\psi_\gamma(x,D')u = \widetilde{\mathscr{N}}^1(x,D+i\gamma\vartheta;z^{0\prime};\gamma)\big(\exp[\widehat{\Lambda}']\big)(x,D')v - r_\psi(x,D+i\gamma\vartheta;\gamma)u$$
$$- \widetilde{\mathscr{N}}^1(x,D+i\gamma\vartheta;z^{0\prime};\gamma)\tilde{r}(x,D')\widetilde{\mathscr{M}}(x,D+i\gamma\vartheta;z^{0\prime};\gamma)\psi_\gamma(x,D')u.$$

$${}^t\mathscr{L}_{\widehat{\Lambda}'}(x,D+i\gamma\vartheta;z^{0\prime})v$$
$$= \big(\exp[-\widehat{\Lambda}']\big)(x,D')\big\{\widetilde{\mathscr{N}}^2(x,D+i\gamma\vartheta;z^{0\prime};\gamma)\psi_\gamma(x,D')^tL(x,D+i\gamma\vartheta)u$$
$$+ \widetilde{\mathscr{N}}^2(x,D+i\gamma\vartheta;z^{0\prime};\gamma)[{}^tL(x,D+i\gamma\vartheta),\psi_\gamma(x,D')I_N]u$$
$$+ \big({}^t\mathscr{L}(x,D+i\gamma\vartheta;z^{0\prime}) - \widetilde{\mathscr{N}}^2(x,D+i\gamma\vartheta;z^{0\prime};\gamma)\,{}^tL(x,D+i\gamma\vartheta)$$
$$\times\widetilde{\mathscr{N}}^1(x,D+i\gamma\vartheta;z^{0\prime};\gamma)\big)(1+\tilde{r}(x,D'))$$
$$\times\widetilde{\mathscr{M}}(x,D+i\gamma\vartheta;z^{0\prime};\gamma)\psi_\gamma(x,D')u$$
$$+ \widetilde{\mathscr{N}}^2(x,D+i\gamma\vartheta;z^{0\prime};\gamma)\,{}^tL(x,D+i\gamma\vartheta)$$
$$\times\big(\widetilde{\mathscr{N}}^1(x,D+i\gamma\vartheta;z^{0\prime};\gamma)\tilde{r}(x,D')$$
$$\times\widetilde{\mathscr{M}}(x,D+i\gamma\vartheta;z^{0\prime};\gamma)\psi_\gamma(x,D') + r_\psi(x,D+i\gamma\vartheta;\gamma)\big)u\big\},$$

respectively. Hence (4.72)–(4.74) in [14] are replaced by

$$\big\|\big(\exp[-\widehat{\Lambda}']\big)(x,D')\widetilde{\mathscr{N}}^2(x,D+i\gamma\vartheta;z^{0\prime};\gamma)\psi_\gamma(x,D')\,{}^tL(x,D+i\gamma\vartheta)u\big\|_{+,\gamma,\{l_{\nu,j}\},(l,0)}$$
$$\le C_{a',l}\|{}^tL(x,D+i\gamma\vartheta)u\|_{+,\gamma,\{-n_j\},(l+m',a'(\delta(z^{0\prime})+c(z^{0\prime}))-m')},$$

$$\big\|\big(\exp\big[-\widehat{\Lambda}'\big]\big)(x,D')\widetilde{\mathscr{N}}^2(x,D+i\gamma\vartheta;z^{0\prime};\gamma)[{}^tL(x,D+i\gamma\vartheta),\psi_\gamma(x,D')I_N]u\big\|_{+,\gamma,\{l_{\nu,j}\},(l,0)}$$
$$\le C_{a'}\|u\|_{+,\gamma,\{m_j\},(l+m',-2a'\delta(z^{0\prime})-1-m')},$$
$$\big\|\big(\exp\big[-\widehat{\Lambda}'\big]\big)(x,D')\big({}^t\mathscr{L}(x,D+i\gamma\vartheta;z^{0\prime})-\widetilde{\mathscr{N}}^2(x,D+i\gamma\vartheta;z^{0\prime};\gamma)$$
$$\times\,{}^tL(x,D+i\gamma\vartheta)\widetilde{\mathscr{N}}^1(x,D+i\gamma\vartheta;z^{0\prime};\gamma)\big)(1+\tilde r(x,D'))$$
$$\times(1-\tilde\psi_\gamma(x,D'))\widetilde{\mathscr{M}}(x,D+i\gamma\vartheta;z^{0\prime};\gamma)\psi_\gamma(x,D')u\big\|_{+,\gamma,\{l_{\nu,j}\},(l,0)}$$
$$\le C_{a',l,M}\|u\|_{+,\gamma,\{m_j\},(l+2m'+m'',-M)}\quad(M\in\mathbb{Z}_+),$$

respectively. $\widetilde{\mathscr{R}}(x,\xi;z^{0\prime})\equiv\big(\widetilde{\mathscr{R}}_{i,j}(x,\xi;z^{0\prime})\big)$ defined in [14], with obvious modifications, satisfies

$$\big|\widetilde{\mathscr{R}}^{(\alpha)}_{i,j(\beta)}(x,\xi;z^{0\prime})\big|\le C_{M,\alpha,\beta}\langle\xi\rangle_\gamma^{-l_{\nu,i}+l_{0,j}+2m'-\alpha_1}\langle\xi'\rangle_\gamma^{-M}.$$

Instead of (4.75) and (4.76) in [14], we have

$$\big\|\big(\exp\big[-\widehat{\Lambda}'\big]\big)(x,D')\big\{\widetilde{\mathscr{R}}(x,D;z^{0\prime})\widetilde{\mathscr{M}}(x,D+i\gamma\vartheta;z^{0\prime};\gamma)\psi_\gamma(x,D')$$
$$+\widetilde{\mathscr{N}}^2(x,D+i\gamma\vartheta;z^{0\prime};\gamma)\ {}^tL(x,D+i\gamma\vartheta)$$
$$\times\big(\widetilde{\mathscr{N}}^1(x,D+i\gamma\vartheta;z^{0\prime};\gamma)\tilde r(x,D')$$
$$\times\widetilde{\mathscr{M}}(x,D+i\gamma\vartheta;z^{0\prime};\gamma)\psi_\gamma(x,D')+r_\psi(x,D+i\gamma\vartheta;\gamma)\big)\big\}u\big\|_{+,\gamma,\{l_{\nu,j}\},(l,0)}$$
$$\le C_{a',l,M}\|u\|_{+,\gamma,\{m_j\},(l+\bar m',-M)}$$
$$\|v\|_{+,\gamma,\{l_{0,j}\},(l,0)}$$
$$\le C\big\{\|{}^tL(x,D+i\gamma\vartheta)u\|_{+,\gamma,\{-n_j\},(l+l'+m',a'\delta'(z^{0\prime})-m')}+\|u\|_{+,\gamma,\{m_j\},(l+\tilde l',-s)}\big\},$$

respectively, where $\bar m'=\max\{2m'+m'',m'+\bar m+\max_j m_j-\min_j m_j\}$, $\tilde l'=\max\{l'+2m'+m'',l'+\bar m',l''+m''\}$ and $s\le 2a'\delta(z^{0\prime})+1+m'$. Modifying l' if necessary, instead of (4.76)–(4.78) in [14] we have

$$\|v\|_{+,\gamma,\{l_{0,j}\},(l,0)}$$
$$\le C\{\|{}^tL(x,D+i\gamma\vartheta)u\|_{+,\gamma,\{-n_j\},(l+l',a'\delta'(z^{0\prime})-m')}+\|u\|_{+,\gamma,\{m_j\},(l+l',-s)}\},$$
$$\|\chi_\gamma(x,D')u\|_{+,\gamma,\{m_j\},(l-m'-(3a'\delta(z^{0\prime})/2-m')_+,0)}$$
$$\le\|\chi_\gamma(x,D')u\|_{+,\gamma,\{m_j\},(l-m',-3a'\delta(z^{0\prime})/2+m')}$$
$$\le C_{a',l}\|v\|_{+,\gamma,\{l_{0,j}\},(l,0)}+C_{a',l,M}\|u\|_{+,\gamma,\{m_j\},(l+\bar m'-2m',-M)},$$
$$\|\chi_\gamma(x,D')u\|_{+,\gamma,\{m_j\},(l,0)}$$
$$\le C\{\|{}^tL(x,D+i\gamma\vartheta)u\|_{+,\gamma,\{-n_j\},(l+l_1,0)}+\|u\|_{+,\gamma,\{m_j\},(l+l_2,-l_2-1)}\},$$

respectively, where $s\le 2a'\delta(z^{0\prime})+1+m'$, $l_1=l'+(3a'\delta(z^{0\prime})/2-m')_++a'\delta'(z^{0\prime})$, $l_2=\max\big\{\tilde l'+m',\bar m'-m'\big\}+(3a'\delta(z^{0\prime})/2-m')_+$, $a'=\max\big\{a_0\delta(z^{0\prime}),m',2\big(\tilde l'-m'\big),2(\bar m-$

$3m')\}/\delta(z^{0\prime})$ and $a_+ = \max\{a, 0\}$. With these modifications, Lemma 4.14 and the arguments after Lemma 4.14 in §4 of [14] are valid and Theorem 1.4 of [14] is valid under the conditions (N-1) and (N-2) on $\mathscr{N}^1(x, D; z^{0\prime})$, $\mathscr{N}^2(x, D; z^{0\prime})$ and $R(x, D; z^{0\prime})$.

Now we return to the proof of the assertions (i) of Theorem 1.1. We fix $t > 0$. We shall show that for every $z^{0\prime} = (x^0, \xi^{0\prime}) \in \mathbb{R}^n \times S^{n-2}$ with $x_1^0 = t$ the condition $(\text{E-2})_{z^{0\prime},1}$ of [14] is satisfied. We should remark that $\nu(z^{0\prime}) = 1$ in our case. Let $\Theta(s)$ be a function in $C_0^\infty(\mathbb{R})$ such that $0 \le \Theta(s) \le 1$, $\Theta(s) = 1$ if $|s| \le 1$, and $\operatorname{supp}\Theta \subset (-2, 2)$. Let $\varphi_1(\xi')$ and $\varphi_2(x, \xi')$ be symbols in $\mathscr{S}^0_{1,0}$ such that $0 \le \varphi_1(\xi'), \varphi_2(x, \xi') \le 1$, $\varphi_1(\xi')$ and $\varphi_2(x, \xi')$ are positively homogeneous of degree 0 for $|\xi'| \ge 1$, $\varphi_1(\xi') = 1$ in a conic neighborhood of $\operatorname{supp}\varphi_2$ if $|\xi'| \ge 1$, $\operatorname{supp}\varphi_1 \subset \{\xi' \in \mathscr{C}'; |\xi'| \ge 1/4\}$, $\varphi_2(x, \xi') = 1$ in a conic neighborhood of $(x^0, \xi^{0\prime})$ if $|\xi'| \ge 1$, and $\operatorname{supp}\varphi_2 \subset \{(x, \xi') \in V \times \mathscr{C}'; |\xi'| \ge 1/2\}$, where V and $\mathscr{C}'$ are a neighborhood of x^0 and a conic neighborhood of $\xi^{0\prime}$, respectively, as in Lemma 2.4. We put

$$\begin{aligned} &P_j(x, \xi) \\ &= p_j(\xi) + \Theta\left(\frac{2|x - x^0|}{t}\right)\varphi_2(x, \xi') \sum_{k=1}^{m_j} \big(l_{j,k}(x, \xi') - l^0_{j,k}(\xi')\big)(\xi_1 + \lambda_j \xi_n)^{m_j - k}, \\ &p_j(\xi) = (\xi_1 + \lambda_j \varphi_1(\xi')\xi_n)^{m_j} + \sum_{k=1}^{m_j} l^0_{j,k}(\xi')\varphi_1(\xi')^k (\xi_1 + \lambda_j \varphi_1(\xi')\xi_n)^{m_j - k} \end{aligned}$$

$(1 \le j \le r)$ and $\tilde{p}(\xi) = \prod_{j=1}^r p_j(\xi)$, where the λ_j and the $l_{j,k}(x, \xi')$ are as in Lemma 2.4. We note that $p(\xi) = \tilde{p}(\xi)$ if $\varphi_2(x, \xi') \ne 0$ and that $p_j(\xi)$ is hyperbolic with respect to ϑ. Put

$$\widetilde{Q}(x, \xi) = \prod_{j=1}^r P_j(x, \xi) - \tilde{p}(\xi).$$

Then the condition (L) implies that there is $C > 0$ satisfying

$$\big|\widetilde{Q}(x, \xi)\big| \le C|\tilde{p}(\xi - i\vartheta)| \quad \text{for} \quad (x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n, \tag{3.1}$$

and, therefore,

$$|P_j(x, \xi) - p_j(\xi)| \le C|p_j(\xi - i\vartheta)| \quad \text{for} \quad (x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n.$$

So, in order to verify $(\text{E-2})_{z^{0\prime},1}$, it suffices to prove $(\text{E-2})_{z^{0\prime},1}$ with $\mathscr{L}^1(x, \xi; z^{0\prime})$ replaced by $P_j(x, \xi)$ for $1 \le j \le r$. Fix j so that $1 \le j \le r$. It is obvious that the conditions (i)–(v) of $(\text{E-2})_{z^{0\prime},1}$ are satisfied with a conic neighborhood $\mathscr{C}'(z^{0\prime})$ of $z^{0\prime}$ in $\mathbb{R} \times (T^*\mathbb{R}^{n-1} \setminus 0)$, $\Omega(z^{0\prime}) = \{x \in \mathbb{R}^n; |x - x^0| < t\}$ and $\kappa = m_j/(m_j - 1)$. Let $z^1 = (x^1, \xi^1)$ be a point in $\overline{\Omega(z^{0\prime})} \times S^{n-1}$ such that $(dp_j)(\xi^1) = 0$. Since we can easily obtain microlocal *a priori* estimates for $P_j(x, D) = D_1^{m_j}$, we can assume that $\xi^{1\prime} \in \mathscr{C}'$ (see, e.g., Lemma 2.10 of [14]). Define the localization polynomial $p_{j,\xi}(\eta)$ of p_j at ξ by

$$p_j(\xi+s\eta)=s^{\mu}(p_{j,\xi}(\eta)+o(1)) \quad \text{as} \quad s\downarrow 0, \quad p_{j,\xi}(\eta)\not\equiv 0 \quad \text{in} \quad \eta.$$

Then $p_{j,\xi}(\eta)$ is hyperbolic with respect to ϑ. (see, e.g., [7]). Let $t_{\pm}(x,\xi)$ and $t(x,\xi)$ are positively homogeneous of degree 0, $(\nabla_x t)(z^1)\in\Gamma(p_{j,\xi^1},\vartheta)$ and

$$t_{\pm}(x,\xi)=x_1-x_1^1\pm|x-x^1|^2\pm\left|\frac{\xi}{|\xi|}\mp\xi^1\right|^2 \quad \text{near} \quad z^1.$$

Now we can prove that $P_j(x,D-i\gamma\vartheta)$ satisfies the condition $\bigl(E;z^1,\mathscr{C}_1,\mathscr{C}_2,\{at_+(x,\xi)+a't(x,\xi)\log\langle\xi'\rangle/\log\langle\xi\rangle\}_{a\ge a_0,\,a'\ge a_0'},m/(m-1),\infty\bigr)$, where $\mathscr{C}_1$ and $\mathscr{C}_2$ are conic neighborhoods of z^1 satisfying $\mathscr{C}_1\cap\mathbb{R}^n\times S^{n-1}\Subset\mathscr{C}_2$, i.e., there are $\psi_k(x,\xi)\in C^\infty(T^*\mathbb{R}^n\setminus 0)$ $(k=1,2)$, $l_k\in\mathbb{R}$ $(1\le k\le 4)$ such that the $\psi_k(x,\xi)$ are positively homogeneous of degree 0, $\operatorname{supp}\psi_2\cap\mathbb{R}^n\times S^{n-1}\Subset\mathscr{C}_2$, $\psi_k(x,\xi)=1$ in a conic neighborhood of $\mathscr{C}_1$ $(k=1,2)$ and for any $a\ge 1$, $a'\ge 1$ and $b\in\mathbb{R}$ there are $\gamma_0\ge 1$, $K\ge 1$ and $C>0$ satisfying

$$\begin{aligned}\bigl\|\langle D\rangle_\gamma^{l_1}u\bigr\| \le C\bigl\{&\bigl\|\langle D\rangle_\gamma^{l_2}P_{j,\Lambda}(x,D;\gamma)u\bigr\|\\ &+\bigl\|\langle D\rangle_\gamma^{l_3}(1-\psi_{1,h}(x,D))u\bigr\|+\bigl\|\langle D\rangle_\gamma^{l_4}\Theta_h(D)u\bigr\|\bigr\}\end{aligned}$$

if $u\in H^\infty(\mathbb{R}^n)$, $\gamma\ge\gamma_0$, $h=K\gamma^{m/(m-1)}$ and

$$\begin{aligned}&\Lambda(x,\xi)\ (\equiv\Lambda_{a,a',b,h}(x,\xi))\\ &=\{(at_+(x,\xi)-b)\log\langle\xi\rangle+a't(x,\xi)\log\langle\xi'\rangle\}(1-\Theta_{h/4}(\xi))\psi_2(x,\xi),\end{aligned}$$

where $\Theta_h(\xi)=\Theta(|\xi|/h)$, $\psi_{1,h}(x,\xi)=(1-\Theta_{h/2}(\xi))\psi_1(x,\xi)$ and $P_{j,\Lambda}(x,D;\gamma)=(e^{-\Lambda})(x,D)P_j(x,D-i\gamma\vartheta)(e^{\Lambda})(x,D)$. Here $A\Subset B$ means that the closure $\overline{A}$ of A is compact and included in the interior $\mathring{B}$ of B, and $\|f\|=\bigl(\int_{\mathbb{R}^n}|f(x)|^2\,dx\bigr)^{1/2}$. Indeed, it follows from (3.1), Lemma 2.5 and Lemma 2.7 of [28] that $P_j(x,D)$ satisfies the condition (A-2)$_{z^1}$ of [15] (see, also, Lemma 2.1 of [15]). Similarly, we can prove that ${}^tP_j(x,D+i\gamma\vartheta)$ satisfies the condition $\bigl(E;(x^1,-\xi^1),\check{\mathscr{C}}_1,\check{\mathscr{C}}_2,\{-at_-(x,\xi)-a't(x,-\xi')\times\log\langle\xi'\rangle/\log\langle\xi\rangle\}_{a\ge a_0,\,a'\ge a_0'},m/(m-1),\infty\bigr)$, where $\check{\mathscr{C}}_k=\{(x,-\xi);(x,\xi)\in\mathscr{C}_k\}$ and ${}^tP_j(x,\xi)=\sigma({}^tP_j(x,D))$. Here $\sigma(a(x,D))$ $(=\sigma(a(x,D))(x,\xi))$ denotes the symbol of $a(x,D)$. Therefore, the condition (E-2)$_{z^{0\prime},1}$ of [14] is satisfied for every $z^{0\prime}=(x^0,\xi^{0\prime})\in\mathbb{R}^n\times S^{n-2}$ with $x^0=t$. By the same argument we can prove that the condition (E-1)$_{z^0}$ of [14] is satisfied for every $z^0=(x^0,\xi^0)\in\mathbb{R}^n\times S^{n-1}$ with $x_1^0>0$ and $(dp)(\xi^0)=0$. For the condition (U) of [14] we need the following

Lemma 3.1. *Let $N\in\mathbb{N}$ and $\mathscr{C}$ be an open conic subset of $T^*\mathbb{R}^n\setminus 0$. Assume that*

$$L(x,D)=M_1(x,D)l(x,D)M_2(x,D)+R(x,D) \quad \textit{in} \quad \mathscr{C},$$

where $M_k(x,\xi) \in M_m\big(S^{m-1}_{1,0}\big)$ $(k = 1,2)$, $l(x,\xi) \in M_m\big(S^{m}_{1,0}\big)$, $R(x,\xi) \in M_m\big(S^{m-N}_{1,0}\big)$ *and the entries of* $M_k(x,\xi)$ $(k = 1,2)$, $l(x,\xi)$ *and* $R(x,\xi)$ *are classical symbols. Here* $A(x, D) = B(x, D)$ *in* $\mathscr{C}$ *means that* $A(x,\xi) = B(x,\xi)$ *in* $\mathscr{C}$. *Let* $\zeta(x) \in C^\infty(\mathbb{R}^n)$ *satisfy* $\partial_k\zeta(x) \in \mathscr{B}(\mathbb{R}^n)$ $(1 \le k \le n)$. *Then there are* $r_{N,k}(x,\xi) \in M_m\big(S^{3m-2-N}_{1,0}\big)$ $(0 \le k \le 3(N-1))$ *such that*

$$\begin{aligned} &e^{-\gamma\zeta(x)}L(x, D)e^{\gamma\zeta(x)} \;(= L(x, D) + \gamma L_1(-i\nabla\zeta(x))) \\ &= M_{1,N-1}(x, D; \gamma)l_{N-1}(x, D; \gamma)M_{2,N-1}(x, D; \gamma) + \sum_{k=0}^{3(N-1)} \gamma^k r_{N,k}(x, D) \end{aligned} \tag{3.2}$$

in $\mathscr{C}$, *where*

$$M_{\mu,N-1}(x,\xi;\gamma) = \sum_{|\alpha|\le N-1} \frac{M^{(\alpha)}_{\mu}(x,\xi)\omega_\alpha(\gamma\zeta(x))}{\alpha!} \quad (\mu = 1, 2),$$

$$l_{N-1}(x,\xi;\gamma) \sum_{|\alpha|\le N-1} \frac{l^{(\alpha)}(x,\xi)\omega_\alpha(\gamma\zeta(x))}{\alpha!}$$

and $\omega_\alpha(\gamma\zeta(x)) = e^{-\gamma\zeta(x)}D^\alpha e^{\gamma\zeta(x)}$.

Proof. We can write

$$\sigma(e^{-\gamma\zeta(x)}M_\mu(x, D)e^{\gamma\zeta(x)})(x,\xi) = M_{\mu,N-1}(x,\xi;\gamma) + \widetilde{M}_{\mu,N}(x,\xi;\gamma) \quad (\mu = 1, 2),$$
$$\sigma(e^{-\gamma\zeta(x)}l(x, D)e^{\gamma\zeta(x)})(x,\xi) = l_{N-1}(x,\xi;\gamma) + \tilde{l}_N(x,\xi;\gamma),$$

where $\widetilde{M}_{\mu,N}(x,\xi;\gamma) \in M_m\big(S^{m-1-N}_{1,0}\big)$ $(\mu = 1,2)$ and $\tilde{l}_N(x,\xi;\gamma) \in M_m\big(S^{m-N}_{1,0}\big)$. So we have

$$\begin{aligned} &L(x, D) + \gamma L_1(-i\nabla\zeta(x)) \\ &= M_{1,N-1}(x, D; \gamma)l_{N-1}(x, D; \gamma)M_{2,N-1}(x, D; \gamma) + \widetilde{R}_N(x, D; \gamma) \\ &= \sum_{k=0}^{3(N-1)} \gamma^k(A_k(x, D) + B_k(x, D)) + \widetilde{R}_N(x, D; \gamma) \quad \text{in} \quad \mathscr{C}, \\ &A_k(x,\xi) \sim \sum_{\mu=0}^{\infty} A^\mu_k(x,\xi) \quad \text{if} \quad k \le N-1, \end{aligned}$$

where $\widetilde{R}_N(x,\xi;\gamma) \in M_m\big(S^{3m-2-N}_{1,0}\big)$, the $A^\mu_k(x,\xi)$ are positively homogeneous of degree $3m-2-k-\mu$, $A_k(x,\xi) \equiv 0$ for $N \le k \le 3(N-1)$ and $B_k(x,\xi) \in M_m\big(S^{3m-2-N}_{1,0}\big)$. So, modifying the $A_k(x,\xi)$ for $|\xi| \le 1$, we may assume that $A_0(x,\xi) = L(x,\xi)$ in $\mathscr{C}$, $A_1(x,\xi) = L_1(-i\nabla\zeta(x))$ in $\mathscr{C}$, and $A_k(x,\xi) = 0$ in $\mathscr{C}$ if $2 \le k \le 3(N-1)$. This gives

$$\widetilde{R}(x,\xi;\gamma) = -\sum_{k=0}^{3(N-1)} \gamma^k B_k(x,\xi) \quad \text{in} \quad \mathscr{C}.$$

Taking $r_{N,k}(x,\xi) = -B_k(x,\xi)$ we have (3.2). □

Applying the same arguments as in §3 of [15], we can prove that the condition (U) of [14] is satisfied. Then the assertion (i) of Theorem 1.1 easily follows from Theorem 1.4 of [14].

4. Proof of the assertion (ii) of Theorem 1.1

In this section we assume that the Cauchy problem $(\mathrm{CP})_t$ is C^∞ well-posed for any $t > 0$. We can assume without loss of generality that $L(x,\xi) \in M_m\big(\mathscr{S}^1_{1,0}\big)$. For $\tau \in \mathbb{R}$, $p \in \mathbb{Z}_+$, $q \in \mathbb{R}$ and $u \in C^\infty(\mathbb{R}^n)$ we define

$$\|u\|_{\tau,-,(p,q)} = \sum_{|\alpha|\le p} \|\langle D'\rangle^q D^\alpha u\|_{\tau,-},$$

$$\|u\|_{\tau,-} = \left(\int_{x_1<\tau} |u(x)|^2\, dx\right)^{1/2}.$$

From Banach's closed graph theorem or the Baire category theorem we have the following lemma (see, e.g., [10] and [27]).

Lemma 4.1. *Let $t > 0$. Then for every compact subset K of $\{x \in \mathbb{R}^n; x_1 > t\}$ and $p \in \mathbb{Z}_+$ there are $C \equiv C_{p,K} > 0$ and $q \in \mathbb{Z}_+$ such that*

$$\|u\|_{\tau,-,(p,0)} \le C\|L(x,D)u\|_{\tau,-,(q,0)}$$

for any $\tau > t$ and $u \in C_0^\infty(\mathbb{R}^n;\mathbb{C}^m)$ with $\operatorname{supp} u \subset K$.

Let $t > 0$, $x^0 \in \mathbb{R}^n$ and $\xi^{0\prime} \in S^{n-2}$ such that $x_1^0 = t$, and let $V \equiv V(x^0,\xi^{0\prime})$ and $\mathscr{C}' \equiv \mathscr{C}'(x^0,\xi^{0\prime})$ be as in the condition (L) and Lemmas 2.2 and 2.4. Moreover, let r, m_j $(1 \le j \le r)$, $N_k(x,\xi)$ $(k = 1,2)$, $l^j(x,\xi)$ $(1 \le j \le r)$ and $R(x,\xi)$ be as in Lemma 2.4. Choose an open subset V_0 of V, an open conic subset $\mathscr{C}_0'$ of $\mathscr{C}'$ and $\chi \in C_0^\infty(V)$ and $\Psi(\xi') \in C^\infty(\mathbb{R}^{n-1})$ so that $V_0 \Subset V$, $\mathscr{C}_0' \cap S^{n-2} \Subset \mathscr{C}'$, $\chi(x) = 1$ in V_0, $\Psi(\xi')$ is positively homogeneous of degree 0 for $|\xi'| \ge 1$, $\Psi(\xi) = 1$ for $\xi' \in \mathscr{C}_0'$ with $|\xi'| \ge 1$ and $\operatorname{supp}\Psi \subset \{\xi' \in \mathscr{C}'; |\xi'| \ge 1/2\}$. We put $\Psi_\pm(\xi') = \Psi(\pm\xi')$. Then from Lemma 2.4 we have

$$\begin{aligned} &N_1(x,D)L(x,D)N_2(x,D)\Psi_\pm(D')(\chi(x)u(x)) \\ &\quad = l(x,D)\Psi_\pm(D')(\chi(x)u(x)) + R_\pm(x,D)u, \end{aligned} \tag{4.1}$$

where $l(x,\xi) = \operatorname{diag}(1,\ldots,1,l^1(x,\xi),\ldots,l^r(x,\xi))$ and $R_\pm(x,\xi) \in M_m\big(\mathscr{S}^{2m-1,-\infty}_{1,0}\big)$.

Lemma 4.2. *Let $t > 0$. Then for every compact subset K of $\{x \in V_0; x_1 > t\}$ there are $C_{K,M} > 0$ $(M \in \mathbb{Z}_+)$ and $p_j \in \mathbb{Z}_+$ $(1 \le j \le 3)$ such that*

$$\max_{x_1 \le \tau} |u(x)| \le C_{K,M}(\|l(x, D)u\|_{\tau,-,(p_1,0)} + \|u\|_{\tau,-,(p_2,-M)} + \|(1 - \Psi_\pm(D'))u\|_{\tau,-,(p_3,0)})$$

for any $\tau > t$, $u \in C_0^\infty(\mathbb{R}^n; \mathbb{C}^m)$ with $\operatorname{supp} u \subset K$ and $M \in \mathbb{Z}_+$.

Proof. Let $\tau > t$, and let K and K_j $(j = 1, 2)$ be compact subsets of $\{x \in V_0; x_1 > t\}$ satisfying $K \Subset K_1 \Subset K_2$. We choose $\chi_j \in C^\infty(\mathbb{R}^n)$ $(j = 1, 2)$ so that $\chi_1(x) = 1$ near K, $\operatorname{supp} \chi_1 \subset K_1$, $\chi_2(x) = 1$ near K_1 and $\operatorname{supp} \chi_2 \subset K_2$. Let $u \in C_0^\infty(\mathbb{R}^n; \mathbb{C}^m)$ satisfy $\operatorname{supp} u \subset K$. Then by (4.1) we have

$$\begin{aligned} &N_1(x, D)L(x, D)\chi_2(x)N_2(x, D)\Psi_\pm(D')u = l(x, D)\Psi_\pm(D')u + R_{1,\pm}(x, D)u, \\ &R_{1,\pm}(x, D) = N_1(x, D)L(x, D)(\chi_2(x) - 1)N_2(x, D)\Psi_\pm(D')\chi_1(x) \\ &\qquad + R_\pm(x, D) \in \mathscr{L}^{2m-1,-\infty}_{1,0}. \end{aligned} \tag{4.2}$$

Let $p \in \mathbb{Z}_+$ satisfy $p \ge m - 1$. From Lemma 4.1 it follows that there are $C > 0$ and $q \in \mathbb{Z}_+$ satisfying

$$\begin{aligned} &\|\chi_2(x)N_2(x, D)\Psi_\pm(D')u\|_{\tau,-,(p,0)} \\ &\le C\|L(x, D)\chi_2(x)N_2(x, D)\Psi_\pm(D')u\|_{\tau,-,(q,0)}. \end{aligned} \tag{4.3}$$

Let $M_j(x, D) \in \mathscr{L}^{m-1}_{1,0}$ be a parametrix of $N_j(x, D)$ $(j = 1, 2)$. Then we have

$$\begin{aligned} &L(x, D)\chi_2(x)N_2(x, D)\Psi_\pm(D')u \\ &= M_1(x, D)N_1(x, D)L(x, D)\chi_2(x)N_2(x, D)\Psi_\pm(D')u + R_{2,\pm}(x, D)u, \\ &R_{2,\pm}(x, \xi) \in \mathscr{L}^{3m-2,-\infty}_{1,0}. \end{aligned}$$

This, together with (4.2) and (4.3), yields

$$\begin{aligned} &\|\chi_2(x)N_2(x, D)\Psi_\pm(D')u\|_{\tau,-,(p,0)} \\ &\le C_M(\|l(x, D)\Psi_\pm(D')u\|_{\tau,-,(q+m-1,0)} + \|u\|_{\tau,-,(q+3m-2,-M)}) \\ &\le C'_M(\|l(x, D)u\|_{\tau,-,(q+m-1,0)} + \|u\|_{\tau,-,(q+3m-2,-M)} \\ &\qquad + \|(1 - \Psi_\pm(D'))u\|_{\tau,-,(q+2m-1,0)}) \end{aligned} \tag{4.4}$$

$(M \in \mathbb{Z}_+)$. On the other hand, we have

$$\begin{aligned} u &= \Psi_\pm(D')u + (1 - \Psi_\pm(D'))u \\ &= M_2(x, D)\chi_2(x)N_2(x, D)\Psi_\pm(D')u + (1 - \Psi_\pm(D'))u + R_{3,\pm}(x, D)u, \end{aligned} \tag{4.5}$$

where $R_{3,\pm}(x,\xi) \in \mathscr{S}^{2m-2,-\infty}_{1,0}$. From (4.4) and (4.5) we have

$$\|u\|_{\tau,-,(p-m+1,0)} \le C_M(\|l(x,D)u\|_{\tau,-,(q+m-1,0)} + \|u\|_{\tau,-,(p_2,-M)} + \|(1-\Psi_\pm(D'))u\|_{\tau,-,(p_3,0)}),$$

where $M \in \mathbb{Z}_+$, $p_2 = \max\{q+3m-2, p+m-1\}$ and $p_3 = \max\{q+2m-1, p-m+1\}$. This proves the lemma. □

Let $x^0 \in \mathbb{R}^n$ and $\xi^{0\prime} \in S^{n-2}$, and let V and $\mathscr{C}'$ be an open neighborhood of x^0 and an open conic neighborhood of $\xi^{0\prime}$, respectively, such that $\mathscr{C}'$ is semi-algebraic. Moreover, let $q(x,\xi)$ be an *SA*-symbol in $V \times \mathscr{C}'$ such that

$$q(x,\lambda\xi) = \lambda^\mu q(x,\xi) \quad \text{for} \quad (x,\xi) \in V \times \mathbb{R} \times (\mathscr{C}' \cup (-\mathscr{C}')) \quad \text{and} \quad \lambda \in \mathbb{R} \setminus \{0\},$$

where $\mu \in \mathbb{N}$. For a conic subset Γ of $\mathbb{R} \times \mathscr{C}'$ and a fixed $x \in V$ we define

$$n(p, q(x,\,\cdot\,); \Gamma) = \inf\left\{\rho \in \mathbb{R};\ \sup_{\xi\in\Gamma,\,|\xi'|\ge 1} \langle\xi\rangle^{-\rho}\left|\frac{q(x,\xi)}{p(\xi - i\vartheta)}\right| < \infty\right\}.$$

Let $\eta^0 \in \mathbb{R} \times \mathscr{C}' \cap S^{n-1}$, and put

$$N_{\eta^0} = \left\{\hat\eta(s) = s^{-1}\left(\eta^0 + \sum_{j=1}^\infty s^{j/l}\eta^j\right),\ \eta^j \in \mathbb{R}^n,\ l \in \mathbb{N} \text{ and } \hat\eta(s) \text{ is convergent for } 0 < s \ll 1\right\}.$$

For $x \in V$ and $\hat\eta \in N_{\eta^0}$ we define $n(x;\hat\eta) \equiv n(p, q(x,\,\cdot\,); \hat\eta) \in \mathbb{R}$ by

$$\left|\frac{q(x,\hat\eta(s))}{p(\hat\eta(s) - i\vartheta)}\right| = s^{-n(x;\hat\eta)}(c(x,\hat\eta) + o(1)) \quad \text{as} \quad s \downarrow 0,$$

where $c(x;\hat\eta) \ne 0$. Here we have defined $n(x;\hat\eta) = -\infty$ if $q(x,\hat\eta(s)) \equiv 0$ in s. Moreover, we define

$$n(p, q(x,\,\cdot\,); \eta^0) = \sup\{n(p, q(x,\,\cdot\,); \hat\eta); \hat\eta \in N_{\eta^0}\}.$$

Lemma 4.3. (i) *For a fixed $x \in V$ there is $c_0 > 0$ such that $\Gamma(c) \subset \mathbb{R} \times \mathscr{C}'$ and $n(p, q(x,\,\cdot\,); \eta^0) = n(p, q(x,\,\cdot\,); \Gamma(c))$ for any $c > 0$ with $c \le c_0$, where $\Gamma(c) = \{\xi \in \mathbb{R}^n \setminus \{0\}; |\xi/|\xi| - \eta^0| \le c\}$.*
(ii) *Let $c > 0$ satisfy $\Gamma(c) \subset \mathbb{R} \times \mathscr{C}'$, where $\Gamma(c)$ is as in the assertion* (i). *Then $n(p, q(x,\,\cdot\,); \Gamma(c)) \in \mathbb{Q}$ for $x \in V$, and for a fixed $x \in V$ there is $C > 0$ satisfying*

$$\left|\frac{q(x,\xi)}{p(\xi - i\vartheta)}\right| \le C\langle\xi\rangle^{n(p,q(x,\cdot);\Gamma(c))} \quad \textit{for} \quad \xi \in \Gamma(c).$$

(iii) *The set* $\{n(p,q(x,\cdot);\xi);(x,\xi)\in V\times\mathbb{R}\times\mathscr{C}'$ *and* $|\xi|=1\}$ *is finite, with a modification of* $\mathscr{C}'$ *if necessary.*

(iv) *Let* $\hat\eta\in N_{\eta^0}$. *Then for any open subset* U *of* V *there is an open subset* U_0 *of* U *such that*

$$\max_{x\in U} n(p,q(x,\ \cdot);\hat\eta)=n(p,q(y,\ \cdot);\hat\eta)\quad \textit{for}\quad y\in U_0.$$

(v) *Assume that* $n(p,q(x,\cdot);\xi)\le 0$ *for every* $x\in K$ *and* $\xi\in\omega$, *where* K *is a compact subset of* V *and* ω *is a compact subset of* $\mathbb{R}\times\mathscr{C}'\cap S^{n-1}$. *Then there is* $C>0$ *such that*

$$\left|\frac{q(x,\xi)}{p(\xi-i\vartheta)}\right|\le C\quad \textit{if}\quad x\in K,\quad \frac{\xi}{|\xi|}\in\omega\quad \textit{and}\quad |\xi'|\ge 1. \tag{4.6}$$

Proof. The assertion (i) can be proved by the same arguments as in the proof of Lemma 1.2.3 of [26] (see, also, the proof of Lemma 1.2 of [23]). Let us prove the assertion (ii). Put

$$\begin{aligned}\mathscr{E}=\{(\phi,\xi,r)\in\mathbb{R}^{n+2};\ &\phi|p(\xi-i\vartheta)|^2=|q(x,\xi)|^2,\\ &|\xi|^2=r^2,\ r\ge 1 \text{ and } |\xi-r\eta^0|^2\le c^2r^2\},\end{aligned}$$

where $x\in V$ is fixed. By assumption $\mathscr{E}$ is a semi-algebraic set. It follows from Corollary A.2.6 of [8] that there are $a\in\mathbb{Q}$ and $A\in\mathbb{C}\setminus\{0\}$ such that

$$\sup\{\phi:\text{ there is }\xi\in\mathbb{R}^n\text{ satisfying }(\phi,\xi,r)\in\mathscr{E}\}=r^a(A+o(1))\quad\text{as}\quad r\to\infty.$$

This gives $n(p,q(x,\ \cdot);\Gamma(c))=a/2$ and proves the assertion (ii). We omit the proof of the assertion (iii) since we will not use it in this paper and it can be proved by the same arguments as in the beginning of §2.1 of [26] and in the proof of Lemma 1.1.3 of [26]. Next let us prove the assertion (iv). By definition there are $N\in\mathbb{N}$, a semi-algebraic set $\mathscr{A}$ in $\mathbb{R}^{N+n-1}$ and a polynomial $F(x,\xi,\lambda)$ of (ξ,λ) ($\equiv(\xi,\lambda_1,\ldots,\lambda_N)$) and a polynomial $G(x,\xi',\lambda)$ of (ξ',λ), whose coefficients belong to $C^\infty(V)$, satisfying the conditions (1)–(3) of Definition 2.1 (ii) with $f(x,\xi)$ replaced by $q(x,\xi)$. Put

$$\tilde q(x,\xi)=F(x,\xi_1,\pi^{-1}(\xi'))|\xi'|^{-\kappa},$$

where π is as in the condition (1) of Definition 2.1 (ii) and $G(x,\pi^{-1}(\xi'))$ is homogeneous of degree κ. Let $\hat\eta\in N_{\eta^0}$, and let U be an open subset of V. It is obvious that there are $c_0\in\mathbb{C}\setminus\{0\}$ and $\mu_0\in\mathbb{Q}$ such that $p(\hat\eta(s)-i\vartheta)=s^{-\mu_0}(c_0+o(1))$ as $s\downarrow 0$. We may assume that $\tilde q(x,\hat\eta(s))\not\equiv 0$ in $(x,s)\in U\times(0,s_0]$, where $0<s_0\ll 1$. Then there are $c(x)\in C^\infty(U)$ and $\mu_1\in\mathbb{Q}$ such that $c(x)\not\equiv 0$ in U and $\tilde q(x,\hat\eta(s))=s^{-\mu_1}(c(x)+o(1))$ as $s\downarrow 0$. Therefore, we have $n(p,\tilde q(x,\cdot);\hat\eta)\le\mu_1-\mu_0$ for $x\in U$, and $n(p,\tilde q(x,\cdot);\hat\eta)=\mu_1-\mu_0$ for $x\in U$ with $c(x)\ne 0$. Since $n(p,q(x,\ \cdot);\hat\eta)=n(p,\tilde q(x,\ \cdot);\hat\eta)$, this proves the assertion (iv). The assertion (v) easily follows from the assertions (i) and (ii) and Lemma 2.5. □

Now suppose that the condition (L) is not satisfied. Then it follows from Lemma 4.3 that there are $(x^0, \xi^{0\prime}) \in \mathbb{R}^n \times S^{n-2}$, $(y^0, \eta^0) \in V \times (\mathbb{R} \times \mathscr{C}' \cap S^{n-1})$ and $\hat{\eta}(s) \equiv s^{-1} \sum_{j=0}^{\infty} s^{j/l} \eta^j \in N_{\eta^0}$ such that $y_1^0 \geq 0$ and

$$\max_{1 \leq j \leq m-1} n(p, Q^j(y^0, \cdot); \hat{\eta}) > 0,$$

where $V \equiv V(x^0, \xi^{0\prime})$, $\mathscr{C}' \equiv \mathscr{C}'(x^0, \xi^{0\prime})$ and $Q(x, \xi) \equiv Q(x, \xi; x^0, \xi^{0\prime})$ are as in the condition (L) and Lemmas 2.2 and 2.4, $Q^j(x, \xi)$ is positively homogeneous of degree $m - j$ and $Q(x, \xi) \sim \sum_{j=1}^{\infty} Q^j(x, \xi)$. By the assertion (iv) of Lemma 4.3 we may assume that $y_1^0 > 0$ and

$$\begin{aligned} &\sigma \ (= \sigma(\hat{\eta})) \\ &\equiv \max_{\substack{1 \leq j \leq m-1 \\ x \in V,\, x_1 \geq 0}} \frac{n_+(p, Q^j(x, \cdot); \hat{\eta})}{j + n_+(p, Q^j(x, \cdot); \hat{\eta})} \\ &= \max_{1 \leq j \leq m-1} \frac{n_+(p, Q^j(y^0, \cdot); \hat{\eta})}{j + n_+(p, Q^j(x^0, \cdot); \hat{\eta})}, \end{aligned}$$

where $n_+(p, q; \hat{\eta}) = \max\{n(p, q; \hat{\eta}), 0\}$. We note that $0 < \sigma < 1$. By translation we may assume that $y^0 = 0$. Then there is $t_0 > 0$ such that the Cauchy problem $(\mathrm{CP})_t$ is C^∞ well-posed for any $t > -t_0$. Let $l^\mu(x, \xi)$ $(1 \leq \mu \leq r)$ be as in Lemma 2.4, and write $l^\mu(x, \xi) \sim \sum_{j=0}^{\infty} l^{\mu,j}(x, \xi)$ $(1 \leq \mu \leq r)$, where $l^{\mu,j}(x, \xi)$ is positively homogeneous of degree $m_\mu - j$ $(j \in \mathbb{Z}_+)$. We may also assume that $l^{1,0}(\eta^0) = 0$. Then we have $l^{\mu,0}(\eta^0) \neq 0$ for $2 \leq \mu \leq r$. Recall that

$$Q^j(x, \hat{\eta}(s) - i\vartheta) = \sum_{j_k \geq 0,\ j_1 + \cdots + j_r = j} l^{1,j_1}(x, \hat{\eta}(s) - i\vartheta) \cdots l^{r,j_r}(x, \hat{\eta}(s) - i\vartheta) \tag{4.7}$$

$(j \in \mathbb{N})$.

Lemma 4.4. *Modifying $\hat{\eta} \in N_{\eta^0}$ if necessary, we have*

$$\begin{aligned} \sigma &= \max_{\substack{x \in V,\, x_1 \geq -t_0 \\ \xi \in \mathbb{R}^n,\, j \geq 1}} \frac{n_+(l^{1,0}, l^{1,j}(x, \cdot); \hat{\eta} + \xi)}{j + n_+(l^{1,0}, l^{1,j}(x, \cdot); \hat{\eta} + \xi)} \\ &= \max_{1 \leq j \leq m-1} \frac{n_+(l^{1,0}, l^{1,j}(0, \cdot); \hat{\eta})}{j + n_+(l^{1,0}, l^{1,j}(0, \cdot); \hat{\eta})}, \end{aligned} \tag{4.8}$$

and

$$n(l^{1,0}, l^{1,j}(0, \cdot); \hat{\eta}) \geq n(l^{1,0}, l^{1,j}(0, \cdot); \hat{\eta}_a^\lambda) \tag{4.9}$$

if $\sigma = n_+(l^{1,0}, l^{1,j}(0, \cdot); \hat{\eta})/(j + n_+(l^{1,0}, l^{1,j}(0, \cdot); \hat{\eta}))$, $0 \leq \lambda < 1$, $a \in \mathbb{R}$ and $1 \leq j \leq m_1 - 1$, where $\hat{\eta}_a^\lambda(s) = s^{\lambda/(1-\lambda)} \hat{\eta}(s^{1/(1-\lambda)}) + a\vartheta$.

Proof. From (4.7) we have

$$n(p, Q^j(x,\ \cdot\ ); \hat{\eta}) \le \max_{1\le k\le j}(n(l^{1,0}, l^{1,k}(x,\ \cdot\ ); \hat{\eta}) + k - j)$$

$(1 \le j \le m-1)$. This yields

$$\begin{aligned}\sigma &\le \max_{1\le j\le m-1}\ \max_{1\le k\le j} \frac{n_+(l^{1,0}, l^{1,k}(0,\ \cdot\ ); \hat{\eta})}{k + n_+(l^{1,0}, l^{1,k}(0,\ \cdot\ ); \hat{\eta})} \\ &\le \max_{1\le k\le m_1-1} \frac{n_+(l^{1,0}, l^{1,k}(0,\ \cdot\ ); \hat{\eta})}{k + n_+(l^{1,0}, l^{1,k}(0,\ \cdot\ ); \hat{\eta})} \equiv \hat{\sigma},\end{aligned}$$

since $n(l^{1,0}, l^{1,k}(x,\ \cdot\ ); \hat{\eta}) \le 0$ for $k \ge m_1$. Put

$$k_0 = \max\left\{k \in \mathbb{N};\, k \le m_1 - 1 \text{ and } \frac{n_+(l^{1,0}, l^{1,k}(0,\ \cdot\ ); \hat{\eta})}{k + n_+(l^{1,0}, l^{1,k}(0,\ \cdot\ ); \hat{\eta})} = \hat{\sigma}\right\}.$$

Then there is $c > 0$ such that

$$\begin{aligned}\left|\frac{Q^{k_0}(0, \hat{\eta}(s) - i\vartheta)}{p(\hat{\eta}(s) - i\vartheta)}\right| &\ge \left|\frac{l^{1,k_0}(0, \hat{\eta}(s) - i\vartheta)}{l^{1,0}(\hat{\eta}(s) - i\vartheta)}\right| \\ &\quad - \sum_{\substack{0\le\mu_1<k_0\\ \mu_1+\cdots+\mu_r=k_0}} \prod_{\nu=1}^{r}\left|\frac{l^{\nu,\mu_\nu}(0, \hat{\eta}(s) - i\vartheta)}{l^{\nu,0}(\hat{\eta}(s) - i\vartheta)}\right| \\ &\ge cs^{-k_0\hat{\sigma}/(1-\hat{\sigma})} \quad \text{for} \quad 0 < s \ll 1,\end{aligned}$$

since $n(l^{1,0}, l^{1,k_0}(0,\ \cdot\ ); \hat{\eta}) = k_0\hat{\sigma}/(1-\hat{\sigma}) > k\hat{\sigma}/(1-\hat{\sigma}) - (k_0 - k) \ge n(l^{1,0}, l^{1,k}(0,\ \cdot\ ); \hat{\eta}) - (k_0 - k)$ for $0 \le k < k_0$. This gives (4.8). (4.9) can be proved by the same argument as in the proof of Lemma 1.2.4 of [26]. □

Let $\sigma_j > 0$ $(j = 1, 2)$ satisfy $0 < \sigma_j < \sigma$, and define

$$l(y, \eta; s) = l(s^{\sigma_1}y_1, s^{\sigma_2}y', s^{-\sigma_1}\eta_1, s^{-\sigma_2}\eta').$$

We shall determine σ_1 and σ_2 later. It is obvious that

$$(l(x, D_x)u)(s^{\sigma_1}y_1, s^{\sigma_2}y') = l(y, D_y; s)u_s(y),$$

where $u_s(y) = u(s^{\sigma_1}y_1, s^{\sigma_2}y')$. The following lemma easily follows from Lemma 4.2.

Lemma 4.5. *For any open bounded neighborhood W of the origin in $\mathbb{R}^n$ there are $s(W) > 0$, $C_M \equiv C_M(W) > 0$ $(M \in \mathbb{Z}_+)$ and $p_j \in \mathbb{Z}_+$ $(j = 1, 2, 3)$ such that $W \Subset$*

$\{x \in \mathbb{R}^n; (s(W)^{\sigma_1}x_1, s(W)^{\sigma_2}x') \in V$ *and* $s(W)^{\sigma_1}x_1 > -t_0\}$ *and*

(4.10)

$$|u(0)| \le s^{a(\sigma_1,\sigma_2)} C_M \max_{y_1 \le 0} \Big\{ \sum_{|\alpha| \le p_1} s^{-\sigma_1\alpha_1 - \sigma_2|\alpha'|} \big| D_y^\alpha l^1(y, D_y; s) u(y) \big| + \sum_{|\alpha| \le p_2} s^{-\sigma_1\alpha_1 - \sigma_2|\alpha'|} \| \langle s^{-\sigma_2} D_{y'} \rangle^{-M} D_y^\alpha u(y_1, y') \|(y_1) + \sum_{|\alpha| \le p_3} s^{-\sigma_1\alpha_1 - \sigma_2|\alpha'|} \| (1 - \Psi_\pm(D_{y'}; s)) D_y^\alpha u(y_1, y') \|(y_1) \Big\}$$

for $u \in C_0^\infty(\mathbb{R}^n)$ *with* $\operatorname{supp} u \subset W$, $0 < s \le s(W)$ *and* $M \in \mathbb{Z}_+$, *where* $a(\sigma_1, \sigma_2) = (\sigma_1 + (n-1)\sigma_2)/2$, $\Psi_\pm(\eta'; s) = \Psi_\pm(s^{-\sigma_2}\eta')$ *and* $\| f(y) \|(y_1) = \left(\int | f(y_1, y')|^2 \, dy' \right)^{1/2}$.

Let $N \in \mathbb{N}$ and $\gamma \in \mathbb{R} \setminus \{0\}$, and put

$$P_N(x, \xi; s; \gamma) = \sum_{j=0}^{N-1} \sum_{|\alpha| \le n(N-j;\sigma)} \frac{s^{-m_1+j+(1-\sigma)|\alpha|} \gamma^{-m_1+j+|\alpha|} l^{1,j(\alpha)}(x, s\eta(s))\xi^\alpha}{\alpha!},$$
$$R^N(x, \xi; s; \gamma) = l^1(x, \gamma^{-1}\eta(s) + s^{-\sigma}\xi) - P_N(x, \xi; s; \gamma),$$

where $\eta(s) = s^{-\sigma}\hat{\eta}(s^{1-\sigma})$ and $n(N; \sigma) = [N/(1-\sigma)]$. Then $R^N(x, \xi; s; \gamma) \equiv \sum_{j=0}^{m_1} R^{N,j}(x, \xi'; s; \gamma)\xi_1^j$ satisfies

$$\begin{aligned} &\big| R_{(\beta)}^{N,j(\alpha')}(x, \xi'; s; \gamma) \big| \\ &\le C_{N,\alpha',\beta}(\gamma) \{ s^{-\sigma j - \sigma|\alpha'|}(s^{-1} + s^{-\sigma}|\xi'|)^{m_1 - j} \\ &\qquad + s^{-m_1}\langle \xi' \rangle^{-j-|\alpha'|}(1 + s^N \langle \xi' \rangle^{N/(1-\sigma)}) \}. \end{aligned} \tag{4.11}$$

Choose $c_0 > 0$ and $s_0 > 0$ so that $\eta^{0\prime} + \xi' \in \mathscr{C}'$ for $\xi' \in \mathbb{R}^{n-1}$ with $|\xi'| \le c_0$ and $\left| \sum_{j=1}^\infty s^{(1-\sigma)j/l} \eta^j \right| \le c_0/4$ for $0 < s \le s_0$. For $< s \le s_0$ and $\xi' \in \mathbb{R}^{n-1}$ with $|\xi'| \le c_0|\gamma^{-1}s^{\sigma-1}|/4$ we write

$$R^N(x, \xi; s; \gamma) = R_1^N(x, \xi; s; \gamma) + R_2^N(x, \xi; s; \gamma),$$

where

$$R_1^N(x, \xi; s; \gamma) = l^1(x, \gamma^{-1}\eta(s) + s^{-\sigma}\xi) - \sum_{j=0}^{N-1} l^{1,j}(x, \gamma^{-1}\eta(s) + s^{-\sigma}\xi),$$
$$R_2^N(x, \xi; s; \gamma) = \sum_{j=0}^{N-1} l^{1,j}(x, \gamma^{-1}\eta(s) + s^{-\sigma}\xi) - P_N(x, \xi; s; \gamma).$$

Then it is easy to see that

$$\left|R^{N,j(\alpha')}_{1(\beta)}(x,\xi';s;\gamma)\right| \le C_{N,\alpha',\beta}(\gamma)s^{N-m_1+(1-\sigma)(j+|\alpha'|)}, \tag{4.12}$$

$$\left|R^{N,j(\alpha')}_{2(\beta)}(x,\xi';s;\gamma)\right| \le C_{N,\alpha',\beta}(\gamma)s^{N-m_1}\langle\xi'\rangle^{N/(1-\sigma)+1} \tag{4.13}$$

if $|\xi'| \le c_0|\gamma^{-1}s^{\sigma-1}|/4$, $0 < s \le s_0$ and $0 \le j \le m_1$, where $R^N_k(x,\xi;s;\gamma) = \sum_{j=0}^{m_1} R^{N,j}_k(x,\xi';s;\gamma)\xi_1^j$ $(k = 1, 2)$. Write

$$l^{1,j(\alpha)}(x, s\eta(s)) = s^{\mu_{j,\alpha}}(C_{j,\alpha}(x) + o(1)) \quad \text{as} \quad s \downarrow 0, \quad C_{j,\alpha}(x) \not\equiv 0$$

for $x \in V$ with $x_1 \ge -t_0$, if $l^{1,j(\alpha)}(x, s\eta(s)) \not\equiv 0$ in (x,s) for $x \in V$ with $x_1 \ge -t_0$. Note that $C_{0,\alpha}(x) \equiv C_{0,\alpha}$. We put $C_{j,\alpha}(x) \equiv 0$ and $\mu_{j,\alpha} = \infty$ if $l^{1,j(\alpha)}(x, s\eta(s)) \equiv 0$ in (x,s) for $x \in V$ with $x_1 \ge -t_0$. Moreover, we put $\mu_j = \max_\alpha(m_1 - j - \mu_{j,\alpha} - (1-\sigma)|\alpha|)$. Then we can write

$$\begin{aligned}
&l^{1,0}(\hat\eta(s) + \xi) = s^{-(\mu_0 - \sigma m_1)/(1-\sigma)}(c_0(\xi) + o(1)),\\
&l^{1,j}(x, \hat\eta(s) + \xi) = s^{-(\mu_j - \sigma(m_1 - j))/(1-\sigma)}(c_j(x,\xi) + o(1)),\\
&\text{if} \quad j \ge 1, \quad \mu_j > -\infty, \quad (x,\xi) \in V \times \mathbb{R}^n \quad \text{and} \quad x_1 \ge -t_0
\end{aligned}$$

as $s \downarrow 0$, where $c_0(\xi) \not\equiv 0$ and $c_j(x,\xi) \not\equiv 0$ in $(x,\xi) \in V \times \mathbb{R}^n$ with $x_1 \ge -t_0$. Then we have

$$\begin{aligned}
c_0(\xi) &= \sum_{m_1 - \mu_{0,\alpha} - (1-\sigma)|\alpha| = \mu_0} \frac{C_{0,\alpha}\xi^\alpha}{\alpha!},\\
c_j(x,\xi) &= \sum_{m_1 - j - \mu_{j,\alpha} - (1-\sigma)|\alpha| = \mu_j} \frac{C_{j,\alpha}(x)\xi^\alpha}{\alpha!}
\end{aligned}$$

for $j \ge 1$ with $\mu_j > -\infty$ and $(x,\xi) \in V \times \mathbb{R}^n$ with $x_1 \ge -t_0$. From (4.8) it follow that

$$\begin{aligned}
0 &= \max_{\substack{x\in V,\, x_1 \ge -t_0\\ \xi\in\mathbb{R}^n,\, j\ge 1}} \left\{ n(l^{1,0}, l^{1,j}(x,\ \cdot\ ); \hat\eta + \xi) - \frac{\sigma j}{1-\sigma} \right\}\\
&= \max_{1\le j<m_1,\, \mu_j > -\infty} \frac{\mu_j - \mu_0}{1-\sigma}.
\end{aligned}$$

This yields

$$\mu_0 = \max_{1\le j\le m_1 - 1} \mu_j \quad \left(= \max_{j\ge 1} \mu_j\right).$$

Put

$$T_\gamma(x,\xi) = \gamma^{-m_1}c_0(\gamma\xi) + \sum_{\substack{1\le j\le m_1-1\\ \mu_j = \mu_0}} \gamma^{-m_1+j}c_j(x,\gamma\xi)$$

for $(x,\xi) \in V \times \mathbb{R}^n$ with $x_1 \geq -t_0$. By assumption we have $T_\gamma(0,\xi) \not\equiv 0$ in ξ. Choose $N \in \mathbb{N}$ so that $N > (1-\sigma)m_1$, which yields $N > m_1 - \mu_0$. Then there are a positive constant $\delta_0 \in \mathbb{Q}$ and a polynomial $r_N(x,\xi;s;\gamma)$ of (ξ,γ) such that

$$P_N(x,\xi;s;\gamma) = s^{-\mu_0}(T_\gamma(x,\xi) + s^{\delta_0} r_N(x,\xi;s;\gamma)), \tag{4.14}$$

$$|r_{N,\alpha(\beta)}(x;s;\gamma)| \leq C_{N,\alpha,\beta}(\gamma) s^{((1-\sigma)|\alpha|+\mu_0-m_1-\delta_0)_+} \tag{4.15}$$

for $(x,\xi) \in V \times \mathbb{R}^n$ with $x_1 \geq -t_0$, $0 < s \leq s_0$ and $\alpha, \beta \in (\mathbb{Z}_+)^n$ with $|\alpha| \leq [N/(1-\sigma)]$, where $r_N(x,\xi;s;\gamma) = \sum_{|\alpha| \leq [N/(1-\sigma)],\, \alpha_1 \leq m_1} r_{N,\alpha}(x;s;\gamma)\xi^\alpha$. From the arguments as in the proof of Theorem 1.2.5 of [26] there are $\zeta^{0\prime} \in \mathbb{R}^{n-1}$, a neighborhood U of $\zeta^{0\prime}$, $\gamma \in \mathbb{R} \setminus \{0\}$ and $q \in \mathbb{N}$ such that the equation $T_\gamma(0,\tau,\eta') = 0$ in τ has a root with negative imaginary part for $\eta' \in U$ and its multiplicity is equal to q for $\eta' \in U$ (see, also, §§2.2 and 2.3 of [26]). We fix γ as above. Then there are real analytic functions $\tau(\eta')$ and $\widetilde{T}(\tau,\eta')$ defined for $\eta' \in U$ such that $\widetilde{T}(\tau,\eta')$ is a polynomial of τ, $\widetilde{T}(\tau(\eta'),\eta') \neq 0$, $\operatorname{Im}\tau(\eta') < 0$ and $T_\gamma(0,\tau,\eta') = (\tau - \tau(\eta'))^q \widetilde{T}(\tau,\eta')$ for $\eta' \in U$. Let $\varphi(x)$ be a solution of

$$\frac{\partial \varphi}{\partial x_1} = \tau(\nabla_{x'}\varphi(x)), \quad \varphi(0,x') = x' \cdot \zeta^{0\prime} + i|x'|^2$$

in a neighborhood V_1 of 0 in V. We choose σ_0, σ_1 and σ_2 so that $0 < \sigma_0 < \sigma/3$, $\sigma_1 = \sigma - \sigma_0$ and $\sigma_2 = \sigma - 2\sigma_0$. We shall impose further conditions on σ_0. It follows from §3 of Chapter VI of [24] that

$$\begin{aligned} &T_\gamma(0, s^{3\sigma_0}D)(\exp[is^{-3\sigma_0}\varphi(x)]u(x)) \\ &= \exp[is^{-3\sigma_0}\varphi(x)] \sum_{|\alpha| \geq q} T_\gamma^{(\alpha)}(0, \nabla\varphi(x))\mathfrak{N}_\alpha(x,D;s)u(x), \end{aligned}$$

where $\Phi(x,y) = \varphi(y) - \varphi(x) - (y-x)\cdot\nabla\varphi(x)$ and

$$\mathfrak{N}_\alpha(x,D;s)u(x) = \frac{1}{\alpha!} s^{3\sigma_0|\alpha|}\big[D_y^\alpha(\exp[is^{-3\sigma_0}\Phi(x,y)]u(y))\big]_{y=x}$$

(see, also, Lemma 3.1 of [2]). It is easy to see that

$$\begin{aligned} &\mathfrak{N}_\alpha(x,\xi;s) = \frac{s^{3\sigma_0|\alpha|}\xi^\alpha}{\alpha!} + \sum_{\beta<\alpha} s^{3\sigma_0|\beta|} b_{\alpha,\beta}(x;s)\xi^\beta, \\ &|b_{\alpha,\beta(\beta^1)}(x;s)| \leq C_{\alpha,\beta,\beta^1} s^{3\sigma_0(|\alpha|-|\beta|-[(|\alpha|-|\beta|)/2])}, \\ &b_{\alpha,\beta}(x;s) \equiv 0 \quad \text{if} \quad |\alpha|-|\beta| = 1 \quad \text{and} \quad \beta<\alpha \end{aligned}$$

for $x \in V_1$. Put $\varphi(y;s) = \varphi(s^{2\sigma_0}y_1, s^{\sigma_0}y')$. Then a simple calculation yields

$$(4.16)\quad \begin{aligned} &T_\gamma(0, s^{\sigma_0}D_{y_1}, s^{2\sigma_0}D_{y'})(\exp[is^{-3\sigma_0}\varphi(y;s)]u(y)) \\ &= \left[T_\gamma(0, s^{3\sigma_0}D_x)(\exp[is^{-3\sigma_0}\varphi(x)]u(s^{-2\sigma_0}x_1, s^{-\sigma_0}x'))\right]_{x_1=s^{2\sigma_0}y_1,\, x'=s^{\sigma_0}y'} \\ &= \exp[is^{-3\sigma_0}\varphi(y;s)] \sum_{|\alpha|\ge q} T_\gamma^{(\alpha)}(0, (\nabla\varphi)(s^{2\sigma_0}y_1, s^{\sigma_0}y')) \\ &\qquad \times \left\{ s^{\sigma_0(3|\alpha|-2\alpha_1-|\alpha'|)} \frac{D_y^\alpha u(y)}{\alpha!} \right. \\ &\qquad\quad \left. + \sum_{\beta<\alpha} s^{\sigma_0(3|\beta|-2\beta_1-|\beta'|)} b_{\alpha,\beta}(s^{2\sigma_0}y_1, s^{\sigma_0}y';s) D_y^\beta u(y) \right\} \\ &= s^{\sigma_0 q} \exp[is^{-3\sigma_0}\varphi(y;s)] \left\{ T_\gamma^{(qe_1)}(0, \tau(\zeta^{0\prime}), \zeta^{0\prime}) \frac{D_{y_1}^q u(y)}{q!} \right. \\ &\qquad \left. + s^{\sigma_0} \sum_{|\alpha|\le q_0} c_\alpha(y;s) D_y^\alpha u(y) \right\}, \\ &|c_{\alpha(\beta)}(y;s)| \le C_{\alpha,\beta} \end{aligned}$$

for $y \in \mathbb{R}^n$ with $(s^{2\sigma_0}y_1, s^{\sigma_0}y') \in V_1$, where $q_0 = \deg_\xi T_\gamma(0,\xi)$. Put

$$E(y;s) = \exp[i\gamma^{-1}(s^{\sigma_1}y_1\eta_1(s) + s^{\sigma_2}y'\cdot\eta'(s))],$$

where $\eta(s) = (\eta_1(s), \eta'(s))$. Let W be an open bounded neighborhood of 0 in $\mathbb{R}^n$. Then we have

$$(4.17)\quad \begin{aligned} &E(y;s)^{-1} l^1(y, D_y;s)(E(y;s)\exp[is^{-3\sigma_0}\varphi(y;s)]u(y)) \\ &= l^1(y, \gamma^{-1}s^{\sigma_1}\eta_1(s) + D_{y_1}, \gamma^{-1}s^{\sigma_2}\eta'(s) + D_{y'};s)(\exp[is^{-3\sigma_0}\varphi(y;s)]u(y)) \\ &= \{P_N(s^{\sigma_1}y_1, s^{\sigma_2}y', s^{\sigma_0}D_{y_1}, s^{2\sigma_0}D_{y'};s;\gamma) \\ &\qquad + R^N(s^{\sigma_1}y_1, s^{\sigma_2}y', s^{\sigma_0}D_{y_1}, s^{2\sigma_0}D_{y'};s;\gamma)\}(\exp[is^{-3\sigma_0}\varphi(y;s)]u(y)) \end{aligned}$$

for $u \in C_0^\infty(W)$ and $0 < s \le s(W)$, where $s(W)$ is a positive constant satisfying $\{(s^{2\sigma_0}y_1, s^{\sigma_0}y');\, y \in W,\ 0 < s \le s(W)\} \subset V_1 \cap \{x \in \mathbb{R}^n; x_1 \ge -t_0\}$. Now we take $\sigma_0 = \min\{\delta_0/(1+q), \sigma/(3+q)\}$. From (4.14)–(4.16) we have

$$\begin{aligned} &P_N(s^{\sigma_1}y_1, s^{\sigma_2}y', s^{\sigma_0}D_{y_1}, s^{2\sigma_0}D_{y'};s;\gamma)(\exp[is^{-3\sigma_0}\varphi(y;s)]u(y)) \\ &= s^{-\mu_0+\sigma_0 q}\exp[is^{-3\sigma_0}\varphi(y;s)]\left\{a_0 D_{y_1}^q u(y) - s^{\sigma_0}H_N(y, D_y;s)u(y)\right\}, \end{aligned}$$

where $a_0 = T_\gamma^{(qe_1)}(0, \tau(\zeta^{0\prime}), \zeta^{0\prime})/q!$ $(= \widetilde{T}(\tau(\zeta^{0\prime}), \zeta^{0\prime}) \ne 0)$ and $H_N(y,\eta;s) \equiv \sum_{|\alpha|\le[N/(1-\sigma)],\, \alpha_1\le m_1} H_{N,\alpha}(y;s)\eta^\alpha$ satisfies

$$|H_{N,\alpha,(\beta)}(y;s)| \le C_{N,\alpha,\beta} \quad \text{for} \quad y \in W \quad \text{and} \quad 0 < s \le s(W).$$

We define $\{u_{N,j}(y;s)\}_{0\le j\le[N/\sigma_0]}$ by

$$\begin{cases} u_{N,0}(y;s)=1, \\ D_{y_1}^q u_{N,j}(y;s)=a_0^{-1}H_N(y,D_y;s)u_{N,j-1}(y;s), \\ D_1^k u_{N,j}(0,y';s)=0 \quad (0\le k\le q-1), \\ \left(1\le j\le \left[\dfrac{N}{\sigma_0}\right]\right), \end{cases}$$

for $y\in W$ and $0<s\le s(W)$. Note that

$$\left|D_y^\alpha u_{N,j}(y;s)\right|\le C_{N,\alpha} \tag{4.18}$$

for $y\in W$, $0<s\le s(W)$, $0\le j\le[N/\sigma_0]$ and $\alpha\in(\mathbb{Z}_+)^n$. It is easy to see that

$$\varphi(y;s)=s^{\sigma_0}y'\cdot\zeta^{0\prime}+is^{2\sigma_0}|y'|^2+s^{2\sigma_0}\tau(\zeta^{0\prime})y_1+O(s^{3\sigma_0}) \quad \text{as} \quad s\downarrow 0, \tag{4.19}$$

$$\operatorname{Im} s^{-3\sigma_0}\varphi(y;s)\ge\frac{s^{-\sigma_0}(\operatorname{Im}\tau(\zeta^{0\prime})y_1+|y'|^2)}{2} \tag{4.20}$$

for $y\in W$ with $y_1\le 0$ and $0<s\le s(W)$, modifying $s(W)$ if necessary. Let $\chi(x)$ be a function in $C_0^\infty(W)$ such that $\chi(x)=1$ near 0, and put

$$u_N(y;s)=\sum_{j=0}^{[N/\sigma_0]}s^{\sigma_0 j}u_{N,j}(y;s)\chi(y).$$

Then, by standard arguments we have

$$\begin{aligned} \max_{y_1\le 0,\,|\alpha|\le p_1} s^{-\sigma_1\alpha_1-\sigma_2|\alpha'|}\big|D_y^\alpha\{E(y;s)P_N(s^{\sigma_1}y_1,s^{\sigma_2}y',s^{\sigma_0}D_{y_1},s^{2\sigma_0}D_{y'};s;\gamma) \\ \times(\exp[is^{-3\sigma_0}\varphi(y;s)]u_N(y;s))\}\big|\le C_N s^{N-\mu_0+\sigma_0 q-p_1}. \end{aligned} \tag{4.21}$$

From (4.18)–(4.20) it follows that for $k,l\in\mathbb{Z}_+$ and $y_1\le 0$

$$\left|D_1^k\mathscr{F}_{y'}[\exp[is^{-3\sigma_0}\varphi(y;s)]u_N(y;s)](\eta')\right|\le C_{N,k,l}s^{-\sigma_0 k}\langle s^{2\sigma_0}\eta'\rangle^{-l}, \tag{4.22}$$

where $\mathscr{F}_{y'}[u](\eta')=\int_{\mathbb{R}^{n-1}}e^{-iy'\cdot\eta'}u(y)\,dy'$. Indeed, we have

$$\begin{aligned} &\left|D_1^k(\eta')^{\alpha'}\mathscr{F}_{y'}[\exp[is^{-3\sigma_0}\varphi(y;s)]w(y)](\eta')\right| \\ &\le\int\left|D_1^kD_{y'}^{\alpha'}\exp[is^{-3\sigma_0}\varphi(y;s)]w(y)\,dy'\right| \\ &\le C_{k,\alpha'}(w)s^{-\sigma_0 k-2\sigma_0|\alpha'|} \quad \text{for} \quad w\in C_0^\infty(\mathbb{R}^n). \end{aligned}$$

Choose $\psi(\xi') \in C_0^\infty(\mathbb{R}^{n-1})$ so that $\psi(\xi') = 1$ for $|\xi'| \le c_0/(8|\gamma|)$ and $\psi(\xi') = 0$ for $|\xi'| \ge c_0/(4|\gamma|)$. From (4.11)–(4.13) we have

$$\begin{aligned}&\left|D_y^\beta R^{N,j}(s^{\sigma_1}y_1, s^{\sigma_2}y', s^{2\sigma_0}\eta'; s; \gamma)\psi(s^{1-\sigma_2}\eta')\right|\\ &\le C_{N,\beta}s^{N-m_1}\langle s^{2\sigma_0}\eta'\rangle^{N/(1-\sigma)+1},\\ &\left|D_y^\beta R^{N,j}(s^{\sigma_1}y_1, s^{\sigma_2}y', s^{2\sigma_0}\eta'; s; \gamma)(1-\psi(s^{1-\sigma_2}\eta'))\right|\\ &\le C_{N,\beta}s^{N-m_1}\langle s^{2\sigma_0}\eta'\rangle^{m_1-j+N/(1-\sigma)},\end{aligned}$$

since $Cs^{1-\sigma}\langle s^{2\sigma_0}\eta'\rangle \ge 1$ if $1-\psi(s^{1-\sigma_2}\eta') \ne 0$. This, together with (4.22), yields

(4.23)
$$\begin{aligned}\max_{y_1\le 0,\,|\alpha|\le p_1} s^{-\sigma_1\alpha_1-\sigma_2|\alpha'|}\Big|D_y^\alpha\{E(y;s)R^N(s^{\sigma_1}y_1, s^{\sigma_2}y', s^{\sigma_0}D_{y_1}, s^{2\sigma_0}D_{y'}; s; \gamma)\\ \times(\exp[is^{-3\sigma_0}\varphi(y;s)]u_N(y;s))\}\Big| \le C_{N,p_1}s^{N-m_1-2\sigma_0 n-p_1},\end{aligned}$$

since $\langle s^{2\sigma_0}\eta'\rangle^{-n} \le s^{-2\sigma_0 n}\langle\eta'\rangle^{-n}$ if $0 < s \le 1$. It is obvious that

$$\begin{aligned}&\langle s^{-\sigma_2}D_{y'}\rangle^{-M}D_y^\alpha(E(y;s)\exp[is^{-3\sigma_0}\varphi(y;s)]u_N(y;s))\\ &= E(y;s)(\gamma^{-1}s^{\sigma_1}\eta_1(s)+D_{y_1})^{\alpha_1}\langle\gamma^{-1}\eta'(s)+s^{-\sigma_2}D_{y'}\rangle^{-M}\\ &\quad\times(\gamma^{-1}s^{\sigma_2}\eta'(s)+D_{y'})^{\alpha'}(\exp[is^{-3\sigma_0}\varphi(y;s)]u_N(y;s)),\\ &|\langle\gamma^{-1}\eta'(s)+s^{-\sigma_2}\eta'\rangle^{-M}(\gamma^{-1}s^{\sigma_2}\eta'(s)+\eta')^{\alpha'}\psi(s^{1-\sigma_2}\eta')|\\ &\le C_{M,\alpha'}s^{\sigma_2|\alpha'|+M-|\alpha'|},\\ &|\langle\gamma^{-1}\eta'(s)+s^{-\sigma_2}\eta'\rangle^{-M}(\gamma^{-1}s^{\sigma_2}\eta'(s)+\eta')^{\alpha'}(1-\psi(s^{1-\sigma_2}\eta'))|\\ &\le C_l s^{\sigma_2|\alpha'|+(1-\sigma)l}\langle s^{2\sigma_0}\eta'\rangle^l \quad (l\in\mathbb{Z}_+)\end{aligned}$$

if $M \ge |\alpha'|$. So, (4.22) gives

(4.24)
$$\begin{aligned}\max_{y_1\le 0,\,|\alpha|\le p_2} s^{-\sigma_1\alpha_1-\sigma_2|\alpha'|}\big\|\langle s^{-\sigma_2}D_{y'}\rangle^{-M}D_y^\alpha\\ \times(E(y;s)\exp[is^{-3\sigma_0}\varphi(y;s)]u_N(y;s))\big\|(y_1) \le C_{N,M,p_2}s\end{aligned}$$

if $M \ge p_2+\sigma_0 n+1$. Put $\varepsilon(\gamma) = +$ if $\gamma > 0$ and $\varepsilon(\gamma) = -$ if $\gamma < 0$. We modify $\mathscr{C}_0'$ so that $\eta^{0\prime} \in \mathscr{C}_0'$. Then $\Psi(\xi') = 1$ if ξ' belongs to a conic neighborhood of $\eta^{0\prime}$ and $|\xi'| \ge 1$. Since $\sigma = \sigma_2+2\sigma_0$, and $1 \le Cs^{1-\sigma_2}\langle\eta'\rangle$ and $\langle s^{2\sigma_0}\eta'\rangle^{-1} \le Cs^{-2\sigma_0}\langle\eta'\rangle^{-1}$ if $1-\Psi_{\varepsilon(\gamma)}(\gamma^{-1}\eta'(s)+s^{-\sigma_2}\eta') \ne 0$, by (4.22) we have

$$\begin{aligned}&\left|D_1^k(1-\Psi_{\varepsilon(\gamma)}(\gamma^{-1}\eta'(s)+s^{-\sigma_2}\eta'))(\gamma^{-1}s^{\sigma_2}\eta'(s)+\eta')^{\alpha'}\right.\\ &\left.\times\mathscr{F}_{y'}[\exp[is^{-3\sigma_0}\varphi(y;s)]u_N(y;s)](\eta')\right|\\ &\le C_{N,k,l}s^{-\sigma_0 k+(1-\sigma)l-2\sigma_0(|\alpha'|+n/2)}\langle\eta'\rangle^{-n/2}\end{aligned}$$

for $y_1 \le 0$ and $l \in \mathbb{Z}_+$. Therefore, we have

$$(4.25)\quad \max_{y_1 \le 0,\, |\alpha| \le p_3} s^{-\sigma_1\alpha_1-\sigma_2|\alpha'|}\|(1-\Psi_{\varepsilon(\gamma')}(D_{y'};s)) \times (E(y;s)\exp[is^{-3\sigma_0}\varphi(y;s)]u_N(y;s))\|(y_1) \le C_{N,p_3}s.$$

By (4.17) the estimates (4.21) and (4.23)–(4.25) with $N \ge \max\{\mu_0 + p_1 + 1 - \sigma_0 q, m_1 + 2\sigma_0 n + p_1 + 1\}$ and $M \ge p_2 + \sigma_0 n + 1$ contradict (4.10), since $E(0;s)\exp[is^{-3\sigma_0}\varphi(0;s)] \times u_N(0;s) = 1$. This proves the assertion (ii) of Theorem 1.1.

5. Some remarks and examples

First we remark that we can obtain the same results on propagation of singularities for operators satisfying the conditions (H), (R) and (L) as given in the colloray of Theorem 3.1 of [25], combining the arguments in this paper with results in [13].

In the case where the characterisic polynomial depends on x, we can also prove C^∞ well-posedness of the Cauchy problem under the maximal rank condition if for each $(x^0, \xi^{0\prime}) \in \mathbb{R}^n \times S^{n-2}$ the reduced operators $l^\mu(x, D)$ satisfy microlocal *a priori* estimates in [14], where the $l^\mu(x, D)$ are defined as in the form of (2.6).

EXAMPLE 5.1. Let $n = 3$, and let

$$L(x,\xi) = \begin{pmatrix} \xi_1 + b(x) & \xi_3 & -\xi_2 \\ a(x) & \xi_1 + \xi_2 & \xi_3 \\ 0 & -a(x) & \xi_1 \end{pmatrix}.$$

It is easy to see that $L(x,D)$ satisfies the conditions (H) and (R). For each $\xi^{0\prime} \in S^1$ with $\xi_3^0 \neq 0$ there are a conic neighborhood $\mathscr{C}'(\xi^{0\prime})$ of $\xi^{0\prime}$, $N_k(x,\xi) \in M_3\big(\mathscr{S}^2_{1,0}\big)$ $(k = 1, 2)$ and $l(x,\xi) \in \mathscr{S}^3_{1,0}$ such that the $N_k(x, D)$ have parametrices in $M_3\big(\mathscr{L}^2_{1,0}\big)$ and

$$\begin{aligned}
&N_1(x,D)L(x,D)N_2(x,D) \equiv \operatorname{diag}(1,1,l(x,D)) \bmod \mathscr{L}^{5,-\infty}_{1,0} \quad \text{in} \quad V \times \mathscr{C}'(\xi^{0\prime}),\\
&l(x,\xi)\\
&\equiv \xi_1^2(\xi_1+\xi_2) + b(x)\xi_1(\xi_1+\xi_2) + (a(x)b(x) - D_1a(x))\xi_3 + (D_1b(x))(2\xi_1+\xi_2)\\
&\quad + a(x)^2\xi_2 + (D_2b(x))\xi_1 + (D_1a(x))\xi_1\xi_2\xi_3^{-1} + (D_2a(x))\xi_1^2\xi_3^{-1}\\
&\quad - \Big\{a(x)b(x)\xi_1(\xi_1+\xi_2)\xi_2^2\xi_3^{-1} + 2\big((D_2b(x))\xi_2 + (D_3b(x))\xi_3\big)\xi_1(\xi_1+\xi_2)\\
&\qquad - \big((D_1a(x))(2\xi_1+\xi_2) - (D_2a(x))\xi_1\big)\xi_1\xi_2^2\xi_3^{-1} - 2(D_2a(x))\xi_1\xi_2\xi_3\\
&\qquad + (D_3a(x))\xi_1\big(\xi_2^2 - \xi_3^2\big)\Big\}\big(\xi_2^2+\xi_3^2\big)^{-1} \bmod S^0_{1,0} \quad \text{in} \quad V \times \mathscr{C}'(\xi^{0\prime}),
\end{aligned}$$

where V is a bounded open subset of $\mathbb{R}^3$. Let $\xi^{0\prime} \in S^1$ satisfy $\xi_2^0 \neq 0$. First we can reduce $L(x,\xi)$ to

$$\begin{pmatrix} \xi_1 + b(x) - a(x)\xi_2^{-1}\xi_3 & -\xi_2^{-1}\left(\xi_2^2 + \xi_3^2\right) - a(x)\xi_2^{-3}\xi_3^3 & 0 \\ 0 & \xi_1 + a(x)\xi_2^{-1}\xi_3 & 0 \\ 0 & 0 & \xi_1 + \xi_2 \end{pmatrix}$$

mod $S_{1,0}^{-1}$ in $V \times \mathscr{C}'(\xi^{0\prime})$ (block-diagonalization). Then, by elementary transformations we can transform $L(x, D)$ to

$$\operatorname{diag}\left(1, D_1^2 + b(x)D_1, D_1 + D_2\right) \bmod S_{1,0}^0 \quad \text{in} \quad V \times \mathscr{C}'(\xi^{0\prime}).$$

Therefore, it follows from Theorem 1.1 that the Cauchy problem $(\mathrm{CP})_t$ for $L(x, D)$ is C^∞ well-posed for any $t > 0$ if and only if $D_1 a(x) = a(x)b(x)$ for any $x \in \mathbb{R}^3$ with $x_1 \geq 0$. On the other hand, for a fixed $x \in \mathbb{R}^3$ $\det L(x,\xi)$ is hyperbolic with respect to ϑ, i.e., $\det L(x, \xi - i\vartheta) \neq 0$ for any $\xi \in \mathbb{R}^3$, if and only if $a(x)b(x) = 0$.

EXAMPLE 5.2. Let $n = 3$, and let

$$L(x,\xi) = \begin{pmatrix} \xi_1 + a(x) & \alpha(x)\xi_2 + \xi_3 + b(x) \\ c(x) & \xi_1 + \xi_2 + d(x) \end{pmatrix}.$$

Then $p(\xi) \equiv \det L_1(x,\xi) = \xi_1(\xi_1 + \xi_2)$ and $L(x, D)$ satisfies the conditions (H) and (R). Note that $(dp)(\xi_1, \xi^{0\prime}) \neq 0$ for any $\xi_1 \in \mathbb{R}$ if $\xi_2^0 \neq 0$, where $\xi^{0\prime} = \left(\xi_2^0, \xi_3^0\right) \in S^1$. Let $x^0 \in \mathbb{R}^3$ and $\xi^{0\prime} = (0, \pm 1) \in S^1$. Then there are a neighborhood $V(x^0)$ of x^0, a conic neighborhood $\mathscr{C}'(\xi^{0\prime})$ of $\xi^{0\prime}$, $N_k(x,\xi) \in M_2\left(\mathscr{S}_{1,0}^1\right)$ $(k = 1,2)$ and $l(x,\xi) \in \mathscr{S}_{1,0}^2$ such that the $N_k(x, D)$ have parametrices in $M_2\left(\mathscr{L}_{1,0}^1\right)$ and

$$\begin{aligned} &N_1(x, D)L(x, D)N_2(x, D) \equiv \operatorname{diag}(1, l(x, D)) \bmod \mathscr{L}_{1,0}^{3,-\infty} \quad \text{in} \quad V(x^0) \times \mathscr{C}'(\xi^{0\prime}), \\ &l(x,\xi) \equiv \xi_1(\xi_1 + \xi_2) + (a(x) + d(x))\xi_1 + a(x)\xi_2 \\ &\qquad + (D_1\alpha(x))\xi_2(\xi_1 + \xi_2)(\xi_3 + \alpha(x)\xi_2)^{-1} - c(x)\xi_3 - \alpha(x)c(x)\xi_2 \end{aligned}$$

mod $S_{1,0}^0$ in $V(x^0) \times \mathscr{C}'(\xi^{0\prime})$. Therefore, it follows from Theorem 1.1 that the Cauchy problem $(\mathrm{CP})_t$ for $L(x, D)$ is C^∞ well-posed for any $t > 0$ if and only if $c(x) = 0$ for $x \in \mathbb{R}^3$ with $x_1 \geq 0$. For $x^0 \in \mathbb{R}^3$ and $\xi^{0\prime} = (0, \pm 1) \in S^1$ we have

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 + \alpha(x)\xi_2\xi_3^{-1} \end{pmatrix} L_1(x,\xi) \begin{pmatrix} 1 & 0 \\ 0 & (1 + \alpha(x)\xi_2\xi_3^{-1})^{-1} \end{pmatrix} = \begin{pmatrix} \xi_1 & \xi_3 \\ 0 & \xi_1 + \xi_2 \end{pmatrix}$$

in a conic neighborhood of $(x^0, \xi^{0\prime})$. This implies that $L(x, D)$ can be reduced microlocally to a system with constant coefficient principal part. However, we can not directly apply the results of [29] to $L(x,D)$ since the lower order terms become pseudo-differential operators.

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