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Infinite speed of propagation and unique continuation for solutions of the drift-diffusion equation via Carleman estimates

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Abstract

Through the use of Carleman estimates, we investigate the drift-diffusion equation and demonstrate the infinite propagation of its solutions, considering both the parabolic-parabolic and parabolic-elliptic forms. A primary result is a *Unique Continuation Theorem*, which states that if a solution vanishes on a non-empty open set, it must vanish throughout the whole space $\mathbb{R}^n$. The proof relies on the *Two-Sphere One-Cylinder Inequality*, derived from the Carleman estimates. As a consequence, we show the infinite speed of propagation for solutions: even if the initial data has compact support, the solution corresponding to this initial data will instantly extend to cover the whole space $\mathbb{R}^n$. This result highlights a solution structure typical of linear diffusion equations of the heat type, where solutions exhibit immediate and global propagation.

Keywords: Drift-diffusion equation; Infinite speed of propagation; Carleman estimates; Unique continuation theorem; Two-sphere one-cylinder inequality

1. Introduction

We consider the drift-diffusion equation, which includes both parabolic–elliptic and parabolic–parabolic types in a semi-linear form posed on the whole space $\mathbb{R}^n$ with $n \geq 1$:

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$$
\text{(DD)} \quad \begin{cases} \dfrac{\partial u}{\partial t} = \Delta u + \chi \nabla \cdot (|u|^{q-2} u \nabla v) & \text{in } \mathbb{R}^n \times (0, T), \\ \tau \dfrac{\partial v}{\partial t} = \Delta v - \gamma v + u & \text{in } \mathbb{R}^n \times (0, T), \\ u(x, 0) = u_0(x), \ \tau v(x, 0) = \tau v_0(x) & \text{in } \mathbb{R}^n, \end{cases}
$$

where $q \geq 2$, $\gamma \geq 0$, $\chi = \pm 1$, and $\tau = 0$ or 1. Here, the functions $u = u(x,t)$ and $v = v(x,t)$ represent the density of the relevant quantity and the associated potential, respectively. The precise interpretation of the functions is determined by the modeling context, which may involve physical, biological, or other types of systems.

We investigate sign-changing solutions to (DD) that emerge from initial data with changing sign. The interest in such solutions stems from the case $\chi = +1$, under which (DD) exhibits structural features closely related to a specific parameter regime. In this setting, (DD) is partially derived from the system (S), referred to as the bi-polar drift-diffusion model, which is widely used in semiconductor simulation:

$$
\text{(S)} \quad \begin{cases} \dfrac{\partial n}{\partial t} - \Delta n + \nabla \cdot (n \nabla \psi) = 0 & \text{in } \mathbb{R}^n \times (0, T), \\ \dfrac{\partial p}{\partial t} - \Delta p - \nabla \cdot (p \nabla \psi) = 0 & \text{in } \mathbb{R}^n \times (0, T), \\ \tau \dfrac{\partial \psi}{\partial t} - \Delta \psi = n - p & \text{in } \mathbb{R}^n \times (0, T), \\ n(x, 0) = n_0(x), \ p(x, 0) = p_0(x) & \text{in } \mathbb{R}^n. \end{cases}
$$

By introducing the variable $u := n - p$, the system (S) reduces to (DD) when $q = 2$, $\gamma = 0$, and $\chi = +1$. This correspondence highlights the role of (DD) as a canonical model for capturing dynamics involving sign-changing behavior.

Both cases $\chi = \pm 1$ are of structural interest. In particular, the case $\chi = -1$ is also known to give rise to aggregation phenomena, despite the repulsive form of the drift term. The mechanism of self-organization in this case is more subtle and model-dependent, and the resulting solution behavior differs qualitatively from that observed in the case $\chi = +1$. Throughout this paper, we consider both signs of χ within a unified framework, with a particular focus on sign-changing solutions and their structural properties.

Our initial aim is to construct solutions to (DD) that originate from initial data exhibiting sign changes, with particular emphasis on those satisfying the following condition:

$$
\int_{\mathbb{R}^n} u_0(x)\, dx = \int_{\mathbb{R}^n} \big(n_0(x) - p_0(x)\big)\, dx \neq 0. \tag{1.1}
$$

This assumption reflects a non-vanishing total charge and plays a key role in the qualitative analysis of the solution structure. To clarify this point, we recall a classical principle from electromagnetism and semiconductor theory, known as the **Principle of Charge Conservation**:

- **Principle of Charge Conservation:** In an isolated system, the total net charge—defined as the sum of all positive and negative charges—remains invariant in time.

In view of this principle, it is natural to consider the case $\chi = +1$, where the total charge associated with the quantity $u := n - p$ is expected to be conserved. Under appropriate assumptions, this consideration leads to the identity:

$$\int_{\mathbb{R}^n} u(x,t)\,dx = \int_{\mathbb{R}^n} \big(n(x,t) - p(x,t)\big)\,dx = \int_{\mathbb{R}^n} \big(n_0(x) - p_0(x)\big)\,dx = \int_{\mathbb{R}^n} u_0(x)\,dx \neq 0, \quad t > 0.$$

In particular, if the initial net charge is nonzero, then:

$$\int_{\mathbb{R}^n} u(x,t)\,dx \neq 0, \quad t > 0. \tag{1.2}$$

This implies that $u(x,t)$ does not vanish identically in $\mathbb{R}^n$ at any positive time.

Building on this observation, the first objective of the present paper is to establish the *Unique Continuation Theorem* for solutions of the drift-diffusion equation. The theorem asserts that if the solution u vanishes at every point of a non-empty open set D_0, then it must vanish throughout the whole space $\mathbb{R}^n$.

The key to proving the *Unique Continuation Theorem* is the derivation of the *Two-Sphere One-Cylinder Inequality*, which is an application of the Carleman estimates. This inequality plays a crucial role in controlling the behavior of solutions within specific domains. Specifically, we derive the *Two-Sphere One-Cylinder Inequality* by introducing integrals over a cylindrical domain involving both space and time variables. This allows us to bound the $L^2(B_\rho(0))$ integral of the solution u over a sphere of radius ρ from above by the $L^2(B_r(0))$ integral of u over a sphere of radius r for $0 < r \leq \rho$. More precisely, the following holds: there exist constants $0 < \eta_1 < 1$ and $C \geq 1$ such that:

$$\int_{B_\rho(0)} u^2(x,0)\,dx \leq \frac{CR}{\rho}\Big(R^{-2}\int_{(0,R^2)}\int_{B_R(0)} u^2\,dxdt\Big)^{1-\theta_1}\Big(\int_{B_r(0)} u^2(x,0)\,dx\Big)^{\theta_1}$$

for all r, ρ, R with $0 < r \leq \rho \leq \eta_1 R$, where $\theta_1 = \big(C\log\frac{R}{r}\big)^{-1}$.

By shifting the center and applying the *Two-Sphere One-Cylinder Inequality* (for details, see Step 2 of the proof of Lemma 5.4), we can ensure that the solution u vanishes at a given time over the whole space $\mathbb{R}^n$ through the *Unique Continuation Theorem*. Specifically, we consider $u \in W^{2,1}_{2,\mathrm{loc}}(Q_T)$ and fix an arbitrary $\widehat{T} \in (0,T)$. To control lower–order contributions, we further assume that there exists a positive constant M such that

$$\Big|\Delta u(x,t) - \partial_t u(x,t)\Big| \leq M\big(|\nabla u(x,t)| + |u(x,t)|\big) \quad \text{for a.e. } (x,t) \in \mathbb{R}^n \times (0,\widehat{T}).$$

Consequently, if there exists an open set $D_0 \subset \mathbb{R}^n$ such that $u(\cdot,\widehat{T}) \equiv 0$ in D_0, then the solution u must vanish identically throughout $\mathbb{R}^n$ at time $\widehat{T}$.

Once this unique continuation property is established, it enables the proof of *Identical Vanishing for the Backward Problem*. In particular, assuming a function satisfies the following two conditions: (1) its growth in the spatial direction is limited to exponential, and (2) when acted upon by a parabolic operator, the function is bounded above by a constant multiple of the sum of its zeroth and first derivatives. Under these conditions, we demonstrate that if the function vanishes identically everywhere in $\mathbb{R}^n$ at some time $\widehat{T}$, then it must remain identically zero throughout $\mathbb{R}^n$ for all times from the initial time up to $\widehat{T}$.

Based on the above (1.2), the second objective of this paper is to establish the property of infinite speed of propagation for solutions of the drift-diffusion equation. This property underpins the main theorem presented in this paper and is derived as a consequence of the *Identical Vanishing for the Backward Problem*. More precisely, the infinite speed of propagation refers to the phenomenon where the support of the solution u instantaneously spreads across the whole space $\mathbb{R}^n$ when the initial data is a non-trivial function with compact support. Thus, the *Unique Continuation Theorem*, along with the *Identical Vanishing for the Backward Problem*, plays a fundamental role in revealing the structure of the solutions to the equation.

To provide further clarification, we explore the Carleman estimate. Initially introduced by Carleman [2], the Carleman estimate is a fundamental tool in proving unique continuation for two-dimensional elliptic equations. A detailed explanation of the Carleman estimate can be found in [27] and [28]. In addition to the works by Carleman [2] and Vessella [26], many studies have employed Carleman estimates to prove unique continuation. For instance, Koch and Tataru [10] established it for elliptic equations in higher dimensions, while Escauriaza and Fernández [7] extended it to parabolic operators. Further relevant works are referenced in Banerjee and Manna [1]. Beyond their role in demonstrating unique continuation properties, the Carleman estimates have applications in various fields of inverse problems, including control theory and stability estimates.

2. Results

In what follows, we introduce the following simplified notations:

(1). $B_r(a) := \{x \in \mathbb{R}^n \mid |x-a| < r,\ r > 0, a \in \mathbb{R}^n\}$, $B_r := B_r(0)$.

(2). $Q_T := \mathbb{R}^n \times (0,T)$.

(3). $\partial_i = \frac{\partial}{\partial x_i}$, $\partial_{ij}^2 = \partial_i\partial_j$, $\partial_{ijk}^3 = \partial_i\partial_j\partial_k$, $\nabla^2 = (\partial_{11}^2, \partial_{12}^2, \dots)$, $\nabla^3 = (\partial_{111}^3, \partial_{112}^3, \dots)$, $\partial_t = \frac{\partial}{\partial t}$, $i, j, k = 1, 2, \dots, n$.

(4). Let D be a domain in $\mathbb{R}^n$ and let I be an interval in $(0,T)$. When the weak derivatives ∇u, $\nabla^2 u$ and $\partial_t u$ are in $L^p(D\times I)$ for some $1 \le p \le \infty$, we say $u \in W_p^{2,1}(D\times I)$. Specifically, this means:

$$W_p^{2,1}(D\times I) := \{u \in L^p(I; W^{2,p}(D)) \cap W^{1,p}(I; L^p(D)) \mid \|u\|_{W_p^{2,1}(D\times I)} < \infty\},$$

where the norm is defined as:

$$\|u\|_{W_p^{2,1}(D\times I)} := \|u\|_{L^p(I;W^{2,p}(D))} + \|u\|_{W^{1,p}(I;L^p(D))}.$$

Similarly, when u and its weak derivatives ∇u, $\nabla^2 u$ and $\partial_t u$ belong to $L^p_{\mathrm{loc}}(Q_T)$ for some $1 \le p \le \infty$, we say $u \in W^{2,1}_{p,\mathrm{loc}}(Q_T)$, which is defined as:

$$W^{2,1}_{p,\mathrm{loc}}(Q_T) := L^p(0,T; W^{2,p}_{\mathrm{loc}}(\mathbb{R}^n)) \cap W^{1,p}(0,T; L^p_{\mathrm{loc}}(\mathbb{R}^n)).$$

(5). For $T > 0$ and $q \geq 2$, we define the function space $W(Q_T)$ as follows:

$$W(Q_T) := \begin{cases} W^{2,1}_{n+q}(Q_T) \times L^{n+q}(0,T; W^{2,n+q}(\mathbb{R}^n)) & \text{for } \tau = 0, \\ W^{2,1}_{n+q}(Q_T) \times W^{2,1}_{n+q}(Q_T) & \text{for } \tau = 1 \end{cases}$$

(6). For $T > 0$ and $q \geq 2$, we define the function space X_T as follows:

$$X_T := \Big\{ u \in L^\infty(0,T; W^{2,n+q}(\mathbb{R}^n)) \;\Big|\; \partial_t u \in L^{n+q}(Q_T), \\ \|\partial_t u\|_{L^{n+q}(Q_T)} + \|u\|_{L^\infty(0,T;W^{2,n+q}(\mathbb{R}^n))} \leq 2\|u_0\|_{W^{2,n+q}(\mathbb{R}^n)} + 1 \Big\}.$$

(7). For $s > 0$, the set of all Lebesgue measurable function f on $\mathbb{R}^n$ satisfies the following condition:

$$|f|_{s,\infty} := \sup\left\{\lambda > 0 \mid \lambda\, m_f(\lambda)^{\frac{1}{s}}\right\} < \infty, \tag{2.1}$$

where $m_f(\lambda)$ represents the Lebesgue measure of the set $\{x \in \mathbb{R}^n \mid |f(x)| > \lambda\}$. This space is denoted by $L^{s,\infty}(\mathbb{R}^n)$, and is referred to as the Lorentz space. It is well known that:

$$|f|_{s,\infty} \leq \|f\|_{L^s(\mathbb{R}^n)},$$

and thus, $L^s(\mathbb{R}^n) \subset L^{s,\infty}(\mathbb{R}^n)$ for $1 \leq s < \infty$. Moreover, we have $L^\infty(\mathbb{R}^n) = L^{\infty,\infty}(\mathbb{R}^n)$.

Throughout this paper, we impose the following assumptions.

Assumption 2.1.

(I). **parabolic-elliptic type (A)**:
 (i). Let $n \geq 1$ and $\tau = 0$.
 (ii). Let $q = 2$ or $q \geq 3$; and $\gamma > 0$.
(II). **parabolic-elliptic type (B)**:
 (i). Let $n \geq 2$ and $\tau = 0$.
 (ii). Let $q = 2$ or $q \geq 3$; and $\gamma = 0$.
(III). **parabolic-parabolic type**:
 (i). Let $n \geq 1$ and $\tau = 1$.
 (ii). Let $q = 2$ or $q \geq 3$; and $\gamma \geq 0$.

Assumption 2.2.

(I). **parabolic-elliptic type (A)**:
 Let $u_0 \in W^{2,n+q}(\mathbb{R}^n)$.
(II). **parabolic-elliptic type (B)**:
 Let $u_0 \in L^\theta(\mathbb{R}^n) \cap W^{2,n+q}(\mathbb{R}^n)$ for some $1 < \theta < n$.
(III). **parabolic-parabolic type**:
 (i). Let $u_0 \in W^{2,n+q}(\mathbb{R}^n)$.
 (ii). Let $v_0 \in W^{3,n+q}(\mathbb{R}^n)$.

The definition of a possibly sign-changing strong solution to (DD) is introduced next.

Definition 2.1. Let $1 \le p \le \infty$, and let r satisfy the following conditions:

$$\begin{cases} p \le r \le r^* & \text{for } p < \frac{n}{2} \quad \text{with} \quad \frac{1}{r^*} = \frac{1}{p} - \frac{2}{n}; \\ \frac{n}{2} \le r < \infty & \text{for } p = \frac{n}{2}; \\ p \le r \le \infty & \text{for } p > \frac{n}{2}. \end{cases}$$

We assume that $u_0 \in W^{2,p}(\mathbb{R}^n)$, and that $v_0 \in W^{2,r}(\mathbb{R}^n)$ when $\tau = 1$. A pair of functions (u, v) on Q_T is called a possibly sign-changing strong solution of (DD) on $[0, T)$ in the class $S_{p,r}(0, T)$ if the following conditions are satisfied:

(i). $u \in W_p^{2,1}(Q_T)$,
(ii-a). $v \in L^r(0, T; W^{2,r}(\mathbb{R}^n))$ for $\tau = 0$,
(ii-b). $v \in W_r^{2,1}(Q_T)$ for $\tau = 1$,
(iii). (u, v) satisfies (DD) in Q_T.

The following theorem regarding the local existence of a possibly sign-changing strong solution can be proven by suitably modifying the arguments presented in Sugiyama and Kunii [23].

Theorem 2.1. *Let Assumptions 2.1 and 2.2 hold. Then, the following statements hold*:

(I). **(Existence of Time Local Solution)**
Let $\tau = 0,\ 1$. Then, there exists a positive time T_1 depending only on n, q, γ, $\|u_0\|_{W^{2,n+q}(\mathbb{R}^n)}$, and $\tau\|v_0\|_{W^{3,n+q}(\mathbb{R}^n)}$ such that (DD) *has a possibly sign-changing strong solution (u, v) on $[0, T_1)$, uniquely in the class $W(Q_{T_1})$ with $u \in X_{T_1}$.*
Since $u \in X_{T_1}$, the solution $u(t)$ satisfies the following estimate: there exists a positive constant C depending only on n and q such that:

$$\sup_{0<t<T_1} \|u(t)\|_{L^\infty(\mathbb{R}^n)} \le C\left(\|u_0\|_{W^{2,n+q}(\mathbb{R}^n)} + 1\right). \tag{2.2}$$

(II). **(Extension Criterion)**
Let $\tau = 0, 1$. If the solution $u(t)$ obtained from Theorem 2.1 (I) *satisfies*:

$$\sup_{0<t<T_0} \|u(t)\|_{L^\infty(\mathbb{R}^n)} < \infty, \tag{2.3}$$

then, there is $T_0' > T_0$ such that (u, v) can be extended as a unique strong solution of (DD) *in $W(Q_{T_0'})$. Furthermore, if the maximal existence time $T_{\max}$ of the above strong solution (u, v) is finite, then we have*:

$$\limsup_{t \to T_{\max}-0} \|u(t)\|_{L^\infty(\mathbb{R}^n)} = \infty.$$

(III). **(Extended Existence of Solutions up to Maximal Time)**
Let $\tau = 0, 1$. Then, (DD) *has a possibly sign-changing strong solution (u, v) on $[0, T_{\max})$,*

which is unique in the class $W(Q_{T_{\max}})$ with $u \in X_{T_{\max}}$, where $T_{\max}$ is defined as in Theorem 2.1 (II).

In addition, let $\widehat{T}$ be an arbitrary positive number with $0 < \widehat{T} < T_{\max}$. Then, the following holds:

(i). **parabolic-elliptic type (A)**: $\tau = 0$, $\gamma > 0$.
There exists a positive constant C depending only on n, q, γ, $\widehat{T}$ and $\|u_0\|_{W^{2,n+q}(\mathbb{R}^n)}$ such that:

$$\sup_{0<t<\widehat{T}} \|u(t)\|_{W^{2,n+q}(\mathbb{R}^n)} \leq C. \tag{2.4}$$

(ii). **parabolic-elliptic type (B)**: $\tau = 0$, $\gamma = 0$.
There exists a positive constant C depending only on n, q, $\widehat{T}$, $\|u_0\|_{L^\theta(\mathbb{R}^n)}$ and $\|u_0\|_{W^{2,n+q}(\mathbb{R}^n)}$ such that:

$$\sup_{0<t<\widehat{T}} \|u(t)\|_{L^\theta(\mathbb{R}^n)} + \sup_{0<t<\widehat{T}} \|u(t)\|_{W^{2,n+q}(\mathbb{R}^n)} \leq C \tag{2.5}$$

for some $1 < \theta < n$, where θ is introduced in Assumption 2.2.

(iii). **parabolic-parabolic type**: $\tau = 1$, $\gamma \geq 0$.
There exists a positive constant C depending only on n, q, γ, $\widehat{T}$, $\|u_0\|_{W^{2,n+q}(\mathbb{R}^n)}$, and $\|v_0\|_{W^{3,n+q}(\mathbb{R}^n)}$ such that:

$$\sup_{0<t<\widehat{T}} \|u(t)\|_{W^{2,n+q}(\mathbb{R}^n)} + \sup_{0<t<\widehat{T}} \|v(t)\|_{W^{3,n+q}(\mathbb{R}^n)} \leq C. \tag{2.6}$$

Furthermore, for all $n + q \leq r \leq \infty$, it holds:

$$u \in C([0, \widehat{T}]; L^r_{\mathrm{loc}}(\mathbb{R}^n)). \tag{2.7}$$

(IV). **(Charge Conservation Law)**
Let $\tau = 0, 1$. We impose the assumption that the initial data u_0 satisfies $u_0 \in L^1(\mathbb{R}^n)$. Let $T_{\max}$ be the maximal existence time of the strong solution (u, v) obtained from Theorem 2.1 (I), (II) *and* (III). *Let $\widehat{T}$ be an arbitrary positive number with $0 < \widehat{T} < T_{\max}$. Then the strong solution u belongs to $L^\infty(0, \widehat{T}; L^1(\mathbb{R}^n))$ and satisfies*:

$$\int_{\mathbb{R}^n} u(x,t)\, dx = \int_{\mathbb{R}^n} u_0(x)\, dx \qquad a.e.\ 0 \leq t < \widehat{T}. \tag{2.8}$$

(V). **(Non-Negativity)**
Let $\tau = 0, 1$. We assume that the initial data $u_0 \in L^2(\mathbb{R}^n)$ satisfy:

$$u_0 \geq 0 \text{ and } u_0 \not\equiv 0 \quad \text{for a.e. } x \in \mathbb{R}^n.$$

Let $T_{\max}$ be the maximal existence time of the strong solution (u, v) obtained from Theorem 2.1 (I), (II) *and* (III). *Let $\widehat{T}$ be an arbitrary positive number with $0 < \widehat{T} < T_{\max}$. Then the strong solution u satisfies the non-negativity property*:

$$u(x,t) \geq 0 \qquad \textit{for a.e. } (x,t) \in \mathbb{R}^n \times (0, \widehat{T}). \tag{2.9}$$

Remark 1. The extension criterion for the solution is derived from (2.2) in Theorem 2.1 (I), and follows as a consequence of Theorem 2.1 (II). This criterion, which is crucial for understanding the behavior of solutions near potential blow-up points, is discussed in detail in Section 4. In that section, we conduct a thorough analysis of the conditions under which the solution can be extended beyond its initial interval of existence.

Remark 2. The existence of mild solutions in critical function spaces for various types of drift-diffusion equations (DD) has been studied extensively; see, for example, [11–14,16,19,21,22].

Furthermore, the existence and uniqueness of weak solutions have been addressed for a broad class of nonlinear extensions of (DD), including degenerate and semilinear systems; see, for instance, [3], [5], [9], [15] and [20]. In the critical and supercritical cases, the asymptotic profile of solutions is given by the fundamental solution of the heat equation or the porous medium equation without advection; see, for example, [17] and [18]. In contrast, solutions are known to approach a nontrivial stationary state in the subcritical regime [4].

The following is one of the main results in this paper: the *Identical Vanishing for the Backward Problem*. This theorem asserts that, under specific conditions, the solution to a given problem can be uniquely determined by its past behavior, meaning that if the solution vanishes identically at a certain time, it must vanish identically for all previous times as well. The proof of this theorem is based on the *Unique Continuation Theorem* and the principles of identical vanishing.

Theorem 2.2 *(Identical Vanishing for the Backward Problem). Let Assumption 2.1 and 2.2 hold, and let $T_{\max}$ be the maximal existence time of the strong solution (u, v) obtained from Theorem 2.1. Let $\widehat{T}$ be an arbitrary positive number with $0 < \widehat{T} < T_{\max}$. If there exists a non-empty open set $D_0 \subset \mathbb{R}^n$ such that $u(\cdot, \widehat{T}) \equiv 0$ in D_0, then, $u \equiv 0$ in $Q_{\widehat{T}}$.*

The following theorem follows from Theorem 2.2, which establishes the *Identical Vanishing for the Backward Problem* as a consequence of the *Unique Continuation Theorem*. As a result, we obtain the theorem stated below.

Theorem 2.3 *(The Property of Infinite Speed of Propagation). Let Assumptions 2.1 and 2.2 hold. We impose the assumption that the initial data u_0 satisfies $u_0 \in L^1(\mathbb{R}^n)$, and assume that*:

$$\int_{\mathbb{R}^n} u_0(x)\, dx \neq 0. \tag{2.10}$$

We denote by $T_{\max}$ the maximal existence time of the strong solution (u, v) obtained from Theorem 2.1. Let $\widehat{T}$ be an arbitrary positive number with $0 < \widehat{T} < T_{\max}$. Then, the support of $u(\cdot, t)$ coincides with $\mathbb{R}^n$ for all $0 < t < \widehat{T}$.

Remark 3. If (2.10) holds, then the support of u_0 is non-empty. Hence, to derive a contradiction by applying the unique continuation theorem in the proof of the infinite speed of propagation, it is not necessary to assume that $\operatorname{supp} u_0$ is non-empty: this fact is already ensured by (2.10).

Remark 4. Theorem 2.3 concerns sign-changing strong solutions of (DD) in the class $W(Q_{\widehat{T}})$. However, when the initial data $u_0(x)$ is non-negative and not identically zero, the corresponding solution u satisfies the non-negativity property stated in Theorem 2.1 (V). In this case, Theorem 2.3 implies that the strong solution u is positive at $x \in \mathbb{R}^n$ almost everywhere for each $0 < t < \widehat{T}$, even if $u_0(x)$ has compact support. Indeed, non-negative solutions are expected to remain positive for all positive times. While the positivity of solutions is fundamental to the structure of the solution, its full mathematical justification remains open and presents a significant problem to be addressed.

Remark 5. For strong solutions satisfying the non-negativity property (V) stated in Theorem 2.1, Theorem 2.3 can be deduced from the general theory of the strong maximum principle and unique continuation for parabolic equations.

In the following section, we will introduce several lemmas that will be frequently referenced throughout this paper. Section 4 will be dedicated to organizing the proofs of Theorem 2.1. In Section 5, we will present the proof of Theorem 2.2, which establishes the *Unique Continuation Theorem*. Additionally, Section 6 will explore the topic of infinite speed of propagation and provide the proof of Theorem 2.3.

3. Preliminary

In this section, we introduce several lemmas that will be frequently employed in the subsequent sections. First, we define a cut-off function in the following lemma.

Lemma 3.1. *Let $R \geq 1$. We define $\tilde{\phi}_R$ by*:

$$\tilde{\phi}_R(r) := \begin{cases} 1 & 0 \leq r < R, \\ 1 - \frac{2}{R^2}(r-R)^2 & R \leq r < \frac{3}{2}R, \\ \frac{2}{R^2}(r-2R)^2 & \frac{3}{2}R \leq r < 2R, \\ 0 & 2R \leq r \end{cases}$$

and set $\phi_R(x)$ as $\tilde{\phi}_R(|x|)$ for $x \in \mathbb{R}^n$. Then, the following estimates hold:

$$|\nabla \phi_R(x)| \leq \frac{2\sqrt{n}}{R}, \qquad |\Delta \phi_R(x)| \leq \frac{12n}{R^2} \qquad \textit{for all } x \in \mathbb{R}^n$$

and:

$$|\nabla \phi_R(x)| \leq \frac{2\sqrt{2}}{R}(\phi_R(x))^{\frac{1}{2}} \qquad \textit{for all } x \in \mathbb{R}^n.$$

In addition, the following estimates are satisfied:

$$\|\nabla \phi_R\|_{L^p(\mathbb{R}^n)} \leq CR^{-1+\frac{n}{p}} \quad \textit{and} \quad \|\Delta \phi_R\|_{L^p(\mathbb{R}^n)} \leq CR^{-2+\frac{n}{p}} \quad \textit{for all } 1 \leq p \leq \infty, \tag{3.1}$$

where C depends only on n and p.

The following estimates are obtained from Duoandikoetxea [6, p. 110].

Lemma 3.2. *Let $w \in W^{2,r}(\mathbb{R}^n)$. Then, the following estimate holds*:

$$\|\nabla^2 w\|_{L^r(\mathbb{R}^n)} \le C\left(\frac{r^2}{r-1}\right)^2 \|\Delta w\|_{L^r(\mathbb{R}^n)} \quad \text{for all } 1 < r < \infty, \tag{3.2}$$

where C depends only on n.

The following lemma provides a variant of the Gagliardo-Nirenberg inequality, which was derived from [25, Lemma 2.4]. This inequality will be frequently used in the next section as a key component of our argument.

Lemma 3.3 *(Gagliardo-Nirenberg inequality). Let $n \ge 1$, $m \ge 1$, $a > 2$, and let $f \in L^{q_1}(\mathbb{R}^n)$ with $q_1 \ge 1$, and $|f|^{\frac{r+m-3}{2}} f \in W^{1,2}(\mathbb{R}^n)$ with $r > 0$. If $q_1 \in [1, r+m-1]$, $q_2 \in [\frac{r+m-1}{2}, \frac{a(r+m-1)}{2}]$, and*:

$$\begin{cases} 1 \le q_1 \le q_2 \le \infty & \text{when } n = 1, \\ 1 \le q_1 \le q_2 < \infty & \text{when } n = 2, \\ 1 \le q_1 \le q_2 \le \frac{(r+m-1)n}{n-2} & \text{when } n \ge 3, \end{cases}$$

then, the following estimate holds:

$$\|f\|_{L^{q_2}(\mathbb{R}^n)} \le C^{\frac{2}{r+m-1}} \|f\|_{L^{q_1}(\mathbb{R}^n)}^{1-\Theta} \|\nabla(|f|^{\frac{r+m-3}{2}} f)\|_{L^2(\mathbb{R}^n)}^{\frac{2\Theta}{r+m-1}},$$

with:

$$\Theta = \frac{r+m-1}{2}\Big(\frac{1}{q_1} - \frac{1}{q_2}\Big)\Big(\frac{1}{n} - \frac{1}{2} + \frac{r+m-1}{2q_1}\Big)^{-1},$$

where:

$$\begin{cases} C \text{ depends only on } n \text{ and } a & \text{when } q_1 \ge \frac{r+m-1}{2}, \\ C = c_0^{\frac{1}{\beta}} \text{ with } c_0 \text{ depending only on } n \text{ and } a & \text{when } 1 \le q_1 < \frac{r+m-1}{2}, \end{cases}$$

and:

$$\beta = \frac{q_2 - \frac{r+m-1}{2}}{q_2 - q_1}\Big[\frac{2q_1}{r+m-1} + \Big(1 - \frac{2q_1}{r+m-1}\Big)\frac{2n}{n+2}\Big].$$

We now define the kernel G_γ of the Bessel potential using the following expression:

$$G_\gamma(x) = \begin{cases} \gamma^{\frac{n}{2}-1} a_n e^{-\sqrt{\gamma}|x|} \displaystyle\int_0^\infty e^{-\sqrt{\gamma}|x|s}\Big(s + \frac{s^2}{2}\Big)^{\frac{n-3}{2}} ds & \text{for } n \ge 2, \\ a_1 \gamma^{-\frac{1}{2}} e^{-\sqrt{\gamma}|x|} & \text{for } n = 1, \end{cases}$$

where the constant a_n is given by:

$$a_n := \begin{cases} \frac{1}{2(2\pi)^{\frac{n-1}{2}}\Gamma(\frac{n-1}{2})} & \text{for } n \geq 2, \\ \frac{1}{2} & \text{for } n = 1, \end{cases}$$

where $\Gamma(\cdot)$ denotes the gamma function. We also introduce the Kernel G_0 of the Newtonian potential, given the expression:

$$G_0(x) = \begin{cases} \frac{1}{n(n-2)\omega_n}|x|^{2-n} & \text{for } n \geq 3, \\ -\frac{1}{2\pi}\log|x| & \text{for } n = 2, \\ -\frac{1}{2}|x| & \text{for } n = 1, \end{cases}$$

where ω_n represents the volume of unit ball in $\mathbb{R}^n$. The kernel G_γ tends to the kernel of the Newtonian potential as $\gamma \to +0$ for $n \geq 3$. However, it is well known that G_γ becomes discontinuous as $\gamma \to +0$ for $n = 1$ and $n = 2$.

We provide $L^1(\mathbb{R}^n)$-estimates for G_γ and ∇G_γ when $\gamma > 0$, which can be derived through direct calculation.

Lemma 3.4. *For $\gamma > 0$, the following estimates hold*:

$$\|G_\gamma\|_{L^1(\mathbb{R}^n)} = \frac{1}{\gamma} \quad \textit{and} \quad \|\nabla G_\gamma\|_{L^1(\mathbb{R}^n)} \leq C, \tag{3.3}$$

where C depends only on n and γ.

Remark 6. In the case where $\gamma > 0$, we observe:

$$\begin{cases} \|\nabla G_\gamma\|_{L^s(\mathbb{R}^n)} < \infty & \text{for } n \geq 2,\ 1 \leq s < \frac{n}{n-1}, \\ \|\nabla G_\gamma\|_{L^s(\mathbb{R}^n)} < \infty & \text{for } n = 1,\ 1 \leq s \leq \infty. \end{cases}$$

On the other hand, in the case where $\gamma = 0$, we find:

$$\begin{cases} \|\nabla G_\gamma\|_{L^{\frac{n}{n-1},\infty}(\mathbb{R}^n)} < \infty & \text{for } n \geq 2, \\ \|\nabla G_\gamma\|_{L^\infty(\mathbb{R}^n)} < \infty & \text{for } n = 1, \end{cases}$$

where $L^{s,\infty}(\mathbb{R}^n)$ denotes the Lorentz space equipped with (2.1).

In the following, we define the function z by:

$$z(x) = \int_{\mathbb{R}^n} G_\gamma(x-y) f(y)\, dy \quad \text{for } f \in L^p(\mathbb{R}^n), \tag{3.4}$$

where $1 \le p \le \infty$. It is well known that the potential z of f in (3.4) satisfies the Poisson equation as follows:

Lemma 3.5. *Let $n \ge 1$, $\gamma > 0$, and $1 \le p \le \infty$. Then, $z \in W^{2,p}(\mathbb{R}^n)$ and satisfies the following equation*:

$$-\Delta z = -\gamma z + f \qquad in\ \mathbb{R}^n.$$

We now provide the $L^p(\mathbb{R}^n)$-estimate for ∇z, where z is the potential of f in (3.4).

Lemma 3.6. *Let $n \ge 1$, $\gamma > 0$, and let z be defined by (3.4). In addition, we assume that $f \in L^\infty(0,T; L^p(\mathbb{R}^n))$ for $1 \le p \le \infty$. Then, the following estimates hold*:

$$\sup_{0<t<T} \|z(t)\|_{L^p(\mathbb{R}^n)} \le \frac{1}{\gamma} \sup_{0<t<T} \|f(t)\|_{L^p(\mathbb{R}^n)}, \tag{3.5}$$

and:

$$\sup_{0<t<T} \|\nabla z(t)\|_{L^p(\mathbb{R}^n)} \le C \sup_{0<t<T} \|f(t)\|_{L^p(\mathbb{R}^n)}, \tag{3.6}$$

where C depends only on n and γ.

Proof of Lemma 3.6. Since $G_\gamma, \nabla G_\gamma \in L^1(\mathbb{R}^n)$, we apply the Young inequality and (3.3) to obtain:

$$\|z(t)\|_{L^p(\mathbb{R}^n)} \le \|G_\gamma\|_{L^1(\mathbb{R}^n)} \|f(t)\|_{L^p(\mathbb{R}^n)} = \frac{1}{\gamma} \|f(t)\|_{L^p(\mathbb{R}^n)}$$

and:

$$\|\nabla z(t)\|_{L^p(\mathbb{R}^n)} \le \|\nabla G_\gamma\|_{L^1(\mathbb{R}^n)} \|f(t)\|_{L^p(\mathbb{R}^n)} \le C \|f(t)\|_{L^p(\mathbb{R}^n)}$$

for a.e. $0 < t < T$, where C depends only on n and γ. This completes the proof of Theorem 3.6. □

Next, we define the function z by:

$$z(x) = \int_{\mathbb{R}^n} G_0(x-y) f(y) dy \qquad \text{for } f \in L^p(\mathbb{R}^n), \tag{3.7}$$

where $1 \le p \le \infty$. It is well known that the potential z of f in (3.7) satisfies the Poisson equation as follows:

Lemma 3.7. *Let $n \ge 1$, $\gamma = 0$, and $1 \le p \le \infty$. Then, $z \in W^{2,p}_{\mathrm{loc}}(\mathbb{R}^n)$ with $\partial^2_{ij} z \in L^p(\mathbb{R}^n)$, and satisfies the following equation*:

$$-\Delta z = f \quad \text{in } \mathbb{R}^n,$$

for $n \geq 2$*. In the case where* $n = 1$*, for* $f \in L^1(\mathbb{R})$ *with* $|x|f \in L^1(\mathbb{R})$*, it holds* $z \in C^1(\mathbb{R})$*. Furthermore,* $\partial_1 z$ *is bounded and absolutely continuous on* $\mathbb{R}$*, and* z *satisfies*:

$$-\partial_1^2 z = f \quad \text{in } \mathbb{R}.$$

Here, we consider the following Cauchy problem:

$$(\mathrm{P}) \qquad \begin{cases} \dfrac{\partial w}{\partial t} = \Delta w - \gamma w + f & \text{in} \quad \mathbb{R}^n \times (0, T), \\ w(x, 0) = w_0(x) & \text{in} \quad \mathbb{R}^n. \end{cases}$$

The following definition is a standard one from semi-group theory.

Definition 3.1. Let $1 \leq p \leq \infty$, and let $w_0 \in L^p(\mathbb{R}^n)$. If $f \in L^1(0, T; L^p(\mathbb{R}^n))$, then (P) has a unique mild solution w in $C([0, T); L^p(\mathbb{R}^n))$, which is given by:

$$w(t) = e^{-\gamma t} e^{t\Delta} w_0 + \int_0^t e^{-\gamma(t-s)} e^{(t-s)\Delta} f(s)\, ds \tag{3.8}$$

for all $0 \leq t < T$, where $(e^{t\Delta} f)(x, t) = \int_{\mathbb{R}^n} (4\pi t)^{-\frac{n}{2}} e^{-\frac{|x-y|^2}{4t}} f(y)\, dy$.

The following lemma is crucial for establishing the *a priori* estimates of v in (DD), which are derived using the $L^p(\mathbb{R}^n)$–$L^q(\mathbb{R}^n)$ estimates for the heat kernel.

Lemma 3.8. *Let* $1 \leq p^* \leq p \leq \infty$, $\frac{1}{p^*} - \frac{1}{p} < \frac{1}{n}$*, and let* $w_0 \in W^{1,p}(\mathbb{R}^n)$*. We suppose that* w *is the mild solution given by (3.8) in Definition 3.1. If* $f \in L^\infty(0, \infty; L^{p^*}(\mathbb{R}^n))$*, then the following estimates hold*:

$$\|w(t)\|_{L^p(\mathbb{R}^n)} \leq \|w_0\|_{L^p(\mathbb{R}^n)} + \frac{C t^{1-\frac{n}{2}\left(\frac{1}{p^*}-\frac{1}{p}\right)}}{1 - \frac{n}{2}\left(\frac{1}{p^*} - \frac{1}{p}\right)} \sup_{0<s<t} \|f(s)\|_{L^{p^*}(\mathbb{R}^n)}, \tag{3.9}$$

and:

$$\|\nabla w(t)\|_{L^p(\mathbb{R}^n)} \leq \|\nabla w_0\|_{L^p(\mathbb{R}^n)} + \frac{C t^{\frac{1}{2}-\frac{n}{2}\left(\frac{1}{p^*}-\frac{1}{p}\right)}}{\frac{1}{2} - \frac{n}{2}\left(\frac{1}{p^*} - \frac{1}{p}\right)} \sup_{0<s<t} \|f(s)\|_{L^{p^*}(\mathbb{R}^n)} \tag{3.10}$$

for all $0 \leq t < T$*, where* C *depends only on* n*.*

In addition, let $|\nabla^i w_0| \in L^p(\mathbb{R}^n)$ *and let* $f \in L^2(0, T; W^{i-1,p}(\mathbb{R}^n))$ *for* $i = 1, 2, 3$*. Then, it holds* :

$$\|\nabla^i w(t)\|^2_{L^p(\mathbb{R}^n)} \leq \|\nabla^i w_0\|^2_{L^p(\mathbb{R}^n)} + 2(p + n - 2) \int_0^t \|\nabla^{i-1} f(s)\|^2_{L^p(\mathbb{R}^n)}\, ds \tag{3.11}$$

for all $0 \le t < T$.

4. Proof of Theorem 2.1

4.1. Proof of Theorem 2.1 (I): existence of time local solution

To establish the local existence of solutions for (DD), we refer to Sugiyama and Kunii [23, Proposition 8] and Sugiyama and Yahagi [24]. In these works, the existence of a non-negative solution to the Keller-Segel system was established under both quasilinear and semilinear diffusion structures, each of which involves a uniformly elliptic leading term.

We present the following modification of the formal statement of the result provided by Sugiyama and Kunii [23, Proposition 8]:

Proposition 4.1. *Let $q \ge 2$ and $\tau = 0$ or 1. We assume that the initial data u_0 satisfies $u_0 \in W^{2,n+q}(\mathbb{R}^n)$. In the case $\tau = 1$, we additionally assume that $v_0 \in W^{3,n+q}(\mathbb{R}^n)$. Then there exists a positive time $T_1 = T_1\left(n, q, \|u_0\|_{W^{2,n+q}},\ \tau\|v_0\|_{W^{3,n+q}}\right)$ such that* (DD) *has a unique strong solution (u, v) in the space $\mathcal{W}(Q_{T_1})$. Here, $\mathcal{W}(Q_{T_1})$ is defined as follows*:

(i). **parabolic-elliptic type**: $\tau = 0$.

$$\mathcal{W}(Q_{T_1}) := W^{2,1}_{n+q}(Q_{T_1}) \times L^{n+q}(0, T_1; W^{2,n+q}(\mathbb{R}^n)).$$

(ii). **parabolic-parabolic type**: $\tau = 1$.

$$\mathcal{W}(Q_{T_1}) := W^{2,1}_{n+q}(Q_{T_1}) \times W^{2,1}_{n+q}(Q_{T_1}).$$

By applying the same argument as in [23, Proposition 8], we obtain the existence and uniqueness of a strong solution (u, v) to (DD) on $[0, T_1)$, uniquely in $W(Q_{T_1})$ with $u \in X_{T_1}$, where the initial data u_0 (and v_0, if applicable) is specified in Assumption 2.2. This completes the proof of Theorem 2.1 (I). $\square$

4.2. Proof of Theorem 2.1 (II): extension criterion

We now establish the extension criterion for the solution of (DD). To proceed, we present the following Lemma:

Lemma 4.2. *Let Assumptions 2.1 and 2.2 hold. Let (u, v) be the strong solution of* (DD) *on $[0, T_1)$ obtained from Theorem 2.1* (I) *with the property* (2.2)*. Then, the strong solution (u, v) on $[0, T_1)$ satisfies the following properties*:

(i). **parabolic-elliptic type (A)**: *Let $\tau = 0$ and $\gamma > 0$. There exists a positive constant C depending only on n, q, γ, T_1, and $\|u_0\|_{W^{2,n+q}(\mathbb{R}^n)}$ such that*:

$$\sup_{0<t<T_1} \|u(t)\|_{W^{2,n+q}(\mathbb{R}^n)} \le C.$$

(ii). **parabolic-elliptic type (B)**: *Let $\tau = 0$ and $\gamma = 0$. There exists a positive constant C depending only on n, q, T_1, $\|u_0\|_{L^\theta(\mathbb{R}^n)}$ and $\|u_0\|_{W^{2,n+q}(\mathbb{R}^n)}$ such that:*

$$\sup_{0<t<T_1} \|u(t)\|_{L^\theta(\mathbb{R}^n)} \le C, \quad \text{and} \quad \sup_{0<t<T_1} \|u(t)\|_{W^{2,n+q}(\mathbb{R}^n)} \le C$$

for some $1 < \theta < n$, where θ is introduced in Assumption 2.2.

(iii). **parabolic-parabolic type**: *Let $\tau = 1$ and $\gamma \ge 0$. There exists a positive constant C depending only on n, q, γ, T_1, $\|u_0\|_{W^{2,n+q}(\mathbb{R}^n)}$ and $\|v_0\|_{W^{3,n+q}(\mathbb{R}^n)}$ such that:*

$$\sup_{0<t<T_1} \|u(t)\|_{W^{2,n+q}(\mathbb{R}^n)} \le C, \quad \text{and} \quad \sup_{0<t<T_1} \|v(t)\|_{W^{3,n+q}(\mathbb{R}^n)} \le C.$$

Proof of Lemma 4.2.

(i). parabolic-elliptic type (A): $\tau = 0$, $\gamma > 0$.

We establish the following regularities:

$$\nabla v,\ \Delta v \in L^\infty(0, T_1; L^\infty(\mathbb{R}^n)), \tag{4.1}$$

$$u \in L^\infty(0, T_1; L^{n+q}(\mathbb{R}^n)), \tag{4.2}$$

$$\nabla u \in L^\infty(0, T_1; L^{n+q}(\mathbb{R}^n)), \tag{4.3}$$

$$\nabla u \in L^\infty(0, T_1; L^\infty(\mathbb{R}^n)), \tag{4.4}$$

$$\partial_i \nabla v, \partial_i \Delta v \in L^\infty(0, T_1; L^\infty(\mathbb{R}^n)) \qquad \text{for all } i = 1, 2, \dots, n, \tag{4.5}$$

$$\partial_i \nabla u \in L^\infty(0, T_1; L^{n+q}(\mathbb{R}^n)) \quad \text{for all } i = 1, 2, \dots, n. \tag{4.6}$$

We first prove (4.1). By applying (3.5) and (3.6) in Lemma 3.6 and using the second equation of (DD), we derive the following estimates:

$$\sup_{0<t<T_1} \|\nabla v(t)\|_{L^\infty(\mathbb{R}^n)} \le C \sup_{0<t<T_1} \|u(t)\|_{L^\infty(\mathbb{R}^n)}, \tag{4.7}$$

and:

$$\begin{aligned}\sup_{0<t<T_1} \|\Delta v(t)\|_{L^\infty(\mathbb{R}^n)} &\le \gamma \sup_{0<t<T_1} \|v(t)\|_{L^\infty(\mathbb{R}^n)} + \sup_{0<t<T_1} \|u(t)\|_{L^\infty(\mathbb{R}^n)} \\ &\le 2 \sup_{0<t<T_1} \|u(t)\|_{L^\infty(\mathbb{R}^n)},\end{aligned} \tag{4.8}$$

where C depends only on n and γ.

Next, we establish (4.2). Let $1 < r < \infty$. By multiplying both sides of the first equation of (DD) by $|u|^{r-2}u$ and integrating over $\mathbb{R}^n$, we obtain:

$$\frac{1}{r}\frac{d}{dt}\|u(t)\|_{L^r(\mathbb{R}^n)}^r = -(r-1)\int_{\mathbb{R}^n} |u|^{r-2}|\nabla u|^2\,dx - \frac{\chi(r-1)}{r+q-2}\int_{\mathbb{R}^n} \nabla v \cdot \nabla |u|^{r+q-2}\,dx$$

$$
\begin{aligned}
&= -(r-1)\int_{\mathbb{R}^n} |u|^{r-2}|\nabla u|^2\,dx + \frac{\chi(r-1)}{r+q-2}\int_{\mathbb{R}^n} \Delta v\,|u|^{r+q-2}\,dx \\
&\le \frac{r-1}{r+q-2}\|\Delta v(t)\|_{L^\infty(\mathbb{R}^n)}\|u(t)\|_{L^\infty(\mathbb{R}^n)}^{q-2}\|u(t)\|_{L^r(\mathbb{R}^n)}^{r} \\
&\le \frac{2(r-1)}{r+q-2}\|u(t)\|_{L^\infty(\mathbb{R}^n)}^{q-1}\|u(t)\|_{L^r(\mathbb{R}^n)}^{r},
\end{aligned}
$$

which yields:

$$
\begin{aligned}
\|u(t)\|_{L^r(\mathbb{R}^n)} &\le \|u_0\|_{L^r(\mathbb{R}^n)} + \frac{2(r-1)}{r+q-2}\int_0^t \|u(s)\|_{L^\infty(\mathbb{R}^n)}^{q-1}\|u(s)\|_{L^r(\mathbb{R}^n)}\,ds \\
&\le \|u_0\|_{L^r(\mathbb{R}^n)} + \frac{2(r-1)}{r+q-2}\sup_{0<t<T_1}\|u(t)\|_{L^\infty(\mathbb{R}^n)}^{q-1}\int_0^t \|u(s)\|_{L^r(\mathbb{R}^n)}\,ds
\end{aligned}
$$

for a.e. $0 < t < T_1$. By applying the Gronwall inequality, we obtain the following:

$$
\sup_{0<t<T_1}\|u(t)\|_{L^r(\mathbb{R}^n)} \le \|u_0\|_{L^r(\mathbb{R}^n)} \exp\left\{\frac{2(r-1)}{r+q-2}T_1 \sup_{0<t<T_1}\|u(t)\|_{L^\infty(\mathbb{R}^n)}^{q-1}\right\}. \tag{4.9}
$$

Therefore, since $q = 2$ or $q \ge 3$, and $n + q > 1$, by taking $r = n + q$, we obtain (4.2).

We move on to proving (4.3). Let $2 \le r < \infty$. To establish (4.3), we differentiate both sides of the first equation of (DD) with respect to x once and multiply by $|\nabla u|^{r-2}\nabla u$. This gives us the following inequality:

$$
\begin{aligned}
\frac{1}{r}\frac{d}{dt}\|\nabla u(t)\|_{L^r(\mathbb{R}^n)}^{r} &\le (r+n-2)(q-1)^2\|u(t)\|_{L^\infty(\mathbb{R}^n)}^{2(q-2)}\|\nabla v(t)\|_{L^\infty(\mathbb{R}^n)}^{2}\|\nabla u(t)\|_{L^r(\mathbb{R}^n)}^{r} \\
&\quad + (r+n-2)\|u(t)\|_{L^\infty(\mathbb{R}^n)}^{2(q-2)}\|u(t)\|_{L^r(\mathbb{R}^n)}^{2}\|\Delta v(t)\|_{L^\infty(\mathbb{R}^n)}^{2}\|\nabla u(t)\|_{L^r(\mathbb{R}^n)}^{r-2},
\end{aligned}
$$

which leads to the following estimate:

$$
\begin{aligned}
&\sup_{0<t<T_1}\|\nabla u(t)\|_{L^r(\mathbb{R}^n)} \\
&\le \Bigg(\|\nabla u_0\|_{L^r(\mathbb{R}^n)} + \sqrt{2(r+n-2)T_1}\sup_{0<t<T_1}\|u(t)\|_{L^\infty(\mathbb{R}^n)}^{q-2} \\
&\qquad \times \sup_{0<t<T_1}\|u(t)\|_{L^r(\mathbb{R}^n)}\sup_{0<t<T_1}\|\Delta v(t)\|_{L^\infty(\mathbb{R}^n)}\Bigg) \\
&\qquad \times \exp\left\{(r+n-2)(q-1)^2T_1\sup_{0<t<T_1}\|u(t)\|_{L^\infty(\mathbb{R}^n)}^{2(q-2)}\sup_{0<t<T_1}\|\nabla v(t)\|_{L^\infty(\mathbb{R}^n)}^{2}\right\}.
\end{aligned} \tag{4.10}
$$

Thus, since $q = 2$ or $q \ge 3$, and $n + q \ge 2$, by taking $r = n + q$, we conclude the proof of (4.3).

We now proceed to establish (4.4) by applying the Moser iteration technique. Let $n+q \le r < \infty$. Differentiating both sides of (DD) with respect to x and multiplying by $|\nabla u|^{r-2}\nabla u$, we obtain the following identity:

$$\begin{aligned}\frac{1}{r}\frac{d}{dt}\|\nabla u(t)\|_{L^r(\mathbb{R}^n)}^r &= \int_{\mathbb{R}^n} (\nabla \Delta u)\cdot |\nabla u|^{r-2}\nabla u\, dx + \chi \int_{\mathbb{R}^n} \nabla(\nabla\cdot(|u|^{q-2}u\nabla v))\cdot |\nabla u|^{r-2}\nabla u\, dx \\ &=: I_1 + I_2. \end{aligned} \tag{4.11}$$

By performing integration by parts once, we derive the following expression for I_1:

$$\begin{aligned} I_1 &= \int_{\mathbb{R}^n} \sum_{i=1}^n \partial_i \sum_{j=1}^n \partial_j^2 u\, |\nabla u|^{r-2}\partial_i u\, dx \\ &= -\int_{\mathbb{R}^n} \sum_{i=1}^n\sum_{j=1}^n (\partial_i\partial_j u)\, \partial_j(|\nabla u|^{r-2}\partial_i u)\, dx \\ &= -(r-2)\int_{\mathbb{R}^n} |\nabla u|^{r-4}\sum_{j=1}^n |\partial_j \nabla u\cdot \nabla u|^2\, dx - \int_{\mathbb{R}^n} |\nabla u|^{r-2}\sum_{j=1}^n |\partial_j\nabla u|^2\, dx \\ &=: -(r-2)J_1 - J_2. \end{aligned} \tag{4.12}$$

Next, we establish the bound for I_2:

$$\begin{aligned} I_2 &= \chi \int_{\mathbb{R}^n} \sum_{i=1}^n \partial_i(\nabla\cdot(u^{q-1}\nabla v))\, |\nabla u|^{r-2}\partial_i u\, dx \\ &= -\chi\int_{\mathbb{R}^n} (\nabla\cdot(|u|^{q-2}u\nabla v))\sum_{i=1}^n \partial_i(|\nabla u|^{r-2}\partial_i u)\, dx \\ &= -\chi \int_{\mathbb{R}^n} \big((q-1)|u|^{q-2}\nabla u\cdot\nabla v + |u|^{q-2}u\Delta v\big) \\ &\qquad \times \left\{(r-2)|\nabla u|^{r-4}\sum_{i=1}^n\sum_{j=1}^n(\partial_i\partial_j u)(\partial_j u)(\partial_i u) + |\nabla u|^{r-2}\sum_{i=1}^n \partial_i^2 u\right\} dx \\ &\le \frac{r-2}{4}\int_{\mathbb{R}^n} |\nabla u|^{r-4}\sum_{j=1}^n|\partial_j\nabla u\cdot\nabla u|^2\, dx \\ &\qquad + (r-2)(q-1)^2\int_{\mathbb{R}^n} |u|^{2(q-2)}|\nabla u|^2|\nabla v|^2|\nabla u|^{r-4}|\nabla u|^2\, dx \\ &\qquad + \frac{r-2}{4}\int_{\mathbb{R}^n}|\nabla u|^{r-4}\sum_{j=1}^n|\partial_j\nabla u\cdot\nabla u|^2\, dx + (r-2)\int_{\mathbb{R}^n} |u|^{2(q-1)}|\Delta v|^2|\nabla u|^{r-4}|\nabla u|^2\, dx \end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{4}\int_{\mathbb{R}^n}|\nabla u|^{r-2}\sum_{j=1}^{n}|\partial_j\nabla u|^2\,dx+n(q-1)^2\int_{\mathbb{R}^n}|u|^{2(q-2)}|\nabla u|^2|\nabla v|^2|\nabla u|^{r-2}\,dx\\
&+\frac{1}{4}\int_{\mathbb{R}^n}|\nabla u|^{r-2}\sum_{j=1}^{n}|\partial_j\nabla u|^2\,dx+n\int_{\mathbb{R}^n}|u|^{2(q-1)}|\Delta v|^2|\nabla u|^{r-2}\,dx\\
\le{}&\frac{r-2}{2}J_1+\frac{1}{2}J_2+(r+n-2)(q-1)^2\|u(t)\|^{2(q-2)}_{L^\infty(\mathbb{R}^n)}\|\nabla v(t)\|^2_{L^\infty(\mathbb{R}^n)}\|\nabla u(t)\|^r_{L^r(\mathbb{R}^n)}\\
&+(r+n-2)\|u(t)\|^{2(q-2)}_{L^\infty(\mathbb{R}^n)}\|\Delta v(t)\|^2_{L^\infty(\mathbb{R}^n)}\|u(t)\|^2_{L^r(\mathbb{R}^n)}\|\nabla u(t)\|^{r-2}_{L^r(\mathbb{R}^n)}\\
\le{}&\frac{r-2}{2}J_1+\frac{1}{2}J_2+(r+n-2)\sup_{0<t<T_1}\|u(t)\|^{2(q-2)}_{L^\infty(\mathbb{R}^n)}\sup_{0<t<T_1}\|\Delta v(t)\|^2_{L^\infty(\mathbb{R}^n)}\|u(t)\|^r_{L^r(\mathbb{R}^n)}\\
&+(r+n-2)(q-1)^2\sup_{0<t<T_1}\|u(t)\|^{2(q-2)}_{L^\infty(\mathbb{R}^n)}\\
&\quad\times\left(\sup_{0<t<T_1}\|\nabla v(t)\|^2_{L^\infty(\mathbb{R}^n)}+\sup_{0<t<T_1}\|\Delta v(t)\|^2_{L^\infty(\mathbb{R}^n)}\right)\|\nabla u(t)\|^r_{L^r(\mathbb{R}^n)}
\end{aligned}
$$

for a.e. $0<t<T_1$. This yields that I_2 can be bounded as follows:

$$
I_2=\frac{r-2}{2}J_1+\frac{1}{2}J_2+(r+n-2)\Big(M_1\|u(t)\|^r_{L^r(\mathbb{R}^n)}+M_2\|\nabla u(t)\|^r_{L^r(\mathbb{R}^n)}\Big) \tag{4.13}
$$

for a.e. $0<t<T_1$, where M_1 and M_2 are determined from the estimates already derived in (2.2) in Theorem 2.1 (I), (4.7), and (4.8) as follows:

$$
M_1:=\sup_{0<t<T_1}\|u(t)\|^{2(q-2)}_{L^\infty(\mathbb{R}^n)}\sup_{0<t<T_1}\|\Delta v(t)\|^2_{L^\infty(\mathbb{R}^n)}
$$

and:

$$
M_2:=(q-1)^2\sup_{0<t<T_1}\|u(t)\|^{2(q-2)}_{L^\infty(\mathbb{R}^n)}\left(\sup_{0<t<T_1}\|\nabla v(t)\|^2_{L^\infty(\mathbb{R}^n)}+\sup_{0<t<T_1}\|\Delta v(t)\|^2_{L^\infty(\mathbb{R}^n)}\right).
$$

Thus, from (4.11) to (4.13), we obtain the following differential inequality:

$$
\begin{aligned}
\frac{1}{r}\frac{d}{dt}\|\nabla u(t)\|^r_{L^r(\mathbb{R}^n)}\le{}&-\frac{r-2}{2}J_1-\frac{1}{2}J_2\\
&+(r+n-2)\Big(M_1\|u(t)\|^r_{L^r(\mathbb{R}^n)}+M_2\|\nabla u(t)\|^r_{L^r(\mathbb{R}^n)}\Big)
\end{aligned} \tag{4.14}
$$

for a.e. $0<t<T_1$.

For $n+q\le r<\infty$, applying Lemma 3.3 with $m=1$, $a=3$, $q_1=\frac{r}{n+q}\in[1,r]$, $q_2=r\in[\frac{r}{2},\frac{ar}{2}]$, $f=|\nabla u|$, $\Theta=\frac{n(n+q-1)}{n(n+q-1)+2}=1-\frac{2}{n(n+q-1)+2}$ and $\beta=\frac{n(n+q-1)+2}{(n+q-1)(n+2)}$, we obtain:

$$\begin{aligned}\|\nabla u(t)\|^r_{L^r(\mathbb{R}^n)} &\le \left(c_0^{\frac{1}{\beta}\cdot\frac{2}{r}}\right)^r \|\nabla u(t)\|^{r(1-\Theta)}_{L^{\frac{r}{n+q}}(\mathbb{R}^n)} \left\|\nabla|\nabla u|^{\frac{r}{2}}(t)\right\|^{2\Theta}_{L^2(\mathbb{R}^n)}\\
&\le c_0^{\frac{2(n+q-1)(n+2)}{n(n+q-1)+2}} \|\nabla u(t)\|^{\frac{2r}{n(n+q-1)+2}}_{L^{\frac{r}{n+q}}(\mathbb{R}^n)} \left\|\nabla|\nabla u|^{\frac{r}{2}}(t)\right\|^{\frac{2n(n+q-1)}{n(n+q-1)+2}}_{L^2(\mathbb{R}^n)},\end{aligned}$$

where c_0 depends only on n. Thus, we observe from the Young inequality, with $\ell = \frac{n(n+q-1)+2}{n(n+q-1)}$ and $\ell' = \frac{n(n+q-1)+2}{2}$:

$$\|\nabla u(t)\|^r_{L^r(\mathbb{R}^n)} \le C_\varepsilon\left(c_0^{\frac{2(n+q-1)(n+2)}{n(n+q-1)+2}} \|\nabla u(t)\|^{\frac{2r}{n(n+q-1)+2}}_{L^{\frac{r}{n+q}}(\mathbb{R}^n)}\right)^{\ell'} + \varepsilon\left\|\nabla|\nabla u|^{\frac{r}{2}}(t)\right\|^2_{L^2(\mathbb{R}^n)},$$

where $\varepsilon = \frac{r-2}{r^2(r+n-2)M_2}$ and $C_\varepsilon = \frac{1}{\ell'}(\varepsilon\ell)^{-\frac{\ell'}{\ell}} = \frac{2}{n(n+q-1)+2}\left(M_2\frac{n(n+q-1)}{n(n+q-1)+2}\cdot\frac{r^2(r+n-2)}{r-2}\right)^{\frac{n(n+q-1)}{2}}$. Since $r+n-2<2r$, noting:

$$\left\|\nabla|\nabla u|^{\frac{r}{2}}(t)\right\|^2_{L^2(\mathbb{R}^n)} = \frac{r^2}{4}J_1 \quad \text{and} \quad C_\varepsilon \le (2M_2)^{\frac{n(n+q-1)}{2}} r^{n(n+q-1)},$$

we obtain the following inequality:

$$\begin{aligned}&(r+n-2)M_2\|\nabla u(t)\|^r_{L^r(\mathbb{R}^n)}\\
&\le 2^{\frac{n(n+q-1)+2}{2}} c_0^{(n+q-1)(n+2)} M_2^{\frac{n(n+q-1)+2}{2}} r^{n(n+q-1)+1}\|\nabla u(t)\|^r_{L^{\frac{r}{n+q}}(\mathbb{R}^n)} + \frac{r-2}{4}J_1. \qquad (4.15)\end{aligned}$$

Hence, for all $n+q\le r<\infty$, we see from (4.14) and (4.15):

$$\begin{aligned}\frac{1}{r}\frac{d}{dt}\|\nabla u(t)\|^r_{L^r(\mathbb{R}^n)} &\le (r+n-2)M_1\|u(t)\|^r_{L^r(\mathbb{R}^n)}\\
&\quad + 2^{\frac{n(n+q-1)+2}{2}} c_0^{(n+q-1)(n+2)} M_2^{\frac{n(n+q-1)+2}{2}} r^{n(n+q-1)+1}\|\nabla u(t)\|^r_{L^{\frac{r}{n+q}}(\mathbb{R}^n)}\\
&\le M_* r^{n(n+q-1)+1}\left(\|u(t)\|^r_{L^r(\mathbb{R}^n)} + \|\nabla u(t)\|^r_{L^{\frac{r}{n+q}}(\mathbb{R}^n)}\right) \qquad (4.16)\end{aligned}$$

for a.e. $0<t<T_1$, where M_* is defined as follows:

$$M_* := \max\left\{2M_1,\ 2^{\frac{n(n+q-1)+2}{2}} c_0^{(n+q-1)(n+2)} M_2^{\frac{n(n+q-1)+2}{2}}\right\}.$$

Therefore, integrating both sides of (4.16) from 0 to t, we have, for all $n+q\le r<\infty$:

$$\begin{aligned}&\|\nabla u(t)\|^r_{L^r(\mathbb{R}^n)}\\
&\le \|\nabla u_0\|^r_{L^r(\mathbb{R}^n)} + M_* r^{n(n+q-1)+2}\int_0^t \left(\|u(s)\|^r_{L^r(\mathbb{R}^n)} + \|\nabla u(s)\|^r_{L^{\frac{r}{n+q}}(\mathbb{R}^n)}\right) ds\end{aligned}$$

$$
\begin{aligned}
&\le \|\nabla u_0\|^r_{L^{n+q}(\mathbb{R}^n)} + \|\nabla u_0\|^r_{L^\infty(\mathbb{R}^n)} \\
&\quad + M_* r^{n(n+q-1)+2} \int_0^t \left(\|u(s)\|^r_{L^{n+q}(\mathbb{R}^n)} + \|u(s)\|^r_{L^\infty(\mathbb{R}^n)} + \|\nabla u(s)\|^r_{L^{\frac{r}{n+q}}(\mathbb{R}^n)} \right) ds \\
&\le (M_*+1) r^{n(n+q-1)+2} \Bigg[2 \max\left\{ \|\nabla u_0\|_{L^{n+q}(\mathbb{R}^n)}, \|\nabla u_0\|_{L^\infty(\mathbb{R}^n)} \right\}^r \\
&\quad + 3 \max\left\{ \sup_{0<t<T_1} \|u(t)\|_{L^{n+q}(\mathbb{R}^n)}, \sup_{0<t<T_1} \|u(t)\|_{L^\infty(\mathbb{R}^n)}, \sup_{0<t<T_1} \|\nabla u(t)\|_{L^{\frac{r}{n+q}}(\mathbb{R}^n)} \right\}^r T_1 \Bigg] \\
&\le 5(M_*+1)(T_1+1) r^{n(n+q-1)+2} \max\Bigg\{ \|\nabla u_0\|_{L^{n+q}(\mathbb{R}^n)}, \|\nabla u_0\|_{L^\infty(\mathbb{R}^n)}, \\
&\quad \sup_{0<t<T_1} \|u(t)\|_{L^{n+q}(\mathbb{R}^n)}, \sup_{0<t<T_1} \|u(t)\|_{L^\infty(\mathbb{R}^n)}, \sup_{0<t<T_1} \|\nabla u(t)\|_{L^{\frac{r}{n+q}}(\mathbb{R}^n)} \Bigg\}^r.
\end{aligned}
$$

This yields:

$$
\begin{aligned}
\|\nabla u(t)\|_{L^r(\mathbb{R}^n)} \le \left\{ 5(M_*+1)(T_1+1) r^{n(n+q-1)+2} \right\}^{\frac{1}{r}} \max\Bigg\{ \|\nabla u_0\|_{L^{n+q}(\mathbb{R}^n)}, \|\nabla u_0\|_{L^\infty(\mathbb{R}^n)}, \\
\sup_{0<t<T_1} \|u(t)\|_{L^{n+q}(\mathbb{R}^n)}, \sup_{0<t<T_1} \|u(t)\|_{L^\infty(\mathbb{R}^n)}, \sup_{0<t<T_1} \|\nabla u(t)\|_{L^{\frac{r}{n+q}}(\mathbb{R}^n)} \Bigg\}
\end{aligned}
$$

for a.e. $0 < t < T_1$. Let $r = (n+q)^k$ with $k \ge 1$. Then, we find:

$$
\begin{aligned}
&\|\nabla u(t)\|_{L^{(n+q)^k}(\mathbb{R}^n)} \\
&\le \left\{ 5(M_*+1)(T_1+1)(n+q)^{n(n+q-1)+2} \right\}^{\frac{k}{(n+q)^k}} \max\Bigg\{ \|\nabla u_0\|_{L^{n+q}(\mathbb{R}^n)}, \|\nabla u_0\|_{L^\infty(\mathbb{R}^n)}, \\
&\quad \sup_{0<t<T_1} \|u(t)\|_{L^{n+q}(\mathbb{R}^n)}, \sup_{0<t<T_1} \|u(t)\|_{L^\infty(\mathbb{R}^n)}, \sup_{0<t<T_1} \|\nabla u(t)\|_{L^{(n+q)^{k-1}}(\mathbb{R}^n)} \Bigg\} \\
&\le \left\{ 5(M_*+1)(T_1+1)(n+q)^{n(n+q-1)+2} \right\}^{\frac{k}{(n+q)^k}} \left\{ 5(M_*+1)(T_1+1)(n+q)^{n(n+q-1)+2} \right\}^{\frac{k-1}{(n+q)^{k-1}}} \\
&\quad \times \max\Bigg\{ \|\nabla u_0\|_{L^{n+q}(\mathbb{R}^n)}, \|\nabla u_0\|_{L^\infty(\mathbb{R}^n)}, \sup_{0<t<T_1} \|u(t)\|_{L^{n+q}(\mathbb{R}^n)}, \\
&\qquad \times \sup_{0<t<T_1} \|u(t)\|_{L^\infty(\mathbb{R}^n)}, \sup_{0<t<T_1} \|\nabla u(t)\|_{L^{(n+q)^{k-2}}(\mathbb{R}^n)} \Bigg\} \\
&\le \prod_{\ell=1}^{k-1} \left\{ 5(M_*+1)(T_1+1)(n+q)^{n(n+q-1)+2} \right\}^{\frac{k-\ell+1}{(n+q)^{k-\ell+1}}} \max\Bigg\{ \|\nabla u_0\|_{L^{n+q}(\mathbb{R}^n)}, \|\nabla u_0\|_{L^\infty(\mathbb{R}^n)}, \\
&\quad \sup_{0<t<T_1} \|u(t)\|_{L^{n+q}(\mathbb{R}^n)}, \sup_{0<t<T_1} \|u(t)\|_{L^\infty(\mathbb{R}^n)}, \sup_{0<t<T_1} \|\nabla u(t)\|_{L^{n+q}(\mathbb{R}^n)} \Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
&= \left\{5(M_* + 1)(T_1 + 1)(n+q)^{n(n+q-1)+2}\right\}^{\sum_{\ell=1}^{k-1} \frac{k-\ell+1}{(n+q)^{k-\ell+1}}} \max\Big\{\|\nabla u_0\|_{L^{n+q}(\mathbb{R}^n)}, \|\nabla u_0\|_{L^\infty(\mathbb{R}^n)}, \\
&\qquad \sup_{0<t<T_1} \|u(t)\|_{L^{n+q}(\mathbb{R}^n)}, \sup_{0<t<T_1} \|u(t)\|_{L^\infty(\mathbb{R}^n)}, \sup_{0<t<T_1} \|\nabla u(t)\|_{L^{n+q}(\mathbb{R}^n)}\Big\} \\
&= \left\{5(M_* + 1)(T_1 + 1)(n+q)^{n(n+q-1)+2}\right\}^{\sum_{\ell=2}^{k} \frac{\ell}{(n+q)^{\ell}}} \max\Big\{\|\nabla u_0\|_{L^{n+q}(\mathbb{R}^n)}, \|\nabla u_0\|_{L^\infty(\mathbb{R}^n)}, \\
&\qquad \sup_{0<t<T_1} \|u(t)\|_{L^{n+q}(\mathbb{R}^n)}, \sup_{0<t<T_1} \|u(t)\|_{L^\infty(\mathbb{R}^n)}, \sup_{0<t<T_1} \|\nabla u(t)\|_{L^{n+q}(\mathbb{R}^n)}\Big\} \\
&\le \left\{5(M_* + 1)(T_1 + 1)(n+q)^{n(n+q-1)+2}\right\}^{2} \max\Big\{\|\nabla u_0\|_{L^{n+q}(\mathbb{R}^n)}, \|\nabla u_0\|_{L^\infty(\mathbb{R}^n)}, \\
&\qquad \sup_{0<t<T_1} \|u(t)\|_{L^{n+q}(\mathbb{R}^n)}, \sup_{0<t<T_1} \|u(t)\|_{L^\infty(\mathbb{R}^n)}, \sup_{0<t<T_1} \|\nabla u(t)\|_{L^{n+q}(\mathbb{R}^n)}\Big\} \qquad (4.17)
\end{aligned}
$$

for a.e. $0 < t < T_1$. Here, we notice that $\sum_{\ell=2}^{k} \frac{\ell}{(n+q)^\ell} \le \sum_{\ell=1}^{\infty} \frac{\ell}{2^\ell} = 2$.

At this point, we introduce the fundamental theorem regarding the limiting norm of $\|\cdot\|_{L^p(\mathbb{R}^n)}$. Specifically, let (X, μ) be a measure space. If $f \in L^{p_0}(X, \mu)$ for some $p_0 < \infty$, the following holds:

$$
\lim_{p\to\infty} \|f\|_{L^p(X)} = \|f\|_{L^\infty(X)}. \qquad (4.18)
$$

See Grafakos [8, p. 11, Exercise 1.1.3] for further details. Therefore, taking the limit as $k \to \infty$ on the left-hand side of (4.17) and applying the result from (4.18), we conclude:

$$
\begin{aligned}
&\sup_{0<t<T_1} \|\nabla u(t)\|_{L^\infty(\mathbb{R}^n)} \\
&\le \left\{5(M_* + 1)(T_1 + 1)(n+q)^{n(n+q-1)+2}\right\}^{2} \max\Big\{\|\nabla u_0\|_{L^{n+q}(\mathbb{R}^n)}, \|\nabla u_0\|_{L^\infty(\mathbb{R}^n)}, \\
&\qquad \sup_{0<t<T_1} \|u(t)\|_{L^{n+q}(\mathbb{R}^n)}, \sup_{0<t<T_1} \|u(t)\|_{L^\infty(\mathbb{R}^n)}, \sup_{0<t<T_1} \|\nabla u(t)\|_{L^{n+q}(\mathbb{R}^n)}\Big\}.
\end{aligned}
$$

We now turn to the proof of (4.5). Applying the Young inequality, we have:

$$
\begin{aligned}
\sup_{0<t<T_1} \|\partial_i \nabla v(t)\|_{L^\infty(\mathbb{R}^n)} &\le \|\nabla G_\gamma\|_{L^1(\mathbb{R}^n)} \sup_{0<t<T_1} \|\nabla u(t)\|_{L^\infty(\mathbb{R}^n)} \\
&\le C \sup_{0<t<T_1} \|\nabla u(t)\|_{L^\infty(\mathbb{R}^n)}, \qquad (4.19)
\end{aligned}
$$

where C depends only on n and γ. In addition, from the second equation of (DD), we have:

$$
\sup_{0<t<T_1} \|\partial_i \Delta v(t)\|_{L^\infty(\mathbb{R}^n)} \le \gamma \sup_{0<t<T_1} \|\nabla v(t)\|_{L^\infty(\mathbb{R}^n)} + \sup_{0<t<T_1} \|\nabla u(t)\|_{L^\infty(\mathbb{R}^n)}. \qquad (4.20)
$$

This inequality directly leads to the proof of (4.5).

We now address the proof of (4.6). Let $2 \le r < \infty$. By differentiating both sides of the first equation of (DD) with respect to x twice and multiplying by $|\partial_i \nabla u|^{r-2}\partial_i \nabla u$, we derive the following from (4.7), (4.8), (4.10), (4.19) and (4.20):

$$\begin{aligned}
&\frac{1}{r}\frac{d}{dt}\|\partial_i \nabla u(t)\|_{L^r(\mathbb{R}^n)}^r \\
&\le 2(r+n-2)(q-1)^2\|u(t)\|_{L^\infty(\mathbb{R}^n)}^{2(q-2)}\|\nabla v(t)\|_{L^\infty(\mathbb{R}^n)}^2\|\partial_i \nabla u(t)\|_{L^r(\mathbb{R}^n)}^r \\
&\qquad + 2(r+n-2)(q-1)^2(q-2)^2\|u(t)\|_{L^\infty(\mathbb{R}^n)}^{2(q-3)}\|\nabla u(t)\|_{L^\infty(\mathbb{R}^n)}^2\|\nabla v(t)\|_{L^\infty(\mathbb{R}^n)}^2 \\
&\qquad\qquad \times \|\nabla u(t)\|_{L^r(\mathbb{R}^n)}^2\|\partial_i \nabla u(t)\|_{L^r(\mathbb{R}^n)}^{r-2} \\
&\qquad + 2(r+n-2)(q-1)^2\|u(t)\|_{L^\infty(\mathbb{R}^n)}^{2(q-2)}\|\partial_i \nabla v(t)\|_{L^\infty(\mathbb{R}^n)}^2\|\nabla u(t)\|_{L^r(\mathbb{R}^n)}^2\|\partial_i \nabla u(t)\|_{L^r(\mathbb{R}^n)}^{r-2} \\
&\qquad + 2(r+n-2)(q-1)^2\|u(t)\|_{L^\infty(\mathbb{R}^n)}^{2(q-2)}\|\Delta v(t)\|_{L^\infty(\mathbb{R}^n)}^2\|\nabla u(t)\|_{L^r(\mathbb{R}^n)}^2\|\partial_i \nabla u(t)\|_{L^r(\mathbb{R}^n)}^{r-2} \\
&\qquad + 2(r+n-2)\|u(t)\|_{L^\infty(\mathbb{R}^n)}^{2(q-2)}\|\partial_i \Delta v(t)\|_{L^\infty(\mathbb{R}^n)}^2\|u(t)\|_{L^r(\mathbb{R}^n)}^2\|\partial_i \nabla u(t)\|_{L^r(\mathbb{R}^n)}^{r-2},
\end{aligned} \tag{4.21}$$

which leads to:

$$\begin{aligned}
&\sup_{0<t<T_1}\|\partial_i \nabla u(t)\|_{L^r(\mathbb{R}^n)} \\
&\le \Bigg(\|\partial_i \nabla u_0\|_{L^r(\mathbb{R}^n)} + 2(q-1)(q-2)\sqrt{(r+n-2)T_1}\sup_{0<t<T_1}\|u(t)\|_{L^\infty(\mathbb{R}^n)}^{q-3}\sup_{0<t<T_1}\|\nabla u(t)\|_{L^\infty(\mathbb{R}^n)} \\
&\qquad \times \sup_{0<t<T_1}\|\nabla v(t)\|_{L^\infty(\mathbb{R}^n)}\sup_{0<t<T_1}\|\nabla u(t)\|_{L^r(\mathbb{R}^n)} \\
&\qquad + 2(q-1)\sqrt{(r+n-2)T_1}\sup_{0<t<T_1}\|u(t)\|_{L^\infty(\mathbb{R}^n)}^{q-2}\sup_{0<t<T_1}\|\partial_i \nabla v(t)\|_{L^\infty(\mathbb{R}^n)}\sup_{0<t<T_1}\|\nabla u(t)\|_{L^r(\mathbb{R}^n)} \\
&\qquad + 2(q-1)\sqrt{(r+n-2)T_1}\sup_{0<t<T_1}\|u(t)\|_{L^\infty(\mathbb{R}^n)}^{q-2}\sup_{0<t<T_1}\|\Delta v(t)\|_{L^\infty(\mathbb{R}^n)}\sup_{0<t<T_1}\|\nabla u(t)\|_{L^r(\mathbb{R}^n)} \\
&\qquad + 2\sqrt{(r+n-2)T_1}\sup_{0<t<T_1}\|u(t)\|_{L^\infty(\mathbb{R}^n)}^{q-2}\sup_{0<t<T_1}\|\partial_i \Delta v(t)\|_{L^\infty(\mathbb{R}^n)}\sup_{0<t<T_1}\|u(t)\|_{L^r(\mathbb{R}^n)}\Bigg) \\
&\qquad \times \exp\left\{2(q-1)^2(r+n-2)T_1\sup_{0<t<T_1}\|u(t)\|_{L^\infty(\mathbb{R}^n)}^{2(q-2)}\sup_{0<t<T_1}\|\nabla v(t)\|_{L^\infty(\mathbb{R}^n)}^2\right\}.
\end{aligned} \tag{4.22}$$

Therefore, since $q = 2$ or $q \ge 3$, and $n + q \ge 2$, and since $\partial_i \nabla u_0 \in L^{n+q}(\mathbb{R}^n)$, from (4.22), we have $\partial_i \nabla u \in L^\infty(0, T_1; L^{n+q}(\mathbb{R}^n))$. Thus, we conclude the proof of (4.6).

(ii). parabolic-elliptic type (B): $\tau = 0$, $\gamma = 0$.

We establish the following regularities:

$$\Delta v \in L^\infty(0, T_1; L^\infty(\mathbb{R}^n)), \tag{4.23}$$

$$u \in L^\infty(0, T_1; L^\theta(\mathbb{R}^n)) \cap L^\infty(0, T_1; L^{n+q}(\mathbb{R}^n)), \tag{4.24}$$

$$\nabla v \in L^\infty(0, T_1; L^\infty(\mathbb{R}^n)), \tag{4.25}$$

$$\nabla u \in L^\infty(0, T_1; L^{n+q}(\mathbb{R}^n)), \tag{4.26}$$

$$\nabla u \in L^\infty(0, T_1; L^\infty(\mathbb{R}^n)), \tag{4.27}$$

$$\partial_i \Delta v \in L^\infty(0, T_1; L^\infty(\mathbb{R}^n)) \quad \text{for all } i = 1, 2, \ldots, n, \tag{4.28}$$

$$\partial_i \nabla u \in L^\infty(0, T_1; L^{n+q}(\mathbb{R}^n)) \quad \text{for all } i = 1, 2, \ldots, n, \tag{4.29}$$

where θ is the exponent introduced in Assumption 2.2 as part of the function space imposed on the initial data u_0.

With regard to (4.23), by applying (2.2) from Theorem 2.1 (I), and the second equation of (DD), we obtain:

$$\sup_{0<t<T_1} \|\Delta v(t)\|_{L^\infty(\mathbb{R}^n)} \leq \sup_{0<t<T_1} \|u(t)\|_{L^\infty(\mathbb{R}^n)}. \tag{4.30}$$

Therefore, (4.23) follows from (4.30).

Let $1 < r < \infty$. Regarding (4.24), by multiplying both sides of the first equation of (DD) by $|u|^{r-2}u$ and integrating over $\mathbb{R}^n$, we obtain:

$$\begin{aligned}
\frac{1}{r}\frac{d}{dt}\|u(t)\|^r_{L^r(\mathbb{R}^n)} &= -(r-1)\int_{\mathbb{R}^n} |\nabla u|^2 |u|^{r-2}\, dx - \chi(r-1)\int_{\mathbb{R}^n} |u|^{q-2}u\nabla v \cdot |u|^{r-2}\nabla u\, dx \\
&= -(r-1)\int_{\mathbb{R}^n} |\nabla u|^2 |u|^{r-2}\, dx + \frac{\chi(r-1)}{r+q-2}\int_{\mathbb{R}^n} \Delta v\, |u|^{r+q-2}\, dx \\
&\leq \frac{r-1}{r+q-2}\|u(t)\|^{q-2}_{L^\infty(\mathbb{R}^n)}\|\Delta v(t)\|_{L^\infty(\mathbb{R}^n)}\|u(t)\|^r_{L^r(\mathbb{R}^n)}.
\end{aligned}$$

By applying (4.30), we have:

$$\begin{aligned}
\sup_{0<t<T_1}\|u(t)\|_{L^r(\mathbb{R}^n)} &\leq \|u_0\|_{L^r(\mathbb{R}^n)} \exp\left\{T_1 \sup_{0<t<T_1}\|u(t)\|^{q-2}_{L^\infty(\mathbb{R}^n)} \sup_{0<t<T_1}\|\Delta v(t)\|_{L^\infty(\mathbb{R}^n)}\right\} \\
&\leq \|u_0\|_{L^r(\mathbb{R}^n)} \exp\left\{T_1 \sup_{0<t<T_1}\|u(t)\|^{q-1}_{L^\infty(\mathbb{R}^n)}\right\}.
\end{aligned} \tag{4.31}$$

Therefore, since $q = 2$ or $q \geq 3$, $1 < \theta < n$ and $n + q > 1$, by choosing $r = \theta$ and $r = n + q$, we deduce (4.24), where θ is the exponent introduced in Assumption 2.2.

Next, we establish (4.25). Considering that $\nabla G_0 \in L^{\frac{n}{n-1},\infty}(\mathbb{R}^n)$ for $n \geq 2$, we apply the weak-type Young inequality (see Grafakos, for example, [8, p. 73 Theorem 1.4.25]). Consequently, we obtain the following estimate:

$$\|\nabla v(t)\|_{L^r(\mathbb{R}^n)} \leq \|\nabla G_0\|_{L^{\frac{n}{n-1},\infty}(\mathbb{R}^n)}\|u(t)\|_{L^\theta(\mathbb{R}^n)} \tag{4.32}$$

for all $\frac{n}{n-1} < r < \infty$ and $1 < \theta < n$, with $\frac{1}{r} = \frac{n-1}{n} + \frac{1}{\theta} - 1 = -\frac{1}{n} + \frac{1}{\theta}$. Then, for all $n < r < \infty$, using the embedding theorem, (3.2) in Lemma 3.2, (4.31) and (4.32), and applying the second equation of (DD), we derive the following:

$$\begin{aligned}\|\nabla v(t)\|_{L^\infty(\mathbb{R}^n)} &\le C\|\nabla v(t)\|_{W^{1,r}(\mathbb{R}^n)}\\ &\le C\Big(\|\nabla v(t)\|_{L^r(\mathbb{R}^n)}+\|\Delta v(t)\|_{L^r(\mathbb{R}^n)}\Big)\\ &\le C\Big(\|u(t)\|_{L^\theta(\mathbb{R}^n)}+\|u(t)\|_{L^r(\mathbb{R}^n)}\Big)\end{aligned}$$

for a.e. $0<t<T_1$, where $\frac{1}{r}=-\frac{1}{n}+\frac{1}{\theta}$, and C depends only on n and θ. Thus, since $u_0\in L^\theta(\mathbb{R}^n)\cap L^\infty(\mathbb{R}^n)$, we conclude that (4.25) holds.

The regularities (4.26), (4.27), (4.28) and (4.29) follow from similar arguments used in the parabolic-elliptic case with $\tau=0$ and $\gamma>0$ as in (4.10) through (4.22).

(iii). parabolic-parabolic type: $\tau=1$, $\gamma\ge 0$.

We establish the following regularities:

$$\nabla v\in L^\infty(0,T_1;L^\infty(\mathbb{R}^n)), \tag{4.33}$$

$$u,v,\nabla v\in L^\infty(0,T_1;L^r(\mathbb{R}^n)) \qquad \text{for all } n+q\le r<\infty, \tag{4.34}$$

$$\nabla u,\Delta v\in L^\infty(0,T_1;L^r(\mathbb{R}^n)) \qquad \text{for all } n+q\le r<\infty, \tag{4.35}$$

$$\nabla^2 u,\nabla^3 v\in L^\infty(0,T_1;L^{n+q}(\mathbb{R}^n)), \tag{4.36}$$

$$\nabla u,\Delta v\in L^\infty(0,T_1;L^\infty(\mathbb{R}^n)). \tag{4.37}$$

We first prove (4.33) and (4.34). From (3.9) and (3.10) in Lemma 3.8, we obtain the following estimates:

$$\sup_{0<t<T_1}\|\nabla v(t)\|_{L^\infty(\mathbb{R}^n)}\le\|\nabla v_0\|_{L^\infty(\mathbb{R}^n)}+C\sup_{0<t<T_1}\|u(t)\|_{L^\infty(\mathbb{R}^n)}, \tag{4.38}$$

where C depends only on n and T_1. Additionally, let $1<r<\infty$. By multiplying both sides of the first equation of (DD) by $|u|^{r-2}u$ and integrating over $\mathbb{R}^n$, we obtain:

$$\begin{aligned}\frac{1}{r}\frac{d}{dt}\|u(t)\|^r_{L^r(\mathbb{R}^n)} &= -(r-1)\int_{\mathbb{R}^n}|u|^{r-2}|\nabla u|^2\,dx-\chi(r-1)\int_{\mathbb{R}^n}|u|^{q-2}u\nabla v\cdot|u|^{r-2}\nabla u\,dx\\ &\le -(r-1)\int_{\mathbb{R}^n}|u|^{r-2}|\nabla u|^2\,dx\\ &\qquad+\frac{r-1}{4}\int_{\mathbb{R}^n}|u|^{r-2}|\nabla u|^2\,dx+(r-1)\int_{\mathbb{R}^n}|\nabla v|^2|u|^{r+2q-4}\,dx\\ &\le (r-1)\|u(t)\|^{2(q-2)}_{L^\infty(\mathbb{R}^n)}\|\nabla v(t)\|^2_{L^\infty(\mathbb{R}^n)}\|u(t)\|^r_{L^r(\mathbb{R}^n)},\end{aligned}$$

which, by applying (2.2) from Theorem 2.1 (I), (4.38) and the Gronwall inequality, leads to:

$$\sup_{0<t<T_1}\|u(t)\|_{L^r(\mathbb{R}^n)}$$

$$\leq \|u_0\|_{L^r(\mathbb{R}^n)} \exp\left\{(r-1)T_1 \sup_{0<t<T_1} \|u(t)\|_{L^\infty(\mathbb{R}^n)}^{2(q-2)} \sup_{0<t<T_1} \|\nabla v(t)\|_{L^\infty(\mathbb{R}^n)}^2\right\}.$$

Since $q = 2$ or $q \geq 3$, and $n + q > 1$, we have $u \in L^\infty(0, T_1; L^r(\mathbb{R}^n))$ for all $n + q \leq r < \infty$.

By applying (3.9) and (3.10) in Lemma 3.8, we obtain the following estimates:

$$\sup_{0<t<T_1} \|v(t)\|_{L^r(\mathbb{R}^n)} \leq \|v_0\|_{L^r(\mathbb{R}^n)} + C \sup_{0<t<T_1} \|u(t)\|_{L^r(\mathbb{R}^n)}$$

and:

$$\sup_{0<t<T_1} \|\nabla v(t)\|_{L^r(\mathbb{R}^n)} \leq \|\nabla v_0\|_{L^r(\mathbb{R}^n)} + C \sup_{0<t<T_1} \|u(t)\|_{L^r(\mathbb{R}^n)},$$

where C depends only on n and T_1. Thus, we conclude (4.34).

We now turn to the proof of (4.35). Let $2 \leq r < \infty$. By applying (3.11) in Lemma 3.8, we obtain the following estimate:

$$\begin{aligned}\|\Delta v(t)\|_{L^r(\mathbb{R}^n)}^2 &\leq C\|\nabla^2 v(t)\|_{L^r(\mathbb{R}^n)}^2 \\ &\leq C\|\nabla^2 v_0\|_{L^r(\mathbb{R}^n)}^2 + C(r+n-2)\int_0^t \|\nabla u(s)\|_{L^r(\mathbb{R}^n)}^2\, ds \qquad (4.39)\end{aligned}$$

for a.e. $0 < t < T_1$, where C depends only on n. Additionally, by differentiating both sides of the first equation of (DD) with respect to x once and multiplying by $|\nabla u|^{r-2}\nabla u$, we obtain:

$$\begin{aligned}\frac{1}{r}\frac{d}{dt}\|\nabla u(t)\|_{L^r(\mathbb{R}^n)}^r &\leq (r+n-2)(q-1)^2\|u(t)\|_{L^\infty(\mathbb{R}^n)}^{2(q-2)}\|\nabla v(t)\|_{L^\infty(\mathbb{R}^n)}^2\|\nabla u(t)\|_{L^r(\mathbb{R}^n)}^r \\ &\quad + (r+n-2)\|u(t)\|_{L^\infty(\mathbb{R}^n)}^{2(q-1)}\|\Delta v(t)\|_{L^r(\mathbb{R}^n)}^2\|\nabla u(t)\|_{L^r(\mathbb{R}^n)}^{r-2},\end{aligned}$$

which, by using (4.39), yields:

$$\begin{aligned}\sup_{0<t<T_1}\|\nabla u(t)\|_{L^r(\mathbb{R}^n)} &\leq \left(\|\nabla u_0\|_{L^r(\mathbb{R}^n)} + \sqrt{C(r+n-2)T_1}\sup_{0<t<T_1}\|u(t)\|_{L^\infty(\mathbb{R}^n)}^{q-1}\|\nabla^2 v_0\|_{L^r(\mathbb{R}^n)}\right) \\ &\quad \times \exp\Bigg\{C(r+n-2)^2(q-1)^2(T_1+1)^2\sup_{0<t<T_1}\|u(t)\|_{L^\infty(\mathbb{R}^n)}^{2(q-2)} \\ &\quad \times\left(\sup_{0<t<T_1}\|u(t)\|_{L^\infty(\mathbb{R}^n)}^2 + \sup_{0<t<T_1}\|\nabla v(t)\|_{L^\infty(\mathbb{R}^n)}^2\right)\Bigg\}, \qquad (4.40)\end{aligned}$$

where C depends only on n. Thus, since $q = 2$ or $q \geq 3$, and $n + q \geq 2$, we observe that $\nabla u \in L^\infty(0, T_1; L^r(\mathbb{R}^n))$ for all $n + q \leq r < \infty$. Moreover, from (4.39) together with (4.40), we infer $\Delta v \in L^\infty(0, T_1; L^r(\mathbb{R}^n))$. Thus, we conclude (4.35).

We now establish (4.36). Using a similar calculation as in (4.21), we derive the following inequality:

$$
\begin{aligned}
&\frac{1}{r}\frac{d}{dt}\|\partial_i \nabla u(t)\|_{L^r(\mathbb{R}^n)}^r \\
&\le 2(r+n-2)(q-1)^2\|u(t)\|_{L^\infty(\mathbb{R}^n)}^{2(q-2)}\|\nabla v(t)\|_{L^\infty(\mathbb{R}^n)}^2\|\partial_i \nabla u(t)\|_{L^r(\mathbb{R}^n)}^r \\
&\quad + 2(r+n-2)(q-1)^2(q-2)^2\|u(t)\|_{L^\infty(\mathbb{R}^n)}^{2(q-3)}\|\nabla v(t)\|_{L^\infty(\mathbb{R}^n)}^2\|\nabla u(t)\|_{L^{2r}(\mathbb{R}^n)}^4\|\partial_i \nabla u(t)\|_{L^r(\mathbb{R}^n)}^{r-2} \\
&\quad + 2(r+n-2)(q-1)^2\|u(t)\|_{L^\infty(\mathbb{R}^n)}^{2(q-2)}\|\partial_i \nabla v(t)\|_{L^{2r}(\mathbb{R}^n)}^2\|\nabla u(t)\|_{L^{2r}(\mathbb{R}^n)}^2\|\partial_i \nabla u(t)\|_{L^r(\mathbb{R}^n)}^{r-2} \\
&\quad + 2(r+n-2)(q-1)^2\|u(t)\|_{L^\infty(\mathbb{R}^n)}^{2(q-2)}\|\Delta v(t)\|_{L^{2r}(\mathbb{R}^n)}^2\|\nabla u(t)\|_{L^{2r}(\mathbb{R}^n)}^2\|\partial_i \nabla u(t)\|_{L^r(\mathbb{R}^n)}^{r-2} \\
&\quad + 2(r+n-2)\|u(t)\|_{L^\infty(\mathbb{R}^n)}^{2(q-1)}\|\partial_i \Delta v(t)\|_{L^r(\mathbb{R}^n)}^2\|\partial_i \nabla u(t)\|_{L^r(\mathbb{R}^n)}^{r-2} \qquad (4.41)
\end{aligned}
$$

for a.e. $0<t<T_1$ and for all $2 \le r < \infty$. Additionally, taking $i=3$ in (3.11) in Lemma 3.8, we obtain:

$$
\begin{aligned}
\|\partial_i \Delta v(t)\|_{L^r(\mathbb{R}^n)}^2 &\le C\|\nabla^3 v(t)\|_{L^r(\mathbb{R}^n)}^2 \\
&\le C\|\nabla^3 v_0\|_{L^r(\mathbb{R}^n)}^2 + C(r+n-2)\int_0^t \|\nabla^2 u(s)\|_{L^r(\mathbb{R}^n)}^2\, ds \qquad (4.42)
\end{aligned}
$$

for a.e. $0<t<T_1$, where C depends only on n. Combining (4.41) with (4.42), we observe the following inequality:

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\partial_i \nabla u(t)\|_{L^r(\mathbb{R}^n)}^2 \\
&\le (r+n-2)\Big(2(q-1)^2(q-2)^2\|u(t)\|_{L^\infty(\mathbb{R}^n)}^{2(q-3)}\|\nabla v(t)\|_{L^\infty(\mathbb{R}^n)}^2\|\nabla u(t)\|_{L^{2r}(\mathbb{R}^n)}^4 \\
&\qquad + 2(q-1)^2\|u(t)\|_{L^\infty(\mathbb{R}^n)}^{2(q-2)}\|\partial_i \nabla v(t)\|_{L^{2r}(\mathbb{R}^n)}^2\|\nabla u(t)\|_{L^{2r}(\mathbb{R}^n)}^2 \\
&\qquad + 2(q-1)^2\|u(t)\|_{L^\infty(\mathbb{R}^n)}^{2(q-2)}\|\Delta v(t)\|_{L^{2r}(\mathbb{R}^n)}^2\|\nabla u(t)\|_{L^{2r}(\mathbb{R}^n)}^2 \\
&\qquad + \|\nabla^3 v_0\|_{L^r(\mathbb{R}^n)}^2\|u(t)\|_{L^\infty(\mathbb{R}^n)}^{2(q-1)}\Big) \\
&\quad + 2(r+n-2)(q-1)^2\|u(t)\|_{L^\infty(\mathbb{R}^n)}^{2(q-2)}\|\nabla v(t)\|_{L^\infty(\mathbb{R}^n)}^2\|\partial_i \nabla u(t)\|_{L^r(\mathbb{R}^n)}^2 \\
&\quad + C(r+n-2)^2\|u(t)\|_{L^\infty(\mathbb{R}^n)}^{2(q-1)}\int_0^t \|\nabla^2 u(s)\|_{L^r(\mathbb{R}^n)}^2\, ds \qquad (4.43)
\end{aligned}
$$

for a.e. $0<t<T_1$, where C depends only on n. Noting:

$$
\int_0^t\int_0^\tau \|\nabla^2 u(s)\|_{L^r(\mathbb{R}^n)}^2\, ds d\tau \le \int_0^t\int_0^t \|\nabla^2 u(s)\|_{L^r(\mathbb{R}^n)}^2\, ds d\tau = T_1\int_0^t \|\nabla^2 u(s)\|_{L^r(\mathbb{R}^n)}^2\, ds,
$$

and:

$$n^{-\frac{r}{2}}\|\nabla^2 u(t)\|_{L^r(\mathbb{R}^n)}^r \le \sum_{i=1}^{n} \|\partial_i \nabla u(t)\|_{L^r(\mathbb{R}^n)}^r,$$

by integrating both sides of (4.43) from 0 to t, we have, for all $2 \le r < \infty$:

$$\begin{aligned}
&\|\nabla^2 u(t)\|_{L^r(\mathbb{R}^n)}^2 \\
&\le \|\nabla^2 u_0\|_{L^r(\mathbb{R}^n)}^2 + 2n^{\frac{r}{2}}(r+n-2)T_1 \\
&\qquad \times \Big(2(q-1)^2(q-2)^2 \sup_{0<t<T_1}\|u(t)\|_{L^\infty(\mathbb{R}^n)}^{2(q-3)} \sup_{0<t<T_1}\|\nabla v(t)\|_{L^\infty(\mathbb{R}^n)}^2 \sup_{0<t<T_1}\|\nabla u(t)\|_{L^{2r}(\mathbb{R}^n)}^4 \\
&\qquad\quad + 2(q-1)^2 \sup_{0<t<T_1}\|u(t)\|_{L^\infty(\mathbb{R}^n)}^{2(q-2)} \sup_{0<t<T_1}\|\nabla^2 v(t)\|_{L^{2r}(\mathbb{R}^n)}^2 \sup_{0<t<T_1}\|\nabla u(t)\|_{L^{2r}(\mathbb{R}^n)}^2 \\
&\qquad\quad + 2(q-1)^2 \sup_{0<t<T_1}\|u(t)\|_{L^\infty(\mathbb{R}^n)}^{2(q-2)} \sup_{0<t<T_1}\|\Delta v(t)\|_{L^{2r}(\mathbb{R}^n)}^2 \sup_{0<t<T_1}\|\nabla u(t)\|_{L^{2r}(\mathbb{R}^n)}^2 \\
&\qquad\quad + \|\nabla^3 v_0\|_{L^r(\mathbb{R}^n)}^2 \sup_{0<t<T_1}\|u(t)\|_{L^\infty(\mathbb{R}^n)}^{2(q-1)}\Big) \\
&\quad + 4n^{\frac{r}{2}}(r+n-2)(q-1)^2 \sup_{0<t<T_1}\|u(t)\|_{L^\infty(\mathbb{R}^n)}^{2(q-2)} \sup_{0<t<T_1}\|\nabla v(t)\|_{L^\infty(\mathbb{R}^n)}^2 \int_0^t \|\nabla^2 u(s)\|_{L^r(\mathbb{R}^n)}^2\,ds \\
&\quad + Cn^{\frac{r}{2}}(r+n-2)^2 T_1 \sup_{0<t<T_1}\|u(t)\|_{L^\infty(\mathbb{R}^n)}^{2(q-1)} \int_0^t \|\nabla^2 u(s)\|_{L^r(\mathbb{R}^n)}^2\,ds \\
&=: \|\nabla^2 u_0\|_{L^r(\mathbb{R}^n)}^2 \\
&\quad + 2n^{\frac{r}{2}}(r+n-2)T_1(M_3^2+M_4^2+M_5^2+M_6^2) + (M_7+M_8)\int_0^t \|\nabla^2 u(s)\|_{L^r(\mathbb{R}^n)}^2\,ds
\end{aligned}$$

for a.e. $0 < t < T_1$, where C depends only on n. Here, $M_3, M_4, M_5, M_6, M_7, M_8$ are defined as follows:

$$\begin{aligned}
M_3 &:= \sqrt{2}(q-1)(q-2) \sup_{0<t<T_1}\|u(t)\|_{L^\infty(\mathbb{R}^n)}^{q-3} \sup_{0<t<T_1}\|\nabla v(t)\|_{L^\infty(\mathbb{R}^n)} \sup_{0<t<T_1}\|\nabla u(t)\|_{L^{2r}(\mathbb{R}^n)}^2, \\
M_4 &:= \sqrt{2}(q-1) \sup_{0<t<T_1}\|u(t)\|_{L^\infty(\mathbb{R}^n)}^{q-2} \sup_{0<t<T_1}\|\nabla^2 v(t)\|_{L^{2r}(\mathbb{R}^n)} \sup_{0<t<T_1}\|\nabla u(t)\|_{L^{2r}(\mathbb{R}^n)}, \\
M_5 &:= \sqrt{2}(q-1) \sup_{0<t<T_1}\|u(t)\|_{L^\infty(\mathbb{R}^n)}^{q-2} \sup_{0<t<T_1}\|\Delta v(t)\|_{L^{2r}(\mathbb{R}^n)} \sup_{0<t<T_1}\|\nabla u(t)\|_{L^{2r}(\mathbb{R}^n)}, \\
M_6 &:= \|\nabla^3 v_0\|_{L^r(\mathbb{R}^n)} \sup_{0<t<T_1}\|u(t)\|_{L^\infty(\mathbb{R}^n)}^{q-1}, \\
M_7 &:= 4n^{\frac{r}{2}}(r+n-2)(q-1)^2 \sup_{0<t<T_1}\|u(t)\|_{L^\infty(\mathbb{R}^n)}^{2(q-2)} \sup_{0<t<T_1}\|\nabla v(t)\|_{L^\infty(\mathbb{R}^n)}^2,
\end{aligned}$$

$$M_8 := Cn^{\frac{r}{2}}(r+n-2)^2 T_1 \sup_{0<t<T_1} \|u(t)\|_{L^\infty(\mathbb{R}^n)}^{2(q-1)},$$

where C depends only on n. This, together with the Gronwall inequality, leads to:

$$\sup_{0<t<T_1} \|\nabla^2 u(t)\|_{L^r(\mathbb{R}^n)} \le \left(\|\nabla^2 u_0\|_{L^r(\mathbb{R}^n)} + \sqrt{2n^{\frac{r}{2}}(r+n-2)T_1}(M_3+M_4+M_5+M_6) \right) \times \exp\left\{ \frac{1}{2}(M_7+M_8)T_1 \right\} \tag{4.44}$$

for a.e. $0<t<T_1$, where C depends only on n. Since $\nabla^2 u_0, \nabla^3 v_0 \in L^{n+q}(\mathbb{R}^n)$, from (4.42) and (4.44), we have $\nabla^2 u, \nabla^3 v \in L^\infty(0, T_1; L^{n+q}(\mathbb{R}^n))$. Thus, we conclude (4.36).

We prove (4.37) as follows. By the Morrey inequality, there exists a positive constant C depending only on n and q such that:

$$\sup_{0<t<T_1} \|\nabla u(t)\|_{L^\infty(\mathbb{R}^n)} \le C \sup_{0<t<T_1} \|\nabla u(t)\|_{W^{1,n+q}(\mathbb{R}^n)}$$

and:

$$\sup_{0<t<T_1} \|\Delta v(t)\|_{L^\infty(\mathbb{R}^n)} \le C \sup_{0<t<T_1} \|\Delta v(t)\|_{W^{1,n+q}(\mathbb{R}^n)}$$

From (4.35) and (4.36), we have the boundedness of $\|\nabla u(t)\|_{W^{1,n+q}(\mathbb{R}^n)}$ and $\|\Delta v(t)\|_{W^{1,n+q}(\mathbb{R}^n)}$. Consequently, (4.37) is obtained. This completes the proof of Lemma 4.2. □

Continuation of the Proof of Theorem 2.1 (II). We are now ready to prove Theorem 2.1 (II). From the construction of the solution described in Subsection 4.1, we observe that the local existence time T_1 depends on n, q, γ and u_0 (and on v_0, if applicable). Specifically, the dependencies for each case are as follows:

(i). **parabolic-elliptic type (A)**: $\tau = 0, \gamma > 0$.
The time T_1 depends only on n, q, γ, and $\|u_0\|_{W^{2,n+q}(\mathbb{R}^n)}$.

(ii). **parabolic-elliptic type (B)**: $\tau = 0, \gamma = 0$.
The time T_1 depends only on n, q, $\|u_0\|_{L^\theta(\mathbb{R}^n)}$, and $\|u_0\|_{W^{2,n+q}(\mathbb{R}^n)}$ for some $1<\theta<n$, where θ is introduced in Assumption 2.2.

(iii). **parabolic-parabolic type**: $\tau = 1, \gamma \ge 0$.
The time T_1 depends only on n, q, γ, $\|u_0\|_{W^{2,n+q}(\mathbb{R}^n)}$, and $\|v_0\|_{W^{3,n+q}(\mathbb{R}^n)}$.

Our objective is to extend the strong solution (u, v) from $[0, T_1)$ to $[0, \widehat{T})$, where $T_1 < \widehat{T} < T_{\max}$. Here, $T_{\max}$ refers to the maximal existence time, the upper bound for the interval during which the solution remains bounded in the $L^\infty(\mathbb{R}^n)$-norm. In other words, $T_{\max}$ is characterized by the property:

$$\limsup_{t\to T_{\max}-0} \|u(t)\|_{L^\infty(\mathbb{R}^n)} = \infty,$$

indicating that the solution $u(t)$ becomes unbounded in the $L^\infty(\mathbb{R}^n)$-norm as t approaches $T_{\max}$.

To achieve this extension, we assume (2.3). Under this condition, we derive the following estimates from Lemma 4.2:

(i). **parabolic-elliptic type (A)**: $\tau = 0$, $\gamma > 0$.
There exists a positive constant C depending only on n, q, γ, T_0 and $\|u_0\|_{W^{2,n+q}(\mathbb{R}^n)}$ such that:

$$\sup_{0<t<T_0} \|u(t)\|_{W^{2,n+q}(\mathbb{R}^n)} \le C. \tag{4.45}$$

(ii). **parabolic-elliptic type (B)**: $\tau = 0$, $\gamma = 0$.
There exists a positive constant C depending only on n, q, T_0, $\|u_0\|_{L^\theta(\mathbb{R}^n)}$ and $\|u_0\|_{W^{2,n+q}(\mathbb{R}^n)}$ such that:

$$\sup_{0<t<T_0} \|u(t)\|_{L^\theta(\mathbb{R}^n)} + \sup_{0<t<T_0} \|u(t)\|_{W^{2,n+q}(\mathbb{R}^n)} \le C \tag{4.46}$$

for some $1 < \theta < n$, where θ is introduced in Assumption 2.2.

(iii). **parabolic-parabolic type**: $\tau = 1$, $\gamma \ge 0$.
There exists a positive constant C depending only on n, q, γ, T_0, $\|u_0\|_{W^{2,n+q}(\mathbb{R}^n)}$, and $\|v_0\|_{W^{3,n+q}(\mathbb{R}^n)}$ such that:

$$\sup_{0<t<T_0} \|u(t)\|_{W^{2,n+q}(\mathbb{R}^n)} + \sup_{0<t<T_0} \|v(t)\|_{W^{3,n+q}(\mathbb{R}^n)} \le C. \tag{4.47}$$

From (4.45), (4.46) and (4.47), the solution $u(T_0)$ (and on $v(T_0)$, if applicable) belongs to the same function space as the initial data specified in Assumption 2.2.

We then consider T_0 as an initial time and apply the construction method outlined in Subsection 4.1, using $u(T_0)$ (and on $v(T_0)$, if applicable) as initial data. This enables us to extend the strong solution (u, v) over $[T_0, T_1^{(1)})$. Here, the existence time $T_1^{(1)}$ is determined by n, q, γ and $u(T_0)$ (and on $v(T_0)$, if applicable), as specified in Assumption 2.2. By applying Lemma 4.2, we derive the same estimates as in (4.45), (4.46), and (4.47), but now over $[T_0, T_0^{(1)}]$, where $T_0^{(1)} < T_1^{(1)}$.

From the estimates obtained over $[T_0, T_0^{(1)}]$, we ensure that the solution $u(T_0^{(1)})$ (and on $v(T_0^{(1)})$, if applicable) belongs to the same function space as the initial data specified in Assumption 2.2. Consequently, we are able to reapply the construction method from Subsection 4.1, treating $u(T_0^{(1)})$ (and on $v(T_0^{(1)})$, if applicable) as the initial data. This allows us to construct the strong solution (u, v) on $[T_0^{(1)}, T_1^{(2)})$.

Repeating this procedure iteratively, we define sequences $\{T_0^{(k)}\}$ and $\{T_1^{(k)}\}$ for $k = 1, 2, \ldots$. In addition, we set $T_0^{(0)} := T_0$, and construct solutions on $[T_0^{(k-1)}, T_1^{(k)})$, ensuring at each step that:

- The same estimates as in (4.45), (4.46), and (4.47) hold over $[T_0^{(k-1)}, T_0^{(k)}]$, where $T_0^{(k)} < T_1^{(k)}$. These estimates are guaranteed by Lemma 4.2.
- The construction method from Subsection 4.1 can be reapplied using $u(T_0^{(k)})$ (and on $v(T_0^{(k)})$, if applicable) as initial data

- We extend the strong solution (u, v) over $[T_0^{(k)}, T_1^{(k+1)})$.

Therefore, by this method of iterative extension, we have successfully extended the strong solution (u, v) to $[0, \widehat{T})$ for any $\widehat{T} < T_{\max}$.

Based on the above facts, the following conclusion follows: the solution can be extended to the maximal existence time $T_{\max}$, at which time the solution may become unbounded in the $L^\infty(\mathbb{R}^n)$-norm. Specifically, $T_{\max}$ is characterized by the condition:

$$\limsup_{t \to T_{\max}-0} \|u(t)\|_{L^\infty(\mathbb{R}^n)} = \infty,$$

indicating that the solution may not cease to exist in the strong sense beyond $T_{\max}$. This condition suggests that a blow-up in the $L^\infty(\mathbb{R}^n)$-norm is a potential cause for the termination of the solution's existence at $T_{\max}$. This completes the proof of Theorem 2.1 (II). $\square$

4.3. Proof of Theorem 2.1 (III): extended existence of solutions up to maximal time

To establish Theorem 2.1 (III), we first apply the local existence result in Theorem 2.1 (I), which guarantees the existence of a strong solution on a small time interval $[0, T_1)$ for some $T_1 > 0$, depending on the norms of the initial data u_0 (and on v_0, if applicable) introduced in Assumption 2.2. The solution is constructed in the appropriate function spaces $W(Q_{T_1}) \cap X_{T_1}$, and by the *a priori* estimates derived in Subsection 4.2, we ensure that, as long as the $L^\infty(\mathbb{R}^n)$-norm of $u(t)$ is bounded, the $W^{2,n+q}(\mathbb{R}^n)$-norm of $u(t)$ (and the $W^{3,n+q}(\mathbb{R}^n)$-norm of $v(t)$, if applicable) remains finite.

Furthermore, the uniqueness of the solution in the class $W(Q_{T_{\max}}) \cap X_{T_{\max}}$ follows from the contraction mapping principle applied in the proof of Theorem 2.1 (II), ensuring that no other solutions with the same initial data exist in this class. Therefore, the strong solution (u, v) is unique on $[0, T_{\max})$.

In addition, we aim to establish (2.7) in Theorem 2.1 (III):

$$u \in C([0, T]; L^r_{\mathrm{loc}}(\mathbb{R}^n)) \quad \text{for all} \quad n + q \le r \le \infty. \tag{4.48}$$

From Lemma 4.2, the following regularity properties hold:

$$u \in L^\infty(0, T; W^{1,n+q}(\mathbb{R}^n)) \quad \text{and} \quad \partial_t u \in L^\infty(0, T; L^{n+q}(\mathbb{R}^n)).$$

These regularity conditions, combined with the compact embedding $W^{1,n+q}(\Omega) \subset C(\Omega)$ for any bounded subset $\Omega \subset \mathbb{R}^n$, imply that:

$$u \in C([0, T]; C(\Omega))$$

for all bounded subset Ω. Since this embedding is valid only on bounded domains, we restrict our analysis to such regions. By invoking the Aubin-Lions Lemma, we conclude that u is continuous in the local $L^r(\mathbb{R}^n)$-spaces, leading to the desired result (4.48). Thus, we confirm that u resides in the appropriate continuity space for all $n + q \le r \le \infty$. This completes the proof of Theorem 2.1 (III). $\square$

4.4. Proof of Theorem 2.1 (IV): charge conservation law

We now proceed with the proof of Theorem 2.1 (IV). Let $0 < \widehat{T} < T_{\max}$, where $T_{\max}$ denotes the maximal existence time. We assume that the initial data satisfies $u_0 \in L^1(\mathbb{R}^n)$. Since $u_0 \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, by modifying Lemma 4.2, there exists a constant C such that the following bound holds:

$$\sup_{0<t<\widehat{T}} \|u(t)\|_{L^p(\mathbb{R}^n)} \leq C \quad \text{for all } 1 < p \leq \infty, \tag{4.49}$$

and:

$$\sup_{0<t<\widehat{T}} \|\nabla v(t)\|_{L^\infty(\mathbb{R}^n)} + \sup_{0<t<\widehat{T}} \|\Delta v(t)\|_{L^\infty(\mathbb{R}^n)} \leq C. \tag{4.50}$$

Here, the constant C depends as follows:

(i). **parabolic-elliptic type (A)**: $\tau = 0$, $\gamma > 0$.
The constant C depends only on n, q, γ, $\|u_0\|_{L^1(\mathbb{R}^n)}$ and $\|u_0\|_{W^{2,n+q}(\mathbb{R}^n)}$.
(ii). **parabolic-elliptic type (B)**: $\tau = 0$, $\gamma = 0$.
The constant C depends only on n, q, $\|u_0\|_{L^1(\mathbb{R}^n)}$ and $\|u_0\|_{W^{2,n+q}(\mathbb{R}^n)}$.
(iii). **parabolic-parabolic type**: $\tau = 1$, $\gamma \geq 0$.
The constant C depends only on n, q, γ, $\|u_0\|_{L^1(\mathbb{R}^n)}$, $\|u_0\|_{W^{2,n+q}(\mathbb{R}^n)}$ and $\|v_0\|_{W^{3,n+q}(\mathbb{R}^n)}$.

To establish Theorem 2.1 (IV), we first present the following lemma:

Lemma 4.3. *We suppose that all assumptions of Theorem 2.1* (IV) *hold. We assume, moreover, that the initial data u_0 belongs to $L^1(\mathbb{R}^n)$. Then, the strong solution u satisfies*:

$$u \in L^\infty(0, \widehat{T}; L^1(\mathbb{R}^n)).$$

Proof of Lemma 4.3. We begin by defining the positive and negative parts of a function f as follows:

$$[f]^+(x) := \max\{0, f(x)\}, \quad [f]^-(x) := -\min\{0, f(x)\}. \tag{4.51}$$

Then, we have the identity:

$$|f(x)| = [f]^+(x) + [f]^-(x) = [f]^+(x) + [-f]^+(x). \tag{4.52}$$

We next focus on the contribution arising from the positive part of u, which constitutes the first step toward establishing its $L^1(\mathbb{R}^n)$-integrability. To this end, we introduce a cut-off function $\eta \in C^1(\mathbb{R})$ satisfying $0 \leq \eta(s) \leq 1$ and $0 \leq \eta'(s) \leq 2$ for all $s \in \mathbb{R}$, and specifically set:

$$\eta(s) := \begin{cases} 0 & \text{for } s \leq 0, \\ 1 & \text{for } s \geq 1. \end{cases} \tag{4.53}$$

We further define, for each parameter $m = 1, 2, \ldots$, the cut-off function:

$$\eta_m : \mathbb{R} \longrightarrow [0, 1], \qquad \eta_m(r) := \eta(mr) \quad \text{for all } r \in \mathbb{R},$$

where r is the variable and m indexes the family $\{\eta_m\}_{m\in\mathbb{N}}$. Then, it holds that:

$$0 \le \eta_m'(r) \le 2m \quad \text{for all } r \in \mathbb{R}. \tag{4.54}$$

Multiplying the first equation of (DD) by $\eta_m(u)\,\phi_R$ and integrating over $\mathbb{R}^n \times (t_0, t_1)$, we obtain:

$$\int_{t_0}^{t_1}\int_{\mathbb{R}^n} \partial_s u\, \eta_m(u)\,\phi_R\, dxds = I + II, \tag{4.55}$$

for a.e. $0 < t_0 < t_1 < \widehat{T}$, where we set:

$$I := -\int_{t_0}^{t_1}\int_{\mathbb{R}^n} \nabla u \cdot \nabla\big(\eta_m(u)\,\phi_R\big)\, dxds,$$

$$II := \chi \int_{t_0}^{t_1}\int_{\mathbb{R}^n} \nabla \cdot \big(|u|^{q-2}u\,\nabla v\big)\,\eta_m(u)\,\phi_R\, dxds.$$

We first bound the term I. Applying the Leibniz rule, we obtain:

$$I = -\int_{t_0}^{t_1}\int_{\mathbb{R}^n} |\nabla u|^2\,\eta_m'(u)\,\phi_R\, dxds - \int_{t_0}^{t_1}\int_{\mathbb{R}^n} \nabla u \cdot \eta_m(u)\,\nabla\phi_R\, dxds =: -I_1 + I_2. \tag{4.56}$$

Since $\eta_m'(r) \ge 0$ for all $r \in \mathbb{R}$, as stated in (4.54), we have $I_1 \ge 0$. Hence, to obtain an upper bound for I, it suffices to estimate I_2 from above. To this end, we introduce:

$$D_m := \Big\{(x, t) \in \mathbb{R}^n \times (0, \widehat{T}) \;\Big|\; 0 < u(x, t) < \tfrac{1}{m}\Big\}.$$

Using (4.54) and Lemma 3.1 yields:

$$\begin{aligned} I_2 &= \int_{t_0}^{t_1}\int_{\mathbb{R}^n} u\,\eta_m'(u)\,\nabla u \cdot \nabla\phi_R\, dxds + \int_{t_0}^{t_1}\int_{\mathbb{R}^n} u\,\eta_m(u)\,\Delta\phi_R\, dxds \\ &\le \frac{1}{4} I_1 + \int_{t_0}^{t_1}\int_{D_m \cap \operatorname{supp}\phi_R} |u|^2 \eta_m'(u)\,|\nabla\phi_R|^2 \frac{1}{\phi_R}\, dxds + \int_{t_0}^{t_1}\int_{\mathbb{R}^n} u\,\eta_m(u)\,\Delta\phi_R\, dxds \end{aligned}$$

$$\le \frac{1}{4} I_1 + \int_{t_0}^{t_1} \int_{D_m \cap \operatorname{supp}\phi_R} \frac{16n}{mR^2}\, dxds + \frac{12n}{R^2} \int_{t_0}^{t_1} \|u(s)\|_{L^1(\mathbb{R}^n)}\, ds. \tag{4.57}$$

Combining (4.56) and (4.57) gives:

$$I \le -\frac{3}{4} I_1 + \int_{t_0}^{t_1} \int_{\operatorname{supp}\phi_R} \frac{16n}{mR^2}\, dxds + \frac{12n}{R^2} \int_{t_0}^{t_1} \|u(s)\|_{L^1(\mathbb{R}^n)}\, ds. \tag{4.58}$$

We now bound the term II. Applying the Leibniz rule, we obtain:

$$\begin{aligned} II &= \chi(q-1) \int_{t_0}^{t_1} \int_{\mathbb{R}^n} |u|^{q-2} \nabla u \cdot \nabla v\, \eta_m(u)\, \phi_R\, dxds \\ &\quad + \chi \int_{t_0}^{t_1} \int_{\mathbb{R}^n} |u|^{q-2} u\, \Delta v\, \eta_m(u)\, \phi_R\, dxds \\ &=: II_1 + II_2. \end{aligned} \tag{4.59}$$

We next bound II_1 using the identity:

$$(q-1)\, |u|^{q-2} \nabla u\, \eta_m(u) = \nabla |u|^{q-1}\, \eta_m(u).$$

We apply the Leibniz rule and perform integration by parts. Together with Lemma 3.1, this yields:

$$\begin{aligned} II_1 &= \chi \int_{t_0}^{t_1} \int_{\mathbb{R}^n} \nabla |u|^{q-1} \cdot \nabla v\, \eta_m(u)\, \phi_R\, dxds \\ &= -\chi \int_{t_0}^{t_1} \int_{\mathbb{R}^n} |u|^{q-1} \Delta v\, \eta_m(u)\, \phi_R\, dxds - \chi \int_{t_0}^{t_1} \int_{\mathbb{R}^n} |u|^{q-1} \nabla v \cdot \eta_m'(u)\, \nabla u\, \phi_R\, dxds \\ &\quad - \chi \int_{t_0}^{t_1} \int_{\mathbb{R}^n} |u|^{q-1} \nabla v \cdot \eta_m(u)\, \nabla \phi_R\, dxds \\ &\le \sup_{0<t<\widehat{T}} \|u(t)\|_{L^\infty(\mathbb{R}^n)}^{q-2} \left(\frac{2\sqrt{n}}{R} \sup_{0<t<\widehat{T}} \|\nabla v(t)\|_{L^\infty(\mathbb{R}^n)} + \sup_{0<t<\widehat{T}} \|\Delta v(t)\|_{L^\infty(\mathbb{R}^n)} \right) \\ &\quad \times \int_{t_0}^{t_1} \|u(s)\|_{L^1(\mathbb{R}^n)}\, ds - \chi \int_{t_0}^{t_1} \int_{\mathbb{R}^n} |u|^{q-1} \nabla v \cdot \eta_m'(u)\, \nabla u\, \phi_R\, dxds. \end{aligned} \tag{4.60}$$

We consider the third term on the right-hand side of (4.60). Since $\eta_m'(r) \le 2m$ for all $r \in \mathbb{R}$, as stated in (4.54), we apply Lemma 3.1 to obtain:

$$
\begin{aligned}
&-\chi \int_{t_0}^{t_1} \int_{\mathbb{R}^n} |u|^{q-1} \nabla v \cdot \eta_m'(u)\, \nabla u\, \phi_R \, dx ds \\
&\leq \frac{1}{4} I_1 + \int_{t_0}^{t_1} \int_{\mathbb{R}^n} |u|^{2q-3}\, |\nabla v|^2\, \eta_m'(u)\, \phi_R \, dx ds \\
&\leq \frac{1}{4} I_1 + \frac{2m}{m^{2q-3}} \sup_{0<t<\widehat{T}} \|\nabla v(t)\|_{L^\infty(\mathbb{R}^n)}^2 \int_{t_0}^{t_1} \|u(s)\|_{L^1(\mathbb{R}^n)} \, ds. \qquad (4.61)
\end{aligned}
$$

Combining (4.60) and (4.61), we deduce:

$$
\begin{aligned}
II_1 \leq \frac{1}{4} I_1 + \Bigg\{ &\frac{2}{m^{2q-4}} \sup_{0<t<\widehat{T}} \|\nabla v(t)\|_{L^\infty(\mathbb{R}^n)}^2 + \sup_{0<t<\widehat{T}} \|u(t)\|_{L^\infty(\mathbb{R}^n)}^{q-2} \\
&\times \left(\frac{2\sqrt{n}}{R} \sup_{0<t<\widehat{T}} \|\nabla v(t)\|_{L^\infty(\mathbb{R}^n)} + \sup_{0<t<\widehat{T}} \|\Delta v(t)\|_{L^\infty(\mathbb{R}^n)} \right) \Bigg\} \int_{t_0}^{t_1} \|u(s)\|_{L^1(\mathbb{R}^n)} \, ds.
\end{aligned}
\qquad (4.62)
$$

We now turn to the estimate of II_2, and observe that:

$$
II_2 \leq \sup_{0<t<\widehat{T}} \|u(t)\|_{L^\infty(\mathbb{R}^n)}^{q-2} \sup_{0<t<\widehat{T}} \|\Delta v(t)\|_{L^\infty(\mathbb{R}^n)} \int_{t_0}^{t_1} \|u(s)\|_{L^1(\mathbb{R}^n)} \, ds. \qquad (4.63)
$$

Combining (4.62) and (4.63), we conclude that:

$$
\begin{aligned}
II \leq \frac{1}{4} I_1 + \Bigg\{ &\frac{2}{m^{2q-4}} \sup_{0<t<\widehat{T}} \|\nabla v(t)\|_{L^\infty(\mathbb{R}^n)}^2 + \sup_{0<t<\widehat{T}} \|u(t)\|_{L^\infty(\mathbb{R}^n)}^{q-2} \\
&\times \left(\frac{2\sqrt{n}}{R} \sup_{0<t<\widehat{T}} \|\nabla v(t)\|_{L^\infty(\mathbb{R}^n)} + 2 \sup_{0<t<\widehat{T}} \|\Delta v(t)\|_{L^\infty(\mathbb{R}^n)} \right) \Bigg\} \int_{t_0}^{t_1} \|u(s)\|_{L^1(\mathbb{R}^n)} \, ds
\end{aligned}
\qquad (4.64)
$$

for a.e. $0 < t_0 < t_1 < \widehat{T}$. We then combine (4.55), (4.58), and (4.64) to obtain:

$$
\begin{aligned}
&\int_{t_0}^{t_1} \int_{\mathbb{R}^n} \partial_s u\, (\eta_m(u)) \phi_R \, dx ds \\
&\leq \int_{t_0}^{t_1} \int_{\operatorname{supp}\phi_R} \frac{16n}{mR^2} \, dx ds + \Bigg\{ \frac{12n}{R^2} + 2 \sup_{0<t<\widehat{T}} \|\nabla v(t)\|_{L^\infty(\mathbb{R}^n)}^2 + \sup_{0<t<\widehat{T}} \|u(t)\|_{L^\infty(\mathbb{R}^n)}^{q-2}
\end{aligned}
$$

$$\times\left(\frac{2\sqrt{n}}{R}\sup_{0<t<\widehat{T}}\|\nabla v(t)\|_{L^\infty(\mathbb{R}^n)}+2\sup_{0<t<\widehat{T}}\|\Delta v(t)\|_{L^\infty(\mathbb{R}^n)}\right)\Bigg\}\int_{t_0}^{t_1}\|u(s)\|_{L^1(\mathbb{R}^n)}\,ds \tag{4.65}$$

for a.e. $0<t_0<t_1<\widehat{T}$. It follows from the Lebesgue dominated convergence theorem that:

$$\begin{aligned}\int_{t_0}^{t_1}\int_{\mathbb{R}^n}\partial_s u(x,s)\,\eta_m(u(x,s))\,\phi_R(x)\,dxds &\to \int_{t_0}^{t_1}\int_{\mathbb{R}^n}\partial_s u(x,s)\ \mathrm{sign}(u(x,s))\,\phi_R(x)\,dxds\\ &=\int_{t_0}^{t_1}\int_{\mathbb{R}^n}\partial_s[u(x,s)]^+\,\phi_R(x)\,dxds,\end{aligned}$$

as $m\to\infty$, where $[\cdot]^+$ denotes the positive part defined in (4.51), and:

$$\mathrm{sign}(s):=\begin{cases}0 & \text{for } s\le 0,\\ 1 & \text{for } s>0.\end{cases}$$

By passing to the limit $m\to\infty$ in both sides of (4.65), we obtain:

$$\begin{aligned}&\int_{\mathbb{R}^n}[u(x,t_1)]^+\,\phi_R(x)\,dx\\ &\le\int_{\mathbb{R}^n}[u(x,t_0)]^+\,\phi_R(x)\,dx+\Bigg\{\frac{12n}{R^2}+2\sup_{0<t<\widehat{T}}\|\nabla v(t)\|^2_{L^\infty(\mathbb{R}^n)}+\sup_{0<t<\widehat{T}}\|u(t)\|^{q-2}_{L^\infty(\mathbb{R}^n)}\\ &\qquad\times\left(\frac{2\sqrt{n}}{R}\sup_{0<t<\widehat{T}}\|\nabla v(t)\|_{L^\infty(\mathbb{R}^n)}+2\sup_{0<t<\widehat{T}}\|\Delta v(t)\|_{L^\infty(\mathbb{R}^n)}\right)\Bigg\}\int_{t_0}^{t_1}\|u(s)\|_{L^1(\mathbb{R}^n)}\,ds.\end{aligned} \tag{4.66}$$

We then let $t_0\to 0$ and subsequently $R\to\infty$ in (4.66). Applying the Fatou lemma to the integral on the left-hand side and the monotone convergence theorem to the first term on the right-hand side, we obtain:

$$\begin{aligned}\int_{\mathbb{R}^n}[u(x,t_1)]^+\,dx\le\int_{\mathbb{R}^n}[u_0(x)]^+\,dx+2\Bigg(&\sup_{0<t<\widehat{T}}\|\nabla v(t)\|^2_{L^\infty(\mathbb{R}^n)}\\ &+\sup_{0<t<\widehat{T}}\|u(t)\|^{q-2}_{L^\infty(\mathbb{R}^n)}\sup_{0<t<\widehat{T}}\|\Delta v(t)\|_{L^\infty(\mathbb{R}^n)}\Bigg)\int_0^{t_1}\|u(s)\|_{L^1(\mathbb{R}^n)}\,ds\end{aligned} \tag{4.67}$$

for a.e. $0<t_1<\widehat{T}$.

We now turn to the contribution arising from the negative part of u, in order to establish the $L^1(\mathbb{R}^n)$-integrability of u. Recall from (4.53) that:

$$\eta_m(-u) = \eta(m(-u)) = \begin{cases} 0 & \text{for } u \geq 0, \\ 1 & \text{for } u \leq -\frac{1}{m}, \end{cases} \quad \text{and} \quad 0 < \eta_m'(-u) < 2m \quad \text{for } u < 0. \tag{4.68}$$

We multiply the first equation of (DD) by $\eta_m(-u)\,\phi_R$ to obtain:

$$\begin{aligned}
&\int_{t_0}^{t_1}\int_{\mathbb{R}^n} \partial_s(-u)\,\eta_m(-u)\,\phi_R\,dxds \\
&= -\int_{t_0}^{t_1}\int_{\mathbb{R}^n} \nabla(-u)\cdot\nabla\big(\eta_m(-u)\,\phi_R\big)\,dxds \\
&\quad + \chi\int_{t_0}^{t_1}\int_{\mathbb{R}^n} \nabla\cdot\big(|u|^{q-2}(-u)\nabla v\big)\,\eta_m(-u)\,\phi_R\,dxds \\
&=: \widehat{I} + \widehat{II}
\end{aligned} \tag{4.69}$$

for a.e. $0 < t_0 < t_1 < \widehat{T}$.

We now estimate $\widehat{I}$. Applying the Leibniz rule, we obtain:

$$\begin{aligned}
\widehat{I} &= -\int_{t_0}^{t_1}\int_{\mathbb{R}^n} |\nabla(-u)|^2\,\eta_m'(-u)\,\phi_R\,dxds - \int_{t_0}^{t_1}\int_{\mathbb{R}^n} \nabla(-u)\cdot\eta_m(-u)\,\nabla\phi_R\,dxds \\
&=: -\widehat{I_1} + \widehat{I_2}.
\end{aligned} \tag{4.70}$$

Since (4.68) ensures $\eta_m'(-u) \geq 0$, it follows that $\widehat{I_1} \geq 0$.

We proceed to an estimate of $\widehat{I_2}$. To this end, we introduce the set:

$$\widehat{D}_m := \Big\{(x,t) \in \mathbb{R}^n \times (0,\widehat{T}) \;\Big|\; -\tfrac{1}{m} < u(x,t) < 0\Big\}.$$

Applying Lemma 3.1, we obtain:

$$\begin{aligned}
\widehat{I_2} &= \int_{t_0}^{t_1}\int_{\mathbb{R}^n} (-u)\,\eta_m'(-u)\,\nabla(-u)\cdot\nabla\phi_R\,dxds + \int_{t_0}^{t_1}\int_{\mathbb{R}^n} (-u)\,\eta_m(-u)\,\Delta\phi_R\,dxds \\
&\leq \frac{1}{4}\,\widehat{I_1} + \int_{t_0}^{t_1}\int_{\widehat{D}_m\cap\operatorname{supp}\phi_R} |u|^2\,\eta_m'(-u)\,|\nabla\phi_R|^2\,\frac{1}{\phi_R}\,dxds \\
&\quad + \int_{t_0}^{t_1}\int_{\mathbb{R}^n} (-u)\,\eta_m(-u)\,\Delta\phi_R\,dxds
\end{aligned}$$

$$\le \frac{1}{4}\widehat{I}_1 + \int_{t_0}^{t_1} \int_{\widehat{D}_m \cap \operatorname{supp}\phi_R} \frac{16n}{mR^2}\, dxds + \frac{12n}{R^2}\int_{t_0}^{t_1} \|u(s)\|_{L^1(\mathbb{R}^n)}\, ds. \tag{4.71}$$

Combining (4.70) with (4.71) yields:

$$\widehat{I} \le -\frac{3}{4}\widehat{I}_1 + \int_{t_0}^{t_1} \int_{\operatorname{supp}\phi_R} \frac{16n}{mR^2}\, dxds + \frac{12n}{R^2}\int_{t_0}^{t_1} \|u(s)\|_{L^1(\mathbb{R}^n)}\, ds. \tag{4.72}$$

We next bound $\widehat{II}$. Proceeding in the same way as for II, we obtain:

$$\begin{aligned}\widehat{II} \le \frac{1}{4}\widehat{I}_1 &+ \Bigg\{ \frac{2}{m^{2q-4}} \sup_{0<t<\widehat{T}} \|\nabla v(t)\|^2_{L^\infty(\mathbb{R}^n)} + \sup_{0<t<\widehat{T}} \|u(t)\|^{q-2}_{L^\infty(\mathbb{R}^n)} \\ &\times \left(\frac{2\sqrt{n}}{R} \sup_{0<t<\widehat{T}} \|\nabla v(t)\|_{L^\infty(\mathbb{R}^n)} + 2 \sup_{0<t<\widehat{T}} \|\Delta v(t)\|_{L^\infty(\mathbb{R}^n)} \right) \Bigg\} \int_{t_0}^{t_1} \|u(s)\|_{L^1(\mathbb{R}^n)}\, ds.\end{aligned} \tag{4.73}$$

Combining (4.69), (4.72), and (4.73), we deduce:

$$\begin{aligned}&\int_{t_0}^{t_1}\int_{\mathbb{R}^n} \partial_s(-u)\,(\eta_m(-u))\phi_R\, dxds \\ &\le \int_{t_0}^{t_1} \int_{\operatorname{supp}\phi_R} \frac{16n}{mR^2}\, dxds + \Bigg\{ \frac{12n}{R^2} + 2\sup_{0<t<\widehat{T}} \|\nabla v(t)\|^2_{L^\infty(\mathbb{R}^n)} + \sup_{0<t<\widehat{T}} \|u(t)\|^{q-2}_{L^\infty(\mathbb{R}^n)} \\ &\quad \times \left(\frac{2\sqrt{n}}{R} \sup_{0<t<\widehat{T}} \|\nabla v(t)\|_{L^\infty(\mathbb{R}^n)} + 2 \sup_{0<t<\widehat{T}} \|\Delta v(t)\|_{L^\infty(\mathbb{R}^n)} \right) \Bigg\} \int_{t_0}^{t_1} \|u(s)\|_{L^1(\mathbb{R}^n)}\, ds\end{aligned} \tag{4.74}$$

for a.e. $0 < t_0 < t_1 < \widehat{T}$. Adapting the argument in (4.65)–(4.67), we obtain:

$$\begin{aligned}\int_{\mathbb{R}^n} [-u(x,t_1)]^+\, dx \le \int_{\mathbb{R}^n} [-u_0(x)]^+\, dx &+ 2\Bigg(\sup_{0<t<\widehat{T}} \|\nabla v(t)\|^2_{L^\infty(\mathbb{R}^n)} \\ &+ \sup_{0<t<\widehat{T}} \|u(t)\|^{q-2}_{L^\infty(\mathbb{R}^n)} \sup_{0<t<\widehat{T}} \|\Delta v(t)\|_{L^\infty(\mathbb{R}^n)} \Bigg) \int_0^{t_1} \|u(s)\|_{L^1(\mathbb{R}^n)}\, ds.\end{aligned} \tag{4.75}$$

Therefore, combining (4.52), (4.67), and (4.75), we obtain:

$$\|u(t_1)\|_{L^1(\mathbb{R}^n)} \le \|u_0\|_{L^1(\mathbb{R}^n)} + 4\Bigg(\sup_{0<t<\widehat{T}} \|\nabla v(t)\|^2_{L^\infty(\mathbb{R}^n)}$$

$$+ \sup_{0<t<\widehat{T}} \|u(t)\|_{L^\infty(\mathbb{R}^n)}^{q-2} \sup_{0<t<\widehat{T}} \|\Delta v(t)\|_{L^\infty(\mathbb{R}^n)}\Bigg) \int_0^{t_1} \|u(s)\|_{L^1(\mathbb{R}^n)}\, ds.$$

The Gronwall inequality then yields:

$$\begin{aligned}
&\|u(t_1)\|_{L^1(\mathbb{R}^n)} \\
&\le \|u_0\|_{L^1(\mathbb{R}^n)} \exp\left\{4\widehat{T}\left(\sup_{0<t<\widehat{T}} \|\nabla v(t)\|_{L^\infty(\mathbb{R}^n)}^2 + \sup_{0<t<\widehat{T}} \|u(t)\|_{L^\infty(\mathbb{R}^n)}^{q-2} \sup_{0<t<\widehat{T}} \|\Delta v(t)\|_{L^\infty(\mathbb{R}^n)}\right)\right\}
\end{aligned}$$

for a.e. $0 < t_1 < \widehat{T}$. This completes the proof of Lemma 4.3. □

With these preparations in place, we now turn to the proof of Theorem 2.1(IV).

Proof of Theorem 2.1 (IV). Let ϕ_R be the cut-off function defined in Lemma 3.1. Multiplying both sides of the first equation of (DD) by $\phi_R = \phi_R(x)$ and integrating it over B_{2R}, we obtain:

$$\frac{d}{dt} \int_{B_{2R}} u\phi_R\, dx = \int_{B_{2R}} \Delta u \phi_R\, dx + \chi \int_{B_{2R}} \nabla \cdot (|u|^{q-2} u \nabla v)\phi_R\, dx. \tag{4.76}$$

Regarding the first term on the right-hand side of (4.76), by applying the Gauss divergence theorem twice, we observe from $u \in W^{2,1}_{n+q}(Q_{\widehat{T}})$:

$$\int_{B_{2R}} \Delta u \phi_R\, dx = \int_{B_{2R}} u \Delta \phi_R\, dx. \tag{4.77}$$

Indeed, we have:

$$\begin{aligned}
&\|\nabla u(t)\phi_R\|_{L^1(B_{2R})} + \|\nabla \cdot (\nabla u(t)\phi_R)\|_{L^1(B_{2R})} \\
&\le \|\nabla u(t)\phi_R\|_{L^1(B_{2R})} + \|\nabla u(t) \cdot \nabla \phi_R\|_{L^1(B_{2R})} + \|\Delta u(t)\phi_R\|_{L^1(B_{2R})} \\
&\le \|\nabla u(t)\|_{L^{n+q}(B_{2R})} \|\phi_R\|_{L^{\frac{n+q}{n+q-1}}(B_{2R})} + \|\nabla u(t)\|_{L^{n+q}(B_{2R})} \|\nabla \phi_R\|_{L^{\frac{n+q}{n+q-1}}(B_{2R})} \\
&\quad + \|\Delta u(t)\|_{L^{n+q}(B_{2R})} \|\phi_R\|_{L^{\frac{n+q}{n+q-1}}(B_{2R})} < \infty
\end{aligned}$$

for a.e. $0 < t < \widehat{T}$, which implies that (4.77) holds.

Additionally, concerning the second term on the right-hand side of (4.76), noting (4.50), and using $u \in W^{2,1}_{n+q}(Q_{\widehat{T}})$, we find, by the Gauss divergence theorem, the following equality:

$$\chi \int_{B_{2R}} \nabla \cdot (|u|^{q-2} u \nabla v)\phi_R\, dx = -\chi \int_{B_{2R}} (|u|^{q-2} u \nabla v) \cdot \nabla \phi_R\, dx. \tag{4.78}$$

Indeed, it follows:

$$\|(|u|^{q-2}u\nabla v)(t)\phi_R\|_{L^1(B_{2R})} \le \|u(t)\|_{L^\infty(B_{2R})}^{q-1}\|\nabla v(t)\|_{L^\infty(B_{2R})}\|\phi_R\|_{L^1(B_{2R})} < \infty$$

and:

$$\begin{aligned}
&\|\nabla\cdot((|u|^{q-2}u\nabla v)(t)\phi_R)\|_{L^1(B_{2R})}\\
&\le \|((q-1)|u|^{q-2}\nabla u\cdot\nabla v)(t)\phi_R\|_{L^1(B_{2R})} + \|(|u|^{q-2}u\Delta v)(t)\phi_R\|_{L^1(B_{2R})}\\
&\quad + \|(|u|^{q-2}u\nabla v)(t)\cdot\nabla\phi_R\|_{L^1(B_{2R})}\\
&\le (q-1)\|u(t)\|_{L^\infty(B_{2R})}^{q-2}\|\nabla u(t)\|_{L^{n+q}(B_{2R})}\|\nabla v(t)\|_{L^\infty(B_{2R})}\|\phi_R\|_{L^{\frac{n+q}{n+q-1}}(B_{2R})}\\
&\quad + \|u(t)\|_{L^\infty(B_{2R})}^{q-1}\|\Delta v(t)\|_{L^\infty(B_{2R})}\|\phi_R\|_{L^1(B_{2R})}\\
&\quad + \|u(t)\|_{L^\infty(B_{2R})}^{q-1}\|\nabla v(t)\|_{L^\infty(B_{2R})}\|\nabla\phi_R\|_{L^1(B_{2R})} < \infty
\end{aligned}$$

for a.e. $0 < t < \widehat{T}$, which yields that (4.78) holds.

Combining (4.76), (4.77) and (4.78), we obtain the following equality:

$$\frac{d}{dt}\int_{B_{2R}} u\phi_R\,dx = \int_{B_{2R}} u\Delta\phi_R\,dx - \chi\int_{B_{2R}}(|u|^{q-2}u\nabla v)\cdot\nabla\phi_R\,dx. \tag{4.79}$$

By integrating both sides of (4.79) from 0 to t, we obtain the following expression:

$$\begin{aligned}
&\int_{B_{2R}} u\phi_R\,dx - \int_{B_{2R}} u_0\phi_R\,dx\\
&= \int_0^t\int_{B_{2R}} u\Delta\phi_R\,dxds - \chi\int_0^t\int_{B_{2R}}(|u|^{q-2}u\nabla v)\cdot\nabla\phi_R\,dxds
\end{aligned} \tag{4.80}$$

for a.e. $0 < t < \widehat{T}$. According to (4.80), we obtain the following expression:

$$\begin{aligned}
&\left|\int_{B_{2R}} u\phi_R\,dx - \int_{B_{2R}} u_0\phi_R\,dx\right|\\
&\le \left|\int_0^t\int_{B_{2R}} u\Delta\phi_R\,dxds\right| + \left|\int_0^t\int_{B_{2R}}(|u|^{q-2}u\nabla v)\cdot\nabla\phi_R\,dxds\right| =: I_R + II_R.
\end{aligned} \tag{4.81}$$

Noting that (4.49), (4.50) and (3.1) in Lemma 3.1, we have:

$$I_R \le \int_0^{\widehat{T}}\|u(s)\|_{L^{\frac{2n}{2n-1}}(B_{2R})}\|\Delta\phi_R\|_{L^{2n}(B_{2R})}\,ds \le C\widehat{T}R^{-\frac{3}{2}}\sup_{0<t<\widehat{T}}\|u(t)\|_{L^{\frac{2n}{2n-1}}(\mathbb{R}^n)}, \tag{4.82}$$

and:

$$
\begin{aligned}
II_R &\le \int_0^{\widehat{T}} \|(|u|^{q-2}u\nabla v)(s)\cdot\nabla\phi_R\|_{L^1(B_{2R})}\,ds \\
&\le \sup_{0<t<\widehat{T}} \|\nabla v(t)\|_{L^\infty(\mathbb{R}^n)} \int_0^{\widehat{T}} \|u(s)\|^{q-1}_{L^{\frac{2n(q-1)}{2n-1}}(\mathbb{R}^n)} \|\nabla\phi_R\|_{L^{2n}(B_{2R})}\,ds \\
&\le C\widehat{T}R^{-\frac{1}{2}} \sup_{0<t<\widehat{T}} \|u(t)\|^{q-1}_{L^{\frac{2n(q-1)}{2n-1}}(\mathbb{R}^n)} \sup_{0<t<\widehat{T}} \|\nabla v(t)\|_{L^\infty(\mathbb{R}^n)}.
\end{aligned}
\tag{4.83}
$$

Therefore, concerning (4.81), (4.82), (4.83), we obtain:

$$
\Big|\int_{\mathbb{R}^n} u\phi_R\,dx - \int_{\mathbb{R}^n} u_0\phi_R\,dx\Big| \le C\widehat{T}\big(R^{-\frac{3}{2}} + R^{-\frac{1}{2}}\big).
$$

Thus, applying Lemma 4.3 and the Lebesgue dominated convergence theorem, we have:

$$
\begin{aligned}
&\Big|\int_{\mathbb{R}^n} u\,dx - \int_{\mathbb{R}^n} u_0\,dx\Big| \\
&\le \Big|\int_{\mathbb{R}^n} u\,dx - \int_{\mathbb{R}^n} u\phi_R\,dx\Big| + \Big|\int_{\mathbb{R}^n} u_0\,dx - \int_{\mathbb{R}^n} u_0\phi_R\,dx\Big| + \Big|\int_{\mathbb{R}^n} u\phi_R\,dx - \int_{\mathbb{R}^n} u_0\phi_R\,dx\Big| \\
&\le \Big|\int_{\mathbb{R}^n} u\,dx - \int_{\mathbb{R}^n} u\phi_R\,dx\Big| + \Big|\int_{\mathbb{R}^n} u_0\,dx - \int_{\mathbb{R}^n} u_0\phi_R\,dx\Big| + C\widehat{T}\big(R^{-\frac{3}{2}} + R^{-\frac{1}{2}}\big) \to 0
\end{aligned}
$$

as $R \to \infty$. This completes the proof of Theorem 2.1 (IV). □

4.5. *Proof of Theorem 2.1 (V): non-negativity*

We turn to the proof of Theorem 2.1 (V). Let $0 < \widehat{T} < T_{\max}$, where $T_{\max}$ denotes the maximal existence time. By Lemma 4.2, there exists a constant C such that the following bound holds:

$$
\sup_{0<t<\widehat{T}} \|u(t)\|_{L^\infty(\mathbb{R}^n)} \le C, \tag{4.84}
$$

and:

$$
\sup_{0<t<\widehat{T}} \|\Delta v(t)\|_{L^\infty(\mathbb{R}^n)} \le C. \tag{4.85}
$$

Here, the constant C depends as follows:

(i). **parabolic-elliptic type (A)**: $\tau = 0$, $\gamma > 0$.
The constant C depends only on n, q, γ, $\|u_0\|_{L^2(\mathbb{R}^n)}$ and $\|u_0\|_{W^{2,n+q}(\mathbb{R}^n)}$.

(ii). **parabolic-elliptic type (B)**: $\tau = 0$, $\gamma = 0$.
The constant C depends only on n, q, $\|u_0\|_{L^2(\mathbb{R}^n)}$ and $\|u_0\|_{W^{2,n+q}(\mathbb{R}^n)}$.

(iii). **parabolic-parabolic type**: $\tau = 1, \gamma \geq 0$.
The constant C depends only on $n, q, \gamma, \|u_0\|_{L^2(\mathbb{R}^n)}, \|u_0\|_{W^{2,n+q}(\mathbb{R}^n)}$ and $\|v_0\|_{W^{3,n+q}(\mathbb{R}^n)}$.

Proof of Theorem 2.1 (V). Multiplying both sides of the first equation of (DD) by $u^- := -\min\{0, u\}$ and integrating over $\mathbb{R}^n$, we obtain:

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|u^-(t)\|^2_{L^2(\mathbb{R}^n)}\\&\leq -\int_{\mathbb{R}^n} |\nabla u^-|^2 dx + \frac{1}{q}\sup_{0<t<\widehat{T}}\|\Delta v(t)\|_{L^\infty(\mathbb{R}^n)}\sup_{0<t<\widehat{T}}\|u(t)\|^{q-2}_{L^\infty(\mathbb{R}^n)}\|u^-(t)\|^2_{L^2(\mathbb{R}^n)}.\end{aligned}$$

By the Gronwall inequality, it follows that:

$$\sup_{0<t<\widehat{T}}\|u^-(t)\|^2_{L^2(\mathbb{R}^n)} \leq \|u_0^-\|^2_{L^2(\mathbb{R}^n)} \cdot \exp\left\{\frac{2\widehat{T}}{q}\sup_{0<t<\widehat{T}}\|\Delta v(t)\|_{L^\infty(\mathbb{R}^n)}\sup_{0<t<\widehat{T}}\|u(t)\|^{q-2}_{L^\infty(\mathbb{R}^n)}\right\} = 0$$

since $u_0^-(x) = 0$ for a.e. $x \in \mathbb{R}^n$. Thus, we conclude:

$$u(x,t) \geq 0 \qquad \text{for a.e. } (x,t) \in \mathbb{R}^n \times (0, \widehat{T}).$$

This completes the proof of Theorem 2.1 (V). □

5. Proof of Theorem 2.2

5.1. *Unique continuation theorem*

Let us consider the symmetric matrix-valued function $\{g^{ij}(x,t)\}_{i,j=1}^n$, which satisfies a uniform ellipticity condition. Let $g(x,t) = \{g_{ij}(x,t)\}_{i,j=1}^n$ represent the inverse of the matrix $\{g^{ij}(x,t)\}_{i,j=1}^n$. Consequently, we have:

$$g^{-1}(x,t) = \{g^{ij}(x,t)\}_{i,j=1}^n. \tag{5.1}$$

In this section, we introduce the following notation for a function w and a vector field ξ with g and g^{-1} used as weights:

Notation.

(1). $|\xi|_g^2(x,t) = \sum_{i,j=1}^n g_{ij}(x,t)\xi_i\xi_j$,
(2). $\nabla w = \left(\frac{\partial w}{\partial x_1}, \ldots, \frac{\partial w}{\partial x_n}\right)$,
(3). $\nabla_g w(x,t) = g^{-1}(x,t)\nabla w(x,t)$.

The following lemma, known as the "*Two-Sphere One-Cylinder Inequality*", was obtained by Vessella [26, Theorem 4.2.6]. This inequality pertains to the evaluation of the square integral over a large integration domain through integrals over smaller regions, including spheres and cylindrical domains.

Proposition 5.1 *(Vessella [26], Two-Sphere One-Cylinder Inequality for General Parabolic Operators). Let* λ, Λ *and* R *be positive numbers, with* $0 < \lambda \leq 1$. *Let* P *be the parabolic operator*:

$$P = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(g^{ij}(x,t)\frac{\partial}{\partial x_j}\right) + \frac{\partial}{\partial t}, \tag{5.2}$$

where $\{g^{ij}(x,t)\}_{i,j=1}^{n}$ *is a real symmetric* $n \times n$ *matrix. When* $\xi \in \mathbb{R}^n$, $(x,t), (y,t) \in \mathbb{R}^{n+1}$, *we assume* :

$$\lambda|\xi|^2 \leq \sum_{i,j=1}^{n} g^{ij}(x,t)\xi_i\xi_j \leq \lambda^{-1}|\xi|^2 \tag{5.3}$$

and:

$$\left(\sum_{i,j=1}^{n}(g^{ij}(x,t) - g^{ij}(y,\tau))^2\right)^{1/2} \leq \frac{\Lambda}{R}\left(|x-y|^2 + |t-\tau|^2\right)^{1/2}. \tag{5.4}$$

Let $w \in W_2^{2,1}(B_R \times (0, R^2))$ *satisfy the inequality*:

$$|Pw(x,t)| \leq \Lambda\left(\frac{|\nabla_g w|_g(x,t)}{R} + \frac{|w(x,t)|}{R^2}\right) \qquad for\ a.e.\ (x,t) \in B_R \times (0, R^2). \tag{5.5}$$

Then, there exist constants $0 < \eta < 1$ *and* $C \geq 1$, *which depend only on* λ *and* Λ, *such that for all* r *and* ρ *with* $0 < r \leq \rho \leq \eta R$, *the following inequality holds*:

$$\int_{B_\rho} w^2(x,0)\,dx \leq \frac{CR}{\rho}\left(R^{-2}\int_0^{R^2}\int_{B_R} w^2(x,t)\,dxdt\right)^{1-\theta}\left(\int_{B_r} w^2(x,0)\,dx\right)^{\theta}, \tag{5.6}$$

where $\theta = (C\log\frac{R}{r})^{-1}$.

Remark 7. Vessella [26, Theorem 5.2] considered a parabolic equation of semi-linear type, given by (5.2), where $\{g^{ij}(x,t)\}_{1\leq i,j\leq n}$ represents a real symmetric positive definite matrix with sufficiently smooth components. The analysis was carried out using the Carleman estimate.

In Proposition 5.1, by setting $g^{ij}(x,t)$ as the Kronecker delta δ_{ij}, we derive the following lemma.

Lemma 5.2 *(Two-Sphere One-Cylinder Inequality for the Classical Heat Operator). Let* R *be a positive number, and let* $w \in W_2^{2,1}(B_R \times (0, R^2))$. *We assume that there exists a positive constant* Λ *such that* w *satisfies the following inequality* :

$$\left|\Delta w(x,t) + \frac{\partial w(x,t)}{\partial t}\right| \leq \Lambda\left(\frac{|\nabla w(x,t)|}{R} + \frac{|w(x,t)|}{R^2}\right) \quad for\ a.e.\ (x,t) \in B_R \times (0, R^2). \tag{5.7}$$

Then, there exist constants $0 < \eta < 1$ *and* $C \geq 1$*, which depend only on* Λ*, such that for all* r *and* ρ *with* $0 < r \leq \rho \leq \eta R$*, the following inequality holds*:

$$\int_{B_\rho} w^2(x,0)\,dx \leq \frac{CR}{\rho}\Big(R^{-2}\int_0^{R^2}\int_{B_R} w^2(x,t)\,dxdt\Big)^{1-\theta}\Big(\int_{B_r} w^2(x,0)\,dx\Big)^{\theta}, \tag{5.8}$$

where $\theta = (C\log\frac{R}{r})^{-1}$.

Proof of Lemma 5.2. Let $i, j = 1, 2, \dots, n$, and define $g^{ij}(x,t)$ as follows:

$$g^{ij}(x,t) = \delta_{ij}. \tag{5.9}$$

By setting $\{g^{ij}(x,t)\}_{i,j=1}^n = \delta_{ij}$ in Proposition 5.1, we can prove Lemma 5.2. Indeed, $\{g^{ij}(x,t)\}_{i,j=1}^n$ forms the identity matrix, making $\{g^{ij}(x,t)\}_{i,j=1}^n$ it a real symmetric $n \times n$ matrix. Thus, (5.3) holds with $\lambda = 1$ since $g^{ij}(x,t)$ satisfies the following equality:

$$\sum_{i,j=1}^n g^{ij}(x,t)\xi_i\xi_j = \sum_{i,j=1}^n \delta_{ij}\xi_i\xi_j = \sum_{i=1}^n \xi_i^2 = |\xi|^2.$$

Additionally, we observe from (5.9):

$$\left(\sum_{i,j=1}^n (g^{ij}(x,t) - g^{ij}(y,\tau))^2\right)^{1/2} = \left(\sum_{i,j=1}^n (\delta_{ij} - \delta_{ij})^2\right)^{1/2} = 0.$$

Therefore, for every positive constant Λ, inequality (5.4) in Proposition 5.1 holds.

We verify that (5.5) in Proposition 5.1 is satisfied. Using (5.9) in (5.2) from Proposition 5.1, P can be expressed as:

$$P = \sum_{i,j=1}^n \frac{\partial}{\partial x_i}\Big(g^{ij}(x,t)\frac{\partial}{\partial x_j}\Big) + \frac{\partial}{\partial t} = \sum_{i,j=1}^n \frac{\partial}{\partial x_i}\Big(\delta_{ij}\frac{\partial}{\partial x_j}\Big) + \frac{\partial}{\partial t} = \Delta + \frac{\partial}{\partial t}. \tag{5.10}$$

By applying the definition of ∇_g and (5.1), we obtain:

$$\nabla_g w(x,t) := g^{-1}(x,t)\nabla w(x,t) = \nabla w(x,t).$$

Thus, from the definition of $|\cdot|_g$, we have:

$$\begin{aligned} |\nabla_g w|_g^2(x,t) = |\nabla w|_g^2(x,t) &= \sum_{i,j=1}^n g_{ij}(x,t)\frac{\partial w}{\partial x_i}\frac{\partial w}{\partial x_j} \\ &= \sum_{i,j=1}^n \delta_{ij}\frac{\partial w}{\partial x_i}\frac{\partial w}{\partial x_j} = \sum_{i=1}^n \Big(\frac{\partial w}{\partial x_i}\Big)^2 = |\nabla w(x,t)|^2. \end{aligned} \tag{5.11}$$

By utilizing (5.10) and (5.11) on the left-hand side and right-hand side of (5.5) in Proposition 5.1, we deduce (5.7). Thus, the conclusion of Proposition 5.1, namely (5.6), ensures the conclusion of Lemma 5.2, specifically (5.8). Therefore, Lemma 5.2 is established. This completes the proof of Lemma 5.2. $\square$

We introduce the following lemma, known as the *Unique Continuation Theorem* in bounded domains. Specifically, this lemma states that if a function u with sufficient regularity vanishes in a ball of radius r at time T, then it also vanishes in the ball of radius $2r$ centered at the same point.

Lemma 5.3 *(Unique Continuation Theorem in bounded domains). Let T be a positive number, $a \in \mathbb{R}^n$ and let $u \in W_2^{2,1}(B_{\sqrt{T}}(a) \times (0,T)) \cap C([0,T]; L^2(B_{\sqrt{T}}(a)))$. We assume that there exists a positive constant Λ such that u satisfies the following inequality*:

$$\left| \Delta u(x,t) - \frac{\partial u(x,t)}{\partial t} \right| \leq \Lambda \left(\frac{|\nabla u(x,t)|}{\sqrt{T}} + \frac{|u(x,t)|}{T} \right) \quad \textit{for a.e. } (x,t) \in B_{\sqrt{T}}(a) \times (0,T). \tag{5.12}$$

Then, for r satisfying $0 < r < \frac{1}{2}\eta\sqrt{T}$, if $u(\cdot,T) \equiv 0$ in $B_r(a)$, it follows that $u(\cdot,T) \equiv 0$ in $B_{2r}(a)$.

Proof of Lemma 5.3. We define $w_a(x,t) := u(x+a, T-t)$. From this definition of w_a, together with the fact that $u \in W_2^{2,1}(B_{\sqrt{T}}(a) \times (0,T))$, it follows that $w \in W_2^{2,1}(B_{\sqrt{T}}(0) \times (0,T))$. In addition, from (5.12), we derive the following:

$$\begin{aligned}
\left| \Delta w_a(x,t) + \frac{\partial w_a}{\partial t}(x,t) \right| &= \left| (\Delta u)(x+a, T-t) - \left(\frac{\partial u}{\partial t}\right)(x+a, T-t) \right| \\
&\leq \Lambda \left(\frac{|(\nabla u)(x+a, T-t)|}{\sqrt{T}} + \frac{|(u)(x+a, T-t)|}{T} \right) \\
&= \Lambda \left(\frac{|\nabla w_a(x,t)|}{\sqrt{T}} + \frac{|w_a(x,t)|}{T} \right)
\end{aligned}$$

for a.e. $(x,t) \in B_{\sqrt{T}}(0) \times (0,T)$. From this, we see that (5.7) in Lemma 5.2 holds with $R = \sqrt{T}$.

Therefore, by applying Lemma 5.2, there exist constants $0 < \eta < 1$ and $C \geq 1$, which depend only on Λ, such that for all r and ρ with $0 < r \leq \rho \leq \eta\sqrt{T}$, we have:

$$\int_{B_\rho(a)} w_a^2(x,0)\,dx \leq \frac{C\sqrt{T}}{\rho} \Big(T^{-1} \int_0^T \int_{B_{\sqrt{T}}(a)} w_a^2(x,t)\,dxdt \Big)^{1-\theta} \Big(\int_{B_r(a)} w_a^2(x,0)\,dx \Big)^{\theta}, \tag{5.13}$$

where $\theta := (C \log \frac{\sqrt{T}}{r})^{-1}$.

Let $0 < r < \frac{1}{2}\eta\sqrt{T}$. Then, since $0 < r < 2r < \eta\sqrt{T}$, taking $\rho = 2r$ in (5.13), we obtain the following inequality:

$$\int\limits_{B_{2r(a)}} w_a^2(x,0)\,dx \le \frac{C\sqrt{T}}{2r}\Big(T^{-1}\int\limits_0^T\int\limits_{B_{\sqrt{T}}(a)} w_a^2(x,t)\,dxdt\Big)^{1-\theta}\Big(\int\limits_{B_r(a)} w_a^2(x,0)\,dx\Big)^{\theta},$$

which is equivalent to the following, based on the definition of w_a:

$$\int\limits_{B_{2r}(a)} u^2(x,T)\,dx \le \frac{C\sqrt{T}}{2r}\Big(T^{-1}\int\limits_0^T\int\limits_{B_{\sqrt{T}}(a)} u^2(x,t)\,dxdt\Big)^{1-\theta}\Big(\int\limits_{B_r(a)} u^2(x,T)\,dx\Big)^{\theta},$$

where $\theta := (C\log\frac{\sqrt{T}}{r})^{-1}$. Thus, assuming $u(\cdot,T)\equiv 0$ in $B_r(a)$, it follows that $u(\cdot,T)\equiv 0$ in $B_{2r}(a)$. This completes the proof of Lemma 5.3. □

We extend Lemma 5.3, which guarantees the *Unique Continuation Theorem in bounded domains*, to cases where the function u vanishes in the whole space $\mathbb{R}^n$. The following lemma demonstrates that if u satisfies certain conditions and vanishes in an open subset of $\mathbb{R}^n$ at a specific time T, then u must also vanish throughout the whole space $\mathbb{R}^n$ at the same time.

Lemma 5.4 *(Unique Continuation Theorem in $\mathbb{R}^n$). Let T be a positive number, and let $u \in W^{2,1}_{2,\mathrm{loc}}(Q_T)\cap C([0,T];L^2_{\mathrm{loc}}(\mathbb{R}^n))$. We assume that there exists a positive constant Λ such that u satisfies the following inequality*:

$$\left|\Delta u(x,t)-\frac{\partial u(x,t)}{\partial t}\right| \le \Lambda\left(\frac{|\nabla u(x,t)|}{\sqrt{T}}+\frac{|u(x,t)|}{T}\right) \quad \textit{for a.e. } (x,t)\in Q_T.$$

Then, if there exists an open set $D_0\subset\mathbb{R}^n$ such that $u(\cdot,T)\equiv 0$ in D_0, we conclude that u is identically zero in the whole space $\mathbb{R}^n$ at $t=T$, i.e., $u(\cdot,T)\equiv 0$ in $\mathbb{R}^n$.

Proof of Lemma 5.4. Let $T>0$ and $a\in\mathbb{R}^n$. Lemma 5.3 guarantees the existence of a constant $0<\eta<1$, depending only on Λ, such that for every $0<r<\frac{1}{2}\eta\sqrt{T}$ the following holds: if $u(\cdot,T)\equiv 0$ in $B_r(a)$, it follows that $u(\cdot,T)\equiv 0$ in $B_{2r}(a)$.

Assume now that $u(\cdot,T)\equiv 0$ on a non-empty open set $D_0\subset\mathbb{R}^n$. Applying Lemma 5.3, we obtain a point $a_*\in D_0$ and a radius $0<r_*<\frac{1}{2}\eta\sqrt{T}$ such that $B_{r_*}(a_*)\subset D_0$. Consequently,

$$u(\cdot,T)\equiv 0 \quad \text{in } B_{2r_*}(a_*).$$

Next, we define the enlarged set:

$$D_1 := D_0\cup\big\{x\in\mathbb{R}^n \;\big|\; \mathrm{dist}(x,D_0)<r_*\big\}.$$

Since the choice of a_* is arbitrary in D_0, the same argument applied to every point of D_0 yields:

$$u(\cdot,T)\equiv 0 \quad \text{in } D_1.$$

We now repeat this enlargement procedure countably many times. For each $k\in\mathbb{N}$, we set:

$$D_{k+1} := D_k\cup\big\{x\in\mathbb{R}^n \;\big|\; \mathrm{dist}(x,D_k)<r_*\big\}.$$

The sets D_k are increasing, and by construction $u(\cdot, T) \equiv 0$ in every D_k. Moreover, it holds:

$$\bigcup_{k=0}^{\infty} D_k = \mathbb{R}^n,$$

so passing to the union we conclude

$$u(\cdot, T) \equiv 0 \quad \text{in } \mathbb{R}^n.$$

This completes the proof of Lemma 5.4. □

5.2. *Proof of Theorem 2.2*

We now introduce the backward uniqueness property. Although Vessella [26] established this property in more general function spaces, we present it here under the assumptions specific to our main theorems.

Let H denote the set of functions w defined on Q_T, such that for every positive number R, its restriction $w|_{B_R\times(0,T)}$ belongs to $W_2^{2,1}(B_R \times (0, T))$. Furthermore, for a given positive constant K, we define:

$$\mathcal{H}_{K,T} := \Big\{ w \in H \mid w(x, 0) = 0,\ we^{-K|x|^2} \in L^\infty(Q_T) \Big\}.$$

In Vessella [26, Theorem 3.0.2], the time endpoint is set to 1. By changing the time endpoint to T, the theorem remains valid with minor modifications, as the method used in the proof iteratively extends the vanishing region, as stated in the following lemma.

Lemma 5.5 *(Vessella [26]). Let T be a positive number, and let $w \in \mathcal{H}_{K,T}$. We assume that there exists a positive constant Λ such that w satisfies the following inequality*:

$$\left| \Delta w(x, t) + \frac{\partial w(x, t)}{\partial t} \right| \le \Lambda \Big(|\nabla w(x, t)| + |w(x, t)| \Big) \quad \textit{for a.e. } (x, t) \in Q_T.$$

Then, we conclude:

$$w \equiv 0 \qquad in\ Q_T.$$

Proof of Theorem 2.2. Let $0 < \widehat{T} < T_{\max}$. From (2.4) to (2.6) in Theorem 2.1, in combination with the second equation of (DD) and by applying the Young inequality, there exists a constant C such that the following bounds hold:

$$\sup_{0<t<\widehat{T}} \|u(t)\|_{L^\infty(\mathbb{R}^n)} \le C, \tag{5.14}$$

$$\sup_{0<t<\widehat{T}} \|\nabla v(t)\|_{L^\infty(\mathbb{R}^n)} + \sup_{0<t<\widehat{T}} \|\Delta v(t)\|_{L^\infty(\mathbb{R}^n)} \le C. \tag{5.15}$$

Here, the constant C depends as follows:

(i). **Parabolic-elliptic type (A)**: $\tau = 0, \gamma > 0$.
The constant C depends only on n, q, γ, and $\|u_0\|_{W^{2,n+q}(\mathbb{R}^n)}$.

(ii). **Parabolic-elliptic type (B)**: $\tau = 0, \gamma = 0$.
The constant C depends only on n, q, $\|u_0\|_{L^\theta(\mathbb{R}^n)}$, and $\|u_0\|_{W^{2,n+q}(\mathbb{R}^n)}$ for some $1 < \theta < n$, where θ is introduced in Assumption 2.2.

(iii). **Parabolic-parabolic type**: $\tau = 1, \gamma \geq 0$.
The constant C depends only on n, q, γ, $\|u_0\|_{W^{2,n+q}(\mathbb{R}^n)}$ and $\|v_0\|_{W^{3,n+q}(\mathbb{R}^n)}$.

On the flux term of (DD), since the following identity holds:

$$\nabla \cdot (|u|^{q-2}u\nabla v) = (q-1)|u|^{q-2}\nabla u \cdot \nabla v + |u|^{q-2}u\Delta v,$$

from (5.14) and (5.15), together with the first equation of (DD), we derive the following estimate:

$$\begin{aligned}\left|\Delta u(x,t) - \frac{\partial u(x,t)}{\partial t}\right| &\leq (q-1)|u|^{q-2}|\nabla v||\nabla u| + |u|^{q-2}|\Delta v||u| \\ &\leq (q-1)\sup_{0<t<\widehat{T}}\|u(t)\|^{q-2}_{L^\infty(\mathbb{R}^n)}\sup_{0<t<\widehat{T}}\|\nabla v(t)\|_{L^\infty(\mathbb{R}^n)}|\nabla u| \\ &\quad + \sup_{0<t<\widehat{T}}\|u(t)\|^{q-2}_{L^\infty(\mathbb{R}^n)}\sup_{0<t<\widehat{T}}\|\Delta v(t)\|_{L^\infty(\mathbb{R}^n)}|u|\end{aligned}$$

for a.e. $(x,t) \in Q_{\widehat{T}}$. This yields:

$$\left|\Delta u(x,t) - \frac{\partial u(x,t)}{\partial t}\right| \leq C_*\Big(|u(x,t)| + |\nabla u(x,t)|\Big) \tag{5.16}$$

for a.e. $(x,t) \in Q_{\widehat{T}}$, where C_* is defined as:

$$C_* := \sup_{0<t<\widehat{T}}\|u(t)\|^{q-2}_{L^\infty(\mathbb{R}^n)}\max\left\{(q-1)\sup_{0<t<\widehat{T}}\|\nabla v(t)\|_{L^\infty(\mathbb{R}^n)}, \sup_{0<t<\widehat{T}}\|\Delta v(t)\|_{L^\infty(\mathbb{R}^n)}\right\}. \tag{5.17}$$

By defining Λ as $\Lambda = C_* \max\left\{\sqrt{\widehat{T}}, \widehat{T}\right\}$, we obtain the following inequality:

$$\left|\Delta u(x,t) - \frac{\partial u(x,t)}{\partial t}\right| \leq \Lambda\left(\frac{|\nabla u(x,t)|}{\sqrt{\widehat{T}}} + \frac{|u(x,t)|}{\widehat{T}}\right) \quad \text{for a.e. } (x,t) \in Q_{\widehat{T}}. \tag{5.18}$$

In addition, using Theorem 2.1 (III) and (2.7) therein, we observe that $u \in W^{2,1}_{2,\mathrm{loc}}(Q_{\widehat{T}}) \cap C([0,\widehat{T}]; L^2_{\mathrm{loc}}(\mathbb{R}^n))$. Furthermore, under the assumptions of Theorem 2.2, there exists a non-empty open set $D_0 \subset \mathbb{R}^n$ such that $u(\cdot,\widehat{T}) \equiv 0$ in D_0. Therefore, (5.18) allows us to apply Lemma 5.4. Consequently, we conclude:

$$u(\cdot,\widehat{T}) \equiv 0 \quad \text{in } \mathbb{R}^n. \tag{5.19}$$

To extend the vanishing property in (5.19) to the time direction, we apply Lemma 5.5. For this purpose, we define w as follows:

$$w(x,t) := u\big(x, \widehat{T} - t\big).$$

From $u \in W^{2,1}_{2,\mathrm{loc}}(Q_{\widehat{T}}) \cap C([0,\widehat{T}]; L^2_{\mathrm{loc}}(\mathbb{R}^n))$, it follows that $w \in W^{2,1}_{2,\mathrm{loc}}(Q_{\widehat{T}}) \cap C([0,\widehat{T}]; L^2_{\mathrm{loc}}(\mathbb{R}^n))$. Furthermore, we have $w \in \mathcal{H}_{K,\widehat{T}}$. Indeed, from (5.19), we observe that $w(\cdot,0) \equiv 0$ in $\mathbb{R}^n$. In addition, let $M > 0$. Then, the following inequality holds:

$$\left|w(x,t)e^{-M|x|^2}\right| = \left|u(x,\widehat{T}-t)\right| \left|e^{-M|x|^2}\right| \le \sup_{0<t<\widehat{T}} \|u(t)\|_{L^\infty(\mathbb{R}^n)}$$

for a.e. $(x,t) \in Q_{\widehat{T}}$. This shows that $w \in \mathcal{H}_{K,\widehat{T}}$. Moreover, since $0 < \widehat{T} - t < \widehat{T}$ for $0 < t < \widehat{T}$, we deduce from (5.16):

$$\begin{aligned}
\left|\Delta w(x,t) + \frac{\partial w}{\partial t}(x,t)\right| &= \left|(\Delta u)(x,\widehat{T}-t) - \Big(\frac{\partial u}{\partial t}\Big)(x,\widehat{T}-t)\right| \\
&\le C_*\Big(\big|u(x,\widehat{T}-t)\big| + \big|\nabla u(x,\widehat{T}-t)\big|\Big) \\
&= C_*\Big(\big|w(x,t)\big| + \big|\nabla w(x,t)\big|\Big)
\end{aligned}$$

for a.e. $(x,t) \in Q_{\widehat{T}}$, where C_* is defined in (5.17). Thus, by applying Lemma 5.5, we conclude that $w \equiv 0$ in $Q_{\widehat{T}}$. Consequently, it follows that $u \equiv 0$ in $Q_{\widehat{T}}$. This completes the proof of Theorem 2.2. $\square$

6. Proof of Theorem 2.3

We assume that there exists $0 < t_0 < \widehat{T}$ such that the support of $u(\cdot,t_0)$ is not the whole space $\mathbb{R}^n$. We define $D_0 := \mathbb{R}^n \setminus \operatorname{supp} u(\cdot,t_0)$. Then, D_0 is a non-empty open set, and $u(\cdot,t_0) \equiv 0$ in D_0. Combined with Theorem 2.2, this implies that $u \equiv 0$ in $\mathbb{R}^n \times (0,t_0)$.

On the other hand, according to (2.8), we have:

$$\int_{\mathbb{R}^n} u_0(x)\,dx = \int_{\mathbb{R}^n} u(x,t_0)\,dx = 0,$$

which contradicts the assumption (2.10). This completes the proof of Theorem 2.3. $\square$

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Data availability

No data was used for the research described in the article.

References

[1] A. Banerjee, R. Manna, Space-like strong unique continuation for sublinear parabolic equations, J. Lond. Math. Soc. (2) 102 (2020) 205–228.
[2] T. Carleman, Sur un problème d'unicité pour les systèmes d'équations aux dérivées partielles à deux variables indépendentes, Ark. Mat. Astron. Fys. 26 (1939) 1–9.
[3] J.A. Carrillo, S. Lisini, E. Mainini, Uniqueness for Keller-Segel-type chemotaxis models, Discrete Contin. Dyn. Syst. 34 (2014) 1319–1338.
[4] J.A. Carrillo, Y. Sugiyama, Compactly supported stationary states of the degenerate Keller-Segel system in the diffusion-dominated regime, Indiana Univ. Math. J. 67 (2018) 2279–2312.
[5] L. Corrias, B. Perthame, H. Zaag, Global solutions of some chemotaxis and angiogenesis systems in high space dimensions, Milan J. Math. 72 (2004) 1–28.
[6] J. Duoandikoetxea, Fourier Analysis, Grad. Stud. Math., vol. 29, Amer. Math. Soc., Providence, RI, 2000.
[7] L. Escauriaza, F.J. Fernández, Unique continuation for parabolic operators, Ark. Mat. 41 (2003) 35–60.
[8] L. Grafakos, Classical Fourier Analysis, 3rd ed., Graduate Texts in Mathematics, Springer, New York, 2010.
[9] T. Kawakami, Y. Sugiyama, Uniqueness theorem on weak solutions to the Keller-Segel system of degenerate and singular types, J. Differ. Equ. 260 (2016) 4683–4716.
[10] H. Koch, D. Tataru, Carleman estimates and unique continuation for second-order elliptic equations with nonsmooth coefficients, Commun. Pure Appl. Math. 54 (2001) 339–360.
[11] H. Kozono, M. Miura, Y. Sugiyama, Time global existence and finite time blow-up criterion for solutions to the Keller-Segel system coupled with the Navier-Stokes fluid, J. Differ. Equ. 267 (2019) 5410–5492.
[12] H. Kozono, Y. Sugiyama, Local existence and finite time blow-up of solutions in the 2-D Keller-Segel system, J. Evol. Equ. 8 (2008) 353–378.
[13] H. Kozono, Y. Sugiyama, Global strong solution to the semi-linear Keller-Segel system of parabolic-parabolic type with small data in scale invariant spaces, J. Differ. Equ. 247 (2009) 1–32.
[14] H. Kozono, Y. Sugiyama, Strong solutions to the Keller-Segel system with the weak $L^{\frac{n}{2}}$ initial data and its application to the blow-up rate, Math. Nachr. 283 (2010) 732–751.
[15] H. Kozono, Y. Sugiyama, Y. Yahagi, Existence and uniqueness theorem on weak solutions to the parabolic-elliptic Keller-Segel system, J. Differ. Equ. 253 (2012) 2295–2313.
[16] M. Kurokiba, T. Ogawa, Finite time blow up for a solution to system of the drift-diffusion equations in higher dimensions, Math. Z. 284 (2016) 231–253.
[17] S. Luckhaus, Y. Sugiyama, Large time behavior of solutions in super-critical cases to degenerate Keller-Segel systems, Math. Model. Numer. Anal. 40 (2006) 597–621.
[18] S. Luckhaus, Y. Sugiyama, Asymptotic profile with the optimal convergence rate for a parabolic equation of chemotaxis in super-critical cases, Indiana Univ. Math. J. 56 (2007) 1279–1297.
[19] T. Nagai, Global existence and decay estimates of solutions to a parabolic elliptic system of drift-diffusion type in $\mathbb{R}^2$, Differ. Integral Equ. 24 (2011) 29–68.
[20] M. Miura, Y. Sugiyama, On uniqueness theorem on weak solutions to the parabolic-parabolic Keller-Segel system of degenerate and singular types, J. Differ. Equ. 257 (2014) 4064–4086.
[21] T. Ogawa, S. Shimizu, End-point maximal regularity and its application to two-dimensional Keller-Segel system, Math. Z. 264 (2010) 601–628.
[22] O. Sawada, On the Spatial Analyticity of Solutions to the Keller-Segel Equations, Parabolic and Navier-Stokes Equations, Part 2, Banach Center Publ., vol. 81, Polish Acad. Sci. Inst. Math., Warszawa, 2008, pp. 421–431.
[23] Y. Sugiyama, H. Kunii, Global existence and decay properties for a degenerate Keller-Segel model with a power factor in the drift term, J. Differ. Equ. 227 (2006) 333–364.
[24] Y. Sugiyama, Y. Yahagi, Extinction, decay and blow-up for Keller-Segel systems of fast diffusion type, J. Differ. Equ. 250 (2011) 3047–3087.
[25] Y. Sugiyama, Global existence in sub-critical cases and finite time blow-up in super-critical cases to degenerate Keller-Segel systems, Differ. Integral Equ. 19 (2006) 841–876.
[26] S. Vessella, Unique continuation properties and quantitative estimates of unique continuation for parabolic equations, Handb. Differ. Equ: Evol. Equ. 5 (2009) 421–500.
[27] M. Yamamoto, Carleman estimates for parabolic equations and applications, Inverse Probl. 25 (2009) 14.
[28] M. Yamamoto, Spatial analyticity of solutions to the drift-diffusion equation with generalized dissipation, Arch. Math. 97 (2011) 261–270.