European Journal of Operational Research 330 (2026) 279–297

Contents lists available at ScienceDirect

European Journal of Operational Research

journal homepage: www.elsevier.com/locate/eor

Innovative Applications of O.R.

Certainty equivalent and uncertainty premium of time-to-build

Haejun Jeon *,[1], Michi Nishihara

The University of Osaka, Japan

ARTICLE INFO

JEL classification:
D25
D81
G31

Keywords:
Time-to-build
Investment lags
Certainty equivalent
Uncertainty premium
Real options

ABSTRACT

Time-to-build of an investment project induces a difference between the timing of investment and that of revenue generation. Jeon (2024b) showed that uncertainty in the time-to-build *always* accelerates investment and enhances pre-investment firm value, regardless of its distribution. This study examines the extent to which the uncertainty advances the timing of investment and improves firm value. Specifically, we show that there *always* exists a unique certainty equivalent of an uncertain time-to-build and derive it in an analytic form. This enables us to derive the investment strategy with an uncertain time-to-build in the form of the one that would have been adopted in the absence of such uncertainty. Even without full knowledge of the uncertainty, the firm can approximate the optimal investment strategy using only the mean and variance of time-to-build. Furthermore, we show that there *always* exists an uncertainty equivalent of fixed time-to-build. This enables firms to evaluate the level of risk implicitly assumed by their investment strategies established without accounting for uncertainty in time-to-build. Lastly, we illustrate the practical application of our findings using some representative probability distributions and analyze the effects of the variance of time-to-build. In particular, we contrast the effects of uncertainty in demand with those of uncertainty in time-to-build, deriving the level of variance in time-to-build that offsets the negative impact of increased demand volatility on investment.

1. Introduction

In 2015, Elon Musk made a bold promise that Tesla's vehicles would drive themselves in two years. In 2019, he made another promise, claiming that there would be a million robotaxis on the road in a year. After nearly a decade, neither have we seen fully self-driving technology from Tesla, nor do we see any of their robotaxis on the road.[2] At an event in October 2024, which had been postponed several times, Tesla revealed some prototypes of their robotaxis. Elon claimed that they would be available before 2027, which seems highly unlikely, given his notorious record of unmet promises regarding timelines, as well as the complex regulatory requirements the technology must meet.

Elon and his company are not the only ones. Olkiluoto 3 in Finland is one of the largest nuclear reactors in Europe. Its construction began in 2005 with an estimated completion date in 2009, but finalized in 2022, resulting in a 13-year delay. Flamanville 3 in France is another example of a significant delay in constructing a power plant. It started in 2007 with the aim of completing by 2012; however, it still has not been finished yet (White, 2024). As seen from these examples, time-to-build is prevalent in real-world investment projects, and uncertainty is one of its inherent attributes.[3] Time-to-build has a significant impact on firm value because it introduces a difference between the timing of investment and that of revenue generation. When its duration is uncertain, the firm's investment strategy must be established even more meticulously. Nevertheless, the effects of uncertainty in time-to-build on corporate investment are underexplored.

To the best of our knowledge, Nishihara (2018) is the first study that shed light on the effects of uncertainty in time-to-build on investment. The paper analyzed a firm's research and development (R&D) investment decision, assuming that duration follows a uniform distribution, and numerically showed that uncertainty in the duration, compared to

* Corresponding author.Correspondence to: Graduate School of Economics, The University of Osaka, 1-7 Machikaneyama, Toyonaka, Osaka, 560-0043, Japan.
E-mail address: jeon@econ.osaka-u.ac.jp (H. Jeon).

[1] This study was supported by the JSPS KAKENHI (grant numbers 22K13434, 23K20613, and 24K00272) and Tokyo University of Science Research Grants. The authors would like to express their gratitude to Masaaki Fukasawa for his constructive comments on this paper.

[2] Although Tesla has provided a driver-assistance system called Full Self-Driving, the name is misleading, as it remains at Level 2 automation, with Level 5 being fully autonomous driving according to the standards of the Society of Automotive Engineers International.

[3] Examples are abundant. Recent issues include sluggish capacity expansion in the semiconductor industry, where demand surged during the COVID-19 pandemic, and significant production delays by new automakers following the rise of the electric vehicle market.

[4] Jeon (2024b) considered not only time-to-build but also regulation as internal and external factors that hinder immediate revenue generation after the investment, respectively. This study excludes the latter to simplify the model and its solution.

https://doi.org/10.1016/j.ejor.2025.07.051
Received 13 March 2025; Accepted 22 July 2025
Available online 14 August 2025
0377-2217/

a fixed equivalent, leads to earlier investment. Jeon (2024b) investigated the effects of uncertainty in time-to-build without any assumption on its distribution and analytically showed that uncertainty in time-to-build *always* accelerates investment and increases pre-investment firm value.[4] Jeon (2024a) extended this framework by incorporating the firm's investment size decision in addition to the timing decision, confirming that the positive impacts of uncertainty in time-to-build on investment remain intact. However, these studies did not demonstrate the extent to which the uncertainty advances the timing of investment and improves firm value. This study addresses this unanswered yet significant problem by clarifying the impacts of uncertainty in time-to-build on firm value in more detail.

First, we show that there always exists a unique *certainty equivalent* for uncertain time-to-build, regardless of its distribution. That is, there is a fixed time-to-build whose duration is shorter than the uncertain counterpart but yields the same firm value. Furthermore, we derive the certainty equivalent in an analytic form. This enables us to determine the optimal investment strategy with uncertain time-to-build in the form of the investment strategy that would have been adopted without such uncertainty. Even without the full knowledge of the uncertainty in time-to-build (i.e., probability distribution), the certainty equivalent can be approximated with only a few moments, such as mean and variance, which significantly enhances practicality. Moreover, we derive the tight upper and lower bounds of the certainty equivalent for given mean and variance of time-to-build. We also show that the extent of investment acceleration due to uncertainty in time-to-build decreases with the expected growth rate of revenue and is independent of its volatility.

Second, we show that there always exists an *uncertainty equivalent* for fixed time-to-build. That is, there is an uncertain time-to-build whose expected duration is longer than the fixed counterpart but induces the identical firm value. Unlike the certainty equivalent, there can be many uncertainty equivalents for a given fixed time-to-build. This enables firms to evaluate the level of risk implicitly assumed by their investment strategies established without accounting for uncertainty in time-to-build. We also show that for a given fixed time-to-build, there always exists an uncertain counterpart whose expected duration is longer yet yields higher firm value, which verifies the positive impacts of uncertainty in time-to-build.

Lastly, we apply the above arguments to representative probability distributions to demonstrate their practicality. Specifically, we applied the main results to positively skewed and unimodal distributions with nonzero mode, which is consistent with empirical evidence, and we find that the mean and variance of time-to-build are often sufficient to approximate its certainty equivalent. Furthermore, we contrast the effects of demand uncertainty with those of uncertainty in time-to-build. The former delays investment because it increases the value of waiting, whereas the latter accelerates investment because it increases the expected profits from the investment by the convexity of the discount factor with respect to the revenue generation timing. With these arguments, we derive the variance of time-to-build that offsets the negative impacts of increased demand volatility on investment.

The remainder of this study is organized as follows. Section 2 reviews the literature on uncertainty-investment relationship and time-to-build. Section 3 introduces the model setup and Section 4 derives its solution. Specifically, Section 4.1 presents the preliminary results based on a standard real options model, and Section 4.2 contrasts the effects of uncertainty in time-to-build with those of uncertainty in demand. Section 4.3 derives the certainty equivalent of uncertain time-to-build and analyzes its sensitivity, while Section 4.4 derives the uncertainty equivalent of a fixed time-to-build. Section 5 applies the arguments discussed in Section 4 to representative probability distributions. Specifically, Sections 5.1–5.4 correspond, respectively, to the following distributions: triangular distribution, log-normal distribution, gamma distribution, and scaled beta distribution. Section 5.5 focuses on the mean and variance of time-to-build and compares the effects of uncertainty in time-to-build and those of uncertainty in demand. Section 6 summarizes the main findings and suggests possible future work, and Appendix A presents all proofs. In the Online Appendix, we discuss the certainty equivalent from the perspective of entropic risk measure and consider different cost structures and alternative probability distributions for time-to-build. It also provides a summary of the characteristics of the distributions discussed in the paper.

2. Literature review

Majd and Pindyck (1987) was one of the first studies to examine the impact of time-to-build on corporate investment. They assumed a maximum rate at which a firm can invest and showed that such friction results in delays in investment. Bar-Ilan and Strange (1996a, 1996b) supposed that a certain period of time must elapse before revenue from investment can be generated, showing that uncertainty in demand can hasten investment in the presence of lags. They assumed a fixed time-to-build, and it is uncertainty in demand, not that in time-to-build, that accelerates investment. Furthermore, they assumed the firm's option to abandon the ongoing project, which truncates the downside risk of the project and yields a stronger incentive for investment. Bar-Ilan and Strange (1998) extended their previous work to a two-stage investment project and found that the investment can be sequential when the firm has an option to suspend the ongoing project. Pacheco-de Almeida and Zemsky (2003) also studied a multi-stage investment in the presence of time-to-build and duopoly. They found that the firm's investment behavior can be either incremental or lumpy depending on the duration of time-to-build. However, these studies only considered a fixed time-to-build, leaving the effects of the uncertain counterpart unaddressed.

Some studies adopted uncertain time-to-build in the discussion of corporate investment decisions. Weeds (2002) examined R&D competition in a duopoly market, assuming random discovery time for new technologies, and found negative impacts of uncertain lags on investment decisions. Alvarez and Keppo (2002) examined a firm's irreversible investment with delivery lags in a generalized setup in which they are interdependent. Specifically, they assumed that the lags increase with the level of demand shock and showed that the investment might be suboptimal depending on the level of demand shock, primarily because higher demands imply longer delivery lags. Jeon (2021a) investigated the effects of uncertain time-to-build on a levered firm's investment and financing decision and showed that the default probability can be lower than the case without time-to-build. Jeon (2021b) studied a duopolistic market with asymmetric uncertain time-to-build and found the equilibrium in which the dominated firm with a longer expected time-to-build becomes a leader. Jeon (2023) took account of learning effects in the discussion of capacity expansion with uncertain time-to-build.

Although these studies considered uncertain time-to-build in their discussion, the *sheer* effects of uncertainty in time-to-build were not addressed. To our knowledge, Nishihara (2018) is the first to discuss this issue. This study investigated a firm's R&D investment decision with uncertainty in market demands, competition, and R&D duration, and numerically showed that uncertainty in the duration, described by a uniform distribution, leads to earlier investment than in the case of fixed duration. Jeon (2024b) compared the optimal investment strategy and firm value with fixed time-to-build and those with uncertain time-to-build whose expected duration is identical with the fixed counterpart, without any assumption on the distribution of time-to-build. The comparison showed that uncertainty in time-to-build always accelerates investment and improves pre-investment firm value, regardless of its distribution. Jeon (2024a) found that the positive impact of uncertainty in time-to-build remains robust even when the investment size decision is taken into account in addition to the timing decision.

Despite the difficulties of collecting data, some studies have empirically analyzed the effects and determinants of time-to-build. Jorgenson and Stephenson (1967) investigated investment behavior in U.S. manufacturing and found that the average lag between the determinants of investment and actual investment expenditures ranges from 1.5 to 3 years. Montgomery (1995) examined the construction duration of U.S. nonresidential structures and found that the value-weighted construction period is approximately 1.5 years, with significant variation over time. Koeva (2000) analyzed plant investment from various industries and found that time-to-build, averaging two years, is not sensitive to business cycles. Zhou (2000) empirically showed that time-to-build can explain the positive correlation of investment. Salomon and Martin (2008) reported that in the semiconductor industry, the duration of time-to-build is associated with market competition, firm ownership, and firm/industry experience. Tsoukalas (2011) showed that in the presence of time-to-build, a firm's investment decision is significantly affected by the firm's cash flows. Kalouptsidi (2014) found that time-varying time-to-build decreases the level and volatility of investment in the bulk shipping industry. Oh and Yoon (2020) showed that in the 2002–2011 U.S. housing boom-bust cycle, the increase of time-to-build during the boom is due to construction bottlenecks whereas that during the bust is due to an increase of uncertainty. Oh et al. (2024) utilized data on U.S. residential land development and showed that time-to-build introduces a significant difference between short-run and long-run housing supply elasticities. Glancy et al. (2024) analyzed data on U.S. commercial construction projects and found that roughly one-third of projects are abandoned during the planning phase, while over 99% of those that reach the construction phase are completed. They also found that property price appreciation reduces the likelihood of abandonment. Charoenwong et al. (2024) utilized Japanese dataset to show that information acquisition and investment flexibility can reduce the negative impacts of time-to-build significantly. Fernandes and Rigato (2025) utilized Indian project-level data to measure time-to-build and found that firms accelerate ongoing projects rather than start new ones when credit dries up.

3. Setup

Suppose that a risk-neutral firm is considering an investment project with demand shocks that follows a geometric Brownian motion:

$$\mathrm{d}X(t) = \mu X(t)\mathrm{d}t + \sigma X(t)\mathrm{d}W(t), \tag{1}$$

where μ and σ are positive constants and $(W(t))_{t\geq 0}$ is a standard Brownian motion on a filtered space $(\Omega, \mathcal{F}, \mathbb{F} := (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ satisfying the usual conditions. For simplicity, we assume that the demand is price-inelastic such that the monopolistic firm's revenue flow from this project coincides with (1).[5] The investment incurs lump-sum costs I and the variable costs of production are normalized to zero.[6] The discount rate is $r(>\mu)$ to ensure a finite value function, which is a standard assumption in real options literature.

The investment project does not yield revenue immediately after the investment because of the project's time-to-build. This can arise from R&D for new technologies or large-scale construction of manufacturing facilities. Due to its inherent uncertainty, the size of time-to-build is a nonnegative random variable τ, which is assumed to be independent of $X(t)$ for simplicity.

4. Models and solutions

4.1. Preliminary results

The pre-investment firm value is evaluated as the expected present value of revenue from the investment project less its costs. Thus, the firm value with an option to invest in a project having a time-to-build of τ is expressed as follows:

$$V_\tau(X) = \max_{T\geq 0} \mathbb{E}\Big[\int_{\hat{T}}^{\infty} e^{-rt} X(t)\mathrm{d}t - e^{-rT} I \Big| X(0) = X\Big]. \tag{2}$$

It is optimal for the firm to invest in the project as the demand shock reaches an upper threshold.[7] Thus, the investment timing can be characterized by the level of demand shock at which the firm invests in the project, and $T := \inf\{t > 0 | X(t) \geq X_\tau\}$ and $\hat{T} := T + \tau$ denote the timing of investment and revenue generation, respectively, where X_τ represents the corresponding investment threshold.

Due to the Markov property, the firm value at the investment timing for given demand shock X is

$$\mathbb{E}\Big[\int_{\tau}^{\infty} e^{-rt} X(t)\mathrm{d}t - I \Big| X(0) = X\Big] = \frac{X\delta(\tau)}{r-\mu} - I, \tag{3}$$

where $\delta(\tau) := \mathbb{E}[e^{-(r-\mu)\tau}]$ represents the discount factor for the timing of revenue generation, which plays a pivotal role in the following discussion. Note that it is the Laplace transform of the time-to-build. Following the standard argument of real options, the firm value in (2) can be calculated as follows[8]:

$$V_\tau(X) = \begin{cases} \Big[\frac{X_\tau \delta(\tau)}{r-\mu} - I\Big]\Big(\frac{X}{X_\tau}\Big)^{\gamma}, & \text{if } X < X_\tau, \\ \frac{X\delta(\tau)}{r-\mu} - I, & \text{if } X \geq X_\tau, \end{cases} \tag{4}$$

where the optimal investment threshold is

$$X_\tau = \frac{\gamma(r-\mu)I}{(\gamma-1)\delta(\tau)}, \tag{5}$$

and

$$\gamma := \frac{1}{2} - \frac{\mu}{\sigma^2} + \sqrt{\Big(\frac{1}{2} - \frac{\mu}{\sigma^2}\Big)^2 + \frac{2r}{\sigma^2}} \ (>1). \tag{6}$$

4.2. Effects of uncertainty on investment

The effects of uncertainty in time-to-build on investment and firm value can be described as follows:

Lemma 1. *If τ_{n+1} is a mean-preserving spread of τ_n for $n \geq 0$ with a constant $\tau_0 = \bar{\tau}$, the following always holds:*

$$X_{\tau_{n+1}} < X_{\tau_n} \quad \text{and} \quad V_{\tau_{n+1}}(X) > V_{\tau_n}(X) \qquad \text{for all } n \geq 0. \tag{7}$$

Proof. See Appendix A.1.

Lemma 1 implies that uncertainty in time-to-build *always* accelerates investment and improves pre-investment firm value, which was first shown by Jeon (2024b). This is essentially due to the convexity of the discount factor with respect to the timing of revenue generation. Specifically, the gain from earlier revenue generation (i.e., $\tau < \mathbb{E}[\tau]$) is discounted over a relatively short period of time, while the loss from delayed revenue generation (i.e., $\tau > \mathbb{E}[\tau]$) is discounted over a longer period of time, resulting in the asymmetric effects of uncertainty in time-to-build on firm value. Note that this argument is independent of the distribution of time-to-build τ.[9]

[5] This simplification can also be found from Jeon (2024b), among many others.

[6] We assume that the lump-sum investment costs are incurred at the investment timing, but the main results of this paper remain intact even when running costs are incurred throughout the period of time-to-build. See the Online Appendix for this discussion.

[7] See Dixit and Pindyck (1994, Chapter 4) and Peskir and Shiryaev (2006, Chapter 4) for the discussion regarding the optimality of threshold policy.

[8] The derivation of the optimal investment threshold based on the real options framework can be found in Dixit and Pindyck (1994, Section 5), among many others.

[9] Jeon (2024b) verified the robustness of this result, showing that it still holds even when there are running costs during the phase of time-to-build and the firm has an option to abandon the ongoing project. Jeon (2024a) showed that the positive impacts of uncertainty in time-to-build persist even when the firm's investment size decision is considered in addition to the timing decision.

Hartman (1972, 1973) and Abel (1983) demonstrated that uncertainty *can* accelerate investment, focusing on uncertainties in the *state space*, such as market demands and output prices. In their studies, the convexity of marginal profitability of capital, resulting from the optimal labor adjustment, leads to the positive impacts of uncertainty in demands. Jeon (2024a, 2024b) and this study shed light on the uncertainty in the *time dimension*, showing that uncertainty in revenue generation timing *always* accelerates investment, and it is also the convexity that drives the positive impacts of uncertainty in time-to-build.

In the standard real options literature in which the investment timing decision is mainly discussed, it is well-known that an increase in demand volatility (i.e., σ) delays investment. This negative impact of demand uncertainty on investment is in sharp contrast with the positive impact of uncertainty in time-to-build, and the economic intuition behind these opposing effects is as follows. When demand is uncertain, the firm obtains more information and resolves the uncertainty by waiting to invest. In other words, the value of the option to wait increases with demand uncertainty, and therefore, the firm delays investment as the market becomes more volatile. This can be seen from the fact that $\partial\gamma/\partial\sigma < 0$, and thereby $\gamma/(\gamma-1)$ in (5), which represents the option value, increases with σ. Note that the expected profits from the investment in (3) are independent of σ. This implies that the option value is the sole channel through which demand uncertainty negatively affects the investment decision.

By contrast, the firm can acquire more information regarding the timing of revenue generation and resolve the uncertainty only after the investment, but the amount and quality of this information is independent of the investment timing. Thus, earlier investment due to the uncertainty in time-to-build is not associated with the value of the option to wait. This can be seen from the fact that γ in (6) is independent of τ. Note that the expected profits at the investment timing in (3) depends on τ. This implies that the expected profits from the investment, which depend on the convexity of the discount factor regarding the revenue generation timing, are the sole channel through which uncertainty in time-to-build positively impacts the investment decision. This argument is illustrated with numerical examples in Section 5.5.

Most empirical studies on the uncertainty-investment relationship indicate a negative link between them (e.g., Guiso and Parigi (1999), Leahy and Whited (1996) and Meinen and Roehe (2017)), but there are a few exceptions. For instance, Driver et al. (2008) used panel data from the British survey to test the effects of uncertainty on investment and found positive impacts in industries with high R&D and advertising intensities. Marmer and Slade (2018) analyzed U.S. copper mining industry and reported a positive impact of uncertainty on investment when the project involved time-to-build. These studies suggest that time-to-build might drive the positive impacts of uncertainty on investment, although this hypothesis requires further empirical testing.

4.3. Certainty equivalent of uncertain time-to-build

Now we examine the extent to which uncertainty in time-to-build advances the investment timing and improves the firm value.

Proposition 1 (*Certainty Equivalent*). *For any uncertain time-to-build τ, there always exists a unique constant $\bar{\tau}_c (< \mathbb{E}[\tau])$ such that $\delta(\tau) = \delta(\bar{\tau}_c)$, or equivalently, $X_\tau = X_{\bar{\tau}_c}$ and $V_\tau(X) = V_{\bar{\tau}_c}(X)$. The certainty equivalent is derived as*

$$\bar{\tau}_c = -\frac{K_\tau(-(r-\mu))}{r-\mu}, \tag{8}$$

where $K_\tau(t)$ is the cumulant-generating function of τ[10]*:*

$$K_\tau(t) = \ln \mathbb{E}[e^{t\tau}] = \sum_{n=1}^{\infty} \frac{t^n \kappa_n}{n!}, \tag{9}$$

with κ_n denoting the nth cumulant of τ.

Proof. See Appendix A.2.

Proposition 1 offers a practical framework for deriving the optimal investment strategy in the presence of uncertain time-to-build in a straightforward manner. Given the prior knowledge of the uncertainty in time-to-build τ, the firm can derive the corresponding certainty equivalent $\bar{\tau}_c$ in (8) and apply it to the optimal investment strategy that would have been adopted in the absence of such uncertainty (i.e., $X_\tau = X_{\bar{\tau}_c}$). This tractable framework is applicable to *any* τ that has its probability density function.

Proposition 1 implies that the firm value with *longer* and *uncertain* time-to-build (i.e., τ) is same as that with *shorter* and *fixed* time-to-build (i.e., $\bar{\tau}_c$) and that the unique correspondence (i.e., $X_\tau = X_{\bar{\tau}_c}$ and $V_\tau(X) = V_{\bar{\tau}_c}(X)$) *always* exists, regardless of the distribution of stochastic time-to-build. The degree to which uncertainty in time-to-build accelerates investment and thus improves firm value is measured by $\mathbb{E}[\tau] - \bar{\tau}_c$ (> 0), which is referred to as *uncertainty premium of time-to-build*.

Fig. 1 graphically illustrates the positive impacts of uncertainty in time-to-build and the existence of the certainty equivalent. To facilitate understanding, Fig. 1(a) reviews the well-known negative impacts of uncertainty in consumption on utility. For a risk-averse investor, her utility function $U(x)$ is a function of consumption level x with $U' \geq 0$ and $U'' \leq 0$.[11] Given possible outcomes of x_1 and x_2, the concavity of the utility function ensures $\mathbb{E}[U(x)] \leq U(\mathbb{E}[x])$ always holds, and there exists the certainty equivalent $\bar{x}_c$ such that $\mathbb{E}[U(x)] = U(\bar{x}_c)$ and $\bar{x}_c \leq \mathbb{E}[x]$. Fig. 1(b) follows similar arguments. Firm value $V(\tau)$ is a function of time-to-build τ with $V' < 0$ and $V'' > 0$, and given possible outcomes of τ_1 and τ_2, the convexity ensures $\mathbb{E}[V(\tau)] > V(\mathbb{E}[\tau])$; there exists the certainty equivalent $\bar{\tau}_c$ such that $\mathbb{E}[V(\tau)] = V(\bar{\tau}_c)$ and $\bar{\tau}_c > \mathbb{E}[\tau]$.

The sensitivity of the certainty equivalent of uncertain time-to-build with respect to market demands is addressed as follows:

Corollary 1. *The certainty equivalent of time-to-build increases with the expected growth rate of demand (i.e., μ). In other words, the uncertainty premium of time-to-build decreases with it. Both are independent of demand volatility (i.e., σ).*

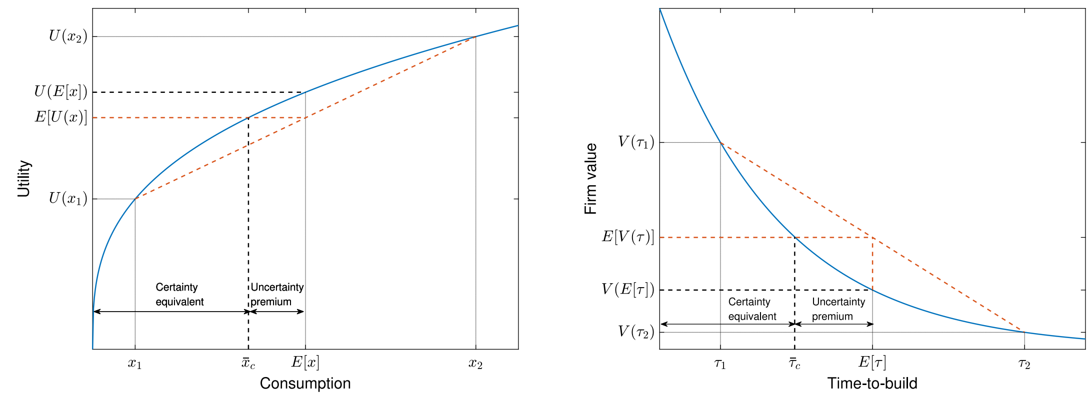
Proof. See Appendix A.3.

This result implies that uncertainty in time-to-build accelerates investment significantly when market demand is expected to grow slowly. Technically speaking, this is because the convexity of the discount factor with respect to the timing of revenue generation — the main driver of the positive effects of uncertainty in time-to-build — decreases with the expected growth rate of demand (i.e., μ).[12] That is, when μ is low, the firm heavily discounts the future cash flow, and thus, earlier completion of the project is more appreciated, and losses from the delay are significantly discounted when μ is low. In summary, the adjustment

[10] The cumulant-generating function is the natural logarithm of the moment-generating function $M_\tau(t) = \mathbb{E}[e^{t\tau}] = \sum_{n=0}^{\infty} \frac{t^n \mathbb{E}[\tau^n]}{n!}$. Since $r > \mu$, $M_\tau(-(r-\mu))(< 1)$ always exists and so does $K_\tau(-(r-\mu))(< 0)$, provided that the probability density function exists.

[11] This includes a broad class of utility functions, including those with constant absolute risk aversion and constant relative risk aversion (i.e., $U(x) = -e^{-\gamma x}$ and $U(x) = (x^{1-\gamma}-1)/(1-\gamma)$, respectively, where $\gamma > 0$ denotes the degree of risk aversion).

[12] For $f(\tau) = \exp(-(r-\mu)\tau)$, the degree of convexity, measured by $\frac{1}{f}\frac{\partial^2 f}{\partial \tau^2} = (r-\mu)^2$, decreases with μ.



(a) Uncertain consumption and utility (b) Uncertain time-to-build and firm value

Fig. 1. Positive impacts of uncertainty in time-to-build on firm value.

in the investment strategy due to the uncertainty of revenue generation timing needs to consider how much the demand is expected to grow over time but it should not reflect how volatile the demand is.

Proposition 1 allows us to summarize the direct relationship between time-to-build and firm value as follows:

Corollary 2. *Suppose the initial demand X is sufficiently low such that the investment is not triggered instantly. The firm value $V_\tau(X)$ being greater than $\bar{X}$ is equivalent to*

$$\bar{\tau}_c \leq \frac{\ln(A(X)/\bar{X})}{(r-\mu)\gamma}, \tag{10}$$

where

$$A(X) = \left(\frac{\gamma-1}{I}\right)^{\gamma-1}\left(\frac{X}{\gamma(r-\mu)}\right)^{\gamma}. \tag{11}$$

Proof. See Appendix A.4.

It is obvious that the right-hand side of (10) decreases with $\bar{X}$. This implies that the certainty equivalent of time-to-build must be smaller, or equivalently, the uncertainty premium of time-to-build must be greater to achieve a higher firm value.

Proposition 1 implies that the firm needs *perfect* prior information regarding time-to-build (i.e., probability distribution) to derive the optimal investment strategy based on the certainty equivalent. However, in practice, firms rarely have such perfect prior information regarding the uncertainty in time-to-build. Nevertheless, even without the full knowledge regarding such uncertainty, the firm can approximate the certainty equivalent using only a few moments of the time-to-build as follows:

Corollary 3. *Given the mean and variance of time-to-build τ, denoted by m and v, respectively, the certainty equivalent of τ is approximated as follows:*

$$\tilde{\tau}_{c,2} := m - \frac{(r-\mu)v}{2}. \tag{12}$$

With the addition of skewness and excess kurtosis, denoted by s and e, respectively, it can be approximated more precisely as follows:

$$\tilde{\tau}_{c,3} := m - \frac{(r-\mu)v}{2}\left(1 - \frac{(r-\mu)s\sqrt{v}}{3}\right), \tag{13}$$

$$\tilde{\tau}_{c,4} := m - \frac{(r-\mu)v}{2}\left(1 - \frac{(r-\mu)s\sqrt{v}}{3} + \frac{(r-\mu)^2 ev}{12}\right), \tag{14}$$

where the approximation error is $\tilde{\tau}_{c,i} - \bar{\tau}_c$ for $i \in \{2,3,4\}$.

Proof. See Appendix A.5.

Corollary 3 implies that if the firm has estimates of the mean and variance of time-to-build from prior investment experiences in similar fields, it can derive the optimal investment strategy that considers the uncertainty in time-to-build without any additional information (i.e., exact distribution). As shown in Section 5, the mean and variance are often sufficient to approximate the certainty equivalent of uncertain time-to-build.

Based on Proposition 1 and Corollary 3, we can easily obtain the following result:

Corollary 4. *The certainty equivalent of time-to-build decreases with its dispersion. In other words, the uncertainty premium of time-to-build increases with its dispersion. Specifically, with the approximation up to the third moment in (13), the uncertainty premium of time-to-build increases with the variance v if $s < 3/((r-\mu)\sqrt{v})$. With the approximation up to the fourth moment in (14), it increases with the variance v if $s < 3/((r-\mu)\sqrt{v}) + (r-\mu)e\sqrt{v}/4$.*

Proof. See Appendix A.6.

The positive impact of the variance of time-to-build on its uncertainty premium is straightforward; as noted in Lemma 1, the more dispersed time-to-build, the stronger incentive the firm's investment incentive. The result in (13) shows that, all else being equal, the skewness of time-to-build negatively impacts its uncertainty premium. This is because the positively-skewed time-to-build implies that the distribution has a longer tail for the likelihood of longer time-to-build, which reduces the firm's incentive to invest. The result in (14) implies that, all else being equal, the excess kurtosis of time-to-build positively affects its uncertainty premium. This is because a greater excess kurtosis implies fatter tails, which increases the firm's incentive to invest due to the convexity effect described in Lemma 1.

Furthermore, for given mean and variance of time-to-build, we can derive the distribution-free upper and lower bounds of the certainty equivalent as follows:

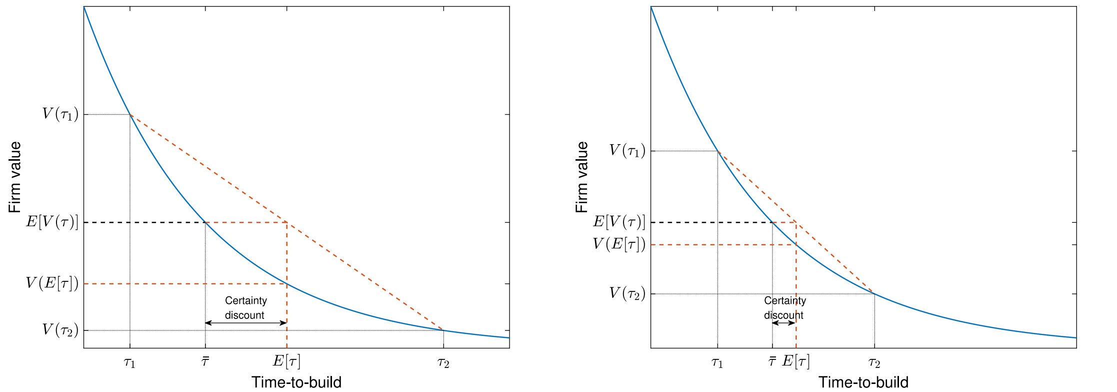
Proposition 2. *For given mean m and variance v of time-to-build, the certainty equivalent of time-to-build is bounded as follows:*

$$-\frac{1}{r-\mu}\ln\left(\frac{e^{-(r-\mu)(m+v/m)}m^2 + v}{m^2+v}\right) \leq \bar{\tau}_c \leq m. \tag{15}$$

In particular, both the upper and lower bounds are tight, and the lower bound strictly decreases with v.

Proof. See Appendix A.7.

Corollary 4 suggests that the uncertainty premium of time-to-build might not increase with its variance. In fact, the following result can be obtained:



(a) Uncertainty equivalent with greater mean and variance (b) Uncertainty equivalent with smaller mean and variance

Fig. 2. Examples of multiple uncertainty equivalents for a fixed time-to-build.

Proposition 3. *The certainty equivalent of time-to-build does not strictly decrease with variance of time-to-build. In other words, the uncertainty premium of time-to-build does not strictly increase with its variance.*

Proof. See Appendix A.8.

At first glance, the result of Proposition 3 might seem to contradict Lemma 1, but that is not the case. A mean-preserving spread of a given time-to-build always accelerates investment, as addressed in Lemma 1, and it has a greater variance than the given time-to-build.[13] However, this does not imply that time-to-build with a greater variance always accelerates investment. This is because a random variable with the same mean but greater variance is not necessarily a mean-preserving spread of the counterpart.[14]

4.4. Uncertainty equivalent of fixed time-to-build

Proposition 1 presents the firm's optimal investment decision, assuming precise knowledge of the uncertainty in time-to-build (i.e., probability distribution), in the form of the investment strategy that would have been implemented without such uncertainty. However, in practice, the opposite scenario is more likely; the firm establishes its investment strategy without considering uncertainty in time-to-build and remains unaware of the level of uncertainty that such an investment strategy implicitly assumes.

From this perspective, we can derive the following result:

Proposition 4 (*Uncertainty Equivalent*)**.** *For any fixed time-to-build $\bar{\tau}$ (>0), there always exists a nonnegative random variable τ_u with $\mathbb{E}[\tau_u] > \bar{\tau}$ such that $\delta(\bar{\tau}) = \delta(\tau_u)$, or equivalently, $X_{\bar{\tau}} = X_{\tau_u}$ and $V_{\bar{\tau}}(X) = V_{\tau_u}(X)$. Specifically, the uncertainty equivalent is derived from*

$$K_{\tau_u}(-(r-\mu)) = -(r-\mu)\bar{\tau}. \tag{16}$$

Proof. See Appendix A.9.

Essentially, Proposition 4 reexamines the argument of Proposition 1 from a reversed standpoint, and it can be read in the same context: the firm value with *longer* and *uncertain* time-to-build (i.e., τ_u) is same as that with *shorter* and *fixed* time-to-build (i.e., $\bar{\tau}$). While seemingly paradoxical, certainty of time-to-build *disencourages* investment. The extent to which it does so — relative to the case of uncertain time-to-build τ_u — is measured by $\bar{\tau} - \mathbb{E}[\tau_u] (< 0)$; its magnitude, $\mathbb{E}[\tau_u] - \bar{\tau}\ (>0)$, is referred to as *certainty discount of time-to-build*. This result provides firms with a structured framework to evaluate the implicit risks embedded in investment strategies developed without accounting for uncertainty in time-to-build, thereby clarifying the equivalent level of risk such policies entail.

Note that Proposition 4 proves the existence of the uncertainty equivalent, but unlike Proposition 1, its uniqueness is not guaranteed. Namely, there can be multiple uncertainty equivalents that satisfy (16) for a given fixed time-to-build, and they need not follow the same distribution. Fig. 2 graphically illustrates the multiplicity of the uncertainty equivalent for a fixed time-to-build. We can see that the uncertainty equivalent in Fig. 2(a) is more dispersed and has a longer expected duration than the one in Fig. 2(b), even though both yield the same pre-investment firm value. In fact, this relationship can be formalized as follows:

Corollary 5. *The expected value of the uncertainty equivalent τ_u for a fixed time-to-build $\bar{\tau}$ increases with the dispersion of τ_u. In other words, the certainty discount of the fixed time-to-build increases with the dispersion of τ_u.*

Proof. See Appendix A.10.

Meanwhile, following similar arguments from Proposition 4, we can also obtain the following result:

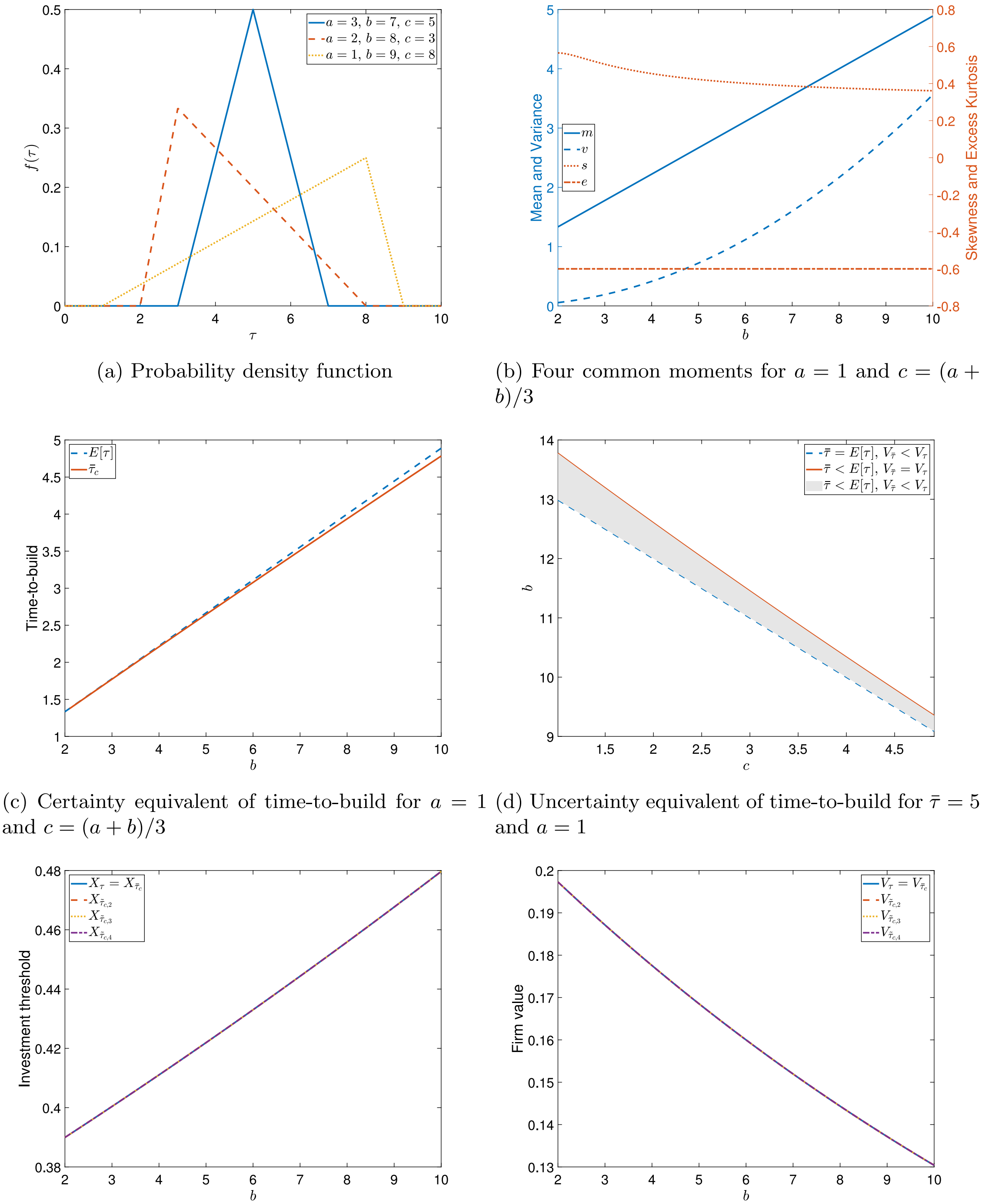
Corollary 6. *For any fixed time-to-build $\bar{\tau}$ (>0), there always exists a nonnegative random variable τ_w with $\mathbb{E}[\tau_w] > \bar{\tau}$ such that $\delta(\bar{\tau}) < \delta(\tau_w)$, or equivalently, $X_{\bar{\tau}} > X_{\tau_w}$ and $V_{\bar{\tau}}(X) < V_{\tau_w}(X)$.*

Proof. See Appendix A.11.

This implies that for any fixed time-to-build, there always exists an *uncertain* time-to-build with a *longer* expected duration that nonetheless yields a *higher* firm value. This can be easily extended to comparisons between uncertain time-to-builds with different degrees of dispersion. Specifically, a more dispersed time-to-build with a longer expected duration can yield a higher firm value than a less dispersed one with a shorter expected duration. This argument offers insight into how investment projects might be evaluated in the real world. Fernandes and Rigato (2025) provides ample evidence on time-to-build using Indian project-level data, showing that the distribution of time-to-build varies across sectors. For instance, time-to-build in logistics is more dispersed

[13] As in Lemma 1, suppose $\tau_{n+1} = \tau_n + \epsilon_{n+1}$ where $\mathbb{E}[\epsilon_{n+1}|\tau_n] = 0$. By the law of iterated expectations, $\mathbb{E}[\epsilon_{n+1}] = \mathbb{E}[\mathbb{E}[\epsilon_{n+1}|\tau_n]] = 0$ holds, and it is straightforward to show that $\mathrm{Var}(\tau_{n+1}) = \mathrm{Var}(\tau_n) + \mathrm{Var}(\epsilon_{n+1}) > \mathrm{Var}(\tau_n)$ since $\mathrm{Cov}(\tau_n, \epsilon_{n+1}) = \mathbb{E}[\tau_n\epsilon_{n+1}] - \mathbb{E}[\tau_n]\mathbb{E}[\epsilon_{n+1}] = \mathbb{E}[\tau_n\mathbb{E}[\epsilon_{n+1}|\tau_n]] = 0$.

[14] Namely, τ_{n+1} is not necessarily a mean-preserving spread of τ_n even if $\mathbb{E}[\tau_{n+1}] = \mathbb{E}[\tau_n]$ and $\mathrm{Var}(\tau_{n+1}) > \mathrm{Var}(\tau_n)$ hold. Refer to Appendix A.8 for a specific example.



(a) Probability density function

(b) Four common moments for $a = 1$ and $c = (a + b)/3$

(c) Certainty equivalent of time-to-build for $a = 1$ and $c = (a + b)/3$

(d) Uncertainty equivalent of time-to-build for $\bar{\tau} = 5$ and $a = 1$

(e) Investment threshold and its approximation for $a = 1$ and $c = (a + b)/3$

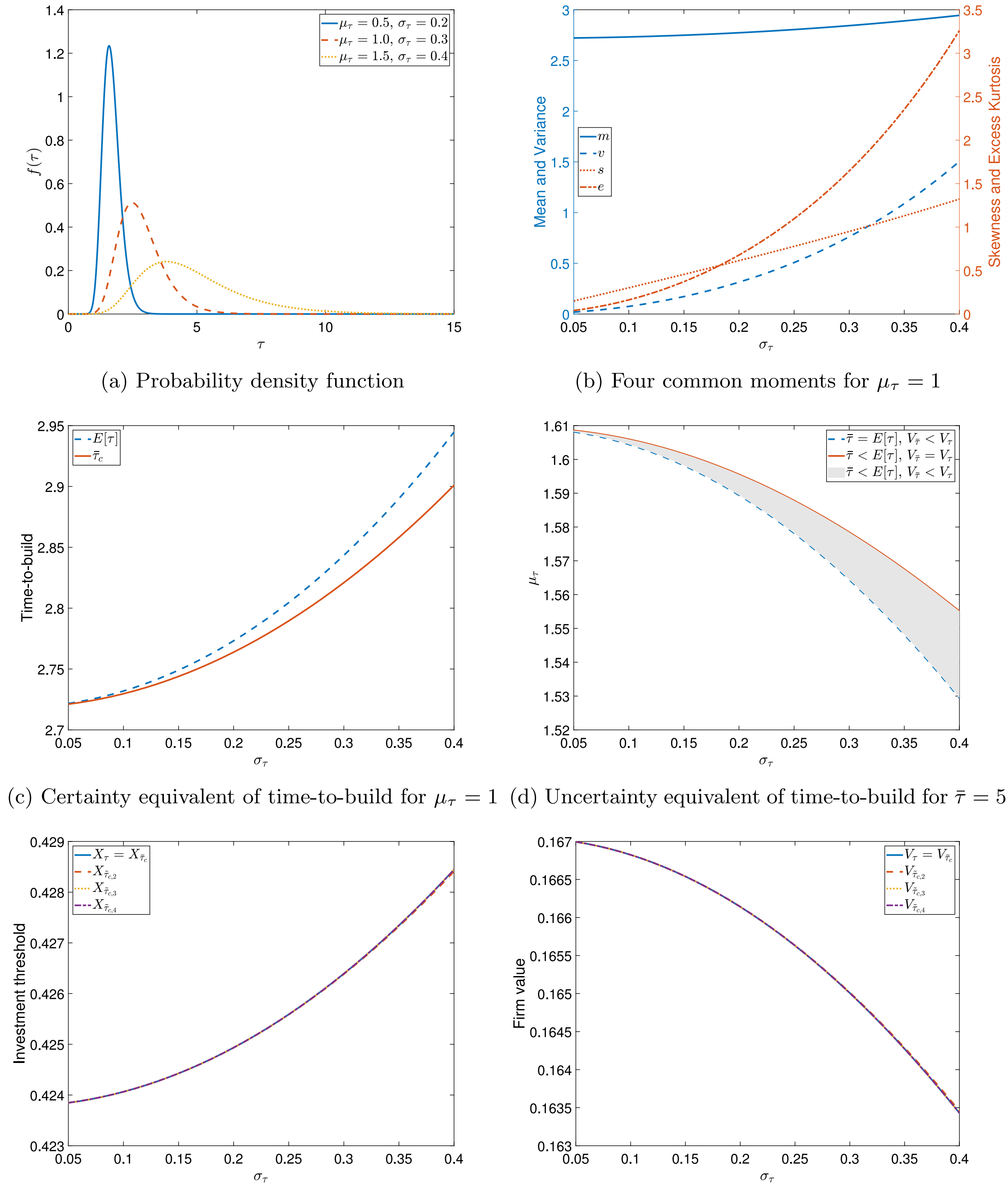
(f) Firm value and its approximation for $a = 1$ and $c = (a + b)/3$

Fig. 3. When time-to-build follows a triangular distribution with minimum a, maximum b, and mode c.

and has a longer expected duration than in construction. However, investment in the former may have been more incentivized by time-to-build than in the latter due to its wider dispersion. Glancy et al. (2024) presents empirical evidence on time-to-build in U.S. commercial construction, showing that the construction lags for hotels are, on average, longer and more dispersed than those for office buildings. Following the same reasoning, investment in the former may have been strongly driven by these lags than in the latter, although this hypothesis requires further empirical testing.

5. Probability distributions of time-to-build

This section shows the practical application of the results from Section 4 using representative probability distributions. Many empirical evidence shows that the distribution of time-to-build is *unimodal* with *nonzero mode* and *positively skewed*; the evidence includes U.S. data from manufacturing industry (Jorgenson & Stephenson, 1967), nonresidential structures (Montgomery, 1995), residential investment (Oh et al., 2024; Oh & Yoon, 2020), commercial construction projects (Glancy



(a) Probability density function

(b) Four common moments for $\mu_\tau = 1$

(c) Certainty equivalent of time-to-build for $\mu_\tau = 1$

(d) Uncertainty equivalent of time-to-build for $\bar{\tau} = 5$

(e) Investment threshold and its approximation for $\mu_\tau = 1$

(f) Firm value and its approximation for $\mu_\tau = 1$

Fig. 4. When time-to-build follows a log-normal distribution with parameters μ_τ and σ_τ.

et al., 2024), and project-level data from India (Fernandes & Rigato, 2025).[15] For this reason, we discuss the distributions that can exhibit the aforementioned properties: triangular distribution, log-normal distribution, gamma distribution, and scaled beta distribution. The cases in which time-to-build follows uniform distribution, which is neither unimodal nor skewed, and exponential distribution, of which mode is 0, are discussed in the Online Appendix. Throughout this section, we adopt the parameters in Table 1 for describing the investment project.

[15] It is obvious that the investment lags are positively associated with the size of investment projects. However, Oh and Yoon (2020) measured economic time-to-build of residential investment that cannot be captured by the characteristics of the projects including their square footage, location, and building methods, and it is found unimodal with nonzero mode and positively skewed. Specifically, the distribution of economic time-to-build in their manuscript is symmetric because it is based on the *log* of time-to-build; the one based on the *level* of time-to-build, which is positively skewed, can be found in their Online Appendix.

Table 1
Benchmark parameters for numerical calculation.

Notation	Value	Description
r	0.08	Risk-free rate
μ	0.02	Expected growth rate of demand shock
σ	0.2	Volatility of demand shock
I	3	Lump-sum investment costs
X	0.1	Initial demand shock

They are in a moderate range that can be easily found in the real options literature.[16]

5.1. Triangular distribution

Suppose that the firm knows the minimum, maximum, and mode of the time-to-build of its project, denoted by a, b, and c, respectively, and that its likelihood is unimodal and piece-wise linear. That is, assume that τ follows a triangular distribution with parameters (a, b, c) with $0 \le a \le c \le b$ and $a < b$. This corresponds to the case in which an investment project requires a certain amount of time to be finished even in its best-case scenario (i.e., nonzero minimum) but the worst-case scenario is bounded (i.e., finite maximum) with the most likely scenario between them. Note that the minimum a, and thus, the mode c, can be nonzero, and the mode can be chosen such that it is positively skewed. Its probability density function is

$$f(\tau) = \begin{cases} \frac{2(\tau-a)}{(b-a)(c-a)} & \text{if } a \le \tau < c, \\ \frac{2}{b-a} & \text{if } \tau = c, \\ \frac{2(b-\tau)}{(b-a)(b-c)} & \text{if } c < \tau \le b, \\ 0 & \text{otherwise,} \end{cases} \tag{17}$$

which is depicted in Fig. 3(a), and its moment-generating function is

$$M_\tau(t) = \frac{2\{(b-c)e^{at} - (b-a)e^{ct} + (c-a)e^{bt}\}}{(b-a)(c-a)(b-c)t^2}, \tag{18}$$

which amounts to the cumulant-generating function $K_\tau(t) = \ln M_\tau(t)$.

Recall that Proposition 3 showed an increase of variance does not always accelerate investment. However, it always has the positive impacts on investment when time-to-build follows a symmetric triangular distribution:

Proposition 5. *When time-to-build follows a symmetric triangular distribution, the certainty equivalent strictly decreases with its variance. In other words, the uncertainty premium strictly increases with its variance.*

Proof. See Appendix A.12.

The certainty equivalent of time-to-build following a triangular distribution can be derived by (8) with (18). Fig. 3(c) presents the certainty equivalent assuming the minimum $a = 1$ and the mode $c = (a + b)/3$, which exhibits a positive skewness. A comparison of Figs. 3(b) and 3(c) shows that the uncertainty premium increases with the variance of time-to-build (Proposition 5).

The uncertainty equivalent of fixed time-to-build, assumed to follow a triangular distribution, can be found by determining (a, b, c) that satisfies (16) with (18). If the firm assumes that time-to-build follows this distribution and is certain of its minimum and mode (i.e., a and c), it can instantly deduce the worst-case scenario of the uncertainty equivalent (i.e., b) corresponding to the fixed time-to-build. As noted in Section 4.4, there can exist many uncertainty equivalents for a fixed time-to-build, which is described by the solid line in Fig. 3(d) for $\bar{\tau} = 5$. We can see that the worst-case scenario (i.e, b) decreases with the most likely scenario (i.e., c) for a given best-case scenario (i.e., $a = 1$) and fixed time-to-build (i.e., $\bar{\tau} = 5$). The shaded area represents the uncertain time-to-build whose expected duration is longer than the fixed counterpart yet induces higher firm value (i.e., $\bar{\tau} < \mathbb{E}[\tau]$ and $V_{\bar{\tau}} < V_\tau$), supporting Corollary 6.

Figs. 3(e) and 3(f) present the optimal investment threshold and firm value along with their approximation based on the moments described in Fig. 3(b), and they indicate that the approximation error is negligible. That is, the firm can essentially establish the optimal investment strategy taking account of uncertain time-to-build based solely on its mean and variance. Note that the approximation described in Figs. 3(e) and 3(f) is solely based on the moments of the corresponding distribution without assuming any specific distribution; this applies to other figures regarding the approximation hereafter.

5.2. Log-normal distribution

Suppose the firm knows that time-to-build τ follows a log-normal distribution with parameters μ_τ and $\sigma_\tau (> 0)$. That is, $\ln \tau$ follows a normal distribution with mean μ_τ and variance σ_τ^2. It is a positively skewed, unimodal distribution with nonzero mode on $(0, \infty)$. This corresponds to the case in which an investment project can generate no revenue in its worst-case scenario (i.e., $\tau \to \infty$), such as the failure of an R&D project, and there is a slight chance that the project is finished instantly in its best-case scenario (i.e., $\tau \to 0$), although the most probable scenario is in between them. Its probability density function is

$$f(\tau) = \begin{cases} \frac{1}{\tau\sigma_\tau\sqrt{2\pi}} \exp\left(-\frac{(\ln\tau-\mu_\tau)^2}{2\sigma_\tau^2}\right) & \text{if } \tau > 0, \\ 0, & \text{otherwise,} \end{cases} \tag{19}$$

which is described in Fig. 4(a). Its moment-generating function (i.e., $M_\tau(t) = \mathbb{E}[e^{t\tau}]$) does not exist for $t \ge 0$ since the defining integral diverges. Although $\mathbb{E}[e^{t\tau}]$ converges for $t < 0$ due to $\tau \in (0, \infty)$, its closed-form expression has not been found yet.[17] Asmussen et al. (2016) suggested the following approximation of the moment-generating function:

$$M_\tau(t) \approx \frac{\exp\left(-\frac{\{W(-t\sigma_\tau^2 e^{\mu_\tau})\}^2 + 2W(-t\sigma_\tau^2 e^{\mu_\tau})}{2\sigma_\tau^2}\right)}{\sqrt{1 + W(-t\sigma_\tau^2 e^{\mu_\tau})}}, \tag{20}$$

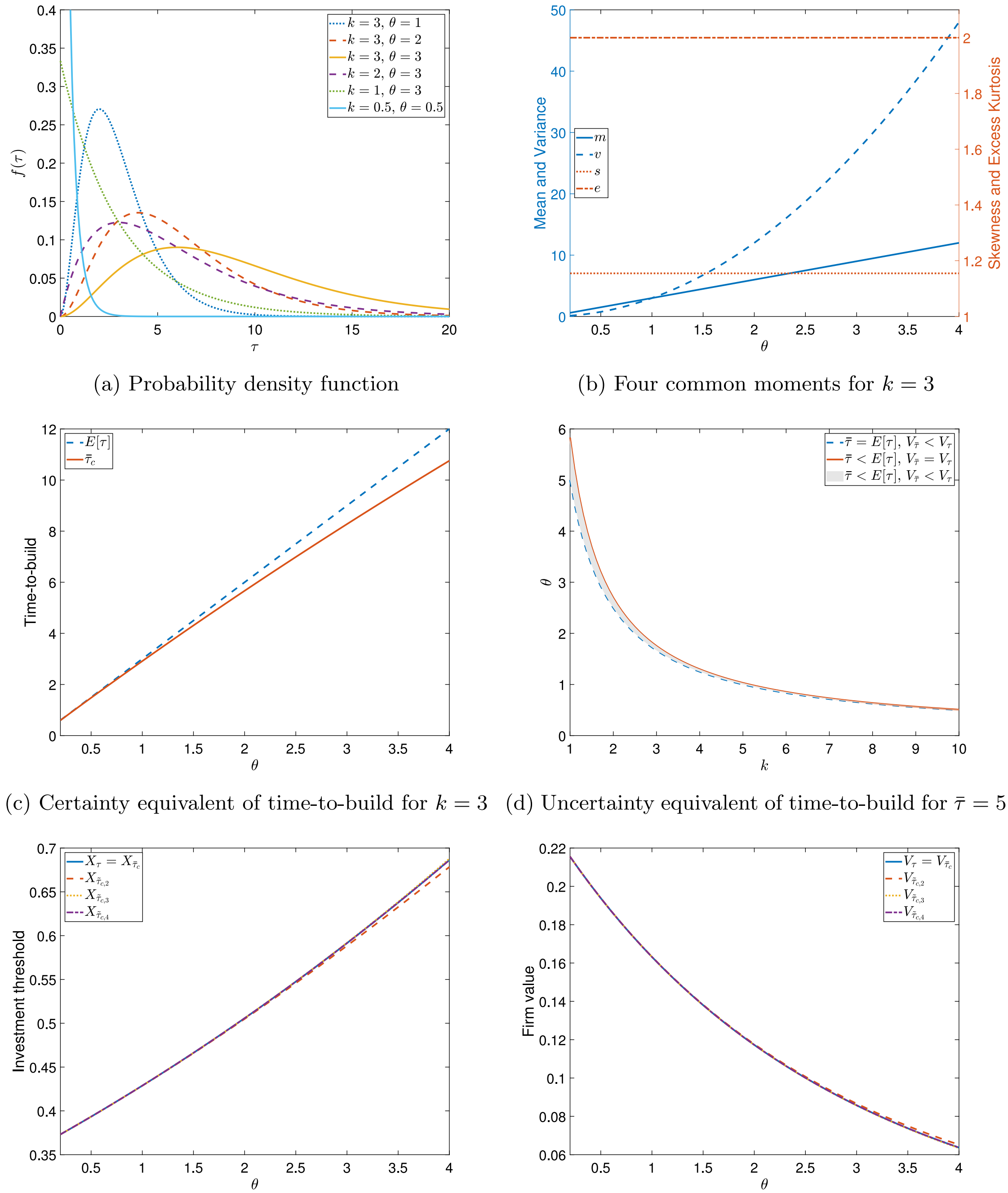
where $W(x)$ is the Lambert W function defined as the solution of $W(x)e^{W(x)} = x$, and we adopt this approximation to derive the certainty equivalent and uncertainty premium of time-to-build following a log-normal distribution.

The certainty equivalent of time-to-build following a log-normal distribution can be found by (8) with (20), which is described in Fig. 4(c) for $\mu_\tau = 1$. Its comparison with Fig. 4(b) numerically shows that the uncertainty premium of time-to-build increases with variance.

The uncertainty equivalent of time-to-build, assumed to follow a log-normal distribution, can be found by obtaining (μ_τ, σ_τ) that satisfies (16) with (20). If the firm supposes that the uncertainty equivalent follows this distribution and is certain of the mean of time-to-build, it can specify the candidates of the uncertainty equivalent. Fig. 4(d)

[16] For the demand shocks, similar parameters can be found in seminal works in real options theory (e.g., Dixit and Pindyck (1994), Huisman and Kort (2015) and Leland (1994)). For the risk-free rate, Leland (1994, 1998) and Leland and Toft (1996) chose $r = 0.06$ and $r = 0.075$, respectively, while Dixit and Pindyck (1994, Chapter 5) and Huisman and Kort (2015) adopted $r = 0.1$. Jeon (2024a, 2024b), which investigated the effects of uncertainty in time-to-build, were based on the mid-range value $r = 0.08$. For consistency and comparability among papers within the same research theme, we adopt the same value.

[17] All moments of the log-normal distribution exist (i.e., $\mathbb{E}[\tau^n] = e^{n\mu_\tau + n^2\sigma_\tau^2/2}$), but the log-normal distribution is not determined by its moments (e.g., Heyde (1963)). This implies that it cannot have a defined moment-generating function in a neighborhood of zero.



(a) Probability density function (b) Four common moments for $k=3$

(c) Certainty equivalent of time-to-build for $k=3$ (d) Uncertainty equivalent of time-to-build for $\bar{\tau}=5$

(e) Investment threshold and its approximation for $k=3$ (f) Firm value and its approximation for $k=3$

Fig. 5. When time-to-build follows a gamma distribution with shape parameter k and scale parameter θ.

presents the uncertainty equivalent of fixed time-to-build $\bar{\tau}=5$, and we can see that μ_τ of the uncertainty equivalent decreases with σ_τ. The shaded area in Fig. 4(d) represents the uncertain time-to-build whose expected duration is longer than the fixed counterpart yet yields higher firm value (i.e., $\bar{\tau}<\mathbb{E}[\tau]$ and $V_{\bar{\tau}}<V_\tau$), consistent with Corollary 6.

Figs. 4(e) and 4(f) present the optimal investment threshold and firm value along with their approximation based on the moments illustrated in Fig. 4(b), and we can see that the approximation errors are negligible.

5.3. Gamma distribution

Suppose the firm knows that the time-to-build τ follows a gamma distribution with parameters (k,θ) where $k>0$ is a shape parameter and $\theta>0$ is a scale parameter. This versatile two-parameter distribution encompasses many distributions as its special cases.[18] It is a positively skewed, unimodal distribution with possibly nonzero mode.[19] Its support is $(0,\infty)$, which, as a log-normal distribution in Section 5.2, allows

[18] With $k=1$, it becomes the exponential distribution discussed in the Online Appendix. When k is an integer, it is known as the Erlang distribution. When

us to model the potential for R&D failure (i.e., $\tau \to \infty$). Its probability density function is

$$f(\tau) = \begin{cases} \frac{\tau^{k-1}e^{-\tau/\theta}}{\Gamma(k)\theta^k} & \text{if } \tau \geq 0, \\ 0 & \text{otherwise,} \end{cases} \tag{21}$$

where $\Gamma(\cdot)$ is gamma function, which is described in Fig. 5(a), and its moment-generating function is

$$M_\tau(t) = (1-\theta t)^{-k} \qquad \text{for } t < \frac{1}{\theta}. \tag{22}$$

The certainty equivalent of time-to-build following a gamma distribution can be found from (8) with (22), which is described in Fig. 5(c). Its comparison with Fig. 5(b) numerically shows that the uncertainty premium of time-to-build increases with its variance.

The uncertainty equivalent of fixed time-to-build, assumed to follow this distribution, can be found by determining (k, θ) that satisfies (16) with (22). If the firm supposes that the uncertainty equivalent follows this distribution without knowing the parameters but is certain of the mean of time-to-build (i.e., $k\theta$), it can specify the candidates of the uncertainty equivalent. Fig. 5(d) presents the combination of k and θ that yields the uncertainty equivalent of fixed time-to-build $\bar{\tau} = 5$, and the shaded area corresponds to the case discussed in Corollary 6.

Figs. 5(e) and 5(f) illustrate the optimal investment threshold and firm value taking uncertain time-to-build into account along with their approximation based on the moments in Fig. 5(b). They show that the approximation based solely on the mean and variance of time-to-build yields a nonnegligible error, but it becomes insignificant when the skewness is considered.

5.4. Scaled beta distribution

Despite its flexibility, the gamma distribution might not be suitable for describing the time-to-build of some investment projects, primarily because of its semi-infinite support. That is, an investment project's time-to-build might be nonzero in its best-case scenario and yet finite even in its worst-case scenario. This can be described by a triangular distribution in Section 5.1, but its piecewise linear density is not be suitable for describing the gradual change in the likelihood. For this reason, we consider a scaled beta distribution.[20]

Suppose v follows a beta distribution with parameters (α, β) where $\alpha, \beta > 0$ are shape parameters. Since $v \in [0, 1]$, we can scale it to $\tau := (c-a)v + a$ such that $\tau \in [a, c]$ where $0 \leq a < c$. The probability density function of v is

$$f(v) = \begin{cases} \frac{v^{\alpha-1}(1-v)^{\beta-1}}{B(\alpha,\beta)} & \text{if } 0 \leq \tau \leq 1, \\ 0 & \text{otherwise,} \end{cases} \tag{23}$$

where $B(\cdot,\cdot)$ is a beta function, and that of τ is $f(\tau) = f(v)/(c-a)$ on its support $[a, c]$, which is described in Fig. 6(a). The moment-generating function of τ following the scaled beta distribution with parameters (α, β, a, c) is

$$\begin{aligned} M_\tau(t) &= \int_a^c e^{t\tau} f(\tau)\mathrm{d}\tau \\ &= \frac{1}{B(\alpha,\beta)} \int_0^1 e^{t\{(c-a)v+a\}} v^{\alpha-1}(1-v)^{\beta-1}\mathrm{d}v \\ &= \frac{1}{B(\alpha,\beta)} \sum_{n=0}^{\infty} \frac{(ta)^n}{n!} \sum_{k=0}^{\infty} \frac{\{t(c-a)\}^k}{k!} \int_0^1 v^{\alpha+k-1}(1-v)^{\beta-1}\mathrm{d}v \\ &= \sum_{n=0}^{\infty} \frac{(ta)^n}{n!} \sum_{k=0}^{\infty} \frac{\{t(c-a)\}^k}{k!} \frac{B(\alpha+k,\beta)}{B(\alpha,\beta)} \\ &= \sum_{n=0}^{\infty} \frac{(ta)^n}{n!} \left[1 + \sum_{k=1}^{\infty} \left(\prod_{s=0}^{k-1} \frac{\alpha+s}{\alpha+\beta+s}\right) \frac{\{t(c-a)\}^k}{k!}\right]. \end{aligned} \tag{24}$$

The certainty equivalent of time-to-build following the scaled beta distribution can be found from (8) with (24). Fig. 6(d) presents the certainty equivalent of time-to-build following this distribution on $[1, 10]$. Although the skewness and excess kurtosis greatly vary depending on α and β (Fig. 6(c)), the comparison of Figs. 6(b) and 6(e) reveals that the uncertainty premium of time-to-build is mainly driven by its variance. Fig. 6(e) also shows that the uncertainty premium of time-to-build is highest when both α and β are below 1 so that it becomes bimodal with peaks at both ends of the support $[a, c]$.

The uncertainty equivalent of fixed time-to-build, assumed to follow this distribution, can be found by determining (α, β, a, c) that satisfies (16) with (24). Fig. 6(f) presents the uncertainty equivalent of time-to-build following this distribution on $[1, 10]$ for fixed time-to-build $\bar{\tau} = 5$. The shaded area in Fig. 6(f) corresponds to the case discussed in Corollary 6.

Figs. 6(g) and 6(h) present the optimal investment threshold and firm value with uncertain time-to-build following this distribution along with their approximation, showing that the approximation error is insignificant.

PERT distribution, developed for program evaluation and review technique, is a special case of the scaled beta distribution. Specifically, if τ follows the PERT distribution with parameters (a, b, c) with $0 \leq a < b < c < \infty$ where a and c are the minimum and maximum of time-to-build, respectively, and b is its mode, its probability density function coincides with that of the scaled beta distribution with $\alpha = 1 + 4(b-a)/(c-a)$ and $\beta = 1 + 4(c-b)/(c-a)$, which is described in Fig. 7(a). Unlike the triangular distribution discussed in Section 5.1, it is a *smooth* unimodal distribution, and it can be suitable for describing time-to-build having gradual changes in likelihoods with a single mode between a potentially nonzero minimum (i.e., a) and a finite worst-case scenario (i.e., c).

The certainty equivalent of time-to-build following the PERT distribution can be found following the same manner as before, which is illustrated in Fig. 7(c). As can be seen from Fig. 7(b), the variance does not vary significantly in accordance with the mode b within the fixed support $[a, c]$. Consequently, the uncertainty premium (i.e., $\mathbb{E}[\tau] - \bar{\tau}_c$) in Fig. 7(c) shows insignificant change.

The uncertainty equivalent of fixed time-to-build, assumed to follow the PERT distribution, can be found by determining (a, b, c) that satisfies (16) and (24) with $\alpha = 1+4(b-a)/(c-a)$ and $\beta = 1+4(c-b)/(c-a)$. If the firm supposes that the uncertainty equivalent follows this distribution and is certain of the best-case scenario (i.e., a) and its most likely one (i.e., b), it can specify the worst-case scenario (i.e., c) of the uncertainty equivalent, which is presented in Fig. 7(d). Figs. 7(e) and 7(f) describe the optimal investment strategy and firm value taking uncertain time-to-build into account, along with their approximation, and they reveal that the approximation error is negligible.

5.5. Mean and variance of time-to-build

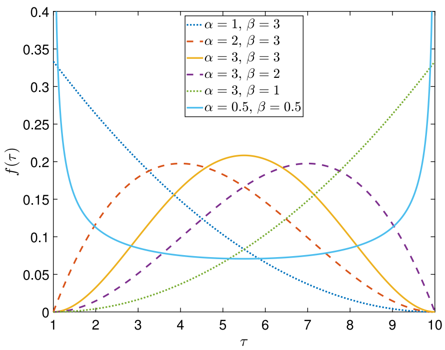
Now we focus on the two most important moments of time-to-build: its mean and variance. Proposition 3 showed that without assuming the distribution of time-to-build, an increase in the variance of time-to-build does not always accelerate investment. However, as seen from Sections 5.1–5.4, such an increase leads to earlier investment for the well-known probability distributions.

Fig. 8 illustrates the tight upper and lower bounds of the certainty equivalent of time-to-build for given mean and variance, which are demonstrated in Proposition 2, along with the corresponding certainty equivalent for representative probability distributions and its approximation based on the mean and variance (i.e., $\tilde{\tau}_{c,2}$ in (12)). Specifically,

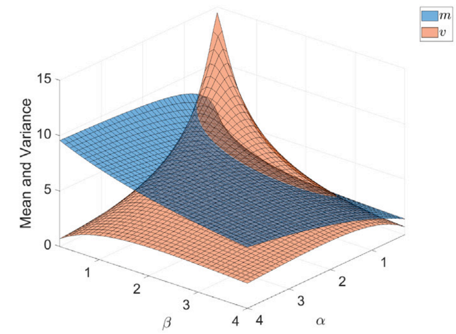
$k = \nu/2$ and $\theta = 2$, it corresponds to the chi-squared distribution with a parameter ν.

[19] Specifically, the mode is 0 when $k < 1$; otherwise, it is nonzero.

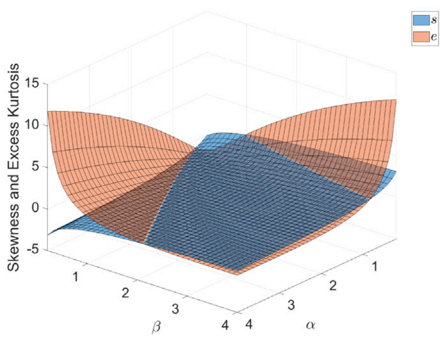
[20] Jung (2013) also adopted a beta distribution to infer the length of aggregate time-to-build in a dynamic stochastic general equilibrium model, mainly due to its versability.



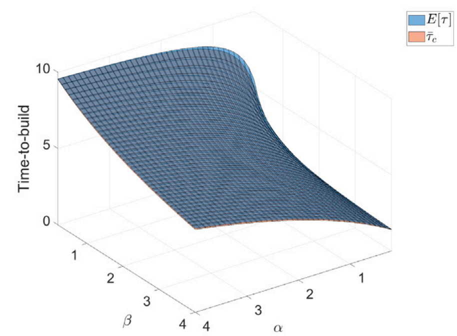
(a) Probability density function for $a = 1$ and $c = 10$



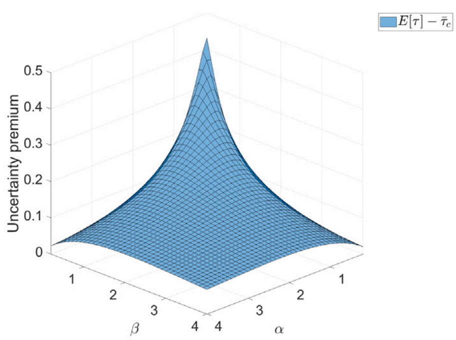
(b) Mean and variance for $a = 1$ and $c = 10$



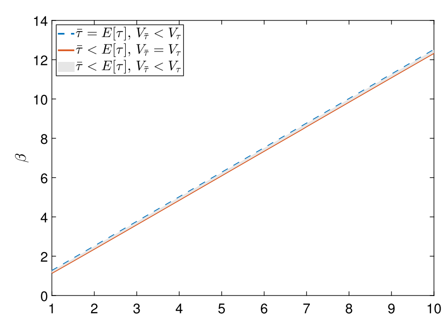
(c) Skewness and excess kurtosis for $a = 1$ and $c = 10$



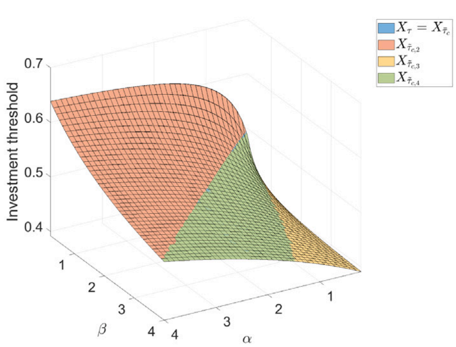
(d) Certainty equivalent of time-to-build for $a = 1$ and $c = 10$



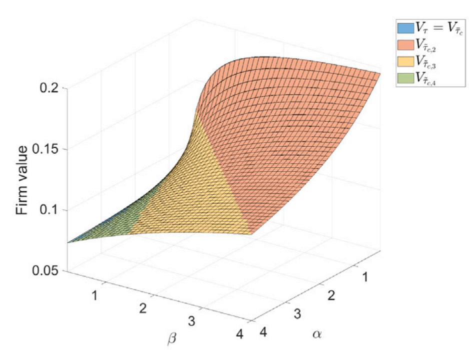
(e) Uncertainty premium of time-to-build for $a = 1$ and $c = 10$



(f) Uncertainty equivalent of time-to-build for $\bar{\tau} = 5$, $a = 1$, and $c = 10$

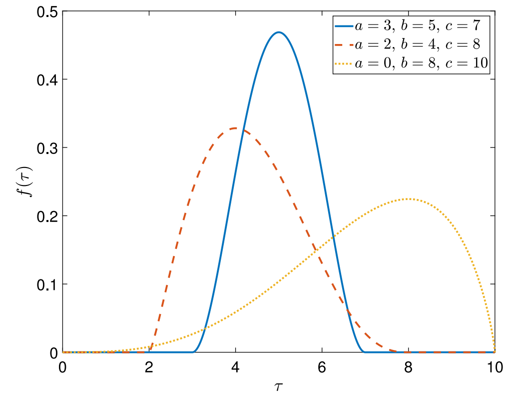


(g) Investment threshold and its approximation for $a = 1$ and $c = 10$

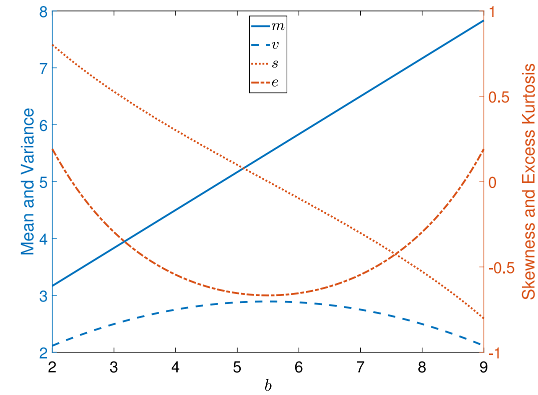


(h) Firm value and its approximation for $a = 1$ and $c = 10$

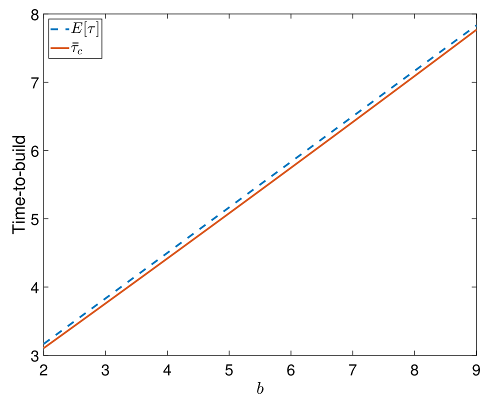
Fig. 6. When time-to-build follows a scaled beta distribution with minimum a, maximum c, and shape parameters α and β.



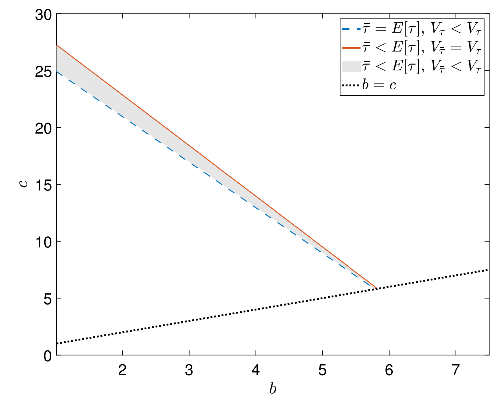
(a) Probability density function



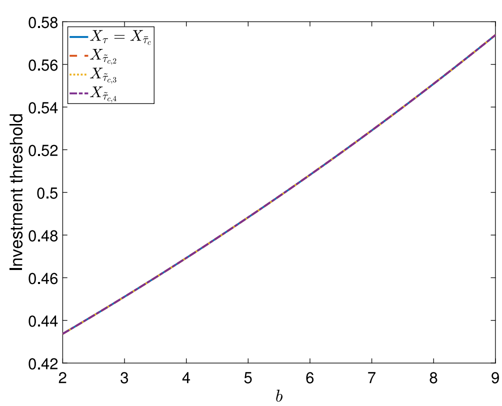
(b) Four common moments for $a = 1$ and $c = 10$



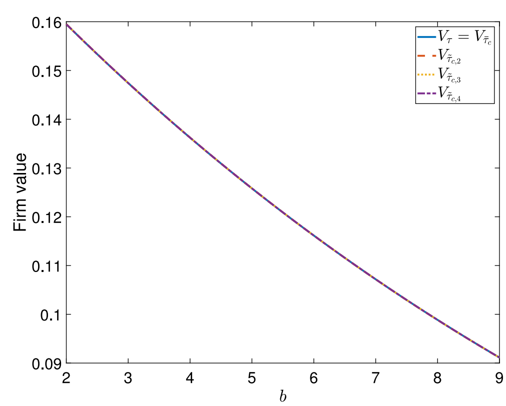
(c) Certainty equivalent of time-to-build for $a = 1$ and $c = 10$



(d) Uncertainty equivalent of time-to-build for $\bar{\tau} = 5$ and $a = 1$



(e) Investment threshold and its approximation for $a = 1$ and $c = 10$



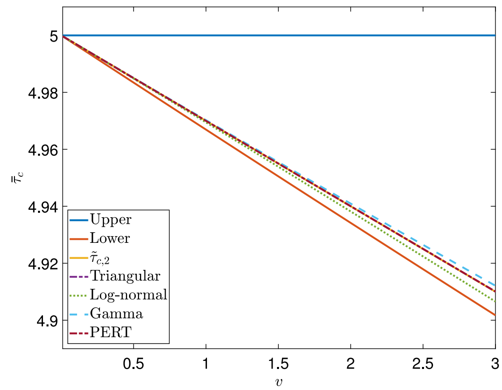
(f) Firm value and its approximation for $a = 1$ and $c = 10$

Fig. 7. When time-to-build follows a PERT distribution with minimum a, mode b, and maximum c.

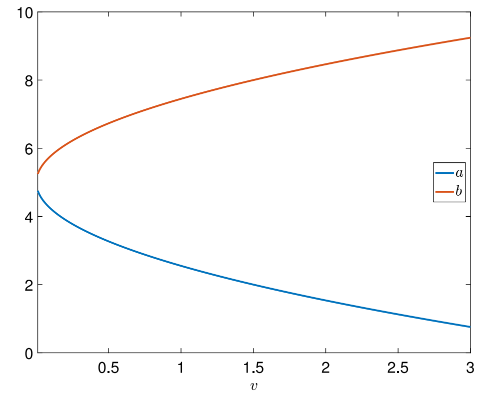
we fix the mean of time-to-build m and vary its variance v, demonstrating how the bounds change along with the certainty equivalent. Due to the degree of freedom, we choose distributions that can be characterized by two parameters: symmetric triangular distribution, log-normal distribution, gamma distribution, and symmetric PERT distribution.

Fig. 8(a) shows that the lower bound decreases with the variance v while the upper bound remains constant. It also clarifies that the certainty equivalent of time-to-build decreases with variance for these distributions, although this might not hold in an extreme case as the counterexample from the proof of Proposition 3. Note that the accuracy of the approximation of the certainty equivalent based on the mean and variance is significantly high. Its approximation error is essentially zero for the symmetric triangular and PERT distributions, whereas it is nonnegligible for the log-normal and gamma distributions due to their semi-infinite supports. Figs. 8(b)–8(e) describe the corresponding parameters for each distribution that satisfy the mean m and variance v.

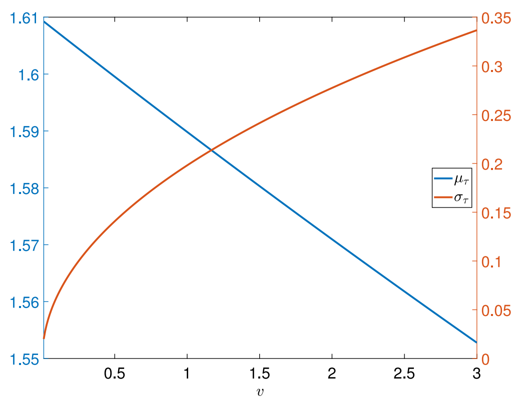
Fig. 9 presents the examples of the uncertainty equivalent discussed in Proposition 4. Specifically, Fig. 9(a) depicts the level of variance v, combined with the mean m, that induces the same optimal investment threshold and firm value as the ones with a fixed time-to-build $\bar{\tau}$. This figure, along with Figs. 9(b) and 9(c), clarifies that even if the expected duration of time-to-build lengthens, the optimal investment timing and firm value can remain the same if its uncertainty increases



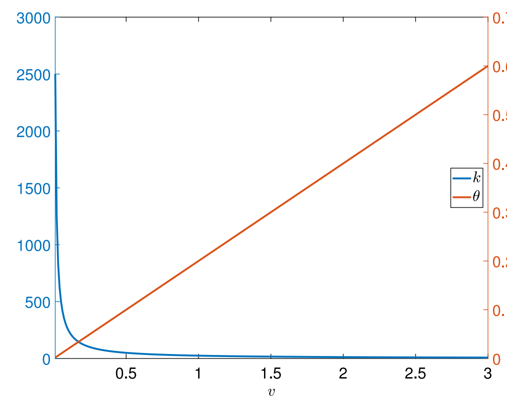
(a) Upper and lower bounds of certainty equivalent



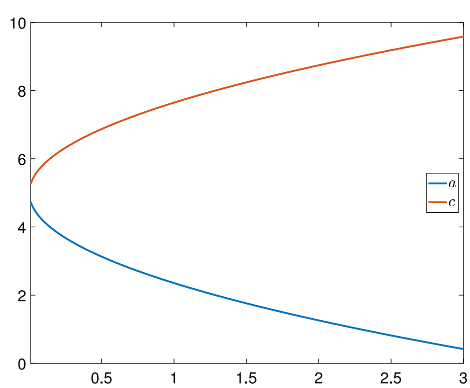
(b) Corresponding parameters for symmetric triangular distribution (a and b)



(c) Corresponding parameters for log-normal distribution (μ_τ and σ_τ)



(d) Corresponding parameters for gamma distribution (k and θ)



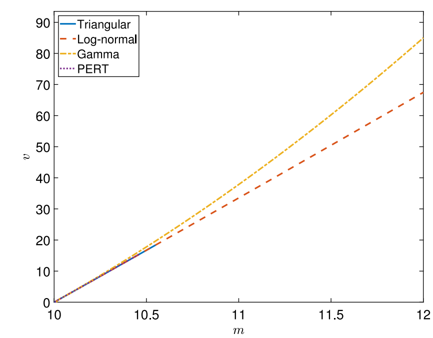
(e) Corresponding parameters for symmetric PERT distribution (a and c)

Fig. 8. Tight upper and lower bounds of certainty equivalent of time-to-build and the certainty equivalents for representative probability distributions with mean $m = 5$ and different levels of variance v.

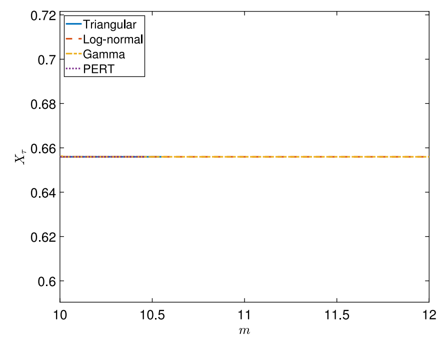
significantly. It also demonstrates that the variance of the uncertainty equivalent depends on the distribution of time-to-build. The variance of the gamma distribution is found to be significantly higher than that of other distributions, primarily due to its semi-infinite support.

Figs. 9(d)–9(g) present the corresponding parameters for each distribution. Note that some distributions cannot yield an uncertainty equivalent for mean m substantially greater than $\bar{\tau}$ due to parameter restrictions, such as a nonnegative minimum. Log-normal and gamma distributions are relatively flexible for yielding the uncertainty equivalent owing to their versatility. Note that the uncertainty equivalents need not follow the distributions illustrated in Fig. 9; they are only a fraction of many alternatives that can result in the same investment decision and firm value for a given $\bar{\tau}$.

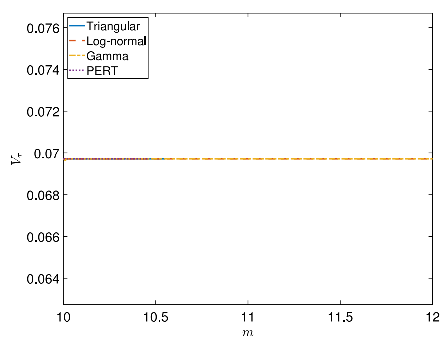
As discussed in Section 4.2, a higher demand uncertainty (i.e., σ) delays investment but increases firm value. This is because the firm's option to wait becomes more valuable when market demands are uncertain. By contrast, Lemma 1 demonstrates that a higher uncertainty



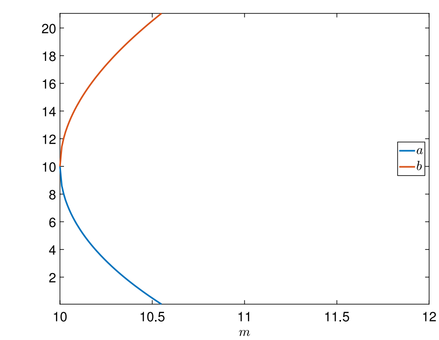
(a) Variance v of uncertainty equivalent with mean m for given $\bar{\tau}$



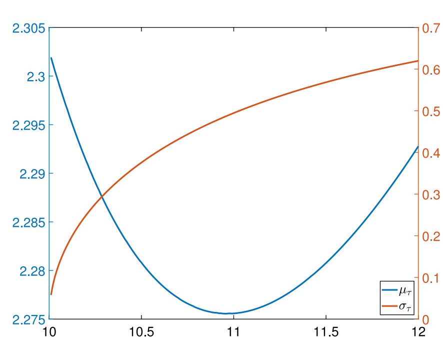
(b) Investment threshold with uncertainty equivalent following each distribution



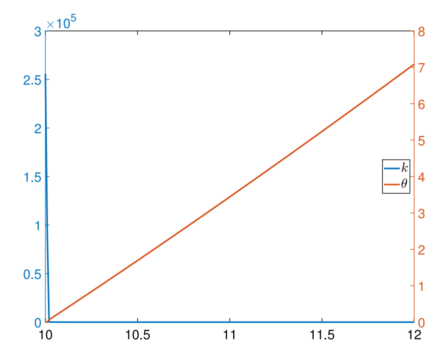
(c) Firm value with uncertainty equivalent following each distribution



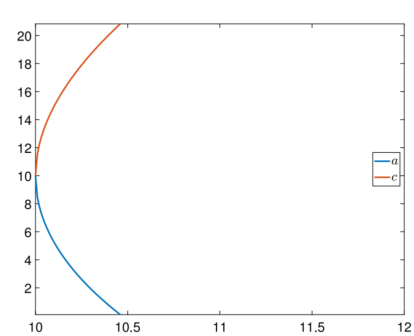
(d) Corresponding parameters for symmetric triangular distribution (a and b)



(e) Corresponding parameters for log-normal distribution (μ_τ and σ_τ)

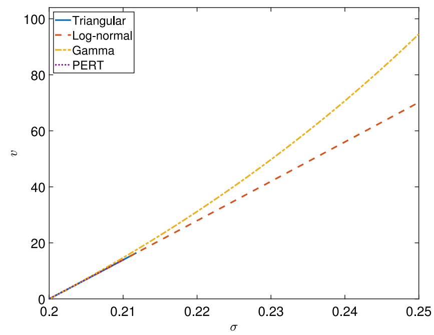


(f) Corresponding parameters for gamma distribution (k and θ)

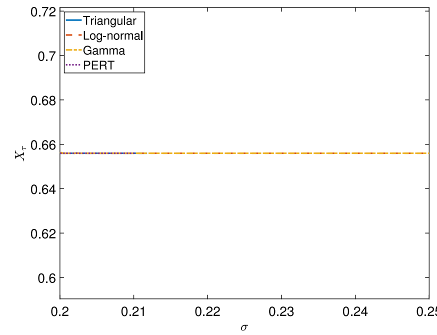


(g) Corresponding parameters for symmetric PERT distribution (a and c)

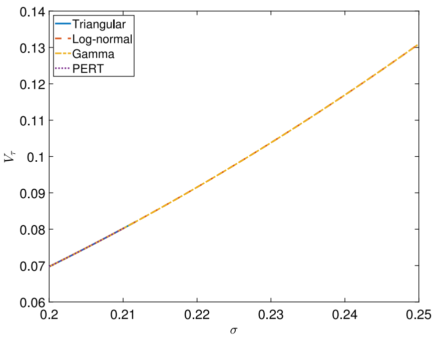
Fig. 9. Examples of uncertainty equivalent with mean m for $\bar{\tau} = 10$.



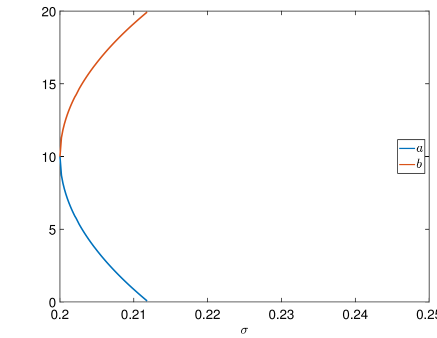
(a) Variance v that induces the same investment threshold for σ



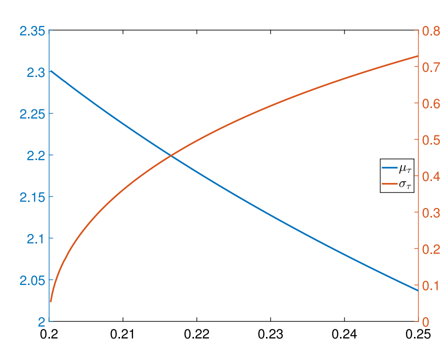
(b) Investment threshold for each distribution



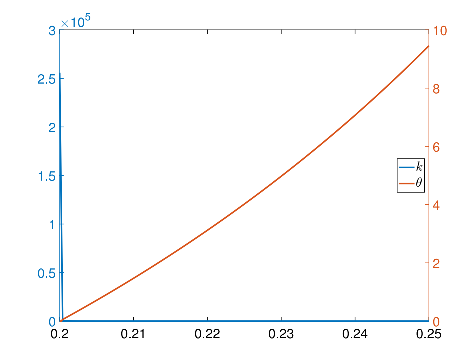
(c) Firm value for each distribution



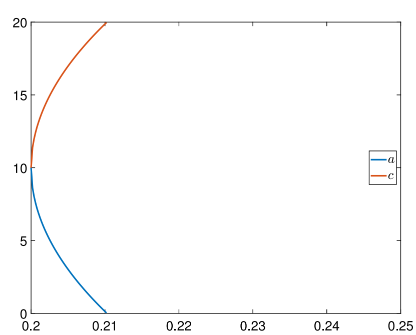
(d) Corresponding parameters for symmetric triangular distribution (a and b)



(e) Corresponding parameters for log-normal distribution (μ_τ and σ_τ)



(f) Corresponding parameters for gamma distribution (k and θ)



(g) Corresponding parameters for symmetric PERT distribution (a and c)

Fig. 10. The level of variance v that induces the same optimal investment threshold for mean $m = 10$ and different levels of demand volatility σ with the base line $\sigma = 0.2$.

in time-to-build advances the investment timing and improves firm value. That is, the uncertainty of demand and that of time-to-build induce the opposite effects on investment timing but both yields the positive impacts on the firm value.

With these arguments, Fig. 10 highlights the contrasting effects of the two different types of uncertainty. Specifically, Fig. 10(a) describes the level of variance v that offsets the negative impacts of an increase in σ on investment timing. It shows that the variance of time-to-build required to offset the impacts of increased demand volatility varies according to the distribution of time-to-build. A gamma distribution is found to require significantly higher variance than other distributions, mainly due to its semi-infinite support. Figs. 10(b) and 10(c) clarify that the effects of uncertainty on investment timing from the two different channels are canceled out, while the firm value significantly improves due to the uncertainty from both channels. Figs. 10(d)–10(g) present the corresponding parameters for each distribution. Note that some distributions cannot offset the negative impacts of a significant increase in demand volatility due to restrictions on the parameters.

6. Conclusion

This study investigated the impacts of uncertainty in time-to-build on corporate investment, clarifying the extent to which the uncertainty accelerates investment and improves firm value. We showed that there always exists a unique certainty equivalent of uncertain time-to-build, regardless of its distribution, and derived it in an analytic form. This allows firms to establish the optimal investment strategy with uncertain time-to-build in the form of the investment strategy that would have been adopted without such uncertainty. Even without knowing the exact distribution, the certainty equivalent can be approximated using only a few moments, such as mean and variance, which significantly enhances its practicality. Furthermore, we showed that for a given fixed time-to-build, an uncertainty equivalent always exists. This enables firms to evaluate the level of risk implicitly assumed by their investment strategies established without accounting for uncertainty in time-to-build. Lastly, we applied these arguments to representative probability distributions to demonstrate the practicality and analyzed the effects of variance of time-to-build on investment. In particular, we derived the variance of time-to-build that offsets the negative impacts of demand uncertainty on investment.

Many problems still remain to be explored. For instance, we focused on a monopolistic firm for simplicity. Preemptive incentive due to market competition will significantly alter firms' optimal investment strategies as well as the impacts of time-to-build on them. However, introducing competition would substantially reduce the model's tractability. This is because an additional layer of decision-making occurs after the investment but before its completion, which directly depends on the remaining time-to-build. Thus, an analytic solution, which allows us to discuss the explicit structure of equilibrium strategies, is unlikely, unless the underlying distribution exhibits the memoryless property (i.e., exponential distribution). We also assumed an all-equity firm, but we need to address the impacts of uncertainty in time-to-build on financing and default decisions for a levered firm. Jeon (2021a) investigated the effects of uncertain time-to-build on a firm's investment and default decisions, showing that it can lead to a lower default probability compared to the case without time-to-build, mainly due to more conservative investment decisions. However, the study did not clarify the pure effects of uncertainty in time-to-build by comparing it with the case of a fixed time-to-build. Future research must address this question, although it will encounter the same technical difficulties mentioned earlier. More importantly, we assumed the independence between the demand shock and time-to-build for tractability. Follow-up research should test if the same results hold without the independence assumption. Lastly, despite data collection challenges, empirical analysis is necessary to validate the theoretical results discussed in this study. It is hoped that this study will serve as a platform for investigating these issues in the future.

CRediT authorship contribution statement

Haejun Jeon: Conceptualization, Formal analysis, Investigation, Validation, Methodology, Visualization, Writing – original draft, Writing – review & editing, Software, Funding acquisition, Project administration, Resources. **Michi Nishihara:** Formal analysis, Investigation, Validation, Methodology, Writing – review & editing, Funding acquisition.

Declaration of Generative AI and AI-assisted technologies in the writing process

During the preparation of this work, the authors used ChatGPT to improve the language and readability. After using ChatGPT, the authors carefully reviewed and edited the content as needed and take full responsibility for the publication.

Appendix A. Proofs

A.1. Proof of Lemma 1

By the definition, $\tau_{n+1} = \tau_n + \epsilon_{n+1}$ where $\mathbb{E}[\epsilon_{n+1}|\tau_n] = 0$. Suppose that τ_n has a cumulative distribution function F_n for $n \geq 1$. Since $f(\tau) := e^{-(r-\mu)\tau}$ is a strictly convex function, Jensen's inequality ensures the following always holds for all $n \geq 0$:

$$\begin{aligned}
&\delta(\tau_{n+1}) = \mathbb{E}[f(\tau_{n+1})] \\
&= \int f(\tau_{n+1})\mathrm{d}F_{n+1}(\tau_{n+1}) = \int \mathbb{E}[f(\tau_n + \epsilon_{n+1}|\tau_n)]\mathrm{d}F_n(\tau_n) \\
&> \int f(\mathbb{E}[\tau_n + \epsilon_{n+1}|\tau_n])\mathrm{d}F_n(\tau_n) = \int f(\tau_n)\mathrm{d}F_n(\tau_n) = \mathbb{E}[f(\tau_n)] = \delta(\tau_n).
\end{aligned} \tag{25}$$

With $\delta(\tau_{n+1}) > \delta(\tau_n)$, it is straightforward that $X_{\tau_{n+1}} < X_{\tau_n}$ and $V_{\tau_{n+1}}(X) > V_{\tau_n}(X)$.

A.2. Proof of Proposition 1

By Lemma 1, $f(\mathbb{E}[\tau]) < \mathbb{E}[f(\tau)] = \delta(\tau)$ always holds, and $f(\tau)$ strictly decreases with τ. Thus, there exists a constant $\bar{\tau}_c(< \mathbb{E}[\tau])$ such that $\delta(\bar{\tau}_c) = \delta(\tau)$. The monotonicity of $f(\tau)$ ensures its uniqueness.

Meanwhile, the definition of $\delta(\tau)$ and that of the moment-generating function in (9) imply $\delta(\tau) = M_\tau(-(r-\mu))$, from which we obtain (8).

A.3. Proof of Corollary 1

Let us define $u(z) := K_\tau(z)/z$ for $z < 0$. It is straightforward that $u'(z) = v(z)/z^2$ where $v(z) := K'_\tau(z)z - K_\tau(z)$. Due to the convexity of the cumulant-generating function, we have $v'(z) = K''_\tau(z)z \leq 0$ for $z < 0$, which amounts to $v(z) \geq v(0) = 0$ for $z < 0$. Therefore, we obtain $u'(z) \geq 0$. Note that $\bar{\tau}_c = u(-(r-\mu))$, and thus, $\bar{\tau}$ increases with μ, and the independence with respect to σ is evident.

A.4. Proof of Corollary 2

Plugging (5) into (4), it is straightforward that $V_\tau(X) = A(X)(\delta(\tau))^\gamma$ where $A(X)$ is given by (11). Thus, $V_\tau(X) \geq \bar{X}$ is equivalent to $\delta(\tau) \geq (\bar{X}/A(X))^{1/\gamma}$. By definition, $\delta(\tau) = \delta(\bar{\tau}_c) = \exp(-(r-\mu)\bar{\tau}_c)$ and $\delta(\tau) = M_\tau(-(r-\mu)) = \exp(K_\tau(-(r-\mu)))$, which amounts to (10).

A.5. Proof of Corollary 3

Combining the cumulants $\kappa_1 = \mathbb{E}[\tau]$, $\kappa_2 = \mathbb{E}[(\tau - \mathbb{E}[\tau])^2]$, $\kappa_3 = \mathbb{E}[(\tau - \mathbb{E}[\tau])^3]$, and $\kappa_4 = \mathbb{E}[(\tau - \mathbb{E}[\tau])^4] - 3(\mathbb{E}[(\tau - \mathbb{E}[\tau])^2])^2$ with (8) and (9), we can easily obtain (12) through (14).

A.6. Proof of Corollary 4

Suppose $\tau_{n+1} = \tau_n + \epsilon_{n+1}$ with $\mathbb{E}[\epsilon_{n+1}|\tau_n] = 0$. Proposition 1 implies that for τ_n, there exists a unique constant $\bar{\tau}_{c,n} = \mathbb{E}[\tau_n] - c_n$ where $c_n > 0$ such that $\delta(\bar{\tau}_{n,c}) = \delta(\tau_n)$. By Lemma 1, $\delta(\tau_n) < \delta(\tau_{n+1})$, or equivalently, $f(\mathbb{E}[\tau_n] - c_n) < f(\mathbb{E}[\tau_{n+1}] - c_{n+1})$. Because $f(\tau)$ strictly decreases with τ, we have $c_{n+1} > c_n$, which implies that the uncertainty premium of time-to-build increases with its dispersion.

The approximation of the certainty equivalent in (13) and (14) decrease with v when the terms in the parentheses are positive.

A.7. Proof of Proposition 2

Suppose τ is a nonnegative random variable with mean m and variance v. Jensen's inequality ensures the following holds:

$$e^{-(r-\mu)m} \le \mathbb{E}[e^{-(r-\mu)\tau}]. \tag{26}$$

The tightness of (26) can be shown as follows. Suppose τ follows a two-point distribution with possible outcomes of $m+n$ and $m-l(n)$ with probabilities vn^{-2} and $1-vn^{-2}$, respectively, where n is sufficiently large and $l(n)$ is chosen such that $\mathbb{E}[\tau] = m$ (i.e., $l(n) = (m+n)v/(n^2-v)$). Since $\lim_{n\to\infty} l(n) = 0$, the following holds:

$$\mathbb{E}[(\tau-m)^2] = n^2 \cdot \frac{v}{n^2} + (l(n))^2\left(1-\frac{v}{n^2}\right) \xrightarrow{n\to\infty} v, \tag{27}$$

$$\mathbb{E}[e^{-(r-\mu)\tau}] = e^{-(r-\mu)(m+n)}\frac{v}{n^2} + e^{-(r-\mu)(m-l(n))}\left(1-\frac{v}{n^2}\right) \xrightarrow{n\to\infty} e^{-(r-\mu)m}. \tag{28}$$

Meanwhile, let us define a quadratic function $g(\tau) := a\tau^2 + b\tau + 1$ where

$$a = \frac{1 - e^{-(r-\mu)\tau_0}(1+(r-\mu)\tau_0)}{\tau_0^2}, \tag{29}$$

$$b = \frac{-2 + e^{-(r-\mu)\tau_0}(2+(r-\mu)\tau_0)}{\tau_0}, \tag{30}$$

$$\tau_0 = m + \frac{v}{n} (>0). \tag{31}$$

Then, for $f(\tau) := e^{-(r-\mu)\tau}$, it is straightforward to show the following:

$$g(0) - f(0) = 0, \tag{32}$$

$$g(\tau_0) - f(\tau_0) = 0, \tag{33}$$

$$g'(\tau_0) - f'(\tau_0) = 0. \tag{34}$$

We can also show that $g''(\tau) - f''(\tau) = 2a - (r-\mu)^2 e^{-(r-\mu)\tau}$ is an increasing function of τ and that

$$g''(\tau_0) - f''(\tau_0) = \frac{2 - e^{-(r-\mu)\tau_0}\{2 + 2(r-\mu)\tau_0 + (r-\mu)^2\tau_0^2\}}{\tau_0^2} > 0. \tag{35}$$

The inequality in (35) holds because for $\tau > 0$,

$$h(\tau) := 2 - e^{-(r-\mu)\tau}\{2 + 2(r-\mu)\tau + (r-\mu)^2\tau^2\} > h(0) = 0 \tag{36}$$

since $h'(\tau) > 0$ for $\tau > 0$. From (32), (33), (34), and the monotonic increase of $g''(\tau) - f''(\tau)$, we can show that $g(\tau) - f(\tau)$ for $\tau \ge 0$ takes the minimum value of 0 at $\tau = 0$ and $\tau = \tau_0$, implying that $g(\tau) \ge f(\tau)$ for $\tau \ge 0$. Thus, we have

$$\mathbb{E}[e^{-(r-\mu)\tau}] = \mathbb{E}[f(\tau)]$$

$$\le \mathbb{E}[g(\tau)] = a(m^2+v) + bm + 1 = \frac{e^{-(r-\mu)(m+v/m)}m^2 + v}{m^2+v}. \tag{37}$$

The tightness of (37) can be shown as follows. Suppose τ follows a two-point distribution with possible outcomes of 0 and $m + v/m$ with probabilities $v/(m^2+v)$ and $m^2/(m^2+v)$, respectively. This satisfies $\mathbb{E}[\tau] = m$ and $\mathbb{E}[\tau^2] = v + m^2$, and $\mathbb{E}[e^{-(r-\mu)\tau}]$ coincides the right-hand side of (37).

By combining (26) and (37) and rewriting them in terms of the certainty equivalent in (8), we can obtain (15). The left-hand side of (15) is $-\ln(p(v))/(r-\mu)$ where $p(v) = (e^{-(r-\mu)(m+v/m)}m^2 + v)/(m^2+v)$, and it is straightforward to show $\partial p/\partial v = \{m^2(1-e^{-c}(1+c))\}/(m^2+v)^2$ where $c = (r-\mu)(m+v/m) > 0$. For $q(c) := e^{-c}(1+c)$, $\partial q/\partial c < 0$ and $q(0) = 1$, and thus, $q(c) < 1$ for $c > 0$. This implies $\partial p/\partial v > 0$, and thus, the left-hand side of (15) strictly decreases with v.

A.8. Proof of Proposition 3

Suppose that τ follows a two-point distribution with possible outcomes of n and $m-l(n)$ with probabilities $n^{-1.5}$ and $1-n^{-1.5}$, respectively, where n is sufficiently large and $l(n)$ is chosen such that $\mathbb{E}[\tau] = m$ (i.e., $l(n) = (n-m)/(n^{1.5}-1)$). Since $\lim_{n\to\infty} l(n) = 0$, the following holds:

$$\mathbb{E}[\tau^2] = \frac{n^2}{n^{1.5}} + (m-l(n))^2\left(1-\frac{1}{n^{1.5}}\right) \xrightarrow{n\to\infty} \infty, \tag{38}$$

$$\mathbb{E}[e^{-(r-\mu)\tau}] = \frac{e^{-(r-\mu)n}}{n^{1.5}} + e^{-(r-\mu)(m-l(n))}\left(1-\frac{1}{n^{1.5}}\right) \xrightarrow{n\to\infty} e^{-(r-\mu)m}. \tag{39}$$

That is, as n increases, variance of τ increases, but its certainty equivalent also increases, converging to m. In other words, it is possible that uncertainty premium of time-to-build can decrease and converge to 0 as its variance increases.

A.9. Proof of Proposition 4

By Lemma 1, $f(\bar{\tau}) = f(\mathbb{E}[\bar{\tau}+\epsilon]) < \mathbb{E}[f(\bar{\tau}+\epsilon)]$ where $\mathbb{E}[\epsilon] = 0$. Since $f(\tau)$ strictly decreases with τ, $\mathbb{E}[f(\bar{\tau}+\epsilon+u)] < \mathbb{E}[f(\bar{\tau}+\epsilon)]$ holds for a constant $u > 0$. Therefore, there always exists $\tau_u := \bar{\tau} + \epsilon + u$ such that $f(\bar{\tau}) = \mathbb{E}[f(\tau_u)]$, or equivalently, $\delta(\bar{\tau}) = \delta(\tau_u)$. This, combined with the definitions of $\delta(\bar{\tau})$ and the moment-generating function, amounts to (16).

A.10. Proof of Corollary 5

Suppose τ_u is the uncertainty equivalent of $\bar{\tau}$. That is, $\delta(\bar{\tau}) = \delta(\tau_u)$, or equivalently, $f(\bar{\tau}) = \mathbb{E}[f(\tau_u)] = \mathbb{E}[f(\bar{\tau}+\epsilon+u)]$ where $\mathbb{E}[\epsilon] = 0$ and $u > 0$ is a constant.

Meanwhile, suppose τ_u' is a mean-preserving spread of τ_u. That is, $\tau_u' = \tau_u + \epsilon'$ where $\mathbb{E}[\epsilon'|\tau_u] = 0$. By Lemma 1, $\mathbb{E}[f(\tau_u)] < \mathbb{E}[f(\tau_u')]$ always holds. Since $f(\tau)$ strictly decreases with τ, there always exists a constant $u' > 0$ such that $\mathbb{E}[f(\tau_u)] = \mathbb{E}[f(\tau_u'+u')]$ holds, which implies $\delta(\bar{\tau}) = \delta(\tau_u) = \delta(\hat{\tau}_u)$ where $\hat{\tau}_u := \tau_u' + u'$. Namely, both τ_u and $\hat{\tau}_u$ are the uncertainty equivalents of $\bar{\tau}$. By the definition, $\mathbb{E}[\tau_u] = \bar{\tau} + u$ and $\mathbb{E}[\hat{\tau}_u] = \bar{\tau} + u + u'$, which completes the proof.

A.11. Proof of Corollary 6

By Proposition 4, for any $\bar{\tau} \ge 0$, there always exists $\tau_u := \bar{\tau} + \epsilon + u$ where $\mathbb{E}[\epsilon] = 0$ and $u > 0$ is a constant such that $f(\bar{\tau}) = \mathbb{E}[f(\tau_u)]$. Because $f(\tau)$ strictly decreases with τ, there always exists a constant $w \in (0,u)$ such that $\mathbb{E}[f(\tau_u)] < \mathbb{E}[f(\tau_w)]$ where $\tau_w := \bar{\tau} + \epsilon + w$, which implies $\delta(\bar{\tau}) = \delta(\tau_u) < \delta(\tau_w)$ and $\mathbb{E}[\tau_w] = \bar{\tau} + w$.

A.12. Proof of Proposition 5

Suppose τ follows a symmetric triangular distribution on $[a-d, b+d]$ with its mode $c = (a+b)/2$. It is obvious that its variance increases while its mean remains the same as d increases. With (18), one can easily show that $\partial M_\tau(t)/\partial d > 0$. That is, an increase of d, which increases its variance, results in a decrease of $\bar{\tau}_c$, and thus, an increase of $\mathbb{E}[\tau] - \bar{\tau}_c$.

Appendix B. Supplementary data

Supplementary material related to this article can be found online at https://doi.org/10.1016/j.ejor.2025.07.051.

References

Abel, A. (1983). Optimal investment under uncertainty. *American Economic Review*, *73*(1), 228–233.

Pacheco-de Almeida, G., & Zemsky, P. (2003). The effect of time-to-build on strategic investment under uncertainty. *RAND Journal of Economics*, *34*(1), 166–182.

Alvarez, L., & Keppo, J. (2002). The impact of delivery lags on irreversible investment under uncertainty. *European Journal of Operational Research*, *136*(1), 173–180.

Asmussen, S., Jensen, J., & Rojas-Nandayapa, L. (2016). On the Laplace transform of the lognormal distribution. *Methodology and Computing in Applied Probability*, *18*, 441–458.

Bar-Ilan, A., & Strange, W. (1996a). Investment lags. *American Economic Review*, *86*(3), 610–622.

Bar-Ilan, A., & Strange, W. (1996b). Urban development with lags. *Journal of Urban Economics*, *39*(1), 87–113.

Bar-Ilan, A., & Strange, W. (1998). A model of sequential investment. *Journal of Economic Dynamics & Control*, *22*(3), 437–463.

Charoenwong, B., Kimura, Y., Kwan, A., & Tan, E. (2024). Capital budgeting, uncertainty, and misallocation. *Journal of Financial Economics*, *153*, Article 103779.

Dixit, A., & Pindyck, R. (1994). *Investment under uncertainty*. New Jersey: Princeton University Press.

Driver, C., Temple, P., & Urga, G. (2008). Real options — delay vs. pre-emption: Do industrial characteristics matter? *International Journal of Industrial Organization*, *26*(2), 532–545.

Fernandes, A., & Rigato, R. (2025). *K wasnt built in a day*: *IMF working paper*.

Glancy, D., Kurtzman, R., & Loewenstein, L. (2024). *On commercial construction activitys long and variable lags*: *FEDS working paper*.

Guiso, L., & Parigi, G. (1999). Investment and demand uncertainty. *Quarterly Journal of Economics*, *114*(1), 185–227.

Hartman, R. (1972). The effects of price and cost uncertainty on investment. *Journal of Economic Theory*, *5*(2), 258–266.

Hartman, R. (1973). Adjustment costs, price and wage uncertainty, and investment. *Review of Economic Studies*, *40*(2), 259–267.

Heyde, C. (1963). On a property of the lognormal distribution. *Journal of the Royal Statistical Society. Series B. Statistical Methodology*, *25*(2), 392–393.

Huisman, K., & Kort, P. (2015). Strategic capacity investment under uncertainty. *RAND Journal of Economics*, *46*(2), 376–408.

Jeon, H. (2021a). Investment and financing decisions in the presence of time-to-build. *European Journal of Operational Research*, *288*(3), 1068–1084.

Jeon, H. (2021b). Investment timing and capacity decisions with time-to-build in a duopoly market. *Journal of Economic Dynamics & Control*, *122*, Article 104028.

Jeon, H. (2023). Time-to-build and capacity expansion. *Annals of Operations Research*, *328*, 1461–1494.

Jeon, H. (2024a). The effects of time-to-build and regulation on investment timing and size. *International Review of Financial Analysis*, *96*, 103741.

Jeon, H. (2024b). Time-to-build, regulation, and investment. *European Journal of Operational Research*, *319*(3), 999–1019.

Jorgenson, D., & Stephenson, J. (1967). The time structure of investment behavior in united states manufacturing, 1947–1960. *Review of Economics and Statistics*, *49*(1), 16–27.

Jung, Y. (2013). An inference about the length of the time-to-build period. *Economic Modelling*, *33*, 42–54.

Kalouptsidi, M. (2014). Time to build and fluctuations in bulk shipping. *American Economic Review*, *104*(2), 564–608.

Koeva, P. (2000). *The facts about time-to-build*: *IMF working paper*.

Leahy, J., & Whited, T. (1996). The effect of uncertainty on investment: Some stylized facts. *Journal of Money, Credit and Banking*, *28*(1), 64–83.

Leland, H. (1994). Corporate debt value, bond covenants, and optimal capital structure. *Journal of Finance*, *49*(4), 1213–1252.

Leland, H. (1998). Agency costs, risk management, and capital structure. *Journal of Finance*, *53*(4), 1213–1243.

Leland, H., & Toft, K. (1996). Optimal capital structure, endogenous bankruptcy, and the term structure of credit spreads. *Journal of Finance*, *51*(3), 987–1019.

Majd, S., & Pindyck, R. (1987). Time to build, option value, and investment decisions. *Journal of Financial Economics*, *18*(1), 7–27.

Marmer, V., & Slade, M. (2018). Investment and uncertainty with time to build: Evidence from entry into U.S. copper mining. *Journal of Economic Dynamics & Control*, *95*, 233–254.

Meinen, P., & Roehe, O. (2017). On measuring uncertainty and its impact on investment: Cross-country evidence from the euro area. *European Economic Review*, *92*, 161–179.

Montgomery, M. (1995). Time-to-build completion patters for nonresidential structures, 1961–1991. *Economics Letters*, *48*(2), 155–163.

Nishihara, M. (2018). Valuation of an R & D project with three types of uncertainty. *EURO Journal on Decision Processes*, *6*, 93–113.

Oh, H., Yang, C., & Yoon, C. (2024). *Land development and frictions to housing supply over the business cycle*: *FEDS working paper*.

Oh, H., & Yoon, C. (2020). Time to build and the real-options channel of residential investment. *Journal of Financial Economics*, *135*(1), 255–269.

Peskir, G., & Shiryaev, A. (2006). *Optimal stopping and free-boundary problems*. Basel: Birkhäuser.

Salomon, R., & Martin, X. (2008). Learning, knowledge transfer, and technology implementation performance: A study of time-to-build in the global semiconductor industry. *Management Science*, *54*(7), 1266–1280.

Tsoukalas, J. (2011). Time to build capital: Revisiting investment-cash-flow sensitivities. *Journal of Economic Dynamics & Control*, *35*(7), 1000–1016.

Weeds, H. (2002). Strategic delay in a real options model of R & D competition. *Review of Economic Studies*, *69*(3), 729–747.

White, S. (2024). EDFs hinkley point woes pile pressure on global nuclear push. *Financial Times*, 2024.

Zhou, C. (2000). Time-to-build and investment. *Review of Economics and Statistics*, *82*(2), 273–282.