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Author(s)	Seki, Yukihiro; Shibata, Kosuke; Sugiyama, Yoshie
Citation	Journal of Differential Equations. 2026, 450, p. 113745
Version Type	VoR
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Existence of axis-symmetric blow-up solution with multiple peak aggregations for the 2-*D* Keller-Segel systems coupled bipolar source and sink flow

Yukihiro Seki ^a, Kosuke Shibata ^b, Yoshie Sugiyama ^{b,*}

^a Department of Mathematics, Tokyo Metropolitan University, Hachioji, Tokyo 192-0397, Japan

^b Department of Pure and Applied Mathematics, Graduate School of Information Science and Technology, The University of Osaka, Suita, Osaka, 565-0871, Japan

Received 25 May 2025; revised 23 August 2025; accepted 25 August 2025

Abstract

We consider the Keller-Segel systems in $\mathbb{R}^d$, coupled with a bipolar source and sink flow. Focusing on the two-dimensional case ($d = 2$), we establish finite-time blow-up of solutions under an axis-symmetric setting, without requiring the solutions to be radial. In particular, we prove that multiple blow-up points appear in pairs (i.e., in even numbers) away from the origin, lying on the x_1 -axis and exhibiting axis-symmetry about the x_2 -axis. This result holds for initial data with total mass strictly greater than 16π , and stands in contrast to the classical radial setting, where blow-up is confined to the origin.

A crucial part of our analysis is a sharp ε -regularity theorem, originally developed for the classical Keller-Segel systems and first established by Luckhaus–Sugiyama–Velázquez [12]. This theorem states that if the local mass around x_1 is sufficiently small at some time t_1 , then the solution remains locally bounded in a suitable parabolic cylinder in space–time centered at (x_1, t_1) . Compared to the classical ε -regularity theorem, it requires weaker assumptions and yields weaker conclusions, making it a form of partial regularity that is particularly essential for analyzing blow-up singularities.

Based on this sharp ε -regularity theorem, we further prove that only finitely many blow-up points appear as singular sets, and the asymptotic profile is characterized as the sum of a finite number of δ -functions and a regular part in $L^1(\mathbb{R}^2)$. Moreover, our results reveal that multi-peak blow-up phenomena can occur with or without the presence of non-trivial flow, highlighting the intricate interplay between diffusion, chemotaxis, and persistent advection. By accounting for non-decaying flow and employing precise blow-up criteria, we establish that the blow-up time can be bounded above by any prescribed threshold. These findings are

* Corresponding author.

E-mail address: sugiyama.ist@osaka-u.ac.jp (Y. Sugiyama).

justified through the construction of a time-local existence and extension theory for strong solutions, which incorporates both advection and mass conservation.

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1. Introduction

Let us consider the Keller-Segel systems coupled bipolar source and sink flow:

$$(KSF) \quad \begin{cases} \frac{\partial n}{\partial t} + \nabla \cdot un = \Delta n - \nabla \cdot (n \nabla v) & \text{in } \mathbb{R}^d \times (0, T), \\ 0 = \Delta v - \gamma v + n & \text{in } \mathbb{R}^d \times (0, T), \\ n(x, 0) = n_0(x) & \text{in } \mathbb{R}^d, \end{cases}$$

where $d \geq 1$ and $\gamma \geq 0$. Here, the unknown functions $n = n(x, t)$ and $v = v(x, t)$ denote, respectively, the density of the relevant species and the potential, while $u = u(x)$ is a prescribed fluid vector field in $\mathbb{R}^d$. The parameter $\gamma \geq 0$ represents the decay rate of the attractant. This mathematical model arises in a broad range of biological and physical contexts, where external flow or stirring interacts with the chemotactic behavior of micro-organisms.

In this paper, we introduce a specific axis-symmetric flow field:

$$u(x) := Q \left(\frac{x - \mathbf{b}}{|x - \mathbf{b}|^2 + \lambda} \chi(x - \mathbf{b}) + \frac{x + \mathbf{b}}{|x + \mathbf{b}|^2 + \lambda} \chi(x + \mathbf{b}) \right), \quad (1.1)$$

where $Q \in \mathbb{R}$ and $\lambda > 0$ are constants, $\mathbf{b} \in \mathbb{R}^d$ is a prescribed shift vector, and χ is a smooth cut-off function. The non-trivial presence of u enriches the dynamics: depending on the direction and magnitude of the flow, chemotactic aggregation may be either enhanced or suppressed.

The Keller-Segel systems and its variants have long been studied as a canonical model for chemotaxis, whereby cells (or organisms) move in response to chemical gradients. In recent years, considerable attention has been devoted to *flow-modified* Keller-Segel systems (commonly referred to as Keller–Segel–Fluid Coupling Systems), in which an additional flow field or external forcing term is introduced. These systems arise naturally in settings such as micro-fluidics or population dynamics, where fluid flow significantly influences chemotactic aggregation. Understanding the interplay among diffusion, chemotaxis, and flow remains mathematically challenging, despite their broad significance in various applied and theoretical contexts. This difficulty is particularly pronounced when analyzing solutions with the flow u given by (1.1), which does not exhibit time decay. The lack of decay introduces persistent effects that significantly complicate the analysis.

Prior to studying the (KSF) with non-decaying flow, Kozono–Miura–Sugiyama [8,9] demonstrated that solutions to the Keller–Segel–Navier–Stokes systems exhibit time decay in $L^r(\mathbb{R}^d)$, where the decaying flow aids in controlling the long-term behavior of solutions. By contrast, the flow considered in the present paper remains non-decaying, requiring us to continuously address its substantial influence over time. In fact, starting with large initial data, the solution may blow

up in finite time, and since the flow does not diminish, controlling its direct impact and isolating its individual effects become particularly challenging. This underscores the uniqueness and inherent difficulty of our problem.

In view of these challenges, one particularly intriguing phenomenon is finite-time blow-up, where the density n relevant species becomes unbounded in $L^\infty(\mathbb{R}^2)$ at certain points. This behavior has been extensively studied in the absence of any fluid flow, particularly in the radial case. Early works by Herrero and Velázquez [6,7] demonstrated that, under specific conditions, solutions concentrate at the origin and asymptotically resemble a sum of δ -functions. These investigations were motivated, in part, by conjectures such as those proposed by Nanjundiah [15] and explored further by Childress and Percus [2], which suggested the formation of point-mass blow-ups in chemotaxis models.

Recent studies, building on these foundational results, have increasingly turned to more general systems where fluid flow interacts with chemotactic processes. A key question in this paper in the presence of flow (i.e. within (KSF)) is whether the flow prevents blow-up entirely or instead facilitates finite-time singularities by concentrating mass in highly localized regions, and whether multiple blow-up points can occur. Even in the classical Keller–Segel systems without flow, Seki, Sugiyama and Velázquez [18,19] have investigated conditions leading to multiple blow-up phenomena; however, in flow-coupled equations, the phenomenon of multiple blow-up points remains not fully understood.

In response to these open questions, we first develop an existence and extension theory for strong solutions and establish what is commonly referred to as “mass conservation” in $L^1(\mathbb{R}^d)$, recognized as a key conservation law. We then establish the so-called ε -regularity theorem, laying the groundwork for a detailed singularity analysis. Specifically, we prove the existence of a suitably small constant such that, if the local mass concentration remains below this threshold, the center of mass stays regular and avoids singularity formation.

The ε -regularity theorem has played a pivotal role in the study of partial differential equations and geometric analysis. Its origins can be traced back to the pioneering work of Morrey [16], which provided fundamental tools for handling harmonic maps and established partial regularity results under suitable energy bounds. A significant breakthrough in the formalization and generalization of these principles came through the work of K. Uhlenbeck in the late 1970s. In her seminal contributions to harmonic maps and gauge theory, K. Uhlenbeck [24] established what is now often referred to as the ε -regularity theorem. She showed that if the local L^2 -norm of the curvature (in the setting of gauge fields) or the energy (in the context of harmonic maps) is sufficiently small, then the solution is smooth in that region. This framework provided a powerful method for controlling singularities and understanding the structure of solutions at multiple scales, influencing subsequent research by Schoen–K. Uhlenbeck [17], Freed–K. Uhlenbeck [3], and many others. Over time, this line of investigation has evolved into a cornerstone of modern geometric analysis, with ε -regularity techniques being routinely employed to study minimal surfaces, Yang–Mills connections, Einstein manifolds, and a variety of geometric and analytic problems where localized control on energy or curvature provides a gateway to global structure.

A version of the ε -regularity theorem for the Keller–Segel system has been established in scale-invariant function spaces, in particular:

$$L^\infty(0, T; L^1(\mathbb{R}^2)).$$

As an application, this framework allows for a comprehensive singularity analysis of the Keller–Segel system. Indeed, building on this ε -regularity theorem, we further clarify the singular structure of the solutions. In particular, the ε -regularity theorem reveals how the interplay between local mass concentration and singularity formation is regulated. For instance, in the classical two-dimensional Keller–Segel system without flow, if the initial mass is strictly greater than 8π , blow-up occurs. By applying the ε -regularity theorem, we also deduce that for radially symmetric initial data, the resulting singularities arise exclusively at the origin. This result underscores the critical role of symmetry in determining the blow-up structure. To further substantiate this claim, in this paper, we extend the ε -regularity theorem to cases where radial symmetry is broken or additional effects, such as fluid flow, are introduced. This extension provides a foundation for analyzing the formation of multiple singularities or spatially distributed blow-up patterns. Of particular significance is that analyzing such multi-point blow-ups requires addressing several complex factors, including:

- the absence of radial symmetry, which complicates the mathematical structure of the solutions;
- the influence of fluid flow, which introduces additional nonlinear interactions; and
- the emergence of axis-symmetrical spatial patterns in the arrangement of blow-up points, which demand detailed investigation to fully understand their formation and underlying dynamics.

The aforementioned ε -regularity theorem, originally developed for the classical Keller–Segel systems by Senba–Suzuki [20] and Nagai–Senba–Suzuki [13], ensures local regularity under assumptions on the local mass concentration of the solution. This theorem has played a pivotal role in advancing our understanding of blow-up phenomena in these systems. Specifically, it has been shown that solutions to the classical Keller–Segel systems can blow-up in finite time at a finite number of points. If $\{x_i\}_{i=1}^k$ denotes the blow-up points at time T_b , the solution develops δ -function singularities at each x_i with corresponding masses $\{M_i\}_{i=1}^k$. More concretely, there exist a function $f \in L^1(\mathbb{R}^2)$ and a sequence $\{t_j\}_{j=1}^\infty \subset (0, T_b)$ such that $t_j \rightarrow T_b$ and:

$$n(\cdot, t_j) \longrightarrow \sum_{i=1}^k M_i \delta_{x_i}(\cdot) + f(\cdot) \quad \text{as } j \rightarrow \infty$$

in the sense of distributions on $\mathbb{R}^2$.

Furthermore, the third author refined the ε -regularity theorem into a sharper form and utilized it to analyze time-global solutions as measure-valued solutions beyond the blow-up time. This sharp ε -regularity theorem, originally developed for the classical Keller–Segel systems, was first established by Luckhaus–Sugiyama–Velázquez [12]. This theorem asserts that if the local mass around x_1 is sufficiently small at some time t_1 , then the solution remains locally bounded within a suitable parabolic cylinder in space–time centered at (x_1, t_1) . Compared to the classical ε -regularity theorem, this refined version requires weaker assumptions and provides weaker conclusions, positioning it as a form of partial regularity that is particularly crucial for analyzing blow-up singularities even after the blow-up time.

In light of the progress made on the Keller–Segel systems, the primary goal of this paper is to construct a blow-up solution exhibiting multiple peak aggregations for (KSF). More precisely,

we establish the occurrence of multiple blow-up points, which has not yet well-understood even for the classical Keller-Segel systems. Although it was known that the number of blow-up points could be finite, it remained possible that only a single blow-up location might occur. Our results characterize the initial condition to exclude this possibility and confirm the existence of multiple blow-up points. Specifically, we relax the strictly radial assumption by allowing the initial density n_0 to be merely axis-symmetric, under the condition:

$$\|n_0\|_{L^1(\mathbb{R}^2)} > 16\pi,$$

with sufficiently separated concentration peaks. For more details, see Remark 6. Under these assumptions, we prove the following:

- **Finite-time blow-up.** We derive an explicit upper bound for the blow-up time T_b . The bound depends on the L^1 -mass and the second moment of the initial data.
- **Even number and non-origin location of blow-up points.** None of the blow-up points lies at the origin. Instead, they appear at an even number of distinct points in $\mathbb{R}^2$:

$$\{x^{(1)}, \dots, x^{(m)}, x^{(m+1)}, \dots, x^{(2m)}\},$$

where each pair $(x^{(j)}, x^{(m+j)})$ is symmetric with respect to the coordinate axes.

These findings not only confirm that multiple blow-up points do indeed occur in (KSF) but also shed light on how symmetry and initial mass distribution play a critical role in blow-up formation. This result stands in contrast to what happens in the classical radial setting, where blow-up is often confined to the origin, and it illustrates the diversity of multi-peak formation that can emerge from non-radial, axis-symmetric initial data.

The proof is carried out in several steps. First, we introduce suitable function spaces and construct a local-in-time solution (n, v) of (KSF) within these spaces, ensuring that it qualifies as a strong solution. Next, we establish a sharp ε -regularity theorem, providing uniform estimates near potential singularities. We then apply standard blow-up criteria for parabolic equations, making use of the mass conservation of a strong solution n and the second moment:

$$\int_{\mathbb{R}^2} n_0(x) |x|^2 dx$$

to confirm that blow-up occurs in finite time. Subsequently, we relate the geometry of blow-up points to the structure of the flow field u . We show that if the initial peaks are sufficiently far apart, then blow-up happens away from the origin and occurs in pairs along the axes. Finally, we prove that only finitely many blow-up points can occur and that, at the blow-up time T_b , the strong solution admits an asymptotic form consisting of a finite sum of delta functions and a regular part in $L^1(\mathbb{R}^2)$. Through these steps, we construct explicit examples of initial data which make multiple blow-up phenomena in solutions of (KSF) having axis-symmetric but non-radial initial data. This work thus advances the study of blow-up analysis for chemotaxis systems under fluid flow.

This paper is organized as follows. Section 2 presents the essential notation, definitions, and main theorems. In Section 3, we collect several preliminary lemmas that will be used in later

proofs. Section 4 establishes the local-in-time existence of solutions in an appropriate strong formulation, along with the extension criterion and the mass conservation law. Section 5 is devoted to proving a sharp ε -regularity theorem. Section 6 provides an upper bound for the maximal existence time and constructs blow-up solutions, showing in particular that blow-up points occur in an even number of symmetrically positioned locations rather than infinitely many. Finally, Section 7 demonstrates that there are only finitely many blow-up points and that, at the blow-up time, the asymptotic form of the solution is given by a finite sum of delta functions plus a regular part in $L^1(\mathbb{R}^2)$.

2. Results

In what follows, we adopt the following notations:

- (1). $B_r(a) := \{x \in \mathbb{R}^d \mid |x - a| < r, r > 0, a \in \mathbb{R}^d\}$, $B_r := B_r(\mathbf{0})$.
- (2). $\mathcal{Q}_T := \mathbb{R}^d \times (0, T)$.
- (3). $e_1 := (1, 0)$, $e_2 := (0, 1)$.
- (4). $\partial_i = \frac{\partial}{\partial x_i}$, $\partial_{ij}^2 = \partial_i \partial_j$, $\nabla^2 = (\partial_{11}^2, \partial_{12}^2, \dots)$, $\partial_t = \frac{\partial}{\partial t}$, $i, j = 1, 2, \dots, d$.
- (5). $(\mathcal{F}f)(\xi) = \hat{f}(\xi) := \int_{\mathbb{R}^d} e^{-2\pi i x \cdot \xi} f(x) dx$, $(\mathcal{F}^{-1}f)(x) = \check{f}(x) := \int_{\mathbb{R}^d} e^{2\pi i x \cdot \xi} f(\xi) d\xi$.
- (6). $\|f\|_{L^p} = \|f\|_{L^p(\mathbb{R}^d)}$, $\|f\|_{W^{m,p}} = \|f\|_{W^{m,p}(\mathbb{R}^d)}$, $1 \leq p \leq \infty$, $m \in \mathbb{N}$.
- (7). For $0 < p < \infty$ and $0 < q \leq \infty$, let $L^{p,q}(\mathbb{R}^d) = L^{p,q}$ denote the Lorentz space on $\mathbb{R}^d$ equipped with the norm:

$$\|f\|_{L^{p,q}(\mathbb{R}^d)} = \|f\|_{L^{p,q}} := \begin{cases} p^{\frac{1}{q}} \left(\int_0^\infty (s\mu(|f| > s)^{\frac{1}{p}})^q \frac{ds}{s} \right)^{\frac{1}{q}} & \text{for } 0 < q < \infty, \\ \sup_{s>0} s\mu(|f| > s)^{\frac{1}{p}} & \text{for } q = \infty, \end{cases} \quad (2.1)$$

where μ denotes the Lebesgue measure.

- (8). When n and its weak derivatives ∇n , $\nabla^2 n$ and $\partial_t n$ belong to $L^p(\mathcal{Q}_T)$ for some $1 \leq p \leq \infty$, we say $n \in W_p^{2,1}(\mathcal{Q}_T)$, which is defined as:

$$W_p^{2,1}(\mathcal{Q}_T) := \{n \in L^p(0, T; W^{2,p}(\mathbb{R}^d)) \cap W^{1,p}(0, T; L^p(\mathbb{R}^d)) \mid \|n\|_{W_p^{2,1}(\mathcal{Q}_T)} < \infty\}$$

with the corresponding norm:

$$\|n\|_{W_p^{2,1}(\mathcal{Q}_T)} := \|n\|_{L^p(0,T;W^{2,p}(\mathbb{R}^d))} + \|n\|_{W^{1,p}(0,T;L^p(\mathbb{R}^d))}.$$

- (9). For $T > 0$, we define the function space $\mathbf{W}(\mathcal{Q}_T)$ as follows:

$$\mathbf{W}(\mathcal{Q}_T) := W_{d+2}^{2,1}(\mathcal{Q}_T) \times L^{d+2}(0, T; W^{2,d+2}(\mathbb{R}^d)). \quad (2.2)$$

- (10). For $T > 0$, we define the function space X_T as follows:

$$X_T := \left\{ n \in L^\infty(0, T; W^{2,d+2}(\mathbb{R}^d)) \mid \partial_t n \in L^{d+2}(\mathcal{Q}_T), n \geq 0 \text{ in } \mathcal{Q}_T, \right.$$

$$\|n\|_{L^\infty(0,T;W^{2,d+2}(\mathbb{R}^d))} + \|\partial_t n\|_{L^{d+2}(\mathcal{Q}_T)} \leq 4\|n_0\|_{W^{2,d+2}(\mathbb{R}^d)} + 1 \Big\}. \quad (2.3)$$

(11). For $r_1, r_2 > 0$ and $\mathbf{p}, \mathbf{q} \in \mathbb{R}^d$, we define the function $\tau_{r_1, r_2}(\mathbf{p}, \mathbf{q})$ as follows:

$$\tau_{r_1, r_2}(\mathbf{p}, \mathbf{q}) := \begin{cases} 1 & \text{if } \mathbf{p} = \mathbf{q}, \\ 0 & \text{if } |\mathbf{p} - \mathbf{q}| \geq r_1 + r_2. \end{cases} \quad (2.4)$$

Throughout this paper, we impose the following assumptions.

Assumption 2.1.

- (i). The parameter $\gamma \geq 0$.
- (ii). The initial data n_0 satisfies $n_0 \in W^{2,d+2}(\mathbb{R}^d)$ with $n_0 \geq 0$.

Under Assumption 2.1, we introduce the axis-symmetric function u .

Definition 2.1. Let $Q \in \mathbb{R}$ and $\lambda > 0$. We suppose that $\mathbf{a} \in \mathbb{R}^d$ satisfy:

$$\mathbf{a} = k(1, 0, \dots, 0) \quad \text{with } k > 0.$$

We consider the following two cases for the vector $\mathbf{b} \in \mathbb{R}^d$:

- (i). $\mathbf{b} = \mathbf{a}$,
- (ii). $\mathbf{b} = \widehat{k}(0, 0, \dots, 1)$ with $\widehat{k} > 0$.

Then, the axis-symmetric fluid vector field u is defined as follows:

$$u(x) := Q \left(\frac{x - \mathbf{b}}{|x - \mathbf{b}|^2 + \lambda} \chi(x - \mathbf{b}) + \frac{x + \mathbf{b}}{|x + \mathbf{b}|^2 + \lambda} \chi(x + \mathbf{b}) \right), \quad (2.5)$$

where $\chi \in C_c^\infty(\mathbb{R}^d)$ is an axis-symmetric function satisfying the following conditions:

$$0 \leq \chi(x) \leq 1 \quad \text{for all } x \in \mathbb{R}^d \quad \text{and} \quad \text{supp } \chi \subset B_h(\mathbf{0}) \quad \text{with } 0 < h < \min\{k, \widehat{k}\}.$$

Remark 1. By Definition 2.1, the following estimate holds:

$$\|u\|_{L^\infty(\mathbb{R}^d)} \leq 2|Q| \max \left\{ 1, \frac{1}{\lambda} \right\}. \quad (2.6)$$

In addition, since $\chi \in C_c^\infty(\mathbb{R}^d)$, there exists a positive constant $C = C(Q, \lambda)$ such that:

$$\|\nabla \cdot u\|_{L^\infty(\mathbb{R}^d)} \leq C \quad \text{and} \quad \|\partial_i(\nabla \cdot u)\|_{L^\infty(\mathbb{R}^d)} \leq C. \quad (2.7)$$

Moreover, since $\text{supp } \chi(x) \subset B_h(\mathbf{0})$ with $0 < h < \min\{k, \widehat{k}\}$, it follows:

$$\text{supp } \chi(x - \mathbf{b}) \cap \text{supp } \chi(x + \mathbf{b}) = \emptyset.$$

To proceed, we introduce the definition of a strong solution to (KSF).

Definition 2.2. Let $1 \leq p \leq \infty$. We assume that $n_0 \in W^{2,p}(\mathbb{R}^d)$ with $n_0 \geq 0$. A pair of non-negative functions (n, v) on $\mathcal{Q}_T$ is called a strong solution of (KSF) on $[0, T)$ if the following conditions are satisfied:

- (i). $n \in W_p^{2,1}(\mathcal{Q}_T)$,
- (ii). $v \in L^p(0, T; W^{2,p}(\mathbb{R}^d))$,
- (iii). The pair of (n, v) satisfies (KSF) on $(0, T)$.

Next, we provide the definitions of *blow-up time* and *blow-up point*.

Definition 2.3. Let (n, v) be a strong solution of (KSF) on $[0, T)$ in the sense of Definition 2.2.

- (i). **(Blow-up Time)** The strong solution n is said to blow-up at time $T_b < \infty$ if:

$$n(t) \in L^\infty(\mathbb{R}^d) \quad \text{for a.e. } t \in (0, T_b) \quad \text{and} \quad \limsup_{t \rightarrow T_b-0} \|n(t)\|_{L^\infty(\mathbb{R}^d)} = \infty. \quad (2.8)$$

Such a time T_b is called the *blow-up time* of n .

- (ii). **(Blow-up Point)** Let T_b be the blow-up time of n . A point $x_0 \in \mathbb{R}^d$ is called a blow-up point of n at time T_b if for any sequence $\{t_j\}_{j=1}^\infty \subset (0, T_b)$ with $t_j \rightarrow T_b$ as $j \rightarrow \infty$, there exists a sequence $\{x_j\}_{j=1}^\infty \subset \mathbb{R}^d$ with $x_j \rightarrow x_0$, such that:

$$n(x_j, t_j) \rightarrow \infty \quad \text{as } j \rightarrow \infty.$$

Our first results concerning the strong solution are presented in the following theorem:

Theorem 2.1. Let Assumption 2.1 hold. Then, the following assertions hold:

(I). **(Existence of Time Local Solution)**

There exists a positive time $T_1 = T_1(d, \gamma, Q, \lambda, \|n_0\|_{W^{2,d+2}(\mathbb{R}^d)})$ such that (KSF) has a non-negative strong solution (n, v) on $[0, T_1)$ in the sense of Definition 2.2, which is unique in the class $\mathbf{W}(\mathcal{Q}_{T_1})$ with $n \in X_{T_1}$. Moreover, the following estimate holds:

$$\sup_{t \in (0, T_1)} \|n(t)\|_{L^\infty(\mathbb{R}^d)} \leq C (\|n_0\|_{W^{2,d+2}(\mathbb{R}^d)} + 1), \quad (2.9)$$

where $C = C(d)$.

(II). **(Extension Criterion)**

If the strong solution n obtained from Theorem 2.1 (I) satisfies:

$$\sup_{t \in (0, T_0)} \|n(t)\|_{L^\infty(\mathbb{R}^d)} < \infty, \quad (2.10)$$

then, there exists a time $T'_0 > T_0$ such that (n, v) can be extended as a unique strong solution of (KSF) in $\mathbf{W}(\mathcal{Q}_{T'_0})$. Furthermore, if the maximal existence time $T_{\max}$ of the extended strong solution (n, v) is finite, the following holds:

$$\limsup_{t \rightarrow T_{\max} - 0} \|n(t)\|_{L^\infty(\mathbb{R}^d)} = \infty, \quad (2.11)$$

which implies:

$$T_{\max} = T_b. \quad (2.12)$$

Here, T_b is the blow-up time of n defined in Definition 2.3. Furthermore, it holds:

$$n \in C([0, T_{\max}); C(\Omega)) \quad (2.13)$$

for any bounded subset $\Omega \subset \mathbb{R}^d$.

(III). **(Mass Conservation Law)**

In addition, we assume that the initial data n_0 belongs to $L^1(\mathbb{R}^d)$. Let $T_{\max}$ be the maximal existence time of the strong solution (n, v) obtained from Theorem 2.1 (I) and (II). Let $\widehat{T}$ be an arbitrary positive number such that $\widehat{T} \in (0, T_{\max})$. Then, the strong solution n belongs to $C([0, \widehat{T}]; L^1(\mathbb{R}^d))$ and satisfies the mass conservation law:

$$\|n(t)\|_{L^1(\mathbb{R}^d)} = \|n_0\|_{L^1(\mathbb{R}^d)} \quad \text{for all } t \in [0, \widehat{T}]. \quad (2.14)$$

We shall state an ε -regularity theorem for strong solutions of (KSF).

Theorem 2.2. (ε -regularity Theorem) Let Assumption 2.1 hold and suppose that $n_0 \in L^1(\mathbb{R}^2)$. Let $T_{\max}$ be the maximal existence time of the strong solution (n, v) obtained from Theorem 2.1 (I) and (II). There exist an absolute positive constant m_* and a positive constant $c_1 = c_1(\gamma, Q, \lambda, \|n_0\|_{L^1(\mathbb{R}^2)})$ such that if:

$$\int_{B_{2\rho}(x_0)} n(x, t_1) dx \leq m_* \quad (2.15)$$

holds for some $x_0 \in \mathbb{R}^2$, $0 < \rho \leq 1$, and $t_1 \in [0, T_{\max})$, then the following estimates are valid:

(i). In the case $2c_1\rho^2 < T_{\max}$ and $2c_1\rho^2 \leq t_1 < T_{\max}$, the following estimate holds:

$$\sup_{s \in (t_1 - c_1\rho^2, \min\{t_1 + 2c_1\rho^2, T_{\max}\})} \|n(s)\|_{L^\infty(B_{\frac{\rho}{4}}(x_0))} \leq C, \quad (2.16)$$

where $C = C(\gamma, Q, \lambda, \|n_0\|_{L^1(\mathbb{R}^2)}, \rho)$.

(ii). In the case $2c_1\rho^2 < T_{\max}$ and $0 \leq t_1 < \min\{2c_1\rho^2, T_{\max}\} = 2c_1\rho^2$, for any $0 < \varepsilon \leq 1$, the following estimate holds:

$$\sup_{s \in (0, \min\{t_1 + 2c_1\rho^2, T_{\max}\})} s^{1+\frac{\varepsilon}{4}} \|n(s)\|_{L^\infty(B_{\frac{\rho}{4}}(x_0))} \leq C, \quad (2.17)$$

where $C = C(\gamma, Q, \lambda, \|n_0\|_{L^1(\mathbb{R}^2)}, \rho, \varepsilon)$.

(iii). In the case $2c_1\rho^2 \geq T_{\max}$ and $0 \leq t_1 < \min\{2c_1\rho^2, T_{\max}\} = T_{\max}$, provided that $n_0 \in L^2_{\text{loc}}(\mathbb{R}^2)$, the following estimate holds:

$$\sup_{s \in (0, \min\{t_1 + 2c_1\rho^2, T_{\max}\})} \|n(s)\|_{L^\infty(B_{\frac{\rho}{4}}(x_0))} \leq C, \quad (2.18)$$

where $C = C(\gamma, Q, \lambda, \|n_0\|_{L^1(\mathbb{R}^2)}, \|n_0\|_{L^\infty_{\text{loc}}(\mathbb{R}^2)}, \rho)$.

Remark 2. In view of the results concerning the global-in-time existence of solutions with spatially global estimates, it is conjectured that the constant m_* in Theorem 2.2 can be improved to 8π . For detailed statements and proofs, refer to the work by Nagai, Senba, and Yoshida [14].

Theorem 2.2 provides a necessary lower bound for the concentration of n near x_0 , uniformly in time. Furthermore, Theorem 2.2 leads to the following result.

Theorem 2.3. Let Assumption 2.1 hold and suppose that $n_0 \in L^1(\mathbb{R}^2)$. Let $T_{\max}$ denote the maximal existence time of the strong solution (n, v) obtained from Theorem 2.1 (I) and (II). Let m_* be the positive constant provided in Theorem 2.2, and let $x_0 \in \mathbb{R}^2$. Then, the following estimate holds:

$$\sup_{t \in (0, T_b)} \int_{B_r(x_0)} n(x, t) \, dx \geq m_* \quad \text{for all } r > 0. \quad (2.19)$$

Remark 3. By Theorem 2.3, the mass at each blow-up point is at least m_* , as described in (2.19). As mentioned in Remark 2, it is conjectured that m_* can be refined to 8π . A significant aspect of blow-up analysis is determining whether the aggregation mass of a blow-up solution can exceed 8π . This question was investigated by Seki, Sugiyama, and Velázquez [18], who constructed a blow-up solution with an aggregation mass strictly greater than 8π through the method of matched asymptotics.

We construct an initial data such that the corresponding solution exhibits blow-up at two or more distinct points in finite time. The main result is stated as follows:

Theorem 2.4. Let Assumption 2.1 hold and suppose that $n_0 \in L^1(\mathbb{R}^2)$. We suppose $\delta > 0$, $0 < \alpha < 1$, and let $\mathbf{a} := k\mathbf{e}_1 = k(1, 0)$ with $k > 0$. We assume that the initial data n_0 is given by:

$$n_0(x) = \phi(x - \mathbf{a}) + \phi(x + \mathbf{a}) \quad \text{with} \quad \|n_0\|_{L^1(\mathbb{R}^2)} > 16\pi. \quad (2.20)$$

Here, we assume that ϕ satisfies the following properties (i), (ii) and (iii):

- (i). The function ϕ is axis-symmetric with respect to both $\mathbf{e}_1 = (1, 0)$ and $\mathbf{e}_2 = (0, 1)$.
- (ii). The support of ϕ satisfies $\text{supp } \phi = B_\delta(\mathbf{0})$.
- (iii). In addition, ϕ satisfies the following inequality:

$$\int_{\mathbb{R}^2} \phi(x) |x|^2 \, dx$$

$$\begin{aligned}
&< \varepsilon_0 \min \left\{ \frac{1}{\gamma} \cdot \frac{(1-\alpha)N_0^{\frac{3}{2}}\sqrt{\pi}}{512\sqrt{2}(\sqrt{2\pi N_0}+8\pi)\|n_0\|_{L^1(\mathbb{R}^2)}} \left(\log \frac{N_0}{32\pi} \right)^2, \frac{\alpha N_0}{16(\sqrt{2\pi N_0}+8\pi)} \right\} \\
&=: \Phi_0,
\end{aligned} \tag{2.21}$$

where ε_0 and N_0 are defined as follows:

$$\varepsilon_0 := \|n_0\|_{L^1(\mathbb{R}^2)} - 16\pi, \quad \text{and} \quad N_0 := \|n_0\|_{L^1(\mathbb{R}^2)} + 16\pi. \tag{2.22}$$

For the flow field u defined by (2.5) in Definition 2.1, we impose the following condition on the coefficient Q of u :

$$|Q| \leq \frac{\varepsilon_0 N_0}{32\|n_0\|_{L^1(\mathbb{R}^2)}(\sqrt{2\pi N_0}+8\pi)} =: Q_0 \quad \text{if } \mathbf{a} = \mathbf{b}. \tag{2.23}$$

Let T^* be defined by:

$$T^* := \frac{32\pi \int_{\mathbb{R}^2} \phi(x) |x|^2 dx}{\alpha(\|n_0\|_{L^1(\mathbb{R}^2)} + 16\pi)(\sqrt{2\pi(\|n_0\|_{L^1(\mathbb{R}^2)} + 16\pi)} - 8\pi)}. \tag{2.24}$$

Then, there exists a mapping:

$$G = G(\varepsilon_0, \tau) : \mathbb{R}_+ \times \mathbb{R}_+ \longrightarrow \mathbb{R}_+,$$

satisfying the following properties:

- (a). For every fixed $\varepsilon_0 > 0$, $\lim_{\tau \rightarrow 0^+} G(\varepsilon_0, \tau) = 6\delta \left(1 + \frac{1}{\varepsilon_0} \cdot \frac{32}{\pi} \left(\frac{\sqrt{2}}{e} + 1 \right) \|n_0\|_{L^1(\mathbb{R}^2)}^2 \cdot \frac{\sqrt{2\pi N_0} + 8\pi}{N_0} \right)$,
- (b). For every fixed $\varepsilon_0 > 0$, $\lim_{\tau \rightarrow \infty} G(\varepsilon_0, \tau) = \infty$,

such that if the following conditions hold:

$$|\mathbf{a}| = k \geq G(\varepsilon_0, T^*),$$

and:

$$|\mathbf{a} - \mathbf{b}| \geq G(\varepsilon_0, T^*) \left(1 + \frac{1}{\varepsilon_0} \cdot \frac{32}{\pi} \left(\frac{\sqrt{2}}{e} + 1 \right) \|n_0\|_{L^1(\mathbb{R}^2)}^2 \cdot \frac{\sqrt{2\pi N_0} + 8\pi}{N_0} \right)^{-1} + h \quad \text{if } \mathbf{a} \neq \mathbf{b},$$

with a positive number h as defined in Definition 2.1, then the strong solution (n, v) of (KSF) on $[0, T_{\max})$, obtained in Theorem 2.1, satisfies the following properties (I) and (II):

(I). **(Maximal Existence Time Estimate)** The maximal existence time $T_{\max}$ is bounded above by an estimate that depends on the initial data, as follows:

$$T_{\max} < T^* := \frac{32\pi \int_{\mathbb{R}^2} \phi(x)|x|^2 dx}{\alpha(\|n_0\|_{L^1(\mathbb{R}^2)} + 16\pi)(\sqrt{2\pi(\|n_0\|_{L^1(\mathbb{R}^2)} + 16\pi)} - 8\pi)}. \quad (2.25)$$

(II). **(Blow-up Configuration)** The blow-up points of $n(t)$ at the blow-up time T_b are not located at the origin. Instead, these points consist of an even number of points $x_1, \dots, x_m, x_{m+1}, \dots, x_{2m}$. These points are organized into m pairs (x_j, x_{m+j}) for $j = 1, \dots, m$, where each pair x_j and x_{m+j} is symmetric with respect to both the e_1 - and e_2 -axes. Consequently, all blow-up points lie on one of these axes.

Remark 4. The support of $n_0(x) = \phi(x - \mathbf{a}) + \phi(x + \mathbf{a})$ described in Theorem 2.4 is depicted, as follows in Fig. 1.

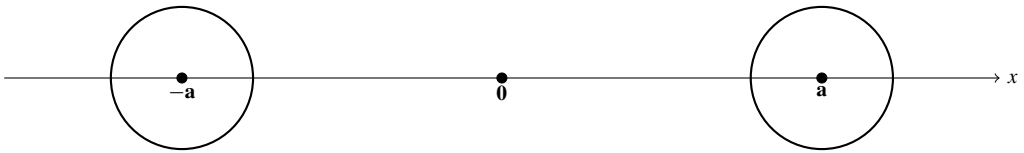


Fig. 1. Solid circles represent $\text{supp } \phi$ centered at $-\mathbf{a} = -k(1, 0)$ and $\mathbf{a} = k(1, 0)$.

Remark 5. For the flow field u defined by (2.5) in Definition 2.1, Theorem 2.4 establishes the existence of solutions exhibiting multiple blow-up behavior in two cases: (i) when $\mathbf{a} = \mathbf{b}$, and (ii) when $|\mathbf{a} - \mathbf{b}| > 2\ell_0 + h$, where ℓ_0 and h are positive constants defined in (2.27) and Definition 2.1, respectively. The second case requires that $|Q|$ is sufficiently small. This paper does not address the case $|\mathbf{a} - \mathbf{b}| \leq 2\ell_0 + h$. Nevertheless, we conjecture that blow-up may also occur in this setting, provided $|Q|$ remains small.

Remark 6. The function $G(\varepsilon_0, T^*)$, whose existence is ensured in Theorem 2.4 (I), can be explicitly defined as follows:

$$G(\varepsilon_0, T^*) := 2\ell_0(\varepsilon_0, T^*) \left(1 + \frac{32}{\pi \varepsilon_0} \left(\frac{\sqrt{2}}{e} + 1 \right) \|n_0\|_{L^1(\mathbb{R}^2)}^2 \cdot \frac{\sqrt{2\pi N_0} + 8\pi}{N_0} \right), \quad (2.26)$$

where ε_0 and N_0 are given in (2.22). In addition, the parameter $\ell_0 = \ell_0(\varepsilon_0, T^*)$ is derived from (2.27) and (2.28). The numbers N_0 , Φ_0 , Q_0 , G , and ℓ_0 satisfy the following properties:

- (1). $\sqrt{2\pi N_0} - 8\pi > 0$ for $\|n_0\|_{L^1(\mathbb{R}^2)} > 16\pi$,
- (2). $\sqrt{2\pi N_0} - 8\pi \rightarrow 0$ as $\|n_0\|_{L^1(\mathbb{R}^2)} \rightarrow 16\pi$,
- (3). $\Phi_0, Q_0 \rightarrow 0$ as $\|n_0\|_{L^1(\mathbb{R}^2)} \rightarrow 16\pi$,
- (4). For every fixed $\tau > 0$, $G(\|n_0\|_{L^1(\mathbb{R}^2)} - 16\pi, \tau) \rightarrow \infty$ as $\|n_0\|_{L^1(\mathbb{R}^2)} \rightarrow 16\pi$,
- (5). For every fixed $\tau > 0$, $G(\|n_0\|_{L^1(\mathbb{R}^2)} - 16\pi, \tau) \rightarrow \infty$ as $\|n_0\|_{L^1(\mathbb{R}^2)} \rightarrow \infty$,
- (6). $\ell_0(\|n_0\|_{L^1(\mathbb{R}^2)} - 16\pi, T^*) \rightarrow \infty$ as $\|n_0\|_{L^1(\mathbb{R}^2)} \rightarrow 16\pi$,

(7). $C_* \rightarrow \infty$ as $\|n_0\|_{L^1(\mathbb{R}^2)} \rightarrow 16\pi$, and $C_* \rightarrow \infty$ as $\|n_0\|_{L^1(\mathbb{R}^2)} \rightarrow \infty$.

Here, the parameter $\ell_0 = \ell_0(\varepsilon_0, T^*)$ is explicitly given by:

$$\begin{aligned} \ell_0(\varepsilon_0, T^*) := \max \left\{ 3\delta, 4\sqrt{\frac{6\|n_0\|_{L^1(\mathbb{R}^2)}\left(2 + \frac{1}{\pi}\left(\frac{\sqrt{2}}{e} + 1\right)\|n_0\|_{L^1(\mathbb{R}^2)}\right)T^*}{\varepsilon_0}}, \right. \\ \frac{16\sqrt{2}\|n_0\|_{L^1(\mathbb{R}^2)}|Q|\max\left\{1, \frac{1}{\lambda}\right\}T^*}{\varepsilon_0}, \\ 2\delta + 32\sqrt{\frac{3\pi C_*\|n_0\|_{L^1(\mathbb{R}^2)}\left(2 + \frac{1}{\pi}\left(\frac{\sqrt{2}}{e} + 1\right)\|n_0\|_{L^1(\mathbb{R}^2)}\right)T^*}{N_0(\sqrt{2\pi N_0} - 8\pi)}}, \\ 2\delta + \frac{256\sqrt{2}\pi C_*\|n_0\|_{L^1(\mathbb{R}^2)}|Q|\max\left\{1, \frac{1}{\lambda}\right\}T^*}{N_0(\sqrt{2\pi N_0} - 8\pi)}, \\ \delta + 4\sqrt{\frac{6\|n_0\|_{L^1(\mathbb{R}^2)}\left(2 + \frac{1}{\pi}\left(\frac{\sqrt{2}}{e} + 1\right)\|n_0\|_{L^1(\mathbb{R}^2)}\right)T^*}{m_*}}, \\ \left. \delta + \frac{8\sqrt{2}\|n_0\|_{L^1(\mathbb{R}^2)}|Q|\max\left\{1, \frac{1}{\lambda}\right\}T^*}{m_*} \right\} \end{aligned} \quad (2.27)$$

with m_* obtained in Theorem 2.2, where C_* is given by:

$$\begin{aligned} C_* := 16(\sqrt{2} + 6) + 4\sqrt{2}|Q|\tau_{2\ell_0, h}(\mathbf{a}, \mathbf{b}) \\ + \frac{1}{\pi}\left(\frac{\sqrt{2}}{e} + 1\right)\|n_0\|_{L^1(\mathbb{R}^2)} \left[8\sqrt{2}\left(2 + \frac{N_0(\sqrt{2\pi N_0} - 8\pi)}{128\left(\frac{\sqrt{2}}{e} + 1\right)\|n_0\|_{L^1(\mathbb{R}^2)}^2}\right) \right. \\ \left. + 48\left(1 + \frac{64\left(\frac{\sqrt{2}}{e} + 1\right)\|n_0\|_{L^1(\mathbb{R}^2)}^2}{N_0(\sqrt{2\pi N_0} - 8\pi)}\right)^2 + 8\sqrt{2}\left(1 + \frac{64\left(\frac{\sqrt{2}}{e} + 1\right)\|n_0\|_{L^1(\mathbb{R}^2)}^2}{N_0(\sqrt{2\pi N_0} - 8\pi)}\right) + 3 \right]. \end{aligned} \quad (2.28)$$

The quantities $G(\varepsilon_0, T^*)$, ℓ_0 , and C_* have the following roles:

- $G(\varepsilon_0, T^*)$: quantifies the separation condition between peaks and ensures that blow-up points remain well separated.
- ℓ_0 : defined in terms of $\|n_0\|_{L^1(\mathbb{R}^2)}$, $|Q|$, ε_0 , and T^* , serves as a local threshold for applying the ε -regularity theorem; in particular, a large value of ℓ_0 prevents excessive concentration at the origin.
- C_* : used in the definition of ℓ_0 as an auxiliary constant, consolidating several constants arising in the proofs and making their parameter dependence explicit.

Remark 7. The term $\sqrt{2\pi N_0} - 8\pi$, which appears in the definitions of ℓ_0 and C_* , is directly related to ε_0 as defined in (2.22). In fact, the following equality holds:

$$(\sqrt{2\pi N_0} - 8\pi)(\sqrt{2\pi N_0} + 8\pi) = 2\pi N_0 - 64\pi^2 = 2\pi(\|n_0\|_{L^1(\mathbb{R}^d)} - 16\pi) = 2\pi \varepsilon_0.$$

Remark 8. Theorem 2.4 can be proved in a more straightforward manner when $\gamma = 0$. In this paper, however, we address the case $\gamma > 0$, which requires additional techniques for the proof.

The following theorem establishes that the number of blow-up points of n at the blow-up time T_b is finite. It provides a quantitative bound on the number of blow-up points in terms of the initial mass of n and a constant m_* determined by Theorem 2.2.

Theorem 2.5. (Finiteness of the Blow-up Points) *Let Assumption 2.1 hold and suppose that $n_0 \in L^1(\mathbb{R}^2)$. Let $T_{\max}$ denote the maximal existence time of the strong solution (n, v) obtained from Theorem 2.1 (I) and (II). Then, the number k of the blow-up points of n of (KSF) at the blow-up time $T_b (= T_{\max})$ is finite. More precisely, k is bounded by:*

$$k \leq \frac{\|n_0\|_{L^1(\mathbb{R}^2)}}{m_*},$$

where m_* is the same constant given by Theorem 2.2.

Remark 9. Theorem 2.5 states that if the initial data is radially symmetric, then the solution must blow up exclusively at the origin. In contrast, our result addresses the non-radial case and provides an example where the solution does not necessarily blow up at the origin.

Next, we present a definition that characterizes the formation of a δ -function singularity at the blow-up points of the solution $n(x, t)$. It describes how $n(x, t)$ converges, as t approaches the blow-up time T_b , to a combination of δ -functions centered at the blow-up points, each associated with a specified mass.

Definition 2.4. Let T_b be the blow-up time of a strong solution n of (KSF) as defined in Definition 2.3. Let $\{x_i\}_{i=1}^k$ denote the blow-up points of n for (KSF) at time T_b . We say that n forms a δ -function singularity at $\{x_i\}_{i=1}^k$ and at time T_b with masses $\{M_i\}_{i=1}^k$ if the following property holds: there exists a function $f \in L^1(\mathbb{R}^2)$ and a sequence $\{t_j\}_{j=1}^\infty \subset (0, T_b)$ with $\lim_{j \rightarrow \infty} t_j = T_b$ such that:

$$n(\cdot, t_j) \rightarrow \sum_{i=1}^k M_i \delta_{x_i}(\cdot) + f(\cdot) \quad \text{as } j \rightarrow \infty \text{ in the sense of distributions in } \mathbb{R}^2,$$

i.e.,

$$\lim_{j \rightarrow \infty} \int_{\mathbb{R}^2} n(x, t_j) \psi(x) \, dx = \sum_{i=1}^k M_i \psi(x_i) + \int_{\mathbb{R}^2} f(x) \psi(x) \, dx$$

for all $\psi \in C_0^\infty(\mathbb{R}^2)$.

The following Theorem establishes that at the blow-up time $T_b = T_{\max}$, the solution $n(x, t)$ develops a δ -function singularity at its blow-up points. Each of these singularities is associated with a mass that is bounded below by a constant m_* , which is determined by Theorem 2.3.

Theorem 2.6. (δ -function Singularity) *Let Assumption 2.1 hold and suppose that $n_0 \in L^1(\mathbb{R}^2)$. Let $T_{\max}$ denote the maximal existence time of the strong solution (n, v) obtained from Theorem 2.1 (I) and (II). Let $\{x_i\}_{i=1}^k$ be the blow-up points of n at time $T_b (= T_{\max})$. We suppose that m_* is the constant provided by Theorem 2.3. Then, there exist k constants $M_i \geq m_*$ ($1 \leq i \leq k$) such that n forms the δ -function singularity at $\{x_i\}_{i=1}^k$ and at time $T_b (= T_{\max})$ with masses $\{M_i\}_{i=1}^k$.*

3. Preliminaries

In this section, we introduce notation and preliminary results to support the analysis developed in subsequent sections.

We begin by defining the Bessel potential operator. Let $\gamma > 0$ and $\alpha > 0$. For $f \in L^p(\mathbb{R}^d)$ with $1 \leq p \leq \infty$, the operator $(\gamma - \Delta)^{-\frac{\alpha}{2}}$ is defined as:

$$(\gamma - \Delta)^{-\frac{\alpha}{2}} f := \mathcal{F}^{-1} \left[(\gamma + 4\pi^2 |\xi|^2)^{-\frac{\alpha}{2}} \hat{f} \right], \quad (3.1)$$

where $\mathcal{F}^{-1}$ denotes the inverse Fourier transform and $\hat{f}$ is the Fourier transform of f .

The corresponding Bessel kernel $G_{\gamma, \alpha}$ is given by:

$$G_{\gamma, \alpha}(x) := \gamma^{\frac{d-\alpha}{2}} a_d e^{-\sqrt{\gamma}|x|} \int_0^\infty e^{-\sqrt{\gamma}|x|s} \left(s + \frac{s^2}{2} \right)^{\frac{d-\alpha-1}{2}} ds, \quad (3.2)$$

where the constant a_d is defined as:

$$a_d = \frac{1}{2(2\pi)^{\frac{d-1}{2}} \Gamma\left(\frac{d-1}{2}\right)},$$

and Γ is the gamma function.

Remark 10. The Bessel kernel $G_{\gamma, \alpha}$ has the following properties:

- (i). For $\alpha > 0$, the function $G_{\gamma, \alpha}(x)$ is smooth for all $x \in \mathbb{R}^d \setminus \{0\}$.
- (ii). The kernel satisfies the scaling property:

$$G_{\gamma, \alpha}(x) = \gamma^{\frac{d-\alpha}{2}} G_{1, \alpha}(\sqrt{\gamma}x),$$

where $G_{1, \alpha}$ corresponds to the case $\gamma = 1$.

- (iii). The Fourier transform of $G_{\gamma, \alpha}$ is:

$$\widehat{G_{\gamma, \alpha}}(\xi) = (\gamma + 4\pi^2 |\xi|^2)^{-\frac{\alpha}{2}}. \quad (3.3)$$

(iv). The $L^1(\mathbb{R}^d)$ -norm of $G_{\gamma,\alpha}$ satisfies:

$$\|G_{\gamma,\alpha}\|_{L^1(\mathbb{R}^d)} = \gamma^{-\frac{\alpha}{2}}.$$

In particular, for $\gamma = 1$, we have $\|G_{1,\alpha}\|_{L^1(\mathbb{R}^d)} = 1$. This follows from the Fourier transform property (3.3), as:

$$\int_{\mathbb{R}^d} G_{\gamma,\alpha}(x) dx = \widehat{G_{\gamma,\alpha}}(0) = \gamma^{-\frac{\alpha}{2}}.$$

(v). The operator $(\gamma - \Delta)^{-\frac{\alpha}{2}}$ can be represented in terms of convolution with the Bessel kernel:

$$(\gamma - \Delta)^{-\frac{\alpha}{2}} f = \mathcal{F}^{-1}[\widehat{G_{\gamma,\alpha}} \hat{f}] = G_{\gamma,\alpha} * f. \quad (3.4)$$

For further details, see Grafakos [5, p.13] or Stein [21, p.132].

When $\alpha = 2$ in (3.2), we denote $G_{\gamma,2}$ by G_γ . The following proposition provides estimates for the inner product between the gradient of the Bessel kernel, ∇G_γ , and x . Specifically, we establish upper bounds for $x \cdot \nabla G_\gamma(x)$ and $|x| |\nabla G_\gamma(x)|$ that hold uniformly for all $\gamma > 0$.

Lemma 3.1. *Let $d = 2$ and $\gamma > 0$. Then, the following estimates hold:*

(i). *For all $x \in \mathbb{R}^2$, the following estimate holds:*

$$x \cdot \nabla G_\gamma(x) \leq -\frac{1}{2\pi} e^{-\sqrt{\gamma}|x|} < 0. \quad (3.5)$$

(ii). *For all $x \in \mathbb{R}^2$, the following estimate holds:*

$$|x| |\nabla G_\gamma(x)| \leq \frac{1}{\pi} \left(\frac{\sqrt{2}}{e} + 1 \right) < \frac{1}{2}. \quad (3.6)$$

Proof of Lemma 3.1. For the estimate (3.5), see Kozono-Sugiyama [10, p.368, Lemma 3.1]. To establish (3.6), we start from the representation in (3.2):

$$\nabla G_\gamma(x) = -\frac{\sqrt{\gamma}}{2\sqrt{2}\pi} \cdot \frac{x}{|x|} e^{-\sqrt{\gamma}|x|} \int_0^\infty e^{-\sqrt{\gamma}|x|s} (1+s) \left(s + \frac{s^2}{2} \right)^{-\frac{1}{2}} ds,$$

which leads to:

$$\begin{aligned} |\nabla G_\gamma(x)| &= \frac{\sqrt{\gamma}}{2\sqrt{2}\pi} e^{-\sqrt{\gamma}|x|} \int_0^\infty e^{-\sqrt{\gamma}|x|s} (1+s) \left(s + \frac{s^2}{2} \right)^{-\frac{1}{2}} ds \\ &= \frac{\sqrt{\gamma}}{2\sqrt{2}\pi} e^{-\sqrt{\gamma}|x|} (I_1 + I_2), \end{aligned} \quad (3.7)$$

where the terms I_1 and I_2 are defined as:

$$I_1 := \int_0^1 e^{-\sqrt{\gamma}|x|s} (1+s) \left(s + \frac{s^2}{2}\right)^{-\frac{1}{2}} ds, \quad I_2 := \int_1^\infty e^{-\sqrt{\gamma}|x|s} (1+s) \left(s + \frac{s^2}{2}\right)^{-\frac{1}{2}} ds.$$

Considering I_1 , we obtain:

$$I_1 \leq \int_0^1 2s^{-\frac{1}{2}} ds = 4. \quad (3.8)$$

For I_2 , using the inequality:

$$(1+s) \left(s + \frac{s^2}{2}\right)^{-\frac{1}{2}} \leq 2s \cdot \frac{\sqrt{2}}{s} = 2\sqrt{2} \quad \text{for all } s > 0,$$

we deduce:

$$I_2 \leq 2\sqrt{2} \int_1^\infty e^{-\sqrt{\gamma}|x|s} ds = \frac{2\sqrt{2}e^{-\sqrt{\gamma}|x|}}{\sqrt{\gamma}|x|}. \quad (3.9)$$

Combining (3.7) through (3.9), it follows that:

$$|\nabla G_\gamma(x)| \leq \frac{\sqrt{\gamma}}{\sqrt{2}\pi} e^{-\sqrt{\gamma}|x|} \left(2 + \frac{\sqrt{2}e^{-\sqrt{\gamma}|x|}}{\sqrt{\gamma}|x|}\right),$$

which further implies:

$$|x| |\nabla G_\gamma(x)| \leq \frac{\sqrt{2\gamma}}{\pi} e^{-\sqrt{\gamma}|x|} |x| + \frac{1}{\pi} e^{-2\sqrt{\gamma}|x|}.$$

Since $e^{-\sqrt{\gamma}|x|} |x| \leq \frac{1}{\sqrt{\gamma}e}$ for all $x \in \mathbb{R}^2$, it follows that:

$$|x| |\nabla G_\gamma(x)| \leq \frac{\sqrt{2\gamma}}{\pi} \cdot \frac{1}{\sqrt{\gamma}e} + \frac{1}{\pi} = \frac{1}{\pi} \left(\frac{\sqrt{2}}{e} + 1\right).$$

This completes the proof of Lemma 3.1. $\square$

In the following lemma, we establish a representation formula for elliptic equations of Poisson type. Specifically, we show that for a function $f \in L^p(\mathbb{R}^d)$, the solution z , defined via convolution, lies in the Sobolev space $W^{2,p}(\mathbb{R}^d)$. For further details, refer to Grafakos [5, Chapter 1, Sec. 1.2.2], among others.

Lemma 3.2. Let $d \geq 1$, and let $\gamma > 0$ and $1 \leq p \leq \infty$. We define the function z as:

$$z(x) = \int_{\mathbb{R}^d} G_\gamma(x-y)f(y) dy \quad \text{for } f \in L^p(\mathbb{R}^d). \quad (3.10)$$

Then:

$$z \in W^{2,p}(\mathbb{R}^d) \quad (3.11)$$

and satisfies the elliptic equation:

$$-\Delta z + \gamma z = f \quad \text{in } \mathbb{R}^d. \quad (3.12)$$

The function z defined by (3.10) is the unique solution of (3.12) in $W^{2,p}(\mathbb{R}^d)$, where $f \in L^p(\mathbb{R}^d)$ for $1 < p < \infty$.

In addition, let $1 < p < \infty$, and we suppose that $f \in W^{2,p}(\mathbb{R}^d)$. Then, the following estimate holds:

$$\|z\|_{W^{4,p}(\mathbb{R}^d)} \leq C \|f\|_{W^{2,p}(\mathbb{R}^d)}, \quad (3.13)$$

where $C = C(d, \gamma, p)$.

Proof of Lemma 3.2. The solution z provided by (3.10) is addressed in Grafakos [5, Chapter 1, Sec. 1.2.2]. To demonstrate the uniqueness of the solution, we proceed as follows. Let z_1 be a solution of (3.12) distinct from z . Then, the following equation holds:

$$\Delta(z - z_1) - \gamma(z - z_1) = 0. \quad (3.14)$$

For $1 < p < \infty$, multiplying both sides of (3.14) by $|z - z_1|^{p-2}(z - z_1)$ and integrating over $\mathbb{R}^d$, we obtain:

$$\frac{4(p-1)}{p^2} \|\nabla(z - z_1)\|^{\frac{p}{2}}_{L^p} + \gamma \|z - z_1\|_{L^p} = 0.$$

This implies $z(x) = z_1(x)$ for a.e. $x \in \mathbb{R}^d$.

To prove (3.13), we assume $f \in W^{2,p}(\mathbb{R}^d)$ and define the function z by (3.10). The second derivatives of z are given by:

$$\partial_{ij}^2 z(x) = \int_{\mathbb{R}^d} G_\gamma(x-y) \partial_{ij}^2 f(y) dy \quad \text{for } i, j = 1, 2, \dots, d, \quad (3.15)$$

where $\partial_{ij}^2 f \in L^p(\mathbb{R}^d)$. From (3.11) and (3.12) in Lemma 3.2, it follows that $\partial_{ij}^2 z \in W^{2,p}(\mathbb{R}^d)$, and z satisfies:

$$-\Delta \partial_{ij}^2 z + \gamma \partial_{ij}^2 z = \partial_{ij}^2 f \quad \text{in } \mathbb{R}^d. \quad (3.16)$$

This implies that the unique solution $Z \in W^{2,p}(\mathbb{R}^d)$ of the equation:

$$-\Delta Z + \gamma Z = \partial_{ij}^2 f \quad \text{in } \mathbb{R}^d$$

is explicitly given by:

$$Z(x) := \int_{\mathbb{R}^d} G_\gamma(x-y) \partial_{ij}^2 f(y) dy = \partial_{ij}^2 z(x),$$

where $\partial_{ij}^2 f \in L^p(\mathbb{R}^d)$. This yields by (3.10), (3.12) and (3.16):

$$\begin{aligned} \|z\|_{W^{4,p}} &\leq C \sum_{i,j=1}^d \|\Delta \partial_{ij}^2 z\|_{L^p} \leq C \sum_{i,j=1}^d \left(\gamma \|\partial_{ij}^2 z\|_{L^p} + \|\partial_{ij}^2 f\|_{L^p} \right) \\ &\leq C (\|z\|_{W^{2,p}} + \|f\|_{W^{2,p}}) \leq C (\|z\|_{L^p} + \|\Delta z\|_{L^p} + \|f\|_{W^{2,p}}) \leq C \|f\|_{W^{2,p}} \end{aligned}$$

for all $1 < p < \infty$, where $C = C(d, \gamma, p)$. $\square$

The following lemma establishes an $L^p(\mathbb{R}^d)$ -estimate for the gradient of the potential z , defined by (3.10). This result highlights the boundedness of ∇z in terms of the $L^p(\mathbb{R}^d)$ -norm of the source term f . It plays a crucial role in the analysis of regularity properties for solutions to elliptic equations.

Lemma 3.3. *Let $d \geq 1$, and let z be defined by (3.10). We suppose $f \in L^p(\mathbb{R}^d)$ with $1 \leq p \leq \infty$. Then, the following estimate holds:*

$$\|\nabla z\|_{L^p(\mathbb{R}^d)} \leq C \|f\|_{L^p(\mathbb{R}^d)}, \quad (3.17)$$

where $C = C(\gamma, p)$.

Proof of Theorem 3.3. Since $\nabla G_\gamma \in L^1(\mathbb{R}^d)$, it follows from the Young inequality:

$$\|\nabla z\|_{L^p} \leq \|\nabla G_\gamma\|_{L^1} \|f\|_{L^p} \leq C \|f\|_{L^p},$$

where $C = C(\gamma, p)$. This completes the proof of Lemma 3.3. $\square$

The following lemma provides a local $L^p(\mathbb{R}^2)$ -estimate for the function z given by (3.10), derived from potential theory.

Lemma 3.4. *Let $d = 2$ and let $2 \leq p < \infty$. Let $\rho_1, \rho_2 > 0$ with $\rho_2 > \rho_1$. For $x_0 \in \mathbb{R}^2$ and a function $f \in L^1(\mathbb{R}^2) \cap L^2(B_{\rho_2}(x_0))$, we define z as in (3.10). Then, there exists a positive constant $C = C(\gamma, p)$ such that:*

$$\|\nabla z\|_{L^p(B_{\rho_1}(x_0))} \leq C \left(\frac{\rho_1^{\frac{2}{p}}}{\rho_2 - \rho_1} \|f\|_{L^1(\mathbb{R}^2)} + \|f\|_{L^2(B_{\rho_2}(x_0))} \right). \quad (3.18)$$

Proof of Lemma 3.4. We decompose the function z given by (3.10) as follows:

$$z(x) = z_1(x) + z_2(x) \quad \text{for } x \in \mathbb{R}^2, \quad (3.19)$$

with:

$$\begin{aligned} z_1(x) &:= \int_{\mathbb{R}^2} G_\gamma(x-y) f(y) \chi_{B_{\rho_2}(x_0)}(y) dy, \\ z_2(x) &:= \int_{\mathbb{R}^2} G_\gamma(x-y) (f(y) - f(y) \chi_{B_{\rho_2}(x_0)}(y)) dy \end{aligned}$$

for $x \in \mathbb{R}^2$, where $\chi_{B_{\rho_2}(x_0)}$ is the characteristic function of $B_{\rho_2}(x_0)$. By Lemma 3.2, we have:

$$-\Delta z_1 + \gamma z_1 = f \chi_{B_{\rho_2}(x_0)} \quad \text{in } \mathbb{R}^2.$$

By the Sobolev inequality and the classical $L^2(\mathbb{R}^2)$ -estimate, we obtain for $2 \leq p < \infty$:

$$\|z_1\|_{W^{1,p}(B_{\rho_1}(x_0))} \leq C \|z_1\|_{W^{2,2}(\mathbb{R}^2)} \leq C \|f\|_{L^2(B_{\rho_2}(x_0))}, \quad (3.20)$$

where $C = C(\gamma, p)$.

As for the function z_2 , we have:

$$\nabla z_2(x) = \int_{\mathbb{R}^2} \nabla_x G_\gamma(x-y) (f(y) - f(y) \chi_{B_{\rho_2}(x_0)}(y)) dy. \quad (3.21)$$

From (3.6) in Lemma 3.1, it follows:

$$|\nabla G_\gamma(x)| \leq \frac{1}{\pi|x|} \left(\frac{\sqrt{2}}{e} + 1 \right) \quad \text{for all } x \in \mathbb{R}^2 \text{ with } x \neq 0.$$

In particular, for $x \in B_{\rho_1}(x_0)$ and $y \in \mathbb{R}^2 \setminus B_{\rho_2}(x_0)$, we have:

$$|\nabla G_\gamma(x-y)| \leq \frac{1}{\rho_2 - \rho_1} \cdot \frac{1}{\pi} \left(\frac{\sqrt{2}}{e} + 1 \right).$$

Combining this estimate with (3.21), we deduce:

$$\|\nabla z_2\|_{L^\infty(B_{\rho_1}(x_0))} \leq \frac{1}{\rho_2 - \rho_1} \cdot \frac{1}{\pi} \left(\frac{\sqrt{2}}{e} + 1 \right) \|f\|_{L^1(\mathbb{R}^2)},$$

and hence:

$$\|\nabla z_2\|_{L^p(B_{\rho_1}(x_0))} \leq |B_{\rho_1}(x_0)|^{\frac{1}{p}} \|\nabla z_2\|_{L^\infty(B_{\rho_1}(x_0))} \leq \frac{C\rho_1^{\frac{2}{p}}}{\rho_2 - \rho_1} \|f\|_{L^1(\mathbb{R}^2)}, \quad (3.22)$$

where $C = C(p)$. Combining (3.20) and (3.22), we obtain the desired estimate (3.18). $\square$

The operator $(\gamma - \Delta)^{-\frac{\alpha}{2}}$ exhibits mapping properties analogous to those of classical fractional integrals, such as the Riesz potentials. In particular, it sends $L^p(\mathbb{R}^d)$ functions to spaces of higher integrability and admits corresponding weak-type estimates. For details, see Grafakos [5, Chapter 1, Corollary 1.2.6]

Lemma 3.5.

- (i). Let $0 < \alpha < d$, and let $1 \leq p < q < \infty$ satisfy $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{d}$. Then, there exist constants $C = C(d, p, q, \alpha)$ such that:

$$\|(\gamma - \Delta)^{-\frac{\alpha}{2}} f\|_{L^q(\mathbb{R}^d)} \leq C \|f\|_{L^p(\mathbb{R}^d)} \quad \text{for all } f \in L^p(\mathbb{R}^d) \text{ with } p > 1,$$

and:

$$\|(\gamma - \Delta)^{-\frac{\alpha}{2}} f\|_{L^{q,\infty}(\mathbb{R}^d)} \leq C \|f\|_{L^1(\mathbb{R}^d)} \quad \text{for all } f \in L^1(\mathbb{R}^d).$$

- (ii). Let $\alpha > 0$. Then the operator $(\gamma - \Delta)^{-\frac{\alpha}{2}}$ is bounded from $L^r(\mathbb{R}^d)$ to itself for all $1 \leq r \leq \infty$.

The following lemma introduces a cut-off function based on a quadratic polynomial, with estimates on its derivatives.

Lemma 3.6. Let $d \geq 1$, and let $\rho_1, \rho_2 > 0$ with $\rho_2 > \rho_1$. We define $\tilde{\psi}$ by:

$$\tilde{\psi}(r) = \begin{cases} 1 & \text{if } 0 \leq r < \rho_1, \\ 1 - \frac{2}{(\rho_2 - \rho_1)^2} (r - \rho_1)^2 & \text{if } \rho_1 \leq r < \frac{\rho_1 + \rho_2}{2}, \\ \frac{2}{(\rho_2 - \rho_1)^2} (r - \rho_2)^2 & \text{if } \frac{\rho_1 + \rho_2}{2} \leq r < \rho_2, \\ 0 & \text{if } \rho_2 \leq r. \end{cases}$$

Set $\psi(x) := \tilde{\psi}(|x|)$ for $x \in \mathbb{R}^d$. The function ψ satisfies the following derivative estimates:

$$|\partial_j \psi(x)| \leq \frac{2}{\rho_2 - \rho_1} \quad \text{and} \quad |\partial_i \partial_j \psi(x)| \leq \frac{12}{(\rho_2 - \rho_1)^2} \quad (3.23)$$

for all $i, j = 1, 2, \dots, d$ and all $x \in \mathbb{R}^d$. Consequently:

$$|\nabla \psi(x)| \leq \frac{2\sqrt{d}}{\rho_2 - \rho_1} \quad \text{and} \quad |\Delta \psi(x)| \leq \frac{12d}{(\rho_2 - \rho_1)^2} \quad \text{for all } x \in \mathbb{R}^d. \quad (3.24)$$

Furthermore, there exists an absolute positive constant C such that:

$$|\nabla \psi(x)| \leq \frac{C}{\rho_2 - \rho_1} \psi(x)^{\frac{1}{2}} \quad \text{for all } x \in \mathbb{R}^d. \quad (3.25)$$

In addition, for any $1 \leq p \leq \infty$, there exists a positive constant $C = C(d, p)$ such that:

$$\|\nabla \psi\|_{L^p(\mathbb{R}^d)} \leq C(\rho_2 - \rho_1)^{-1+\frac{d}{p}} \quad \text{and} \quad \|\Delta \psi\|_{L^p(\mathbb{R}^d)} \leq C(\rho_2 - \rho_1)^{-2+\frac{d}{p}}. \quad (3.26)$$

The following lemma provides a local mass estimate for (KSF), giving an inequality that links the rate of localized mass change to the initial mass $\|n_0\|_{L^1(\mathbb{R}^2)}$, the spatial scales ρ_1 and ρ_2 , and other parameters. This quantifies how the mass distribution evolves over time within a prescribed region.

Lemma 3.7. *Let $d = 2$ and let Assumption 2.1 hold. Let $\rho_1, \rho_2 > 0$ with $\rho_2 > \rho_1$. We suppose that ψ be the cut-off function from Lemma 3.6 with ρ_1 and ρ_2 . In addition, we suppose that the function u is defined in Definition 2.1. Let (n, v) denote the strong solution of (KSF) on $[0, T)$ as obtained in Theorem 2.1 and let $\mathbf{p} \in \mathbb{R}^2$. Then the following estimate holds:*

$$\left| \frac{d}{dt} \int_{\mathbb{R}^2} n(x, t) \psi(x - \mathbf{p}) \, dx \right| \leq 2\|n_0\|_{L^1(\mathbb{R}^2)} \left(\frac{6\left(2 + \frac{1}{\pi}\left(\frac{\sqrt{2}}{e} + 1\right)\|n_0\|_{L^1(\mathbb{R}^2)}\right)}{(\rho_2 - \rho_1)^2} + \frac{\sqrt{2}|Q| \max\left\{1, \frac{1}{\lambda}\right\} \tau_{\rho_2, h}(\mathbf{p}, \mathbf{b})}{\rho_2 - \rho_1} \right), \quad (3.27)$$

where $\tau_{\rho_2, h}(\mathbf{p}, \mathbf{b})$ is defined in (2.4).

Proof of Lemma 3.7. Multiplying both sides of the first equation of (KSF) by $\psi(x - \mathbf{p})$ and integrating over $\mathbb{R}^2$, we obtain:

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}^2} n(x, t) \psi(x - \mathbf{p}) \, dx &= \int_{\mathbb{R}^2} n(x, t) \Delta \psi(x - \mathbf{p}) \, dx + \int_{\mathbb{R}^2} (n \nabla v)(x, t) \cdot \nabla \psi(x - \mathbf{p}) \, dx \\ &\quad + \int_{\mathbb{R}^2} u(x) n(x, t) \cdot \nabla \psi(x - \mathbf{p}) \, dx \\ &=: I + J + K. \end{aligned} \quad (3.28)$$

Regarding the integral I , we use (3.24) to obtain:

$$I \leq \frac{24}{(\rho_2 - \rho_1)^2} \int_{\mathbb{R}^2} n(x, t) \, dx = \frac{24\|n_0\|_{L^1}}{(\rho_2 - \rho_1)^2}. \quad (3.29)$$

Next, we consider J . Since the Bessel kernel G_γ is symmetric, we have:

$$\nabla_x G_\gamma(x - y) = -\nabla_y G_\gamma(x - y) \quad \text{for all } x, y \in \mathbb{R}^2 \text{ with } x \neq y.$$

This implies:

$$\begin{aligned} J &= \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} n(x, t) n(y, t) \nabla G_\gamma(x - y) \cdot \nabla \psi(x - \mathbf{p}) \, dx dy \\ &= \frac{1}{2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} n(x, t) n(y, t) \nabla G_\gamma(x - y) \cdot \nabla \psi(x - \mathbf{p}) \, dx dy \\ &\quad + \frac{1}{2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} n(y, t) n(x, t) \nabla G_\gamma(y - x) \cdot \nabla \psi(y - \mathbf{p}) \, dy dx \\ &= \frac{1}{2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} n(x, t) n(y, t) \nabla G_\gamma(x - y) \cdot (\nabla \psi(x - \mathbf{p}) - \nabla \psi(y - \mathbf{p})) \, dx dy, \end{aligned} \quad (3.30)$$

since $\nabla v = \nabla G_\gamma * n$.

By applying (3.6) in Lemma 3.1, (3.24) in Lemma 3.6, and the fundamental theorem of calculus, we have:

$$|J| \leq \frac{1}{2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} n(x, t) n(y, t) |\nabla G_\gamma(x - y)| \frac{24}{(\rho_2 - \rho_1)^2} |x - y| \, dx dy \leq \frac{\frac{12}{\pi} \left(\frac{\sqrt{2}}{e} + 1 \right) \|n_0\|_{L^1}^2}{(\rho_2 - \rho_1)^2}. \quad (3.31)$$

Considering K , by (2.6), we have:

$$\begin{aligned} |K| &\leq \int_{\mathbb{R}^2} |Q| \left(\frac{|x - \mathbf{b}|}{|x - \mathbf{b}|^2 + \lambda} \chi(x - \mathbf{b}) + \frac{|x + \mathbf{b}|}{|x + \mathbf{b}|^2 + \lambda} \chi(x + \mathbf{b}) \right) n(x, t) |\nabla \psi(x - \mathbf{p})| \, dx \\ &\leq \frac{2\sqrt{2} \|n_0\|_{L^1} |Q| \max \left\{ 1, \frac{1}{\lambda} \right\} \tau_{\rho_2, h}(\mathbf{p}, \mathbf{b})}{\rho_2 - \rho_1}, \end{aligned} \quad (3.32)$$

where $\tau_{\rho_2, h}(\mathbf{p}, \mathbf{b})$ is defined in (2.4). Combining (3.28), (3.29), (3.31) and (3.32), we obtain (3.27). This completes the proof of Lemma 3.7. $\square$

4. Proof of Theorem 2.1

4.1. Proof of Theorem 2.1 (I): existence of time local solution

In this subsection, we construct a time local solution of (KSF). To this end, we introduce the metric on X_T by:

$$\mathcal{D}(n_1, n_2) := \|n_1 - n_2\|_{L^\infty(0, T; L^{d+2}(\mathbb{R}^d))}. \quad (4.1)$$

Then, it is readily seen that $(X_T, \mathcal{D})$ is the complete metric space.

To prove Theorem 2.1 (I), we consider the following modified problem $^h(\text{KSF})$, derived from (KSF). In this modified problem, (KSF) is transformed into a set of independent equations by replacing the unknown functions in the advection and reaction terms with given functions:

$$^h(\text{KSF}) \quad \begin{cases} \frac{\partial n}{\partial t} + \nabla \cdot (un) = \Delta n - \nabla \cdot (n \nabla h) & \text{in } \mathbb{R}^d \times (0, T), & (1)_h \\ 0 = \Delta v - \gamma v + f & \text{in } \mathbb{R}^d \times (0, T), & (2)_f \\ n(x, 0) = n_0(x) & \text{in } \mathbb{R}^d, \end{cases}$$

where $h \in L^\infty(0, T; W^{2,\infty}(\mathbb{R}^d))$ and $0 \leq f \in L^r(Q_T)$ with some $1 < r < \infty$.

We remark that generalizing Theorem 9.1 in Ladyzhenskaya, Solonnikov, and Ural'tseva [11] to the Cauchy problem is straightforward. A more modern approach, such as the maximal regularity theorem in L^p , can be found in Amann [1, Chapter IV, Theorem 1.5.1]. By virtue of [1, Theorem 1.5.1] or [11, Theorem 9.1], we obtain the following lemma:

Lemma 4.1. *Let the same assumptions as in Theorem 2.1 hold. We assume that there exist positive number B_1 and B_2 such that:*

$$\|u\|_{L^\infty(\mathbb{R}^d)} + \|\nabla \cdot u\|_{L^\infty(\mathbb{R}^d)} + \|\nabla h\|_{L^\infty(Q_T)} + \|\Delta h\|_{L^\infty(Q_T)} \leq B_1 \quad (4.2)$$

and:

$$\|f\|_{L^\infty(0,T;L^{d+2}(\mathbb{R}^d))} \leq B_2. \quad (4.3)$$

Then, $^h(\text{KSF})$ has a unique non-negative strong solution (n, v) belonging to $\mathbf{W}(Q_T)$, where the function space $\mathbf{W}$ is defined in (2.2). Moreover, (n, v) satisfies the following estimates:

$$\|n\|_{W_{d+2}^{2,1}(Q_T)} \leq C_1 T^{\frac{1}{d+2}} \|n_0\|_{W^{2,d+2}(\mathbb{R}^d)} \quad (4.4)$$

and:

$$\|v\|_{L^\infty(0,T;W^{2,d+2}(\mathbb{R}^d))} \leq C_2 \|f\|_{L^\infty(0,T;L^{d+2}(\mathbb{R}^d))} \quad (4.5)$$

for some positive constants $C_1 = C_1(B_1, d, p)$ and $C_2 = C_2(d)$.

Proof of Theorem 2.1. Let X_T be the space defined in (2.3), and let $f \in X_T$. By Lemma 4.1, there exists a strong solution $\bar{v}^f$ of $(2)_f$ in $L^{d+2}(0, T; W^{2,d+2}(\mathbb{R}^d))$. Furthermore, since $f \in X_T$, it follows from the Sobolev embedding theorem, the second equation of $^h(\text{KSF})$, and (3.13) in Lemma 3.2 that the following holds:

$$\begin{aligned} & \|\bar{v}^f\|_{L^\infty(0,T;W^{3,\infty}(\mathbb{R}^d))} \\ & \leq C \|\bar{v}^f\|_{L^\infty(0,T;W^{4,d+2}(\mathbb{R}^d))} \leq C \|f\|_{L^\infty(0,T;W^{2,d+2}(\mathbb{R}^d))} \leq C(4\|n_0\|_{W^{2,d+2}} + 1), \end{aligned} \quad (4.6)$$

where $C = C(d, \gamma)$. Then, Lemma 4.1 guarantees that ${}^h(\text{KSF})$ with $h = \bar{v}^f$ (denoted by ${}^{h=\bar{v}^f}(\text{KSF})$) has a unique strong solution $(\bar{n}^{h=\bar{v}^f}, \bar{v}^f) \in \mathbf{W}(\mathcal{Q}_T)$, since $\bar{v}^f \in L^\infty(0, T; W^{3,\infty}(\mathbb{R}^d))$. Moreover, by (4.4) of Lemma 4.1, it follows:

$$\|\bar{n}^{h=\bar{v}^f}\|_{W_{d+2}^{2,1}(\mathcal{Q}_T)} \leq CT^{\frac{1}{d+2}} \|n_0\|_{W^{2,d+2}}, \quad (4.7)$$

where $C = C(d, \gamma)$.

For the sake of simplicity, let us denote $(\bar{n}^{h=\bar{v}^f}, \bar{v}^f)$ by (n, v) . It follows that the strong solution (n, v) is a non-negative. This non-negativity can be verified by multiplying both sides of the first equation of ${}^{h=\bar{v}^f}(\text{KSF})$ by $n^- := -\min\{0, n\}$ and integrating over $\mathbb{R}^d$, which leads to:

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|n^-(t)\|_{L^2}^2 &\leq - \int_{\mathbb{R}^d} |\nabla n^-|^2 dx + \int_{\mathbb{R}^d} |n^-| (|\nabla v| + |u|) |\nabla n^-| dx \\ &\leq -\frac{1}{2} \|\nabla n^-(t)\|_{L^2}^2 + \left(\sup_{t \in (0, T)} \|\nabla v(t)\|_{L^\infty}^2 + \|u\|_{L^\infty}^2 \right) \|n^-(t)\|_{L^2}^2 \end{aligned}$$

Applying (4.6), we derive the inequality:

$$\frac{d}{dt} \|n^-(t)\|_{L^2}^2 + \|\nabla n^-(t)\|_{L^2}^2 \leq C \|n^-(t)\|_{L^2}^2,$$

where $C = C(d, \gamma, Q, \lambda, \|n_0\|_{W^{2,d+2}})$. By the Gronwall inequality, we find:

$$\sup_{t \in (0, T)} \|n^-(t)\|_{L^2}^2 \leq \|n^-(x, 0)\|_{L^2}^2 \cdot \exp\{CT\} = 0$$

since $n^-(x, 0) = 0$. This implies:

$$n(x, t) \geq 0 \quad \text{for a.e. } (x, t) \in \mathbb{R}^d \times (0, T).$$

Next, we define the operator Φ by:

$$\Phi : X_T \ni f \mapsto \bar{n}^{h=\bar{v}^f} \in W_{d+2}^{2,1}(\mathcal{Q}_T).$$

The existence of a strong solution to (KSF) is established by applying the Banach fixed-point theorem. Specifically, there exists $T_1 > 0$ such that the operator Φ maps X_{T_1} into itself as a contraction. Since X_{T_1} , equipped with the metric defined by (4.1), forms a complete metric space, the Banach fixed-point theorem guarantees the existence of a unique solution $(n^{h=v^f}, v^f)$ in X_{T_1} , satisfying $f = n^{h=v^f}$ over $[0, T_1]$. This can be shown in the following steps.

To proceed, we demonstrate that for sufficiently small $T^* > 0$, the operator Φ maps X_{T^*} into itself.

We further establish that $n \in L^\infty(0, T; L^{d+2}(\mathbb{R}^d))$. For any $1 < r < \infty$, multiplying both sides of the first equation of ${}^{h=\bar{v}^f}(\text{KSF})$ by n^{r-1} and integrating by parts, it follows:

$$\begin{aligned}
\frac{1}{r} \frac{d}{dt} \|n(t)\|_{L^r}^r &\leq -(r-1) \int_{\mathbb{R}^d} n^{r-2} |\nabla n|^2 dx + (r-1) \int_{\mathbb{R}^d} n^{r-1} (|\nabla v| + |u|) |\nabla n| dx \\
&\leq -\frac{r-1}{2} \int_{\mathbb{R}^d} n^{r-2} |\nabla n|^2 dx + \frac{r-1}{2} \left(\sup_{t \in (0, T)} \|\nabla v(t)\|_{L^\infty}^2 + \|u\|_{L^\infty}^2 \right) \|n(t)\|_{L^r}^r \\
&\leq -\frac{r-1}{2} \int_{\mathbb{R}^d} n^{r-2} |\nabla n|^2 dx + C \|n(t)\|_{L^r}^r,
\end{aligned}$$

where $C = C(d, \gamma, Q, \lambda, \|n_0\|_{W^{2,d+2}}, r)$. This yields:

$$\sup_{t \in (0, T)} \|n(t)\|_{L^r} \leq \|n_0\|_{L^r} \exp\{CT\} \quad (4.8)$$

for all $1 < r < \infty$, where $C = C(d, \gamma, Q, \lambda, \|n_0\|_{W^{2,d+2}}, r)$. Therefore, by setting $r = d + 2 > 1$, it follows that $n \in L^\infty(0, T; L^{d+2}(\mathbb{R}^d))$.

Next, we show that $\nabla n \in L^\infty(0, T; L^{d+2}(\mathbb{R}^d))$. Let $2 \leq r < \infty$. Differentiating both sides of the first equation of $^{h=\bar{v}^f}$ (KSF) with respect to x and multiplying by $|\nabla n|^{r-2} \nabla n$, we obtain:

$$\begin{aligned}
\frac{1}{r} \frac{d}{dt} \|\nabla n(t)\|_{L^r}^r &= \int_{\mathbb{R}^d} \nabla \Delta n \cdot |\nabla n|^{r-2} \nabla n dx - \int_{\mathbb{R}^d} \nabla (\nabla \cdot (n \nabla v)) \cdot |\nabla n|^{r-2} \nabla n dx \\
&\quad - \int_{\mathbb{R}^d} \nabla (\nabla \cdot (un)) \cdot |\nabla n|^{r-2} \nabla n dx \\
&=: I_1^{(1)} + I_2^{(1)} + I_3^{(1)}.
\end{aligned} \quad (4.9)$$

Applying integrating by parts to $I_1^{(1)}$, we find:

$$\begin{aligned}
I_1^{(1)} &= \int_{\mathbb{R}^d} \sum_{i=1}^d \sum_{j=1}^d \partial_i \partial_j^2 n |\nabla n|^{r-2} \partial_i n dx \\
&= -(r-2) \int_{\mathbb{R}^d} |\nabla n|^{r-4} \sum_{j=1}^d |(\partial_j \nabla n) \cdot (\nabla n)|^2 dx - \int_{\mathbb{R}^d} |\nabla n|^{r-2} \sum_{j=1}^d |\partial_j \nabla n|^2 dx \\
&=: -(r-2) J_1^{(1)} - J_2^{(1)}.
\end{aligned} \quad (4.10)$$

As for $I_2^{(1)}$, it follows from (4.6):

$$I_2^{(1)} = - \int_{\mathbb{R}^d} \sum_{j=1}^d \partial_j (\nabla \cdot (n \nabla v)) |\nabla n|^{r-2} \partial_j n dx$$

$$\begin{aligned}
&= \int_{\mathbb{R}^d} \nabla \cdot (n \nabla v) \sum_{j=1}^d \partial_j (|\nabla n|^{r-2} \partial_j n) \, dx \\
&\leq (r-2) \int_{\mathbb{R}^d} (|\nabla n| |\nabla v| + n |\Delta v|) |\nabla n|^{r-4} \sum_{j=1}^d |(\partial_j \nabla n) \cdot (\nabla n)| |\partial_j n| \, dx \\
&\quad + \int_{\mathbb{R}^d} (|\nabla n| |\nabla v| + n |\Delta v|) |\nabla n|^{r-2} \sum_{j=1}^d |\partial_j^2 n| \, dx \\
&\leq \frac{r-2}{4} J_1^{(1)} + \frac{1}{4} J_2^{(1)} + 2(r-2+d) \sup_{t \in (0,T)} \|\nabla v(t)\|_{L^\infty}^2 \|\nabla n(t)\|_{L^r}^r \\
&\quad + 2(r-2+d) \sup_{t \in (0,T)} \|\Delta v(t)\|_{L^\infty}^2 \|\nabla n(t)\|_{L^r}^{r-2} \|n(t)\|_{L^r}^2 \\
&\leq \frac{r-2}{4} J_1^{(1)} + \frac{1}{4} J_2^{(1)} + C \|\nabla n(t)\|_{L^r}^r + C \|\nabla n(t)\|_{L^r}^{r-2} \|n(t)\|_{L^r}^2, \tag{4.11}
\end{aligned}$$

where $C = C(d, \gamma, \|n_0\|_{W^{2,d+2}}, r)$. Furthermore, similar to (4.11), we observe:

$$\begin{aligned}
I_3^{(1)} &\leq \frac{r-2}{4} J_1^{(1)} + \frac{1}{4} J_2^{(1)} + 2(r-2+d) \|u\|_{L^\infty}^2 \|\nabla n(t)\|_{L^r}^r \\
&\quad + 2(r-2+d) \|\nabla \cdot u\|_{L^\infty}^2 \|\nabla n(t)\|_{L^r}^{r-2} \|n(t)\|_{L^r}^2 \\
&\leq \frac{r-2}{4} J_1^{(1)} + \frac{1}{4} J_2^{(1)} + C \|\nabla n(t)\|_{L^r}^r + C \|\nabla n(t)\|_{L^r}^{r-2} \|n(t)\|_{L^r}^2, \tag{4.12}
\end{aligned}$$

where $C = C(d, \gamma, Q, \lambda, r)$. Combining (4.9)-(4.12), we obtain the following expression:

$$\frac{d}{dt} \|\nabla n(t)\|_{L^r}^2 \leq -\frac{r-2}{2} J_1^{(1)} - \frac{1}{2} J_2^{(1)} + C \|n(t)\|_{L^r}^2 + C \|\nabla n(t)\|_{L^r}^2,$$

which, by (4.8), yields:

$$\begin{aligned}
\sup_{t \in (0,T)} \|\nabla n(t)\|_{L^r} &\leq \left(\|\nabla n_0\|_{L^r} + CT^{\frac{1}{2}} \sup_{t \in (0,T)} \|n(t)\|_{L^r} \right) \exp\{CT\} \\
&\leq \left(\|\nabla n_0\|_{L^r} + CT^{\frac{1}{2}} \|n_0\|_{L^r} \right) \exp\{CT\} \tag{4.13}
\end{aligned}$$

for all $2 \leq r < \infty$, where $C = C(d, \gamma, Q, \lambda, \|n_0\|_{W^{2,d+2}}, r)$. Therefore, by setting $r = d+2 > 1$, it follows that $\nabla n \in L^\infty(0, T; L^{d+2}(\mathbb{R}^d))$.

Next, we demonstrate that $\partial_i \nabla n \in L^\infty(0, T; L^{d+2}(\mathbb{R}^d))$ for all $i = 1, \dots, d$. Let $2 \leq r < \infty$. We begin by differentiating both sides of the first equation of ${}^h\tilde{v}^f$ (KSF) with respect to x and then multiply by $|\partial_i \nabla n|^{r-2} \partial_i \nabla n$, which gives:

$$\frac{1}{r} \frac{d}{dt} \|\partial_i \nabla n(t)\|_{L^r}^r = \int_{\mathbb{R}^d} \partial_i \nabla \Delta n \cdot |\partial_i \nabla n|^{r-2} \partial_i \nabla n \, dx - \int_{\mathbb{R}^d} \partial_i \nabla (\nabla \cdot (n \nabla v)) \cdot |\partial_i \nabla n|^{r-2} \partial_i \nabla n \, dx$$

$$\begin{aligned}
& - \int_{\mathbb{R}^d} \partial_i \nabla (\nabla \cdot (un)) \cdot |\partial_i \nabla n|^{r-2} \partial_i \nabla n \, dx \\
& =: I_1^{(2)} + I_2^{(2)} + I_3^{(2)}.
\end{aligned} \tag{4.14}$$

We now proceed to handle $I_1^{(2)}$. By integrating by parts, we obtain the following:

$$\begin{aligned}
I_1^{(2)} &= \int_{\mathbb{R}^d} \sum_{j=1}^d \partial_i \nabla \partial_j^2 n \cdot |\partial_i \nabla n|^{r-2} \partial_i \nabla n \, dx \\
&= -(r-2) \int_{\mathbb{R}^d} |\partial_i \nabla n|^{r-4} \sum_{j=1}^d |(\partial_i \partial_j \nabla n) \cdot (\partial_i \nabla n)|^2 \, dx - \int_{\mathbb{R}^d} |\partial_i \nabla n|^{r-2} \sum_{j=1}^d |\partial_i \partial_j \nabla n|^2 \, dx \\
&=: -(r-2)J_1^{(2)} - J_2^{(2)}.
\end{aligned} \tag{4.15}$$

For $I_2^{(2)}$, using (4.6), we obtain:

$$\begin{aligned}
I_2^{(2)} &= - \int_{\mathbb{R}^d} \sum_{j=1}^d \partial_i \partial_j (\nabla \cdot (n \nabla v)) |\partial_i \nabla n|^{r-2} \partial_i \partial_j n \, dx \\
&= \int_{\mathbb{R}^d} \partial_i (\nabla \cdot (n \nabla v)) \sum_{j=1}^d \partial_j (|\partial_i \nabla n|^{r-2} \partial_i \partial_j n) \, dx \\
&\leq (r-2) \int_{\mathbb{R}^d} (|\partial_i \nabla n| |\nabla v| + |\nabla n| |\partial_i \nabla v| + |\partial_i n| |\Delta v| + n |\partial_i \Delta v|) \\
&\quad \times |\partial_i \nabla n|^{r-4} \sum_{j=1}^d |(\partial_i \partial_j \nabla n) \cdot (\partial_i \nabla n)| |\partial_i \partial_j n| \, dx \\
&\quad + \int_{\mathbb{R}^d} (|\partial_i \nabla n| |\nabla v| + |\nabla n| |\partial_i \nabla v| + |\partial_i n| |\Delta v| + n |\partial_i \Delta v|) |\partial_i \nabla n|^{r-2} \sum_{j=1}^d |\partial_i \partial_j^2 n| \, dx \\
&\leq \frac{r-2}{4} J_1^{(4)} + \frac{1}{4} J_2^{(4)} + 4(r-2+d) \sup_{t \in (0,T)} \|\nabla v(t)\|_{L^\infty}^2 \|\partial_i \nabla n(t)\|_{L^r}^r \\
&\quad + 4(r-2+d) \left(\sup_{t \in (0,T)} \|\partial_i \nabla v(t)\|_{L^\infty}^2 + \sup_{t \in (0,T)} \|\Delta v(t)\|_{L^\infty}^2 \right) \|\partial_i \nabla n(t)\|_{L^r}^{r-2} \|\nabla n(t)\|_{L^r}^2 \\
&\quad + 4(r-2+d) \sup_{t \in (0,T)} \|\partial_i \Delta v(t)\|_{L^\infty}^2 \|\partial_i \nabla n(t)\|_{L^r}^{r-2} \|n(t)\|_{L^r}^2 \\
&\leq \frac{r-2}{4} J_1^{(2)} + \frac{1}{4} J_2^{(2)} + C \left(\|n(t)\|_{L^r}^2 + \|\nabla n(t)\|_{L^r}^2 \right) \|\partial_i \nabla n(t)\|_{L^r}^{r-2} + C \|\partial_i \nabla n(t)\|_{L^r}^r,
\end{aligned} \tag{4.16}$$

where $C = C(d, \gamma, \|n_0\|_{W^{2,d+2}}, r)$. Furthermore, similar to (4.16), we observe:

$$\begin{aligned} I_3^{(2)} &\leq \frac{r-2}{4} J_1^{(2)} + \frac{1}{4} J_2^{(2)} + 4(r-2+d) \|u\|_{L^\infty}^2 \|\partial_i \nabla n(t)\|_{L^r}^r \\ &\quad + 4(r-2+d) (\|\partial_i u\|_{L^\infty}^2 + \|\nabla \cdot u\|_{L^\infty}^2) \|\partial_i \nabla n(t)\|_{L^r}^{r-2} \|\nabla n(t)\|_{L^r}^2 \\ &\quad + 4(r-2+d) \|\partial_i (\nabla \cdot u)\|_{L^\infty}^2 \|\partial_i \nabla n(t)\|_{L^r}^{r-2} \|n(t)\|_{L^r}^2 \\ &\leq \frac{r-2}{4} J_1^{(2)} + \frac{1}{4} J_2^{(2)} C \left(\|n(t)\|_{L^r}^2 + \|\nabla n(t)\|_{L^r}^2 \right) \|\partial_i \nabla n(t)\|_{L^r}^{r-2} + C \|\partial_i \nabla n(t)\|_{L^r}^r, \quad (4.17) \end{aligned}$$

where $C = C(d, \gamma, Q, \lambda, r)$. It follows from (4.14)-(4.17):

$$\frac{d}{dt} \|\partial_i \nabla n(t)\|_{L^r}^2 \leq -\frac{r-2}{2} J_1^{(2)} - \frac{1}{2} J_2^{(2)} + C \left(\|n(t)\|_{L^r}^2 + \|\nabla n(t)\|_{L^r}^2 \right) + C \|\partial_i \nabla n(t)\|_{L^r}^2,$$

where $C = C(d, \gamma, Q, \lambda, \|n_0\|_{W^{2,d+2}}, r)$. Choosing r as $d+2$, we obtain from (4.8) and (4.13):

$$\begin{aligned} &\sup_{t \in (0, T)} \|\partial_i \nabla n(t)\|_{L^{d+2}} \\ &\leq \left\{ \|\partial_i \nabla n_0\|_{L^{d+2}} + CT^{\frac{1}{2}} \left(\sup_{t \in (0, T)} \|n(t)\|_{L^{d+2}} + \sup_{t \in (0, T)} \|\nabla n(t)\|_{L^{d+2}} \right) \right\} \exp\{CT\} \\ &\leq \left\{ \|\partial_i \nabla n_0\|_{L^{d+2}} + CT^{\frac{1}{2}} \|\nabla n_0\|_{L^{d+2}} + C(T^{\frac{1}{2}} + T) \|n_0\|_{L^{d+2}} \right\} \exp\{CT\} \quad (4.18) \end{aligned}$$

where $C = C(d, \gamma, Q, \lambda, \|n_0\|_{W^{2,d+2}})$. Hence, combining (4.8), (4.13) and (4.18), we conclude that $n \in L^\infty(0, T; W^{2,d+2}(\mathbb{R}^d))$.

Therefore, we observe from (4.7), (4.8), (4.13) and (4.18):

$$\begin{aligned} &\|n\|_{L^\infty(0, T; W^{2,d+2}(\mathbb{R}^d))} + \|\partial_t n\|_{L^{d+2}(\mathcal{Q}_T)} \\ &\leq (1 + C_1 T^{\frac{1}{2}} + C_1 T) \exp\{C_1 T\} \|n_0\|_{W^{2,d+2}} + C_1 T^{\frac{1}{d+2}} \|n_0\|_{W^{2,d+2}}, \quad (4.19) \end{aligned}$$

where $C_1 = C_1(d, \gamma, Q, \lambda, \|n_0\|_{W^{2,d+2}})$. Taking T_* by:

$$T_* \leq \min \left\{ \left(\frac{1}{C_1} \right)^2, \frac{1}{C_1} \log \frac{7}{6}, \left(\frac{1}{2C_1} \right)^{d+2} \right\},$$

we observe from (4.19):

$$\|n\|_{L^\infty(0, T_*; W^{2,d+2}(\mathbb{R}^d))} + \|\partial_t n\|_{L^{d+2}(\mathcal{Q}_{T_*})} \leq 4\|n_0\|_{W^{2,d+2}} + 1. \quad (4.20)$$

This implies that the operator Φ maps X_{T_*} into itself.

It remains to prove that the operator Φ is a contraction mapping. For the sake of simplicity, for $i = 1, 2$, we denote $(\bar{n}^{h=\bar{v}^{f_i}}, \bar{v}^{f_i})$ by (n_i, v_i) . In addition, we define w by $w := n_1 - n_2$. Then, we observe:

$$\begin{aligned}
\partial_t w &= \Delta w - \nabla \cdot (n_1 \nabla v_1 - n_2 \nabla v_2) - \nabla \cdot (uw) \\
&= \Delta w - \nabla \cdot (n_1 \nabla (v_1 - v_2)) - \nabla \cdot (w \nabla v_2) - \nabla \cdot (uw).
\end{aligned} \tag{4.21}$$

We obtain from (4.8) and (4.13):

$$\sup_{t \in (0, T)} \|n_i(t)\|_{L^\infty} \leq C \sup_{t \in (0, T)} \|n_i(t)\|_{W^{1, d+2}} \leq C(1 + T^{\frac{1}{2}}) \exp\{CT\}$$

for $i = 1, 2$, where $C = C(d, \gamma, \|n_0\|_{W^{2, d+2}})$. Then, multiplying both sides of (4.21) by $|w|^d w$ and integrating over $\mathbb{R}^d$, we observe:

$$\begin{aligned}
&\frac{1}{d+2} \frac{d}{dt} \|w(t)\|_{L^{d+2}}^{d+2} \\
&= -(d+1) \int_{\mathbb{R}^d} |w|^d |\nabla w|^2 dx + (d+1) \int_{\mathbb{R}^d} (n_1 \nabla (v_1 - v_2)) \cdot |w|^d \nabla w dx \\
&\quad + (d+1) \int_{\mathbb{R}^d} (w \nabla v_2) \cdot |w|^d \nabla w dx + (d+1) \int_{\mathbb{R}^d} (uw) |w|^d \cdot \nabla w dx \\
&\leq -\frac{d+1}{4} \int_{\mathbb{R}^d} |w|^d |\nabla w|^2 dx + C \|w(t)\|_{L^{d+2}}^d \sup_{t \in (0, T)} \|n_1(t)\|_{L^\infty}^2 \|\nabla (v_1 - v_2)(t)\|_{L^{d+2}}^2 \\
&\quad + C \left(\sup_{t \in (0, T)} \|\nabla v_2(t)\|_{L^\infty}^2 + \|u\|_{L^\infty}^2 \right) \|w(t)\|_{L^{d+2}}^{d+2} \\
&\leq -\frac{d+1}{4} \int_{\mathbb{R}^d} |w|^d |\nabla w|^2 dx + C(1 + T^{\frac{1}{2}}) \exp\{CT\} \|w(t)\|_{L^{d+2}}^d \|\nabla (v_1 - v_2)(t)\|_{L^{d+2}}^2 \\
&\quad + C \|w(t)\|_{L^{d+2}}^{d+2},
\end{aligned}$$

which yields:

$$\|w(t)\|_{L^{d+2}}^2 \leq C(1 + T^{\frac{1}{2}}) \exp\{CT\} \|\nabla (v_1 - v_2)\|_{L^2(0, T; L^{d+2}(\mathbb{R}^d))}^2, \tag{4.22}$$

where $C = C(d, \gamma, Q, \lambda, \|n_0\|_{W^{2, d+2}})$. In addition, by virtue of (3.17), it holds:

$$\begin{aligned}
&\|\nabla (v_1 - v_2)(t)\|_{L^2(0, T; L^{d+2}(\mathbb{R}^d))}^2 \\
&= \int_0^T \|\nabla (v_1 - v_2)(s)\|_{L^{d+2}}^2 ds \leq CT \|f_1 - f_2\|_{L^\infty(0, T; L^{d+2}(\mathbb{R}^d))}^2,
\end{aligned} \tag{4.23}$$

where $C = C(\gamma)$. Hence, from (4.22) and (4.23), we obtain:

$$\|w\|_{L^\infty(0, T; L^{d+2}(\mathbb{R}^d))} \leq (C_2 T)^{\frac{1}{2}} (1 + T^{\frac{1}{2}})^{\frac{1}{2}} \exp\{C_2 T\} \|f_1 - f_2\|_{L^\infty(0, T; L^{d+2}(\mathbb{R}^d))}, \tag{4.24}$$

where $C_2 = C_2(d, \gamma, Q, \lambda, \|n_0\|_{W^{2,d+2}})$. Taking T_{**} by:

$$T_{**} \leq \min \left\{ \frac{1}{64C_2}, 9, \frac{1}{C_2} \log 2 \right\},$$

we observe from (4.24):

$$\|w\|_{L^\infty(0, T_{**}; L^{d+2}(\mathbb{R}^d))} \leq \frac{1}{2} \|f_1 - f_2\|_{L^\infty(0, T_{**}; L^{d+2}(\mathbb{R}^d))}.$$

Now, we apply the Banach fixed-point theorem, which guarantees the existence of a positive number $T_1 := \min\{T_*, T_{**}\}$ such that Φ becomes a contraction from X_{T_1} into itself. Thus, Φ has a fixed point $\tilde{f} = \Phi(\tilde{f}) = \tilde{n}^{h=\tilde{v}\tilde{f}}$. Hence, we construct the desired solution $(\tilde{n}^{h=\tilde{v}\tilde{f}}, \tilde{v}^{\tilde{f}})$ in Theorem 2.1. $\square$

4.2. Proof of Theorem 2.1 (II): extension criterion

We now establish the extension criterion for the solution of (KSF). To proceed, we present the following Lemma:

Lemma 4.2. *Let Assumptions 2.1 hold. Let (n, v) be the strong solution of (KSF) on $[0, T_1]$ obtained from Theorem 2.1 (I) with the property (2.10). Then, there exists a positive constant $C = C(d, \gamma, Q, \lambda, T_1, \|n_0\|_{W^{2,d+2}(\mathbb{R}^d)})$ such that:*

$$\sup_{t \in (0, T_1)} \|n(t)\|_{W^{2,d+2}(\mathbb{R}^d)} \leq C.$$

Proof of Lemma 4.2. We establish the following regularities:

$$n \in L^\infty(0, T_1; L^{d+2}(\mathbb{R}^d)), \quad (4.25)$$

$$\nabla v, \Delta v \in L^\infty(0, T_1; L^\infty(\mathbb{R}^d)), \quad (4.26)$$

$$\nabla n \in L^\infty(0, T_1; L^{d+2}(\mathbb{R}^d)), \quad (4.27)$$

$$\nabla n \in L^\infty(0, T_1; L^\infty(\mathbb{R}^d)), \quad (4.28)$$

$$\partial_i \nabla v, \partial_i \Delta v \in L^\infty(0, T_1; L^\infty(\mathbb{R}^d)) \quad \text{for all } i = 1, 2, \dots, d, \quad (4.29)$$

$$\partial_i \nabla n \in L^\infty(0, T_1; L^{d+2}(\mathbb{R}^d)) \quad \text{for all } i = 1, 2, \dots, d. \quad (4.30)$$

As for (4.25), let $1 < r < \infty$. Multiplying both sides of the first equation of (KSF) by n^{r-1} and integrating over $\mathbb{R}^d$, we obtain:

$$\sup_{t \in (0, T_1)} \|n(t)\|_{L^r} \leq \|n_0\|_{L^r} \exp \left\{ \frac{r-1}{r} T_1 \left(\sup_{t \in (0, T_1)} \|n(t)\|_{L^\infty} + \|\nabla \cdot u\|_{L^\infty} \right) \right\}.$$

Therefore, since $d+2 > 1$, by taking $r = d+2$, we obtain (4.25).

Next, we establish (4.26). By applying Lemma 3.3 and using the second equation of (KSF), we derive the following estimates:

$$\sup_{t \in (0, T_1)} \|\nabla v(t)\|_{L^\infty} \leq C \sup_{t \in (0, T_1)} \|n(t)\|_{L^\infty}, \quad (4.31)$$

and:

$$\sup_{t \in (0, T_1)} \|\Delta v(t)\|_{L^\infty} \leq \gamma \sup_{t \in (0, T_1)} \|v(t)\|_{L^\infty} + \sup_{t \in (0, T_1)} \|n(t)\|_{L^\infty} \leq 2 \sup_{t \in (0, T_1)} \|n(t)\|_{L^\infty}, \quad (4.32)$$

where $C = C(d, \gamma)$.

We move on to proving (4.27). Let $2 \leq r < \infty$. To establish (4.27), we differentiate both sides of the first equation of (KSF) with respect to x once and multiply by $|\nabla n|^{r-2} \nabla n$. This gives us the following estimate:

$$\begin{aligned} & \sup_{t \in (0, T_1)} \|\nabla n(t)\|_{L^r} \\ & \leq \left(\|\nabla n_0\|_{L^r} + \sqrt{2(r+d-2)T_1} \left(\sup_{t \in (0, T_1)} \|\Delta v(t)\|_{L^\infty} + \|\nabla \cdot u\|_{L^\infty} \right) \sup_{t \in (0, T_1)} \|n(t)\|_{L^r} \right) \\ & \quad \times \exp \left\{ (r+d-2)T_1 \left(\sup_{t \in (0, T_1)} \|\nabla v(t)\|_{L^\infty}^2 + \|u\|_{L^\infty}^2 \right) \right\}. \end{aligned} \quad (4.33)$$

Thus, since $d+2 > 1$, by taking $r = d+2$, we deduce that (4.27).

We now proceed to establish (4.28) by applying Moser's iteration technique. Let $d+2 \leq r < \infty$. Differentiating both sides of (KSF) with respect to x and multiplying by $|\nabla n|^{r-2} \nabla n$, we obtain the following estimate:

$$\begin{aligned} \|\nabla n(t)\|_{L^r} & \leq \left\{ 5(M_* + 1)(T_1 + 1)r^{d(d+1)+2} \right\}^{\frac{1}{r}} \max \left\{ \|\nabla n_0\|_{L^{d+2}}, \right. \\ & \quad \left. \|\nabla n_0\|_{L^\infty}, \sup_{t \in (0, T_1)} \|n(t)\|_{L^{d+2}}, \sup_{t \in (0, T_1)} \|n(t)\|_{L^\infty}, \sup_{t \in (0, T_1)} \|\nabla n(t)\|_{L^{\frac{r}{d+2}}} \right\} \end{aligned}$$

for a.e. $t \in (0, T_1)$, where M_* is defined by:

$$M_* := C \left(\sup_{t \in (0, T_1)} \|\nabla v(t)\|_{L^\infty}^2 + \sup_{t \in (0, T_1)} \|\Delta v(t)\|_{L^\infty}^2 + \|u\|_{L^\infty} + \|\nabla \cdot u\|_{L^\infty} \right),$$

where $C = C(d)$. Choosing r as $(d+2)^k$ with $k \geq 1$, we find:

$$\begin{aligned} \|\nabla n(t)\|_{L^{(d+2)^k}} & \leq \left\{ 5(M_* + 1)(T_1 + 1)(d+2)^{d(d+1)+2} \right\}^2 \max \left\{ \|\nabla n_0\|_{L^{d+2}}, \right. \\ & \quad \left. \|\nabla n_0\|_{L^\infty}, \sup_{t \in (0, T_1)} \|n(t)\|_{L^{d+2}}, \sup_{t \in (0, T_1)} \|n(t)\|_{L^\infty}, \sup_{t \in (0, T_1)} \|\nabla n(t)\|_{L^{d+2}} \right\} \end{aligned} \quad (4.34)$$

for a.e. $t \in (0, T_1)$.

At this point, we introduce the fundamental theorem regarding the limiting norm of $\|\cdot\|_{L^p}$. Specifically, let (X, μ) be a measure space. If $f \in L^{p_0}(X, \mu)$ for some $p_0 < \infty$, the following holds:

$$\lim_{p \rightarrow \infty} \|f\|_{L^p(X)} = \|f\|_{L^\infty(X)}. \quad (4.35)$$

See Grafakos [4, p.11, Exercise 1.1.3] for further details. Therefore, taking the limit as $k \rightarrow \infty$ on the left-hand side of (4.34) and applying the result from (4.35), we conclude:

$$\sup_{t \in (0, T_1)} \|\nabla n(t)\|_{L^\infty} \leq \left\{ 5(M_* + 1)(T_1 + 1)(d + 2)^{d(d+1)+2} \right\}^2 \max \left\{ \|\nabla n_0\|_{L^{d+2}}, \right. \\ \left. \|\nabla n_0\|_{L^\infty}, \sup_{t \in (0, T_1)} \|n(t)\|_{L^{d+2}}, \sup_{t \in (0, T_1)} \|n(t)\|_{L^\infty}, \sup_{t \in (0, T_1)} \|\nabla n(t)\|_{L^{d+2}} \right\}$$

We now turn to the proof of (4.29). Applying the Young inequality, we have:

$$\sup_{t \in (0, T_1)} \|\partial_i \nabla v(t)\|_{L^\infty} \leq \|\nabla G_\gamma\|_{L^1} \sup_{t \in (0, T_1)} \|\nabla n(t)\|_{L^\infty} \leq C \sup_{t \in (0, T_1)} \|\nabla n(t)\|_{L^\infty}, \quad (4.36)$$

where $C = C(d, \gamma)$. In addition, from the second equation of (KSF), we have:

$$\sup_{t \in (0, T_1)} \|\partial_i \Delta v(t)\|_{L^\infty} \leq \gamma \sup_{t \in (0, T_1)} \|\nabla v(t)\|_{L^\infty} + \sup_{t \in (0, T_1)} \|\nabla n(t)\|_{L^\infty}. \quad (4.37)$$

This inequality directly leads to the proof of (4.29).

Finally, we address the proof of (4.30). Let $2 \leq r < \infty$. By differentiating both sides of the first equation of (KSF) with respect to x twice and multiplying by $|\partial_i \nabla n|^d \partial_i \nabla n$, we derive the following from (4.31), (4.32), (4.33), (4.36) and (4.37):

$$\sup_{t \in (0, T_1)} \|\partial_i \nabla n(t)\|_{L^r} \\ \leq \left(\|\partial_i \nabla n_0\|_{L^r} + 2\sqrt{(r+d-2)T_1} \left(\sup_{t \in (0, T_1)} \|\partial_i \nabla v(t)\|_{L^\infty} + \|\partial_i u\|_{L^\infty} \right) \sup_{t \in (0, T_1)} \|\nabla n(t)\|_{L^r} \right. \\ \left. + 2\sqrt{(r+d-2)T_1} \left(\sup_{t \in (0, T_1)} \|\Delta v(t)\|_{L^\infty} + \|\nabla \cdot u\|_{L^\infty} \right) \sup_{t \in (0, T_1)} \|\nabla n(t)\|_{L^r} \right. \\ \left. + 2\sqrt{(r+d-2)T_1} \left(\sup_{t \in (0, T_1)} \|\partial_i \Delta v(t)\|_{L^\infty} + \|\partial_i \nabla \cdot u\|_{L^\infty} \right) \sup_{t \in (0, T_1)} \|n(t)\|_{L^r} \right) \\ \times \exp \left\{ 2(r+d-2)T_1 \left(\sup_{t \in (0, T_1)} \|\nabla v(t)\|_{L^\infty}^2 + \|u\|_{L^\infty} \right) \right\}. \quad (4.38)$$

Choosing r as $d+2$, we have (4.30). This completes the proof of Lemma 4.2. $\square$

Continuation of the Proof of Theorem 2.1 (II). We are now ready to prove Theorem 2.1 (II). From the construction of the solution described in Subsection 4.1, we observe that the local existence time T_1 depends on d, γ, Q, λ and $\|n_0\|_{W^{2,d+2}}$.

Our objective is to extend the strong solution (n, v) from $[0, T_1)$ to $[0, \widehat{T})$, where $T_1 < \widehat{T} < T_{\max}$. Here, $T_{\max}$ refers to the maximal existence time, the upper bound for the interval during which the solution remains bounded in the $L^\infty(\mathbb{R}^d)$ -norm. In other words, $T_{\max}$ is characterized by the property:

$$\limsup_{t \rightarrow T_b - 0} \|n(t)\|_{L^\infty} = \infty,$$

indicating that the solution $n(t)$ becomes unbounded in the $L^\infty(\mathbb{R}^d)$ -norm as t approaches $T_{\max}$.

To achieve this extension, we assume (2.10). Then, from Lemma 4.2, there exists a positive constant $C = C(d, \gamma, Q, \lambda, T_1, \|n_0\|_{W^{2,d+2}})$ such that:

$$\sup_{t \in (0, T_1)} \|n(t)\|_{W^{2,d+2}} \leq C. \quad (4.39)$$

From (4.39), the solution $n(T_0)$ with $T_0 < T_1$ belongs to $W^{2,d+2}(\mathbb{R}^d)$.

We then consider T_0 as an initial time and apply the construction method outlined in Subsection 4.1, using $n(T_0)$ as initial data. This enables us to extend the strong solution (n, v) over $[T_0, T_1^{(1)})$. Here, the existence time $T_1^{(1)}$ is determined by d, γ, Q, λ and $n(T_0)$. By applying Lemma 4.2, we derive the same estimates as in (4.39), but now over $[T_0, T_0^{(1)}]$, where $T_0^{(1)} < T_1^{(1)}$.

From the estimates obtained over $[T_0, T_0^{(1)}]$, we ensure that the solution $n(T_0^{(1)})$ belongs to $W^{2,d+2}(\mathbb{R}^d)$. Consequently, we are able to reapply the construction method from Subsection 4.1, treating $n(T_0^{(1)})$ as the initial data. This allows us to construct the strong solution (n, v) on $[T_0^{(1)}, T_1^{(2)})$.

Repeating this procedure iteratively, we define sequences $\{T_0^{(k)}\}$ and $\{T_1^{(k)}\}$ for $k = 1, 2, \dots$. In addition, we set $T_0^{(0)} := T_0$, and construct solutions on $[T_0^{(k-1)}, T_1^{(k)})$, ensuring at each step that:

- The same estimates as in (4.39) hold over $[T_0^{(k-1)}, T_0^{(k)}]$, where $T_0^{(k)} < T_1^{(k)}$. These estimates are guaranteed by Lemma 4.2.
- The construction method from Subsection 4.1 can be reapplied using $n(T_0^{(k)})$ as initial data
- We extend the strong solution (n, v) over $[T_0^{(k)}, T_1^{(k+1)})$.

Therefore, by this method of iterative extension, we have successfully extended the strong solution (n, v) to $[0, \widehat{T})$ for any $\widehat{T} < T_{\max}$.

Based on the above facts, the following conclusion follows: the strong solution n can be extended to the maximal existence time $T_{\max}$, at which time the solution may become unbounded in the $L^\infty(\mathbb{R}^d)$ -norm. Specifically, $T_{\max}$ is characterized by the condition:

$$\limsup_{t \rightarrow T_{\max} - 0} \|n(t)\|_{L^\infty} = \infty. \quad (4.40)$$

This condition suggests that a blow-up in the $L^\infty(\mathbb{R}^d)$ -norm may cause for the termination of the existence of a strong solution at $T_{\max}$. Indeed, $T_{\max} \leq T_b$ by the definition of $T_{\max}$ and T_b , and the case $T_{\max} < T_b$ cannot occur if (4.40) holds. Thus, we conclude that $T_{\max} = T_b$, i.e., (2.12).

In addition, we aim to establish (2.13) in Theorem 2.1 (II). Applying Lemma 4.2, for $\hat{T} \in (0, T_{\max})$, the following regularity properties hold:

$$n \in L^\infty(0, \hat{T}; W^{1,d+2}(\mathbb{R}^d)) \quad \text{and} \quad \partial_t n \in L^\infty(0, \hat{T}; L^{d+2}(\mathbb{R}^d)).$$

These regularities, together with the embedding $W^{1,d+2}(\Omega) \subset C(\Omega)$ for any bounded subset $\Omega \subset \mathbb{R}^d$, imply by the Aubin-Lions Lemma:

$$n \in C([0, \hat{T}]; C(\Omega)).$$

Thus, we conclude (2.13). This completes the proof of Theorem 2.1 (II). $\square$

4.3. Proof of Theorem 2.1 (III): mass conservation law

We now proceed with the proof of Theorem 2.1 (III). Let $\hat{T} \in (0, T_{\max})$, where $T_{\max}$ denotes the maximal existence time. We assume that $n_0 \in L^1(\mathbb{R}^d)$. By Lemma 4.2, there exists a positive constant $C = C(d, \gamma, Q, \lambda, \|n_0\|_{L^1}, \|n_0\|_{W^{2,d+2}})$ such that the following bounds below hold:

$$\sup_{t \in (0, \hat{T})} \|n(t)\|_{L^p} \leq C \quad \text{for all } 1 < p \leq \infty, \quad (4.41)$$

and:

$$\sup_{t \in (0, \hat{T})} \|\nabla v(t)\|_{L^\infty} + \sup_{t \in (0, \hat{T})} \|\Delta v(t)\|_{L^\infty} \leq C. \quad (4.42)$$

Proof of Theorem 2.1 (III). Let $\ell \geq 1$. According to Lemma 3.6, we introduce the cut-off function ψ with $\rho_1 = \ell$ and $\rho_2 = 2\ell$. Multiplying both sides of the first equation of (KSF) by $\psi = \psi(x)$ and integrating over $B_{2\ell}$, we obtain:

$$\frac{d}{dt} \int_{B_{2\ell}} n(t) \psi_\ell \, dx = \int_{B_{2\ell}} \Delta n(t) \psi_\ell \, dx - \int_{B_{2\ell}} \nabla \cdot (n \nabla v)(t) \psi_\ell \, dx - \int_{B_{2\ell}} \nabla \cdot (un(t)) \psi_\ell \, dx. \quad (4.43)$$

Regarding the first term on the right-hand side of (4.43), since $n \in W_{d+2}^{2,1}(\mathcal{Q}_{\hat{T}})$ and the following integrability condition holds:

$$\nabla n(t) \psi_\ell, \nabla \cdot (\nabla n(t) \psi_\ell) \in L^1(B_{2\ell}), \quad (4.44)$$

we can apply the Gauss divergence theorem twice, which leads to the following identity:

$$\int_{B_{2\ell}} \Delta n(t) \psi_\ell \, dx = \int_{B_{2\ell}} n(t) \Delta \psi_\ell \, dx. \quad (4.45)$$

To confirm (4.44), we use the Hölder inequality and $n \in W_{d+2}^{2,1}(\mathcal{Q}_{\widehat{T}})$, obtaining the following estimate:

$$\begin{aligned} & \|\nabla n(t)\psi_\ell\|_{L^1(B_{2\ell})} + \|\nabla \cdot (\nabla n(t)\psi_\ell)\|_{L^1(B_{2\ell})} \\ & \leq \|\nabla n(t)\psi_\ell\|_{L^1(B_{2\ell})} + \|\nabla n(t) \cdot \nabla \psi_\ell\|_{L^1(B_{2\ell})} + \|\Delta n(t)\psi_\ell\|_{L^1(B_{2\ell})} \\ & \leq \|\nabla n(t)\|_{L^{d+2}(B_{2\ell})} \|\psi_\ell\|_{L^{\frac{d+2}{d+1}}(B_{2\ell})} + \|\nabla n(t)\|_{L^{d+2}(B_{2\ell})} \|\nabla \psi_\ell\|_{L^{\frac{d+2}{d+1}}(B_{2\ell})} \\ & \quad + \|\Delta n(t)\|_{L^{d+2}(B_{2\ell})} \|\psi_\ell\|_{L^{\frac{d+2}{d+1}}(B_{2\ell})} \\ & < \infty \end{aligned}$$

for a.e. $t \in (0, \widehat{T})$. This verifies that (4.45) holds, confirming the integrability conditions (4.44).

Concerning the second term on the right-hand side of (4.43), noting (4.42), and using $n \in W_{d+2}^{2,1}(\mathcal{Q}_{\widehat{T}})$, we apply the Gauss divergence theorem to obtain the following equality:

$$-\int_{B_{2\ell}} \nabla \cdot (n \nabla v)(t) \psi_\ell \, dx = \int_{B_{2\ell}} (n \nabla v)(t) \cdot \nabla \psi_\ell \, dx. \quad (4.46)$$

In order to verify (4.46), we conduct the following calculations:

$$\|(n \nabla v)(t) \psi_\ell\|_{L^1(B_{2\ell})} \leq \|n(t)\|_{L^\infty(B_{2\ell})} \|\nabla v(t)\|_{L^\infty(B_{2\ell})} \|\psi_\ell\|_{L^1(B_{2\ell})} < \infty$$

and:

$$\begin{aligned} & \|\nabla \cdot ((n \nabla v)(t) \psi_\ell)\|_{L^1(B_{2\ell})} \\ & \leq \|\nabla n(t)\|_{L^{d+2}(B_{2\ell})} \|\nabla v(t)\|_{L^\infty(B_{2\ell})} \|\psi_\ell\|_{L^{\frac{d+2}{d+1}}(B_{2\ell})} \\ & \quad + \|n(t)\|_{L^{d+2}(B_{2\ell})} \|\Delta v(t)\|_{L^\infty(B_{2\ell})} \|\psi_\ell\|_{L^{\frac{d+2}{d+1}}(B_{2\ell})} \\ & \quad + \|n(t)\|_{L^{d+2}(B_{2\ell})} \|\nabla v(t)\|_{L^\infty(B_{2\ell})} \|\nabla \psi_\ell\|_{L^{\frac{d+2}{d+1}}(B_{2\ell})} < \infty \end{aligned}$$

for a.e. $t \in (0, \widehat{T})$, which yields that (4.46) holds.

As for the third term on the right-hand side of (4.43), since $u \in W^{1,\infty}(\mathbb{R}^d)$, we replace ∇v in (4.46) with u , yielding:

$$-\int_{B_{2\ell}} \nabla \cdot (un(t)) \psi_\ell \, dx = \int_{B_{2\ell}} un(t) \cdot \nabla \psi_\ell \, dx. \quad (4.47)$$

Combining (4.43), (4.45), (4.46) and (4.47), we obtain the following equality:

$$\frac{d}{dt} \int_{B_{2\ell}} n(t) \psi_\ell \, dx = \int_{B_{2\ell}} n(t) \Delta \psi_\ell \, dx + \int_{B_{2\ell}} (n \nabla v)(t) \cdot \nabla \psi_\ell \, dx + \int_{B_{2\ell}} un(t) \cdot \nabla \psi_\ell \, dx. \quad (4.48)$$

Integrating both sides of (4.48) from 0 to t , we obtain the following expression:

$$\begin{aligned} & \int_{B_{2\ell}} n(t) \psi_\ell \, dx - \int_{B_{2\ell}} n_0 \psi_\ell \, dx \\ &= \int_0^t \int_{B_{2\ell}} n(s) \Delta \psi_\ell \, dx \, ds + \int_0^t \int_{B_{2\ell}} (n \nabla v)(s) \cdot \nabla \psi_\ell \, dx \, ds + \int_0^t \int_{B_{2\ell}} u n(s) \cdot \nabla \psi_\ell \, dx \, ds \end{aligned} \quad (4.49)$$

for a.e. $t \in (0, \widehat{T})$. From (4.49), we find the following:

$$\begin{aligned} & \left| \int_{B_{2\ell}} n(t) \psi_\ell \, dx - \int_{B_{2\ell}} n_0 \psi_\ell \, dx \right| \\ & \leq \left| \int_0^t \int_{B_{2\ell}} n(s) \Delta \psi_\ell \, dx \, ds \right| + \left| \int_0^t \int_{B_{2\ell}} (n \nabla v)(s) \cdot \nabla \psi_\ell \, dx \, ds \right| + \left| \int_0^t \int_{B_{2\ell}} u n(s) \cdot \nabla \psi_\ell \, dx \, ds \right| \\ & =: I_\ell + II_\ell + III_\ell. \end{aligned} \quad (4.50)$$

Based on (4.41), (4.42) and (3.26) in Lemma 3.6, we derive the following estimates. For I_ℓ , we have:

$$I_\ell \leq \int_0^{\widehat{T}} \|n(s)\|_{L^{\frac{2d}{2d-1}}(B_{2\ell})} \|\Delta \psi_\ell\|_{L^{2d}(B_{2\ell})} \, ds \leq C \widehat{T} \ell^{-\frac{3}{2}} \sup_{t \in (0, \widehat{T})} \|n(t)\|_{L^{\frac{2d}{2d-1}}}, \quad (4.51)$$

where $C = C(d)$. For II_ℓ , it follows:

$$\begin{aligned} II_\ell & \leq \int_0^{\widehat{T}} \|(n \nabla v)(s) \cdot \nabla \psi_\ell\|_{L^1(B_{2\ell})} \, ds \leq \sup_{t \in (0, \widehat{T})} \|\nabla v(t)\|_{L^\infty} \int_0^{\widehat{T}} \|n(s)\|_{L^{\frac{2d}{2d-1}}} \|\nabla \psi_\ell\|_{L^{2d}(B_{2\ell})} \, ds \\ & \leq C \widehat{T} \ell^{-\frac{1}{2}} \sup_{t \in (0, \widehat{T})} \|n(t)\|_{L^{\frac{2d}{2d-1}}} \sup_{t \in (0, \widehat{T})} \|\nabla v(t)\|_{L^\infty}, \end{aligned} \quad (4.52)$$

where $C = C(d)$. Similarly, for III_ℓ , we obtain:

$$\begin{aligned} III_\ell & \leq \int_0^{\widehat{T}} \|u n(s) \cdot \nabla \psi_\ell\|_{L^1(B_{2\ell})} \, ds \leq \|u\|_{L^\infty} \int_0^{\widehat{T}} \|n(s)\|_{L^{\frac{2d}{2d-1}}} \|\nabla \psi_\ell\|_{L^{2d}(B_{2\ell})} \, ds \\ & \leq C \widehat{T} \ell^{-\frac{1}{2}} \sup_{t \in (0, \widehat{T})} \|n(t)\|_{L^{\frac{2d}{2d-1}}} \|u\|_{L^\infty}, \end{aligned} \quad (4.53)$$

where $C = C(d)$. Therefore, in view of (4.50), (4.51), (4.52) and (4.53), by choosing an arbitrary number $\ell \geq 1$, we obtain the following:

$$\begin{aligned}
& \int_{\mathbb{R}^d} n(t) \psi_\ell \, dx \\
& \leq \int_{\mathbb{R}^d} n_0 \, dx + C \widehat{T} \ell^{-\frac{3}{2}} \sup_{t \in (0, \widehat{T})} \|n(t)\|_{L^{\frac{2d}{2d-1}}} \\
& \quad + C \widehat{T} \ell^{-\frac{1}{2}} \sup_{t \in (0, \widehat{T})} \|n(t)\|_{L^{\frac{2d}{2d-1}}} \sup_{t \in (0, \widehat{T})} \|\nabla v(t)\|_{L^\infty} + C \widehat{T} \ell^{-\frac{1}{2}} \sup_{t \in (0, \widehat{T})} \|n(t)\|_{L^{\frac{2d}{2d-1}}} \|u\|_{L^\infty} \\
& \leq \int_{\mathbb{R}^d} n_0 \, dx + C \widehat{T} \sup_{t \in (0, \widehat{T})} \|n(t)\|_{L^{\frac{2d}{2d-1}}} \\
& \quad + C \widehat{T} \sup_{t \in (0, \widehat{T})} \|n(t)\|_{L^{\frac{2d}{2d-1}}} \sup_{t \in (0, \widehat{T})} \|\nabla v(t)\|_{L^\infty} + C \widehat{T} \sup_{t \in (0, \widehat{T})} \|n(t)\|_{L^{\frac{2d}{2d-1}}} \|u\|_{L^\infty},
\end{aligned}$$

where $C = C(d)$. Since the right-hand side is independent of ℓ , by the monotone convergence theorem of Beppo Levi, we observe:

$$\begin{aligned}
& \left| \int_{\mathbb{R}^d} n(t) \, dx - \int_{\mathbb{R}^d} n_0 \, dx \right| \leq \left| \int_{\mathbb{R}^d} n(t) \, dx - \int_{\mathbb{R}^d} n(t) \psi_\ell \, dx \right| + \left| \int_{\mathbb{R}^d} n_0 \, dx - \int_{\mathbb{R}^d} n_0 \psi_\ell \, dx \right| \\
& \quad + C \widehat{T} \ell^{-\frac{3}{2}} \sup_{t \in (0, \widehat{T})} \|n(t)\|_{L^{\frac{2d}{2d-1}}} \\
& \quad + C \widehat{T} \ell^{-\frac{1}{2}} \sup_{t \in (0, \widehat{T})} \|n(t)\|_{L^{\frac{2d}{2d-1}}} \sup_{t \in (0, \widehat{T})} \|\nabla v(t)\|_{L^\infty} \\
& \quad + C \widehat{T} \ell^{-\frac{1}{2}} \sup_{t \in (0, \widehat{T})} \|n(t)\|_{L^{\frac{2d}{2d-1}}} \|u\|_{L^\infty} \rightarrow 0
\end{aligned}$$

as $\ell \rightarrow \infty$. This completes the proof of Theorem 2.1 (III). $\square$

5. Proof of Theorem 2.2: ε -regularity theorem

We proceed with the proof in four steps.

Step 1. We begin by showing that if the local mass concentration of n at a given time t , as described in (5.1) below, is less than a certain constant m_* , then the local $L^\infty(\mathbb{R}^2)$ -norm of n remains bounded over a suitable time interval around t . In this paper, we refer to such an ε -regularity result as an ε -regularity theorem of sharp type. The concept of a sharp type ε -regularity theorem was first introduced by Luckhaus, Sugiyama, and Velázquez [12, Propositions 10, 14]. Here, we generalize and establish this sharp type ε -regularity theorem for (KSF), which includes the fluid vector field term u as defined in (2.5) of Definition 2.1.

Proposition 5.1. *Let Assumption 2.1 hold and suppose that $n_0 \in L^1(\mathbb{R}^2)$. Let $T_{\max}$ denote the maximal existence time of the strong solution (n, v) of (KSF) obtained in Theorem 2.1 (I)–(III). There exist an absolute positive constant m_* and a positive constant $c_1 = c_1(\gamma, Q, \lambda, \|n_0\|_{L^1(\mathbb{R}^2)})$ such that if:*

$$\int_{B_{2\rho}(x_0)} n(x, t_1) \, dx \leq m_* \quad (5.1)$$

holds for some $x_0 \in \mathbb{R}^2$, $0 < \rho \leq 1$, and $t_1 \in [0, T_{\max})$, then the following estimates are valid:

(i). In the case $2c_1\rho^2 < T_{\max}$ and $2c_1\rho^2 \leq t_1 < T_{\max}$, the following estimate holds:

$$\sup_{s \in (t_1 - c_1\rho^2, \min\{t_1 + 2c_1\rho^2, T_{\max}\})} \int_{B_{\frac{\rho}{2}}(x_0)} n^2(x, s) \, dx \leq \frac{C}{\rho^3}. \quad (5.2)$$

(ii). In the case $2c_1\rho^2 < T_{\max}$ and $0 \leq t_1 < \min\{2c_1\rho^2, T_{\max}\} = 2c_1\rho^2$, the following estimate holds:

$$\sup_{s \in (0, \min\{t_1 + 2c_1\rho^2, T_{\max}\})} \int_{B_{\frac{\rho}{2}}(x_0)} sn^2(x, s) \, dx \leq \frac{C}{\rho^5}. \quad (5.3)$$

(iii). In the case $2c_1\rho^2 \geq T_{\max}$ and $0 \leq t_1 < \min\{2c_1\rho^2, T_{\max}\} = T_{\max}$, provided that $n_0 \in L^2_{\text{loc}}(\mathbb{R}^2)$, the following estimate holds:

$$\sup_{s \in (0, \min\{t_1 + 2c_1\rho^2, T_{\max}\})} \int_{B_{\frac{\rho}{2}}(x_0)} n^2(x, s) \, dx \leq \frac{C}{\rho^7} + \|n_0\|_{L^2(B_\rho(x_0))}^2. \quad (5.4)$$

Here, $C = C(\gamma, Q, \lambda, \|n_0\|_{L^1(\mathbb{R}^2)})$, and it is independent of x_0 and ρ .

Remark 11. The constant c_1 appearing in (5.2), (5.3) and (5.4) of Proposition 5.1 can be explicitly expressed as:

$$c_1 = \frac{1}{8\tilde{C}m_*\|n_0\|_{L^1(\mathbb{R}^2)}(1 + \|n_0\|_{L^1(\mathbb{R}^2)})}, \quad (5.5)$$

where $\tilde{C} = \tilde{C}(\gamma, Q, \lambda, \|n_0\|_{L^1(\mathbb{R}^2)})$.

Proof of Proposition 5.1. Let $0 < \rho \leq 1$, and let ψ be the function with $\rho_1 = \frac{\rho}{2}$ and $\rho_2 = \rho$ as in Lemma 3.6. Let $k > 0$. We choose $t_0 \in (-\infty, T_{\max})$ and take $t \in (0, T_{\max}) \cap [t_0, T_{\max})$.

Noting that:

$$\begin{aligned} \frac{1}{2} \partial_t (n(x, t)^2 (t - t_0)^2 \psi^k(x - x_0)) &= n^2(x, t) (t - t_0) \psi^k(x - x_0) \\ &\quad + n(x, t) \partial_t n(x, t) (t - t_0)^2 \psi^k(x - x_0), \end{aligned}$$

we then apply the first equation of (KSF) to deduce:

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^2} n^2(x, t) (t - t_0)^2 \psi^k(x - x_0) dx \\
&= \int_{\mathbb{R}^2} n^2(x, t) (t - t_0) \psi^k(x - x_0) dx + \int_{\mathbb{R}^2} n(x, t) \partial_t n(x, t) (t - t_0)^2 \psi^k(x - x_0) dx \\
&= \int_{\mathbb{R}^2} n^2(x, t) (t - t_0) \psi^k(x - x_0) dx - \int_{\mathbb{R}^2} \nabla n(x, t) \cdot \nabla \left(n(x, t) (t - t_0)^2 \psi^k(x - x_0) \right) dx \\
&\quad + \int_{\mathbb{R}^2} (n \nabla v)(x, t) \cdot \nabla \left(n(x, t) (t - t_0)^2 \psi^k(x - x_0) \right) dx \\
&\quad + \int_{\mathbb{R}^2} u(x) n(x, t) \cdot \nabla \left(n(x, t) (t - t_0)^2 \psi^k(x - x_0) \right) dx
\end{aligned} \tag{5.6}$$

for all $t \in (0, T_{\max}) \cap [t_0, T_{\max})$.

We define $\phi(x, t)$ by:

$$\phi(x, t) := (t - t_0) \psi^{\frac{k}{2}}(x - x_0). \tag{5.7}$$

Then we have:

$$\begin{aligned}
\nabla n \cdot \nabla (n \times (t - t_0)^2 \psi^k(x - x_0)) &= \nabla n \cdot \nabla (n \phi^2) = (\nabla n \cdot \nabla n) \phi^2 + 2(\nabla n \cdot \nabla \phi) n \phi \\
&= |\phi \nabla n + n \nabla \phi|^2 - n^2 |\nabla \phi|^2 = |\nabla(n \phi)|^2 - n^2 |\nabla \phi|^2,
\end{aligned} \tag{5.8}$$

which implies:

$$\begin{aligned}
& - \int_{\mathbb{R}^2} \nabla n(x, t) \cdot \nabla (n(x, t) \psi^k(x - x_0) (t - t_0)^2) dx \\
&= - \int_{\mathbb{R}^2} |\nabla(n(x, t) \phi(x, t))|^2 dx + \int_{\mathbb{R}^2} n(x, t)^2 |\nabla \phi(x, t)|^2 dx.
\end{aligned} \tag{5.9}$$

Substituting (5.9) into (5.6), we obtain:

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^2} n(x, t)^2 (t - t_0)^2 \psi^k(x - x_0) dx = -g_0(t) + g_1(t) + g_2(t) + g_3(t) + g_4(t) \tag{5.10}$$

for all $t \in (0, T_{\max}) \cap [t_0, T_{\max})$, where:

$$g_0(t) = \int_{\mathbb{R}^2} |\nabla(n(x, t) \phi(x, t))|^2 dx \geq 0,$$

$$\begin{aligned}
g_1(t) &= \frac{1}{t-t_0} \int_{\mathbb{R}^2} n^2(x, t) \phi^2(x, t) \, dx, & g_2(t) &= \int_{\mathbb{R}^2} n^2(x, t) |\nabla \phi(x, t)|^2 \, dx, \\
g_3(t) &= \int_{\mathbb{R}^2} (n \nabla v)(x, t) \cdot \nabla \left(n(x, t) \phi^2(x, t) \right) \, dx, \\
g_4(t) &= \int_{\mathbb{R}^2} u(x) n(x, t) \cdot \nabla \left(n(x, t) \phi^2(x, t) \right) \, dx.
\end{aligned} \tag{5.11}$$

We now derive estimates for g_1 , g_2 , g_3 , and g_4 as indicated above. To achieve this, we establish several auxiliary inequalities based on the Gagliardo–Nirenberg and Hölder inequalities.

Lemma 5.2. *Let $0 < \rho < 1$ and $k \geq 8$. Let $t_0 \in (-\infty, \min\{t_1 + 2c_1\rho^2, T_{\max}\})$ with the constant c_1 defined in (5.5). Let ϕ be the function introduced in (5.7) with ψ satisfying $\rho_1 = \frac{\rho}{2}$ and $\rho_2 = \rho$ as in Lemma 3.6. We assume $n(t) \in L^1_{\text{loc}}(\mathbb{R}^2)$ and $n(t) \in W^{1,2}_{\text{loc}}(\mathbb{R}^2)$ for all $t \in (0, T_{\max}) \cap [t_0, T_{\max})$. Then, the following estimates hold:*

$$\int_{\mathbb{R}^2} n^3(x, t) \phi^3(x, t) \, dx \leq C \|n(t)\|_{L^1(\text{supp } \psi(\cdot - x_0))} (t - t_0) g_0(t), \tag{5.12}$$

$$\int_{\mathbb{R}^2} n^2(x, t) \phi^2(x, t) \, dx \leq C \|n(t)\|_{L^1(\text{supp } \psi(\cdot - x_0))} (t - t_0) g_0^{\frac{1}{2}}(t), \tag{5.13}$$

$$\int_{\mathbb{R}^2} n^2(x, t) |\nabla \phi(x, t)|^2 \, dx \leq C \left(\frac{k}{\rho} \right)^2 \|n(t)\|_{L^1(\text{supp } \psi(\cdot - x_0))} (t - t_0) g_0^{\frac{1}{2}}(t) \tag{5.14}$$

for all $t \in (0, T_{\max}) \cap [t_0, T_{\max})$, where C is an absolute positive constant and $g_0(t)$ is as defined in (5.11).

In addition, there exists an absolute positive constant C such that:

$$\begin{aligned}
& \int_{\mathbb{R}^2} n^3(x, t) \phi^2(x, t) \, dx \\
& \leq m_* \|n(t)\|_{L^1(\text{supp } \psi(\cdot - x_0))} g_0(t) + C \left(\frac{k}{\rho} \right)^2 \|n(t)\|_{L^1(\text{supp } \psi(\cdot - x_0))}^2 (t - t_0) g_0^{\frac{1}{2}}(t)
\end{aligned} \tag{5.15}$$

for all $t \in (0, T_{\max}) \cap [t_0, T_{\max})$. Here, m_* is the constant introduced in (2.15).

Proof of Lemma 5.2. First, we prove (5.12). Applying the Gagliardo–Nirenberg inequality, we obtain:

$$\begin{aligned}
\int_{\mathbb{R}^2} n^3(x, t) \phi^3(x, t) \, dx & \leq C \int_{\mathbb{R}^2} n(x, t) \phi(x, t) \, dx \int_{\mathbb{R}^2} |\nabla n(x, t) \phi(x, t)|^2 \, dx \\
& \leq C \|u(t)\|_{L^1(\text{supp } \psi(\cdot - x_0))} (t - t_0) g_0(t),
\end{aligned}$$

where C is an absolute positive constant.

Next, we establish (5.13). Using (5.7), (5.12) and the interpolation inequality, we have:

$$\begin{aligned} \int_{\mathbb{R}^2} n^2(x, t) \phi^2(x, t) dx &\leq \left(\int_{\mathbb{R}^2} n(x, t) \phi(x, t) dx \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^2} n^3(x, t) \phi^3(x, t) dx \right)^{\frac{1}{2}} \\ &\leq C \|n(t)\|_{L^1(\text{supp } \psi(\cdot - x_0))} (t - t_0) g_0^{\frac{1}{2}}(t) \end{aligned} \quad (5.16)$$

for all $t \in (0, T_{\max}) \cap [t_0, T_{\max})$, where C is an absolute positive constant.

We next address (5.14). Using (5.7) and the Hölder inequality, we obtain:

$$\begin{aligned} \int_{\mathbb{R}^2} n^2(x, t) |\nabla \phi(x, t)|^2 dx \\ \leq \left(\frac{k}{2} \right)^2 \left(\int_{\mathbb{R}^2} n(x, t) |\nabla \psi(x - x_0)|^4 dx \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^2} n^3(x, t) (t - t_0)^4 \psi^{2(k-2)}(x - x_0) dx \right)^{\frac{1}{2}}. \end{aligned} \quad (5.17)$$

Since it holds:

$$(t - t_0)^4 \psi^{2(k-2)}(x - x_0) = \phi^3(x, t) \cdot (t - t_0) \psi^{\frac{k}{2}-4}(x - x_0) \leq (t - t_0) \phi^3(x, t) \quad (5.18)$$

for $k \geq 8$, substituting (5.18) into (5.17), and applying (3.24) in Lemma 3.6 with $\rho_1 = \frac{\rho}{2}$ and $\rho_2 = \rho$, we have:

$$\int_{\mathbb{R}^2} n^2(x, t) |\nabla \phi(x, t)|^2 dx \leq \left(\frac{k}{2} \right)^2 \left(\frac{32}{\rho} \right)^2 (t - t_0)^{\frac{1}{2}} \|n(t)\|_{L^1(\text{supp } \psi(\cdot - x_0))}^{\frac{1}{2}} \left(\int_{\mathbb{R}^2} n^3(x, t) \phi^3(x, t) dx \right)^{\frac{1}{2}}$$

for all $k \geq 8$ and all $t \in (0, T_{\max}) \cap [t_0, T_{\max})$. Combining this with (5.12) yields:

$$\int_{\mathbb{R}^2} n^2(x, t) |\nabla \phi(x, t)|^2 dx \leq C \left(\frac{k}{\rho} \right)^2 \|n(t)\|_{L^1(\text{supp } \psi(\cdot - x_0))} (t - t_0) g_0^{\frac{1}{2}}(t) \quad (5.19)$$

for all $k \geq 8$ and $t \in (0, T_{\max}) \cap [t_0, T_{\max})$, where C is an absolute positive constant. Thus, we obtain (5.14).

To show (5.15), it should be noted by the Sobolev inequality:

$$\|w\|_{L^2} \leq C \|\nabla w\|_{L^1} \quad \text{for } w \in W^{1,1}(\mathbb{R}^2), \quad (5.20)$$

where C is an absolute positive constant. By (5.20), we have:

$$\int_{\mathbb{R}^2} n^3(x, t) \phi^2(x, t) dx = \int_{\mathbb{R}^2} (n^{\frac{3}{2}}(x, t) \phi(x, t))^2 dx \leq C \left(\int_{\mathbb{R}^2} \left| \nabla (u^{\frac{3}{2}}(x, t) \phi(x, t)) \right| dx \right)^2,$$

where C is an absolute positive constant. Applying the product rule, we have:

$$\nabla(n^{\frac{3}{2}}\phi) = \frac{3}{2}n^{\frac{1}{2}}\nabla n\phi + n^{\frac{3}{2}}\nabla\phi = \frac{3}{2}n^{\frac{1}{2}}\nabla(n\phi) - \frac{1}{2}n^{\frac{1}{2}}n\nabla\phi.$$

By the Hölder inequality, and using (5.14), we deduce:

$$\begin{aligned} & \int_{\mathbb{R}^2} n^3(x, t)\phi^2(x, t) \, dx \\ & \leq m_* \left(\int_{\mathbb{R}^2} n^{\frac{1}{2}}(x, t) \left| \nabla(n(x, t)\phi(x, t)) \right| \, dx \right)^2 + C \left(\int_{\mathbb{R}^2} n^{\frac{1}{2}}(x, t) |n(x, t)\nabla\phi(x, t)| \, dx \right)^2 \\ & \leq m_* \|n(t)\|_{L^1(\text{supp } \psi(\cdot - x_0))} \int_{\mathbb{R}^2} \left| \nabla(n(x, t)\phi(x, t)) \right|^2 \, dx \\ & \quad + C \|n(t)\|_{L^1(\text{supp } \psi(\cdot - x_0))} \int_{\mathbb{R}^2} n^2(x, t) |\nabla\phi(x, t)|^2 \, dx \\ & \leq m_* \|n(t)\|_{L^1(\text{supp } \psi(\cdot - x_0))} g_0(t) + C \left(\frac{k}{\rho} \right)^2 \|n(t)\|_{L^1(\text{supp } \psi(\cdot - x_0))}^2 (t - t_0) g_0^{\frac{1}{2}}(t) \end{aligned}$$

for all $t \in (0, T_{\max}) \cap [t_0, T_{\max})$, where m_* is the constant from (2.15), and C is an absolute positive constant. This completes the proof of Lemma 5.2. $\square$

Completion of the proof of Proposition 5.1. Concerning g_1 , using (5.13), we have:

$$g_1(t) \leq C \|n(t)\|_{L^1(\text{supp } \psi(\cdot - x_0))} g_0^{\frac{1}{2}}(t) \quad (5.21)$$

for all $t \in (0, T_{\max}) \cap [t_0, T_{\max})$, where C is an absolute positive constant.

For g_2 , applying (5.14), we deduce:

$$g_2(t) \leq C \left(\frac{k}{\rho} \right)^2 \|n(t)\|_{L^1(\text{supp } \psi(\cdot - x_0))} (t - t_0) g_0^{\frac{1}{2}}(t) \quad (5.22)$$

for all $t \in (0, T_{\max}) \cap [t_0, T_{\max})$, where C is an absolute positive constant.

Regarding g_3 , we have:

$$\begin{aligned} g_3(t) &= \int_{\mathbb{R}^2} (n\nabla v)(x, t) \cdot \nabla(n(x, t)\phi^2(x, t)) \, dx \\ &= \int_{\mathbb{R}^2} n(x, t)\nabla n(x, t) \cdot \nabla v(x, t)\phi^2(x, t) \, dx + 2 \int_{\mathbb{R}^2} (n\nabla v)(x, t) \cdot u(x, t)\phi(x, t)\nabla\phi(x, t) \, dx \end{aligned}$$

By integration by parts, we obtain:

$$\begin{aligned}
g_3(t) = & -\frac{1}{2} \int_{\mathbb{R}^2} n^2(x, t) \Delta v(x, t) \phi^2(x, t) \, dx - \int_{\mathbb{R}^2} n^2(x, t) \nabla v(x, t) \cdot \phi(x, t) \nabla \phi(x, t) \, dx \\
& + 2 \int_{\mathbb{R}^2} (n \nabla v)(x, t) \cdot n(x, t) \phi(x, t) \nabla \phi(x, t) \, dx.
\end{aligned}$$

Using the second equation in (KSF), we obtain:

$$\begin{aligned}
g_3(t) = & -\frac{\gamma}{2} \int_{\mathbb{R}^2} n^2(x, t) v(x, t) \phi^2(x, t) \, dx + \frac{1}{2} \int_{\mathbb{R}^2} n^3(x, t) \phi^2(x, t) \, dx \\
& + \int_{\mathbb{R}^2} n^2(x, t) \nabla v(x, t) \cdot \phi(x, t) \nabla \phi(x, t) \, dx \\
\leq & \frac{1}{2} \int_{\mathbb{R}^2} n^3(x, t) \phi^2(x, t) \, dx - \int_{\mathbb{R}^2} n^2(x, t) v(x, t) \left(|\nabla \phi(x, t)|^2 + \phi \Delta \phi(x, t) \right) \, dx \\
& - 2 \int_{\mathbb{R}^2} n(x, t) \nabla n(x, t) \cdot v(x, t) \phi(x, t) \nabla \phi(x, t) \, dx \\
= & g_{3,1}(t) + g_{3,2}(t) + g_{3,3}(t),
\end{aligned} \tag{5.23}$$

where the terms are defined as follows:

$$\begin{aligned}
g_{3,1}(t) &:= \frac{1}{2} \int_{\mathbb{R}^2} n^3(x, t) \phi^2(x, t) \, dx, \\
g_{3,2}(t) &:= - \int_{\mathbb{R}^2} n^2(x, t) v(x, t) \left(|\nabla \phi(x, t)|^2 + \phi \Delta \phi(x, t) \right) \, dx, \\
g_{3,3}(t) &:= -2 \int_{\mathbb{R}^2} n(x, t) \nabla n(x, t) \cdot v(x, t) \phi(x, t) \nabla \phi(x, t) \, dx.
\end{aligned}$$

Regarding $g_{3,1}$, it follows from (5.15) that:

$$g_{3,1}(t) \leq \frac{m_*}{2} \|n(t)\|_{L^1(\text{supp } \psi(\cdot - x_0))} g_0(t) + C \left(\frac{k}{\rho} \right)^2 \|n(t)\|_{L^1(\text{supp } \psi(\cdot - x_0))}^2 (t - t_0) g_0^{\frac{1}{2}}(t) \tag{5.24}$$

for all $t \in (0, T_{\max}) \cap [t_0, T_{\max})$. The constant m_* is introduced in (2.15), and C is an absolute positive constant.

We now estimate $g_{3,2}$. By its definition, we see:

$$g_{3,2}(t) \leq \int_{\mathbb{R}^2} n^2(x, t) v(x, t) \left(|\nabla \phi(x, t)|^2 + \phi(x, t) |\Delta \phi(x, t)| \right) \, dx =: J. \tag{5.25}$$

Since we find:

$$|\nabla\phi(x, t)|^2 = \left(\frac{k}{2}\right)^2 (t - t_0)^2 \psi^{k-2}(x - x_0) |\nabla\psi(x - x_0)|^2$$

and:

$$\begin{aligned} & \phi(x, t) |\Delta\phi(x, t)| \\ &= (t - t_0) \psi^{\frac{k}{2}}(x - x_0) \cdot (t - t_0) |\Delta\psi^{\frac{k}{2}}(x - x_0)| \\ &\leq (t - t_0)^2 \psi^{\frac{k}{2}}(x - x_0) \frac{k}{2} \left\{ \left(\frac{k}{2} - 1\right) \psi^{\frac{k}{2}-2}(x - x_0) |\nabla\psi(x - x_0)|^2 + \psi^{\frac{k}{2}-1}(x - x_0) |\Delta\psi(x - x_0)| \right\}, \end{aligned}$$

choosing $k \geq 8$ and $\rho_1 = \frac{\rho}{2}$ and $\rho_2 = \rho$ in Lemma 3.6, and using (3.24) and (3.25), we have:

$$\begin{aligned} & |\nabla\phi(x, t)|^2 + \phi(x, t) |\Delta\phi(x, t)| \\ &\leq \frac{k(k-1)}{2} (t - t_0)^2 \psi^{k-2}(x - x_0) \left(|\nabla\psi(x - x_0)|^2 + \psi(x - x_0) |\Delta\psi(x - x_0)| \right) \\ &\leq \frac{Ck(k-1)}{\rho^2} (t - t_0)^2 \psi^{k-1}(x - x_0), \end{aligned} \quad (5.26)$$

where C is an absolute positive constant. Substituting (5.26) into (5.25), we obtain:

$$J \leq \frac{Ck(k-1)}{\rho^2} \int_{\mathbb{R}^2} n^2(x, t) v(x, t) (t - t_0)^2 \psi^{k-1}(x - x_0) dx. \quad (5.27)$$

On the other hand, by the definition of the Lorentz norm as given in (2.1) and the Hardy-Littlewood rearrangement theorem, we have:

$$\int_{\mathbb{R}^2} |\varphi_1(x) \varphi_2(x)| dx \leq \|\varphi_1\|_{L^{p,1}} \|\varphi_2\|_{L^{p',\infty}} \quad (5.28)$$

for all $1 < p < \infty$ with $\frac{1}{p} + \frac{1}{p'} = 1$.

Since it holds:

$$\begin{aligned} (t - t_0)^2 \psi^{k-1}(x - x_0) &= (t - t_0)^{\frac{2}{3}} \phi^{\frac{4}{3}}(x, t) \psi^{\frac{k}{3}-1}(x - x_0) \\ &\leq (t - t_0)^{\frac{2}{3}} \phi^{\frac{4}{3}}(x, t) \chi_{\text{supp } \psi(\cdot - x_0)}(x) \end{aligned} \quad (5.29)$$

for $k \geq 3$, we observe from (5.27), (5.28), (5.29) and the Hölder inequality for weak spaces:

$$\begin{aligned} J &\leq \frac{Ck(k-1)}{\rho^2} \int_{\mathbb{R}^2} n^2(x, t) v(x, t) (t - t_0)^{\frac{2}{3}} \phi^{\frac{4}{3}}(x, t) \chi_{\text{supp } \psi(\cdot - x_0)}(x) dx \\ &\leq \frac{Ck(k-1)}{\rho^2} \|n^2(t) v(t) \phi^{\frac{4}{3}}(t)\|_{L^{\frac{6}{5},\infty}} \|\chi_{\text{supp } \psi(\cdot - x_0)}\|_{L^{6,1}} (t - t_0)^{\frac{2}{3}} \\ &\leq \frac{Ck(k-1)}{\rho^2} \|n^2(t) \phi^{\frac{4}{3}}(t)\|_{L^{\frac{3}{2},\infty}} \|v(t)\|_{L^{6,\infty}} \|\chi_{\text{supp } \psi(\cdot - x_0)}\|_{L^{6,1}} (t - t_0)^{\frac{2}{3}} \end{aligned}$$

$$\leq \frac{Ck(k-1)}{\rho^2} \|n^2(t)\phi^{\frac{4}{3}}(t)\|_{L^{\frac{3}{2}}}^{\frac{2}{3}} \|v(t)\|_{L^{6,\infty}} \|\chi_{\text{supp } \psi(\cdot-x_0)}\|_{L^{6,1}} (t-t_0)^{\frac{2}{3}}, \quad (5.30)$$

where C is an absolute positive constant. By direct computation, we obtain:

$$\|\chi_{\text{supp } \psi(\cdot-x_0)}\|_{L^{6,1}} = \int_0^\infty s \mu(|\chi_{\text{supp } \psi(\cdot-x_0)}| > s)^{\frac{1}{6}} \frac{ds}{s} = \int_0^1 |B_\rho(x_0)|^{\frac{1}{6}} ds = \pi^{\frac{1}{6}} \rho^{\frac{1}{3}}. \quad (5.31)$$

By Lemma 3.5, we find:

$$\|v(t)\|_{L^{6,\infty}} = \|(\gamma - \Delta)^{-\frac{5}{6}} (\gamma - \Delta)^{-\frac{1}{6}} n(t)\|_{L^{6,\infty}} \leq C \|(\gamma - \Delta)^{-\frac{1}{6}} n(t)\|_{L^1} \leq C \|n(t)\|_{L^1}, \quad (5.32)$$

where $C = C(\gamma)$. Substituting (5.31) and (5.32) into (5.30), we have:

$$J \leq \frac{Ck(k-1)}{\rho^{\frac{5}{3}}} \|n(t)\|_{L^1} \|n^3(t)\phi^2(t)\|_{L^1}^{\frac{2}{3}} (t-t_0)^{\frac{2}{3}}, \quad (5.33)$$

where $C = C(\gamma)$. According to (5.15):

$$\begin{aligned} & \left(\int_{\mathbb{R}^2} n^3(x, t) \phi^2(x, t) dx \right)^{\frac{2}{3}} \\ & \leq m_*^{\frac{2}{3}} \left(\|n(t)\|_{L^1(\text{supp } \psi(\cdot-x_0))}^{\frac{2}{3}} g_0^{\frac{2}{3}}(t) + \left(\frac{k}{\rho}\right)^{\frac{4}{3}} \|n(t)\|_{L^1(\text{supp } \psi(\cdot-x_0))}^{\frac{4}{3}} (t-t_0)^{\frac{2}{3}} g_0^{\frac{1}{3}}(t) \right), \end{aligned} \quad (5.34)$$

where m_* is the constant associated with (2.15). Combining (5.34) with (5.25) and (5.33), it follows that:

$$\begin{aligned} g_{3,2}(t) & \leq J \leq \frac{Ck^2}{\rho^{\frac{5}{3}}} \|n(t)\|_{L^1} \|n(t)\|_{L^1}^{\frac{2}{3}} \|\chi_{\text{supp } \psi(\cdot-x_0)}\|_{L^{6,1}}^{\frac{2}{3}} (t-t_0)^{\frac{2}{3}} g_0^{\frac{2}{3}}(t) \\ & \quad + \frac{Ck^{\frac{10}{3}}}{\rho^3} \|n(t)\|_{L^1} \|n(t)\|_{L^1}^{\frac{4}{3}} \|\chi_{\text{supp } \psi(\cdot-x_0)}\|_{L^{6,1}}^{\frac{4}{3}} (t-t_0)^{\frac{4}{3}} g_0^{\frac{1}{3}}(t), \end{aligned} \quad (5.35)$$

where $C = C(\gamma)$.

For $g_{3,3}$, using the identity $(\nabla n)\phi = \nabla(n\phi) - n\nabla\phi$, we obtain:

$$\begin{aligned} g_{3,3}(t) & = -2 \int_{\mathbb{R}^2} n(x, t) \nabla n(x, t) \phi(x, t) \cdot v(x, t) \nabla \phi(x, t) dx \\ & \leq 2 \|\nabla(n\phi)(t)\|_{L^2} \|n(t)v(t)\nabla\phi(t)\|_{L^2} + 2 \int_{\mathbb{R}^2} n^2(x, t) v(x, t) |\nabla\phi(x, t)|^2 dx \\ & \leq 2g_0^{\frac{1}{2}}(t) \|n(t)v(t)\nabla\phi(t)\|_{L^2} + 2J. \end{aligned} \quad (5.36)$$

Applying Lemma 3.6 with $\rho_1 = \frac{\rho}{2}$ and $\rho_2 = \rho$, together with (5.28) and the Hölder inequality for weak spaces, we estimate:

$$\begin{aligned}
 & \|n(t)v(t)\nabla\phi(t)\|_{L^2}^2 \\
 &= \left(\frac{k}{2}\right)^2 \int_{\mathbb{R}^2} n^2(x, t) v^2(x, t) (t-t_0)^2 |\nabla\psi(x-x_0)|^2 \psi^{k-2}(x-x_0) dx \\
 &\leq C \left(\frac{k}{2}\right)^2 \left(\frac{4\sqrt{2}}{\rho}\right)^2 \|n^2(t)v^2(t)\phi^{\frac{4}{3}}(t)\|_{L^{\frac{6}{5}, \infty}} \|\chi_{\text{supp } \psi(\cdot-x_0)}\|_{L^{6,1}} (t-t_0)^{\frac{2}{3}} \\
 &\leq C \left(\frac{k}{2}\right)^2 \left(\frac{4\sqrt{2}}{\rho}\right)^2 \|n^2(t)\phi^{\frac{4}{3}}(t)\|_{L^{\frac{3}{2}}} \|v^2(t)\|_{L^{6,\infty}} \|\chi_{\text{supp } \psi(\cdot-x_0)}\|_{L^{6,1}} (t-t_0)^{\frac{2}{3}} \\
 &= C \left(\frac{k}{2}\right)^2 \left(\frac{4\sqrt{2}}{\rho}\right)^2 \|n^2(t)\phi^{\frac{4}{3}}(t)\|_{L^{\frac{3}{2}}} \|v(t)\|_{L^{12,\infty}}^2 \|\chi_{\text{supp } \psi(\cdot-x_0)}\|_{L^{6,1}} (t-t_0)^{\frac{2}{3}}, \quad (5.37)
 \end{aligned}$$

where C is an absolute positive constant. By Lemma 3.5, we have:

$$\|v(t)\|_{L^{12,\infty}} = \|(\gamma - \Delta)^{-\frac{11}{12}} (\gamma - \Delta)^{-\frac{1}{12}} n(t)\|_{L^{12,\infty}} \leq C \|(\gamma - \Delta)^{-\frac{1}{12}} n(t)\|_{L^1} \leq C \|n(t)\|_{L^1}, \quad (5.38)$$

where $C = C(\gamma)$. Substituting (5.31) and (5.38) into (5.37), we obtain:

$$\|n(t)v(t)\nabla\phi(t)\|_{L^2}^2 \leq \frac{Ck^2}{\rho^{\frac{5}{3}}} \|n^3(t)\varphi^2(t)\|_{L^1}^{\frac{2}{3}} \|n(t)\|_{L^1}^2 (t-t_0)^{\frac{2}{3}}, \quad (5.39)$$

where $C = C(\gamma)$. Further substituting (5.15) into (5.39), we obtain:

$$\begin{aligned}
 & \|n(t)v(t)\nabla\phi(t)\|_{L^2} \leq \frac{Ck}{\rho^{\frac{5}{6}}} \|n(t)\|_{L^1} (t-t_0)^{\frac{1}{3}} \\
 & \times \left(\|n(t)\|_{L^1(\text{supp } \psi(\cdot-x_0))}^{\frac{1}{3}} g_0^{\frac{1}{3}}(t) + \left(\frac{k}{\rho}\right)^{\frac{2}{3}} \|n(t)\|_{L^1(\text{supp } \psi(\cdot-x_0))}^{\frac{2}{3}} (t-t_0)^{\frac{1}{3}} g_0^{\frac{1}{6}}(t) \right), \quad (5.40)
 \end{aligned}$$

where $C = C(\gamma)$. Using (5.35) and (5.40) in (5.36), we estimate $g_{3,3}(t)$ as follows:

$$\begin{aligned}
 g_{3,3}(t) &\leq 2g_0^{\frac{1}{2}}(t) \|u(t)v(t)\nabla\phi(t)\|_2 + 2J \\
 &\leq \frac{Ck}{\rho^{\frac{5}{6}}} \|n(t)\|_1 (t-t_0)^{\frac{1}{3}} g_0^{\frac{1}{2}}(t) \\
 &\quad \times \left(\|n(t)\|_{L^1(\text{supp } \psi(\cdot-x_0))}^{\frac{1}{3}} g_0^{\frac{1}{3}}(t) + \left(\frac{k}{\rho}\right)^{\frac{2}{3}} \|n(t)\|_{L^1(\text{supp } \psi(\cdot-x_0))}^{\frac{2}{3}} (t-t_0)^{\frac{1}{3}} g_0^{\frac{1}{6}}(t) \right) \\
 &\quad + \frac{Ck^2}{\rho^{\frac{5}{3}}} \|n(t)\|_{L^1} \|n(t)\|_{L^1(\text{supp } \psi(\cdot-x_0))}^{\frac{2}{3}} (t-t_0)^{\frac{2}{3}} g_0^{\frac{2}{3}}(t)
 \end{aligned}$$

$$+ \frac{Ck^{\frac{10}{3}}}{\rho^3} \|n(t)\|_{L^1} \|n(t)\|_{L^1(\text{supp } \psi(\cdot - x_0))}^{\frac{4}{3}} (t - t_0)^{\frac{4}{3}} g_0^{\frac{1}{3}}(t), \quad (5.41)$$

where $C = C(\gamma)$.

Combining (5.23), (5.24), (5.35), and (5.41), we obtain:

$$\begin{aligned} g_3(t) &\leq \frac{m_*}{2} \|n(t)\|_{L^1(\text{supp } \psi(\cdot - x_0))} g_0(t) + C \left(\frac{k}{\rho}\right)^2 \|n(t)\|_{L^1(\text{supp } \psi(\cdot - x_0))}^2 (t - t_0) g_0^{\frac{1}{2}}(t) \\ &\quad + \frac{Ck^2}{\rho^{\frac{5}{3}}} \|n(t)\|_{L^1} \|n(t)\|_{L^1(\text{supp } \psi(\cdot - x_0))}^{\frac{2}{3}} (t - t_0)^{\frac{2}{3}} g_0^{\frac{2}{3}}(t) \\ &\quad + \frac{Ck^{\frac{10}{3}}}{\rho^3} \|n(t)\|_{L^1} \|n(t)\|_{L^1(\text{supp } \psi(\cdot - x_0))}^{\frac{4}{3}} (t - t_0)^{\frac{4}{3}} g_0^{\frac{1}{3}}(t) \\ &\quad + \frac{Ck}{\rho^{\frac{5}{6}}} \|n(t)\|_{L^1} \|n(t)\|_{L^1(\text{supp } \psi(\cdot - x_0))}^{\frac{1}{3}} (t - t_0)^{\frac{1}{3}} g_0^{\frac{5}{6}}(t), \end{aligned} \quad (5.42)$$

where m_* is the constant associated with (2.15) and $C = C(\gamma)$.

To evaluate g_4 , we utilize the product rule for differentiation:

$$\begin{aligned} g_4(t) &= \int_{\mathbb{R}^2} u(x) n(x, t) \cdot \nabla \left(n(x, t) \phi^2(x, t) \right) dx \\ &= \int_{\mathbb{R}^2} u(x) n(x, t) \cdot \left(\nabla n(x, t) \phi^2(x, t) \right) dx + 2 \int_{\mathbb{R}^2} u(x) n(x, t) \cdot \left(n(x, t) \phi(x, t) \nabla \phi(x, t) \right) dx \\ &= -\frac{1}{2} \int_{\mathbb{R}^2} n^2(x, t) \nabla \cdot \left(u(x) \phi^2(x, t) \right) dx + 2 \int_{\mathbb{R}^2} n^2(x, t) u(x) \cdot \phi(x, t) \nabla \phi(x, t) dx \\ &= -\frac{1}{2} \int_{\mathbb{R}^2} n^2(x, t) \nabla \cdot u(x) \phi^2(x, t) dx + \int_{\mathbb{R}^2} n^2(x, t) u(x) \cdot \phi(x, t) \nabla \phi(x, t) dx \\ &=: g_{4,1}(t) + g_{4,2}(t). \end{aligned}$$

For $g_{4,1}$, using (2.7) and (5.13), we estimate:

$$g_{4,1}(t) \leq \frac{1}{2} \|\nabla \cdot u\|_{L^\infty} \int_{\mathbb{R}^2} n^2(x, t) \phi^2(x, t) dx \leq C \|n(t)\|_{L^1(\text{supp } \psi(\cdot - x_0))} (t - t_0) g_0^{\frac{1}{2}}(t),$$

where $C = C(Q, \lambda)$.

Next, we consider $g_{4,2}$. Combining (2.7), (5.13), and (5.14), we have:

$$g_{4,2}(t) \leq \|\nabla \cdot u\|_{L^\infty} \int_{\mathbb{R}^2} n^2(x, t) \phi(x, t) |\nabla \phi(x, t)| dx$$

$$\begin{aligned}
&\leq \frac{1}{2} \|\nabla \cdot u\|_{L^\infty} \int_{\mathbb{R}^2} n^2(x, t) \phi^2(x, t) dx + \frac{1}{2} \|\nabla \cdot u\|_{L^\infty} \int_{\mathbb{R}^2} n^2(x, t) |\nabla \phi(x, t)|^2 dx \\
&\leq C \|n(t)\|_{L^1(\text{supp } \psi(\cdot - x_0))} (t - t_0) g_0^{\frac{1}{2}}(t) + C \left(\frac{k}{\rho}\right)^2 \|n(t)\|_{L^1(\text{supp } \psi(\cdot - x_0))} (t - t_0) g_0^{\frac{1}{2}}(t),
\end{aligned}$$

where $C = C(Q, \lambda)$. By summing up the contributions from $g_{4,1}$ and $g_{4,2}$, we obtain:

$$g_4(t) \leq C \|n(t)\|_{L^1(\text{supp } \psi(\cdot - x_0))} (t - t_0) g_0^{\frac{1}{2}}(t) + C \left(\frac{k}{\rho}\right)^2 \|n(t)\|_{L^1(\text{supp } \psi(\cdot - x_0))} (t - t_0) g_0^{\frac{1}{2}}(t), \quad (5.43)$$

where $C = C(Q, \lambda)$.

From (5.10), (5.21), (5.22), (5.42), and (5.43), the following inequality holds:

$$\begin{aligned}
&\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^2} n^2(x, t) (t - t_0)^2 \psi^k(x - x_0) dx \\
&\leq -g_0(t) + C \|n(t)\|_{L^1(\text{supp } \psi(\cdot - x_0))} g_0^{\frac{1}{2}}(t) + C \left(\frac{k}{\rho}\right)^2 \|n(t)\|_{L^1(\text{supp } \psi(\cdot - x_0))} (t - t_0) g_0^{\frac{1}{2}}(t) \\
&\quad + \frac{m_*}{2} \|n(t)\|_{L^1(\text{supp } \psi(\cdot - x_0))} g_0(t) + C \left(\frac{k}{\rho}\right)^2 \|n(t)\|_{L^1(\text{supp } \psi(\cdot - x_0))}^2 (t - t_0) g_0^{\frac{1}{2}}(t) \\
&\quad + \frac{Ck^2}{\rho^{\frac{5}{3}}} \|n(t)\|_{L^1} \|n(t)\|_{L^1(\text{supp } \psi(\cdot - x_0))}^{\frac{2}{3}} (t - t_0)^{\frac{2}{3}} g_0^{\frac{2}{3}}(t) \\
&\quad + \frac{Ck^{\frac{10}{3}}}{\rho^3} \|n(t)\|_{L^1} \|n(t)\|_{L^1(\text{supp } \psi(\cdot - x_0))}^{\frac{4}{3}} (t - t_0)^{\frac{4}{3}} g_0^{\frac{1}{3}}(t) \\
&\quad + \frac{Ck}{\rho^{\frac{5}{6}}} \|n(t)\|_{L^1} \|n(t)\|_{L^1(\text{supp } \psi(\cdot - x_0))}^{\frac{1}{3}} (t - t_0)^{\frac{1}{3}} g_0^{\frac{5}{6}}(t)
\end{aligned}$$

for all $t \in (0, T_{\max}) \cap [t_0, T_{\max})$, where m_* is the constant associated with (2.15), and $C = C(\gamma, Q, \lambda)$. Using the Young inequality and (2.14) in Theorem 2.1 (III), we obtain:

$$\begin{aligned}
&\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^2} n^2(x, t) (t - t_0)^2 \psi^k(x - x_0) dx \\
&\leq -\frac{1}{2} g_0(t) + \frac{m_*}{2} \|n(t)\|_{L^1(\text{supp } \psi(\cdot - x_0))} g_0(t) + \frac{C}{\rho^5} ((t - t_0)^2 + 1) \|n(t)\|_{L^1(\text{supp } \psi(\cdot - x_0))} \quad (5.44)
\end{aligned}$$

for all $0 < \rho \leq 1$ and $t \in (0, T_{\max}) \cap [t_0, T_{\max})$, where m_* is the constant associated with (2.15), and $C = C(\gamma, Q, \lambda, \|n_0\|_{L^1})$.

Now, let $x_0 \in \mathbb{R}^2$ and $t_1 \in [0, T_{\max})$, and assume that (5.1) holds with the absolute positive constant m_* defined in (2.15). Applying Lemma 3.7 with $\rho_1 = \frac{\rho}{2}$ and $\rho_2 = \rho$, we have:

$$\begin{aligned}
\int_{B_\rho(x_0)} n(x, t) \, dx &\leq \frac{\tilde{C}}{\rho^2} \|n_0\|_{L^1} (1 + \|n_0\|_{L^1}) |t_1 - t| + \int_{B_{2\rho}(x_0)} n(x, t_1) \, dx \\
&\leq \frac{\tilde{C}}{\rho^2} \|n_0\|_{L^1} (1 + \|n_0\|_{L^1}) |t_1 - t| + \frac{1}{4m_*}
\end{aligned} \tag{5.45}$$

for all $t \in (t_1, T_{\max})$, where $\tilde{C} = \tilde{C}(\gamma, Q, \lambda)$. Let us define a and b as follows:

$$\begin{aligned}
a &:= \max\{0, t_1 - 2c_1\rho^2\} \quad \text{and} \quad b := \min\{t_1 + 2c_1\rho^2, T_{\max}\} \\
&\quad \text{with } c_1 = \frac{1}{8\tilde{C}m_*\|n_0\|_{L^1}(1 + \|n_0\|_{L^1})}.
\end{aligned} \tag{5.46}$$

Substituting this definition (5.46) into (5.45), we deduce:

$$\int_{B_\rho(x_0)} n(x, t) \, dx \leq \frac{\tilde{C}}{\rho^2} \|n_0\|_{L^1} (1 + \|n_0\|_{L^1}) \cdot 2c_1\rho^2 + \frac{1}{4m_*} = \frac{1}{2m_*} \tag{5.47}$$

for all $t \in (0, T_{\max}) \cap (t_1 - 2c_1\rho^2, t_1 + 2c_1\rho^2)$. Using the result in (5.47) with (5.44), we obtain:

$$\frac{d}{dt} \int_{\mathbb{R}^2} n^2(x, t) (t - t_0)^2 \psi^k(x - x_0) \, dx \leq \frac{C}{2m_*\rho^5} ((t - t_0)^2 + 1) \tag{5.48}$$

for all $t \in (0, T_{\max}) \cap [t_0, T_{\max}) \cap (t_1 - 2c_1\rho^2, t_1 + 2c_1\rho^2)$, where $C = C(\gamma, Q, \lambda, \|n_0\|_{L^1})$.

(i). In the case $2c_1\rho^2 \leq t_1 < T_{\max}$, we choose t_0 as $a = \max\{0, t_1 - 2c_1\rho^2\} = t_1 - 2c_1\rho^2$ in (5.48). Integrating both sides of (5.48) over $[a, s]$ for $s \in (a, b)$, we find:

$$\int_{\mathbb{R}^2} n^2(x, s) (s - a)^2 \psi^k(x - x_0) \, dx \leq \frac{C}{2m_*\rho^5} \left(\frac{(s - a)^3}{3} + (s - a) \right),$$

which implies:

$$\int_{\mathbb{R}^2} n^2(x, s) \psi^k(x - x_0) \, dx \leq \frac{C}{2m_*\rho^5} \left(\frac{s - a}{3} + \frac{1}{s - a} \right). \tag{5.49}$$

Since s satisfies:

$$\begin{aligned}
s \in [a, T_{\max}) \cap (a + c_1\rho^2, b) &= (a + c_1\rho^2, \min\{t_1 + 2c_1\rho^2, T_{\max}\}) \\
&= (t_1 - c_1\rho^2, \min\{t_1 + 2c_1\rho^2, T_{\max}\}),
\end{aligned}$$

it holds:

$$\begin{aligned}
c_1 \rho^2 < s - a < \min\{4c_1 \rho^2, T_{\max} - t_1 + 2c_1 \rho^2\} \\
&= \begin{cases} 4c_1 \rho^2 & \text{when } 2c_1 \rho^2 < T_{\max} - t_1, \\ T_{\max} - t_1 + 2c_1 \rho^2 & \text{when } 2c_1 \rho^2 \geq T_{\max} - t_1, \end{cases} \\
&\leq 4c_1 \rho^2.
\end{aligned}$$

This, together with (5.49), yields:

$$\int_{B_{\frac{\rho}{2}}(x_0)} n^2(x, s) \, dx \leq \frac{C}{\rho^3}$$

for all $s \in (t_1 - c_1 \rho^2, \min\{t_1 + 2c_1 \rho^2, T_{\max}\})$, where $C = C(\gamma, Q, \lambda, \|n_0\|_{L^1})$. This provides the estimate (5.2).

(ii). In the case $0 < t_1 < \min\{2c_1 \rho^2, T_{\max}\}$, we choose t_0 as 0 in (5.48). Integrating both sides of (5.48) over $[0, s]$ for $s \in (0, b)$, we have:

$$\int_{\mathbb{R}^2} n^2(x, s) s^2 \psi^k(x - x_0) \, dx \leq \frac{C}{2m_* \rho^5} \left(\frac{s^3}{3} + s \right).$$

This yields:

$$\int_{\mathbb{R}^2} s n^2(x, s) \psi^k(x - x_0) \, dx \leq \frac{C}{2m_* \rho^5} \left(\frac{s^2}{3} + 1 \right). \quad (5.50)$$

Since $s \in (0, b)$, it follows:

$$0 < s < b \leq t_1 + 2c_1 \rho^2 \leq 4c_1 \rho^2.$$

Thus, combining this with (5.50), we have:

$$\int_{B_{\frac{\rho}{2}}(x_0)} s n^2(x, s) \, dx \leq \frac{C(s^2 + 1)}{\rho^5} \leq \frac{C}{\rho^5}$$

for all $s \in (0, \min\{t_1 + 2c_1 \rho^2, T_{\max}\})$, where $C = C(\gamma, Q, \lambda, \|n_0\|_{L^1})$. This provides the desired estimate (5.3).

(iii). For the case $2c_1 \rho^2 \geq T_{\max}$, the situation $2c_1 \rho^2 \leq t_1 < T_{\max}$ as described in (i) does not arise. As (ii), we consider $0 < t_1 < \min\{2c_1 \rho^2, T_{\max}\} = T_{\max}$ and choose $t_0 = -\rho^2$ in (5.48). Then, integrating both sides of (5.48) over $[0, s]$ for $s \in (0, b)$, we have:

$$\begin{aligned} & \int_{\mathbb{R}^2} n^2(x, s)(s + \rho^2)^2 \psi^k(x - x_0) dx \\ & \leq \frac{C}{2m_* \rho^5} \left(\frac{(s + \rho^2)^3}{3} + (s + \rho^2) \right) + \int_{\mathbb{R}^2} n_0^2(x) \rho^4 \psi^k(x - x_0) dx. \end{aligned} \quad (5.51)$$

Since it holds:

$$0 < s < b \leq t_1 + 2c_1 \rho^2 \leq 4c_1 \rho^2,$$

we deduce from (5.51):

$$\begin{aligned} \int_{B_{\frac{\rho}{2}}(x_0)} n^2(x, s) dx & \leq \frac{C}{2m_* \rho^5} \left(\frac{(s + \rho^2)}{3} + \frac{1}{s + \rho^2} \right) + \int_{\mathbb{R}^2} n_0^2(x) \psi^k(x - x_0) \frac{\rho^4}{(s + \rho^2)^2} dx \\ & \leq \frac{C}{\rho^7} + \|n_0\|_{L^2(B_\rho(x_0))}^2 \end{aligned}$$

for all $s \in (0, \min\{t_1 + 2c_1 \rho^2, T_{\max}\})$, where $C = C(\gamma, Q, \lambda, \|n_0\|_{L^1})$. Thus, the estimate (5.4) is obtained. This completes the proof of Proposition 5.1. $\square$

Step 2. This theorem provides local $L^p(\mathbb{R}^2)$ -estimates for the gradient of v . If the local mass of n satisfies (5.52), bounds for $\|\nabla v\|_{L^p}$ are derived in three cases based on the relationships between t_1 , $T_{\max}$, and $2c_1 \rho^2$, as given in (5.53), (5.54), and (5.55).

Proposition 5.3. Let $2 \leq p < \infty$. Let Assumption 2.1 hold and suppose that $n_0 \in L^1(\mathbb{R}^2)$. Let $T_{\max}$ denote the maximal existence time of the strong solution (n, v) of (KSF) obtained in Theorem 2.1 (I)–(III). There exist an absolute positive constant m_* and a positive constant $c_1 = c_1(\gamma, Q, \lambda, \|n_0\|_{L^1(\mathbb{R}^2)})$ such that if:

$$\int_{B_{2\rho}(x_0)} n(x, t_1) dx \leq m_* \quad (5.52)$$

holds for some $x_0 \in \mathbb{R}^2$, $0 < \rho \leq 1$, and $t_1 \in [0, T_{\max})$, then the following estimates are valid:

(i). In the case $2c_1 \rho^2 < T_{\max}$ and $2c_1 \rho^2 \leq t_1 < T_{\max}$, the following estimate holds:

$$\sup_{s \in (t_1 - c_1 \rho^2, \min\{t_1 + 2c_1 \rho^2, T_{\max}\})} \|\nabla v(s)\|_{L^p(B_{\frac{\rho}{4}}(x_0))} \leq \frac{C}{\rho^{\frac{3}{2}}}. \quad (5.53)$$

(ii). In the case $2c_1 \rho^2 < T_{\max}$ and $0 \leq t_1 < \min\{2c_1 \rho^2, T_{\max}\} = 2c_1 \rho^2$, the following estimate holds:

$$\sup_{s \in (0, \min\{t_1 + 2c_1 \rho^2, T_{\max}\})} s^{\frac{1}{2}} \|\nabla v(s)\|_{L^p(B_{\frac{\rho}{4}}(x_0))} \leq \frac{C}{\rho^{\frac{5}{2}}}. \quad (5.54)$$

(iii). In the case $2c_1\rho^2 \geq T_{\max}$ and $0 \leq t_1 < \min\{2c_1\rho^2, T_{\max}\} = T_{\max}$, provided that $n_0 \in L^2_{\text{loc}}(\mathbb{R}^2)$, the following estimate holds:

$$\sup_{s \in (0, \min\{t_1 + 2c_1\rho^2, T_{\max}\})} \|\nabla v(s)\|_{L^p(B_{\frac{\rho}{4}}(x_0))} \leq C \left(\frac{1}{\rho^{\frac{7}{2}}} + \|n_0\|_{L^2(B_{\rho}(x_0))} \right). \quad (5.55)$$

Here, $C = C(\gamma, Q, \lambda, \|n_0\|_{L^1(\mathbb{R}^2)}, p)$, and it is independent of x_0 and ρ .

Proof of Proposition 5.3. Applying Lemma 3.4 with $\rho_1 = \frac{\rho}{4}$ and $\rho_2 = \frac{\rho}{2}$, we find that there exists a constant $C = C(\gamma, p)$ satisfying:

$$\|\nabla v(s)\|_{L^p(B_{\frac{\rho}{4}}(x_0))} \leq C \left(\frac{\rho^{\frac{2}{p}}}{\rho} \|n(s)\|_{L^1} + \|n(s)\|_{L^2(B_{\frac{\rho}{2}}(x_0))} \right), \quad (5.56)$$

for all $s \in (0, T_{\max})$.

(i). In the case $2c_1\rho^2 < T_{\max}$ and $2c_1\rho^2 \leq t_1 < T_{\max}$, it follows from Proposition 5.1 (i) that:

$$\sup_{s \in (t_1 - c_1\rho^2, \min\{t_1 + 2c_1\rho^2, T_{\max}\})} \int_{B_{\frac{\rho}{2}}(x_0)} n^2(x, s) \, dx \leq \frac{C}{\rho^3}, \quad (5.57)$$

where $C = C(\gamma, Q, \lambda, \|n_0\|_{L^1})$. From (5.56), (5.57) and (2.14) in Theorem 2.1 (III), we have:

$$\sup_{s \in (t_1 - c_1\rho^2, \min\{t_1 + 2c_1\rho^2, T_{\max}\})} \|\nabla v(s)\|_{L^p(B_{\frac{\rho}{4}}(x_0))} \leq C \left(\frac{\rho^{\frac{2}{p}}}{\rho} \|n_0\|_{L^1} + \frac{1}{\rho^{\frac{3}{2}}} \right) \leq \frac{C}{\rho^{\frac{3}{2}}}$$

for $2 \leq p < \infty$, where $C = C(\gamma, Q, \lambda, \|n_0\|_{L^1}, p)$.

(ii). In the case $0 \leq t_1 < \min\{2c_1\rho^2, T_{\max}\} = 2c_1\rho^2$, we obtain from Proposition 5.1 (ii):

$$\sup_{s \in (0, \min\{t_1 + 2c_1\rho^2, T_{\max}\})} \int_{B_{\frac{\rho}{2}}(x_0)} sn^2(x, s) \, dx \leq \frac{C}{\rho^5}, \quad (5.58)$$

where $C = C(\gamma, Q, \lambda, \|n_0\|_{L^1})$. Since $s \in (0, \min\{t_1 + 2c_1\rho^2, T_{\max}\})$, it holds:

$$\begin{aligned} 0 < s < \min\{t_1 + 2c_1\rho^2, T_{\max}\} &= \begin{cases} t_1 + 2c_1\rho^2 & \text{when } 2c_1\rho^2 < T_{\max} - t_1, \\ T_{\max} & \text{when } 2c_1\rho^2 \geq T_{\max} - t_1, \end{cases} \\ &\leq t_1 + 2c_1\rho^2 \leq 4c_1\rho^2. \end{aligned}$$

Using (5.56), (5.58), and (2.14) in Theorem 2.1 (III), we deduce:

$$\begin{aligned}
& \sup_{s \in (0, \min\{t_1 + 2c_1\rho^2, T_{\max}\})} s^{\frac{1}{2}} \|\nabla v(s)\|_{L^p(B_{\frac{\rho}{4}}(x_0))} \\
& \leq C \left(s^{\frac{1}{2}} \frac{\rho^{\frac{2}{p}}}{\rho} \|n_0\|_{L^1} + \left(\sup_{s \in (0, \min\{t_1 + 2c_1\rho^2, T_{\max}\})} \int_{B_{\frac{\rho}{2}}(x_0)} s n^2(x, s) \, dx \right)^{\frac{1}{2}} \right) \\
& \leq C \left(\rho \frac{\rho^{\frac{2}{p}}}{\rho} \|n_0\|_{L^1} + \frac{1}{\rho^{\frac{5}{2}}} \right) \leq \frac{C}{\rho^{\frac{5}{2}}}
\end{aligned}$$

for $2 \leq p < \infty$, where $C = C(\gamma, Q, \lambda, \|n_0\|_{L^1}, p)$.

(iii). In the case $2c_1\rho^2 \geq T_{\max}$ and $0 \leq t_1 < \min\{2c_1\rho^2, T_{\max}\} = T_{\max}$, we have, by Proposition 5.1 (iii):

$$\sup_{s \in (0, \min\{t_1 + 2c_1\rho^2, T_{\max}\})} \int_{B_{\frac{\rho}{2}}(x_0)} n^2(x, s) \, dx \leq \frac{C}{\rho^7} + \|n_0\|_{L^2(B_\rho(x_0))}^2, \quad (5.59)$$

where $C = C(\gamma, Q, \lambda, \|n_0\|_{L^1})$. From (5.56), (5.59) and (2.14) in Theorem 2.1 (III), we find:

$$\begin{aligned}
\sup_{s \in (0, \min\{t_1 + 2c_1\rho^2, T_{\max}\})} \|\nabla v(s)\|_{L^p(B_{\frac{\rho}{4}}(x_0))} & \leq C \left(\frac{\rho^{\frac{2}{p}}}{\rho} \|n_0\|_{L^1} + \frac{C}{\rho^{\frac{7}{2}}} + \|n_0\|_{L^2(B_\rho(x_0))} \right) \\
& \leq C \left(\frac{1}{\rho^{\frac{7}{2}}} + \|n_0\|_{L^2(B_\rho(x_0))} \right)
\end{aligned}$$

for $2 \leq p < \infty$, where $C = C(\gamma, Q, \lambda, \|n_0\|_{L^1}, p)$. This completes the proof of Proposition 5.3. $\square$

Step 3. In this step, we derive the estimates required for the application of Moser's iteration technique.

Proposition 5.4. *Let $2 < p < \infty$ and let $i \in \mathbb{N}$ with $i \geq 2$. Let Assumption 2.1 hold and suppose that $n_0 \in L^1(\mathbb{R}^2)$. Let $T_{\max}$ denote the maximal existence time of the strong solution (n, v) of (KSF) obtained in Theorem 2.1 (I)–(III). There exist an absolute positive constant m_* and a positive constant $c_1 = c_1(\gamma, Q, \lambda, \|n_0\|_{L^1(\mathbb{R}^2)})$ such that if:*

$$\int_{B_{2\rho}(x_0)} n(x, t_1) \, dx \leq m_* \quad (5.60)$$

holds for some $x_0 \in \mathbb{R}^2$, $0 < \rho \leq 1$, and $t_1 \in [0, T_{\max})$, then, for any $0 < \rho_1 < \rho_2 \leq \rho$, the following estimates are valid:

(i). In the case $2c_1\rho^2 < T_{\max}$ and $2c_1\rho^2 \leq t_1 < T_{\max}$, the following estimate holds:

$$\|n(s)\|_{L^{2i}(B_{\rho_1}(x_0))} \leq \left(\frac{Ci}{(\rho_2 - \rho_1)^{\frac{9}{5}}} \right)^{\frac{5}{2i}} \left(\frac{1}{\rho} \right)^{\frac{1}{i}} \sup_{t_1 - c_1\rho^2 < \tau < \min\{t_1 + 2c_1\rho^2, T_{\max}\}} \|n(\tau)\|_{L^i(B_{\rho_2}(x_0))} \quad (5.61)$$

for all $s \in (t_1 - c_1\rho^2, \min\{t_1 + 2c_1\rho^2, T_{\max}\})$, where $C = C(\gamma, Q, \lambda, \|n_0\|_{L^1(\mathbb{R}^2)}, p)$.

(ii). In the case $2c_1\rho^2 < T_{\max}$ and $0 \leq t_1 < \min\{2c_1\rho^2, T_{\max}\} = 2c_1\rho^2$, the following estimate holds:

$$\begin{aligned} & s^{1 - \frac{1}{2i} + \frac{2(i-1)}{i(p-2)}} \|n(s)\|_{L^{2i}(B_{\rho_1}(x_0))} \\ & \leq \left(\frac{Ci}{(\rho_2 - \rho_1)^3} \right)^{\frac{5}{2i}} \sup_{\tau \in (0, \min\{t_1 + 2c_1\rho^2, T_{\max}\})} \tau^{1 - \frac{1}{i} + \frac{2(i-2)}{i(p-2)}} \|n(\tau)\|_{L^i(B_{\rho_2}(x_0))} \end{aligned} \quad (5.62)$$

for all $s \in (0, \min\{t_1 + 2c_1\rho^2, T_{\max}\})$, where $C = C(\gamma, Q, \lambda, \|n_0\|_{L^1(\mathbb{R}^2)}, p)$.

(iii). In the case $2c_1\rho^2 \geq T_{\max}$ and $0 \leq t_1 < \min\{2c_1\rho^2, T_{\max}\} = T_{\max}$, provided that $n_0 \in L^2_{\text{loc}}(\mathbb{R}^2)$, the following estimate holds:

$$\begin{aligned} & \|n(s)\|_{L^{2i}(B_{\rho_1}(x_0))} \\ & \leq \left(\frac{Ci}{(\rho_2 - \rho_1)^{\frac{21}{5}}} \right)^{\frac{5}{2i}} \left(\frac{1}{\rho} \right)^{\frac{1}{i}} \sup_{0 < \tau < \min\{t_1 + 2c_1\rho^2, T_{\max}\}} \|n(\tau)\|_{L^i(B_{\rho_2}(x_0))} + \|n_0\|_{L^{2i}(B_{\rho_2}(x_0))} \end{aligned} \quad (5.63)$$

for all $s \in (0, \min\{t_1 + 2c_1\rho^2, T_{\max}\})$, where $C = C(\gamma, Q, \lambda, \|n_0\|_{L^1(\mathbb{R}^2)}, p)$.

Proof of Proposition 5.4. Let $x_0 \in \mathbb{R}^2$, and let $0 < \rho_1 < \rho_2$. We suppose that ψ be the cut-off function given in Lemma 3.6 corresponding to ρ_1 and $\rho_2 \leq \rho$. We choose $t_0 \in (-\infty, \min\{t_1 + 2c_1\rho^2, T_{\max}\})$ and $t \in [\max\{0, t_0\}, T_{\max})$. Let $\ell \geq 2$ and let $i \in \mathbb{N}$ with $i \geq 2$. We define ϕ by:

$$\phi(x, t) := (t - t_0)^{\frac{\ell}{2}} \psi^4(x - x_0). \quad (5.64)$$

Using the following identity:

$$\frac{1}{2i} \partial_t \left(n^{2i}(x, t) \phi^2(x, t) \right) = \frac{\ell}{2i(t - t_0)} n^{2i}(x, t) \phi^2(x, t) + n^{2i-1}(x, t) \partial_t n(x, t) \phi^2(x, t),$$

we deduce:

$$\begin{aligned} & \frac{1}{2i} \frac{d}{dt} \int_{\mathbb{R}^2} n^{2i}(x, t) \phi^2(x, t) dx \\ & = \frac{\ell}{2i(t - t_0)} \int_{\mathbb{R}^2} n^{2i}(x, t) \phi^2(x, t) dx - \int_{\mathbb{R}^2} \nabla(n^{2i-1}(x, t) \phi^2(x, t)) \cdot \nabla n(x, t) dx \end{aligned}$$

$$+ \int_{\mathbb{R}^2} \nabla(n^{2i-1}(x, t)\phi^2(x, t)) \cdot \nabla v(x, t)n(x, t) dx + \int_{\mathbb{R}^2} \nabla(n^{2i-1}(x, t)\phi^2(x, t)) \cdot u(x)n(x, t) dx.$$

Examining the term $-\nabla(n^{2i-1}(x, t)\phi^2(x, t)) \cdot \nabla n(x, t)$, we find:

$$\begin{aligned} -\nabla(n^{2i-1}(x, t)\phi^2(x, t)) \cdot \nabla n(x, t) &= -\frac{2i-1}{i^2} (|\nabla(n^i(x, t)\phi(x, t))|^2 - |n^i(x, t)\nabla\phi(x, t)|^2) \\ &\quad + \frac{2(i-1)}{i^2} n^i(x, t)\phi(x, t)\nabla n^i(x, t) \cdot \nabla\phi(x, t). \end{aligned} \quad (5.65)$$

For the last term, we have:

$$\begin{aligned} &\frac{2(i-1)}{i^2} n^i(x, t)\phi(x, t)\nabla n^i(x, t) \cdot \nabla\phi(x, t) \\ &= \frac{2(i-1)}{i^2} (\nabla(n^i(x, t)\phi(x, t)) - n^i(x, t)\nabla\phi(x, t)) \cdot n^i(x, t)\nabla\phi(x, t) \\ &\leq \frac{i-1}{i^2} (|\nabla(n^i(x, t)\phi(x, t))|^2 - |n^i(x, t)\nabla\phi(x, t)|^2). \end{aligned} \quad (5.66)$$

Combining (5.65) with (5.66), we have:

$$-\nabla(n^{2i-1}(x, t)\phi^2(x, t)) \cdot \nabla n(x, t) \leq \frac{-1}{i} |\nabla(n^i(x, t)\phi(x, t))|^2 + \frac{1}{i} |n^i(x, t)\nabla\phi(x, t)|^2.$$

This leads to the inequality:

$$\frac{1}{2i} \frac{d}{dt} \int_{\mathbb{R}^2} n^{2i}(x, t)\phi^2(x, t) dx \leq -\tilde{g}_0(t) + \tilde{g}_1(t) + \tilde{g}_2(t) + \tilde{g}_3(t) + \tilde{g}_4(t), \quad (5.67)$$

where the terms are defined as:

$$\begin{aligned} \tilde{g}_0(t) &= \frac{1}{i} \int_{\mathbb{R}^2} |\nabla(n^i(x, t)\phi(x, t))|^2 dx, & \tilde{g}_1(t) &= \frac{\ell}{2i(t-t_0)} \int_{\mathbb{R}^2} |n^i(x, t)\phi(x, t)|^2 dx, \\ \tilde{g}_2(t) &= \frac{1}{i} \int_{\mathbb{R}^2} |n^i(x, t)\nabla\phi(x, t)|^2 dx, & \tilde{g}_3(t) &= \int_{\mathbb{R}^2} \nabla(n^{2i-1}(x, t)\phi^2(x, t)) \cdot \nabla v(x, t)n(x, t) dx, \\ \tilde{g}_4(t) &= \int_{\mathbb{R}^2} \nabla(n^{2i-1}(x, t)\phi^2(x, t)) \cdot u(x)n(x, t) dx. \end{aligned} \quad (5.68)$$

Next, we derive three estimates for n^i in L^2 and L^3 .

Lemma 5.5. *Let $x_0 \in \mathbb{R}^2$, $0 < \rho \leq 1$, and $\ell \geq 2$. We suppose that $i \geq 2$ is an integer, and take $t_0 \in (-\infty, \min\{t_1 + 2c_1\rho^2, T_{\max}\})$, where c_1 is the constant in (5.5). We denote by ϕ the function defined in (5.64), where ψ satisfies $\rho_1 = \rho/2$ and $\rho_2 = \rho$ as in Lemma 3.6. We further assume that $n(t) \in L^i(B_{\rho_2}(x_0))$ for every $t \in (0, T_{\max}) \cap [t_0, T_{\max})$. Then, the following estimates hold:*

$$\int_{\mathbb{R}^2} |n^i(x, t)\phi(x, t)|^3 dx \leq Ci(t - t_0)^{\frac{\ell}{2}} \tilde{g}_0(t) \|n(t)\|_{L^i(B_{\rho_2}(x_0))}^i, \quad (5.69)$$

$$\int_{\mathbb{R}^2} |n^i(x, t)\phi(x, t)|^2 dx \leq Ci^{\frac{1}{2}}(t - t_0)^{\frac{\ell}{2}} \tilde{g}_0^{\frac{1}{2}}(t) \|n(t)\|_{L^i(B_{\rho_2}(x_0))}^i, \quad (5.70)$$

$$\int_{\mathbb{R}^2} |n^i(x, t)\nabla\phi(x, t)|^2 dx \leq \frac{Ci^{\frac{1}{2}}}{(\rho_2 - \rho_1)^2} (t - t_0)^{\frac{\ell}{2}} \tilde{g}_0^{\frac{1}{2}}(t) \|n(t)\|_{L^i(B_{\rho_2}(x_0))}^i \quad (5.71)$$

for all $t \in (0, T_{\max}) \cap [t_0, T_{\max})$, where C is an absolute positive constant, and $\tilde{g}_0$ is as defined in (5.68).

Proof of Lemma 5.5. Concerning (5.69), the Gagliardo–Nirenberg inequality gives:

$$\begin{aligned} \int_{\mathbb{R}^2} |n^i(x, t)\phi(x, t)|^3 dx &\leq C \|\nabla(n^i(t)\phi(t))\|_{L^2}^2 \|n^i(t)\phi(t)\|_{L^1} \\ &\leq Ci(t - t_0)^{\frac{\ell}{2}} \tilde{g}_0(t) \|n(t)\|_{L^i(B_{\rho_2}(x_0))}^i, \end{aligned}$$

where C is an absolute positive constant.

For (5.70), we apply (5.69) to estimate:

$$\begin{aligned} \int_{\mathbb{R}^2} |n^i(x, t)\phi(x, t)|^2 dx &\leq \|n^i(t)\phi(t)\|_{L^3}^{\frac{3}{2}} \|n^i(t)\phi(t)\|_{L^1}^{\frac{1}{2}} \\ &\leq \left(Ci(t - t_0)^{\frac{\ell}{2}} \tilde{g}_0(t) \|n(t)\|_{L^i(B_{\rho_2}(x_0))}^i \right)^{\frac{1}{2}} (t - t_0)^{\frac{\ell}{4}} \|n(t)\|_{L^i(B_{\rho_2}(x_0))}^{\frac{i}{2}} \\ &\leq Ci^{\frac{1}{2}}(t - t_0)^{\frac{\ell}{2}} \tilde{g}_0^{\frac{1}{2}}(t) \|n(t)\|_{L^i(B_{\rho_2}(x_0))}^i, \end{aligned}$$

where C is an absolute positive constant.

Next, we consider (5.71). From (3.24) in Lemma 3.6, it follows:

$$|\nabla\phi(x)|^2 \leq \frac{C}{(\rho_2 - \rho_1)^2} (t - t_0)^\ell \psi^6(x - x_0) = \frac{C}{(\rho_2 - \rho_1)^2} (t - t_0)^{\frac{\ell}{4}} \phi^{\frac{3}{2}}(x, t).$$

This, together with (5.69), leads to the conclusion:

$$\begin{aligned} \int_{\mathbb{R}^2} |n^i(x, t)\nabla\phi(x, t)|^2 dx &\leq \frac{C}{(\rho_2 - \rho_1)^2} (t - t_0)^{\frac{\ell}{4}} \|n^i(t)\phi(t)\|_{L^3}^{\frac{3}{2}} \|n(t)\|_{L^i(B_{\rho_2}(x_0))}^{\frac{i}{2}} \\ &\leq \frac{C}{(\rho_2 - \rho_1)^2} (t - t_0)^{\frac{\ell}{4}} \left(Ci(t - t_0)^{\frac{\ell}{2}} \tilde{g}_0(t) \|n(t)\|_{L^i(B_{\rho_2}(x_0))}^i \right)^{\frac{1}{2}} \|n(t)\|_{L^i(B_{\rho_2}(x_0))}^{\frac{i}{2}} \end{aligned}$$

$$\leq \frac{Ci^{\frac{1}{2}}}{(\rho_2 - \rho_1)^2} (t - t_0)^{\frac{\ell}{2}} \tilde{g}_0^{\frac{1}{2}}(t) \|n(t)\|_{L^i(B_{\rho_2}(x_0))}^i,$$

where C is an absolute positive constant. This completes the proof of Lemma 5.5. $\square$

Continuation of the proof of Proposition 5.4. As for $\tilde{g}_1$ and $\tilde{g}_2$, using (5.69) in Lemma 5.5, we obtain:

$$\tilde{g}_1(t) \leq Ci^{-\frac{1}{2}} \ell(t - t_0)^{\frac{\ell}{2}-1} \tilde{g}_0^{\frac{1}{2}}(t) \|n(t)\|_{L^i(B_{\rho_2}(x_0))}^i \leq \frac{\tilde{g}_0(t)}{8} + C\ell^2(t - t_0)^{\ell-2} \|n(t)\|_{L^i(B_{\rho_2}(x_0))}^{2i}, \quad (5.72)$$

where C is an absolute positive constant. Similarly, applying (5.70) in Lemma 5.5, we obtain the following estimate:

$$\tilde{g}_2(t) \leq \frac{Ci^{-\frac{1}{2}}}{\rho^2} (t - t_0)^{\frac{\ell}{2}} \tilde{g}_0^{\frac{1}{2}}(t) \|n(t)\|_{L^i(B_{\rho_2}(x_0))}^i \leq \frac{\tilde{g}_0(t)}{8} + \frac{C}{(\rho_2 - \rho_1)^4} (t - t_0)^{\ell} \|n(t)\|_{L^i(B_{\rho_2}(x_0))}^{2i}, \quad (5.73)$$

where C is an absolute positive constant.

Next, we consider $\tilde{g}_3$. Let $\varepsilon_0 > 0$. Then, we have:

$$\begin{aligned} \tilde{g}_3(t) &\leq (2i - 1) \int_{\mathbb{R}^2} n^{2i-1}(x, t) |\nabla n(x, t)| |\nabla v(x, t)| \phi^2(x, t) \, dx \\ &\quad + 2 \int_{\mathbb{R}^2} n^{2i}(x, t) |\nabla v(x, t)| |\nabla \phi(x, t)| \phi(x, t) \, dx \\ &= \frac{2i - 1}{i} \int_{\mathbb{R}^2} n^i(x, t) \phi(x, t) |\nabla(n^i(x, t) \phi(x, t)) - n^i(x, t) \nabla \phi(x, t)| |\nabla v(x, t)| \, dx \\ &\quad + 2 \int_{\mathbb{R}^2} n^{2i}(x, t) |\nabla v(x, t)| |\nabla \phi(x, t)| \phi(x, t) \, dx \\ &\leq \frac{2i - 1}{i} \|\nabla(n^i(t) \phi(t))\|_{L^2} \|n^i(t) \phi(t) \nabla v(t)\|_{L^2} \\ &\quad + \frac{4i - 1}{i} \|n^i(t) \phi(t) \nabla v(t)\|_{L^2} \|n^i(t) \nabla \phi(t)\|_{L^2} \\ &\leq \varepsilon_0 \|\nabla(n^i(t) \phi(t))\|_{L^2}^2 + \left(\frac{1}{\varepsilon_0} + 4\right) \|n^i(t) \phi(t) \nabla v(t)\|_{L^2}^2 + \|n^i(t) \nabla \phi(t)\|_{L^2}^2. \end{aligned} \quad (5.74)$$

In addition, let $\varepsilon_1 > 0$. Then, we have the following estimate:

$$\begin{aligned} \|n^i(t) \phi(t) \nabla v(t)\|_{L^2}^2 &\leq \|n^i(t) \phi(t)\|_{L^{\frac{2p}{p-2}}}^2 \|\nabla v(t)\|_{L^p(B_{\rho_2}(x_0))}^2 \\ &\leq C \|\nabla(n^i(t) \phi(t))\|_{L^2}^{\frac{p+2}{p}} \|n^i(t) \phi(t)\|_{L^1}^{\frac{p-2}{p}} \|\nabla v(t)\|_{L^p(B_{\rho_2}(x_0))}^2 \end{aligned}$$

$$\leq \varepsilon_1 i \tilde{g}_0(t) + C \varepsilon_1^{-\frac{p+2}{p-2}} \|n^i(t)\phi(t)\|_{L^1}^2 \|\nabla v(t)\|_{L^p(B_{\rho_2}(x_0))}^{\frac{4p}{p-2}} \quad (5.75)$$

for all $p > 2$, where $C = C(p)$. Combining (5.71), (5.74) and (5.75), we derive:

$$\begin{aligned} \tilde{g}_3(t) &\leq \varepsilon_0 i \tilde{g}_0(t) + \left(\frac{1}{\varepsilon_0} + 4\right) \varepsilon_1 i \tilde{g}_0(t) \\ &\quad + C \left(\frac{1}{\varepsilon_0} + 4\right) \varepsilon_1^{-\frac{p+2}{p-2}} \|\nabla v(t)\|_{L^p(B_{\rho_2}(x_0))}^{\frac{4p}{p-2}} (t - t_0)^\ell \|n(t)\|_{L^i(B_{\rho_2}(x_0))}^{2i} \\ &\quad + \frac{\tilde{g}_0(t)}{8} + \frac{Ci}{(\rho_2 - \rho_1)^4} (t - t_0)^\ell \|n(t)\|_{L^i(B_{\rho_2}(x_0))}^{2i}, \end{aligned} \quad (5.76)$$

where $C = C(p)$.

We proceed to estimate $\tilde{g}_4$. According to Definition 2.1, for any $1 \leq r < \infty$, the following holds:

$$\begin{aligned} \|u\|_{L^r(B_{\rho_2}(x_0))}^r &= |Q|^r \int_{B_{\rho_2}(x_0)} \left| \frac{x - x_0}{|x - x_0|^2 + \lambda} \chi(x - x_0) + \frac{x + x_0}{|x + x_0|^2 + \lambda} \chi(x + x_0) \right|^r dx \\ &\leq 2^{r-1} |Q|^r \left(2 \max \left\{ 1, \frac{1}{\lambda^r} \right\} |B_{\rho_2}(x_0)| \right) = 2^r |Q|^r \pi(\rho_2)^2 \max \left\{ 1, \frac{1}{\lambda^r} \right\}, \end{aligned}$$

which yields:

$$\|u\|_{L^r(B_{\rho_2}(x_0))} \leq 2|Q| \left(\pi(\rho_2)^2 \right)^{\frac{1}{r}} \max \left\{ 1, \frac{1}{\lambda} \right\} \leq C, \quad (5.77)$$

where $C = C(Q, \lambda, r)$, since $0 < \rho_2 \leq 1$. Following a similar argument to that in (5.76), we obtain:

$$\begin{aligned} \tilde{g}_4(t) &\leq \varepsilon_0 i \tilde{g}_0(t) + \left(\frac{1}{\varepsilon_0} + 4\right) \varepsilon_1 i \tilde{g}_0(t) + C \left(\frac{1}{\varepsilon_0} + 4\right) \varepsilon_1^{-\frac{p+2}{p-2}} (t - t_0)^\ell \|n(t)\|_{L^i(B_{\rho_2}(x_0))}^{2i} \\ &\quad + \frac{\tilde{g}_0(t)}{8} + \frac{Ci}{(\rho_2 - \rho_1)^4} (t - t_0)^\ell \|n(t)\|_{L^i(B_{\rho_2}(x_0))}^{2i}, \end{aligned} \quad (5.78)$$

where $C = C(Q, \lambda, p)$.

Combining (5.67), (5.72), (5.73), (5.76), and (5.78), we deduce:

$$\begin{aligned} \frac{1}{i} \frac{d}{dt} \int_{\mathbb{R}^2} n^{2i}(x, t) \phi^2(x, t) dx &\leq -2\tilde{g}_0(t) + 2\tilde{g}_1(t) + 2\tilde{g}_2(t) + 2\tilde{g}_3(t) + 2\tilde{g}_4(t) \\ &\leq -2\tilde{g}_0(t) + \left(1 + 4i\varepsilon_0 + 4\left(\frac{1}{\varepsilon_0} + 4\right)\varepsilon_1 i\right) \tilde{g}_0(t) + C \ell^2 (t - t_0)^{\ell-2} \|n(t)\|_{L^i(B_{\rho_2}(x_0))}^{2i} \end{aligned}$$

$$+ \left(\frac{Ci}{(\rho_2 - \rho_1)^4} + C \left(\frac{1}{\varepsilon_0} + 4 \right) \frac{1}{\varepsilon_1^{\frac{p+2}{p-2}}} \left(\|\nabla v(t)\|_{L^p(B_{\rho_2}(x_0))}^{\frac{4p}{p-2}} + 1 \right) \right) (t - t_0)^\ell \|n(t)\|_{L^i(B_{\rho_2}(x_0))}^{2i}, \quad (5.79)$$

where $C = C(\gamma, Q, \lambda, p)$. Setting $\varepsilon_0 = \frac{1}{16i}$ and $\varepsilon_1 = \frac{1}{64(4i+1)i}$ in (5.79), we obtain:

$$\begin{aligned} \frac{1}{i} \frac{d}{dt} \int_{\mathbb{R}^2} n^{2i}(x, t) \phi^2(x, t) dx &\leq C \ell^2 (t - t_0)^{\ell-2} \|n(t)\|_{L^i(B_{\rho_2}(x_0))}^{2i} \\ &+ Ci^{\frac{3p-2}{p-2}} \left(\frac{1}{(\rho_2 - \rho_1)^4} + \|\nabla v(t)\|_{L^p(B_{\rho_2}(x_0))}^{\frac{4p}{p-2}} + 1 \right) (t - t_0)^\ell \|n(t)\|_{L^i(B_{\rho_2}(x_0))}^{2i}, \end{aligned} \quad (5.80)$$

where, $C = C(\gamma, Q, \lambda, p)$, as $\frac{3p-2}{p-2} > 3$ for $p > 2$.

(i). In the case $2c_1\rho^2 < T_{\max}$ and $2c_1\rho^2 \leq t_1 < T_{\max}$, we set $t_0 = a = t_1 - 2c_1\rho^2$ and $\ell = 2$ with $2 < p < \infty$ in (5.80). Let $s \in (t_1 - c_1\rho^2, \min\{t_1 + 2c_1\rho^2, T_{\max}\})$. Integrating both sides of (5.80) over $[a, s]$ for $s \in (a, \min\{t_1 + 2c_1\rho^2, T_{\max}\})$, and using (5.53) in Lemma 5.3, we deduce:

$$\begin{aligned} \frac{(s-a)^2}{i} \int_{B_{\rho_1}(x_0)} n^{2i}(x, s) dx &\leq \frac{1}{i} \int_{\mathbb{R}^2} n^{2i}(x, s) \phi^2(x, s) dx \\ &\leq C \left\{ (s-a) + i^{\frac{3p-2}{p-2}} \left(\frac{1}{(\rho_2 - \rho_1)^4} + \left(\frac{1}{(\rho_2 - \rho_1)^{\frac{3}{2}}} \right)^{\frac{4p}{p-2}} + 1 \right) (s-a)^3 \right\} \\ &\quad \times \sup_{\tau \in (t_1 - c_1\rho^2, \min\{t_1 + 2c_1\rho^2, T_{\max}\})} \|n(\tau)\|_{L^i(B_{\rho_2}(x_0))}^{2i}, \end{aligned} \quad (5.81)$$

where $C = C(\gamma, Q, \lambda, \|n_0\|_{L^1}, p)$.

By selecting $p \geq 6$ in (5.81), it follows that $\frac{3p-2}{p-2} \leq 4$, $\frac{4p}{p-2} \leq 6$, and the following inequality holds:

$$\begin{aligned} \frac{(s-a)^2}{i} \int_{B_{\rho_1}(x_0)} n^{2i}(x, s) dx &\leq \frac{1}{i} \int_{\mathbb{R}^2} n^{2i}(x, s) \phi^2(x, s) dx \\ &\leq C \left\{ (s-a) + \frac{i^4}{(\rho_2 - \rho_1)^9} (s-a)^3 \right\} \sup_{\tau \in (t_1 - c_1\rho^2, \min\{t_1 + 2c_1\rho^2, T_{\max}\})} \|n(\tau)\|_{L^i(B_{\rho_2}(x_0))}^{2i} \\ &\leq \frac{Ci^4}{(\rho_2 - \rho_1)^9} \left((s-a) + (s-a)^3 \right) \sup_{\tau \in (t_1 - c_1\rho^2, \min\{t_1 + 2c_1\rho^2, T_{\max}\})} \|n(\tau)\|_{L^i(B_{\rho_2}(x_0))}^{2i}, \end{aligned}$$

where $C = C(\gamma, Q, \lambda, \|n_0\|_{L^1}, p)$. This yields the inequality:

$$\|n(s)\|_{L^{2i}(B_{\rho_1}(x_0))}^{2i} \leq \frac{Ci^5}{(\rho_2 - \rho_1)^9} \left(\frac{1}{s-a} + s-a \right) \sup_{t_1 - c_1\rho^2 < \tau < \min\{t_1 + 2c_1\rho^2, T_{\max}\}} \|n(\tau)\|_{L^i(B_{\rho_2}(x_0))}^{2i}$$

for all $s \in (t_1 - c_1\rho^2, \min\{t_1 + 2c_1\rho^2, T_{\max}\})$, where $C = C(\gamma, Q, \lambda, \|n_0\|_{L^1}, p)$. Moreover, it holds that:

$$\begin{aligned} c_1\rho^2 < s - a = s - (t_1 - 2c_1\rho^2) < \min\{4c_1\rho^2, T_{\max} - t_1 + 2c_1\rho^2\} \\ &= \begin{cases} 4c_1\rho^2 & \text{when } 2c_1\rho^2 < T_{\max} - t_1, \\ T_{\max} - t_1 + 2c_1\rho^2 & \text{when } 2c_1\rho^2 \geq T_{\max} - t_1, \end{cases} \\ &\leq 4c_1\rho^2. \end{aligned}$$

Consequently, we obtain:

$$\|n(s)\|_{L^{2i}(B_{\rho_1}(x_0))} \leq \left(\frac{Ci}{(\rho_2 - \rho_1)^{\frac{9}{5}}}\right)^{\frac{5}{2i}} \left(\frac{1}{\rho}\right)^{\frac{1}{i}} \sup_{\tau \in (t_1 - c_1\rho^2, \min\{t_1 + 2c_1\rho^2, T_{\max}\})} \|n(\tau)\|_{L^i(B_{\rho_2}(x_0))}$$

for all $s \in (t_1 - c_1\rho^2, \min\{t_1 + 2c_1\rho^2, T_{\max}\})$, where $C = C(\gamma, Q, \lambda, \|n_0\|_{L^1}, p)$. This establishes the desired estimate (5.61).

(ii). In the case $2c_1\rho^2 < T_{\max}$ and $0 \leq t \leq \min\{2c_1\rho^2, T_{\max}\}$, we take $t_0 = 0$, $\ell = 2i - 2 + \frac{2p}{p-2} + \frac{4(i-2)}{p-2}$ with $2 < p < \infty$ in (5.80). Let $s \in (0, \min\{t_1 + 2c_1\rho^2, T_{\max}\})$. Substituting t_0 and ℓ into (5.80), we have:

$$\begin{aligned} &\frac{1}{i} \frac{d}{dt} \int_{\mathbb{R}^2} n^{2i}(x, t) \phi^2(x, t) dx \\ &\leq \left\{ C \left(2i - 2 + \frac{2p}{p-2} + \frac{4(i-2)}{p-2} \right)^2 t^{\frac{4}{p-2}} \right. \\ &\quad \left. + Ci^{\frac{3p-2}{p-2}} \left(\frac{1}{(\rho_2 - \rho_1)^4} + (t^{\frac{1}{2}} \|\nabla v(t)\|_{L^p(B_{\rho_2}(x_0))})^{\frac{4p}{p-2}} t^{-\frac{2p}{p-2}} + 1 \right) t^{\frac{2p}{p-2}} \right\} \\ &\quad \times t^{(1-\frac{1}{i} + \frac{2(i-2)}{i(p-2)})2i} \|n(t)\|_{L^i(B_{\rho_2}(x_0))}^{2i}, \end{aligned} \quad (5.82)$$

where $C = C(\gamma, Q, \lambda, p)$. Thus, integrating both sides of (5.82) over $[0, s]$ for $s \in (0, \min\{t_1 + 2c_1\rho^2, T_{\max}\})$, and applying (5.54) in Proposition 5.3, we find:

$$\begin{aligned} \frac{1}{i} \int_{\mathbb{R}^2} n^{2i}(x, s) \phi^2(x, s) dx &\leq C \left\{ \left(2i - 2 + \frac{2p}{p-2} + \frac{4(i-2)}{p-2} \right)^2 \frac{p-2}{p+2} s^{\frac{p+2}{p-2}} \right. \\ &\quad \left. + i^{\frac{3p-2}{p-2}} \left(\left(\frac{1}{(\rho_2 - \rho_1)^4} + 1 \right) \frac{p-2}{3p-2} s^{\frac{3p-2}{p-2}} + \frac{1}{(\rho_2 - \rho_1)^{\frac{10p}{p-2}}} s \right) \right\} \\ &\quad \times \sup_{\tau \in (0, \min\{t_1 + 2c_1\rho^2, T_{\max}\})} \left(\tau^{1-\frac{1}{i} + \frac{2(i-2)}{i(p-2)}} \|n(\tau)\|_{L^i(B_{\rho_2}(x_0))} \right)^{2i} \end{aligned}$$

for all $s \in (0, \min\{t_1 + 2c_1\rho^2, T_{\max}\})$, where $C = C(\gamma, Q, \lambda, \|n_0\|_{L^1}, p)$.

Here, we observe:

$$\begin{aligned} s \left(s^{1-\frac{1}{2i}+\frac{2(i-1)}{i(p-2)}} \|n(s)\|_{L^{2i}(B_{\rho_1}(x_0))} \right)^{2i} &= \int_{B_{\rho_1}(x_0)} n^{2i}(x, s) \psi^8(x - x_0) s^{2i-2+\frac{2p}{p-2}+\frac{4(i-2)}{p-2}} dx \\ &= \int_{B_{\rho_1}(x_0)} n^{2i}(x, t) \phi^2(x, t) dx. \end{aligned}$$

Choosing $p \geq 6$, we have $\frac{3p-2}{p-2} \leq 4$ and $\frac{10p}{p-2} \leq 15$. Furthermore, noting that $s \in (0, \min\{t_1 + 2c_1\rho^2, T_{\max}\})$, it follows:

$$\begin{aligned} 0 < s < \min\{t_1 + 2c_1\rho^2, T_{\max}\} &= \begin{cases} t_1 + 2c_1\rho^2 & \text{when } 2c_1\rho^2 < T_{\max} - t_1, \\ T_{\max} & \text{when } 2c_1\rho^2 \geq T_{\max} - t_1, \end{cases} \\ &\leq t_1 + 2c_1\rho^2 \leq 4c_1\rho^2. \end{aligned}$$

As a consequence, we have:

$$\begin{aligned} &\left(s^{1-\frac{1}{2i}+\frac{2(i-1)}{i(p-2)}} \|n(s)\|_{L^{2i}(B_{\rho_1}(x_0))} \right)^{2i} \leq s^{-1} \int_{\mathbb{R}^2} n^{2i}(x, s) \phi^2(x, s) dx \\ &\leq Ci \left\{ \left(2i - 2 + \frac{2p}{p-2} + \frac{4(i-2)}{p-2} \right)^2 \frac{p-2}{p+2} s^{\frac{4}{p-2}} \right. \\ &\quad \left. + i^{\frac{3p-2}{p-2}} \left(\left(\frac{1}{(\rho_2 - \rho_1)^4} + 1 \right) \frac{p-2}{3p-2} s^{\frac{2p}{p-2}} + \frac{1}{(\rho_2 - \rho_1)^{\frac{10p}{p-2}}} \right) \right\} \\ &\quad \times \sup_{\tau \in (0, \min\{t+2c_1\rho^2, T_{\max}\})} \left(\tau^{1-\frac{1}{i}+\frac{2(i-2)}{i(p-2)}} \|n(\tau)\|_{L^i(B_{\rho_2}(x_0))} \right)^{2i} \\ &\leq Ci^5 \left\{ \rho^{\frac{8}{p-2}} + \left(\left(\frac{1}{(\rho_2 - \rho_1)^4} + 1 \right) \rho^{\frac{4p}{p-2}} + \frac{1}{(\rho_2 - \rho_1)^{15}} \right) \right\} \\ &\quad \times \sup_{\tau \in (0, \min\{t+2c_1\rho^2, T_{\max}\})} \left(\tau^{1-\frac{1}{i}+\frac{2(i-2)}{i(p-2)}} \|n(\tau)\|_{L^i(B_{\rho_2}(x_0))} \right)^{2i} \\ &\leq \frac{Ci^5}{(\rho_2 - \rho_1)^{15}} \sup_{\tau \in (0, \min\{t+2c_1\rho^2, T_{\max}\})} \left(\tau^{1-\frac{1}{i}+\frac{2(i-2)}{i(p-2)}} \|n(\tau)\|_{L^i(B_{\rho_2}(x_0))} \right)^{2i} \end{aligned}$$

for all $s \in (0, \min\{t + 2c_1\rho^2, T_{\max}\})$, where $C = C(\gamma, Q, \lambda, \|n_0\|_{L^1}, p)$. This implies:

$$s^{1-\frac{1}{2i}+\frac{2(i-1)}{i(p-2)}} \|n(s)\|_{L^{2i}(B_{\rho_1}(x_0))}$$

$$\leq \left(\frac{Ci}{(\rho_2 - \rho_1)^3} \right)^{\frac{5}{2i}} \sup_{\tau \in (0, \min\{t+2c_1\rho^2, T_{\max}\})} \tau^{1-\frac{1}{i}+\frac{2(i-2)}{i(p-2)}} \|n(\tau)\|_{L^i(B_{\rho_2}(x_0))}$$

for all $s \in (0, \min\{t + 2c_1\rho^2, T_{\max}\})$, where $C = C(\gamma, Q, \lambda, \|n_0\|_{L^1}, p)$. Hence, the desired estimate (5.62) is obtained.

(iii). In the case $2c_1\rho^2 \geq T_{\max}$ and $0 \leq t_1 < \min\{2c_1\rho^2, T_{\max}\} = T_{\max}$, we choose $t_0 = -\rho^2$ and $\ell = 2$ with $2 < p < \infty$ in (5.80). Let $s \in (0, \min\{t + 2c_1\rho^2, T_{\max}\})$. Integrating both sides of (5.80) over $[0, s]$ for $s \in (0, \min\{t_1 + 2c_1\rho^2, T_{\max}\})$, and applying (5.55) in Proposition 5.3, we find:

$$\begin{aligned} & \frac{(s + \rho^2)^2}{i} \int_{B_{\rho_1}(x_0)} n^{2i}(x, s) dx \leq \frac{1}{i} \int_{\mathbb{R}^2} n^{2i}(x, s) \phi^2(x, s) dx \\ & \leq C \left\{ (s + \rho^2) + i^{\frac{3p-2}{p-2}} \left(\frac{1}{(\rho_2 - \rho_1)^4} + \left(\frac{1}{(\rho_2 - \rho_1)^{\frac{7}{2}}} \right)^{\frac{4p}{p-2}} + 1 \right) (s + \rho^2)^3 \right\} \\ & \quad \times \sup_{\tau \in (0, \min\{t_1+2c_1\rho^2, T_{\max}\})} \|n(\tau)\|_{L^i(B_{\rho_2}(x_0))}^{2i} + \frac{\rho^4}{i} \|n_0\|_{L^{2i}(B_{\rho_2}(x_0))}^{2i}, \end{aligned}$$

where $C = C(\gamma, Q, \lambda, \|n_0\|_{L^1}, p)$.

Choosing $p \geq 6$, we find that $\frac{3p-2}{p-2} \leq 4$ and $\frac{4p}{p-2} \leq 6$. Furthermore, noting that $s \in (0, \min\{t_1 + 2c_1\rho^2, T_{\max}\})$, it follows that:

$$\begin{aligned} 0 < s < \min\{t_1 + 2c_1\rho^2, T_{\max}\} &= \begin{cases} t_1 + 2c_1\rho^2 & \text{when } 2c_1\rho^2 < T_{\max} - t_1, \\ T_{\max} & \text{when } 2c_1\rho^2 \geq T_{\max} - t_1, \end{cases} \\ &\leq t_1 + 2c_1\rho^2 \leq 4c_1\rho^2. \end{aligned}$$

Therefore, we deduce:

$$\begin{aligned} & \frac{(s + \rho^2)^2}{i} \int_{B_{\rho_1}(x_0)} n^{2i}(x, s) dx \leq \frac{1}{i} \int_{\mathbb{R}^2} n^{2i}(x, s) \phi^2(x, s) dx \\ & \leq C \left\{ (s + \rho^2) + i^4 \left(\frac{1}{(\rho_2 - \rho_1)^4} + \frac{1}{(\rho_2 - \rho_1)^{21}} + 1 \right) (s + \rho^2)^3 \right\} \\ & \quad \times \sup_{\tau \in (0, \min\{t_1+2c_1\rho^2, T_{\max}\})} \|n(\tau)\|_{L^i(B_{\rho_2}(x_0))}^{2i} + \frac{\rho^4}{i} \|n_0\|_{L^{2i}(B_{\rho_2}(x_0))}^{2i} \\ & \leq \frac{Ci^4}{(\rho_2 - \rho_1)^{21}} \left((s + \rho^2) + (s + \rho^2)^3 \right) \sup_{\tau \in (0, \min\{t_1+2c_1\rho^2, T_{\max}\})} \|n(\tau)\|_{L^i(B_{\rho_2}(x_0))}^{2i} \\ & \quad + \frac{\rho^4}{i} \|n_0\|_{L^{2i}(B_{\rho_2}(x_0))}^{2i}, \end{aligned}$$

where $C = C(\gamma, Q, \lambda, \|n_0\|_{L^1}, p)$. This leads to the following inequality:

$$\begin{aligned}
& \|n(s)\|_{L^{2i}(B_{\rho_1}(x_0))}^{2i} \\
& \leq \frac{Ci^5}{(\rho_2 - \rho_1)^{21}} \left(\frac{1}{s + \rho^2} + s + \rho^2 \right) \sup_{\tau \in (0, \min\{t_1 + 2c_1\rho^2, T_{\max}\})} \|n(\tau)\|_{L^i(B_{\rho_2}(x_0))}^{2i} \\
& \quad + \frac{\rho^4}{(s + \rho^2)^2} \|n_0\|_{L^{2i}(B_{\rho_2}(x_0))}^{2i} \\
& \leq \frac{Ci^5}{(\rho_2 - \rho_1)^{21}} \cdot \frac{1}{\rho^2} \sup_{\tau \in (0, \min\{t_1 + 2c_1\rho^2, T_{\max}\})} \|n(\tau)\|_{L^i(B_{\rho_2}(x_0))}^{2i} + \|n_0\|_{L^{2i}(B_{\rho_2}(x_0))}^{2i}
\end{aligned}$$

for all $s \in (0, \min\{t + 2c_1\rho^2, T_{\max}\})$, where $C = C(\gamma, Q, \lambda, \|n_0\|_{L^1})$. Hence, we conclude:

$$\begin{aligned}
& \|n(s)\|_{L^{2i}(B_{\rho_1}(x_0))} \\
& \leq \left(\frac{Ci}{(\rho_2 - \rho_1)^{\frac{21}{5}}} \right)^{\frac{5}{2i}} \left(\frac{1}{\rho} \right)^{\frac{1}{i}} \sup_{\tau \in (0, \min\{t_1 + 2c_1\rho^2, T_{\max}\})} \|n(\tau)\|_{L^i(B_{\rho_2}(x_0))} + \|n_0\|_{L^{2i}(B_{\rho_2}(x_0))}
\end{aligned}$$

for all $s \in (0, \min\{t + 2c_1\rho^2, T_{\max}\})$, where $C = C(\gamma, Q, \lambda, \|n_0\|_{L^1}, p)$. This establishes the desired estimate.

Step 4. In this step, we complete the proof of Theorem 2.2. To achieve this, we assume that (2.15) holds for some $x_0 \in \mathbb{R}^2$, $0 < \rho \leq 1$, and $0 < t < T_{\max}$. Furthermore, let m_* and c_1 denote the constants given in Proposition 5.1, where the explicit expression for c_1 is provided in (5.5) of Remark 11.

Proof of Theorem 2.2. (i). If $2c_1\rho^2 < T_{\max}$ and $2c_1\rho^2 \leq t_1 < T_{\max}$, Proposition 5.4 with $i \geq 2$ provides a local L^i -estimate of n on:

$$B_\rho(x_0) \times (t_1 - c_1\rho^2, \min\{t_1 + 2c_1\rho^2, T_{\max}\}).$$

Combining this with the reverse Hölder-type inequality in Proposition 5.4, and starting from this inequality, we apply Moser's iteration (cf. [23, p. 3079–3082], [22, Proof of Theorem 2.2]). Iterating with $i \mapsto 2i \mapsto 4i \mapsto \dots$ and letting $i \rightarrow \infty$, we obtain:

$$\sup_{s \in (t_1 - c_1\rho^2, \min\{t_1 + 2c_1\rho^2, T_{\max}\})} \|n(s)\|_{L^\infty(B_{\frac{\rho}{4}}(x_0))} \leq C,$$

where $C = C(\gamma, Q, \lambda, \|n_0\|_{L^1}, \rho)$. This proves (2.16).

(ii). If $2c_1\rho^2 < T_{\max}$ and $0 \leq t_1 < 2c_1\rho^2$, Proposition 5.4 with $i \geq 2$ yields the weighted inequality:

$$\begin{aligned}
& s^{1 - \frac{1}{2i} + \frac{(i-1)\varepsilon}{4i}} \|n(s)\|_{L^{2i}(B_{\rho_1}(x_0))} \\
& \leq \left(\frac{Ci}{(\rho_2 - \rho_1)^3} \right)^{\frac{5}{2i}} \sup_{\tau \in (0, \min\{t_1 + 2c_1\rho^2, T_{\max}\})} \tau^{1 - \frac{1}{i} + \frac{(i-2)\varepsilon}{4i}} \|n(\tau)\|_{L^i(B_{\rho_2}(x_0))}.
\end{aligned}$$

Applying the same Moser's iteration as in (i) and letting $i \rightarrow \infty$, we obtain:

$$\sup_{s \in (0, \min\{t_1 + 2c_1\rho^2, T_{\max}\})} s^{1+\frac{\varepsilon}{4}} \|n(s)\|_{L^\infty(B_{\frac{\rho}{4}}(x_0))} \leq C,$$

where $C = C(\gamma, Q, \lambda, \|n_0\|_{L^1}, \rho, \varepsilon)$. This proves (2.17).

(iii). If $2c_1\rho^2 \geq T_{\max}$ and $0 \leq t_1 < T_{\max}$, Proposition 5.4 with $i \geq 2$ provides a local L^i -estimate for n , including additional terms depending on the initial data, on:

$$B_\rho(x_0) \times (0, \min\{t_1 + 2c_1\rho^2, T_{\max}\}).$$

Together with the reverse Hölder inequality in Proposition 5.4, performing the same Moser's iteration as above and letting $i \rightarrow \infty$ gives:

$$\sup_{s \in (0, \min\{t_1 + 2c_1\rho^2, T_{\max}\})} \|n(s)\|_{L^\infty(B_{\frac{\rho}{4}}(x_0))} \leq C,$$

where $C = C(\gamma, Q, \lambda, \|n_0\|_{L^1}, \|n_0\|_{L^\infty_{\text{loc}}}, \rho)$. This proves (2.18). This completes the proof of Theorem 2.2. $\square$

6. Proof of Theorem 2.4

6.1. Proof of Theorem 2.4 (I): maximal existence time estimate

Let T^* be positive time defined in (2.24), and let ε_0 be given in (2.22). We define $G(\varepsilon_0, T^*)$ as in (2.26) in Remark 6. The constant ℓ_0 in the definition of $G(\varepsilon_0, T^*)$ will be determined later, so that:

$$\ell_0 > \frac{\delta}{2}, \tag{6.1}$$

where, δ denotes the parameter in assumption (ii) concerning ϕ in Theorem 2.4. In addition, we assume $|\mathbf{a}| \geq G(\varepsilon_0, T^*)$.

Using Lemma 3.6, we introduce the cut-off function ψ with $\rho_1 = \ell_0$ and $\rho_2 = 2\ell_0$. To derive an upper bound for the blow-up time $T_{\max}$, we multiply the first equation of (KSF) by $|x - \mathbf{a}|^2 \psi(x - \mathbf{a})$. This yields:

$$\begin{aligned} & \frac{d}{dt} \int_{\mathbb{R}^2} n(x, t) |x - \mathbf{a}|^2 \psi(x - \mathbf{a}) \, dx \\ &= \int_{\mathbb{R}^2} \Delta n(x, t) |x - \mathbf{a}|^2 \psi(x - \mathbf{a}) \, dx - \int_{\mathbb{R}^2} \nabla \cdot ((n \nabla v)(x, t)) |x - \mathbf{a}|^2 \psi(x - \mathbf{a}) \, dx \\ & \quad - \int_{\mathbb{R}^2} \nabla \cdot ((u(x) n(x, t)) |x - \mathbf{a}|^2 \psi(x - \mathbf{a})) \, dx \\ &=: I + J + K. \end{aligned} \tag{6.2}$$

Since the following identity holds:

$$\Delta(|x - \mathbf{a}|^2 \psi(x - \mathbf{a})) = 4\psi(x - \mathbf{a}) + 4(x - \mathbf{a}) \cdot \nabla \psi(x - \mathbf{a}) + |x - \mathbf{a}|^2 \Delta \psi(x - \mathbf{a}),$$

we estimate I as:

$$\begin{aligned} I &\leq 4 \int_{\mathbb{R}^2} n(x, t) \psi(x - \mathbf{a}) \, dx + 4 \int_{B_{2\ell_0}(\mathbf{a}) \setminus B_{\ell_0}(\mathbf{a})} n(x, t) |x - \mathbf{a}| |\nabla \psi(x - \mathbf{a})| \, dx \\ &\quad + \int_{B_{2\ell_0}(\mathbf{a}) \setminus B_{\ell_0}(\mathbf{a})} n(x, t) |x - \mathbf{a}|^2 |\Delta \psi(x - \mathbf{a})| \, dx \\ &\leq 4 \int_{\mathbb{R}^2} n(x, t) \psi(x - \mathbf{a}) \, dx + 16(\sqrt{2} + 6) \int_{B_{2\ell_0}(\mathbf{a}) \setminus B_{\ell_0}(\mathbf{a})} n(x) \, dx. \end{aligned} \quad (6.3)$$

Next, we consider J . By integrating by parts, it follows:

$$\begin{aligned} J &= \int_{\mathbb{R}^2} (n \nabla v)(x, t) \cdot \nabla(|x - \mathbf{a}|^2 \psi(x - \mathbf{a})) \, dx \\ &= 2 \int_{\mathbb{R}^2} (n \nabla v)(x, t) \cdot (x - \mathbf{a}) \psi(x - \mathbf{a}) \, dx + \int_{B_{2\ell_0}(\mathbf{a}) \setminus B_{\ell_0}(\mathbf{a})} (n \nabla v)(x, t) \cdot |x - \mathbf{a}|^2 \nabla \psi(x - \mathbf{a}) \, dx \\ &= 2 \int_{\mathbb{R}^2} (n \nabla v)(x, t) \cdot (x - \mathbf{a}) \psi(x - \mathbf{a}) \psi(y - \mathbf{a}) \, dx \\ &\quad + 2 \int_{\mathbb{R}^2} (n \nabla v)(x, t) \cdot (x - \mathbf{a}) \psi(x - \mathbf{a}) (1 - \psi(y - \mathbf{a})) \, dx \\ &\quad + \int_{B_{2\ell_0}(\mathbf{a}) \setminus B_{\ell_0}(\mathbf{a})} (n \nabla v)(x, t) \cdot |x - \mathbf{a}|^2 \nabla \psi(x - \mathbf{a}) \, dx \\ &=: J_1 + J_2 + J_3. \end{aligned} \quad (6.4)$$

Let R be a sufficiently small positive real number, to be determined later. Since $\nabla v = \nabla G_\gamma * n$, it follows from the symmetry of $\nabla_x G_\gamma(x - y) \cdot (x - y)$ and (3.5):

$$\begin{aligned} J_1 &= \int_{\mathbb{R}^2 \times \mathbb{R}^2} n(x, t) n(y, t) \nabla G_\gamma(x - y) \cdot \{(x - \mathbf{a}) - (y - \mathbf{a})\} \psi(x - \mathbf{a}) \psi(y - \mathbf{a}) \, dx dy \\ &\leq -\frac{1}{2\pi} \int_{\mathbb{R}^2 \times \mathbb{R}^2} n(x, t) n(y, t) e^{-\sqrt{\gamma}|x-y|} \psi(x - \mathbf{a}) \psi(y - \mathbf{a}) \, dx dy \end{aligned}$$

$$\begin{aligned}
&\leq -\frac{e^{-\sqrt{\gamma}R}}{2\pi} \int \int_{|x-y|\leq R} n(x,t)n(y,t)\psi(x-\mathbf{a})\psi(y-\mathbf{a}) \, dx dy \\
&= -\frac{e^{-\sqrt{\gamma}R}}{2\pi} \int \int_{\mathbb{R}^2 \times \mathbb{R}^2} n(x,t)n(y,t)\psi(x-\mathbf{a})\psi(y-\mathbf{a}) \, dx dy \\
&\quad + \frac{e^{-\sqrt{\gamma}R}}{2\pi} \int \int_{|x-y|\geq R} n(x,t)n(y,t)\psi(x-\mathbf{a})\psi(y-\mathbf{a}) \, dx dy \\
&\leq -\frac{e^{-\sqrt{\gamma}R}}{2\pi} \left(\int_{\mathbb{R}^2} n(x,t)\psi(x-\mathbf{a}) \, dx \right)^2 \\
&\quad + \frac{e^{-\sqrt{\gamma}R}}{2\pi} \int \int_{|x-y|\geq R} n(x,t)n(y,t)\psi(x-\mathbf{a})\psi(y-\mathbf{a}) \frac{2}{R^2} (|x-\mathbf{a}|^2 + |y-\mathbf{a}|^2) \, dx dy \\
&\leq -\frac{e^{-\sqrt{\gamma}R}}{2\pi} \left(\int_{\mathbb{R}^2} n(x,t)\psi(x-\mathbf{a}) \, dx \right)^2 + \frac{2e^{-\sqrt{\gamma}R}}{\pi R^2} \|n_0\|_{L^1} \int_{\mathbb{R}^2} n(x,t)|x-\mathbf{a}|^2 \psi(x-\mathbf{a}) \, dx.
\end{aligned} \tag{6.5}$$

We now consider J_2 . Let L be a positive real number, to be determined later. Then, the following holds:

$$\begin{aligned}
J_2 &= 2 \int \int_{\mathbb{R}^2 \setminus B_{\ell_0}(\mathbf{a}) \setminus B_{2\ell_0}(\mathbf{a})} n(x,t)n(y,t) \nabla G_\gamma(x-y) \cdot (x-\mathbf{a})\psi(x-\mathbf{a})(1-\psi(y-\mathbf{a})) \, dx dy \\
&= 2 \int \int_{\mathbb{R}^2 \setminus B_{2\ell_0+L}(\mathbf{a}) \setminus B_{2\ell_0}(\mathbf{a})} n(x,t)n(y,t) \nabla G_\gamma(x-y) \cdot (x-\mathbf{a})\psi(x-\mathbf{a})(1-\psi(y-\mathbf{a})) \, dx dy \\
&\quad + 2 \int \int_{B_{2\ell_0+L}(\mathbf{a}) \setminus B_{\ell_0}(\mathbf{a}) \setminus B_{2\ell_0}(\mathbf{a})} n(x,t)n(y,t) \nabla G_\gamma(x-y) \cdot (x-\mathbf{a})\psi(x-\mathbf{a})(1-\psi(y-\mathbf{a})) \, dx dy \\
&=: J_2^1 + J_2^2.
\end{aligned} \tag{6.6}$$

As for J_2^1 , since $|x-y| \geq L$ and $|x-\mathbf{a}| \leq 2\ell_0$, it follows from (3.6):

$$\begin{aligned}
|J_2^1| &\leq 2 \int \int_{\mathbb{R}^2 \setminus B_{2\ell_0+L}(\mathbf{a}) \setminus B_{2\ell_0}(\mathbf{a})} n(x,t)n(y,t) |\nabla G_\gamma(x-y)| |x-\mathbf{a}| \psi(x-\mathbf{a})(1-\psi(y-\mathbf{a})) \, dx dy \\
&\leq \frac{2}{L} \int \int_{\mathbb{R}^2 \setminus B_{2\ell_0+L}(\mathbf{a}) \setminus B_{2\ell_0}(\mathbf{a})} n(x,t)n(y,t) |\nabla G_\gamma(x-y)| |x-y| 2\ell_0 \, dx dy
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{4\ell_0}{L} \cdot \frac{1}{\pi} \left(\frac{\sqrt{2}}{e} + 1 \right) \int_{\mathbb{R}^2 \setminus B_{2\ell_0+L}(\mathbf{a})} \int_{B_{2\ell_0}(\mathbf{a})} n(x, t) n(y, t) \, dx dy \\
&\leq \frac{4\ell_0}{L} \cdot \frac{1}{\pi} \left(\frac{\sqrt{2}}{e} + 1 \right) \|n_0\|_{L^1}^2.
\end{aligned} \tag{6.7}$$

We define D_{ℓ_0} as:

$$D_{\ell_0} := \left\{ x \in \mathbb{R}^2 \mid \frac{\ell_0}{2} \leq |x - \mathbf{a}| \leq 2\ell_0 + L \right\}. \tag{6.8}$$

Regarding J_2^2 , since $\psi(x - \mathbf{a}) = 0$ for $2\ell_0 \leq |x - \mathbf{a}| \leq 2\ell_0 + L$ and $1 - \psi(y - \mathbf{a}) = 0$ for $\frac{\ell_0}{2} \leq |y - \mathbf{a}| \leq \ell_0$, it holds:

$$\begin{aligned}
J_2^2 &= 2 \int_{B_{2\ell_0+L}(\mathbf{a}) \setminus B_{\ell_0}(\mathbf{a})} \int_{B_{2\ell_0+L}(\mathbf{a})} n(x, t) n(y, t) \nabla G_\gamma(x - y) \cdot (x - \mathbf{a}) \psi(x - \mathbf{a}) (1 - \psi(y - \mathbf{a})) \, dx dy \\
&= 2 \int_{D_{\ell_0}} \int_{D_{\ell_0}} n(x, t) n(y, t) \nabla G_\gamma(x - y) \cdot (x - \mathbf{a}) \psi(x - \mathbf{a}) (1 - \psi(y - \mathbf{a})) \, dx dy \\
&\quad + 2 \int_{B_{2\ell_0+L}(\mathbf{a}) \setminus B_{\ell_0}(\mathbf{a})} \int_{B_{\frac{\ell_0}{2}}(\mathbf{a})} n(x, t) n(y, t) \nabla G_\gamma(x - y) \cdot (x - \mathbf{a}) \psi(x - \mathbf{a}) (1 - \psi(y - \mathbf{a})) \, dx dy \\
&=: J_2^{2,1} + J_2^{2,2}.
\end{aligned} \tag{6.9}$$

We estimate $J_2^{2,1}$ and $J_2^{2,2}$. For $J_2^{2,1}$, using the symmetry of $\nabla_x G_\gamma(x - y) \cdot (x - y)$, it follows:

$$\begin{aligned}
J_2^{2,1} &= \int_{D_{\ell_0}} \int_{D_{\ell_0}} n(x, t) n(y, t) \nabla G_\gamma \\
&\quad \cdot \left((x - \mathbf{a}) \psi(x - \mathbf{a}) (1 - \psi(y - \mathbf{a})) - (y - \mathbf{a}) \psi(y - \mathbf{a}) (1 - \psi(x - \mathbf{a})) \right) \, dx dy.
\end{aligned} \tag{6.10}$$

Using the following identity transformation:

$$\begin{aligned}
&(x - \mathbf{a}) \psi(x - \mathbf{a}) (1 - \psi(y - \mathbf{a})) - (y - \mathbf{a}) \psi(y - \mathbf{a}) (1 - \psi(x - \mathbf{a})) \\
&= -\psi(x - \mathbf{a}) \psi(y - \mathbf{a}) (x - y) + (x - \mathbf{a}) \psi(x - \mathbf{a}) - (y - \mathbf{a}) \psi(y - \mathbf{a}) \\
&= -\psi(x - \mathbf{a}) \psi(y - \mathbf{a}) (x - y) + (x - \mathbf{a}) (\psi(x - \mathbf{a}) - \psi(y - \mathbf{a})) + (x - y) \psi(y - \mathbf{a}) \\
&= (x - y) (1 - \psi(x - \mathbf{a})) \psi(y - \mathbf{a}) + (x - \mathbf{a}) (\psi(x - \mathbf{a}) - \psi(y - \mathbf{a})),
\end{aligned} \tag{6.11}$$

it follows from (6.10), (6.11), (3.6), (3.24), and the fundamental theorem of calculus:

$$\begin{aligned}
|J_2^{2,1}| &\leq \int_{D_{\ell_0}} \int_{D_{\ell_0}} n(x,t)n(y,t)|\nabla G_\gamma(x-y)||x-y|(1-\psi(x-\mathbf{a}))\psi(y-\mathbf{a})\,dx\,dy \\
&\quad + \int_{D_{\ell_0}} \int_{D_{\ell_0}} n(x,t)n(y,t)|\nabla G_\gamma(x-y)||x-\mathbf{a}||\psi(x-\mathbf{a})-\psi(y-\mathbf{a})|\,dx\,dy \\
&\leq \frac{1}{\pi} \left(\frac{\sqrt{2}}{e} + 1 \right) \int_{D_{\ell_0}} \int_{D_{\ell_0}} n(x,t)n(y,t)\,dx\,dy \\
&\quad + \int_{D_{\ell_0}} \int_{D_{\ell_0}} n(x,t)n(y,t)|\nabla G_\gamma(x-y)|(2\ell_0+L)\frac{2\sqrt{2}}{\ell_0}|x-y|\,dx\,dy \\
&\leq \frac{1}{\pi} \left(\frac{\sqrt{2}}{e} + 1 \right) \|n_0\|_{L^1} \left(1 + 2\sqrt{2} \left(2 + \frac{L}{\ell_0} \right) \right) \int_{D_{\ell_0}} n(x,t)\,dx. \tag{6.12}
\end{aligned}$$

As for $J_2^{2,2}$, since $|x-y| \geq \frac{\ell_0}{2}$, it follows from (3.6):

$$\begin{aligned}
|J_2^{2,2}| &\leq 2 \int_{B_{2\ell_0+L}(\mathbf{a}) \setminus B_{\ell_0}(\mathbf{a})} \int_{B_{\frac{\ell_0}{2}}(\mathbf{a})} n(x,t)n(y,t)|\nabla G_\gamma(x-y)||x-\mathbf{a}|\psi(x-\mathbf{a})(1-\psi(y-\mathbf{a}))\,dx\,dy \\
&\leq 2 \int_{B_{2\ell_0+L}(\mathbf{a}) \setminus B_{\ell_0}(\mathbf{a})} \int_{B_{\frac{\ell_0}{2}}(\mathbf{a})} n(x,t)n(y,t)|\nabla G_\gamma(x-y)|\frac{2|x-y|}{\ell_0}\frac{\ell_0}{2}\,dx\,dy \\
&\leq 2 \cdot \frac{1}{\pi} \left(\frac{\sqrt{2}}{e} + 1 \right) \int_{B_{2\ell_0+L}(\mathbf{a}) \setminus B_{\ell_0}(\mathbf{a})} \int_{B_{\frac{\ell_0}{2}}(\mathbf{a})} n(x,t)n(y,t)\,dx\,dy \\
&\leq 2 \cdot \frac{1}{\pi} \left(\frac{\sqrt{2}}{e} + 1 \right) \|n_0\|_{L^1} \int_{D_{\ell_0}} n(x,t)\,dx. \tag{6.13}
\end{aligned}$$

Thus, combining (6.9), (6.12) and (6.13), we have:

$$|J_2^2| \leq \frac{1}{\pi} \left(\frac{\sqrt{2}}{e} + 1 \right) \|n_0\|_{L^1} \left(3 + 2\sqrt{2} \left(2 + \frac{L}{\ell_0} \right) \right) \int_{D_{\ell_0}} n(x)\,dx. \tag{6.14}$$

At this point, substituting (6.7) and (6.14) into (6.6), we obtain:

$$J_2 \leq |J_2^1| + |J_2^2|$$

$$\leq \frac{4\ell_0}{L} \cdot \frac{1}{\pi} \left(\frac{\sqrt{2}}{e} + 1 \right) \|n_0\|_{L^1}^2 + \frac{1}{\pi} \left(\frac{\sqrt{2}}{e} + 1 \right) \|n_0\|_{L^1} \left(3 + 2\sqrt{2} \left(2 + \frac{L}{\ell_0} \right) \right) \int_{D_{\ell_0}} n(x, t) \, dx. \quad (6.15)$$

We now proceed to estimate J_3 . The whole plane $\mathbb{R}^2$ is divided as follows:

$$\mathbb{R}^2 = B_{\frac{\ell_0}{2}}(\mathbf{a}) \cup D_{\ell_0} \cup E_{\ell_0}, \quad \text{where} \quad E_{\ell_0} := \left\{ x \in \mathbb{R}^2 \mid |x - \mathbf{a}| \geq 2\ell_0 + L \right\}.$$

It then follows:

$$\begin{aligned} J_3 &= \int_{\mathbb{R}^2 \setminus B_{2\ell_0}(\mathbf{a}) \setminus B_{\ell_0}(\mathbf{a})} n(x, t) n(y, t) \nabla G_\gamma(x - y) \cdot |x - \mathbf{a}|^2 \nabla \psi(x - \mathbf{a}) \, dx dy \\ &= \left(\int_{B_{\frac{\ell_0}{2}}(\mathbf{a})} + \int_{D_{\ell_0}} + \int_{E_{\ell_0}} \right) \int_{B_{2\ell_0}(\mathbf{a}) \setminus B_{\ell_0}(\mathbf{a})} n(x, t) n(y, t) \nabla G_\gamma(x - y) \cdot |x - \mathbf{a}|^2 \nabla \psi(x - \mathbf{a}) \, dx dy \\ &=: J_3^1 + J_3^2 + J_3^3. \end{aligned} \quad (6.16)$$

We now estimate J_1^3 and J_3^3 simultaneously. Since the support of $|\nabla \psi|$ is contained in D_{ℓ_0} , we deduce from (3.6) that:

$$\begin{aligned} |J_3^1| + |J_3^3| &\leq \int_{B_{\frac{\ell_0}{2}}(\mathbf{a}) \setminus B_{2\ell_0}(\mathbf{a}) \setminus B_{\ell_0}(\mathbf{a})} n(x, t) n(y, t) |\nabla G_\gamma(x - y)| |x - \mathbf{a}|^2 |\nabla \psi(x - \mathbf{a})| \, dx dy \\ &\quad + \int_{E_{\ell_0} \setminus B_{2\ell_0}(\mathbf{a}) \setminus B_{\ell_0}(\mathbf{a})} n(x, t) n(y, t) |\nabla G_\gamma(x - y)| |x - \mathbf{a}|^2 |\nabla \psi(x - \mathbf{a})| \, dx dy \\ &\leq \int_{B_{\frac{\ell_0}{2}}(\mathbf{a}) \setminus B_{2\ell_0}(\mathbf{a}) \setminus B_{\ell_0}(\mathbf{a})} n(x, t) n(y, t) |\nabla G_\gamma(x - y)| \frac{2|x - y|}{\ell_0} 4\ell_0^2 \frac{2\sqrt{2}}{\ell_0} \, dx dy \\ &\quad + \int_{E_{\ell_0} \setminus B_{2\ell_0}(\mathbf{a}) \setminus B_{\ell_0}(\mathbf{a})} n(x, t) n(y, t) |\nabla G_\gamma(x - y)| \frac{|x - y|}{L} 4\ell_0^2 \frac{2\sqrt{2}}{\ell_0} \, dx dy \\ &\leq 8\sqrt{2} \cdot \frac{1}{\pi} \left(\frac{\sqrt{2}}{e} + 1 \right) \|n_0\|_{L^1} \left(2 + \frac{\ell_0}{L} \right) \int_{D_{\ell_0}} n(x, t) \, dx. \end{aligned} \quad (6.17)$$

For J_3^2 , we have:

$$J_3^2 = \frac{1}{2} \int_{D_{\ell_0}} \int_{D_{\ell_0}} n(x, t) n(y, t) \nabla G_\gamma(x - y) \cdot \left(|x - \mathbf{a}|^2 \nabla \psi(x - \mathbf{a}) - |y - \mathbf{a}|^2 \nabla \psi(y - \mathbf{a}) \right) \, dx dy.$$

Using the following identity:

$$\begin{aligned} & |x - \mathbf{a}|^2 \nabla \psi(x - \mathbf{a}) - |y - \mathbf{a}|^2 \nabla \psi(y - \mathbf{a}) \\ &= |x - \mathbf{a}|^2 (\nabla \psi(x - \mathbf{a}) - \nabla \psi(y - \mathbf{a})) + (|x - \mathbf{a}|^2 - |y - \mathbf{a}|^2) \nabla \psi(y - \mathbf{a}) \\ &= |x - \mathbf{a}|^2 (\nabla \psi(x - \mathbf{a}) - \nabla \psi(y - \mathbf{a})) + (|x - \mathbf{a}| - |y - \mathbf{a}|)(|x - \mathbf{a}| + |y - \mathbf{a}|) \nabla \psi(y - \mathbf{a}), \end{aligned}$$

it follows from (3.24), (3.6), and the fundamental theorem of calculus:

$$\begin{aligned} |J_3^2| &\leq \frac{1}{2} \int_{D_{\ell_0}} \int_{D_{\ell_0}} n(x, t) n(y, t) |\nabla G_\gamma(x - y)| |x - \mathbf{a}|^2 |\nabla \psi(x - \mathbf{a}) - \nabla \psi(y - \mathbf{a})| \, dx dy \\ &\quad + \frac{1}{2} \int_{D_{\ell_0}} \int_{D_{\ell_0}} n(x, t) n(y, t) |\nabla G_\gamma(x - y)| |x - y| (|x - \mathbf{a}| + |y - \mathbf{a}|) |\nabla \psi(y - \mathbf{a})| \, dx dy \\ &\leq \frac{1}{2} \int_{D_{\ell_0}} \int_{D_{\ell_0}} n(x, t) n(y, t) |\nabla G_\gamma(x - y)| (2\ell_0 + L)^2 \frac{24}{\ell_0^2} |x - y| \, dx dy \\ &\quad + \frac{1}{2} \int_{D_{\ell_0}} \int_{D_{\ell_0}} n(x, t) n(y, t) |\nabla G_\gamma(x - y)| |x - y| (2(2\ell_0 + L)) \frac{2\sqrt{2}}{\ell_0} \, dx dy \\ &\leq \frac{1}{\pi} \left(\frac{\sqrt{2}}{e} + 1 \right) \|n_0\|_{L^1} \left[12 \left(2 + \frac{L}{\ell_0} \right)^2 + 2\sqrt{2} \left(2 + \frac{L}{\ell_0} \right) \right] \int_{D_{\ell_0}} n(x, t) \, dx. \quad (6.18) \end{aligned}$$

Thus, combining (6.16), (6.17) and (6.18), we find:

$$\begin{aligned} J_3 &\leq 8\sqrt{2} \cdot \frac{1}{\pi} \left(\frac{\sqrt{2}}{e} + 1 \right) \|n_0\|_{L^1} \left(2 + \frac{\ell_0}{L} \right) \int_{D_{\ell_0}} n(x, t) \, dx \\ &\quad + \frac{1}{\pi} \left(\frac{\sqrt{2}}{e} + 1 \right) \|n_0\|_{L^1} \left[12 \left(2 + \frac{L}{\ell_0} \right)^2 + 2\sqrt{2} \left(2 + \frac{L}{\ell_0} \right) \right] \int_{D_{\ell_0}} n(x, t) \, dx \\ &\leq 2\sqrt{2} \cdot \frac{1}{\pi} \left(\frac{\sqrt{2}}{e} + 1 \right) \|n_0\|_{L^1} \left[4 \left(2 + \frac{\ell_0}{L} \right) + 3\sqrt{2} \left(2 + \frac{L}{\ell_0} \right)^2 + \left(2 + \frac{L}{\ell_0} \right) \right] \int_{D_{\ell_0}} n(x, t) \, dx. \quad (6.19) \end{aligned}$$

Consequently, substituting (6.5), (6.15) and (6.19) into (6.4), we have:

$$J \leq -\frac{e^{-\sqrt{\gamma}R}}{2\pi} \left(\int_{\mathbb{R}^2} n(x, t) \psi(x - \mathbf{a}) \, dx \right)^2$$

$$\begin{aligned}
& + \frac{2e^{-\sqrt{\gamma}R}}{\pi R^2} \|n_0\|_{L^1} \int_{\mathbb{R}^2} n(x, t) |x - \mathbf{a}|^2 \psi(x - \mathbf{a}) \, dx + \frac{4\ell_0}{L} \cdot \frac{1}{\pi} \left(\frac{\sqrt{2}}{e} + 1 \right) \|n_0\|_{L^1}^2 \\
& + \frac{1}{\pi} \left(\frac{\sqrt{2}}{e} + 1 \right) \|n_0\|_{L^1} \left[8\sqrt{2} \left(2 + \frac{\ell_0}{L} \right) + 12 \left(2 + \frac{L}{\ell_0} \right)^2 + 4\sqrt{2} \left(2 + \frac{L}{\ell_0} \right) + 3 \right] \\
& \times \int_{D_{\ell_0}} n(x, t) \, dx \tag{6.20}
\end{aligned}$$

Let us now focus on K . By applying integration by parts, it follows:

$$\begin{aligned}
K &= \int_{\mathbb{R}^2} u(x) n(x, t) \cdot \nabla (|x - \mathbf{a}|^2 \psi(x - \mathbf{a})) \, dx \\
&= 2 \int_{B_{2\ell_0}(\mathbf{a})} u(x) n(x, t) \cdot (x - \mathbf{a}) \psi(x - \mathbf{a}) \, dx \\
&\quad + \int_{B_{2\ell_0}(\mathbf{a}) \setminus B_{\ell_0}(\mathbf{a})} u(x) n(x, t) \cdot |x - \mathbf{a}|^2 \nabla \psi(x - \mathbf{a}) \, dx.
\end{aligned}$$

Since $\text{supp } \chi \subset B_h(\mathbf{b})$, it follows:

$$\begin{aligned}
|K| &\leq 2|Q|\tau_{2\ell_0, h}(\mathbf{a}, \mathbf{b}) \int_{B_{2\ell_0}(\mathbf{a}) \cap B_h(\mathbf{b})} \frac{|x - \mathbf{a}||x - \mathbf{b}|}{|x - \mathbf{b}|^2 + \lambda} \chi(x - \mathbf{b}) n(x, t) \psi(x - \mathbf{a}) \, dx \\
&\quad + |Q|\tau_{2\ell_0, h}(\mathbf{a}, \mathbf{b}) \int_{(B_{2\ell_0}(\mathbf{a}) \setminus B_{\ell_0}(\mathbf{a})) \cap B_h(\mathbf{b})} \frac{|x - \mathbf{a}||x - \mathbf{b}|}{|x - \mathbf{b}|^2 + \lambda} \chi(x - \mathbf{b}) n(x, t) |x - \mathbf{a}| |\nabla \psi(x - \mathbf{a})| \, dx \\
&\leq 2|Q|\tau_{2\ell_0, h}(\mathbf{a}, \mathbf{b}) \|n_0\|_{L^1} + |Q|\tau_{2\ell_0, h}(\mathbf{a}, \mathbf{b}) \int_{(B_{2\ell_0}(\mathbf{a}) \setminus B_{\ell_0}(\mathbf{a})) \cap B_h(\mathbf{b})} n(x, t) \min\{2\ell_0, h\} \frac{2\sqrt{2}}{\ell_0} \, dx \\
&\leq 2|Q|\tau_{2\ell_0, h}(\mathbf{a}, \mathbf{b}) \|n_0\|_{L^1} + 4\sqrt{2}|Q|\tau_{2\ell_0, h}(\mathbf{a}, \mathbf{b}) \int_{D_{\ell_0}} n(x, t) \, dx, \tag{6.21}
\end{aligned}$$

where, the function $\tau_{2\ell_0, h}(\mathbf{a}, \mathbf{b})$ is defined in (2.4). Thus, using (6.2), (6.3), (6.20) and (6.21), we derive:

$$\begin{aligned}
& \frac{d}{dt} \int_{\mathbb{R}^2} n(x, t) |x - \mathbf{a}|^2 \psi(x - \mathbf{a}) \, dx \\
& \leq -\frac{1}{2\pi} \int_{\mathbb{R}^2} n(x, t) \psi(x - \mathbf{a}) \, dx \left(e^{-\sqrt{\gamma}R} \int_{\mathbb{R}^2} n(x, t) \psi(x - \mathbf{a}) \, dx - 8\pi \right)
\end{aligned}$$

$$\begin{aligned}
& + \frac{4\ell_0}{L} \cdot \frac{1}{\pi} \left(\frac{\sqrt{2}}{e} + 1 \right) \|n_0\|_{L^1}^2 + C_* \int_{D_{\ell_0}} n(x, t) \, dx + 2|Q|\tau_{2\ell_0, h}(\mathbf{a}, \mathbf{b}) \|n_0\|_{L^1} \\
& + \frac{2e^{-\sqrt{\gamma}R}}{\pi R^2} \|n_0\|_{L^1} \int_{\mathbb{R}^2} n(x, t) |x - \mathbf{a}|^2 \psi(x - \mathbf{a}) \, dx,
\end{aligned} \tag{6.22}$$

where C_* is defined as follows:

$$\begin{aligned}
C_* &:= 16(\sqrt{2} + 6) + 4\sqrt{2}|Q|\tau_{2\ell_0, h}(\mathbf{a}, \mathbf{b}) \\
&+ \frac{1}{\pi} \left(\frac{\sqrt{2}}{e} + 1 \right) \|n_0\|_{L^1} \left[8\sqrt{2} \left(2 + \frac{\ell_0}{L} \right) + 12 \left(2 + \frac{L}{\ell_0} \right)^2 + 4\sqrt{2} \left(2 + \frac{L}{\ell_0} \right) + 3 \right].
\end{aligned}$$

We assume that:

$$T_{\max} \geq \frac{32\pi \int_{\mathbb{R}^2} \phi(x) |x|^2 \, dx}{\alpha(\|n_0\|_{L^1} + 16\pi)(\sqrt{2\pi}(\|n_0\|_{L^1} + 16\pi) - 8\pi)} := T^*. \tag{6.23}$$

Applying Lemma 3.7 with $\rho_1 = \ell_0$ and $\rho_2 = 2\ell_0$, we obtain:

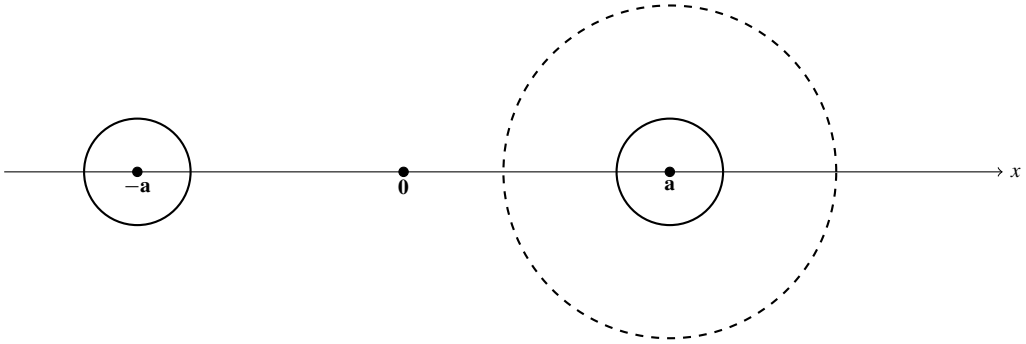
$$\begin{aligned}
& \int_{\mathbb{R}^2} n(x, t) \psi(x - \mathbf{a}) \, dx \geq \int_{\mathbb{R}^2} n_0(x) \psi(x - \mathbf{a}) \, dx \\
& + \begin{cases} -2\|n_0\|_{L^1} \frac{6(2 + \frac{1}{\pi}(\frac{\sqrt{2}}{e} + 1)\|n_0\|_{L^1})}{\ell_0^2} T^* & \text{if } \mathbf{a} = \mathbf{b}, \\ -2\|n_0\|_{L^1} \left(\frac{6(2 + \frac{1}{\pi}(\frac{\sqrt{2}}{e} + 1)\|n_0\|_{L^1})}{\ell_0^2} + \frac{\sqrt{2}|Q|\max\{1, \frac{1}{\lambda}\}}{\ell_0} \right) T^* & \text{if } |\mathbf{a} - \mathbf{b}| \geq 2\ell_0 + h \end{cases}
\end{aligned} \tag{6.24}$$

for a.e. $t \in (0, T^*)$.

We consider the first term on the right-hand side of (6.24). From (2.26) and (6.1), it holds that $\delta < 2\ell_0 < G(\varepsilon_0, T^*) \leq |\mathbf{a}|$, which implies $\text{supp } \phi(x + \mathbf{a}) \cap \text{supp } \psi(x - \mathbf{a}) = \emptyset$. Thus, we obtain:

$$\begin{aligned}
\int_{\mathbb{R}^2} n_0(x) \psi(x - \mathbf{a}) \, dx &= \int_{\mathbb{R}^2} (\phi(x - \mathbf{a}) + \phi(x + \mathbf{a})) \psi(x - \mathbf{a}) \, dx \\
&= \int_{\mathbb{R}^2} \phi(x - \mathbf{a}) \psi(x - \mathbf{a}) \, dx = \int_{B_\delta(\mathbf{0})} \phi(x) \, dx = 8\pi + \frac{\varepsilon_0}{2}.
\end{aligned} \tag{6.25}$$

Here, the supports of $n_0(x) = \phi(x - \mathbf{a}) + \phi(x + \mathbf{a})$ and $\psi(x - \mathbf{a})$ are depicted as follows:



Next, we choose $\ell_0 = \ell_0(\varepsilon_0, T^*)$ sufficiently large to satisfy the following conditions:

$$\begin{cases} -2\|n_0\|_{L^1} \frac{6(2 + \frac{1}{\pi}(\frac{\sqrt{2}}{e} + 1)\|n_0\|_{L^1})}{\ell_0^2} T^* \geq -\frac{\varepsilon_0}{8} & \text{if } \mathbf{a} = \mathbf{b}, \\ -2\|n_0\|_{L^1} \left(\frac{6(2 + \frac{1}{\pi}(\frac{\sqrt{2}}{e} + 1)\|n_0\|_{L^1})}{\ell_0^2} + \frac{\sqrt{2}|Q|\max\{1, \frac{1}{\lambda}\}}{\ell_0} \right) T^* \geq -\frac{\varepsilon_0}{4} \\ \text{if } |\mathbf{a} - \mathbf{b}| \geq 2\ell_0 + h. \end{cases} \quad (6.26)$$

Thus, combining (6.24) with (6.25) and (6.26), we deduce:

$$\int_{\mathbb{R}^2} n(x, t) \psi(x - \mathbf{a}) \, dx \geq 8\pi + \frac{\varepsilon_0}{2} - \frac{\varepsilon_0}{4} = 8\pi + \frac{\varepsilon_0}{4} \quad (6.27)$$

for a.e. $t \in (0, T^*)$. In addition, choosing R as follows:

$$R := \frac{1}{2\sqrt{\gamma}} \log \frac{\|n_0\|_{L^1} + 16\pi}{32\pi} > 0, \quad (6.28)$$

we obtain:

$$e^{-\sqrt{\gamma}R} = \sqrt{\frac{32\pi}{\|n_0\|_{L^1} + 16\pi}}, \quad \frac{2e^{-\sqrt{\gamma}R}}{\pi R^2} = \frac{8\gamma}{\pi} \sqrt{\frac{32\pi}{\|n_0\|_{L^1} + 16\pi}} \left(\log \frac{\|n_0\|_{L^1} + 16\pi}{32\pi} \right)^{-2}. \quad (6.29)$$

Then, from (6.27), we find:

$$e^{-\sqrt{\gamma}R} \int_{\mathbb{R}^2} n(x, t) \psi(x - \mathbf{a}) \, dx - 8\pi \geq \sqrt{2\pi(\|n_0\|_{L^1} + 16\pi)} - 8\pi > 0,$$

which leads to:

$$\begin{aligned}
& -\frac{1}{2\pi} \int_{\mathbb{R}^2} n(x, t) \psi(x - \mathbf{a}) \, dx \left(e^{-\sqrt{\gamma} R} \int_{\mathbb{R}^2} n(x, t) \psi(x - \mathbf{a}) \, dx - 8\pi \right) \\
& \leq -\frac{1}{2\pi} \left(8\pi + \frac{\varepsilon_0}{4} \right) (\sqrt{2\pi}(\|n_0\|_{L^1} + 16\pi) - 8\pi) \\
& = -\frac{1}{8\pi} (\|n_0\|_{L^1} + 16\pi) (\sqrt{2\pi}(\|n_0\|_{L^1} + 16\pi) - 8\pi) =: -\varepsilon_1 < 0.
\end{aligned} \tag{6.30}$$

Substituting (6.28)-(6.30) into (6.22), we find:

$$\begin{aligned}
& \frac{d}{dt} \int_{\mathbb{R}^2} n(x, t) |x - \mathbf{a}|^2 \psi(x - \mathbf{a}) \, dx \\
& \leq -\varepsilon_1 + \frac{4\ell_0}{L} \cdot \frac{1}{\pi} \left(\frac{\sqrt{2}}{e} + 1 \right) \|n_0\|_{L^1}^2 + C_* \int_{D_{\ell_0}} n(x, t) \, dx \\
& \quad + 2|Q|\tau_{2\ell_0, h}(\mathbf{a}, \mathbf{b}) \|n_0\|_{L^1} + \tilde{C} \int_{\mathbb{R}^2} n(x, t) |x - \mathbf{a}|^2 \psi(x - \mathbf{a}) \, dx,
\end{aligned} \tag{6.31}$$

where $\tilde{C}$ is defined by:

$$\tilde{C} := \frac{8\gamma}{\pi} \sqrt{\frac{32\pi}{\|n_0\|_{L^1} + 16\pi}} \left(\log \frac{\|n_0\|_{L^1} + 16\pi}{32\pi} \right)^{-2} \|n_0\|_{L^1}. \tag{6.32}$$

We determine L in (6.6) based on ε_1 in (6.30) as follows:

$$L := \frac{16\ell_0 \cdot \frac{1}{\pi} \left(\frac{\sqrt{2}}{e} + 1 \right)}{\varepsilon_1} \|n_0\|_{L^1}^2. \tag{6.33}$$

From (2.26), we confirm:

$$|\mathbf{a}| \geq G(\varepsilon_0, T^*) = 2\ell_0 + L.$$

Using the identity:

$$\frac{L}{\ell_0} = \frac{128 \left(\frac{\sqrt{2}}{e} + 1 \right) \|n_0\|_{L^1}^2}{(\|n_0\|_{L^1} + 16\pi) (\sqrt{2\pi}(\|n_0\|_{L^1} + 16\pi) - 8\pi)},$$

we observe:

$$\begin{aligned}
C_* & := 16(\sqrt{2} + 6) + 4\sqrt{2}|Q|\tau_{2\ell_0, h}(\mathbf{a}, \mathbf{b}) \\
& \quad + \frac{1}{\pi} \left(\frac{\sqrt{2}}{e} + 1 \right) \|n_0\|_{L^1} \left[8\sqrt{2} \left(2 + \frac{(\|n_0\|_{L^1} + 16\pi) (\sqrt{2\pi}(\|n_0\|_{L^1} + 16\pi) - 8\pi)}{128 \left(\frac{\sqrt{2}}{e} + 1 \right) \|n_0\|_{L^1}} \right) \right]
\end{aligned}$$

$$\begin{aligned}
& + 48 \left(1 + \frac{64 \left(\frac{\sqrt{2}}{e} + 1 \right) \|n_0\|_{L^1}^2}{(\|n_0\|_{L^1} + 16\pi)(\sqrt{2\pi}(\|n_0\|_{L^1} + 16\pi) - 8\pi)} \right)^2 \\
& + 8\sqrt{2} \left(1 + \frac{64 \left(\frac{\sqrt{2}}{e} + 1 \right) \|n_0\|_{L^1}^2}{(\|n_0\|_{L^1} + 16\pi)(\sqrt{2\pi}(\|n_0\|_{L^1} + 16\pi) - 8\pi)} \right) + 3 \Big]. \quad (6.34)
\end{aligned}$$

Combining (6.33) with (6.31), we obtain:

$$\begin{aligned}
& \frac{d}{dt} \int_{\mathbb{R}^2} n(x, t) |x - \mathbf{a}|^2 \psi(x - \mathbf{a}) \, dx \\
& \leq -\varepsilon_1 + \frac{\varepsilon_1}{4} + C_* \int_{D_{\ell_0}} n(x, t) \, dx + 2|Q| \tau_{2\ell_0, h}(\mathbf{a}, \mathbf{b}) \|n_0\|_{L^1} \\
& \quad + \tilde{C} \int_{\mathbb{R}^2} n(x, t) |x - \mathbf{a}|^2 \psi(x - \mathbf{a}) \, dx. \quad (6.35)
\end{aligned}$$

On the other hand, to estimate the integral term on D_{ℓ_0} defined in (6.8), let $\rho_3, \rho_4, \rho_5, \rho_6$ be positive numbers such that $\rho_3 < \rho_4 < \rho_5 < \rho_6$. Using these, we define ζ as follows:

$$\tilde{\zeta}(r) = \begin{cases} 1 & \text{for } 0 \leq r < \rho_3, \\ \frac{2}{(\rho_4 - \rho_3)^2} (r - \rho_3)^2 & \text{for } \rho_3 \leq r < \frac{\rho_3 + \rho_4}{2}, \\ 1 - \frac{2}{(\rho_4 - \rho_3)^2} (r - \rho_4)^2 & \text{for } \frac{\rho_3 + \rho_4}{2} \leq r < \rho_4, \\ 0 & \text{for } \rho_4 \leq r < \rho_5, \\ \frac{2}{(\rho_6 - \rho_5)^2} (r - \rho_5)^2 & \text{for } \rho_5 \leq r < \frac{\rho_5 + \rho_6}{2}, \\ 1 - \frac{2}{(\rho_6 - \rho_5)^2} (r - \rho_6)^2 & \text{for } \frac{\rho_5 + \rho_6}{2} \leq r < \rho_6, \\ 1 & \text{for } \rho_6 \leq r \end{cases}$$

and set $\zeta(x)$ as $\zeta(x) := \tilde{\zeta}(|x|)$ for $x \in \mathbb{R}^2$. We choose $\rho_3 = L + \frac{3}{2}\ell_0$, $\rho_4 = L + 2\ell_0 - \delta$, $\rho_5 = L + 2\ell_0 + \delta$, and $\rho_6 = L + \frac{5}{2}\ell_0$. Then, since $\ell_0 > 2\delta$, we verify that $\rho_3 < \rho_4 < \rho_5 < \rho_6$.

Similar to Lemmas 3.6 and 3.7, we find the following:

$$\begin{aligned}
& \int_{D_{\ell_0}} n(x, t) \, dx \leq \int_{\mathbb{R}^2} n(x, t) \zeta(x - \mathbf{a}) \, dx \\
& \leq \begin{cases} 4\|n_0\|_{L^1} \frac{12(2 + \frac{1}{\pi}(\frac{\sqrt{2}}{e} + 1)\|n_0\|_{L^1})}{(\ell_0 - 2\delta)^2} T^*, & \text{if } \mathbf{a} = \mathbf{b}, \\ 4\|n_0\|_{L^1} \left(\frac{12(2 + \frac{1}{\pi}(\frac{\sqrt{2}}{e} + 1)\|n_0\|_{L^1})}{(\ell_0 - 2\delta)^2} + \frac{\sqrt{2}|Q| \max\{1, \frac{1}{\lambda}\}}{\ell_0 - 2\delta} \right) T^* & \text{if } |\mathbf{a} - \mathbf{b}| \geq 2\ell_0 + h, \end{cases} \quad (6.36)
\end{aligned}$$

for a.e. $t \in (0, T^*)$. Furthermore, we choose $\ell_0 = \ell_0(\varepsilon_0, T^*)$ sufficiently large to satisfy the following conditions:

$$\frac{48C_* \|n_0\|_{L^1} \left(2 + \frac{1}{\pi} \left(\frac{\sqrt{2}}{e} + 1\right) \|n_0\|_{L^1}\right)}{(\ell_0 - 2\delta)^2} T^* \leq \frac{\varepsilon_1}{8}, \quad \frac{4\sqrt{2}C_* \|n_0\|_{L^1} |Q| \max\left\{1, \frac{1}{\lambda}\right\}}{\ell_0 - 2\delta} T^* \leq \frac{\varepsilon_1}{8}. \quad (6.37)$$

Using (6.36) and (6.37), this leads to:

$$C_* \int_{D_{\ell_0}} n(x, t) \, dx \leq \frac{\varepsilon_1}{4}. \quad (6.38)$$

In addition, applying (2.23), we obtain:

$$2|Q| \tau_{2\ell_0, h}(\mathbf{a}, \mathbf{b}) \|n_0\|_{L^1} \leq \frac{\varepsilon_1}{4}. \quad (6.39)$$

Combining (6.35), (6.38) and (6.39), and choosing $\ell_0 = \ell_0(\varepsilon_0, T^*)$ sufficiently large to satisfy (6.1), (6.26) and (6.37), we deduce the following:

$$\frac{d}{dt} \int_{\mathbb{R}^2} n(x, t) |x - \mathbf{a}|^2 \psi(x - \mathbf{a}) \, dx \leq -\frac{\varepsilon_1}{4} + \tilde{C} \int_{\mathbb{R}^2} n(x, t) |x - \mathbf{a}|^2 \psi(x - \mathbf{a}) \, dx,$$

which implies:

$$\begin{aligned} & \int_{\mathbb{R}^2} n(x, t) |x - \mathbf{a}|^2 \psi(x - \mathbf{a}) \, dx \\ & \leq \int_{\mathbb{R}^2} n_0(x) |x - \mathbf{a}|^2 \psi(x - \mathbf{a}) \, dx + \int_0^t \left(-\frac{\varepsilon_1}{4} + \tilde{C} \int_{\mathbb{R}^2} n(x, s) |x - \mathbf{a}|^2 \psi(x - \mathbf{a}) \, dx \right) ds \end{aligned} \quad (6.40)$$

for a.e. $t \in (0, T^*)$.

We define $M(t)$ as:

$$M(t) := \int_{\mathbb{R}^2} n(x, t) |x - \mathbf{a}|^2 \psi(x - \mathbf{a}) \, dx. \quad (6.41)$$

We introduce the linear function $F(M)$ for $M \geq 0$ as:

$$F(M) := -\frac{\varepsilon_1}{4} + \tilde{C}M,$$

where ε_1 and $\tilde{C}$ are given by (6.30) and (6.32), respectively. Then, from (6.40), we have:

$$M(t) \leq M(0) + \int_0^t F(M(s)) \, ds \quad (6.42)$$

for $t \in (0, T^*)$. To prove (2.25), we proceed by contradiction. For this purpose, we first observe that $F(M_*) = 0$ for M_* defined as:

$$M_* := \frac{1}{\gamma} \cdot \frac{(\|n_0\|_{L^1} + 16\pi)^{\frac{3}{2}} (\sqrt{2\pi(\|n_0\|_{L^1} + 16\pi)} - 8\pi)}{1024\sqrt{2\pi}\|n_0\|_{L^1}} \left(\log \frac{\|n_0\|_{L^1} + 16\pi}{32\pi} \right)^2.$$

Next, we note that $|\mathbf{a}| \geq G(\varepsilon_0, T^*) = 2\ell_0 + L > 2\ell_0 > \delta$ by (6.1) and (6.33), which implies that $\text{supp } \phi(x + \mathbf{a}) \cap \text{supp } \psi(x - \mathbf{a}) = \emptyset$. Thus, for $0 < \alpha < 1$, it follows from (2.20) and (2.21):

$$\begin{aligned} M(0) &= \int_{\mathbb{R}^2} n_0(x) |x - \mathbf{a}|^2 \psi(x - \mathbf{a}) \, dx = \int_{B_\delta(\mathbf{a})} \phi(x - \mathbf{a}) |x - \mathbf{a}|^2 \psi(x - \mathbf{a}) \, dx \\ &= \int_{B_\delta(\mathbf{a})} \phi(x - \mathbf{a}) |x - \mathbf{a}|^2 \, dx = \int_{\mathbb{R}^2} \phi(x) |x|^2 \, dx \leq (1 - \alpha) M_*. \end{aligned}$$

Since F is strictly increasing on $[0, \infty)$, we observe:

$$F(M(0)) \leq F((1 - \alpha)M_*) < F(M_*) = 0. \quad (6.43)$$

We define H as:

$$H(t) := M(0) + \int_0^t F(M(\tau)) \, d\tau \quad \text{for } t \in (0, T^*). \quad (6.44)$$

Using (6.42), we obtain:

$$M(t) \leq H(t) \quad \text{for } t \in (0, T^*). \quad (6.45)$$

For the moment, we assume the following estimate for H , which will be proved later:

$$H(t) \leq H(0) \quad \text{for } t \in (0, T^*). \quad (6.46)$$

Then, from (6.43)-(6.46) and monotonicity of F , we have:

$$H'(t) = F(M(t)) \leq F(H(t)) \leq F(H(0)) = F(M(0)) \leq F((1 - \alpha)M_*)$$

for $t \in (0, T^*)$. Consequently, from (6.45), we deduce:

$$M(t) \leq H(t) \leq H(0) + F((1 - \alpha)M_*) \cdot t = M(0) + F((1 - \alpha)M_*) \cdot t$$

for $t \in (0, T^*)$. Noting that $F((1 - \alpha)M_*) < 0$, we conclude:

$$M(t) \leq 0 \quad \text{for } t \geq -\frac{M(0)}{F((1-\alpha)M_*)}, \quad (6.47)$$

where:

$$-\frac{M(0)}{F((1-\alpha)M_*)} = \frac{32\pi \int_{\mathbb{R}^2} \phi(x) |x|^2 dx}{\alpha(\|n_0\|_{L^1} + 16\pi)(\sqrt{2\pi(\|n_0\|_{L^1} + 16\pi)} - 8\pi)}. \quad (6.48)$$

Since the strong solution n is non-negative, one must have $M(t) \geq 0$ on $(0, T_{\max})$. Therefore, the above (6.47) contradicts (6.23), implying:

$$T_{\max} < T^* := \frac{32\pi \int_{\mathbb{R}^2} \phi(x) |x|^2 dx}{\alpha(\|n_0\|_{L^1} + 16\pi)(\sqrt{2\pi(\|n_0\|_{L^1} + 16\pi)} - 8\pi)},$$

which is precisely the bound asserted in Theorem 2.4 (I).

It now remains to establish (6.46). To this end, we proceed by contradiction and assume that there exists $T_0 \in (0, T_*)$ such that:

$$H(T_0) > H(0).$$

By (6.43), we observe:

$$H'(0) = F(M(0)) < 0,$$

implying:

$$H(\tau) \leq H(0) \quad \text{for } \tau \in (0, T') \text{ with some } 0 < T' \leq T^*.$$

We may assume:

$$T' = \sup\{t > 0 \mid H(\tau) \leq H(0) \text{ for } \tau \in (0, t)\}.$$

Since H is a continuous function on $[0, T^*)$, we find that $H(T') = H(0)$. By (6.43)-(6.45) and monotonicity of F , it follows:

$$H'(T') = F(M(T')) \leq F(H(T')) = F(H(0)) = F(M(0)) < 0,$$

which implies:

$$H(\tau) \leq H(0) \quad \text{for } \tau \in (0, T'')$$

for some $T'' > T'$. This contradicts the definition of T' , and thus we conclude (6.46). This completes the proof of Theorem 2.4 (I). $\square$

6.2. Proof of Theorem 2.4 (II): blow-up configuration

First, we show that the strong solution (n, v) of (KSF) on $[0, T_{\max})$ is axis-symmetric with respect to both the e_1 - and e_2 -axes. We define n^+ and n^- as follows:

$$\begin{aligned} n^+(x, t) &:= n(x_1, x_2, t), & n^-(x, t) &:= n(-x_1, x_2, t), \\ v^+(x, t) &:= v(x_1, x_2, t), & v^-(x, t) &:= v(-x_1, x_2, t), \\ u^+(x, t) &:= u(x_1, x_2, t), & u^-(x, t) &:= u(-x_1, x_2, t). \end{aligned}$$

Since the pairs of functions (n^+, v^+) and (n^-, v^-) are strong solutions of (KSF) on $[0, T_{\max})$ with fluid vector fields u^+ and u^- , respectively, we obtain:

$$\begin{aligned} \partial_t(n^+ - n^-) &= \Delta(n^+ - n^-) - \nabla \cdot \left((n^+ - n^-) \nabla v^+ + n^- \nabla(v^+ - v^-) \right) \\ &\quad - \nabla \cdot \left(u^+(n^+ - n^-) + (u^+ - u^-)n^- \right), \end{aligned} \quad (6.49)$$

where v^+ and v^- satisfy $\Delta v^+ - \gamma v^+ + n^+ = 0$ and $\Delta v^- - \gamma v^- + n^- = 0$, respectively. Let $1 < r < \infty$. Multiplying both sides of (6.49) by $|n^+ - n^-|^{r-2}(n^+ - n^-)$ and integrating over $\mathbb{R}^2$, we obtain:

$$\begin{aligned} &\frac{1}{r} \frac{d}{dt} \| (n^+ - n^-)(t) \|_{L^r}^r \\ &\leq (r-1) \| n^- \|_{L^\infty}^2 \left(\| \nabla(v^+ - v^-)(t) \|_{L^r}^2 + \| u^+ - u^- \|_{L^r}^2 \right) \| (n^+ - n^-)(t) \|_{L^r}^{r-2} \\ &\quad + (r-1) \left(\| \nabla v^+(t) \|_{L^\infty}^2 + \| u^+ \|_{L^\infty}^2 \right) \| (n^+ - n^-)(t) \|_{L^r}^r. \end{aligned}$$

Since u is axis-symmetric with respect to the e_2 -axis, it follows that $u^+(x) - u^-(x) = 0$ for a.e. $x \in \mathbb{R}^2$. Thus, we deduce:

$$\frac{1}{r} \frac{d}{dt} \| (n^+ - n^-)(t) \|_{L^r}^r \leq C(r-1) \left(\| n^- \|_{L^\infty}^2 + \| \nabla v^+(t) \|_{L^\infty}^2 + \| u^+ \|_{L^\infty}^2 \right) \| (n^+ - n^-)(t) \|_{L^r}^r, \quad (6.50)$$

where $C = C(\gamma)$. Let $\widehat{T}$ be an arbitrary number satisfying $\widehat{T} \in (0, T_{\max})$. By Theorem 2.1 (II), there exists a positive constant $C = C(\gamma, Q, \lambda, \widehat{T}, \|n_0\|_{W^{2,4}(\mathbb{R}^2)})$ such that:

$$\sup_{t \in (0, \widehat{T})} \| n(t) \|_{W^{2,4}(\mathbb{R}^2)} \leq C.$$

Using the embedding theorem, (3.17) and (2.6), we obtain the following:

$$\begin{aligned} \sup_{t \in (0, \widehat{T})} \| n(t) \|_{L^\infty} &\leq C, & \sup_{t \in (0, \widehat{T})} \| \nabla v(t) \|_{L^\infty} &\leq C & \sup_{t \in (0, \widehat{T})} \| n(t) \|_{L^\infty} &\leq C, \\ \| u \|_{L^\infty} &\leq 2|Q| \max \left\{ 1, \frac{1}{\lambda} \right\}, \end{aligned}$$

where $C = C(\gamma, \widehat{T}, \|n_0\|_{W^{2,4}})$. Thus, choosing $r = 4$ in (6.50), we have:

$$\frac{d}{dt} \|(n^+ - n^-)(t)\|_{L^4} \leq C \|(n^+ - n^-)(t)\|_{L^4}$$

for a.e. $t \in (0, \widehat{T})$, where $C = C(\gamma, Q, \lambda, \widehat{T}, \|n_0\|_{W^{2,4}})$.

Since n_0 is also axis-symmetric with respect to the e_2 -axis, we find $n_0^+(x) - n_0^-(x) = 0$ for a.e. $x \in \mathbb{R}^2$. From the Gronwall inequality, it follows:

$$0 \leq \|(n^+ - n^-)(t)\|_{L^4} \leq \|n_0^+ - n_0^-\|_{L^4} \cdot \exp\{C\widehat{T}\} = 0$$

for a.e. $t \in (0, \widehat{T})$. Therefore, $n^+(x, t) - n^-(x, t) = 0$ holds for a.e. $(x, t) \in \mathbb{R}^2 \times (0, \widehat{T})$, and the strong solution of (KSF) on the two-dimensional whole space $\mathbb{R}^2$, as obtained in Theorem 2.1, is axis-symmetric with respect to the e_2 -axis for a.e. $t \in (0, \widehat{T})$.

Since $\widehat{T}$ is an arbitrary positive number, the strong solution $n(t)$ is axis-symmetric with respect to the e_2 -axis for a.e. $t \in (0, T_{\max})$. Furthermore, since both the initial data n_0 and the fluid vector fields are also symmetric with respect to the e_1 -axis, a similar argument shows that the strong solution $n(t)$ is axis-symmetric with respect to both the e_1 - and e_2 -axes for a.e. $t \in (0, T_{\max})$.

Here, let m_* be the constant satisfying (2.15), and choose $G(\varepsilon_0, T^*)$ such that $G(\varepsilon_0, T^*) > 2\ell_0 \geq \delta + \ell_0$, where $\ell_0 = \ell_0(\varepsilon_0, T^*)$ satisfies:

$$\ell_0 > \max \left\{ 4\sqrt{\frac{6\|n_0\|_{L^1} \left(2 + \frac{1}{\pi} \left(\frac{\sqrt{2}}{e} + 1\right) \|n_0\|_{L^1}\right) T^*}{m_*}}, \frac{8\sqrt{2}\|n_0\|_{L^1} |Q| \max\left\{1, \frac{1}{\lambda}\right\} T^*}{m_*} \right\}. \quad (6.51)$$

Applying Lemmas 3.6 and 3.7 with $\rho_1 = \frac{\ell_0}{2}$ and $\rho_2 = \ell_0$, and noting that $\text{supp } n_0 \cap B_{\ell_0}(\mathbf{0}) = \emptyset$, we obtain:

$$\begin{aligned} \int_{B_{\frac{\ell_0}{2}}(\mathbf{0})} n(x, t) \, dx &\leq \int_{\mathbb{R}^2} n(x, t) \psi(x) \, dx \\ &\leq \begin{cases} 4\|n_0\|_{L^1} \frac{12\left(2 + \frac{1}{\pi} \left(\frac{\sqrt{2}}{e} + 1\right) \|n_0\|_{L^1}\right)}{\ell_0^2} T_*, & \text{if } \mathbf{a} = \mathbf{b}, \\ 4\|n_0\|_{L^1} \left(\frac{12\left(2 + \frac{1}{\pi} \left(\frac{\sqrt{2}}{e} + 1\right) \|n_0\|_{L^1}\right)}{\ell_0^2} + \frac{\sqrt{2}|Q| \max\left\{1, \frac{1}{\lambda}\right\}}{\ell_0} \right) T^* & \text{if } |\mathbf{a} - \mathbf{b}| \geq 2\ell_0 + h, \end{cases} \end{aligned} \quad (6.52)$$

for a.e. $t \in (0, T_{\max})$. Here, we note that $T_{\max} < T^*$, as shown in Theorem 2.4 (I).

Furthermore, by (6.51), we have:

$$4\|n_0\|_{L^1} \left(\frac{12\left(2 + \frac{1}{\pi} \left(\frac{\sqrt{2}}{e} + 1\right) \|n_0\|_{L^1}\right)}{\ell_0^2} + \frac{\sqrt{2}|Q| \max\left\{1, \frac{1}{\lambda}\right\}}{\ell_0} \right) T^* < \frac{m_*}{2} + \frac{m_*}{2} = m_*. \quad (6.53)$$

Thus, combining (6.52) and (6.53), we conclude:

$$\int_{B_{\frac{\ell_0}{2}}(\mathbf{0})} n(x, t) \, dx < m_* \quad (6.54)$$

for a.e. $t \in (0, T_{\max})$. Applying Theorem 2.2 together with (6.54), often referred to as the ε -regularity theorem, we conclude that the origin cannot be a blow-up point. Since $n(t)$ remains axis-symmetric for a.e. $t \in (0, T_{\max})$, the blow-up points of $n(t)$ at the blow-up time T_b ($=T_{\max}$) consist of points $x_1, \dots, x_m, \dots, x_{2m}$, where $2m$ is even and each pair x_j and x_{m+j} (for $j = 1, \dots, m$) is symmetrically located with respect to e_2 -axis. Since the strong solution $n(t)$ is symmetric with respect to both the e_1 - and e_2 -axes up to $T_{\max}$, this implies that all blow-up points lie on one of these axes. This completes the proof of Theorem 2.4 (II). $\square$

7. Proof of Theorem 2.5 and 2.6

7.1. Proof of Theorem 2.5: finiteness of the blow-up points

First, let us take an integer $k \in \mathbb{N}$ such that:

$$km_* > \|n_0\|_{L^1(\mathbb{R}^2)}, \quad (7.1)$$

where m_* is the constant obtained in Theorem 2.2.

From (2.13) in Theorem 2.1 (II), it follows that $n \in C([0, T_{\max}); C(\Omega))$ for every bounded subset $\Omega \subset \mathbb{R}^2$. Here, $T_{\max}$ is the maximal existence time of the strong solution (n, v) constructed in Theorem 2.1 (I) and (II). Moreover, by applying (2.12) in Theorem 2.1 (II), we deduce $T_{\max} = T_b$, which implies $n \in C([0, T_b); C(\Omega))$, where T_b represents the blow-up time of n as defined in (2.8) of Definition 2.3.

We suppose there exist infinitely many blow-up points $\{x_i\}_{i=1}^\infty$. Then we can choose $\ell > 0$ and k blow-up points $\{x_1, x_2, \dots, x_k\} \subset \{x_i\}_{i=1}^\infty$ such that:

$$B_{2\ell}(x_i) \cap B_{2\ell}(x_j) = \emptyset \quad \text{for all } i, j = 1, 2, \dots, k \text{ with } i \neq j. \quad (7.2)$$

For these ℓ and $\{x_i\}_{i=1}^k$, it follows from (2.19) in Theorem 2.3:

$$\sup_{t \in (0, T_b)} \int_{B_\ell(x_i)} n(x, t) \, dx \geq m_* \quad \text{for all } i = 1, 2, \dots, k. \quad (7.3)$$

Since Theorem 2.1 (III) ensures that $\sup_{t \in (0, T_b)} \|n(t)\|_{L^1} = \|n_0\|_{L^1}$, combining the results from (7.1) through (7.3), we obtain:

$$\begin{aligned} km_* &< \sum_{i=1}^k \sup_{t \in (0, T_b)} \int_{B_\ell(x_i)} n(x, t) \, dx = \sup_{t \in (0, T_b)} \sum_{i=1}^k \int_{B_\ell(x_i)} n(x, t) \, dx \\ &\leq \sup_{t \in (0, T_b)} \|n(t)\|_{L^1} = \|n_0\|_{L^1} < km_*, \end{aligned}$$

which is a contradiction. $\square$

7.2. Proof of Theorem 2.6: δ -function singularity

The following lemma ensures that evaluating the solution against any smooth, compactly supported test function yields a time-continuous mapping on the entire interval up to and including the blow-up time T_b .

Lemma 7.1. *Let all assumptions in Theorem 2.6 hold. Then, for every $\psi \in C_c^\infty(\mathbb{R}^2)$, the mapping:*

$$t \mapsto \int_{\mathbb{R}^2} n(x, t) \psi(x) \, dx$$

is continuous on $[0, T_b]$.

Proof of Lemma 7.1. Using the first equation of (KSF), we have:

$$\int_{\mathbb{R}^2} n(x, t) \psi(x) \, dx = f_1(t) + f_2(t) + f_3(t) + \int_{\mathbb{R}^2} n_0(x) \psi(x) \, dx \quad (7.4)$$

for all $t \in (0, T_b)$ and all $\psi \in C_c^\infty(\mathbb{R}^2)$, where:

$$f_1(t) := \int_0^t \int_{\mathbb{R}^2} n \Delta \psi \, dx ds, \quad f_2(t) := \int_0^t \int_{\mathbb{R}^2} (n \nabla v \cdot \nabla \psi) \, dx ds, \quad f_3(t) := \int_0^t \int_{\mathbb{R}^2} (un \cdot \nabla \psi) \, dx ds. \quad (7.5)$$

As for f_1 and f_3 , we obtain from (2.6):

$$|f_1(t) - f_1(s)| \leq \|\Delta \psi\|_{L^\infty} \int_s^t \|n(\tau)\|_{L^1(\text{supp } \psi)} \, d\tau$$

and:

$$|f_3(t) - f_3(s)| \leq 2|Q| \max \left\{ 1, \frac{1}{\lambda} \right\} \|\nabla \psi\|_{L^\infty} \int_s^t \|n(\tau)\|_{L^1(\text{supp } \psi)} \, d\tau$$

for all $0 \leq s < t \leq T_b$. Since $n \in L^1(0, T_b; B)$ for all balls B , we deduce:

$$f_1, f_3 \in C([0, T_b]). \quad (7.6)$$

Next, we establish that $f_2 \in C([0, T_b])$. Choose a ball B such that $\text{supp } \psi \subset B$. Let us choose $p = \frac{2p'_1}{p'_1+2}$. Since $1 < p_1 < 2$, we have $1 < p < \infty$ with:

$$\frac{1}{p'} = \frac{1}{p_1} - \frac{1}{2}.$$

Hence, by Lemma 3.5, it holds:

$$\|\nabla v\|_{L^{p'}(B)} \leq C \|n\|_{L^{p_1}(\mathbb{R}^2)}, \quad (7.7)$$

where $C = C(\gamma, p)$.

According to Theorem 2.1, it holds by $d = 2$:

$$n \in W_4^{2,1}(\mathcal{Q}_{T_{\max}}) \cap L^\infty(0, T_{\max}; L^1(\mathbb{R}^2)).$$

This implies:

$$n \in L^1(0, T_{\max}; L^{\frac{6}{5}}(B)) \cap L^\infty(0, T_{\max}; L^{\frac{3}{2}}(\mathbb{R}^2)).$$

Applying (7.7) with $p = \frac{6}{5}$ and $p_1 = \frac{3}{2}$, we have $p' = 6$ and $p'_1 = 3$, which yields:

$$\begin{aligned} |f_2(t) - f_2(s)| &\leq \|\nabla \psi\|_{L^\infty(B)} \int_s^t \int_B |n \nabla v| \, dx d\tau \\ &\leq \|\nabla \psi\|_{L^\infty(B)} \int_s^t \|n(\tau)\|_{L^{\frac{6}{5}}(B)} \|\nabla v(\tau)\|_{L^6(B)} \, d\tau \\ &\leq C \|\nabla \psi\|_{L^\infty(B)} \int_s^t \|n(\tau)\|_{L^{\frac{6}{5}}(B)} \|n(\tau)\|_{L^{\frac{3}{2}}(\mathbb{R}^2)} \, d\tau \\ &\leq C \|\nabla \psi\|_{L^\infty(B)} \|n\|_{L^1(s,t; L^{\frac{6}{5}}(B))} \|n\|_{L^\infty(s,t; L^{\frac{3}{2}}(\mathbb{R}^2))}, \end{aligned} \quad (7.8)$$

for all $0 \leq s < t \leq T_b$.

Combining (2.12) in Theorem 2.1 and (7.8) with the absolute continuity of the integral, we conclude:

$$f_2 \in C([0, T_b]). \quad (7.9)$$

Thus, applying (7.4)–(7.6) and (7.9), we obtain the desired continuity. This completes the proof of Lemma 7.1. $\square$

Proof of Theorem 2.6. By Theorem 2.5, we may assume that n blows up at exactly k distinct points $x_1, \dots, x_k$.

For each $1 \leq i \leq k$ and $r > 0$, we define:

$$M_{i,r} := \lim_{t \rightarrow T_b} \int_{B_r(x_i)} n(x, t) \eta_i(x) \, dx, \quad (7.10)$$

where $\eta_i \in C_c^\infty(\mathbb{R}^2)$ is a cut-off function chosen so that:

$$\eta_i(x) = \begin{cases} 1 & \text{if } |x - x_i| \leq \frac{r}{2}, \\ 0 & \text{if } |x - x_i| \geq r. \end{cases} \quad (7.11)$$

It follows from Lemma 7.1 that the limit in (7.10) exists. Moreover, for each $i = 1, 2, \dots, k$, the quantity $M_{i,r}$ is monotonically decreasing in r and bounded below by m_* . Therefore, the limit as $r \rightarrow 0$ exists, and we define M_i as:

$$M_i := \lim_{r \rightarrow 0} M_{i,r} < \infty \quad \text{for all } i = 1, 2, \dots, k. \quad (7.12)$$

In the following lemma, we identify and characterize the *regular part* $f(x)$ of $n(x, t)$: that is, the limit to which $n(x, t)$ converges almost everywhere, excluding potential blow-up points.

Lemma 7.2. *We suppose that the assumptions of Theorem 2.6 hold. Then, there exists a sequence $\{t_m\}_{m=1}^\infty$ with $t_m \rightarrow T_b$ as $m \rightarrow \infty$ such that:*

$$\lim_{m \rightarrow \infty} n(x, t_m) =: f(x) < \infty \quad \text{for a.e. } x \in \mathbb{R}^2. \quad (7.13)$$

Furthermore, f belongs to $L^1(\mathbb{R}^2)$.

Proof of Lemma 7.2. We divide the proof into two steps. First, we define the sets Ω_r and $\Omega_{r'}$ by:

$$\Omega_r := \mathbb{R}^2 \setminus \bigcup_{i=1}^k B_r(x_i), \quad \Omega_{r'} := \mathbb{R}^2 \setminus \bigcup_{i=1}^k B_{\frac{r}{2}}(x_i), \quad (7.14)$$

where $0 < r < \rho$ is chosen sufficiently small so that:

$$B_r(x_i) \cap B_r(x_j) = \emptyset \quad \text{for all } i, j = 1, 2, \dots, k \text{ with } i \neq j.$$

For such r , there exist ℓ points $x_1^*, \dots, x_\ell^* \in \Omega_r$ and positive constants ρ_0 and δ with $0 < \delta < \rho_0 < \rho$ such that:

$$\Omega_r \subset \bigcup_{i=1}^{\ell} B_{\rho_0}(x_i^*) \subset \bigcup_{i=1}^{\ell} B_{\rho_0+\delta}(x_i^*) \subset \Omega_{r'}. \quad (7.15)$$

Step 1. We establish (7.13). By (2.13) in Theorem 2.1 (II), we have:

$$n \in C([0, T_b); C(\Omega))$$

for any bounded subset $\Omega \subset \mathbb{R}^2$. Here, T_b is the blow-up time of n specified by (2.8) (see Definition 2.3). Consequently, there exist a function $f_r \in L^2(\Omega_r)$ and a sequence $\{t_m\}_{m=1}^\infty \subset (0, T_b)$ with $t_m \rightarrow T_b$ as $m \rightarrow \infty$, such that:

$$n(x, t_m) \rightarrow f_r(x) \quad \text{for a.e. } x \in \Omega_r \quad \text{as } m \rightarrow \infty.$$

Let $\{r_j\}_{j=1}^\infty$ be a sequence with $r_1 > r_2 > \dots > r_j \rightarrow 0$ as $j \rightarrow \infty$. For each $x \in \mathbb{R}^2 \setminus \bigcup_{i=1}^k \{x_i\}$, there is an integer ℓ such that $x \in \Omega_{r_\ell}$. We define $f(x)$ on $\mathbb{R}^2 \setminus \bigcup_{i=1}^k \{x_i\}$ by $f(x) := f_{r_\ell}(x)$. It is straightforward to verify that $f(x)$ is well-defined on $\mathbb{R}^2 \setminus \bigcup_{i=1}^k \{x_i\}$, since $f_{r_\ell}(x) = f_{r_p}(x)$ for all $p \geq \ell$.

By a standard diagonal argument, we can extract a subsequence $\{t_m\}_{m=1}^\infty$ with $\lim_{m \rightarrow \infty} t_m = T_b$ such that:

$$n(x, t_m) \rightarrow f(x) \quad \text{for a.e. } x \in \mathbb{R}^2 \setminus \bigcup_{i=1}^k \{x_i\} \quad \text{as } m \rightarrow \infty.$$

This yields (7.13) for a.e. $x \in \mathbb{R}^2 \setminus \bigcup_{i=1}^k \{x_i\}$ as $m \rightarrow \infty$.

Step 2. We now establish:

$$f \in L^1(\mathbb{R}^2). \quad (7.16)$$

By choosing r sufficiently small as in the proof of Lemma 7.2 and applying the Lebesgue dominated convergence theorem, we obtain:

$$\int_{\Omega_r} f(x) dx = \lim_{m \rightarrow \infty} \int_{\Omega_r} n(x, t_m) dx \leq \|n_0\|_{L^1} \quad \text{for all } 0 < r < \rho, \quad (7.17)$$

since $n(x, t_m) \leq \sup_{0 < t < T_b} n(x, t)$ for all $x \in \Omega_r$, and $\sup_{0 < t < T_b} n(\cdot, t)$ is bounded on $\mathbb{R}^2 \setminus \bigcup_{i=1}^k B_r(x_i)$. Letting $r \rightarrow 0$ in (7.17), we deduce:

$$\int_{\mathbb{R}^2} f(x) dx \leq \|n_0\|_{L^1},$$

which verifies (7.16). This completes the proof of Lemma 7.2. $\square$

Continuation of the proof of Theorem 2.6. We now prove:

$$\lim_{m \rightarrow \infty} \int_{\mathbb{R}^2} n(x, t_m) \psi(x) dx = \sum_{i=1}^k M_i \psi(x_i) + \int_{\mathbb{R}^2} f(x) \psi(x) dx,$$

for all $\psi \in C_c^\infty(\mathbb{R}^2)$. Let $\eta_i(x)$, $i = 1, \dots, k$, be the cut-off functions introduced in (7.11). Since $1 - \eta_i(x) = 0$ for $x \in B_{\frac{r}{2}}(x_i)$, a straightforward calculation shows:

$$\int_{\mathbb{R}^2} n(x, t) \psi(x) dx - \sum_{i=1}^k M_i \psi(x_i) - \int_{\mathbb{R}^2} f(x) \psi(x) dx$$

$$\begin{aligned}
&= \int_{\mathbb{R}^2 \setminus \bigcup_{i=1}^k B_r(x_i)} (n(x, t) - f(x)) \psi(x) \, dx - \sum_{i=1}^k \int_{B_r(x_i)} f(x) \psi(x) \, dx \\
&\quad + \sum_{i=1}^k \int_{B_r(x_i)} n(x, t) \eta_i(x) \, dx \cdot \psi(x_i) - \sum_{i=1}^k M_i \psi(x_i) \\
&\quad - \sum_{i=1}^k \int_{B_r(x_i)} n(x, t) \eta_i(x) \, dx \cdot \psi(x_i) + \sum_{i=1}^k \int_{B_r(x_i)} n(x, t) \psi(x) \, dx \\
&= \int_{\mathbb{R}^2 \setminus \bigcup_{i=1}^k B_r(x_i)} (n(x, t) - f(x)) \psi(x) \, dx - \sum_{i=1}^k \int_{B_r(x_i)} f(x) \psi(x) \, dx \\
&\quad + \sum_{i=1}^k \left(\int_{B_r(x_i)} n(x, t) \eta_i(x) \, dx - M_i \right) \psi(x_i) \\
&\quad + \sum_{i=1}^k \int_{B_r(x_i) \setminus B_{\frac{r}{2}}(x_i)} (n(x, t) - f(x)) \psi(x) \cdot (1 - \eta_i(x)) \, dx \\
&\quad + \sum_{i=1}^k \int_{B_r(x_i) \setminus B_{\frac{r}{2}}(x_i)} f(x) \psi(x) \cdot (1 - \eta_i(x)) \, dx \\
&\quad + \sum_{i=1}^k \int_{B_r(x_i)} n(x, t) \eta_i(x) \cdot (\psi(x) - \psi(x_i)) \, dx. \tag{7.18}
\end{aligned}$$

Similarly to (7.17), we use the definition of the function f to obtain:

$$\begin{aligned}
&\left| \int_{\mathbb{R}^2 \setminus \bigcup_{i=1}^k B_r(x_i)} (n(x, t_m) - f(x)) \psi(x) \, dx \right| \rightarrow 0, \\
&\sum_{i=1}^k \left| \int_{B_r(x_i) \setminus B_{\frac{r}{2}}(x_i)} (n(x, t_m) - f(x)) \psi(x) \cdot (1 - \eta_i(x)) \, dx \right| \rightarrow 0 \quad \text{as } m \rightarrow \infty.
\end{aligned}$$

Substituting $t = t_m$ in (7.18) and then letting $m \rightarrow \infty$, we obtain from (7.10):

$$\limsup_{n \rightarrow \infty} \left| \int_{\mathbb{R}^2} n(x, t_m) \psi(x) \, dx - \sum_{i=1}^k M_i \psi(x_i) - \int_{\mathbb{R}^2} f(x) \psi(x) \, dx \right|$$

$$\begin{aligned}
& \leq \sum_{i=1}^k \int_{B_r(x_i)} f(x) \, dx \cdot \max_{x \in \mathbb{R}^2} |\psi(x)| + \sum_{i=1}^k |M_{i,r} - M_i| |\psi(x_i)| \\
& \quad + \sum_{i=1}^k \int_{B_r(x_i)} f(x) \, dx \cdot \max_{x \in \mathbb{R}^2} |\psi(x)| + \sum_{i=1}^k \|n_0\|_{L^1} \cdot \max_{x \in B_r(x_i)} |\psi(x) - \psi(x_i)| \\
& =: F(r).
\end{aligned} \tag{7.19}$$

Since $\psi \in C_c^\infty(\mathbb{R}^2)$, we obtain from (7.12) and (7.16) that $\lim_{r \rightarrow 0} F(r) = 0$. Since the left-hand side of (7.19) is independent of r , we conclude:

$$\left| \lim_{m \rightarrow \infty} \int_{\mathbb{R}^2} n(x, t_m) \psi(x) \, dx - \sum_{i=1}^k M_i \psi(x_i) - \int_{\mathbb{R}^2} f(x) \psi(x) \, dx \right| = 0.$$

This completes the proof of Theorem 2.6. $\square$

Acknowledgments

Y. Sugiyama was supported by JST CREST (Grant No. JPMJCR2013). Y. Seki was partly supported by Grant-in-Aid for Scientific Research (22K03387).

The authors are deeply grateful to Juan José López Velázquez for providing the authors with his valuable insights. They also acknowledge that the discussions during Y. Seki's postdoctoral stay and Y. Sugiyama's visits on many occasions to Madrid and Bonn were highly fruitful and beneficial to this study. They are also grateful for his kind hospitality.

Data availability

No data was used for the research described in the article.

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