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THE EIGENVALUE DISTRIBUTION OF ELLIPTIC OPERATORS WITH HÖLDER CONTINUOUS COEFFICIENTS II

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1. Introduction

This is the continuation of the previous paper [11], in which we attempted the improvement of the remainder estimate for the eigenvalue distribution of the elliptic operator of order $2m$ with Hölder continuous coefficients of top order. We use the same notation as in [11] if not specified.

Let us recall the situation. Let Ω be a bounded domain in $\boldsymbol{R}^n$. We consider a symmetric integro-differential sesquilinear form

$$B[u, v]=\int_{\Omega} \sum_{|\alpha|,|\beta| \leqq m} a_{\alpha\beta}(x) D^{\alpha} u(x) \overline{D^{\beta} v(x)}\, dx$$

and a closed subspace V of the Sobolev space $H^m(\Omega)$, and assume the following.

(H1) $H_0^m(\Omega) \subset V \subset H^m(\Omega)$.

(H2) There exist $C_0 \geqq 0$ and $\delta_0>0$ such that

$$B[u, u] \geqq \delta_0\|u\|_m^2-C_0\|u\|_0^2 \qquad \text{for any} \quad u \in V.$$

(H3) The coefficients $a_{\alpha\beta}(x)(|\alpha|+|\beta| \leqq 2m)$ are bounded on Ω, and for some $\tau>0$ the coefficients of top order satisfy

$$a_{\alpha\beta} \in \mathscr{B}^{\tau}(\Omega) \quad (|\alpha|=|\beta|=m).$$

REMARK 1.1. Since an element of $\mathscr{B}^{\tau}(\Omega)$ can be extended to an element of $\mathscr{B}^{\tau}(\boldsymbol{R}^n)$ (see [10], [20]), we may assume that

$$\sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}(x) \xi^{\alpha+\beta} \geqq \delta_0|\xi|^{2m} \qquad \text{for} \quad x \in \boldsymbol{R}^n, \xi \in \boldsymbol{R}^n \tag{1.1}$$

by modifying the values of $a_{\alpha\beta}$ outside Ω and replacing δ_0 with another constant if necessary.

Let $\mathcal{A}$ be the self-adjoint operator associated with the variational triple $\{B, V, L_2(\Omega)\}$, and let $N(t)$ $(=N(t, B, V, L_2(\Omega)))$ denote the number of the eigenvalues of $\mathcal{A}$ not exceeding t. Then in general, the asymptotic behavior of $N(t)$ is given by

$$\text{(WF1)} \qquad N(t) = \mu_{\mathcal{A}}(\Omega)t^{n/2m} + O(t^{(n-\theta)/2m}) \quad \text{as } t \to \infty$$

with an appropriate constant θ, $0<\theta\leqq 1$ or

$$\text{(WF2)} \qquad N(t) = \mu_{\mathcal{A}}(\Omega)t^{n/2m} + O(t^{(n-1)/2m}\log t) \quad \text{as } t \to \infty .$$

In [11] we proved that (WF1) holds with $\theta=\tau$ for $0<\tau\leqq 1$ mainly when $n=1$, improving the known estimate $\theta=\tau/(\tau+1)$. Moreover we derived the asymptotic behavior of the trace $U(t)$ of the heat kernel:

$$\text{(WF1')} \qquad U(t) = \Gamma\Big(\frac{n}{2m}+1\Big)\mu_{\mathcal{A}}(\Omega)t^{-n/2m} + O(t^{(\tau-n)/2m}) \quad \text{as } t \to +0$$

when $0<\tau<1$, and

$$\text{(WF2')} \quad U(t) = \Gamma\Big(\frac{n}{2m}+1\Big)\mu_{\mathcal{A}}(\Omega)t^{-n/2m} + O(t^{(1-n)/2m}\log t^{-1}) \quad \text{as } t \to +0$$

when $\tau=1$ under the assumptions that $2m>n$ and that Ω has the restricted cone property. (WF1') and (WF2') are weak versions of (WF1) with $\theta=\tau$ and (WF2) respectively, for

$$U(t) = \int_{-\infty}^{\infty} e^{-ts}dN(s) .$$

By these results we are tempted to conjecture that (WF1) may hold with $\theta=\tau$ for $0<\tau\leqq 1$ in general and that therefore the optimal estimate $\theta=1$ may be attained when $\tau=1$.

In this paper we want to take a step for solving this conjecture. Here we give main results. We recall that

$$\Gamma_\varepsilon = \{x\in\Omega;\ \mathrm{dist}(x, \partial\Omega)<\varepsilon\}, \quad \tilde{\Gamma}_\varepsilon = \{x\in \boldsymbol{R}^n;\ \mathrm{dist}(x, \partial\Omega)<\varepsilon\}.$$

Theorem A. *Suppose* (H1)-(H3) *and that* Ω *is a bounded domain. In addition we suppose one of the following.*

(a-D) $m=1$; *the coefficients* $a_{\alpha\beta}(x)$ $(|\alpha|=|\beta|=1)$ *are real-valued*; $V=H_0^1(\Omega)$; *and* $\sup_{\varepsilon>0}|\Gamma_\varepsilon|/\varepsilon<\infty$.

(a-N) $m=1$; *the coefficients* $a_{\alpha\beta}(x)$ $(|\alpha|=|\beta|=1)$ *are real-valued*; Ω *has the extension property* (*see* [3]); *and* $sup_{\varepsilon>0}|\tilde{\Gamma}_\varepsilon|/\varepsilon<\infty$.

Then (WF1) *holds with* $\theta=3\tau/(2\tau+3)$ *when* $0<\tau<3$, *and* (WF2) *holds when* $\tau=3$.

Remark 1.2. Theorem A improves the result by Métivier [9] that (WF1) holds with $\theta=\tau/(\tau+1)$ for $0<\tau\leqq 1$, although we restrict ourselves to the case (a-D) or (a-N).

Theorem B. *Suppose* (H1)-(H3) *and that* Ω *is a bounded domain. Under one of the following conditions* (WF1') *holds when* $0<\tau<1$, *and* (WF2') *holds when* $\tau=1$.

(a-D) *or* (a-N) *of Theorem A.*

(b) $2m>n$; *and* Ω *has the restricted cone property.*

(c) $V=H_0^m(\Omega)$; *and* $\partial\Omega$ *is in* C^{2m}*-class.*

REMARK 1.3. As was stated above, in the case (b) Theorem B has already been proved in [11]. Hence the proof will be given only for the case (a-D), (a-N) or (c).

Further we treat the cases where the coefficients of top order satisfy the following condition (H4), which is weaker than that of [11, Theorem A. (ii)-(iii)].

(H4) The coefficients $a_{\alpha\beta}(x)$ ($|\alpha|=|\beta|=m$) can be written in the form $a_{\alpha\beta}(x)=b_{\alpha\beta}p(x)^{\alpha+\beta}$ with some real constants $b_{\alpha\beta}$ and $p(x)=(p_1(x), \cdots, p_n(x))$ where $p_j(x)$ is a function only of x_j for each $1\leqq j\leqq n$.

Theorem C. *Suppose* (H1)-(H3) *and that* Ω *is a bounded domain. In addition we suppose one of the following.*

(d) $m=1$; (H4); $V=H_0^1(\Omega)$; *and* $\sup_{\varepsilon>0}|\Gamma_\varepsilon|/\varepsilon<\infty$.

(e) $2m>n$; (H4); *and* Ω *has the restricted cone property.*

Then in the case (d) (WF1) *holds with* $\theta=\tau$ *when* $0<\tau<1$, *and* (WF2) *holds when* $\tau=1$. *In the case* (e) (WF1) *holds with* $\theta=\tau$ *when* $0<\tau<1$.

REMARK 1.4. In view of Remark 1.1 it is seen that the conditions (H2)-(H4) imply that $p_j(x)$ ($1\leqq j\leqq n$) is a real-valued function in $\mathcal{B}^\tau(\boldsymbol{R}^n)$ and that

$$0<\inf_{x\in \boldsymbol{R}^n} p_j(x)\leqq\sup_{x\in \boldsymbol{R}^n} p_j(x)<\infty\,. \tag{1.2}$$

For the proof of Theorem A, which will be given in Sections 2–6, we approximate $\mathcal{A}$ by operators $\mathcal{A}_\varepsilon$ with C^∞ coefficients and derive the asymptotic behavior for the spectral function of $\mathcal{A}_\varepsilon$. Then using the properties of $N(t, B, V, L_2(\Omega))$ we get Theorem A.

In order to derive the asymptotic behavior of the spectral function we follow the idea of Seeley [19]–we construct the fundamental solution of the Cauchy problem for the wave equation by the Hadamard-Riesz method ([17]), apply the inverse Fourier transform with respect to the time t, and use the Tauberian argument.

For our purpose we need the fundamental solution only in the time before the wave reaches the boundary and expand it up to any order so that the error term is sufficiently smooth. To evaluate the error term, we use not the energy inequality but the rough estimate for $e(t, x, y)$, which can be derived from the estimate for the heat kernel by Davies [3]. We also need to elaborate the Tauberian argument used by Seeley.

For the proof of Theorem B in the case (a-D), (a-N) or (c), which will be given in Section 7, we approximate $\mathcal{A}$ by $\mathcal{A}_\varepsilon$ again and construct a parametrix for the heat equation for $\mathcal{A}_\varepsilon$ by pseudo-differential operators. The global rough estimate for the heat kernel plays an important role. We don't have to pass through the construction of a parametrix for the resolvent kernel, as we did in the case (b) in [11].

The proof of Theorem C, which will be given in Sections 8–10, is based on the fact that the spectral function can be found explicitly for the operator on $L_2(\boldsymbol{R}^n)$ which has the same principal symbol as $\mathcal{A}_\varepsilon$. We use the method of the wave equation in the case (d), and the method of the resolvent kernel ([8]) in the case (e).

In the proofs we use the following fact: if Ω satisfies the conditions in Theorems A-C it follows that

$$\int_{d(x)\geqq t} d(x)^{-p}dx \leqq \begin{cases} C & (0<p<1) \\ C(\log t^{-1}+1) & (p=1) \\ Ct^{1-p} & (p>1) \end{cases} \tag{1.3}$$

for $0<t<1$ and $p>0$ where $C>0$ is a constant depending only on p and Ω.

In concluding this section we define some notations. Let $\boldsymbol{N}$ denote the set of nonnegative integers. Let $d(x)$ denote the Euclidean distance from x to $\partial\Omega$. For $\tau=k+\sigma>0$ with an integer k and $0<\sigma\leqq 1$ and $f\in\mathcal{B}^\tau(\boldsymbol{R}^n)$ we set

$$|f|_0=\sup_{x\in\boldsymbol{R}^n}|f(x)|, \quad |f|_\tau=\sup_{|\alpha|=k}\sup_{\substack{x,y\in\boldsymbol{R}^n\\ x\neq y}}\frac{|\partial^\alpha f(x)-\partial^\alpha f(y)|}{|x-y|^\sigma}.$$

Most of the constants, which will appear in the proofs, depend on one of the following constants:

$$M_0=n+m+\delta_0^{-1}+\sum_{|\alpha|,|\beta|\leqq m}|a_{\alpha\beta}|_0,$$

$$M_\tau=M_0+\sum_{|\alpha|=|\beta|=m}|a_{\alpha\beta}|_\tau.$$

In the proof of each lemma or proposition below we will use one and the same symbol C to denote constants which possesses the same property of the constant stated in the lemma or the proposition. When we distinguish these constants, we write $C_1, C_2, \cdots$.

2. Construction of the fundamental solution of the wave equation

Throughout Sections 2–4 in addition to the assumptions of Theorem A we assume that $V=H_0^1(\Omega)$ or $H^1(\Omega)$ and that $a_{\alpha\beta}\in\mathcal{B}^\infty(\boldsymbol{R}^n)$ $(|\alpha|=|\beta|=1)$ and $a_{\alpha\beta}(x)\equiv 0$ $(|\alpha|+|\beta|<2)$. We set

$$g^{ij}(x)=a_{e_ie_j}(x)\quad(1\leqq i,j\leqq n)$$

and

$$(g_{ij}(x))=(g^{ij}(x))^{-1}\,,\quad G(x)=\det(g_{ij}(x))$$

where e_j is the unit vector whose jth element is 1.

From (1.1) it follows that

$$\delta_0|\xi|^2\leqq\sum_{i,j=1}^n g^{ij}(x)\xi_i\xi_j\leqq\delta_1|\xi|^2\qquad\text{for}\quad x\in\boldsymbol{R}^n,\ \xi\in\boldsymbol{R}^n \tag{2.1}$$

and

$$\delta_1^{-1}|\xi|^2\leqq\sum_{i,j=1}^n g_{ij}(x)\xi_i\xi_j\leqq\delta_0^{-1}|\xi|^2\quad\text{for}\quad x\in\boldsymbol{R}^n,\ \xi\in\boldsymbol{R}^n \tag{2.2}$$

with $\delta_1=nM_0$. We introduce the Riemannian metric $\sum_{i,j=1}^n g_{ij}(x)dx_i\otimes dx_j$ on $\boldsymbol{R}^n$. $\mathcal{A}$ is written in the form

$$\mathcal{A}=-\sum_{i,j=1}^n\partial_i(g^{ij}(x)\partial_j\cdot)\,,\quad\partial_i=\frac{\partial}{\partial x_i}$$

and the Laplace operator is given by

$$\Delta=\frac{1}{\sqrt{G}}\sum_{i,j=1}^n\partial_i(\sqrt{G}g^{ij}(x)\partial_j\cdot)=-\mathcal{A}+\frac{1}{2}\sum_{i,j}\partial_i(\log G)g^{ij}\partial_j\,. \tag{2.3}$$

The exponential mapping $\exp_x$ is defined by $\exp_x v=x(1,v)$ where $v=(v_1,\cdots,v_n)$ denotes a tangent vector $\sum_{j=1}^n v_j(\partial/\partial x_j)_x$ and $x(t,v)$ is the geodesic satisfying

$$x(0,v)=x\,,\quad\frac{dx}{dt}(0,v)=v\,.$$

We also define the mapping e_x by

$$e_x(u)=\exp_x((g_{ij}(x))^{-1/2}u)\,.$$

Then there exists $R_0>0$ independent of x such that the inverse function $\exp_x^{-1}$ exists on $\{y\in\boldsymbol{R}^n;\ |x-y|<R_0\}$ and such that e_x is a diffeomorphism from $\{u\in\boldsymbol{R}^n;\ |u|<R_0\}$ into $\boldsymbol{R}^n$. Hence the geodesic distance $r(x,y)$ from x to y is defined and satisfies

$$c_1^{-1}|x-y|\leqq r(x,y)\leqq c_1|x-y| \tag{2.4}$$

with $c_1=n^{1/2}\max\{\delta_1^{1/2},\delta_0^{-1/2}\}\geqq 1$ when $|x-y|<R_0$. We set

$$d_1(x)=\min\{d(x),c_1^{-1}R_0\}\,. \tag{2.5}$$

Let $x_0 \in \Omega$ be fixed. For a given $f \in C_0^\infty(\Omega)$ with $\operatorname{supp} f \subset \{x;\ |x-x_0| < d_1(x_0)/3\}$ the function $u(t, x) = (\cos t\sqrt{\mathcal{A}}\, f)(x)$ satisfies the Cauchy problem for the wave equation in $L_2(\Omega)$:

$$
\left\{
\begin{aligned}
&\Big(\frac{\partial^2}{\partial t^2}+\mathcal{A}\Big)u(t, x) = 0 \\
&u(0, x) = f(x)\,, \quad \partial_t u(0, x) = 0\,.
\end{aligned}
\right. \tag{2.6}
$$

In order to solve (2.6) by the Hadamard-Riesz method we introduce an analytic family of distributions $\theta^{(\nu)}(s)$ on $\boldsymbol{R}$, which is defined by

$$
\theta^{(\nu)}(s) = \frac{s_+^{-\nu}}{\Gamma(1-\nu)}, \quad s_+ = \begin{cases} s & (s \geqq 0) \\ 0 & (s<0) \end{cases}
$$

for $\mathrm{Re}\,\nu < 1$, and can be analytically continued to the entire plane $\boldsymbol{C}$ by the first equality of the following:

$$
\frac{d}{ds}\theta^{(\nu)}(s) = \theta^{(\nu+1)}(s)\,, \quad s\theta^{(\nu)}(s) = (1-\nu)\theta^{(\nu-1)}(s)\,. \tag{2.7}
$$

Let us find the solution of (2.6) in the form

$$
u(t, x) = (\cos t\sqrt{\mathcal{A}}\, f)(x) = \sum_{l=0}^{N} I_l(f; t, x) + R_N(f; t, x) \tag{2.8}
$$

when $|t| < d_1(x_0)/3c_1$ where

$$
I_l(f; t, x) = \int_\Omega A_l(x, y)\cdot \operatorname{sgn} t\cdot \partial_t \theta^{((n-1)/2-l)}(t^2 - r(x, y)^2) f(y)\, dy \tag{2.9}
$$

and $R_N(f; t, x)$ is the error term. When $(n-1)/2 - l \geqq 0$ the integral in (2.9) is in the distributional sense, and can be justified by

$$
\begin{aligned}
\Gamma(1-\nu+k) &\int_\Omega \operatorname{sgn} t\cdot \partial_t \theta^{(\nu)}(t^2 - r(x, y)^2) F(y)\, dy \\
&= \operatorname{sgn} t\cdot \partial_t\Big(\frac{1}{2t}\partial_t\Big)^k \int_{r(x,y)^2 \leqq t^2} (t^2 - r(x, y)^2)^{-\nu+k} F(y)\, dy\,, \\
&= \partial_t\Big(\frac{1}{2t}\partial_t\Big)^k \int_0^t r^{n-1}(t^2-r^2)^{-\nu+k} dr \int_{S^{n-1}} F(e_x(r\omega))\psi(r\omega)\, d\omega
\end{aligned} \tag{2.10}
$$

with $\nu = (n-1)/2 - l$, an integer k with $k - \nu > 0$ and $F(y) = A_l(x, y) f(y)$ where S^{n-1} denotes the unit sphere and e_x and ψ are defined by

$$
y = e_x(u)\,, \quad dy = \psi(u)du\,.
$$

We have $I_l(f; t, x) = 0$ when $|x - x_0| \geqq 2d_1(x_0)/3$ and $|t| < d_1(x_0)/3c_1$, since

$$
\Big\{x;\ |y-x_0| < \frac{d_1(x_0)}{3},\ r(x, y) < \frac{d_1(x_0)}{3c_1}\Big\} \subset \Big\{x;\ |x - x_0| < \frac{2d_1(x_0)}{3}\Big\}\,.
$$

Therefore $I_l(f; t, \cdot)$ is a function in $C_0^\infty(\Omega)$ when $|t|<d_1(x_0)/3c_1$.

Using (2.3) and

$$\sum_{i,j} g^{ij}(x)\partial_i r\partial_j r = 1 ,$$
$$\frac{dA}{dr} = \sum_{i,j} g^{ij}(x)\partial_i A\partial_j r \qquad \text{for} \quad A\in C_0^\infty(\Omega) ,$$

where d/dr is the derivative along the geodesic, we have for any C^∞ function A,

$$\left(\frac{\partial^2}{\partial t^2}+\mathcal{A}_x\right)\{A(x)\theta^{(\nu)}(t^2-r(x, y)^2)\}$$
$$= (\mathcal{A}_x A)\cdot\theta^{(\nu)}+\left\{4r\frac{dA}{dx}-(\mathcal{A}_x(r^2)+4\nu+2)A\right\}\theta^{(\nu+1)} .$$

On the other hand, an elementary calculation shows that

$$I_0(f; 0, x) = 2\pi^{(n-1)/2}G(x)^{-1/2}A_0(x, x)f(x) , \quad \partial_t I_0(f; 0, x) = 0 ,$$
$$I_l(f; 0, x) = \partial_t I_l(f; 0, x) = 0 \quad (l\geqq 1) .$$

Hence we are led to the following conditions on A_l and R_N. The amplitudes A_l satisfy the transport equations: $A_{-1}\equiv 0$ and

$$4r\frac{dA_l}{dr}-(\mathcal{A}(r^2)+2n-4l)A_l+\mathcal{A}A_{l-1} = 0 \quad (l\in \boldsymbol{N}) \tag{2.11}$$

with the initial value

$$A_0(x, x) = 2^{-1}\pi^{-(n-1)/2}\sqrt{G(x)} . \tag{2.12}$$

The error term $R_N(f; t, x)$ satisfies the Cauchy problem in $L_2(\Omega)$:

$$\begin{cases} \left(\dfrac{\partial^2}{\partial t^2}+\mathcal{A}\right) R_N(f; t, x) = Q_N(f; t, x) \\ R_N(f; 0, x) = \partial_t R_N(f; 0, x) = 0 \end{cases} \tag{2.13}$$

where

$$Q_N(f; t, x) = \int_\Omega Q_E(t, x, y)f(y)\, dy ,$$
$$Q_N(t, x, y) = -\mathcal{A}_x A_N(x, y)\cdot \operatorname{sgn} t\cdot\partial_t\theta^{((n-1)/2-N)}(t^2-r(x, y)^2) .$$

Lemma 2.1. *For* $|x-x_0|<2d_1(x_0)/3$ *and* $|y-x_0|<d_1(x_0)/3$ *we have*

$$A_0(x, y) = 2^{-1}\pi^{-(n-1)/2}\sqrt{G(x)}\left(\det\left(\frac{\partial x}{\partial v}\right)\right)^{-1/2} \tag{2.14}$$

and

$$A_l(x, y) = -A_0(x, y) \int_0^1 \frac{s^{l-1}(\mathcal{A}A_{l-1})(\exp_y sv, y)}{4A_0(\exp_y sv, y)} ds \tag{2.15}$$

for $l \geqq 1$ where v is defined by $x = \exp_y v$.

Proof. Set

$$g^*_{ij}(v) = \sum_{i,j=1}^{n} \frac{\partial x_k}{\partial v_i} g_{kl}(x) \frac{\partial x_l}{\partial v_j}, \quad G^*(v) = \det(g^*_{ij}(v)).$$

By (2.11) and the well known equality

$$\Delta(r^2) = 2n + r\frac{d}{dr}\log G^*,$$

we have

$$4r\frac{dA_0}{dr} - r\frac{d}{dr}\log\frac{G(x)}{G^*(v)}A_0 = 0.$$

This combined with (2.12) gives (2.14).

When $l \geqq 1$ (2.11) is equivalent to

$$\frac{d}{dr}\left(\frac{r^l A_l}{A_0}\right) = -\frac{r^{l-1}\mathcal{A}A_{l-1}}{4A_0},$$

from which (2.15) follows. Q.E.D.

Lemma 2.2. *Let $|t| < d_1(x_0)/3c_1$ and $K = [n/2]+2$. If $N \geqq 3[n/2]+5$, then we have*

$$R_N(f; t, x) = \int_\Omega R_N(t, x, y) f(y)\, dy$$

where R_N is continuous in $(t, x, y) \in (-d_1(x_0)/3c_1, d_1(x_0)/3c_1) \times \Omega \times \{y;\ |y - x_0| < d_1(x_0)/3\}$ and given by

$$R_N(t, x, y) = \int_0^t \left(\frac{\sin(t-s)\sqrt{\mathcal{A}}}{\sqrt{\mathcal{A}}(\mathcal{A}+1)^K}\right)_x (\mathcal{A}_x+1)^K Q_N(s, x, y)\, ds. \tag{2.16}$$

Moreover, for an integer k with $0 \leqq k \leqq N - 3[n/2] - 5$, R_N is a C^k function of t and

$$\partial_t^k R_N(t, x, y) = \int_0^t \left(\frac{\sin(t-s)\sqrt{\mathcal{A}}}{\sqrt{\mathcal{A}}(\mathcal{A}+1)^K}\right)_x (\mathcal{A}_x+1)^K \partial_s^k Q_N(s, x, y)\, ds \tag{2.17}$$

holds.

Proof. We note that $Q_N(f; t, \cdot) \in C_0^\infty(\Omega) \subset \bigcap_{j=1}^\infty D(\mathcal{A}^j)$.

Since $\partial_s \theta^{((n-1)/2-N)}(s^2 - r^2)$ is $N-[n/2]$-1 times differentiable, it follows that

$$\sup\{\|\partial_t^i \mathcal{A}^j Q_N(f; t, \cdot)\|_{L_2(\Omega)};\ |t|<d_1(x_0)/3c_1\}<\infty$$

for $0\leqq i+2j\leqq N-[n/2]-1$, especially for $(i, j)=(0, 0), (0, 1), (1, 0)$. Therefore the solution of (2.13) is given by

$$\begin{aligned}
(2.18)\qquad R_N(f; t, x) &= \int_0^t \frac{\sin(t-s)\sqrt{\mathcal{A}}}{\sqrt{\mathcal{A}}} Q_N(f; s, x)\, ds \\
&= \int_0^t \frac{\sin(t-s)\sqrt{\mathcal{A}}}{\sqrt{\mathcal{A}}(\mathcal{A}+1)^K} (\mathcal{A}+1)^K Q_N(f; s, x)\, ds\,.
\end{aligned}$$

As will be seen later (Lemma 5.7), the operator

$$\frac{\sin(t-s)\sqrt{\mathcal{A}}}{\sqrt{\mathcal{A}}(\mathcal{A}+1)^K}$$

has a bounded integral kernel which is a C^1 function of t. Hence we get (2.16) from (2.18).

(2.17) follows from (2.16) by integration by parts. Q.E.D.

Summing up, we conclude the following.

Lemma 2.3. *Let A_l and R_N be as in Lemmas* 2.1 *and* 2.2. *For $f\in C_0^\infty(\Omega)$ with $\mathrm{supp}\, f\subset\{x\in \boldsymbol{R}^n;\ |x-x_0|<d_1(x_0)/3\}$ the solution of the Cauchy problem for the wave equation* (2.6) *in $L_2(\Omega)$ is given by* (2.8) *and* (2.9) *when $|t|<d_1(x_0)/3c_1$.*

3. Fourier transform of the fundamental solution

In this section we find the inverse Fourier transform of $\partial_t\theta^{(\nu)}(t^2-r(x, y)^2)$ with respect to t by a direct and elementary method. Let us begin with the lemma concerning the Bessel function. Let $J_\nu(z)$ be the Bessel function and let

$$\mathcal{J}_\nu(z) = \left(\frac{z}{2}\right)^{-\nu} J_\nu(z) = \sum_{k=0}^\infty \frac{(-1)^k}{k!\,\Gamma(\nu+k+1)}\left(\frac{z}{2}\right)^{2k}.$$

Lemma 3.1. *For $\nu\in \boldsymbol{N}$*

$$(3.1)_\nu\qquad \int_1^\infty \frac{xt\,\mathcal{J}_{\nu+1/2}(xt)}{\sqrt{t^2-1}}dt = \sqrt{\pi}\,\mathcal{J}_\nu(x)\,.$$

Proof. We proceed by induction on ν. When $\nu=0$, it follows from $\mathcal{J}_{1/2}(z)=2\sin z/(\sqrt{\pi}z)$ and the formula

$$\mathcal{J}_0(x) = \frac{2}{\pi}\int_1^\infty \frac{\sin xt}{\sqrt{t^2-1}}dt$$

([14]). Let us denote the left hand side of $(3.1)_\nu$ by $G_\nu(x)$. Suppose that $(3.1)_{\nu-1}$ has been proved. Differentiating $G_\nu(x)$ under integral sign and using

$(3.1)_{\nu-1}$ and the formula

$$z\mathcal{J}_\nu'(z)+2\nu\mathcal{J}_\nu(z)=2\mathcal{J}_{\nu-1}(z)$$

with $z=xt$ and ν replaced by $\nu+1/2$ we have

$$xG_\nu'(x)+2\nu G_\nu(x)=2\sqrt{\pi}\mathcal{J}_{\nu-1}(x)\,,$$

which implies that $G_\nu(x)$ satisfies the same differential equation of first order as $\sqrt{\pi}\mathcal{J}_\nu(x)$. In addition, it is easily checked that $G_\nu(x)$ is bounded in a neighborhood of $x=0$. Hence we get $(3.1)_\nu$. Q.E.D.

Lemma 3.2. *Let* $l\in \boldsymbol{N}$, $0\leqq l\leqq (n-1)/2$, $|x-x_0|<2d_1(x_0)/3$ *and* $f\in C_0^\infty(\Omega)$ *with* $\operatorname{supp} f\subset \{y\in \boldsymbol{R}^n;\ |y-x_0|<d_1(x_0)/3\}$. *Then*

$$\int_{-\infty}^{\infty} e^{it\lambda}dt\int_\Omega \operatorname{sgn} t\cdot\partial_t\theta^{((n-1)/2-l)}(t^2-r(x,y)^2)f(y)\,dy \tag{3.2}$$
$$=\int_\Omega 2^{-n+2l+2}\sqrt{\pi}\,|\lambda|^{n-2l-1}\mathcal{J}_{n/2-l-1}(\lambda r(x,y))f(y)\,dy$$

for $\lambda\in \boldsymbol{R}$ *where* $i=\sqrt{-1}$.

Proof. By the continuity in λ we have only to prove when $\lambda\neq 0$.

First suppose that n is odd. From (2.10), integration by parts and the formulas

$$\left(-\frac{1}{2t}\partial_t\right)\mathcal{J}_\nu(t\lambda)=\left(\frac{\lambda}{2}\right)^2\mathcal{J}_{\nu+1}(t\lambda)\,,\quad \mathcal{J}_{-1/2}(z)=\frac{\cos z}{\sqrt{\pi}}\,,$$

we see that the left hand side of (3.2) is equal to

$$2\int_0^\infty \cos t\lambda\ dt\ \partial_t\left(\frac{1}{2t}\partial_t\right)^{(n-1)/2-l}\int_0^t r^{n-1}F(r)\,dr$$
$$=2\sqrt{\pi}\int_0^\infty\left(\frac{\lambda}{2}\right)^{2((n-1)/2-l)}\mathcal{J}_{n/2-l-1}(t\lambda)t^{n-1}F(t)\,dt$$

where

$$F(r)=\begin{cases}\int_{S^{n-1}}f(e_x(r\omega))\psi(r\omega)d\omega & (0<r\leqq c_1d_1(x_0)\leqq R_0)\\ 0 & (r>c_1d_1(x_0))\,.\end{cases}$$

Then the lemma immediately follows.

Next suppose that n is even. The left hand side of (3.2) is equal to

$$\frac{2}{\sqrt{\pi}}\int_0^\infty\cos t\lambda\ dt\ \partial_t\left(\frac{1}{2t}\partial_t\right)^{n/2-l-1}\int_0^t\frac{r^{n-1}}{\sqrt{t^2-r^2}}F(r)\,dr$$
$$=2\int_0^\infty 2t\left(\frac{\lambda}{2}\right)^{2(n/2-l)}\mathcal{J}_{(n-1)/2-l}(t\lambda)\,dt\int_0^t\frac{r^{n-1}}{\sqrt{t^2-r^2}}F(r)\,dr$$

$$= 2^{-n+2l+2}|\lambda|^{n-2l-1}\int_0^\infty r^{n-1}F(r)\,dr\int_1^\infty \frac{|\lambda|rt\mathcal{J}_{(n-1)/2-l}(|\lambda|rt)}{\sqrt{t^2-1}}\,dt\,.$$

This combined with Lemma 3.1 yields the lemma. Q.E.D.

REMARK. In the proofs of Lemmas 3.1 and 3.2 we often carried out interchange of integrations and differentiation under integral sign, which are not so difficult to be justified.

Let us choose $\rho\in\mathcal{S}(\boldsymbol{R})$ (the set of rapidly decreasing functions) satisfying

$$(3.3)\qquad \hat{\rho}(0)=1\,,\quad \operatorname{supp}\hat{\rho}\subset(-1,1)\,,\quad \rho(\lambda)\geqq 0\,,$$
$$\inf_{|\lambda|\leqq 1}\rho(\lambda)=c_0>0\,,\quad \rho\colon \text{even}$$

([18]) where $\hat{\rho}(t)$ denotes the Fourier transform of $\rho(\lambda)$:

$$\hat{\rho}(t)=\int_{-\infty}^{\infty}e^{-it\lambda}\rho(\lambda)\,d\lambda\,.$$

For $\delta>0$ we put $\rho_\delta(\lambda)=\delta\rho(\delta\lambda)$ and have $\hat{\rho}_\delta(t)=\hat{\rho}(t/\delta)$. We denote by $e(\lambda, x, y)$ the spectral function of $\mathcal{A}$ and put

$$(3.4)\qquad E(\lambda, x, y)=\operatorname{sgn}\lambda\cdot e(\lambda^2, x, y)\,.$$

For a fixed $x_0\in\Omega$ set $\delta=d_1(x_0)/3c_1$. Taking the inverse Fourier transform of (2.8) multiplied by $\hat{\rho}_\delta(t)$, using Lemma 3.2 and putting $x=y=x_0$ we get the following.

Lemma 3.3. *Let $N\geqq 3[n/2]+5$. For $\lambda\in\boldsymbol{R}$ and $x\in\Omega$ we have*

$$\frac{1}{2}\rho_\delta * dE(\lambda, x, x)=\sum_{0\leqq l\leqq(n-1)/2} b_lA_l(x, x)\rho_0*(|\lambda|^{n-2l-1})$$
$$+\sum_{(n-1)/2<l\leqq N} b_lA_l(x, x)\int_{-\infty}^{\infty}e^{it\lambda}\hat{\rho}_\delta(t)|t|^{2l-n}\,dt$$
$$+(2\pi)^{-1}\int_{-\infty}^{\infty}e^{it\lambda}\hat{\rho}_\delta(t)R_N(t, x, x)\,dt$$

where $\delta=d_1(x)/3c_1$ and

$$b_l=\begin{cases}2^{-n+2l+1}\pi^{-1/2}\Gamma(n/2-l)^{-1} & (0\leqq l\leqq(n-1)/2)\\ \pi^{-1}\Gamma(l-(n-1)/2)^{-1} & (l>(n-1)/2)\,.\end{cases}$$

4. Tauberian argument

In this section we derive the asymptotic behavior of $E(\lambda, x, x)$ from Lemma 3.3 by the Tauberian argument. For simplicity we often write $E(\lambda)$ for

$E(\lambda, x, x)$. Roughly speaking, the Tauberian argument ([5], [6], [15], [18], [19]) states that

$$\rho * dE(\lambda) = na_0\lambda^{n-1}+O(\lambda^{n-2}) \quad \text{as } \lambda\to\infty ,$$

implies

$$E(\lambda) = a_0\lambda^n+O(\lambda^{n-1}) \quad \text{as } \lambda\to\infty .$$

For our purpose we must investigate precisely the effect by δ and the terms of lower power of λ. Our starting point is

$$E(\lambda) = \frac{1}{2}\int_{-\lambda}^{\lambda} (\rho_\delta * dE)(\mu)\, d\mu - \{\rho_\delta * E(\lambda) - E(\lambda)\}. \tag{4.1}$$

Using

$$|E(\lambda\pm\delta^{-1})-E(\lambda)| \leqq c_0^{-1}\delta^{-1}\rho_\delta * dE(\lambda) ,$$

we have

$$\begin{aligned} |\rho_\delta * E(\lambda)-E(\lambda)| &\leqq \int_{-\infty}^{\infty} \rho(\sigma)\,|E(\lambda-\sigma\delta^{-1})-E(\lambda)|\, d\sigma \\ &\leqq c_0^{-1}\delta^{-1}\int_{-\infty}^{\infty} \rho(\sigma) \sum_{l=-[|\sigma|]}^{[|\sigma|]} \rho_\delta * dE(\lambda+l\delta^{-1})\, d\sigma . \end{aligned} \tag{4.2}$$

Here $[\sigma]$ stands for the largest integer which is not greater than σ for a real number σ. For $k\in \boldsymbol{N}$, $N\geqq 3[n/2]+5$ and $x\in\Omega$ put

$$G_1(\lambda, \delta, k) = \rho_\delta * |\lambda|^k , \quad G_2(\lambda, \delta, k) = \int_{-\infty}^{\infty} e^{i\lambda t}\hat{\rho}_\delta(t)\,|t|^k\, dt ,$$

$$G_3(\lambda, \delta, N, x) = \int_{-\infty}^{\infty} e^{i\lambda t}\hat{\rho}_\delta(t) R_N(t, x, x)\, dt .$$

In view of (4.1), (4.2) and Lemma 3.3 we need the following estimates.

Lemma 4.1. *There exists* $C_k>0$ *depending only on* k *such that* (i)-(iv) *hold for* $\lambda>0$, $\delta>0$, $L\in \boldsymbol{N}$ *and* $k\in \boldsymbol{N}$, *and such that* (v)-(vi) *hold for* $\lambda>0$, $x\in\Omega$, $\delta=d_1(x)/3c_1$, $K=[n/2]+2$ *and* $0\leqq k\leqq N-3[n/2]-5$.

(i) $$\Big|\sum_{l=-L}^{L} G_1(\lambda+l\delta^{-1}, \delta, k)\Big| \leqq C_k(L+1)^{k+1}\sum_{l=0}^{k} \delta^{-l}\lambda^{k-l} .$$

(ii) $$\Big|\int_{-\lambda}^{\lambda} G_1(\mu, \delta, k)\, d\mu - \frac{2}{k+1}\lambda^{k+1}\Big| \leqq C_k\sum_{l=0}^{k} \delta^{-l-1}\lambda^{k-l} .$$

(iii) $$\Big|\sum_{l=-L}^{L} G_2(\lambda+l\delta^{-1}, \delta, k)\Big| \leqq C_k(L+1)^{k+1}\delta\lambda^{-k} .$$

(iv) $$\Big|\int_{-\lambda}^{\lambda} G_2(\mu, \delta, k)\, d\mu\Big| \leqq C_k\lambda^{-k} .$$

(v) $$\left|\sum_{l=-L}^{L} G_3(\lambda+l\delta^{-1}, \delta, N, x)\right| \leqq C_k \mathcal{R}_{N,k}(x)(L+1)^{k+1}\delta^{2N-2K-k+2}\lambda^{-k}.$$

(vi) $$\left|\int_{-\lambda}^{\lambda} G_3(\mu, \delta, N, x)\, d\mu\right| \leqq C_k \mathcal{R}_{N,k}(x)\delta^{2N-2K-k+1}\lambda^{-k}.$$

Here and in what follows we set

(4.3) $$\mathcal{R}_{N,k}(x) = \max_{0\leqq l\leqq k} \sup_{|t|<\delta} |t^{-(2N-2K-l+1)}\partial_t^l R_N(t, x, x)|.$$

REMARK. We will show later (Lemma 5.8) that $\mathcal{R}_{N,k}(x)<\infty$.

Proof. (i): Since $\rho\in\mathcal{S}(\boldsymbol{R})$ we have

$$0\leqq \sum_{l=-L}^{L} G_1(\lambda+l\delta^{-1}, \delta, k) = \sum_{l=-L}^{L}\int_{-\infty}^{\infty}\rho(\sigma)\left|\lambda+\frac{l}{\delta}-\frac{\sigma}{\delta}\right|^k d\sigma$$
$$\leqq C\sum_{l=-L}^{L}\int_{-\infty}^{\infty}\rho(\sigma)\sum_{k_1+k_2+k_3=k}\lambda^{k_1}\left(\frac{l}{\delta}\right)^{k_2}\left(\frac{|\sigma|}{\delta}\right)^{k_3} d\sigma$$
$$\leqq C(L+1)^{k+1}\sum_{l=0}^{k}\delta^{-l}\lambda^{k-l},$$

which is the desired result.

(ii): When k is even, (ii) immediately follows from

$$\int_{-\lambda}^{\lambda} G_1(\mu, \delta, k)\, d\mu = \int_{-\lambda}^{\lambda} d\mu\int_{-\infty}^{\infty}\rho(\sigma)\left|\mu-\frac{\sigma}{\delta}\right|^k d\sigma$$
$$= \frac{1}{k+1}\int_{-\infty}^{\infty}\rho(\sigma)\left\{\left(\lambda-\frac{\sigma}{\delta}\right)^{k+1}+\left(\lambda+\frac{\sigma}{\delta}\right)^{k+1}\right\} d\sigma.$$

When k is odd, we have

$$\int_{-\lambda}^{\lambda} G_1(\mu, \delta, k)\, d\mu = 2\int_{0}^{\lambda} G_1(\mu, \delta, k)\, d\mu$$
$$= \frac{2}{k+1}\int_{-\infty}^{\infty}\rho(\sigma)\left\{\left(\lambda-\frac{\sigma}{\delta}\right)^{k+1}-\left(\frac{\sigma}{\delta}\right)^{k+1}\right\} d\sigma+4\int_0^{\lambda} d\mu\int_{\delta\mu}^{\infty}\rho(\sigma)\left(\frac{\sigma}{\delta}-\mu\right)^k d\sigma.$$

The first term is easily evaluated. The second term is evaluated by $C\delta^{-k}\lambda$, since

$$\sup_{\xi>0}\xi^l\int_{\xi}^{\infty}\rho(\sigma)\sigma^{k-l}d\sigma<\infty \qquad \text{for} \quad 0\leqq l\leqq k.$$

Hence (ii) follows.

(iii): By integration by parts we have

$$\left|\lambda^k\sum_{l=-L}^{L} G_2(\lambda+l\,\delta^{-1}, \delta, k)\right| = \left|\int_{-\infty}^{\infty} e^{i\lambda t}\partial_t^k\{\sum_{l=-L}^{L} e^{ilt/\delta}\hat{\rho}_\delta(t)|t|^k\}\, dt\right|$$
$$\leqq C\int_{-\delta}^{\delta}\sum_{k_1+k_2+k_3=k}\sum_{l=-L}^{L}\left(\frac{l}{\delta}\right)^{k_1}\delta^{-k_2}|t|^{k-k_3}\, dt\leqq C(L+1)^{k+1}\delta,$$

from which (iii) follows.

(iv): When $k=0$, (iv) follows from

$$\int_{-\lambda}^{\lambda} G_2(\mu, \delta, 0)\, d\mu = 2\pi \int_{-\lambda}^{\lambda} \delta\rho(\delta\mu)\, d\mu = 2\pi \int_{-\delta\lambda}^{\delta\lambda} \rho(\mu)\, d\mu\,.$$

When $k \geqq 1$, we have

$$\int_{-\lambda}^{\lambda} G_2(\mu, \delta, k)\, d\mu = \int_{-\infty}^{\infty} \frac{e^{i\lambda t} - e^{-i\lambda t}}{it} \hat{\rho}_\delta(t)\, |t|^k\, dt$$
$$= 2i \int_0^{\infty} (e^{-i\lambda t} - e^{i\lambda t}) \hat{\rho}_\delta(t) t^{k-1}\, dt$$

and

$$\left| \lambda^k \int_0^{\infty} e^{\pm i\lambda t} \hat{\rho}_\delta(t) t^{k-1}\, dt \right| \leqq (k-1)! + \left| \int_0^{\infty} e^{\pm i\lambda t} \partial_t^k \{\hat{\rho}_\delta(t) t^{k-1}\}\, dt \right|$$
$$\leqq C + C \int_0^{\delta} \sum_{l=0}^{k-1} \delta^{-(k-l)} t^{k-1-l}\, dt \leqq C\,,$$

from which (iv) follows.

(v): By integration by parts and (4.3) we have

$$\left| \lambda^k \sum_{l=-L}^{L} G_3(\lambda + l\delta^{-1}, \delta, N, x) \right|$$
$$= \left| \int_{-\infty}^{\infty} e^{i\lambda t} \partial_t^k \{ \sum_{l=-L}^{L} e^{ilt/\delta} \hat{\rho}_\delta(t) R_N(t, x, x)\}\, dt \right|$$
$$= C \int_{-\delta}^{\delta} \sum_{k_1+k_2+k_3=k} \sum_{l=-L}^{L} \left(\frac{l}{\delta}\right)^{k_1} \delta^{-k_2} |\partial_t^{k_3} R_N(t, x, x)|\, dt$$
$$\leqq C \int_{-\delta}^{\delta} \sum_{k_1+k_2+k_3=k} (L+1)^{k+1} \delta^{-k_1-k_2} \mathcal{R}_{N,k}(x)\, |t|^{2N-2K-k_3+1}\, dt$$
$$\leqq C\mathcal{R}_{N,k}(x)(L+1)^{2+1} \delta^{2N-2K-k+2}\,,$$

from which (v) follows.

(vi): In the same way as in (iv) we have

$$\int_{-\lambda}^{\lambda} G_3(\mu, \delta, N, x) d\mu = 2i \int_0^{\infty} (e^{-i\lambda t} - e^{i\lambda t}) \hat{\rho}_\delta(t) t^{-1} R_N(t, x, x)\, dt$$

and

$$\left| \lambda^k \int_0^{\infty} e^{\pm i\lambda t} \hat{\rho}_\delta(t) t^{-1} R_N(t, x, x)\, dt \right|$$
$$\leqq \left| \int_0^{\infty} e^{\pm i\lambda t} \partial_t^k \{\hat{\rho}_\delta(t) t^{-1} R_N(t, x, x)\}\, dt \right|$$
$$\leqq C \int_0^{\delta} \sum_{k_1+k_2+k_3=k} \delta^{-k_1} t^{-k_2-1} \mathcal{R}_{N,k}(x) t^{2N-2K+k_3+1}\, dt$$

$$\leqq C\mathcal{R}_{N,k}(x)\delta^{2N-2K-k+1},$$

from which (vi) follows. Q.E.D.

Lemma 4.2. *Let $N\geqq 5n$ and $K=[n/2]+2$. There exists $C_N>0$ depending only on N and M_0 such that for $x\in\Omega$*

$$|E(\lambda, x, x)-\mu_{\mathcal{A}}(x)\lambda^n|\leqq C_N\{|A_0|_0 d_1(x)^{-1}\lambda^{n-1}+\sum_{1\leqq l\leqq N}|A_l|_0\lambda^{n-2l}$$
$$+\mathcal{R}_{N,N-5n}(x)d_1(x)^{N+5n-2K+1}\lambda^{-N+5n}\}$$

when $d_1(x)\lambda\geqq 1$.

Proof. We can take $k=N-5n$ in Lemma 4.1 since $3[n/2]+5\leqq 5n$. The lemma follows from (2.12), (4.1), (4.2), Lemmas 3.3 and 4.1. Q.E.D.

5. Estimates for $A_l(x, y)$ and $\partial_t^k R_N(t, x, y)$

Throughout Sections 5 and 6 we assume the assumptions of Theorem *A*. Moreover we assume that $V=H_0^1(\Omega)$ or $H^1(\Omega)$ in the case (a-N).

First we construct the operator $\mathcal{A}_\varepsilon$ approximating $\mathcal{A}$. For $\tau=k+\sigma>0$ with an integer k and $0<\sigma\leqq 1$ we take a function $\varphi\in C_0^\infty(\boldsymbol{R}^n)$ satisfying $\operatorname{supp}\varphi\subset\{x\in\boldsymbol{R}^n; |x|<1\}$ and

$$\int_{\boldsymbol{R}^n}\varphi(x)\,dx=1\,,\quad \int_{\boldsymbol{R}^n}x^\alpha\varphi(x)\,dx=0\quad(1\leqq|\alpha|\leqq k)$$

([10, Lemma 5.1]), and put $\varphi_\varepsilon(x)=\varepsilon^{-n}\varphi(x/\varepsilon)$.

For $\varepsilon>0$ we consider the form

$$B_\varepsilon[u, v]=\int_\Omega\sum_{i,j=1}^n g_\varepsilon^{ij}(x)D_i u(x)\overline{D_j v(x)}\,dx\,,\quad D_i=-i\frac{\partial}{\partial x_i}$$

where

$$g_\varepsilon^{ij}(x)=\varphi_\varepsilon * g^{ij}(x)\,,\quad g^{ij}(x)=a_{e_ie_j}(x)\,.$$

From the properties of φ it follows that

$$(5.1)\qquad |g_\varepsilon^{ij}(x)-g^{ij}(x)|\leqq C\varepsilon^\tau,\quad |\partial^\alpha g_\varepsilon^{ij}(x)|\leqq C_\alpha\varepsilon^{\min\{\tau-|\alpha|,0\}}.$$

for some constant C depending only on M_τ, and some constant C_α depending only on α and M_τ.

Until the end of Section 6 we take $0<\varepsilon_0<1/2$ sufficiently small and assume $0<\varepsilon<\varepsilon_0$. Then from (2.1) and (5.1) it follows that

$$(5.2)\qquad \frac{1}{2}\delta_0|\xi|^2\leqq\sum_{i,j=1}^n g_\varepsilon^{ij}(x)\xi_i\xi_j\leqq 2\delta_1|\xi|^2$$

and that B_ε is therefore coercive.

We define $(g_{\varepsilon,ij})$, $\mathcal{A}_\varepsilon$, $r_\varepsilon(x,y)$, $c_{\varepsilon,1}$ $d_{\varepsilon,1}(x)$, $A_{\varepsilon,l}(x,y)$, $R_{\varepsilon,N}$ etc. for (g_ε^{ij}) as we defined (g_{ij}), $\mathcal{A}$, $r(x,y)$, c_1, $d_1(x)$, $A_l(x,y)$, R_N etc. for (g^{ij}) in Sections 2–4. In view of (5.2) we have

$$c_{\varepsilon,1} = (2n)^{1/2} \max\{\delta_1^{1/2}, \delta_0^{-1/2}\}. \tag{5.3}$$

In order to evaluate $A_{\varepsilon,l}$ and $\partial_\varepsilon^k R_{\varepsilon,N}$ we begin with estimating the derivatives of the exponential mapping. Let $y\in \boldsymbol{R}^n$ be fixed and v a tangent vector at y. The exponential mapping $\exp_{\varepsilon,y}$ is given by $\exp_{\varepsilon,y} v=x(1,v)$. Here $x(t,v)$ is the geodesic satisfying

$$\begin{cases} \dfrac{d^2x_i}{dt^2}+\sum_{j,k=1}^n \Gamma^i_{\varepsilon,jk}(x(t))\dfrac{dx_j}{dt}\dfrac{dx_k}{dt}=0 \quad (1\leqq i\leqq n) \\ x(0,v)=y\,, \quad \dfrac{dx}{dt}(0,v)=v \end{cases} \tag{5.4}$$

where

$$\Gamma^i_{\varepsilon,jk}=\frac{1}{2}\sum_{h=1}^n\Big(\frac{\partial g_{\varepsilon,hj}}{\partial x_k}+\frac{\partial g_{\varepsilon,hk}}{\partial x_j}-\frac{\partial g_{\varepsilon,jk}}{\partial x_h}\Big)g_\varepsilon^{ih}\,.$$

Lemma 5.1. *Let $0<\tau\leqq 3$ and $|\alpha|\geqq 1$. There exists $C_\alpha>0$ depending only on α and M_τ such that*

$$|\partial_v^\alpha \exp_{\varepsilon,y} v|\leqq C_\alpha \varepsilon^{\min\{\tau/3-|\alpha|+1,0\}}$$

when $|v|\leqq \varepsilon^{1-\tau/3}$.

REMARK. We note that

$$\frac{\partial \exp_{\varepsilon,y} v}{\partial v}\bigg|_{v=0}=I$$

where I is the identity matrix.

Proof. For simplicity of notation we prove the lemma when $n=1$. The proof also works when $n\geqq 2$ only with a little change of notations. We put

$$\xi(t,v)=\frac{dx}{dt}(t,v)$$

and have

$$\frac{d}{dt}\begin{pmatrix}x\\ \xi\end{pmatrix}=\begin{pmatrix}\xi\\ -\Gamma_\varepsilon(x)\xi^2\end{pmatrix}, \quad \begin{pmatrix}x\\ \xi\end{pmatrix}(0)=\begin{pmatrix}y\\ v\end{pmatrix} \tag{5.5}$$

where

$$\Gamma_\varepsilon(x)=\Gamma^1_{\varepsilon,11}(x)\,.$$

By (5.2) and the property of the geodesic we have

(5.6) $$\frac{1}{2}\delta_1^{-1}|\xi|^2 \leqq \sum_{i,j=1}^{n} g_{\varepsilon,ij}(x)\xi_i\xi_j = \sum_{i,j=1}^{n} g_{\varepsilon,ij}(y)v_iv_j \leqq 2\delta_0^{-1}|v|^2,$$

from which it follows that the solution of (5.4) exists on $\boldsymbol{R}$.

Let $|t|\leqq 1$ and $|v|\leqq \varepsilon^{1-\tau/3}$. From (5.5) it follows that

$$\begin{pmatrix}\partial_v x\\ \partial_v \xi\end{pmatrix}(t) = \begin{pmatrix}0\\ 1\end{pmatrix} + \int_0^t \begin{pmatrix}0 & 1\\ -\Gamma_\varepsilon'(x)\xi^2 & -2\Gamma_\varepsilon(x)\xi\end{pmatrix}\begin{pmatrix}\partial_v x\\ \partial_v \xi\end{pmatrix}(s)\,ds.$$

Here we omit v for simplicity. By (5.1) and (5.6) we have

$$|\Gamma_\varepsilon'(x)\xi^2|\leqq C, \quad |\Gamma_\varepsilon(x)\xi|\leqq C.$$

Hence Gronwall's inequality gives

(5.7) $$|\partial_v x(t,v)|\leqq C, \quad |\partial_v \xi(t,v)|\leqq C.$$

By induction we will show that

$(5.8)_k$ $$|\partial_v^k x(t,v)|\leqq C\varepsilon^{\tau/3-k+1}, \quad |\partial_v^k \xi(t,v)|\leqq C\varepsilon^{\tau/3-k+1}$$

when $k\geqq 2$. We have

$$\begin{pmatrix}\partial_v^k x\\ \partial_v^k \xi\end{pmatrix}(t) = \int_0^t \begin{pmatrix}0 & 1\\ -\Gamma_\varepsilon'(x)\xi^2 & -2\Gamma_\varepsilon(x)\xi\end{pmatrix}\begin{pmatrix}\partial_v^k x\\ \partial_v^k \xi\end{pmatrix}(s)\,ds + \int_0^t F_{\varepsilon,k}(s,v)\,ds$$

for $k\geqq 1$ where $F_{\varepsilon,k}(t,v)$ is a (2, 1)-matrix given by

$$\begin{cases} F_{\varepsilon,1}=0 \\ F_{\varepsilon,k+1}=\partial_v(F_{\varepsilon,k})-\begin{pmatrix}0\\ \partial_v(\Gamma_\varepsilon'(x)\xi^2)\partial_v^k x+2\partial_v(\Gamma_\varepsilon(x)\xi)\partial_v^k\xi\end{pmatrix} \quad (k\geqq 1). \end{cases}$$

It is easily seen that

$(5.9)_k$ $$|F_{\varepsilon,k}(t)|\leqq C\varepsilon^{\tau/3-k+1}$$

when $k=2$. This combined with Gronwall's inequality gives $(5.8)_2$.

Suppose $k\geqq 3$ and that $(5.8)_l$ holds for any l, $2\leqq l\leqq k-1$. Then we get $(5.9)_k$ using $(5.8)_l$ for $1\leqq l\leqq k-1$ and noting that $|G|\leqq C\varepsilon^\sigma$ for some $\sigma\in\boldsymbol{R}$ implies $|\partial_v G|\leqq C\varepsilon^{\sigma-1}$ where G stands for either $\Gamma_\varepsilon^{(l)}(x)$ or $\partial_v^\alpha x\ (\alpha\geqq 1)$ or $\partial_v^\beta\xi(\beta\geqq 0)$. Then $(5.9)_k$ and Gronwall's inequality yield $(5.8)_k$.

Finally we obtain the lemma by putting $t=1$ in (5.7) and $(5.8)_k$. Q.E.D.

We need an elaboration of the inverse function theorem.

Lemma 5.2. (i) *Let* $f={}^t(f_1,\cdots,f_n)$: $\{x;\ |x|<r\}\rightarrow\boldsymbol{R}^n$ *be a* C^2 *function satisfying* $f(0)=0$ *and*

$$|f'(0)^{-1}| = K>0\,, \quad |f'(x)-f'(0)| \leqq L|x|$$

for some $K>0$ *and* $L>0$. *Set*

$$r' = \min\left\{r, \frac{1}{2KL}, \frac{|\det f'(0)|}{2^n n! nL |f'(0)|^{n-1}}\right\}, \quad r'' = \frac{r'}{4|f'(0)|K}\,, \quad \delta = \frac{r'}{2K}\,.$$

Then f^{-1} *exists on* $\{y; |y|<\delta\}$ *and maps it into* $\{x; |x|<r'\}$. *Conversely* f *maps* $\{x; |x|<r''\}$ *into* $\{y; |y|<\delta\}$. *Moreover we have*

$$|f'(x)| \leqq 2|f'(0)|, \quad |\det f'(x)| \geqq \frac{1}{2}|\det f'(0)| \tag{5.10}$$

when $|x|<r'$, *and*

$$|(f^{-1})'(y)| \leqq 2K \tag{5.11}$$

when $|y|<\delta$.

(ii) *In addition, suppose that* f *is a* C^∞ *function. Then for a multi-index* α *with* $|\alpha|\geqq 2$ *each element of the derivative* $\partial_y^\alpha f^{-1}$ *is expressed by a finite sum of terms of the form*

$$C_{p,\alpha,\beta_1,\cdots,\beta_n}(\det f'(x))^{-p} \prod_{j=1}^{n} \partial_x^{\beta_j} f_j$$

where $p\geqq 1$ *and* $|\beta_1|+\cdots+|\beta_n|-J=|\alpha|-1$ *with* J *denoting the number of* j *such that* $|\beta_j|\geqq 1$, *and* $C_{p,\alpha,\beta_1,\cdots,\beta_n}$ *is a constant depending only on* p, α *and* $\beta_1, \cdots, \beta_n$.

REMARK. For a matrix $A=(a_{ij})$ we define its norm by

$$|A| = (\sum_{i,j} a_{ij}^2)^{1/2}\,.$$

Proof. Let $|x|<r'$ and $|y|<\delta$. Put

$$F_y(x) = F(x) = x-f'(0)^{-1}(f(x)-y)\,.$$

Then we have

$$\begin{aligned}|F(x)| &= \left|f'(0)^{-1}\left[y-\int_0^1 \{f'(tx)-f'(0)\}\,dt\cdot x\right]\right| \\ &\leqq K|y|+\frac{1}{2}KL|x|^2 \leqq \frac{1}{2}r'+\frac{1}{4}r' = \frac{3}{4}r'\end{aligned}$$

and

$$\begin{aligned}|F(x)-F(x')| &= \left|f'(0)^{-1}\int_0^1 \{f'(0)-f'(tx+(1-t)x')\}\,dt\cdot(x-x')\right| \\ &\leqq KLr'|x-x'| \leqq \frac{1}{2}|x-x'|\,.\end{aligned}$$

Hence F_y is a contraction on $\{x;\ |x|\leqq 3r'/4\}$. Therefore there exists x such that $F_y(x)=x$, that is, $y=f(x)$ with $|x|<3r'/4$. Thus it has been proved that f^{-1} exists on $\{y;\ |y|<\delta\}$ and maps it into $\{x;\ |x|<r'\}$. (5.10) follows from

$$|f'(x)|\leqq L|(x)|+|f'(0)|\leqq LK|f'(0)||x|+|f'(0)|\leqq\frac{3}{2}|f'(0)|$$

and

$$|\det f'(x)-\det f'(0)|\leqq n!nL|x|(2|f'(0)|)^{n-1}.$$

By (5.10) we have

$$|f(x)|=\left|\int_0^1 f'(tx)x\,dt\right|\leqq 2|f'(0)|\,|x|.$$

Hence f maps $\{x;\ |x|<r''\}$ into $\{y;\ |y|<\delta\}$.

Since

$$|I-f'(0)^{-1}f'(x)|\leqq KL|x|\leqq\frac{1}{2},$$

we have

$$f'(x)^{-1}=\sum_{k=0}^{\infty}(I-f'(0)^{-1}f'(x))^k f'(0)^{-1},$$

from which (5.11) follows.

(ii) follows from Cramer's formula and

$$(f^{-1})'(y)=f'(f^{-1}(y))^{-1}. \qquad \text{Q.E.D.}$$

Lemma 5.3. *Let $0<\tau\leqq 3$ and $|\alpha|\geqq 1$. There exist $C_1\geqq 1$ depending only on M_τ, and $C_\alpha>0$ depending only on α and M_τ such that the following holds.*

(i) *$\exp_{\varepsilon,y}^{-1}$ exists on $\{x;\ |x-y|<C_1\varepsilon^{1-\tau/3}\}$ and satisfies*

$$|\partial_x^\alpha \exp_{\varepsilon,y}^{-1}|\leqq C_\alpha\varepsilon^{\min\{\tau/3-|\alpha|+1,0\}}.$$

(ii) *The geodesic distance $r_\varepsilon(x,y)$ can be defined and satisfies*

$$|\partial_x^\alpha[r_\varepsilon(x,y)^2]|\leqq C_\alpha\varepsilon^{\min\{\tau/3-|\alpha|+1,0\}}$$

when $|x-y|<C_1\varepsilon^{1-\tau/3}$.

(iii) *$d_{\varepsilon,1}(x)=\min\{d(x),\ C_1\varepsilon^{1-\tau/3}\}$.*

Proof. (i) and (iii) follow from Lemmas 5.1, 5.2 and (5.3). (ii) follows from (i) and

$$(5.12)\qquad r_\varepsilon(x,y)^2=\sum_{i,j}g_{\varepsilon,ij}(y)v_iv_j,\quad v=\exp_{\varepsilon,y}^{-1}x. \qquad \text{Q.E.D.}$$

For simplicity of notation we put

$$\mathcal{P}_k(\varepsilon)=\varepsilon^{\min\{\tau/3-k,0\}} \quad (k\in \boldsymbol{N}),$$

and have

$$\mathcal{P}_k(\varepsilon)\mathcal{P}_l(\varepsilon)\leqq \mathcal{P}_{k+l}(\varepsilon) \quad (k, l\in \boldsymbol{N}). \tag{5.13}$$

Lemma 5.4. *Let $0<\tau\leqq 3$, $|\alpha|\geqq 0$ and $l\in \boldsymbol{N}$. There exists $C_l>0$ depending only on l and M_τ such that*

$$|\partial_x^\alpha A_{\varepsilon,l}|_0\leqq C_l\mathcal{P}_{2l+|\alpha|}(\varepsilon)$$

when $|x-y|<C_1\varepsilon^{1-\tau/3}$.

Proof. From Lemmas 2.1, 5.2 and 5.3 the lemma follows by induction. Q.E.D.

In order to evaluate $\partial_t^k R_{\varepsilon,N}$ we need the estimate for the heat kernel by Davies.

Lemma 5.5 ([3]). *There exist $C>0$ and $\delta>0$ depending only on M_0 and Ω such that*

$$|\mathcal{K}[\exp(-t\mathcal{A}_\varepsilon)](x,y)|\leqq C\max\{t^{-n/2},1\}\exp\left(-\delta\frac{|x-y|^2}{t}\right)$$

for $t>0$ and $x, y\in\Omega$.

Lemma 5.6. *There exists $C>0$ depending only on M_0 and Ω such that*

$$|e_\varepsilon(\lambda, x, y)|\leqq C\max\{\lambda^{n/2},1\}$$

for $\lambda\geqq 0$ and $x, y\in\Omega$.

Proof. We follow the idea of Hörmander. By Lemma 5.5 we have

$$\begin{aligned} C\max\{t^{-n/2},1\} &\geqq \int_{-0}^{\infty} e^{-t\lambda}d_\lambda e_\varepsilon(\lambda, x, x)\\ &\geqq \int_{-0}^{1/t} e^{-1}d_\lambda e_\varepsilon(\lambda, x, x)=e^{-1}e_\varepsilon(1/t, x, y).\end{aligned}$$

This combined with

$$|e_\varepsilon(\lambda, x, y)|\leqq e_\varepsilon(\lambda, x, x)^{1/2}e_\varepsilon(\lambda, y, y)^{1/2}$$

gives the lemma. Q.E.D.

Lemma 5.7. *Let $K=[n/2]+2$. The integral kernel of*

$$\frac{\sin(t-s)\sqrt{\mathcal{A}_\varepsilon}}{\sqrt{\mathcal{A}_\varepsilon}(\mathcal{A}_\varepsilon+1)^K} \tag{5.14}$$

is uniformly bounded with respect to t, s and ε when (t, s) varies on a compact set T. The bound depends only on M_0, T and Ω. Furthermore it is a C^1 function of (t, s).

Proof. By integration by parts we see that the integral kernel of (5.14) equals

$$-\int_0^\infty F(t, s, \lambda) e_\varepsilon(\lambda, x, y)\, d\lambda$$

where $F(t, s, \lambda)$ satisfies

$$\begin{aligned} F(t, s, \lambda) &= \frac{\partial}{\partial\lambda}\left\{\frac{\sin(t-s)\sqrt{\lambda}}{\sqrt{\lambda}(\lambda+1)^K}\right\} \\ &= (\lambda+1)^{-K}\left\{\frac{(t-s)^3}{2}S(z) - \frac{K \sin z}{\sqrt{\lambda}(\lambda+1)}\right\}\Bigg|_{z=(t-s)\sqrt{\lambda}} \end{aligned}$$

with $S(z) = z^{-3}(z \cos z - \sin z)$ and

$$\begin{aligned} \partial_t F(t, s, \lambda) &= -\partial_s F(t, s, \lambda) \\ &= -(\lambda+1)^{-K}\left\{\frac{(t-s)^2 \sin z}{2z} + \frac{K \cos z}{\lambda+1}\right\}\Bigg|_{z=(t-s)\sqrt{\lambda}}. \end{aligned}$$

Using Lemma 5.6 and

$$\sup_{z \geqq 0} |S(z)| < \infty, \quad \sup_{z \geqq 0} \left|\frac{\sin z}{z}\right| < \infty,$$

we get the lemma. Q.E.D.

Lemma 5.8. *Let $0 < \tau \leqq 3$, $|t| < d_{\varepsilon,1}(x)/3c_{\varepsilon,1}$, $K = [n/2]+2$ and $0 \leqq k \leqq N - 3[n/2] - 5$. Then we have*

$$|\partial_t^k R_{\varepsilon,N}(t, x, x)| \leqq C_N \mathcal{P}_{2N+2K+2}(\varepsilon) t^{2N-2K-k+1},$$

that is,

$$\mathcal{R}_{\varepsilon,N,k}(x) \leqq C_N \mathcal{P}_{2N+2K+2}(\varepsilon)$$

where C_N is a constant depending only on N, M_τ and Ω.

Proof. Noting that $(\mathcal{A}+1)^K$ is written in the form

$$(\mathcal{A}+1)^K = \sum_{|\alpha| \leqq 2K} a_{\varepsilon,\alpha}(x) \partial^\alpha$$

with

$$|a_{\varepsilon,\alpha}(x)| \leqq C \varepsilon^{\min\{\tau - 2K + |\alpha|, 0\}} \leqq C \mathcal{P}_{2K-|\alpha|}(\varepsilon),$$

and using Lemmas 5.3, 5.4 and (5.13) we have

(5.15)
$$|(\mathcal{A}_z+1)^K\partial_s^k Q_{\varepsilon,N}(s,z,x)| \leqq \begin{cases} C\mathcal{P}_{2N+2K+2}(\varepsilon)s^{2N-2K-n-k} & (r_\varepsilon(z,x)^2\leqq s^2)\\ 0 & (r_\varepsilon(z,x)^2\geqq s^2),\end{cases}$$

when $|s|<d_{\varepsilon,1}(x)/3c_{\varepsilon,1}$. Combining (2.17), (5.15) and Lemma 5.7 we get

$$|\partial_t^k R_{\varepsilon,N}(t,x,x)| \leqq \left|\int_0^t ds\int_{r_\varepsilon(z,x)^2\leqq s^2} C\mathcal{P}_{2N+2K+2}(\varepsilon)s^{2N-2K-n-k}\,dz\right| \leqq C\mathcal{P}_{2N+2K+2}(\varepsilon)t^{2N-2K-k+1},$$

which is the desired result. Q.E.D.

6. Proof of Theorem A

Now we are ready to prove Theorem A. Let $0<\tau\leqq 3$. We put $\sigma=1-\tau/3$ and have

$$d_{\varepsilon,1}(x)=\min\{d(x),\, C_1\varepsilon^\sigma\}$$

by Lemma 5.3. We note that $d_{\varepsilon,1}(x)\lambda\geqq 1$ holds when $C_1\varepsilon^\tau\lambda\geqq 1$, $d(x)\geqq C_1\varepsilon^\tau$ and $\tau>\sigma$, or when $C_1\varepsilon^\sigma\lambda\geqq 1$, $d(x)\geqq C_1\varepsilon^\sigma$ and $\tau\leqq\sigma$. Applying Lemma 4.2 to $\mathcal{A}_\varepsilon$ and using (1.3), Lemmas 5.4 and 5.8 we have for $N\geqq 5n$

(6.1)
$$\begin{aligned}|N_\varepsilon(\lambda^2)-\mu_{\mathcal{A}_\varepsilon}(\Omega)\lambda^n| &= \left|\int_\Omega\{E_\varepsilon(\lambda,x,x)-\mu_{\mathcal{A}_\varepsilon}(x)\lambda^n\}dx\right|\\ &= \begin{cases}\left|\int_{d(x)<C_1\varepsilon^\tau}+\int_{C_1\varepsilon^\tau\leqq d(x)<C_1\varepsilon^\sigma}+\int_{C_1\varepsilon^\sigma\leqq d(x)}\right| & (\tau<\sigma)\\ \left|\int_{d(x)<C_1\varepsilon^\sigma}+\int_{C_1\varepsilon^\sigma d\leqq(x)}\right| & (\tau\leqq\sigma)\end{cases}\\ &\leqq C_N\{\varepsilon^\tau\lambda^n+|A_{\varepsilon,0}|_0\lambda^{n-1}\log\varepsilon^{-1}+|A_{\varepsilon,0}|_0\varepsilon^{-\sigma}\lambda^{n-1}\\ &\quad+\sum_{1\leqq l\leqq N}|A_{\varepsilon,l}|_0\lambda^{n-2l}+\mathcal{P}_{2N+2K+2}(\varepsilon)\varepsilon^{\sigma(N+5n-2K+1)}\lambda^{-N+5n}\}\\ &\leqq C_N\{\varepsilon^\tau\lambda^n+\lambda^{n-1}\log\varepsilon^{-1}+\varepsilon^{-\sigma}\lambda^{n-1}+\varepsilon^{(\sigma-2)N-2K-2+\sigma(5n-2K+1)}\lambda^{-N+5n}\}\end{aligned}$$

when $C_1\varepsilon^{\max\{\tau,\sigma\}}\lambda\geqq 1$ and $\varepsilon\lambda\geqq 1$. Here and in what follows we use C_N to denote constants depending only on N, M_τ and Ω.

From (5.1) and the interpolation inequality it follows that

(6.2)
$$B[u,u]\geqq(1+C\varepsilon^\tau)^{-1}(B_\varepsilon[u,u]-C\varepsilon^{1-2}\|u\|_0^2)$$

with some constant C depending only on M_τ and Ω. By the properties of $N(t,B,V,L_2(\Omega))(=N(t,B))$, (5.1), (6.1) and (6.2) we have

(6.3)
$$\begin{aligned}N(\lambda^2,B) &\leqq N(\lambda^2,(1+C\varepsilon^\tau)^{-1}(B_\varepsilon-C\varepsilon^{1-2}))\\ &= N_\varepsilon((1+C\varepsilon^\tau)\lambda^2+C\varepsilon^{1-2})\\ &\leqq \mu_{\mathcal{A}}(\Omega)\lambda^n+C_N\{\varepsilon^\tau\lambda^n+\lambda^{n-1}\log\varepsilon^{-1}+\varepsilon^{-\sigma}\lambda^{n-1}\\ &\quad+\varepsilon^{(\sigma-2)N-2K-2+\sigma(5n-2K+1)}\lambda^{-N+5n}\}\end{aligned}$$

when $C_1\varepsilon^{\max\{\tau,\sigma\}}\lambda\geqq 1$ and $\varepsilon\lambda\geqq 1$. Now we take λ_0 sufficiently large and put $\varepsilon=\lambda^{-3/(2\tau+3)}$ for $\lambda\geqq\lambda_0$ in (6.3). Since

$$1-\frac{3\sigma}{2\tau+3}=\frac{3\tau}{2\tau+3}, \quad -\frac{3(\sigma-2)}{2\tau+3}-1=\frac{-\tau}{2\tau+3}<0,$$

taking N sufficiently large we get

$$N(\lambda^2)-\mu_{\mathcal{A}}(\Omega)\lambda^n\leqq C\{\lambda^{n-2\tau/(2\tau+3)}+\lambda^{n-1}\log\lambda\}$$

for $\lambda\geqq\lambda_0$. In the same way we get the estimate from below. Hence we obtain Theorem A, although we have assumed $V=H_0^1(\Omega)$ or $H^1(\Omega)$ in the case (*a-N*).

For general V in the case (a-N) Theorem A is also valid, since

$$N(t, B, H_0^1(\Omega))\leqq N(t, B, V)\leqq N(t, B, H^1(\Omega)).$$

Thus we complete the proof of Theorem A.

7. Proof of Theorem B

We begin with the proof of the following.

Proposition 7.1. *Let Ω be a bounded domain satisfying* $\sup_{\varepsilon>0}|\Gamma_\varepsilon|/\varepsilon<\infty$. *Let $\{\mathcal{A}_\varepsilon\}_{0<\varepsilon<\varepsilon_0}$ be a family of self-adjoint operators bounded from below on $L_2(\Omega)$ satisfying the following conditions.*

(I) *The operator $\mathcal{A}_\varepsilon$ is a realization of a uniformly elliptic operator. More precisely, the domain of definition $D(\mathcal{A}_\varepsilon)\supset C_0^\infty(\Omega)$ and*

$$\mathcal{A}_\varepsilon u=\sum_{|\alpha|\leqq 2m}a_{\varepsilon,\alpha}(x)D^\alpha u \qquad \text{for} \quad u\in C_0^\infty(\Omega)$$

with $a_{\varepsilon,\alpha}\in\mathcal{B}^\infty(\Omega)$ and

$$\sum_{|\alpha|=2m}a_{\varepsilon,\alpha}(x)\xi^\alpha\geqq\delta_0|\xi|^{2m} \qquad \text{for} \quad x\in\Omega,\ \xi\in\mathbf{R}^n,$$

where $\delta_0>0$ is a constant independent of x, ξ and ε.

(II) *The heat kernel of $\mathcal{A}_\varepsilon$ satisfies*

$$|\mathcal{K}[e^{-t\mathcal{A}_\varepsilon}](x,y)|\leqq C_2t^{-n/2m}\exp\left\{C_3\varepsilon^{-2m}t-\delta\left(\frac{|x-y|^{2m}}{t}\right)^{1/(2m-1)}\right\}$$

for any $x, y\in\Omega$ and $t>0$ where C_2, C_3 and δ are positive constants independent of x, y, t and ε.

(III) *For some τ, $0<\tau\leqq 1$ the coefficients $a_{\varepsilon,\alpha}(x)$ satisfy*

$$|\partial^\beta a_{\varepsilon,\alpha}(x)|\leqq C_{\alpha\beta}\varepsilon^{\min\{\tau-2m+|\alpha|-|\beta|,0\}}$$

where $C_{\alpha\beta}>0$ *are constants independent of* x *and* ε.

Then we get the estimate for the trace $U_\varepsilon(t)$ *of the heat kernel*:

$$|U_\varepsilon(t)-c_{n,m}\mu_{\mathcal{A}_\varepsilon}(\Omega)t^{-n/2m}|\leqq Ct^{(1-n)/2m}(\log t^{-1}+\varepsilon^{\tau-1}) \tag{7.1}$$

for $0<t\leqq\min\{\varepsilon^{2m}, 1/2\}$ *where* $c_{n,m}=\Gamma(n/2m+1)$ *and* $C>0$ *is a constant independent of* t *and* ε.

To prove Proposition 7.1 we construct a parametrix for the heat equation. We set

$$a_j(x, \xi)=\sum_{|\alpha|=j} a_{\varepsilon,\alpha}(x)\xi^\alpha \quad (0\leqq j\leqq 2m), \quad a(x, \xi)=a_{2m}(x, \xi).$$

Let $z\in\Omega$ be fixed. Choose $\kappa\in C_0^\infty(\boldsymbol{R}^n)$ satisfying $\operatorname{supp}\kappa\subset\{x;\ |x|<1\}$ and $\kappa(x)=1$ for $|x|\leqq 1/2$, and set

$$\psi(x)=\kappa\Big(\frac{x-z}{d(z)}\Big).$$

For $t>0$ and $f\in C_0^\infty(\Omega)$ with $\operatorname{supp} f\subset\{x;\ |x-z|<d(z)/2\}$ we define

$$H_t f(x)=(2\pi)^{-n}\iint_{R^{2n}} e^{i(x-y)\xi}e^{-ta(x,\xi)}f(y)\,dy\,d\xi.$$

Then from the theory of pseudo-differential operators ([7]) we have

$$\begin{cases}\Big(\dfrac{\partial}{\partial t}+\mathcal{A}_\varepsilon\Big)(\psi H_t f)=R_{t,z}f \\ \psi H_t f\to\psi f=f \quad \text{as} \quad t\to+0\end{cases} \tag{7.2}$$

in $L_2(\Omega)$ and

$$R_{t,z}f\in C^1([0, \infty), L_2(\Omega)) \tag{7.3}$$

where $R_{t,z}$ is the pseudo-differential operator with symbol

$$\sum_{k,\alpha}\frac{1}{\alpha!}\partial_\xi^\alpha a_{2m-k}D_x^\alpha\{\psi(x)e^{-ta(x,\xi)}\}$$
$$=\sum_{k,\alpha,\beta}\frac{1}{\beta!(\alpha-\beta)!}\partial_\xi^\alpha a_{2m-k}\cdot D_x^\beta\psi(x)\cdot D_x^{\alpha-\beta}\{e^{-ta(x,\xi)}\}$$

with the sum taken over $0\leqq k\leqq 2m$, $0\leqq|\alpha|\leqq 2m-k$, $\beta\leqq\alpha$ and $k+|\alpha|\geqq 1$. When $\alpha>\beta$ we have

$$D_x^{\alpha-\beta}(e^{-ta})=\sum_{\gamma_1+\cdots+\gamma_l=\alpha-\beta}C_{\gamma_1,\cdots,\gamma_l}\prod_{j=1}^{l}D_x^{\gamma_j}(-ta(x, \xi))e^{-ta(x,\xi)}$$

with the sum taken over $|\gamma_j|\geqq 1$ $(1\leqq j\leqq l)$ and $l\geqq 1$ where $C_{\gamma_1,\cdots,\gamma_l}$ is a constant depending only on $\gamma_1, \cdots, \gamma_l$. In view of (7.2), (7.3) and $f\in D(\mathcal{A}_\varepsilon)$ we have

$$e^{-t\mathcal{A}_\varepsilon}f = \psi H_t f - \int_0^t e^{-(t-s)\mathcal{A}_\varepsilon} R_{s,\varepsilon} f\, ds\,,$$

from which it follows that

$$(7.4)\qquad \mathcal{K}[e^{-t\mathcal{A}_\varepsilon}](z, z) = \mathcal{K}[H_t](z, z) - \mathcal{K}\left[\int_0^t e^{-(t-s)\mathcal{A}_\varepsilon} R_{s,\varepsilon} ds\right](z, z)\,.$$

To evaluate the term involving $R_{s,\varepsilon}$ we prepare two lemmas.

Lemma 7.2 ([2]). *For a multi-index α there exist $C>0$ and $\delta>0$ depending only on n, m, α, δ_0 and $C_{\beta 0}$ with $|\beta|=2m$ (see the condition* (III) *of Proposition* 7.1) *such that*

$$\left|\int e^{i(x-y)\xi}\xi^\alpha e^{-ta(x,\xi)}d\xi\right| \leqq Ct^{-(n+|\alpha|)/2m}\exp\left\{-\delta\left(\frac{|x-y|^{2m}}{t}\right)^{1/(2m-1)}\right\}.$$

Lemma 7.3 ([2]). *Let $\alpha>-1$, $\beta>-1$, $\delta>0$ and $q=1/(2m-1)$. Then we have*

$$\int_0^t (t-s)^{-n/2m+\alpha} s^{-n/2m+\beta}\, ds \int_{R^n} \exp\left\{-\delta\left(\frac{|x-z|^{2m}}{t-s}\right)^q - \delta\left(\frac{|z-y|^{2m}}{s}\right)^q\right\} dz$$
$$\leqq Ct^{-n/2m+\alpha+\beta+1}\exp\left\{-\delta'\left(\frac{|x-y|^{2m}}{t}\right)^q\right\}$$

for any δ', $0<\delta'<\delta$ where $C>0$ depends only on, n, m, α, β and δ'.

In particular, when $m=1$ we have

$$\int_0^t (t-s)^{-n/2+\alpha} s^{-n/2+\beta}\, ds \int_{R^n} \exp\left\{-\delta\frac{|x-z|^2}{t-s} - \delta\frac{|z-y|^2}{s}\right\} dz$$
$$= \pi^{n/2}\delta^{-n/2}B(\alpha+1, \beta+1)t^{-n/2+\alpha+\beta+1}\exp\left(-\delta\frac{|x-y|^2}{t}\right)$$

where $B(p, q)$ is the Beta function.

Using Lemma 7.2 we have

$$(7.5)\qquad |\mathcal{K}[R_{t,\varepsilon}](x, y)| \leqq Ct^{-n/2m}\exp\left\{-\delta\left(\frac{|x-y|^{2m}}{t}\right)^{1/(2m-1)}\right\}$$
$$\times \sum_{k,\alpha,\beta} \varepsilon^{\min\{\tau-k,0\}} t^{-(2m-k-|\alpha|)/2m} d(z)^{-|\beta|} \varepsilon^{\min\{\tau-|\alpha|+|\beta|,0\}}\,.$$

We divide the sum in (7.5) into two cases: (i) $k=0$ and $\beta=\alpha$, which imply $|\alpha|\geqq 1$; and (ii) $k>0$ or $\beta<\alpha$. Then (7.5), the condition (II) and Lemma 7.3 yield

$$(7.6)\qquad \mathcal{K}\left[\int_0^t e^{-(t-s)\mathcal{A}_\varepsilon} R_{s,\varepsilon} ds\right](z, z) \leqq Ct^{-n/2m}\exp(C_3\varepsilon^{-2m}t)$$
$$\times\left\{\sum_{|\alpha|\geqq 1} t^{|\alpha|/2m} d(z)^{-|\alpha|} + \sum_{k>0 \text{ or } \beta<\alpha} \varepsilon^{\tau-k-|\alpha|+|\beta|} t^{(k+|\alpha|)/2m} d(z)^{-|\beta|}\right\}.$$

It is easily seen that

$$\mathcal{K}[H_t](z, z) = c_{n,m}\mu_{\mathcal{A}_\varepsilon}(z)t^{-n/2m}. \tag{7.7}$$

By (1.3), (7.4), (7.6), (7.7) and the condition (II) we get

$$\begin{aligned} &|U_\varepsilon(t) - c_{n,m}\mu_{\mathcal{A}_\varepsilon}(\Omega)t^{-n/2m}| \\ &= \left| \int_\Omega \{\mathcal{K}[e^{-t\mathcal{A}_\varepsilon}](z, z) - \mathcal{K}[H_t](z, z)\} dz \right| \\ &\leqq \int_{d(z)<t^{1/2m}} Ct^{-n/2m} + \int_{d(z)\geqq t^{1/2m}} \left| \mathcal{K}\left[\int_0^t e^{-(t-s)\mathcal{A}_\varepsilon} R_{s,z} ds\right](z, z) \right| dz \\ &\leqq Ct^{-n/2m}(t^{1/2m} \log t^{-1} + \varepsilon^{\tau-1}t^{1/2m} + \varepsilon^\tau t^{1/2m} \log t^{-1}) \end{aligned}$$

when $0<t\leqq \min\{\varepsilon^{2m}, 1/2\}$. Then Proposition 7.1 immediately follows.

Now we will derive Theorem B from Proposition 7.1. As was stated in Remark 1.3 we give the proof only for the case (a-D), (a-N) or (c). We may assume that $V=H_0^1(\Omega)$ or $H^1(\Omega)$ in the case (a-N) by the same reason as was stated in the proof of Theorem A. We define the sesquilinear form

$$B_\varepsilon[u, v] = \int_\Omega \sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}^\varepsilon(x) D^\alpha u(x) \overline{D^\beta v(x)}\, dx + (u, v)_{L_2(\Omega)}$$

with

$$a_{\alpha\beta}^\varepsilon = \varphi_\varepsilon * a_{\alpha\beta}$$

where φ_ε is the mollifier defined in Section 5. B_ε is coercive when $0<\varepsilon<\varepsilon_0$ for some $\varepsilon_0>0$. Let $\mathcal{A}_\varepsilon$ be the operator associated with $\{B_\varepsilon, V, L_2(\Omega)\}$. Then it is easily checked that the condition on Ω and the conditions (I), (III) of Proposition 7.1 are satisfied.

In the case (a-D) or (a-N) the condition (II) follows from Lemma 5.5.

In the case (c) we derived (II) in [13], which was not explicitly written, in the process of obtaining the estimate for the kernel of $(\mathcal{A}_\varepsilon + C\varepsilon^{-2m} - \lambda)^{-1}$ by using the L_p-theory ([1], [21]).

Now that all the conditions of Proposition 7.1 have been checked, we get (7.1). As was given in [11], it follows that

$$U(t) \leqq \exp(C\varepsilon^{1-2m}t)U_\varepsilon((1-C\varepsilon^\tau)t).$$

This combined with (7.1) gives

$$U(t) \leqq c_{n,m}\mu_{\mathcal{A}}(\Omega)t^{-n/2m} + C\{\varepsilon^\tau t^{-n/2m} + t^{(1-n)/2m} \log t^{-1}\}$$

when $0<t\leqq \min\{\varepsilon^{2m}, 1/2\}$. Putting $\varepsilon=t^{1/2m}$ we get

$$U(t) - c_{n,m}\mu_{\mathcal{A}}(\Omega)t^{-n/m} \leqq C\{t^{(\tau-n)/2m} + t^{(1-n)/2m} \log t^{-1}\}$$

for sufficiently small t. Similarly we obtain the estimate from below. Thus we complete the proof of Theorem B.

REMARK. The method in this section gives an alternative proof of Theorem B for the case (b) when $V=H_0^m(\Omega)$, since the conditions of Proposition 7.1 are satisfied in this case, too (see [22, Lemma 5.1]).

8. Spectral function in the whole space

This section is devoted to the preparation for the proof of Theorem C. We assume $p_j\in\mathcal{B}^\infty(\boldsymbol{R}^n)$ $(1\leqq j\leqq n)$ in addition to the assumptions of Theorem C, and note that (1.2) holds.

Let us find the spectral function for the operator $\tilde{b}_L$ on $L_2(\boldsymbol{R}^n)$ defined below, which has the same principal symbol as $\mathcal{A}$. We define the differential expression L by

$$L=L(x, D)=(L_1(x, D), \cdots, L_n(x, D)),$$

$$L_j(x, D)=p_j(x)D_j-\frac{i}{2}p_j'(x),\ \ p_j'(x)=\frac{\partial}{\partial x_j}p_j(x)$$

and set

$$b_L(x, D)=\sum_{|\alpha|=|\beta|=m} b_{\alpha\beta}L^{\alpha+\beta},\quad b(\xi)=\sum_{|\alpha|=|\beta|=m} b_{\alpha\beta}\xi^{\alpha+\beta}.$$

From (H2) and the inverse of Gårding's inequality it follows that $b(\xi)>0$ for $\xi\neq0$. Let $\tilde{b}_L$ be the self-adjoint realization of $b_L(x, D)$ on $L_2(\boldsymbol{R}^n)$:

$$D(\tilde{b}_L)=H^{2m}(\boldsymbol{R}^n),\quad \tilde{b}_L u=b_L(x, D)u\qquad \text{for}\quad u\in D(\tilde{b}_L).$$

It is easily seen that the function

$$\mathcal{E}(x, y, \xi)=\prod_{j=1}^n (p_j(x)p_j(y))^{-1/2}\exp\left(i\xi_j\int_{y_j}^{x_j}p_j(s)^{-1}ds\right)$$

satisfies

$$L_j(x, D)\mathcal{E}(x, y, \xi)=\xi_j\mathcal{E}(x, y, \xi)\quad(1\leqq j\leqq n). \tag{8.1}$$

Lemma 8.1. (i) *The resolvent for* $\tilde{b}_L$ *is given by*

$$(\tilde{b}_L-\lambda)^{-1}f(x)=(2\pi)^{-n}\iint_{\boldsymbol{R}^{2n}}\frac{\mathcal{E}(x, y, \xi)}{b(\xi)-\lambda}f(y)\,dy\,d\xi \tag{8.2}$$

for $\lambda\in\boldsymbol{C}\setminus[0, \infty)$ *and* $f\in\mathcal{S}(\boldsymbol{R}^n)$.

(ii) *The spectral function* $\tilde{e}(t, x, y)$ *for* $\tilde{b}_L$ *is given by*

$$\tilde{e}(t, x, y)=(2\pi)^{-n}\int_{b(\xi)<t}\mathcal{E}(x, y, \xi)\,d\xi.$$

In particular, we have

$$\tilde{e}(t, x, x) = \mu_{\mathcal{A}}(x) t^{n/2m} .$$

(iii) *When* $2m>n$,

$$\mathcal{K}[(\tilde{b}_L-\lambda)^{-1}](x, y) = (2\pi)^{-n} \int_{R^n} \frac{\mathcal{E}(x, y, \xi)}{b(\xi)-\lambda} d\xi = \int_0^\infty \frac{d_t \tilde{e}(t, x, y)}{t-\lambda} .$$

Proof. To check the differentiability of the right hand side of (8.2) under integral sings we set

$$F(x, \xi) = \int \mathcal{E}(x, y, \xi) f(y)\, dy .$$

Since $f \in \mathcal{S}(\boldsymbol{R}^n)$ we have

$$\partial_x^\alpha F(x, \xi) = \int \partial_x^\alpha \mathcal{E}(x, y, \xi) f(y)\, dy = \int Q_\alpha(x, \xi) \mathcal{E}(x, y, \xi) f(y)\, dy$$

where $Q_\alpha(x, \xi)$ is a polynomial in ξ satisfying

$$|Q_\alpha(x, \xi)| \leqq C(1+|\xi|^{|\alpha|})$$

for some constant C independent of x and ξ. Using new variables

$$(8.3) \qquad Y_j = \int_0^{y_j} p_j(s)^{-1}\, ds \quad (1 \leqq j \leqq n) ,$$

we easily obtain

$$|\partial_x^\alpha F(x, \xi)| \leqq C(1+|\xi|)^{-n-1} .$$

Hence we can differentiate the right hand side of (8.2) under integral signs. By (8.1), (8.3) and the Fourier inversion formula we have

$$(\tilde{b}_L-\lambda)(2\pi)^{-n} \iint \frac{\mathcal{E}(x, y, \xi)}{b(\xi)-\lambda} f(y)\, dy\, d\xi$$
$$= (2\pi)^{-n} \iint \mathcal{E}(x, y, \xi) f(y)\, dy\, d\xi = f(x) ,$$

from which (8.2) follows.

(ii) follows from (i) and the formula for the resolution of the identity $\{\tilde{E}_t\}$ for $\tilde{b}_L$:

$$\tilde{E}_t = \lim_{\delta \to +0} \lim_{\varepsilon \to +0} \frac{1}{2\pi i} \int_{-\infty}^{t+\delta} \{(\tilde{b}_L-(\sigma+i\varepsilon))^{-1} - (\tilde{b}_L-(\sigma-i\varepsilon))^{-1}\}\, d\sigma .$$

The first equality of (iii) easily follows from (i). By interchanging integrations and integration by parts we have

$$(2\pi)^{-n}\int\frac{\mathcal{E}(x,y,\xi)}{b(\xi)-\lambda}d\xi=(2\pi)^{-n}\int_{R^n}d\xi\int_{b(\xi)}^{\infty}\frac{\mathcal{E}(x,y,\xi)\,dt}{(t-\lambda)^2}$$
$$=\int_0^\infty\frac{\tilde{e}(t,x,y)\,dt}{(t-\lambda)^2}=\int_0^\infty\frac{d_t\tilde{e}(t,x,y)}{t-\lambda},$$

which is the second equality of (iii). Q.E.D.

In the rest of this section we assume $m=1$. Let b_L denote the realization of $b_L(x, D)$ on $L_2(\Omega)$ with the Dirichlet or Neumann boundary condition. Let $x_0\in\Omega$ be fixed. For a function $f\in C_0^\infty(\Omega)$ with $\operatorname{supp} f\subset\{|x-x_0|<d(x_0)/2\}$ let us put

$$\tilde{E}(f;t,x)=\int_0^\infty\cos t\sqrt{\lambda}\,d_\lambda\Big(\int_{R^n}\tilde{e}(\lambda,x,y)f(y)\,dy\Big)$$
$$=(2\pi)^{-n}\iint_{R^{2n}}\cos t\sqrt{b(\xi)}\,\mathcal{E}(x,y,\xi)f(y)\,d\xi\,dy\,. \tag{8.4}$$

The second equality of (8.4) is obtained by letting $\varepsilon\to+0$ in

$$\int_0^\infty e^{-\varepsilon\lambda}\cos t\sqrt{\lambda}\,d_\lambda\Big(\int_{R^n}\tilde{e}(\lambda,x,y)f(y)\,dy\Big)$$
$$=(2\pi)^{-n}\iint_{R^{2n}}e^{-\varepsilon b(\xi)}\cos t\sqrt{b(\xi)}\,\mathcal{E}(x,y,\xi)f(y)\,d\xi\,dy\,,$$

which is verified in the same way as the second equality of Lemma 8.1.(iii).

In view of (8.4) we have $\tilde{E}(f;t,x)\in C^\infty(\boldsymbol{R}\times\boldsymbol{R}^n)$. Further by using new variables (8.3) and applying the theory of pseudo-differential operators it is seen that $\tilde{E}(f;t,x)$ is the solution of the Cauchy problem in $L_2(\boldsymbol{R}^n)$:

$$\begin{cases}\Big(\dfrac{\partial^2}{\partial t^2}+\tilde{b}_L\Big)u(t,x)=0\\ u(0,x)=f(x),\quad \partial_t u(0,x)=0\,.\end{cases} \tag{8.5}$$

From the theory of the wave equation it follows that

$$\operatorname{supp}\tilde{E}(f;t,\cdot)\subset\{x;\,|x-x_0|<d(x_0)\}$$

when $|t|<d(x_0)/2\delta_1$ where δ_1 was defined in (2.1).

Hence $\tilde{E}(f;t,x)$ is also the solution of (8.5) with $\tilde{b}_L$ replaced by b_L. Therefore the spectral function $e(\lambda,x,y)$ for b_L satisfies

$$\int_0^\infty\cos t\sqrt{\lambda}\,d_\lambda\Big(\int_\Omega e(\lambda,x,y)f(y)\,dy\Big)$$
$$=\int_0^\infty\cos t\sqrt{\lambda}\,d_\lambda\Big(\int_\Omega\tilde{e}(\lambda,x,y)f(y)\,dy\Big) \tag{8.6}$$

when $|t|<d(x_0)/2\delta_1$. We define $E(\lambda,x_0,x_0)$ and $\tilde{E}(\lambda,x_0,x_0)$ for b_L and $\tilde{b}_L$ respectively in the same way as in (3.4). Taking the inverse Fourier transform of

(8.6) multiplied by $\hat{\rho}(t)$ with respect to t we obtain

$$\rho_\delta * dE(\lambda, x_0, x_0) = \rho_\delta * d\tilde{E}(\lambda, x_0, x_0) = \mu_{\mathcal{A}}(x_0)\, \rho_\delta * (n|\lambda|^{n-1})$$

where ρ is the function defined in (3.3) and $\delta = d(x_0)/2\delta_1$. Using (4.1), (4.2) and Lemma 4.1 we get the following.

Lemma 8.2. *Let $m=1$. Then*

$$|E(\lambda, x, x) - \mu_{\mathcal{A}}(x)\, \lambda^n| \leqq C d(x)^{-1} \lambda^{n-1}$$

when $d(x)\lambda \geqq 1$ where C is a constant depending only on M_0.

9. Proof of Theorem C in the case (d)

In Sections 9 and 10 we assume the assumptions of Theorem C. We write $L_\varepsilon(x, D)$ for $L(x, D)$ in Section 8 with p_j replaced by $p_{\varepsilon,j} = \varphi_\varepsilon * p_j$ and denote by $\mathcal{A}_\varepsilon$ the operator associated with $\{B_\varepsilon, V, L_2(\Omega)\}$ where

$$B_\varepsilon[u, v] = \int_\Omega \sum_{|\alpha|=|\beta|=m} b_{\alpha\beta}\, L_\varepsilon(x, D)^\alpha u\, \overline{L_\varepsilon(x, D)^\beta v}\, dx\,. \tag{9.1}$$

We note that $p_{\varepsilon,j}$ satisfies

$$|p_{\varepsilon,j}(x) - p_j(x)| \leqq C\varepsilon^\tau, \quad |\partial^\alpha p_{\varepsilon,j}(x)| \leqq C_\alpha \varepsilon^{\min\{\tau - |\alpha|, 0\}} \tag{9.2}$$

for some C depending only on M_τ, and some C_α depending only on α and M_τ.

Lemma 9.1. *In the case* (d) *we have*

$$|\mathcal{K}[e^{-t\mathcal{A}_\varepsilon}](x, y)| \leqq C t^{-n/2} \exp\left(C\varepsilon^{-2} t - \delta \frac{|x-y|^2}{t}\right)$$

where C and δ are positive constants depending only on M_τ and Ω.

Proof. Let $\mathcal{A}'_\varepsilon$ be the operator associated with $\{B'_\varepsilon, V, L_2(\Omega)\}$ where

$$B'_\varepsilon[u, v] = \int_\Omega \sum_{|\alpha,=|\beta|=1} b_{\alpha\beta}\, p(x)^\alpha p(x)^\beta D^\alpha u\, \overline{D^\beta v}\, dx + (u, v)_{L_2(\Omega)}\,.$$

It is easily seen that

$$\mathcal{A}_\varepsilon = \mathcal{A}'_\varepsilon + q \tag{9.3}$$

where

$$q(x) = -\frac{1}{4} \sum_{i,j=1}^n b_{ij}\, p'_{\varepsilon,i}\, p'_{\varepsilon,j} - \frac{1}{2} \sum_{i=1}^n b_{ii}\, p_{\varepsilon,i}\, p''_{\varepsilon,i} - 1\,,$$

$$b_{ij} = b_{e_i e_j} (1 \leqq i, j \leqq n)\,.$$

By Lemma 5.5 we have

$$(9.4)\qquad |\mathcal{K}[\exp(-t\mathcal{A}'_\varepsilon)](x,y)| \leqq C_4 t^{-n/2} \exp\left(-\delta \frac{|x-y|^2}{t}\right).$$

From (9.3) it follows that

$$(9.5)\qquad e^{-t\mathcal{A}_\varepsilon} = e^{-t\mathcal{A}'_\varepsilon} - \int_0^t e^{-(t-s)\mathcal{A}_\varepsilon} q e^{-s\mathcal{A}'_\varepsilon} ds .$$

To solve (9.5) for $e^{-t\mathcal{A}_\varepsilon}$ we set

$$\begin{cases} \Gamma_0(t) = e^{-t\mathcal{A}'_\varepsilon} \\ \Gamma_{k+1}(t) = \int_0^t \Gamma_k(t-s)\, q e^{-s\mathcal{A}'_\varepsilon} ds \quad (k \geqq 0). \end{cases}$$

By induction, (9.4) and Lemma 7.3 we have

$$(9.6)\qquad |\mathcal{K}[\Gamma_k(t)](x,y)| \leqq C_4 \frac{(C_4 \pi^{n/2} \delta^{-n/2} |q|_0)^k}{k!} t^{-n/2+k} \exp\left(-\delta \frac{|x-y|^2}{t}\right).$$

Then it is seen that

$$e^{-t\mathcal{A}_\varepsilon} = \sum_{k=0}^{\infty} (-1)^k \Gamma_k(t)$$

where the series of the right hand side converges in the operator norm and that

$$(9.7)\qquad \mathcal{K}[e^{-t\mathcal{A}_\varepsilon}] = \sum_{k=0}^{\infty} (-1)^k \mathcal{K}[\Gamma_k(t)] .$$

Noting that $|q|_0 \leqq C\varepsilon^{\tau-2}$ and using (9.6) and (9.7) we have

$$|\mathcal{K}[e^{-t\mathcal{A}_\varepsilon}](x,y)| \leqq C_4 t^{-n/2} \exp\left(C\varepsilon^{\tau-2} t - \delta \frac{|x-y|^2}{t}\right),$$

from which the lemma follows. Q.E.D.

Lemma 9.2. *In the case* (d) *we have*

$$|e_\varepsilon(\lambda, x, y)| \leqq C\lambda^{n/2} \exp(C\varepsilon^{-2}\lambda^{-1}).$$

for $\lambda > 0$ *where* $C > 0$ *is a constant depending only on* M_τ *and* Ω.

Proof. The lemma follows from Lemma 9.1 in the same way as Lemma 5.6. Q.E.D.

Now we are ready to prove Theorem C in the case (d). Applying Lemma 8.2 to $\mathcal{A}_\varepsilon$ and using (1.3) and Lemma 9.2 we have

$$\begin{aligned} &|N_\varepsilon(\lambda^2) - \mu_{\mathcal{A}_\varepsilon}(x)\lambda^n| \\ &= \left| \left(\int_{d(x)<\lambda^{-1}} + \int_{d(x)\geqq\lambda^{-1}} \right) \{E_\varepsilon(\lambda, x, x) - \mu_{\mathcal{A}_\varepsilon}(x)\lambda^n\}\, dx \right| \\ &\leqq C(\lambda^{n-1} + \lambda^{n-1}\log\lambda) \end{aligned}$$

for $\lambda \geqq \max\{\varepsilon^{-1}, 2\}$. This combined with

$$B[u, u] \geqq (1+C\varepsilon^{\tau})^{-1}(B_{\varepsilon}[u, u]-C\varepsilon^{\tau-2})$$

yields

$$\begin{aligned} N(\lambda^2) &\leqq N_{\varepsilon}((1+C\varepsilon^{\tau})\lambda^2+C\varepsilon^{\tau-2}) \\ &\leqq \mu_{\mathcal{A}}(\Omega)\lambda^n+C(\varepsilon^{\tau}\lambda^n+\lambda^{n-1}\log\lambda) \end{aligned} \tag{9.8}$$

when $\lambda \geqq \max\{\varepsilon^{-1}, 2\}$. Putting $\varepsilon=\lambda^{-1}$ in (9.8) we get

$$N(\lambda^2)-\mu_{\mathcal{A}}(\Omega)\lambda^n \leqq C(\lambda^{n-1}+\lambda^{n-1}\log\lambda)$$

for sufficiently large λ. Similarly we get the inequality from below. Thus Theorem C has been proved in the case (d).

10. Proof of Theorem C in the case (e)

For the proof in the case (e) we follow the resolvent kernel method.

We write $\tilde{\mathcal{A}}_{\varepsilon}$ for $\tilde{b}_L$ in Section 8 with p_j replaced by $p_{\varepsilon,j}$. Then Lemma 8.1 gives

$$\mathcal{K}[(\tilde{\mathcal{A}}_{\varepsilon}-\lambda)^{-1}](x, x) = \frac{n\pi/2m}{\sin(n\pi/2m)}\mu_{\mathcal{A}_{\varepsilon}}(x)(-\lambda)^{-1+n/2m}. \tag{10.1}$$

If we write (9.1) in the form

$$B_{\varepsilon}[u, v] = \int_{\Omega} \sum_{|\alpha|,|\beta|\leqq m} a^{\varepsilon}_{\alpha\beta}(x)\, D^{\alpha}u\, \overline{D^{\beta}v}\, dx,$$

we have

$$\begin{aligned} &|a^{\varepsilon}_{\alpha\beta}(x)-a_{\alpha\beta}(x)| \leqq C\varepsilon^{\tau} \quad (|\alpha|=|\beta|=m), \\ &|\partial^{\gamma} a^{\varepsilon}_{\alpha\beta}(x)| \leqq C_{\gamma}\,\varepsilon^{\min\{\tau-2m+|\alpha|+|\beta|-|\gamma|,0\}} \quad (|\alpha|, |\beta| \leqq m). \end{aligned} \tag{10.2}$$

By (H2), (10.2) and the interpolation inequality we have

$$B_{\varepsilon}[u, u] \geqq \frac{\delta_0}{2}\|u\|_m^2-C\varepsilon^{\tau-2m}\|u\|_0^2 \tag{10.3}$$

for $0<\varepsilon<\varepsilon_0$ if we choose $\varepsilon_0>0$ sufficiently small. Then it follows that

$$|B_{\varepsilon}[u, u]-\lambda(u, u)| \geqq C_5\frac{|\lambda|}{d(\lambda)}(\|u\|_m+|\lambda|^{1/2}\|u\|_0)^2 \tag{10.4}$$

when $|\lambda| \geqq C_6\,\varepsilon^{\tau-2m}$ for some constants C_5 and C_6 depending only on M_{τ} and Ω. If we replace Ω with $\boldsymbol{R}^n$, (10.3) and (10.4) are also valid.

Lemma 10.1. *For any $p>0$ there exists $C_p>0$ depending only on p, M_{τ} and Ω such that*

$$|\mathcal{K}[(\mathcal{A}_\varepsilon-\lambda)^{-1}](x,x)-\mathcal{K}[(\tilde{\mathcal{A}}_\varepsilon-\lambda)^{-1}](x,x)|$$
$$\leqq C_p\frac{|\lambda|^{n/2m}}{d_2(\lambda)}\left(\frac{|\lambda|^{1-1/2m}}{d(x)\,d_2(\lambda)}\right)^p$$

when $0<\varepsilon<\varepsilon_0$, $\lambda\in\boldsymbol{C}\backslash[0,\infty)$ *and* $|\lambda|\geqq\varepsilon^{-2m}$ *where* $d_2(\lambda)=\mathrm{dist}\,(\lambda,(0,\infty])$.

Proof. The lemma is essentially due to Maruo and Tanabe [8, Lemmas 4.1, 4.2 and 7.2]. Taking care of the dependence on ε, using (10.2) and (10.4) and following the argument of [8] we obtain the lemma if we add the condition $|\lambda|\geqq\varepsilon^{-2m}$.

For example, in the proof of [8, Lemma 4.1] we have to estimate

$$I_1=\left|\int_\Omega\sum_{|\alpha|,|\beta|\leqq m}a^\varepsilon_{\alpha\beta}(x)\sum_{\gamma<\alpha}\binom{\alpha}{\gamma}D^{\alpha-\gamma}\psi\,D^\gamma u\,\overline{D^\beta v}\,dx\right|$$

and

$$I_2=\left|\int_\Omega\sum_{|\alpha|,|\beta|\leqq m}a^\varepsilon_{\alpha\beta}(x)\sum_{\gamma<\beta}\binom{\beta}{\gamma}D^\alpha u\,D^{\beta-\gamma}\psi\,\overline{D^\gamma v}\,dx\right|$$

for a fixed $z\in\Omega$ where ψ was defined in Section 7. Using (10.2) and the interpolation inequality ([8, Lemma 3.4])

$$\|v\|_k\leqq C|\lambda|^{-(m-k)/2m}(\|v\|_m+|\lambda|^{1/2}\|v\|_0)\,,$$

we have

$$\begin{aligned}I_1&\leqq C\sum\varepsilon^{-2m+|\alpha|+|\beta|}\,d(z)^{-|\alpha|+|\gamma|}\,\|u\|_{|\gamma|}\,\|v\|_{|\beta|}\\&\leqq C\sum(\varepsilon|\lambda|^{1/2m})^{-2m+|\alpha|+|\beta|}\,(d(z)|\lambda|^{1/2m})^{-|\alpha|+|\gamma|}\\&\quad\times(\|u\|_m+|\lambda|^{1/2}\|u\|_0)\,(\|v\|_m+|\lambda|^{1/2}\|v\|_0)\\&\leqq C(d(z)|\lambda|^{1/2m})^{-1}(\|u\|_m+|\lambda|^{1/2}\|u\|_0)\,(\|v\|_m+|\lambda|^{1/2}\|v\|_0)\end{aligned}$$

when $|\lambda|\geqq\max\{\varepsilon^{-2m},d(z)^{-2m}\}$. In the similar manner I_2 can be evaluated. Based on these estimates the lemma can be obtained. Q.E.D.

Now we apply Pleijel's formula ([16]) to $e_\varepsilon(\cdot,x,x)$. In its original form Pleijel's formula needs the assumption that the support of $e_\varepsilon(\cdot,x,x)$ is contained in $[0,\infty)$, but we may weaken it to the assumption that the support of $e_\varepsilon(\cdot,x,x)$ is bounded from below. Hence we have

$$\left|e_\varepsilon(t,x,x)-\frac{1}{2\pi i}\int_{L(\lambda)}\mathcal{K}[(\mathcal{A}_\varepsilon-z)^{-1}](x,x)\,dz\right|\leqq2\tau|\mathcal{K}[(\mathcal{A}_\varepsilon-\lambda)^{-1}](x,x)|\tag{10.5}$$

for $t\geqq\varepsilon^{-2m}$, $\tau>0$ and $\lambda=t+i\tau$ where $L(\lambda)$ is an oriented curve in $\boldsymbol{C}$ from $\bar{\lambda}$ to λ not intersecting the interval $[-\varepsilon^{-2m},\infty)$.

By (10.1), (10.5), Lemma 10.1 and the same calculation as in the proof of [8, Lemma 8.3] we get

$$|e_\varepsilon(t, x, x) - \mu_{\mathcal{A}_\varepsilon}(x)\, t^{n/2m}| \leqq C_\theta\, \delta(x)^{-\theta}\, t^{(n-\theta)/2m} \tag{10.6}$$

for any $0<\theta<1$ when $t \geqq \varepsilon^{-2m}$ where C_θ depends only on θ, M_τ and Ω. Putting $\theta=\tau$ and integrating (10.6) on Ω we have

$$|N_\varepsilon(t) - \mu_{\mathcal{A}_\varepsilon}(\Omega)\, t^{n/2m}| \leqq Ct^{(n-\tau)/2m}$$

when $t \geqq \varepsilon^{-2m}$. Then the same calculation as in the case (d) leads us to Theorem C in the case (e).

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