Multiobject operational tasks for measurement incompatibility

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We introduce *multiobject* operational tasks for measurement incompatibility in the form of multiobject quantum subchannel discrimination and exclusion games with prior information, where a player can simultaneously harness the resources contained within both a quantum state and a set of measurements. We show that any fully or partially resourceful pair of objects is useful for a suitably chosen multiobject subchannel discrimination and exclusion game with prior information. The advantage provided by a fully or partially resourceful object against all possible fully free objects in such a game can be quantified in a *multiplicative* manner by the resource quantifiers of generalized robustness and the weight of a resource for discrimination and exclusion games, respectively. These results hold for arbitrary properties of quantum states as well as for arbitrary properties of sets of measurements closed under classical pre- and postprocessing and, consequently, include measurement incompatibility as a particular case. We furthermore show that these results are not exclusive to quantum theory, but can also be extended to the realm of general probabilistic theories.

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I. INTRODUCTION

The theoretical framework of quantum resource theories (QRTs) [1] has consolidated itself during the past two decades as a fruitful approach to quantum information where quantum properties of physical systems are harnessed for the benefit of operational tasks. A quantum resource theory can be specified by first defining a quantum *object* of interest, followed by one property of such objects to be exploited as a *resource* [1]. The two most explored QRTs are arguably those of quantum states [2,3] and measurements [4–6]. QRTs of states explore desirable properties such as: entanglement [3], coherence [7], asymmetry [7], superposition [8], purity [9], magic [10], among many others [11–16]. Similarly, QRTs of measurements explore properties such as: entanglement [5], coherence [5], informativeness [17–19], and nonprojective simulability [20]. There exist, however, additional desirable resources contained within more general types of objects such as: sets of measurements [21,22], behaviors or boxes [23,24], steering assemblages [25], teleportation assemblages [26], and channels [27–30], among many others [1,31–34]. In this work, we focus on QRTs of *sets of measurements*, with the resource of *measurement incompatibility* in particular. Measurement incompatibility [21,22] is a property lying at the foundations of quantum mechanics that acts as a parent resource for the properties of Bell-nonlocality [35] and EPR-steering [36,37] and, as a consequence of this, acts as a prerequisite for practical applications relying on fully device-independent as well as semi-device-independent protocols [38–43].

A broad research area of interest within QRTs concerns the development of operational tasks harnessing the resources contained within quantum objects. There are several results quantifying the advantage provided by resources contained within quantum measurements, many of which address this quantification in terms of *advantage ratios* and connecting this to *resource measures*. Three specific approaches in this regard are the following. First, in the particular case of the QRT of incompatible sets of measurements, it has been found that incompatibility serves as fuel for *single-object* quantum state discrimination tasks [44–49]. Second, operational tasks for QRTs of single measurements have been explored for general resources as well as explored more broadly within general probabilistic theories (GPTs) [50]. Third, it has also been pointed out that there exist operational tasks that can simultaneously exploit the resources contained within *multiple* objects, or *multiobject* operational tasks for short, when considering composite QRTs of states and individual measurements [51–54]. Despite the relevance of these three approaches for the development of QRTs of measurements, they still contain various shortcomings from the point of view of QRTs of POVM sets with general resources. In particular, a major shortcoming concerns with the fact that current *multi-*

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TABLE I. Comparison between this work (last column) and Refs. [44–47,50,51], in terms of the following five features pertaining to the quantification of advantage ratios in quantum resource theories of measurements. As a first feature, we ask whether the statement deals with POVM sets. Second, we ask if the statement is proven for general resources represented by closed convex cones. Third, whether the statement allows for the simultaneous implementation of various quantum objects (multiobject). Fourth, whether the statement is proven to hold for GPTs. Fifth, whether an exclusion dual of the discrimination case is proven.

	Features \ Reference	Refs. [44–47]	Ref. [50]	Ref. [51]	This work (Result 3)
1	POVM sets	✓	✗	✗	✓
2	General resources	✗	✓	✓	✓
3	Multiobject	✗	✗	✓	✓
4	GPT-extension	✗	✓	✗	✓
5	Exclusion dual	✗	✗	✓	✓

object tasks are specifically tailored to exploit the resources contained within states and *individual* measurements and, consequently, they currently do not accommodate for desirable properties of *sets* of measurements such as measurement incompatibility.

In this work we resolve these shortcomings by introducing *multiobject* operational tasks for *general resources* of *POVM sets* (including incompatibility) in the form of *multiobject* quantum subchannel discrimination and exclusion games with prior information. We characterize the advantage provided by resourceful sets of measurements when playing these operational tasks in terms of resource quantifiers, and furthermore extend these results to the framework of GPTs [55–61]. The construction in this paper simultaneously extends the previously described three approaches, thus resolving the shortcomings present in them. In Table I, we compare these approaches and shortcomings thereof, and we further describe this in what follows.

In order to compare these approaches, it is useful to introduce the following five features as points of comparison. First, a statement deals with *POVM sets*, whenever it addresses properties of *sets* of measurements, such as incompatibility, as opposed to *single-POVM* QRTs, where only one POVM is considered at a given time. Second, a statement deals with *general resources*, whenever the operational task in question can potentially harness general resources represented by closed convex cones, sometimes addressed in the literature simply as *convex resources,* for short. Third, a statement is *multiobject*, whenever the operational task allows for the resources contained within multiple quantum objects to be harnessed simultaneously. Fourth, a statement allows for a *GPT-extension*, whenever a similar statement can be derived more generally within the formalism of GPTs. Fifth and finally, a statement involving a discrimination task has an *exclusion dual*, whenever there exists a dual result, which now connects the measure of *weight of resource* to the respective exclusion task opposite to the discrimination case under consideration. In Table I we use these five features to provide a comparison between the results in this work and various other results in the literature that also explore the quantification of advantage ratios in QRTs of measurements.

Having introduced these five features, we now compare Refs. [44–47,50,51], and the present work. First, the results in [44–47] quantify the advantage provided by measurement incompatibility in the operational tasks of discrimination games with prior information. These works clearly deal with *sets* of measurements, but do not address any of the other four features described above. Second, the results in [50] address: QRTs of *single* measurements, *general* resources, extensions to GPTs, but do not explore either the *multiobject* approach or the discrimination-exclusion duality. Third, the results in [51] address: QRTs of quantum states and *single* measurements, *multiobject* operational tasks, the discrimination-exclusion duality, but do not address situations either involving multiple POVMs, or extensions of these connections to GPTs. Finally, the results in the present paper incorporate these five features, and it can therefore be seen as a generalization (unification) of either (all) of the three approaches under consideration.

This paper is organized as follows. We first address measurement incompatibility and classical pre- and postprocessing. We then introduce multiobject operational tasks for states and sets of measurements. We then introduce multiobject operational tasks in the form of discrimination games, exclusion games, and furthermore extend these results to the realm of general probabilistic theories. We end up with conclusions and perspectives.

II. MEASUREMENT INCOMPATIBILITY AND CLASSICAL PRE-AND POSTPROCESSING

Let $\mathcal{L}(\mathcal{H})$, $\mathcal{L}_S(\mathcal{H})$, and $\mathcal{L}_S^+(\mathcal{H})$, be the sets of all bounded, self-adjoint, and positive semidefinite operators, respectively, on a finite-dimensional complex Hilbert space $\mathcal{H}$. A quantum state, or density operator, is a trace one positive semidefinite operator. The set of all states is denoted as $\mathcal{D}(\mathcal{H}) := \{\rho \in \mathcal{L}_S^+(\mathcal{H}) \mid \mathrm{Tr}[\rho] = 1\}$. A quantum measurement or normalized positive operator-valued measure (POVM) is a set of positive semidefinite operators $\{M_a\}_{a=1}^l$ satisfying $\sum_{a\in A} M_a = \mathbb{1}$, with $A = \{1, \dots, l\}$, and $\mathbb{1}$ the identity operator on $\mathcal{H}$. More formally, a measurement M can be stated to be a map from a σ-algebra on an outcome set $A = \{1, \dots, l\}$ to $\mathcal{L}_S^+(\mathcal{H})$, and so $\mathsf{M} = \{M_a\}_{a=1}^l$. The operator $M_a = \mathsf{M}(\{a\}) \in \mathcal{L}_S^+(\mathcal{H})$ is denoted as the POVM element or effect corresponding to the specific outcome $a \in A$. A *POVM set* is a set of POVMs $\{\mathsf{M}_x\}_{x=1}^\kappa$, with each POVM having the same outcome space A and so, for convenience, we write $\{\mathsf{M}_x\}_{x=1}^\kappa \equiv \mathbb{M}_{A|X} = \{M_{a|x}\}_{a,x}$, where $M_{a|x} = \mathsf{M}_x(\{a\})$, and thus $\sum_{a\in A} M_{a|x} = \mathbb{1}$, $\forall x \in \{1, \dots, \kappa\}$. We will use the further simplified notation $\{M_{a|x}\} \equiv \{M_{a|x}\}_{a,x}$.

Let us also invoke the following notation from probability theory. Let $(X, G, \dots)$ be random variables on a finite alphabet $\mathcal{X}$, and the probability mass function (PMF) of a random variable X represented as p_X satisfying: $p_X(x) \geqslant 0$, $\forall x \in \mathcal{X}$, and $\sum_{x\in\mathcal{X}} p_X(x) = 1$. For simplicity, we address $p_X(x)$ as $p(x)$ when evaluating, and omit the alphabet when summing. Joint and conditional PMFs are denoted as p_{XG}, $p_{G|X}$, respectively. With this notation in place, let us now address the property of *incompatibility* of POVM sets. A POVM set $\mathbb{M}_{A|X} = \{M_{a|x}\}$ is compatible whenever there exist a parent POVM $\mathbb{G} = \{G_\lambda\}$

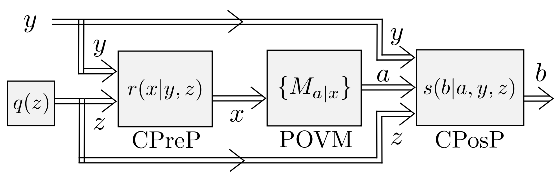

FIG. 1. Classical preprocessing (CPreP) and postprocessing (CPosP) of the POVM set $\mathbb{M}_{A|X} = \{M_{a|x}\}$ into a new POVM set $\mathbb{N}_{B|Y} = \{N_{b|y}\}$ with each $N_{b|y}$ as in (2). A random variable z is generated according to the PMF q_Z. The preprocessing stage consists of choosing a classical input x according to a PMF that depends on the classical input y and z as $r_{X|YZ}$. The postprocessing consists of processing the measurement outcome a according to a PMF that depends on y and z as $s_{B|AYZ}$. We obtain the pre- and postprocessed set of measurements $\mathbb{N}_{B|Y} = \{N_{b|y}\}$.

and a PMF $p_{A|X\Lambda}$ such that

$$M_{a|x} = \sum_{\lambda} p(a|x,\lambda)\, G_\lambda, \quad \forall a, x, \tag{1}$$

and it is called incompatible otherwise. This property of POVM sets is commonly also refereed to as measurement compatibility or joint measurability. For simplicity, in this work we stick to measurement (in)compatibility or simply (in)compatibility. It will be useful to introduce the operation of *simulability* of POVM sets. We say that a POVM set $\mathbb{N}_{B|Y} = \{N_{b|y}\}$, $b \in \{1,\dots,m\}$, $y \in \{1,\dots,\tau\}$ is simulable by the POVM set $\mathbb{M}_{A|X} = \{M_{a|x}\}$, $a \in \{1,\dots,l\}$, $x \in \{1,\dots,\kappa\}$ whenever there exist a triplet of PMFs q_Z, $r_{X|YZ}$, and $s_{B|AYZ}$ such that [20]

$$N_{b|y} = \sum_{a,x,z} s(b|a,y,z)\, M_{a|x}\, r(x|y,z)\, q(z), \quad \forall b, y. \tag{2}$$

Simulability of POVM sets can be thought of as composed of a classical preprocessing (CPreP) stage, represented by the PMFs q_Z and $r_{X|YZ}$, and a classical postprocessing (CPosP) stage, represented by the PMF $s_{B|AYZ}$. The triplet of PMFs allowing the simulation are going to be refereed to as the set of *strategies* $\mathcal{S} = \{q_Z, r_{X|YZ}, s_{B|AYZ}\}$. In Fig. 1, we illustrate the simulability of POVM sets. One can check that the simulability of POVM sets defines a partial order for POVM sets and therefore, this motivates the notation $\mathbb{N}_{B|Y} \preceq \mathbb{M}_{A|X}$, meaning that $\mathbb{N}_{B|Y}$ is simulable by $\mathbb{M}_{A|X}$. It is straightforward to check that CPreP and CPosP are operations that take compatible POVM sets into compatible POVM sets and therefore, in this sense, we say measurement compatibility is closed under simulation of POVM sets. We now move on to introducing the concept of an ensemble of instruments. A quantum channel is a map $\phi : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})$ that is completely positive (CP) and trace-preserving (TP). A quantum subchannel is a map $\phi : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})$ that is completely positive (CP) and trace-nonincreasing (TNI). A quantum instrument is a set of subchannels $\Phi = \{\phi_b\}_{b=1}^m$ such that $\sum_{b=1}^m \phi_b$ is a channel, $\phi_b = \Phi(\{b\})$, $\forall b \in B$. Instruments are mathematical representations of quantum measurement processes. Given an instrument $\Phi = \{\phi_b\}_{b=1}^m$, we can construct a measurement model such that the output probabilities and postmeasurement states for a given target state ρ are respectively $\{\mathrm{Tr}[\phi_b(\rho)]\}_b$ and $\{\phi_b(\rho)/\mathrm{Tr}[\phi_b(\rho)]\}_b$ [62,63]. An instrument set is a set of instruments $\{\Phi_y\}_{y=1}^\tau$ with each instrument having the same outcome set B. Similarly to the case of POVM sets, we can alternatively write an instrument set as $\Phi_{B|Y} = \{\phi_{b|y}\}_{b,y}$, where $\phi_{b|y} = \Phi_y(\{b\})$, and thus $\sum_{b=1}^m \phi_b$ is a channel $\forall y \in \{1,\dots,\tau\}$, $B = \{1,\dots m\}$. Finally, an ensemble of instruments is a pair $(p_Y, \Phi_{B|Y})$ with $\Phi_{B|Y}$ an instrument set and p_Y a PMF. We now introduce multiobject operational tasks that are meant to be played with two quantum objects, a quantum state and a POVM set.

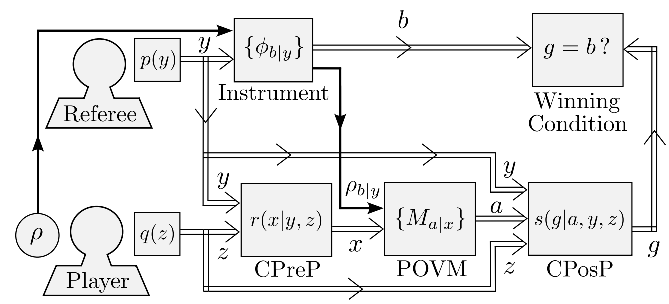

FIG. 2. Multiobject quantum subchannel discrimination with prior information. Solid lines represent quantum information (states), and double lines represent classical information. The player is in possession of a fixed quantum state ρ and a POVM set $\mathbb{M}_{A|X} = \{M_{a|x}\}$. The referee has an instrument set $\Phi_{B|Y} = \{\phi_{b|y}(\cdot)\}$. The player sends state ρ to the referee, who then implements one of the instruments $\{\phi_{b|y}(\cdot)\}$, according to the PMF p_Y. The output of the instrument is the measurement outcome b and a postmeasured state $\rho_{b|y}$. The referee sends both state $\rho_{b|y}$ and label y (prior information) back to the player. The goal of the game is for the player to correctly guess the output b. The player then proceeds to use both y and $\rho_{b|y}$, together with his set of measurements $\mathbb{M}_{A|X} = \{M_{a|x}\}$, in order to generate a guess index g. The player is allowed to do pre- and postprocessing (2). The player then sends the guess g to the referee, who then checks the winning condition $(g = b)$.

III. MULTI-OBJECT OPERATIONAL TASKS FOR STATES AND POVM SETS

We now introduce the operational task of multiobject quantum subchannel discrimination game with prior information (QScD-PI), as a generalization of the single-object state discrimination tasks with prior information first introduced in [64]. We illustrate this multiobject game in Fig. 2 and describe it as follows. The game is played by two parties, Alice (the player) and Bob (the referee). Alice is in possession of two quantum objects: a state ρ and a set of κ POVMs $\{\mathsf{M}_x\}_{x=1}^\kappa \equiv \mathbb{M}_{A|X} = \{M_{a|x}\}$. Here, all the POVMs are considered with the same outcome set $A = \{1,\dots,l\}$ and $M_{a|x} \in \mathcal{L}_S^+(\mathcal{H})$, $\forall a, x$. Bob, on the other hand is in possession of a set of τ instruments $\{\Phi_y\}_{y=1}^\tau \equiv \Phi_{B|Y} = \{\phi_{b|y}\}$. All instruments are considered with the same outcome set $B = \{1,\dots,m\}$, and $\phi_{b|y} : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})$, $\forall b, y$. The first step of the game is for Alice (player) to prepare a quantum state ρ and send it to Bob (referee). Bob then proceeds to implement one of the instruments $\Phi_{B|Y}$, say $\Phi_y = \{\phi_{b|y}\}_y$, according to the PMF q_Y, on the set $Y = \{1,\dots,\tau\}$, for which we assume that $q(y) \neq 0$, $\forall y \in Y$. Bob then conducts the measurement associated to Φ_y on ρ, observing an outcome $b \in B$ and a postmeasurement state $\rho_{b|y} := \phi_{b|y}(\rho)/\mathrm{Tr}[\phi_{b|y}(\rho)]$. After Bob's measurement, Alice is informed of Bob's choice y (prior information), and is also sent the state $\rho_{b|y}$. Alice's goal is to correctly guess the label b of the sub-channel $\phi_{b|y}$. In order to do this, Alice first generates a random variable z according to a PMF q_Z, and uses this to determine the choice of measurement M_x with a probability $r(x|y,z)$. Alice then proceeds to measure $\mathsf{M}_x = \{M_{a|x}\}$ on $\rho_{b|y}$, and observes an outcome a. Alice is allowed to classically postprocess this measurement outcome to generate a guess $g \in B$ of b, according to $s(g|a,y,z)$. Finally, Alice sends her guess g to Bob, and wins the game whenever $g = b$. We address this task as *multiobject* quantum subchannel discrimination with prior information (QScD-PI). The maximum probability of success in such a task is given by

$$P^{\mathrm{D}}_{\mathrm{succ}}(p_Y, \Phi_{B|Y}, \rho, \mathbb{M}_{A|X}) := \max_{\mathcal{S}} \sum_{g,a,x,\mu,b,y} \delta_{g,b}\, s(g|a,y,z) \times \mathrm{Tr}[M_{a|x}\phi_{b|y}(\rho)]\, r(x|y,z)\, q(z)\, p(y), \tag{3}$$

with the maximization over all possible strategies $\mathcal{S} = \{q_Z, r_{X|YZ}, s_{G|AYZ}\}$. In a multiobject quantum subchannel *exclusion* game with prior information (QScE-PI) on the other hand, the goal is for Alice (player) to output a guess $g \in \{1,\dots,m\}$ for a subchannel that did not take place, that is, Alice succeeds at the game if $g \neq b$ and fails when $g = b$. The minimum probability of error in quantum subchannel exclusion with prior information is

$$P^{\mathrm{E}}_{\mathrm{err}}(p_Y, \Phi_{B|Y}, \rho, \mathbb{M}_{A|X}) := \min_{\mathcal{S}} \sum_{g,a,x,\mu,b,y} \delta_{g,b}\, s(g|a,y,z) \times \mathrm{Tr}[M_{a|x}\phi_{b|y}(\rho)]\, r(x|y,z)\, q(z)\, p(y), \tag{4}$$

with the minimization over all possible strategies $\mathcal{S} = \{q_Z, r_{X|YZ}, s_{G|AYZ}\}$. A multiobject quantum subchannel discrimination/exclusion game is specified by the *ensemble of instruments* $\{p_Y, \Phi_{B|Y}\}$. A key point to remark here is that, the object of interest is now the state-POVM set pair $(\rho, \mathbb{M}_{A|X})$, as opposed to the POVM set alone, as is the case in standard single-object state discrimination tasks [64]. We now analyze these multiobject tasks from the point of view of resource theories.

IV. ADVANTAGE PROVIDED BY RESOURCEFUL STATES AND POVM SETS

We start by addressing quantum resource theories (QRTs) of states and sets of measurements with arbitrary resources. In order to do this, we need some elements from conic programming [65,66]. In particular, we need the concept of a closed convex cone (CCC). First, a set $\mathcal{K} \subseteq \mathcal{L}_S(\mathcal{H})$ is a *cone* if $\lambda K \in \mathcal{K}$, $\forall K \in \mathcal{K}$ and $\forall \lambda \geqslant 0$. Second, a cone $\mathcal{K}$ is *convex* when $K_1 + K_2 \in \mathcal{K}$, $\forall K_1, K_2 \in \mathcal{K}$. Third, a *closed* set, which is considered as being closed under some operator topology such as the trace norm. We note that all operator topologies are equivalent in the finite dimensional case [67,68]. Fourth, the *dual* of a cone $\mathcal{K}$ is the set defined as $\mathcal{K}^\circ := \{O \in \mathcal{L}_S(\mathcal{H}) | \langle O, K\rangle_{\mathrm{HS}} \geqslant 0, \forall K \in \mathcal{K}\}$, where $\langle C, D\rangle_{\mathrm{HS}} = \mathrm{Tr}[C^\dagger D]$ is the Hilbert-Schmidt inner product in $\mathcal{L}(\mathcal{H})$. For any cone $\mathcal{K}$ we have $(\mathcal{K}^\circ)^\circ = \overline{conv(\mathcal{K})}$ (the closure of the convex hull), and so for any CCC $\mathcal{K}$ we have $(\mathcal{K}^\circ)^\circ = \mathcal{K}$. We now consider a property of quantum states defining a CCC as $\mathcal{F}(\mathcal{H}) \subseteq \mathcal{L}^+_S(\mathcal{H})$. We will address the set of *free states* as the set $\mathrm{F} \equiv \mathrm{F}(\mathcal{H}) := \{\rho \in \mathcal{F}(\mathcal{H}) \mid \mathrm{Tr}[\rho] = 1\}$. We say a state $\rho \notin \mathrm{F}$ is a *resourceful* state, and *free (resourceless)* otherwise. Desirable properties of quantum states regarded as resources include the likes of entanglement, coherence, amongst others [1]. We now similarly consider a property of measurement sets defining a CCC and use it to introduce a *free set of POVM sets* and denote it as $\mathbb{F}$. We say a POVM set $\mathbb{M}_{A|X} \notin \mathbb{F}$ is a *resourceful* POVM set, and *free (resourceless)* otherwise. Desirable properties of POVM sets include measurement incompatibility as a particular case [1]. We now address resource quantifiers. Consider a QRT of a quantum object O being either a quantum state or a POVM set. The generalized robustness of resource and the weight of resource of O are given by

$$\mathrm{R}_{\mathrm{F}}(O) := \min_{\substack{r \geqslant 0 \\ O^F \in \mathrm{F} \\ O^G}} \{r \mid O + rO^G = (1+r)O^F\}, \tag{5}$$

$$\mathrm{W}_{\mathrm{F}}(O) := \min_{\substack{w \geqslant 0 \\ O^F \in \mathrm{F} \\ O^G}} \{w \mid O = w\, O^G + (1-w)O^F\}. \tag{6}$$

The generalized robustness quantifies the minimum amount of a general quantum object O^G (either a general state ρ^G or a general POVM set $\mathbb{M}^G_{A|X}$) that has to be added to the quantum object O (either state ρ or POVM set $\mathbb{M}_{A|X}$) to get a free object O^F (either a free state ρ^F or a free POVM set $\mathbb{M}^F_{A|X}$). The weight, on the other hand, quantifies the minimum amount of a general object O^G (either a general state ρ^G or a general POVM set $\mathbb{M}^G_{A|X}$) that has to be used for recovering the quantum object O (either state ρ or POVM set $\mathbb{M}_{A|X}$) from a free object O^F. It is well known that these resource quantifiers can be written as conic programs [1,47,50,69–71]. We are now dealing with multiple objects, so it is natural to introduce the following notation. We say that a state-POVM set pair $(\rho, \mathbb{M}_{A|X})$ is: *fully free* when both objects are free, *partially resourceful* when either is resourceful, and *fully resourceful* when both are resourceful. We will be addressing, from now on, QRTs of POVM sets for which CPreP and CPosP are free operations, meaning that the free set of POVM sets is closed under simulability of POVM sets. Having established these elements from conic programming (further details in Appendix A), we are now ready to analyze multiobject tasks from the point of view of QRTs.

The main motivation now is to address the multiobject quantum subchannel discrimination games introduced in the previous section and compare the performance of a player having access to a potentially resourceful pair $(\rho, \mathbb{M}_{A|X})$, against the performance of a player having access only to free resources, $(\sigma, \mathbb{N}_{A|X}) \in \mathrm{F} \times \mathbb{F}$ (fully free player). We want the comparison to be fair, and so both players are compared when playing the same game (i.e., same ensemble of instruments $\{p_Y, \Phi_{B|Y}\}$). We can then compare the performance of both players by analyzing the following ratio:

$$\frac{P^{\mathrm{D}}_{\mathrm{succ}}(p_Y, \Phi_{B|Y}, \rho, \mathbb{M}_{A|X})}{\max_{\sigma \in \mathrm{F}} \max_{\mathbb{N}_{A|X} \in \mathbb{F}} P^{\mathrm{D}}_{\mathrm{succ}}(p_Y, \Phi_{B|Y}, \sigma, \mathbb{N}_{A|X})}. \tag{7}$$

If this ratio is larger than one, it naturally means that the pair $(\rho, \mathbb{M}_{A|X})$ offers an advantage over all fully free pairs, as it leads to larger probability of winning. It is then desirable to derive upper bounds for this ratio, and to explore how large the ratio can get to be, meaning maximising the ratio over all possible games, as this would represent the best case scenario for the pair $(\rho, \mathbb{M}_{A|X})$.

We now establish connections between robustness-based (weight-based) resource quantifiers for states and POVM sets and multiobject subchannel discrimination (exclusion) games with prior information. We proceed to establish a first result comparing the performance of a *fully* or *partially* resourceful pair against all *fully free* pairs.

Result 1. Consider a player with a quantum state ρ and a POVM set $\mathbb{M}_{A|X}$. The advantage provided by these two objects when playing multiobject subchannel discrimination games with prior information $\{p_Y, \Phi_{B|Y}\}$ is

$$\max_{\{p_Y,\Phi_{B|Y}\}} \frac{P^{\rm D}_{\rm succ}(p_Y, \Phi_{B|Y}, \rho, \mathbb{M}_{A|X})}{\max\limits_{\substack{\sigma\in{\rm F}\\ \mathbb{N}_{A|X}\in\mathbb{F}}} P^{\rm D}_{\rm succ}(p_Y, \Phi_{B|Y}, \sigma, \mathbb{N}_{A|X})} = [1+{\rm R}_{\rm F}(\rho)][1+{\rm R}_{\mathbb{F}}(\mathbb{M}_{A|X})], \tag{8}$$

with the maximization over all quantum subchannel discrimination games and the generalized robustness of resource of state and of POVM sets.

Proof. (Sketch) The full proof of this result is in Appendix B. We address here a sketch of the proof. The proof of the statement has two parts. The first part is to show that the right hand side of (8) constitutes an upper bound. This first part employs the primal conic program of the generalized robustness, and it follows from relatively straightforward arguments. The second part of the statement, proving that the upper bound is achievable, is a more involved endeavor, and so we describe it next. Given a pair $(\rho, \mathbb{M}_{A|X})$, it is possible to use the dual conic programs of the generalized robustness to extract positive semidefinite operators Z^ρ and $\{Z^{\mathbb{M}_{A|X}}_{a|x} \equiv Z^{\mathbb{M}}_{a|x}\}$, $a \in \{1,\ldots,l\}$, $x \in \{1,\ldots,\kappa\}$, satisfying desirable optimality properties. Using these operators, we can define the following subchannel discrimination game. Fix a PMF p_Y, and consider the game $\{p_Y, \Psi_{B|Y}\}$, $b \in \{1,\ldots,l+J\}$, $y \in \{1,\ldots,\kappa\}$, as:

$$\Psi^{(\rho,\mathbb{M}_{A|X},p_X,J)}_{b|y}(\eta) := \begin{cases} \alpha {\rm Tr}[Z^\rho \eta]\, Z^{\mathbb{M}}_{b|y}\, p(y)^{-1}, & b = 1,\ldots,l \\ \frac{1}{J}[1-F_y(\eta)]\chi & b = l+1,\ldots,l+J \end{cases} \tag{9}$$

with $J \geqslant 1$ an integer, χ an arbitrary quantum state, α a coefficient, and $\{F_y(\cdot)\}$ functions that depend on $(\rho, \mathbb{M}_{A|X})$. These parameters are all specified in Appendix B. The next step is to analyze the performance of a player using fully free pairs to play this game. In Appendix B we derive the following upper bound respected by all fully free pairs $(\sigma, \mathbb{N}_{A|X})$:

$$P^{\rm D}_{\rm succ}\big(p_Y, \Psi^{(\rho,\mathbb{M}_{A|X},p_X,J)}_{B|Y}, \sigma, \mathbb{N}_{A|X}\big) \leqslant \alpha + \frac{1}{J}. \tag{10}$$

The next step is to analyze the performance of a player using now the fixed pair $(\rho, \mathbb{M}_{A|X})$. In Appendix B we also derive the lower bound,

$$P^{\rm D}_{\rm succ}\big(p_Y, \Psi^{(\rho,\mathbb{M}_{A|X},p_X,J)}_{B|Y}, \rho, \mathbb{M}_{A|X}\big) \geqslant \alpha[1+{\rm R}_{\rm F}(\rho)][1+{\rm R}_{\mathbb{F}}(\mathbb{M}_{A|X})]. \tag{11}$$

Taking the previous two statements with $J \to \infty$ achieves the claim. ■

We first note that this result applies to QRTs of states with *arbitrary resources* and QRTs of POVM sets with *arbitrary resources* for which POVM set simulability is a free operation and therefore, it covers as particular instances, several desirable resources for both states and POVM sets, such as measurement incompatibility. A second point to highlight is that the result holds for any pair $(\rho, \mathbb{M}_{A|X})$ that is either fully resourceful or even partially resourceful. Thirdly, this result generalizes two sets of results from the literature: (i) the single-object results reported in [44–47] as well as (ii) the multiobject case of state-measurement pairs in [51]. Let us address these cases in more detail.

First, the result in [51] can immediately be recovered from (8) by restricting the POVM set to trivially have only one POVM, and similarly the operational task to consist on only one instrument from which we get

$$\max_{\Phi} \frac{P^{\rm D}_{\rm succ}(\Phi, \rho, \mathbb{M})}{\max\limits_{\substack{\sigma\in{\rm F}\\ \mathbb{N}\in\mathbb{F}}} P^{\rm D}_{\rm succ}(\Phi, \sigma, \mathbb{N})} = [1+{\rm R}_{\rm F}(\rho)][1+{\rm R}_{\mathbb{F}}(\mathbb{M})], \tag{12}$$

thus explicitly recovering the result for state-measurement pairs reported in [51].

Second, let us now address how (8) also recovers the single-object results in [44–47]. In [44–47], the player has access only to POVM sets (with no quantum states at their disposal), and the resource being exploited is specifically that of measurement incompatibility. Imposing these restrictions in our setup, our operational task reduces to that of *single-object* quantum *state* discrimination with prior information, and in turn the advantage is given by that of the generalized robustness of incompatibility alone. Let us see this explicitly. Consider the denominator in the ratio of interest,

$$\max_{\sigma\in{\rm F}} \max_{\mathbb{N}_{A|X}\in\mathbb{F}} P^{\rm D}_{\rm succ}(p_Y, \Phi_{B|Y}, \sigma, \mathbb{N}_{A|X}), \tag{13}$$

and the free state and POVM set achieving this as $(\sigma^*, \mathbb{N}^*_{A|X})$. Consider now comparing the performance of a player using such fully free pair $(\sigma^*, \mathbb{N}^*_{A|X})$ against a player using the partially resourceful pair $(\sigma^*, \mathbb{M}_{A|X})$. In this case, because the state is free, the right-hand side in (8) is simply $[1+{\rm R}_{\mathbb{F}}(\mathbb{M}_{A|X})]$, and so we explicitly get

$$\max_{\{p_Y,\Phi_{B|Y}\}} \frac{P^{\rm D}_{\rm succ}(p_Y, \Phi_{B|Y}, \sigma^*, \mathbb{M}_{A|X})}{P^{\rm D}_{\rm succ}(p_Y, \Phi_{B|Y}, \sigma^*, \mathbb{N}^*_{A|X})} = [1+{\rm R}_{\mathbb{F}}(\mathbb{M}_{A|X})]. \tag{14}$$

The left-hand side of the latter expression can now be seen as an ensemble of states with prior information as $\mathcal{E}_{B|Y} := \{\rho_{b|y}, p(y)\}$ with $\rho_{b|y} := \phi_{b|y}(\sigma^*)$, and so we get

$$\max_{\mathcal{E}_{B|Y}} \frac{P^{\rm QSD-PI}_{\rm succ}(\mathcal{E}_{B|Y}, \mathbb{M}_{A|X})}{P^{\rm QSD-PI}_{\rm succ}(\mathcal{E}_{B|Y}, \mathbb{N}^*_{A|X})} = [1+{\rm R}_{\mathbb{F}}(\mathbb{M}_{A|X})], \tag{15}$$

thus explicitly recovering the results in [44–47]. When considering subchannel discrimination games being played with a POVM set alone, the advantage we see becomes $[1 + \mathrm{R}_{\mathbb{F}}(\mathbb{M}_{A|X})]$ [44–47]. In the multiobject scenario considered here we instead get $[1 + \mathrm{R}_{\mathrm{F}}(\rho)][1 + \mathrm{R}_{\mathbb{F}}(\mathbb{M}_{A|X})]$, which can be larger than $[1 + \mathrm{R}_{\mathbb{F}}(\mathbb{M}_{A|X})]$, whenever ρ is resourceful. This increment can be conceptually be understood by considering that, since we are now addressing a composite object, it is natural to expect each object to contribute to the overall advantage. Having said this, however, it is still appealing that the advantage can be quantified in such a straightforward multiplicative manner.

Finally, we present a practical application of Result 1 to the problem of noise detection. In [69], an explicit analysis of noise detection was provided. Consider an instrument set $\Phi_{B|Y} = \{\Phi_y\}_y = \{\phi_{b|y}\}$. To make the argument more intuitive, let us assume that each instrument is a binary ensemble of channels: $\Phi_y = \{\phi_{b=1|y}, \phi_{b=2|y}\}$ with $\phi_{b|y} = \nu_{b|y}\hat{\phi}_{b|y}$, where $\{\nu_{b|y}\}_{b=1,2}$ is a PMF and $\{\hat{\phi}_{b|y}\}_{b=1,2}$ are channels. Assume also $\hat{\phi}_{b=1|y}$ is an ideal (noiseless) channel and interpret the instrument Φ_y as composed of ideal and noisy channels $\{\hat{\phi}_{b=1|y}, \hat{\phi}_{b=2|y}\}$ that occur probabilistically according to $\{\nu_{b|y}\}_b$. The problem of noise detection is to detect the noise (or in other words, to discriminate the noisy channel $\hat{\phi}_{b=2|y}$), for each Φ_y, with the help of a state and measurements. Following [69], we take the action of the non-Clifford unitary $U_{\mathrm{NC}} = \exp(-i\frac{\pi}{4}\frac{X-Y}{\sqrt{2}})$ as the ideal channel $\hat{\phi}_{b=1}$, and its phase-flipped version $U_{\mathrm{NC}}Z$ as the noisy channel $\hat{\phi}_{b=2}$. It was shown in [69], within the QRT of magic, that the most efficient noise detection in the instrument $\Phi = \{\nu_1\hat{\phi}_{b=1}, \nu_2\hat{\phi}_{b=2}\}$ is achieved when considering the magic state (T-state) $|T\rangle := \frac{1}{\sqrt{2}}(|0\rangle + e^{i\pi/4}|1\rangle)$, together with the POVM $\sigma_z = \{|0\rangle\langle 0|, |1\rangle\langle 1|\}$. This state-measurement pair $(|T\rangle, \sigma_z)$ is, however, not always the ideal pair for detecting other types of noise. In fact, when considering the task of noise detection on the instrument $\{\frac{1}{2}U_{\mathrm{NC}}, \frac{1}{2}U_{\mathrm{NC}}X\}$, the pair $(|T\rangle, \mathsf{M})$ provides a better detection than $(|T\rangle, \sigma_z)$, where $\mathsf{M} = \{|M\rangle\langle M|, |\bar{M}\rangle\langle\bar{M}|\}$ with $|M\rangle = \cos\frac{\theta}{2}|0\rangle - \sin\frac{\theta}{2}|1\rangle$, $\theta = \frac{\pi}{2} - \arctan\sqrt{2}$. This can be checked as follows. The success probability for discriminating $\{\frac{1}{2}U_{\mathrm{NC}}, \frac{1}{2}U_{\mathrm{NC}}X\}$ via $(|T\rangle, \mathsf{M})$ is given by

$$\frac{1}{2}|\langle M|U_{NC}|T\rangle|^2 + \frac{1}{2}|\langle\bar{M}|U_{NC}X|T\rangle|^2. \tag{16}$$

Ignoring the coefficient $\frac{1}{2}$, and noting that

$$U_{NC} = \frac{1}{2}\begin{pmatrix} \sqrt{2} & 1-i \\ -1-i & \sqrt{2} \end{pmatrix}, \tag{17}$$

we get

$$\begin{aligned} U_{NC}|T\rangle &= |0\rangle, \\ U_{NC}X|T\rangle &= \frac{1}{\sqrt{2}}(|0\rangle + e^{-\pi/4}|1\rangle). \end{aligned} \tag{18}$$

With this in place, we get

$$\begin{aligned} &|\langle M|U_{NC}|T\rangle|^2 + |\langle\bar{M}|U_{NC}X|T\rangle|^2 \\ &= 1 + \frac{1}{2}\left(\frac{1}{\sqrt{2}}\sin\theta + \cos\theta\right), \end{aligned}$$

which attains its maximum at $\theta = \frac{\pi}{2} - \arctan\sqrt{2}$. In particular, $\theta = \frac{\pi}{2} - \arctan\sqrt{2}$ gives greater success probability than $\theta = 0$ (the σ_z measurement).

Taking this into account, it is thus natural to use multiple measurements (i.e., a POVM set) $\mathbb{M}_{A|X}$ for the sake of more general noise detection schemes, with the performance of each state-POVM set pair $(\rho, \mathbb{M}_{A|X})$ evaluated similarly by the average success probability of discriminating multiple instruments $\Phi_{B|Y}$. From the perspective of the noise detection task, Result 1 then implies that using incompatible observables possibly (and definitely in certain cases) provides better success probability of detection than using compatible observables. This also demonstrates that combining incompatibility of POVM sets with properties of states, such as magic, can further improve the performance in discrimination tasks, and it moreover manifests the theoretically possible upper bound in terms of generalized robustness.

We now proceed to show that Result 1 can also be extended to multiobject quantum subchannel *exclusion* games with prior information, where it is now the *weight* of resource that characterizes the advantage provided by resourceful pairs of states and POVM sets. In this scenario, since the figure of merit is now the probability of *error* when playing the game, the ratio of interest is now

$$\frac{P^{\mathrm{E}}_{\mathrm{err}}(p_Y, \Phi_{B|Y}, \rho, \mathbb{M}_{A|X})}{\min_{\sigma\in\mathrm{F}}\min_{\mathbb{N}_{A|X}\in\mathbb{F}} P^{\mathrm{E}}_{\mathrm{err}}(p_Y, \Phi_{B|Y}, \sigma, \mathbb{N}_{A|X})}. \tag{19}$$

If this ratio is smaller than one, it means the pair $(\rho, \mathbb{M}_{A|X})$ offers an advantage over all fully free pairs, as it leads to smaller probability of error. It is then desirable to derive lower bounds for this ratio, and to explore how small the ratio can get to be, meaning minimizing the ratio over all possible games, as this would represent the best case scenario for the pair $(\rho, \mathbb{M}_{A|X})$.

Result 2. Consider a player with a quantum state ρ and a POVM set $\mathbb{M}_{A|X}$, then, the advantage provided by these two objects when playing subchannel exclusion games with prior information $\{p_Y, \Phi_{B|Y}\}$ is given by

$$\begin{aligned} &\min_{\{p_Y,\Phi_{B|Y}\}} \frac{P^{\mathrm{E}}_{\mathrm{err}}(p_Y, \Phi_{B|Y}, \rho, \mathbb{M}_{A|X})}{\min_{\substack{\sigma\in\mathrm{F}\\ \mathbb{N}_{A|X}\in\mathbb{F}}} P^{\mathrm{E}}_{\mathrm{err}}(p_Y, \Phi_{B|Y}, \sigma, \mathbb{N}_{A|X})} \\ &= [1 - \mathrm{W}_{\mathrm{F}}(\rho)][1 - \mathrm{W}_{\mathbb{F}}(\mathbb{M}_{A|X})], \end{aligned} \tag{20}$$

with the minimization over all quantum subchannel exclusion games and the weight of resource of state and of POVM sets.

The full proof of this result is in Appendix C. Similar to the discrimination case, this result also holds for arbitrary resources of both quantum states and POVM sets. It is illustrative to address a sketch of this proof, so in order to highlight the differences with the case for discrimination.

Proof. (Sketch) Similar to the discrimination case, that the right-hand side of (20) constitutes a lower bound for the ratio of interest follows from the primal conic programs of the weight measures. Showing that such a lower bound is achievable is a more involved endeavor, and so we describe it next. Given a pair $(\rho, \mathbb{M}_{A|X})$, and the dual conic programs of the weight of resource, we can extract positive semidefinite operators Y^ρ and $\{Y^{\mathbb{M}_{A|X}}_{a|x} \equiv Y^{\mathbb{M}}_{a|x}\}$, $a \in \{1, \ldots, l\}$, $x \in \{1, \ldots, \kappa\}$, satisfying desirable optimality properties, so that we can construct the following game. Fix a PMF p_Y, and consider the game $\{p_Y, \Psi_{B|Y}\}$, $b \in \{1, \dots, l+1\}$, $y \in \{1, \dots, \kappa\}$, as

$$\Psi_{b|y}^{(\rho,\mathbb{M}_{A|X},p_X)}(\eta) := \begin{cases} \beta \operatorname{Tr}[Y^{\rho}\eta]\, Y_{b|y}^{\mathbb{M}}\, p(y)^{-1}, & b = 1, \dots, l \\ [1 - G_y(\eta)]\, \xi_y^{\mathbb{M}_{A|X}}, & b = l+1 \end{cases} \tag{21}$$

with $\{\xi_y^{\mathbb{M}_{A|X}}\}$ a set of quantum states, β a coefficient, and $\{G_y(\cdot)\}$ functions that depend on $(\rho, \mathbb{M}_{A|X})$. These parameters are all specified in Appendix C. We derive the following lower bound for all fully free pairs $(\sigma, \mathbb{N}_{A|X})$:

$$\min_{\sigma\in F} \min_{\mathbb{N}_{A|X}\in\mathbb{F}} P_{\rm err}^{\rm E}\big(p_Y, \Psi_{b|y}^{(\rho,\mathbb{M}_{A|X},p_X)}, \sigma, \mathbb{N}_{A|X}\big) \geqslant \beta. \tag{22}$$

In Appendix C we also prove that for the fixed pair $(\rho, \mathbb{M}_{A|X})$ we get the upper bound,

$$P_{\rm err}^{\rm E}\big(p_Y, \Psi_{b|y}^{(\rho,\mathbb{M}_{A|X},p_X)}, \rho, \mathbb{M}_{A|X}\big) \leqslant \beta[1 - {\rm W_F}(\rho)][1 - {\rm W_F}(\mathbb{M}_{A|X})]. \tag{23}$$

These two statements together then achieve the claim. ■

One technical point of comparison regarding the proofs of Result 1 (8) and Result 2 (20) is the nature of the game saturating the bound. In the discrimination case, the subchannel game needed to achieve the upper bound contains an infinite amount of extra subchannels $J \to \infty$, whilst in the exclusion case, the subchannel game needed to achieve the lower bound requires only one extra subchannel. This difference can qualitatively be understood by taking into account that, the goal when maximizing the ratio of interest is to make it difficult for the fully free players to perform well. In the discrimination case, this can be done by *increasing* the amount of objects to discriminate from (as there will be more alternatives that are "bad" options). In the exclusion case, increasing the amount of objects makes it instead actually easier for the player in question to win (as there will be more alternatives that are "good" options), and so in the exclusion case, in order to make it difficult for the fully free players to perform well, *decreasing* the amount of objects to exclude from is desirable.

V. EXTENSION TO GENERAL PROBABILISTIC THEORIES

Our main results (8) and (20) can be extended to general probabilistic theories (GPTs) [72–74]. Let V be a finite-dimensional Euclidean space and V^* be its dual. We often identify V^* with V and the action $f(x)$ of $f \in V^*$ on $x \in V$ with the Euclidean inner product $\langle f, x\rangle$ of two vectors $f, x \in V$ (by means of the Riesz representation theorem [75]). A *GPT* is a pair of sets $(\Omega, \mathscr{E})$ such that Ω is a compact convex subset of V with its affine hull satisfying $aff(\Omega) \not\ni 0$ and linear hull satisfying $lin(\Omega) = V$, and $\mathscr{E} = \{e \in V^* \mid e(\forall\omega) \in [0, 1]\}$. The sets Ω and $\mathscr{E}$ are called the *state space* and *effect space*, and their elements are called *states* and *effects* respectively. We remark that, in this paper, we assume the *no-restriction hypothesis* [59]. Clearly, GPTs are generalizations of quantum theory: taking $\mathcal{L}_S(\mathcal{H})$ as V and $\mathcal{D}(\mathcal{H})$ as Ω, we can recover the description of quantum states. It is also easy to see that effects are generalizations of POVM elements, and the Euclidean inner product $\langle e, \omega\rangle = e(\omega)\, (\omega \in \Omega, e \in \mathscr{E})$ generalizes the Hilbert-Schmidt inner product $\langle M, \rho\rangle_{\rm HS} = \operatorname{Tr}[M\rho]\, (\rho \in \mathcal{D}(\mathcal{H}), M \in \mathcal{L}_S^+(\mathcal{H}))$ for quantum theory. In particular, the *unit effect* $u \in \mathscr{E}$ defined via $u(\omega) = 1 (\forall\omega \in \Omega)$ corresponds to the identity operator $\mathbb{1}$. With similar notations to the quantum case, we define a *measurement* with an outcome set $A = \{1, \dots, l\}$ as a set of effects $\{e_a\}_{a=1}^l$ such that $\sum_{a\in A} e_a = u$. A *measurement set* $\mathbb{E}_{A|X} = \{e_{a|x}\}_{a,x}$ is also introduced as a generalization of a POVM set. The notion of (in)compatibility and simulability for POVMs can be extended naturally to measurements in GPTs by rephrasing (1) and (2) in terms of effects instead of POVM elements. It allows us to use the expression $\mathbb{N}_{B|Y} \preceq \mathbb{E}_{A|X}$ for two measurement sets $\mathbb{N}_{B|Y}$ and $\mathbb{E}_{A|X}$ in GPTs, meaning that $\mathbb{N}_{B|Y}$ is simulable by $\mathbb{E}_{A|X}$. It is also clear that measurement compatibility is closed under simulation in GPTs.

For a GPT $(\Omega, \mathscr{E})$ whose underlying vector space is V, we define the *positive cone* $V_+ \subset V$ by $V_+ = cone(\Omega)$ and the *dual cone* $V_+^\circ \subset V^*$ by $V_+^\circ = cone(\mathscr{E}) = \{f \in V^* \mid f(x) \geqslant 0, \forall x \in V_+\}$, where $cone(\cdot)$ is the conic hull of the set. For the quantum case, these cones correspond to the set of positive semidefinite operators $\mathcal{L}_S^+(\mathcal{H})$. A linear map $\xi : V \to V$ is called *positive* if $\xi(V_+) \subset V_+$. A positive map ξ is called a *channel* if it satisfies $\langle u, \xi(\omega)\rangle = 1, \forall\omega \in \Omega$, and a *subchannel* if $\langle u, \xi(\omega)\rangle \leqslant 1, \forall\omega \in \Omega$. An *instrument* Ξ is defined as a set of subchannels $\Xi = \{\xi_b\}_{b=1}^m$ such that $\sum_{b=1}^m \xi_b$ is a channel, and we also introduce an *instrument set* $\Xi_{B|Y} = \{\Xi_y\}_{y=1}^\tau = \{\xi_{b|y}\}_{b,y}$ in the same way as the quantum case. We remark that here we do not require complete positivity for (sub)channels due to the difficulty in determining composite systems for GPTs [76]. It seems that quantum description for channels is not recovered, but such inconsistency can be overcome by restricting ourselves to a (convex) subset of the set of all channels in GPTs.

Now we are in position to present our third main result. We consider *general-probabilistic* multiobject subchannel discrimination game with prior information (GPScD-PI) as natural extension of QScD-PI to GPTs. The scenario of GPScD-PI is the same as QScD-PI: replacing a quantum state $\rho \in \mathcal{D}(\mathcal{H})$, a POVM set $\mathbb{M}_{A|X} = \{M_{a|x}\}$, and a quantum instrument set $\Phi_{B|Y}$ with a state $\omega \in \Omega$, a measurement set $\mathbb{E}_{A|X} = \{e_{a|x}\}$, and an instrument set $\Xi_{B|Y}$ for a GPT $(\Omega, \mathscr{E})$. The maximum success probability is given in the same way as (3) by

$$P_{\rm succ}^{\rm GPD}(p_Y, \Xi_{B|Y}, \omega, \mathbb{E}_{A|X}) := \max_{\mathcal{S}} \sum_{g,a,x,\mu,b,y} \delta_{g,b}\, s(g|a, y, z) \times \langle e_{a|x}, \xi_{b|y}(\omega)\rangle r(x|y, z)\, q(z)\, p(y) \tag{24}$$

with the maximization over all possible strategies $\mathcal{S} = \{q_Z, r_{X|YZ}, s_{G|AYZ}\}$. Similarly, we can consider general-probabilistic multiobject subchannel exclusion game with prior information (GPScE-PI) generalizing the QScE-PI and

(4), and the minimum error probability is

$$P^{\rm GPE}_{\rm err}(p_Y, \Xi_{B|Y}, \omega, \mathbb{E}_{A|X}) := \min_{\mathcal{S}} \sum_{g,a,x,\mu,b,y} \delta_{g,b}\, s(g|a,y,z) \times \langle e_{a|x},\ \xi_{b|y}(\omega)\rangle r(x|y,z)\, q(z)\, p(y) \tag{25}$$

with the minimization over all possible strategies $\mathcal{S} = \{q_Z, r_{X|YZ}, s_{G|AYZ}\}$. The last step we need is to introduce the notion of resourceful sets in GPTs, and this can be also done by generalizing quantum concepts straightforwardly. In fact, with a CCC $\mathcal{F} \subseteq V_+$ (closed with respect to, e.g., the Euclidean topology in V), we can introduce free states as elements of the set $\mathrm{F} = \{\omega \in \mathcal{F} \mid \langle u, \omega\rangle = 1\}$ and resourceful states as $\Omega \setminus \mathrm{F}$. A set $\mathbb{F}$ of free measurement sets is similarly extended, and $\mathbb{M}_{A|X} \notin \mathbb{F}$ is called resourceful. We can also introduce the generalized robustness of resource and the weight of resource of an object O (either a state or a measurement set) in the GPT $(\Omega, \mathscr{E})$ as

$$\mathrm{R}^{\rm GP}_{\rm F}(O) := \min_{\substack{r \geqslant 0 \\ O^F \in \mathrm{F} \\ O^G}} \{r \mid O + rO^G = (1+r)O^F\}, \tag{26}$$

$$\mathrm{W}^{\rm GP}_{\rm F}(O) := \min_{\substack{w \geqslant 0 \\ O^F \in \mathrm{F} \\ O^G}} \{w \mid O = w\, O^G + (1-w)O^F\}. \tag{27}$$

These quantities are associated with conic programs with the positive cone V_+ and dual cones V_+° generated respectively by Ω and $\mathscr{E}$. Now we can generalize the results from the quantum domain (8) and (20) to GPTs:

Result 3. Consider a player with a state ω and a measurement set $\mathbb{E}_{A|X}$ of a GPT $(\Omega, \mathscr{E})$. The advantage provided by these two objects when playing general-probabilistic multiobject subchannel discrimination and exclusion games with prior information $\{p_Y, \Xi_{B|Y}\}$ is

$$\max_{\{p_Y, \Xi_{B|Y}\}} \frac{P^{\rm GPD}_{\rm succ}(p_Y, \Xi_{B|Y}, \omega, \mathbb{E}_{A|X})}{\max_{\substack{\sigma\in\mathrm{F} \\ \mathbb{N}_{A|X}\in\mathbb{F}}} P^{\rm GPD}_{\rm succ}(p_Y, \Xi_{B|Y}, \sigma, \mathbb{N}_{A|X})} = \big[1 + \mathrm{R}^{\rm GP}_{\rm F}(\omega)\big]\big[1 + \mathrm{R}^{\rm GP}_{\mathbb{F}}(\mathbb{E}_{A|X})\big], \tag{28}$$

and

$$\min_{\{p_Y, \Xi_{B|Y}\}} \frac{P^{\rm GPE}_{\rm err}(p_Y, \Xi_{B|Y}, \omega, \mathbb{E}_{A|X})}{\min_{\substack{\sigma\in\mathrm{F} \\ \mathbb{N}_{A|X}\in\mathbb{F}}} P^{\rm GPE}_{\rm err}(p_Y, \Xi_{B|Y}, \sigma, \mathbb{N}_{A|X})} = \big[1 - \mathrm{W}^{\rm GP}_{\rm F}(\omega)\big]\big[1 - \mathrm{W}^{\rm GP}_{\mathbb{F}}(\mathbb{E}_{A|X})\big], \tag{29}$$

respectively. With the maximization (minimization) over all general-probabilistic subchannel discrimination (exclusion) games and the generalized robustness (weight) of resource of state and of measurement sets.

The full proof of this result is in Appendix D. These proofs proceed in a similar way as the previous ones by appropriately extending the quantum objects to their counterparts in GPTs. Some examples of this include using the Euclidean inner product and the *order unit norm* instead of the Hilbert-Schmidt inner product and the operator norm respectively. Finally, we remark that from the standpoint of GPTs, incompatibility is a genuine nonclassical feature: Any pair of observables are compatible if and only if the theory is classical [77]. Our results then reinforce the value of incompatibility as a nonclassical resource from a higher-level perspective.

VI. CONCLUSIONS

In this work, we have introduced multiobject operational tasks for general resources of POVM sets (including measurement incompatibility) in the form of multiobject quantum subchannel discrimination and exclusion games with prior information, where the player can simultaneously harness the resources contained within two quantum objects; a quantum state, and a set of measurements (POVM set). Specifically, we have shown that *any* fully or partially resourceful pair (state, POVM set) is useful for a suitably chosen multiobject subchannel discrimination and exclusion game with prior information. We have found that, when compared to the best possible strategy using *fully free* state-POVM set pairs, the advantage provided by a pair state-POVM set can be quantified, in a *multiplicative* manner, by the resource quantifiers of generalized robustness and weight of resource of the state and the POVM set, for discrimination and exclusion games respectively. These results hold true for arbitrary resources of quantum states and for resources of POVM sets closed under classical pre- and postprocessing. The results presented here are therefore telling us that *all* sets of incompatible measurement can be useful for multiobject operational tasks. As described in Table I, these results also happen to generalize various previous results in the literature. First, they generalize the results reported in [44–47], where *single-object* operational tasks for measurement incompatibility were characterized. Second, they also generalize the operational tasks for state-measurement pairs introduced in [51]. Third and finally, they generalize the *single-object* results for GPTs reported in [50], now to the *multiobject* regime.

There are various different directions where to further explore these findings. First, quantum resource theories have recently been explored beyond the realm of convexity and so, it would be interesting to explore multiobject operational tasks in such regimes [54,78–80]. Second, whilst we have explored both discrimination and exclusion games, it is also known that these games can be considered more generally as quantum state betting games [81,82], from the point of view of expected utility theory, and so it would be interesting to explore such betting games using the multiobject perspective employed in this paper. Third, it would be interesting to explore these findings from the point of view of cooperative game theory, akin to the setting being explored for multiresource tasks in [54]. Fourth, it could also be relevant to explore multiobject tasks for more general scenarios involving the incompatibility of sets of instruments [83–91].

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DATA AVAILABILITY

No data were created or analyzed in this study.

APPENDIX A: PRELIMINARIES

Consider a set of POVMs $\mathbb{M}_{A|X} := \{M_{a|x}\}$, $M_{a|x} \geqslant 0, \forall a, x$, $\sum_a M_{a|x} = \mathbb{1}, \forall x$, $x = 1, \dots, \kappa$, with each POVM having elements $a = 1, \dots, l$, a set of subchannels $\Phi_{B|Y} := \{\Phi_{b|y}(\cdot)\}$, $\Phi_{b|y}(\cdot)$ a completely positive (CP) trace-nonincreasing map $\forall b, y$, $\sum_b \Phi_{b|y}(\cdot) = \Phi_y(\cdot)$ a trace-preserving (TP) map $\forall y$, $y = 1, \dots, m$, $b = 1, \dots, n$, and a quantum state ρ, $\rho \geqslant 0$, $\text{Tr}(\rho) = 1$. We start by rewriting the probability of success in quantum subchannel discrimination (QScD) with prior information [64] as

$$P^{\text{D}}_{\text{succ}}(p_Y, \Phi_{B|Y}, \rho, \mathbb{M}_{A|X}) := \max_{\mathcal{S}} \sum_{g,a,x,z,b,y} \delta_{g,b}\, s(g|a, y, z)\text{Tr}[M_{a|x} \Phi_{b|y}(\rho)]\, r(x|y, z)\, q(z)\, p(y) \tag{A1}$$

$$= \max_{\mathcal{S}} \sum_{a,x,z,b,y} s(b|a, y, z)\text{Tr}[M_{a|x} \Phi_{b|y}(\rho)]\, r(x|y, z)\, q(z)\, p(y) \tag{A2}$$

$$= \max_{\mathcal{S}} \sum_{b,y} \text{Tr}\left[\left(\sum_{a,x,z} s(b|a, y, z)\, M_{a|x}\, r(x|y, z)\, q(z)\right) \Phi_{b|y}(\rho)\right] p(y) \tag{A3}$$

$$= \max_{\mathbb{N}_{B|Y} \preceq \mathbb{M}_{A|X}} \sum_{b,y} \text{Tr}[N_{b|y} \Phi_{b|y}(\rho)] p(y), \tag{A4}$$

with the set of strategies $\mathcal{S} = \{q_Z, p_{X|YZ}, p_{B|AYZ}\}$ and the simulability of POVM sets $\mathbb{N}_{B|Y} \preceq \mathbb{M}_{A|X}$ defined as

$$N_{b|y} = \sum_{a,x,z} s(b|a, y, z)\, M_{a|x}\, r(x|y, z)\, q(z). \tag{A5}$$

Lemma A 1. (Dual conic programs for the generalized robustness of state and POVM sets) The generalized robustness of resource of a state ρ and a POVM set $\mathbb{M} = \{M_{a|x}\}$, $x \in \{1, \dots, \kappa\}$, $a \in \{1, \dots, l\}$ is given by

$$\text{R}_{\text{F}}(\rho) = \max_{Z} \quad \text{Tr}[Z\rho] - 1, \tag{A6a}$$

$$\text{s.t.} \quad Z \geqslant 0, \tag{A6b}$$

$$\text{Tr}[Z\sigma] \leqslant 1, \quad \forall \sigma \in \text{F}, \tag{A6c}$$

and

$$\text{R}_{\mathbb{F}}(\mathbb{M}_{A|X}) = \max_{\{Z_{a,x}\}} \quad \sum_{x=1}^{\kappa} \sum_{a=1}^{l} \text{Tr}[Z_{a,x} M_{a|x}] - 1, \tag{A7a}$$

$$\text{s.t.} \quad Z_{a,x} \geqslant 0, \quad \forall a, x, \tag{A7b}$$

$$\sum_{x=1}^{\kappa} \sum_{a=1}^{l} \text{Tr}[Z_{a,x} N_{a|x}] \leqslant 1, \quad \forall \mathbb{N}_{A|X} = \{N_{a|x}\} \in \mathbb{F}. \tag{A7c}$$

These are the dual conic formulations of the generalized robustnesses for states and POVM sets, respectively.

Proof. The dual version of the generalized robustness of resource of a state is a known and well-reported case in the literature (see for instance [69,92]), and so we only address here the case for POVM sets. This proof follows similar techniques to that of states, and we present it below for completeness. For simplicity, we use a symbol $\overset{x,a}{\oplus} K$ to represent the direct sum of the identical subset (or element) K of the linear space $\mathcal{L}_S(\mathcal{H})$, i.e., $\overset{x,a}{\oplus} K = \oplus_{x=1}^{\kappa} \oplus_{a=1}^{l} K = K \oplus K \oplus \cdots \oplus K$. A POVM set $\mathbb{M}_{A|X} = \{M_{a|x}\}$ ($x \in \{1, \dots, \kappa\}$, $a \in \{1, \dots, l\}$) is naturally an element of the direct sum $\overset{x,a}{\oplus} \mathcal{L}_S(\mathcal{H})$, and the generalized robustness of resource of $\mathbb{M} = \{M_{a|x}\}$ is given as

$$\text{R}_{\mathbb{F}}(\mathbb{M}_{A|X}) := \min_{\substack{r \geqslant 0 \\ \mathbb{M}^F_{A|X} \in \mathbb{F} \\ \mathbb{M}^G_{A|X}}} \left\{ r \,\middle|\, \mathbb{M}_{A|X} + r\mathbb{M}^G_{A|X} = (1 + r)\mathbb{M}^F_{A|X} \right\}. \tag{A8}$$

Let us choose an arbitrary $\rho_0 \in \mathcal{D}(\mathcal{H})$ to construct $\overset{x,a}{\oplus}\rho_0 \in \overset{x,a}{\oplus}\mathcal{L}(\mathcal{H})$. The direct sum $\overset{x,a}{\oplus}\mathcal{L}(\mathcal{H})$ equips an inner product $\langle\cdot,\cdot\rangle$ induced naturally by the Hilbert-Schmidt inner product $\langle\cdot,\cdot\rangle_{\rm HS}$ on $\mathcal{L}(\mathcal{H})$. Since $\langle \mathbb{M}^F_{A|X}, \overset{x,a}{\oplus}\rho_0\rangle = \sum_{x,a}\langle M^F_{a|x}, \rho_0\rangle_{\rm HS} = \kappa$ holds, we can rewrite (A8) as a conic program with primal variables $O_{A|X} := (1+r)\mathbb{M}^F_{A|X}$,

$$1+{\rm R}_{\mathbb{F}}(\mathbb{M}_{A|X}) = \min_{O_{A|X}} \left\langle O_{A|X}, \frac{1}{\kappa}\overset{x,a}{\oplus}\rho_0\right\rangle \tag{A9}$$

$$\text{s.t. } \mathbb{M}_{A|X} \leqslant_{\overset{x,a}{\oplus}\mathcal{L}_S^+} O_{A|X}, \tag{A10}$$

$$O_{A|X} \in \mathfrak{F}(\mathcal{H}), \tag{A11}$$

where we defined $\mathbb{M}^F_{A|X} \geqslant_{\overset{x,a}{\oplus}\mathcal{L}_S^+} O_{A|X}$ by $O_{A|X} - \mathbb{M}_{A|X} \in \overset{x,a}{\oplus}\mathcal{L}_S^+$ and $\mathfrak{F}(\mathcal{H}) := cone(\mathbb{F})$. Expressing $O_{A|X} = \{O_{a|x}\}$, we explicitly have

$$1+{\rm R}_{\mathbb{F}}(\mathbb{M}_{A|X}) = \min_{O_{A|X}} \ \frac{1}{\kappa}\sum_{a,x}{\rm Tr}[O_{a|x}\rho_0], \tag{A12}$$

$$\text{s.t. } M_{a|x} \leqslant O_{a|x}, \quad \forall a,x, \tag{A13}$$

$$O_{A|X} \in \mathfrak{F}(\mathcal{H}). \tag{A14}$$

The primal constraints can alternatively be written as (i) $O_{a|x} - M_{a|x} \geqslant 0, \forall a,x,$ and (ii) ${\rm Tr}[O_{a|x}Q_{a|x}] \geqslant 0, \forall a,x, \forall Q_{A|X} \in \mathfrak{F}(\mathcal{H})^\circ$. Consider now a set of dual variables $Z_{A|X} := \{Z_{a|x}\} \in \overset{x,a}{\oplus}\mathcal{L}(\mathcal{H})$ with a first dual constraint (i) $Z_{A|X} \geqslant_{\overset{x,a}{\oplus}\mathcal{L}_S^+(\mathcal{H})} 0$. Similarly, for the second primal constraints consider $g(Q_{A|X}) \geqslant 0, \forall Q_{A|X} \in \mathfrak{F}(\mathcal{H})^\circ$. Let us now construct the Lagrangian,

$$\mathcal{L}(O_{A|X}, Z_{A|X}, \mathbb{M}_{A|X}) := \frac{1}{\kappa}\langle O_{A|X}, \overset{x,a}{\oplus}\rho_0\rangle - \langle (O_{A|X} - \mathbb{M}_{A|X}), Z_{A|X}\rangle - \int_{Q_{A|X}\in\mathfrak{F}(\mathcal{H})^\circ} dQ\, g(Q_{A|X})\langle O_{A|X}, Q_{A|X}\rangle \tag{A15}$$

$$= \langle \mathbb{M}_{A|X}, Z_{A|X}\rangle + \left\langle O_{A|X}, \left(\frac{1}{\kappa}\overset{x,a}{\oplus}\rho_0 - Z_{A|X} - Q'_{A|X}\right)\right\rangle, \ Q'_{A|X} := \int_{Q_{A|X}\in\mathfrak{F}(\mathcal{H})^\circ} dQ\, g(Q_{A|X})Q_{A|X} \tag{A16}$$

$$= \sum_{a,x}{\rm Tr}\big[M_{a|x}Z_{a|x}\big] + \sum_{a,x}{\rm Tr}\left[O_{a|x}\left(\frac{1}{\kappa}\rho_0 - Z_{a|x} - Q'_{a|x}\right)\right]. \tag{A17}$$

We have that $Q'_{A|X} := \{Q'_{a|x}\} \in \mathfrak{F}(\mathcal{H})^\circ$ because $Q_{A|X} \in \mathfrak{F}(\mathcal{H})^\circ$ and $\mathfrak{F}(\mathcal{H})^\circ$ is a convex cone. By construction, the Lagrangian satisfies $\mathcal{L}(O_{A|X}, Z_{A|X}, \mathbb{M}_{A|X}) \leqslant 1 + {\rm R}_{\mathbb{F}}(\mathbb{M}_{A|X})$. We can now eliminate the Lagrangian dependence on the primal variables by imposing suitable constraints on the dual variables as ii) $\frac{1}{\kappa}\overset{x,a}{\oplus}\rho_0 - Z_{A|X} - Q'_{A|X} = 0$ (i.e., $\frac{1}{\kappa}\rho_0 - Z_{a|x} - Q'_{a|x} = 0, \forall a,x$) We can then multiply these second dual constraints with $\mathbb{N}_{A|X} = \{N_{a|x}\} \in \mathbb{F}$ to get $1 = \langle Z_{A|X}, N_{A|X}\rangle + \langle Q'_{A|X}, N_{A|X}\rangle$. The last term in the r.h.s is non-negative [since $Q'_{A|X} \in \mathfrak{F}(\mathcal{H})^\circ$ and $N_{A|X} \in \mathfrak{F}(\mathcal{H})$] and so we get $1 \geqslant \langle Z_{A|X}, N_{A|X}\rangle$. Maximising the Lagrangian over the dual variables subject to the dual constraints achieves the upper bound because of strong duality. Strong duality in turn follows from Slatter's condition, since there exists a strictly feasible choice for $Z_{A|X}$, take for instance $Z_{a|x} = \frac{\mathbb{1}}{2d\kappa}, \forall a,x$. This choice satisfies (i) $Z_{a|x} > 0, \forall a,x,$ and (ii) $\langle Z_{A|X}, N_{A|X}\rangle = \sum_{a,x}{\rm Tr}[Z_{a|x}N_{a|x}] = \frac{1}{d\kappa}\sum_{a,x}{\rm Tr}[N_{a|x}] = \frac{1}{2d\kappa}\sum_x {\rm Tr}[\mathbb{1}] = \frac{1}{2} < 1$. Overall, the dual conic program reads

$$\begin{aligned} 1+{\rm R}_{\mathbb{F}}(\mathbb{M}_{A|X}) = \max_{Z_{A|X}} \ & \langle M_{a|x}, Z_{a|x}\rangle, \\ \text{s.t. } & Z_{A|X} \geqslant_{\overset{x,a}{\oplus}\mathcal{L}_S^+} 0, \\ & \langle Z_{A|X}, \mathbb{N}_{A|X}\rangle \leqslant 1, \quad \forall \mathbb{N}_{A|X} \in \mathbb{F}, \end{aligned}$$

or

$$\begin{aligned} 1+{\rm R}_{\mathbb{F}}(\mathbb{M}_{A|X}) = \max_{Z_{A|X}} \ & \sum_{a,x}{\rm Tr}[M_{a|x}Z_{a|x}], \\ \text{s.t. } & Z_{a|x} \geqslant 0, \quad \forall a,x, \\ & \sum_{a,x}{\rm Tr}[Z_{a|x}N_{a|x}] \leqslant 1, \quad \forall \mathbb{N}_{A|X} \in \mathbb{F}, \end{aligned}$$

and thus achieving the claim. ∎

Lemma A 1 can be also rewritten for the exclusion scenario. In short, it reads:

Lemma A 2. (Dual conic programs for the weight of resource of states and POVM sets) The weight of resource of a state ρ and a POVM set $\mathbb{M} = \{M_{a|x}\}$ for $x \in \{1, \dots, \kappa\}$, $a \in \{1, \dots, l\}$ can be written as

$$\begin{aligned} \mathrm{W}_{\mathrm{F}}(\rho) = \max_{Y} \quad & \mathrm{Tr}[(-Y)\rho] + 1, \\ \text{s.t.} \quad & Y \geqslant 0, \\ & \mathrm{Tr}[Y\sigma] \geqslant 1, \quad \forall \sigma \in \mathrm{F}, \end{aligned} \tag{A18}$$

and

$$\begin{aligned} \mathrm{W}_{\mathrm{F}}(\mathbb{M}_{a|x}) = \max_{\{Y_{a,x}\}} \quad & \sum_{x=1}^{\kappa}\sum_{a=1}^{l} \mathrm{Tr}[(-Y_{a,x})M_{a|x}] + 1, \\ \text{s.t.} \quad & Y_{a,x} \geqslant 0, \qquad \forall a, x, \\ & \sum_{x=1}^{\kappa}\sum_{a=1}^{l} \mathrm{Tr}[Y_{a,x}N_{a|x}] \geqslant 1, \quad \forall \mathbb{N}_{A|X} = \{N_{a|x}\} \in \mathbb{F}, \end{aligned} \tag{A19}$$

respectively.

APPENDIX B: PROOF OF RESULT 1

Consider a set of free states F and a set of free POVM assemblages $\mathbb{F}$. The statement we want to prove is

$$\max_{\{p_Y, \Phi_{B|Y}\}} \frac{P^{\mathrm{D}}_{\mathrm{succ}}(p_Y, \Phi_{B|Y}, \rho, \mathbb{M}_{A|X})}{\max_{\substack{\sigma \in \mathrm{F} \\ \mathbb{N}_{A|X} \in \mathbb{F}}} P^{\mathrm{D}}_{\mathrm{succ}}(p_Y, \Phi_{B|Y}, \sigma, \mathbb{N}_{A|X})} = [1 + \mathrm{R}_{\mathrm{F}}(\rho)][1 + \mathrm{R}_{\mathbb{F}}(\mathbb{M}_{A|X})], \tag{B1}$$

with the maximization over all sets of subchannels. We start by proving the upper bond.

Proof. (Upper bound) Given *any* game $(p_Y, \Phi_{B|Y})$ and *any* pair $(\rho, \mathbb{M}_{A|X})$, we have

$$\begin{aligned} P^{\mathrm{D}}_{\mathrm{succ}}(p_Y, \Phi_{B|Y}, \rho, \mathbb{M}_{A|X}) &= \max_{\mathbb{N}_{B|Y} \preceq \mathbb{M}_{A|X}} \sum_{b,y} \mathrm{Tr}[N_{b|y}\Phi_{b|y}(\rho)]\, p(y) \\ &\leqslant [1 + \mathrm{R}_{\mathrm{F}}(\rho)] \max_{\mathbb{N}_{B|Y} \preceq \mathbb{M}_{A|X}} \sum_{b,y} \mathrm{Tr}[N_{b|y}\Phi_{b|y}(\sigma^*)]\, p(y) \\ &\leqslant [1 + \mathrm{R}_{\mathrm{F}}(\rho)] \max_{\sigma \in \mathrm{F}} \max_{\mathbb{N}_{B|Y} \preceq \mathbb{M}_{A|X}} \sum_{b,y} \mathrm{Tr}[N_{b|y}\Phi_{b|y}(\sigma)]p(y) \\ &= [1 + \mathrm{R}_{\mathrm{F}}(\rho)] \max_{\sigma \in \mathrm{F}} \max_{\mathcal{S}} \sum_{b,y} \mathrm{Tr}\left[\left(\sum_{a,x,\mu} p(b|a, y, \mu)\, p(x|y, \mu)\, p(\mu)\, M_{a|x}\right)\Phi_{b|y}(\sigma)\right]p(y) \\ &\leqslant [1 + \mathrm{R}_{\mathrm{F}}(\rho)][1 + \mathrm{R}_{\mathbb{F}}(\mathbb{M}_{A|X})] \max_{\sigma \in \mathrm{F}} \max_{\mathcal{S}} \sum_{b,y} \mathrm{Tr}\left[\left(\sum_{a,x} p(b|a, y, \mu)\, p(x|y, \mu)\, p(\mu)\, M_{a|x}\tilde{N}^*_{a|x}\right)\Phi_{b|y}(\sigma)\right]p(y) \\ &= [1 + \mathrm{R}_{\mathrm{F}}(\rho)][1 + \mathrm{R}_{\mathbb{F}}(\mathbb{M}_{A|X})] \max_{\sigma \in \mathrm{F}} \max_{\mathbb{N}^{\approx}{}_{B|Y} \preceq \tilde{\mathbb{N}}^*_{A|X}} \sum_{b,y} \mathrm{Tr}[\overset{\approx}{N}_{b|y}\Phi_{b|y}(\sigma)]p(y) \\ &\leqslant [1 + \mathrm{R}_{\mathrm{F}}(\rho)][1 + \mathrm{R}_{\mathbb{F}}(\mathbb{M}_{A|X})] \max_{\sigma \in \mathrm{F}} \max_{\tilde{\mathbb{N}}_{A|X} \in \mathbb{F}} \max_{\overset{\approx}{\mathbb{N}}_{B|Y} \preceq \tilde{\mathbb{N}}_{A|X}} \sum_{b,y} \mathrm{Tr}[\overset{\approx}{N}_{b|y}\Phi_{b|y}(\sigma)]p(y) \\ &= [1 + \mathrm{R}_{\mathrm{F}}(\rho)][1 + \mathrm{R}_{\mathbb{F}}(\mathbb{M}_{A|X})] \max_{\sigma \in \mathrm{F}} \max_{\tilde{\mathbb{N}}_{A|X} \in \mathbb{F}} P^{\mathrm{D}}_{\mathrm{succ}}(p_Y, \Phi_{B|Y}, \sigma, \tilde{\mathbb{N}}_{A|X}). \end{aligned} \tag{B2}$$

In the first inequality we use $\Phi_{b|y}(\rho) \leqslant [1 + \mathrm{R}_{\mathrm{F}}(\rho)]\Phi_{b|y}(\sigma^*)$, $\forall x$, which follows from $\rho \leqslant [1 + \mathrm{R}_{\mathrm{F}}(\rho)]\sigma^*$ (σ^* the free state from the definition of the generalized robustness) and $\Phi_{b|y}(\cdot)$ being positive $\forall b, y$. In the second inequality we maximize over all free states. In the third inequality, we use $M_{a|x} \leqslant [1 + \mathrm{R}_{\mathbb{F}}(\mathbb{M}_{A|X})]\tilde{N}^*_{a|x}$, $\forall a, x$, $N^*_{A|X}$ the free measurement assemblage from the definition of the generalized robustness. In the fourth inequality we maximize over all free measurement assemblages. ■

We now prove the upper bound is achievable by using the dual conic programs. We also use the following CPosP operation. Given an arbitrary POVM $\mathbb{N} = \{N_a\}$ with $a \in \{1, \dots, K + N\}$, N and K integers, we then construct the POVM $\tilde{\mathbb{N}} = \{\tilde{N}_x\}$ with K

elements as

$$\tilde{N}_x := N_x, \quad x \in \{1, \dots, K-1\},$$
$$\tilde{N}_K := N_k + \sum_{y=K+1}^{K+N} N_y. \tag{B3}$$

This constitutes a POVM and the operation transforming $\mathbb{N}$ into $\tilde{\mathbb{N}}$ is a CPP operation on the initial measurement $\mathbb{N}$. In short, that any outcome of $\mathbb{N}$ greater or equal than K is now declared as outcome K.

Proof. (Achievability) We start by considering a pair $(\rho, \mathbb{M}_{A|X})$. Using the dual conic formulations, there exist an operator Z^ρ satisfying the conditions (A6a), (A6b), (A6c) and a set of operators $\{Z_{a|x}^{\mathbb{M}_{A|X}}\}$, $x = 1, \dots, \kappa$, $a = 1, \dots, l$, satisfying (A7a), (A7b), and (A7c). Consider also any PMF p_X. We now define a set of maps $\{\Phi_{a|x}^{(\rho,\mathbb{M}_{A|X},p_X)}(\cdot)\}$ such that for any state η,

$$\Phi_{a|x}^{(\rho,\mathbb{M}_{A|X},p_X)}(\eta) := \alpha^{(\rho,\mathbb{M}_{A|X},p_X)}\mathrm{Tr}[Z^\rho \eta] Z_{a|x}^{\mathbb{M}_{A|X}} p(x)^{-1},$$
$$\alpha^{(\rho,\mathbb{M}_{A|X},p_X)} := \frac{1}{\|Z^\rho\|_{\mathrm{op}} \mathrm{Tr}[Z^{\mathbb{M}_{A|X}}]}, \qquad Z^{\mathbb{M}_{A|X}} := \sum_{x=1}^{k}\sum_{a=1}^{l} Z_{a|x}^{\mathbb{M}_{A|X}} p(x)^{-1},$$

with $\|\cdot\|_{\mathrm{op}}$ the operator norm. For simplicity, we use the notation $\alpha \equiv \alpha^{(\rho,\mathbb{M}_{A|X},p_X)}$. We can check that these maps are completely positive, linear, and that they satisfy that $\forall \eta$, $\forall x$,

$$F_x(\eta) := \mathrm{Tr}\left[\sum_{a=1}^{l} \Phi_{a|x}^{(\rho,\mathbb{M}_{A|X},p_X)}(\eta)\right] = \frac{\mathrm{Tr}[Z^\rho \eta]}{\|Z^\rho\|_{\mathrm{op}}} \frac{\mathrm{Tr}\left[\sum_{a=1}^{l} Z_{a|x}^{\mathbb{M}_{A|X}} p(x)^{-1}\right]}{\mathrm{Tr}[Z^{\mathbb{M}_{A|X}}]} \leqslant 1. \tag{B4}$$

The inequality follows from the variational characterization of the operator norm $\|C\|_{\mathrm{op}} = \max_{\rho\in\mathcal{D}(\mathcal{H})}\{|\mathrm{Tr}[C\rho]|\}$ for any Hermitian operator C [93]. We now define an instrument set as follows. Given a pair $(\rho, \mathbb{M}_{A|X})$, $\mathbb{M}_{A|X} = \{M_{a|x}\}$, $x = 1, \dots, \kappa$, $a = 1, \dots, l$, and an integer $J \geqslant 1$, we define the set of subchannels given by $\Psi^{(\rho,\mathbb{M}_{A|X},p_X,J)} = \{\Psi_{b|y}^{(\rho,\mathbb{M}_{A|X},p_X,J)}(\cdot)\}$, $y = 1, \dots, \kappa$, $b = 1, \dots, l+J$, as

$$\Psi_{b|y}^{(\rho,\mathbb{M}_{A|X},p_X J)}(\eta) := \begin{cases} \alpha \mathrm{Tr}[Z^\rho \eta] Z_{b|y}^{\mathbb{M}_{A|X}} p(y)^{-1}, & b = 1, \dots, l \\ \frac{1}{J}[1 - F_y(\eta)]\chi, & b = l+1, \dots, l+J \end{cases} \tag{B5}$$

with χ begin an arbitrary quantum state $\chi \geqslant 0$, $\mathrm{Tr}[\chi] = 1$. We can check that this is a well-defined set of subchannels because they add up to a CPTP linear map as follows. We have that $\forall J$, $\forall \eta$, $\forall y$,

$$\begin{aligned}
\mathrm{Tr}\left[\sum_{b=1}^{l+J} \Psi_{b|y}^{(\rho,\mathbb{M}_{A|X},p_X,J)}(\eta)\right] &= \mathrm{Tr}\left[\sum_{b=1}^{l} \Psi_{b|y}^{(\rho,\mathbb{M}_{A|X},p_X,J)}(\eta)\right] + \mathrm{Tr}\left[\sum_{b=l+1}^{l+J} \Psi_{b|y}^{(\rho,\mathbb{M}_{A|X},p_X,J)}(\eta)\right],\\
&= \mathrm{Tr}\left[\sum_{b=1}^{l} \alpha \mathrm{Tr}[Z^\rho \eta] Z_{b|y}^{\mathbb{M}_{A|X}} p(y)^{-1}\right] + \mathrm{Tr}\left[\sum_{b=l+1}^{l+J} \frac{1}{J}[1 - F_y(\eta)]\chi\right],\\
&= \alpha \mathrm{Tr}[Z^\rho \eta]\mathrm{Tr}\left[\sum_{b=1}^{l} Z_{b|y}^{\mathbb{M}_{A|X}} p(y)^{-1}\right] + \frac{1}{J}[1 - F_y(\eta)]\sum_{b=l+1}^{l+J}\mathrm{Tr}[\chi],\\
&= \alpha \mathrm{Tr}[Z^\rho \eta]\mathrm{Tr}\left[\sum_{b=1}^{l} Z_{b|y}^{\mathbb{M}_{A|X}} p(y)^{-1}\right] + [1 - F_y(\eta)],\\
&= F_y(\eta) + [1 - F_y(\eta)],\\
&= 1.
\end{aligned}$$

We now analyze the multiobject subchannel discrimination game with prior information given by $\Psi_{B|Y}^{(\rho,\mathbb{M}_{A|X},p_X,J)}$ and a PMF p_Y that we specify as $p_Y = p_X$, which can be done since $|Y| = |X| = \kappa$. We start by analyzing the best *fully free* player,

$$\max_{\substack{\sigma\in\mathrm{F}\\ \mathbb{N}_{A|X}\in\mathbb{F}}} P_{\mathrm{succ}}^{\mathrm{D}}\left(p_Y, \Psi_{B|Y}^{(\rho,\mathbb{M}_{A|X},p_X,J)}, \sigma, \mathbb{N}_{A|X}\right) = \max_{\substack{\sigma\in\mathrm{F}\\ \mathbb{N}_{A|X}\in\mathbb{F}\\ \tilde{\mathbb{N}}_{B|Y}\preceq\mathbb{N}_{A|X}}} \sum_{y=1}^{\kappa}\sum_{b=1}^{l+J} \mathrm{Tr}\left[\tilde{N}_{b|y}\Psi_{b|y}^{(\rho,\mathbb{M}_{A|X},p_X,J)}(\sigma)\right]p(y).$$

We are considering QRTs of POVM sets closed under CProP, so the optimal set $\{\tilde{N}_{b|y}\}$ is a free object. Let us now consider, without loss of generality, that this maximization is achieved by the *fully free* pair $(\sigma^*, \mathbb{N}^*_{B|Y})$. We have

$$\max_{\sigma\in\mathrm{F}\,\mathbb{N}_{A|X}\in\mathbb{F}} P^{\mathrm{D}}_{\mathrm{succ}}\big(p_Y, \Psi^{(\rho,\mathbb{M}_{A|X},p_X,J)}_{B|Y}, \sigma, \mathbb{N}_{A|X}\big) = \sum_{y=1}^{\kappa}\sum_{b=1}^{l+J}\mathrm{Tr}\big[N^*_{b|y}\Psi^{(\rho,\mathbb{M}_{A|X},p_X,J)}_{b|y}(\sigma^*)\big]p(y), \tag{B6}$$

$$= \alpha\mathrm{Tr}[Z^\rho\sigma^*]\sum_{y=1}^{\kappa}\sum_{b=1}^{l}\mathrm{Tr}\big[N^*_{b|y}Z^{\mathbb{M}_{A|X}}_{b|y}\big] + \sum_{y=1}^{\kappa}\sum_{b=l+1}^{l+J}\frac{1}{J}[1-F_y(\sigma^*)]\mathrm{Tr}[N^*_{b|y}\chi]p(y). \tag{B7}$$

In the second equality we have replaced the subchannel game (B5). The first term can be upper bounded as

$$\sum_{y=1}^{\kappa}\sum_{b=1}^{l}\mathrm{Tr}\big[N^*_{b|y}Z^{\mathbb{M}_{A|X}}_{b|y}\big] \leqslant \sum_{y=1}^{\kappa}\sum_{b=1}^{l}\mathrm{Tr}\big[\tilde{N}^*_{b|y}Z^{\mathbb{M}_{A|B}}_{b|y}\big] \leqslant 1,$$

with the POVM $\tilde{\mathbb{N}}^*_y$ (with l outcomes) constructed from the POVM $\mathbb{N}^*_y$ (which has $l+J$ outcomes), $\forall y = 1,\dots,\kappa$, as defined in (B3). The first inequality follows from the definition of the POVMs $\tilde{\mathbb{N}}^*_y$ (B3). In the second inequality we use the fact that $\tilde{\mathbb{N}}^*_{B|Y}$ is a free set of POVMs (because it was constructed from a free set of POVMs $\mathbb{N}^*_{B|Y}$ and a CPP operation) and therefore we can use the conic program condition (A7c). We now also use the fact that $1-F_y(\eta)\leqslant 1$, $\forall\eta$, $\forall y$, as well as (A6c) and so equation (B7) becomes

$$\max_{\substack{\sigma\in\mathrm{F}\\ \mathbb{N}_{A|X}\in\mathbb{F}}} P^{\mathrm{D}}_{\mathrm{succ}}\big(p_Y, \Psi^{(\rho,\mathbb{M}_{A|X},p_X,J)}_{B|Y}, \sigma, \mathbb{N}_{A|X}\big) \leqslant \alpha + \frac{1}{J}\sum_{y=1}^{\kappa}p(y)\sum_{b=l+1}^{l+J}\mathrm{Tr}[N^*_{b|y}\chi].$$

The second term in the latter expression can now be upper bounded as

$$\sum_{b=l+1}^{l+J}\mathrm{Tr}[N^*_{b|y}\chi] \leqslant \sum_{b=1}^{l+J}\mathrm{Tr}[N^*_{b|y}\chi] = \mathrm{Tr}\left[\left(\sum_{b=1}^{l+J}N^*_{b|y}\right)\chi\right] = 1, \qquad \forall y.$$

The inequality follows because we added l non-negative terms and the last equality follows from $\mathbb{N}^*_y$ being a POVM, $\sum_{b=1}^{l+J}N^*_{b|y} = \mathbb{1}$, $\forall y$, and χ being a quantum state. We then get

$$\max_{\substack{\sigma\in\mathrm{F}\\ \mathbb{N}_{A|X}\in\mathbb{F}}} P^{\mathrm{D}}_{\mathrm{succ}}(p_Y, \Psi^{(\rho,\mathbb{M}_{A|X},p_X,J)}, \sigma, \mathbb{N}_{A|X}) \leqslant \alpha + \frac{1}{J}.$$

We now choose the subchannel game given by $\Psi^{(\rho,\mathbb{M}_{A|X},p_X,J\to\infty)}$ and therefore we get

$$\max_{\substack{\sigma\in\mathrm{F}\\ \mathbb{N}_{A|X}\in\mathbb{F}}} P^{\mathrm{D}}_{\mathrm{succ}}(p_Y, \Psi^{(\rho,\mathbb{M}_{A|X},p_X,J\to\infty)}, \sigma, \mathbb{N}_{A|X}) \leqslant \alpha. \tag{B8}$$

We now analyze the probability of success of a player using the fully resourceful pair $(\rho, \mathbb{M}_{A|X})$,

$$\begin{aligned}
P^{\mathrm{D}}_{\mathrm{succ}}\big(p_Y, \Psi^{(\rho,\mathbb{M}_{A|X},p_X,J)}_{B|Y}, \rho, \mathbb{M}_{A|X}\big) &= \max_{\mathbb{N}_{B|Y}\preceq\mathbb{M}_{A|X}}\sum_{y=1}^{\kappa}\sum_{b=1}^{l+J}\mathrm{Tr}\big[N_{b|y}\Psi^{(\rho,\mathbb{M}_{A|X},p_X,J)}_{b|y}(\rho)\big]\,p(y)\\
&\geqslant \sum_{y=1}^{\kappa}\sum_{b=1}^{l}\mathrm{Tr}\big[M_{b|y}\Psi^{(\rho,\mathbb{M}_{A|X},p_X,J)}_{b|y}(\rho)\big]\,p(y)\\
&= \alpha\mathrm{Tr}[Z^\rho\rho]\sum_{y=1}^{\kappa}\sum_{b=1}^{l}\mathrm{Tr}\big[M_{b|y}Z^{\mathbb{M}_{A|X}}_{b|y}\big]\\
&= \alpha[1+\mathrm{R}_{\mathrm{F}}(\rho)][1+\mathrm{R}_{\mathbb{F}}(\mathbb{M}_{A|X})].
\end{aligned} \tag{B9}$$

The inequality follows because one can choose to simulate the specific measurement, i.e., $N_{b|y} = M_{b|y}$ for $b\leqslant l$ and $N_{b|y} = 0$ for $l < b < J$. We have replaced the subchannel discrimination game with (B5). The last line follows from (A6a) and (A7a). We now choose the subchannel game given by $\Psi^{(\rho,\mathbb{M}_{A|X},p_X,J\to\infty)}$ and have

$$P^{\mathrm{D}}_{\mathrm{succ}}\big(p_Y, \Psi^{(\rho,\mathbb{M}_{A|X},p_X,J\to\infty)}_{B|Y}, \rho, \mathbb{M}_{A|X}\big) \geqslant \max_{\substack{\sigma\in\mathrm{F}\\ \mathbb{N}_{A|X}\in\mathbb{F}}} P^{\mathrm{D}}_{\mathrm{succ}}\big(p_Y, \Psi^{(\rho,\mathbb{M}_{A|X},p_X,J\to\infty)}_{B|Y}, \sigma, \mathbb{N}_{A|X}\big)[1+\mathrm{R}_{\mathrm{F}}(\rho)][1+\mathrm{R}_{\mathbb{F}}(\mathbb{M}_{A|X})]. \tag{B10}$$

Putting together (B10) and (B2) achieves

$$\frac{P^{\rm D}_{\rm succ}\big(p_Y,\Psi_{B|Y}^{(\rho,\mathbb{M}_{A|X},p_X,J\to\infty)},\rho,\mathbb{M}_{A|X}\big)}{\max_{\substack{\sigma\in{\rm F}\\ \mathbb{N}_{A|X}\in\mathbb{F}}} P^{\rm D}_{\rm succ}\big(p_Y,\Psi_{B|Y}^{(\rho,\mathbb{M}_{A|X},p_X,J\to\infty)},\sigma,\mathbb{N}_{A|X}\big)}=[1+{\rm R}_{\rm F}(\rho)][1+{\rm R}_{\mathbb{F}}(\mathbb{M}_{A|X})].$$

This shows the upper bound is achievable thus completing the proof. ■

APPENDIX C: PROOF OF RESULT 2

Following the same line of reasoning as in Appendix B and the preliminaries as in Appendix A, here we prove the lower bound and the achievability of the statement in Result 2 [Eq. (20)],

$$\min_{\{p_Y,\Phi_{B|Y}\}}\frac{P^{\rm E}_{\rm err}(p_Y,\Phi_{B|Y},\rho,\mathbb{M}_{A|X})}{\min_{\substack{\sigma\in{\rm F}\\ \mathbb{N}_{A|X}\in\mathbb{F}}} P^{\rm E}_{\rm err}(p_Y,\Phi_{B|Y},\sigma,\mathbb{N}_{A|X})}=[1-{\rm W}_{\rm F}(\rho)][1-{\rm W}_{\mathbb{F}}(\mathbb{M}_{A|X})],$$

where the probability of error in quantum subchannel exclusion with prior information reads

$$P^{\rm E}_{\rm err}(p_Y,\Phi_{B|Y},\rho,\mathbb{M}_{A|X}):=\min_{\mathcal{S}}\sum_{g,a,x,z,b,y}\delta_{g,b}\,s(g|a,y,z){\rm Tr}[M_{a|x}\Phi_{b|y}(\rho)]r(x|y,z)\,q(z)\,p(y) \tag{C1}$$

$$=\min_{\mathbb{N}_{B|Y}\preceq\mathbb{M}_{A|X}}\sum_{b,y}{\rm Tr}[N_{b|y}\Phi_{b|y}(\rho)]p(y), \tag{C2}$$

with the minimization over all the possible strategies $\mathcal{S}$ and POVM sets $\mathbb{N}_{B|Y}$ simulable by $\mathbb{M}_{A|X}$, respectively.

Proof. (Lower bound) Given any game $(p_Y,\Phi_{B|Y})$ and any pair $(\rho,\mathbb{M}_{A|X})$, we have

$$\begin{aligned}
P^{\rm E}_{\rm err}(p_Y,\Phi_{B|Y},\rho,\mathbb{M}_{A|X})&=\min_{\mathbb{N}_{B|Y}\preceq\mathbb{M}_{A|X}}\sum_{b,y}{\rm Tr}[N_{b|y}\Phi_{b|y}(\rho)]p(y)\\
&\geqslant[1-{\rm W}_{\rm F}(\rho)]\min_{\sigma\in F}\min_{\mathbb{N}_{B|Y}\preceq\mathbb{M}_{A|X}}\sum_{b,y}{\rm Tr}[N_{b|y}\Phi_{b|y}(\sigma)]p(y),\\
&=[1-{\rm W}_{\rm F}(\rho)]\min_{\sigma\in{\rm F}}\min_{S}\sum_{b,y}{\rm Tr}\left[\sum_{a,x,z}s(b|a,y,z)\,M_{a|x}r(x|y,z)\,q(z)\,\Phi_{b|y}(\sigma)\right]p(y)\\
&\geqslant[1-{\rm W}_{\rm F}(\rho)][1-{\rm W}_{\rm F}(\mathbb{M}_{A|X})]\min_{\sigma\in{\rm F}}\min_{\tilde{\mathbb{N}}_{A|X}\in\mathbb{F}}\min_{S}\sum_{b,y}{\rm Tr}\left[\sum_{a,x,z}s(b|a,y,z)\,\tilde{N}_{a|x}\,r(x|y,z)\,q(z)\,\Phi_{b|y}(\sigma)\right]p(y)\\
&=[1-{\rm W}_{\rm F}(\rho)][1-{\rm W}_{\rm F}(\mathbb{M}_{A|X})]\min_{\sigma\in F}\min_{\tilde{\mathbb{N}}_{A|X}\in\mathbb{F}}\min_{\tilde{\tilde{\mathbb{N}}}_{B|Y}\preceq\tilde{\mathbb{N}}_{A|X}}\sum_{b,y}{\rm Tr}[\tilde{\tilde{N}}_{b|y}\Phi_{b|y}(\sigma)]p(y)\\
&=[1-{\rm W}_{\rm F}(\rho)][1-{\rm W}_{\rm F}(\mathbb{M}_{A|X})]\min_{\sigma\in F}\min_{\tilde{\mathbb{N}}_{A|X}\in\mathbb{F}}P^{\rm E}_{\rm err}(p_Y,\Phi_{B|Y},\sigma,\tilde{\mathbb{N}}_{A|X}).
\end{aligned} \tag{C3}$$

From the definition of the weight of the resource [Eq. (6)] for quantum states and we have $\rho={\rm W}_{\rm F}(\rho)\rho^G+(1-{\rm W}_{\rm F}(\rho))\sigma^*$ for ${\rm W}_{\rm F}(\rho)$ and $\sigma^*\in{\rm F}$ satisfying the minimum in the above definition. The positivity of $\Phi_{b|y}$ then implies $\Phi_{b|y}(\rho)\geqslant(1-{\rm W}_{\rm F}(\rho))\Phi_{b|y}(\sigma^*)$, which shows the first inequality. The equivalent definition of weight for POVM sets leads to the inequality $M_{a|x}\geqslant(1-{\rm W}_{\mathbb{F}}(\mathbb{M}_{A|X}))\tilde{N}^*_{a|x}$ for ${\rm W}_{\mathbb{F}}(\mathbb{M}_{A|X})$ and $\tilde{\mathbb{N}}^*_{A|X}=\{\tilde{N}^*_{a|x}\}\in\mathbb{F}$ that satisfies the minimum weight. Thus the second inequality is proved. ■

Proof. (Achievability) For a pair $(\rho,\mathbb{M}_{A|X})$, the dual cone formulations assures the existence of an operator Y^ρ satisfying conditions (A18) and a set of operators $\{Y^{\mathbb{M}_{A|X}}_{a|x}\}$ $(x=1,\dots,\kappa,\ a=1,\dots,l)$ satisfying (A19). We define a set of subchannels $\{\Phi^{(\rho,\mathbb{M}_{A|X},p_X)}_{a|x}\}_{a,x}$ such that for any state η,

$$\Phi^{(\rho,\mathbb{M}_{A|X},p_X)}_{a|x}(\eta):=\beta^{(\rho,\mathbb{M}_{A|X},p_X)}\,{\rm Tr}[Y^\rho\eta]\,Y^{\mathbb{M}_{A|X}}_{a|x}\,p(x)^{-1},$$

$$\beta\equiv\beta^{(\rho,\mathbb{M}_{A|X},p_X)}:=\frac{1}{2\|Y^\rho\|_{\rm op}\,{\rm Tr}[Y^{\mathbb{M}_{A|X}}]},\quad Y^{\mathbb{M}_{A|X}}:=\sum_{x=1}^{\kappa}\sum_{a=1}^{l}Y^{\mathbb{M}_{A|X}}_{a|x}\,p(x)^{-1}.$$

We can verify that these maps are completely positive and linear, and that they satisfy

$$G_x(\eta):={\rm Tr}\left[\sum_{a=1}^{l}\Phi^{(\rho,\mathbb{M}_{A|X},p_X)}_{a|x}(\eta)\right]=\frac{{\rm Tr}[Y^\rho\rho]}{2\|Y^\rho\|_{\rm op}}\frac{{\rm Tr}\big[\sum_{a=1}^{l}Y^{\mathbb{M}_{A|X}}_{a|x}\,p(x)^{-1}\big]}{{\rm Tr}[Y^{\mathbb{M}_{A|X}}]}\leqslant\frac{1}{2},\quad\forall\,\eta,x. \tag{C4}$$

Given a pair $(\rho, \mathbb{M}_{A|X})$, where $\mathbb{M}_{A|X} = \{M_{a|x}\}, x = 1, \dots, \kappa, a = 1, \dots, l,$ we now define an instrument set $\Psi^{(\rho,\mathbb{M}_{A|X},p_X)}_{B|Y} = \{\Psi^{(\rho,\mathbb{M}_{A|X},p_X)}_{b|y}(\cdot)\}, y = 1, \dots, \kappa, b = 1, \dots, l+1$, as

$$\Psi^{(\rho,\mathbb{M}_{A|X},p_X)}_{b|y}(\eta) := \begin{cases} \beta\,\mathrm{Tr}[Y^\rho \eta] Y^{\mathbb{M}_{A|X}}_{b|y}\, p(y)^{-1}, & b = 1, \dots, l \\ [1 - G_y(\eta)]\xi^{\mathbb{M}_{A|X}}_y, & b = l+1 \end{cases}, \qquad \xi^{\mathbb{M}_{B|Y}}_y := \frac{\sum_{b=1}^l p(b|y) Y^{\mathbb{M}_{A|X}}_{b|y}}{\sum_{b=1}^l p(b|y)\mathrm{Tr}\big[Y^{\mathbb{M}_{A|X}}_{b|y}\big]}, \tag{C5}$$

with $\{p(b|y)\}_{b,y}$ an arbitrary conditional PMF. We can verify that this is a well-defined instrument set because they sum up to a CPTP map as follows, $\forall\eta$, $\forall y$,

$$\begin{aligned} \mathrm{Tr}\left[\sum_{b=1}^{l+1} \Psi^{(\rho,\mathbb{M}_{A|X},p_X)}_{b|y}(\eta)\right] &= \mathrm{Tr}\left[\sum_{b=1}^{l} \Psi^{(\rho,\mathbb{M}_{A|X},p_X)}_{b|y}(\eta)\right] + \mathrm{Tr}\big[\Psi^{(\rho,\mathbb{M}_{A|X},p_X)}_{(l+1)|y}(\eta)\big] \\ &= \mathrm{Tr}\left[\sum_{b=1}^{l} \Phi^{(\rho,\mathbb{M}_{A|X},p_X)}_{b|y}(\eta)\right] + \mathrm{Tr}\big[(1 - G_y(\eta))\xi^{\mathbb{M}_{A|X}}_y\big] \\ &= G_y(\eta) + [1 - G_y(\eta)]\mathrm{Tr}\big[\xi^{\mathbb{M}_{A|X}}_y\big] \\ &= 1. \end{aligned}$$

We now analyse the multiobject subchannel exclusion game with prior information given by $\Psi^{(\rho,\mathbb{M}_{A|X},p_X)}_{B|Y}$ and a PMF $p_Y = p_X$, which can be done because $|Y| = |X| = \kappa$. We start by addressing the best fully free player,

$$\min_{\sigma\in\mathrm{F}} \min_{\mathbb{N}_{A|X}\in\mathbb{F}} P^{\mathrm{E}}_{\mathrm{err}}\big(p_Y, \Psi^{(\rho,\mathbb{M}_{A|X},p_X)}_{B|Y}, \sigma, \mathbb{N}_{A|X}\big) = \min_{\substack{\sigma\in\mathrm{F} \\ \mathbb{N}_{A|X}\in\mathbb{F} \\ \tilde{\mathbb{N}}_{B|Y}\preceq\mathbb{N}_{A|X}}} \sum_{y=1}^{\kappa}\sum_{b=1}^{l+1} \mathrm{Tr}\big[\tilde{N}_{b|y}\Psi^{(\rho,\mathbb{M}_{A|X},p_X)}_{b|y}(\sigma)\big]p(y). \tag{C6}$$

Let us consider that the minimum in the right hand side is achieved with the fully free pair ($\sigma = \sigma^*$, $\tilde{\mathbb{N}}_{B|Y} = \mathbb{N}^*_{B|Y}$). This can be done because the free set $\mathbb{F}$ is assumed to be closed under simulability of POVM sets. We then can write this as

$$\min_{\sigma\in\mathrm{F}} \min_{\mathbb{N}_{A|X}\in\mathbb{F}} P^{\mathrm{E}}_{\mathrm{err}}\big(p_Y, \Psi^{(\rho,\mathbb{M}_{A|X},p_X)}_{B|Y}, \sigma, \mathbb{N}_{A|X}\big) = \sum_{y=1}^{\kappa}\sum_{b=1}^{l+1} \mathrm{Tr}\big[N^*_{b|y}\Psi^{(\rho,\mathbb{M}_{A|X},p_X)}_{b|y}(\sigma^*)\big]p(y) \tag{C7}$$

$$= \beta\,\mathrm{Tr}[Y^\rho\sigma^*]\mathrm{Tr}[\sum_{y=1}^{\kappa}\sum_{b=1}^{l} N^*_{b|y}Y^{\mathbb{M}_{A|X}}_{b|y}] + \sum_{y=1}^{\kappa}[1 - G_y(\sigma^*)]\mathrm{Tr}\big[N^*_{(l+1)|y}\,\xi^{\mathbb{M}_{A|X}}_y\big]p(y). \tag{C8}$$

We now introduce the POVM set $\tilde{\mathbb{N}}^*_{B|Y} = \{\tilde{N}^*_{b|y}\}$ $(y = 1, \dots, \kappa,\ b = 1, \dots, l)$ with $\tilde{N}^*_{b|y} := N^*_{b|y} + p(b|y)N^*_{(l+1)|y}$ via a CPosP of $\{N^*_{b|y}\}$, where $\{p(b|y)\}$ is the PMF from the state in (C5). We now add and subtract the term $\beta\mathrm{Tr}[Y^\rho\sigma^*]\sum_{y=1}^{\kappa}\sum_{b=1}^{l}\mathrm{Tr}[p(b|y)N^*_{(l+1)|y}Y^{\mathbb{M}_{A|X}}_{b|y}]$, and so (C8) can be rewritten as

$$\begin{aligned} \min_{\sigma\in\mathrm{F}} \min_{\mathbb{N}_{A|X}\in\mathbb{F}} P^{\mathrm{E}}_{\mathrm{err}}\big(p_Y, \Psi^{(\rho,\mathbb{M}_{A|X},p_X)}_{B|Y}, \sigma, \mathbb{N}_{A|X}\big) &= \beta\,\mathrm{Tr}[Y^\rho\sigma^*]\mathrm{Tr}\left[\sum_{y=1}^{\kappa}\sum_{b=1}^{l}\tilde{N}^*_{b|y}Y^{\mathbb{M}_{A|X}}_{b|y}\right] + \sum_{y=1}^{\kappa}[1 - G_y(\sigma^*)]\mathrm{Tr}\big[N^*_{(l+1)|y}\,\xi^{\mathbb{M}_{A|X}}_y\big]p(y) \\ &\quad - \beta\mathrm{Tr}[Y^\rho\sigma^*]\sum_{y=1}^{\kappa}\sum_{b=1}^{l} p(b|y)\mathrm{Tr}\big[N^*_{(l+1)|y}\,Y^{\mathbb{M}_{A|X}}_{b|y}\big]. \end{aligned} \tag{C9}$$

The first term in (C9) can be lower bounded by β because $\mathrm{Tr}[Y^\rho\sigma^*] \geqslant 1$ and $\sum_{y=1}^{\kappa}\sum_{b=1}^{l}\mathrm{Tr}[Y^{\mathbb{M}_{A|X}}_{b|y}\tilde{N}^*_{b|y}] \geqslant 1$ for the positive semidefinite operators Y^ρ and $\{Y^{\mathbb{M}_{A|X}}_{b|y}\}$ [as per (A19)]. Thus, we have

$$\begin{aligned} \min_{\sigma\in\mathrm{F}} \min_{\mathbb{N}_{A|X}\in\mathbb{F}} P^{\mathrm{E}}_{\mathrm{err}}\big(p_Y, \Psi^{(\rho,\mathbb{M}_{A|X},p_X)}_{B|Y}, \sigma, \mathbb{N}_{A|X}\big) &\geqslant \beta + \sum_{y=1}^{\kappa}[1 - G_y(\sigma^*)]\mathrm{Tr}\big[N^*_{(l+1)|y}\,\xi^{\mathbb{M}_{A|X}}_y\big]p(y) \\ &\quad - \beta\mathrm{Tr}[Y^\rho\sigma^*]\sum_{y=1}^{\kappa}\sum_{b=1}^{l} p(b|y)\mathrm{Tr}\big[N^*_{(l+1)|y}\,Y^{\mathbb{M}_{A|X}}_{b|y}\big]. \end{aligned} \tag{C10}$$

Let us now prove that the last two terms add up to a non-negative value. The last two terms can be written as

$$\sum_{y=1}^{\kappa}[1-G_y(\sigma^*)]\mathrm{Tr}\big[N^*_{(l+1)|y}\,\xi_y^{\mathbb{M}_{A|X}}\big]p(y)-\beta\mathrm{Tr}[Y^\rho\sigma^*]\sum_{y=1}^{\kappa}\sum_{b=1}^{l}p(b|y)\mathrm{Tr}\big[N^*_{(l+1)|y}Y_{b|y}^{\mathbb{M}_{A|X}}\big]$$
$$=\mathrm{Tr}\left[\sum_{y=1}^{\kappa}N^*_{(l+1)|y}\left([1-G_y(\sigma^*)]\xi_y^{\mathbb{M}_{A|X}}p(y)-\beta\mathrm{Tr}[Y^\rho\sigma^*]\sum_{b=1}^{l}p(b|y)Y_{b|y}^{\mathbb{M}_{A|X}}\right)\right]. \tag{C11}$$

Let us check the operator inside the brackets is positive semidefinite for all y. Let us write

$$[1-G_y(\sigma^*)]\xi_y^{\mathbb{M}_{A|X}}p(y)-\beta\mathrm{Tr}[Y^\rho\sigma^*]\sum_{b=1}^{l}p(b|y)Y_{b|y}^{\mathbb{M}_{A|X}}$$
$$=[1-G_y(\sigma^*)]\frac{\sum_{b'=1}^{l}p(b'|y)Y_{b'|y}^{\mathbb{M}_{A|X}}}{\sum_{b'=1}^{l}p(b'|y)\mathrm{Tr}\big[Y_{b'|y}^{\mathbb{M}_{A|X}}\big]}p(y)-\beta\mathrm{Tr}[Y^\rho\sigma^*]\sum_{b=1}^{l}p(b|y)Y_{b|y}^{\mathbb{M}_{A|X}}.$$

Multiplying by the positive term $A_y:=\sum_{b'=1}^{l}p(b'|y)\mathrm{Tr}[Y_{b'|y}^{\mathbb{M}_{A|X}}]$, we get

$$[1-G_y(\sigma^*)]\left(\left[\sum_{b'=1}^{l}p(b'|y)Y_{b'|y}^{\mathbb{M}_{A|X}}\right]\right)p(y)-\beta\mathrm{Tr}[Y^\rho\sigma^*]\left(\left[\sum_{b=1}^{l}p(b|y)Y_{b|y}^{\mathbb{M}_{A|X}}\right]\right)A_y$$
$$\stackrel{1}{=}([1-G_y(\sigma^*)]p(y)-\beta\mathrm{Tr}[Y^\rho\sigma^*]A_y)\left(\left[\sum_{b=1}^{l}p(b|y)Y_{b|y}^{\mathbb{M}_{A|X}}\right]\right)$$

We now analyze the coefficient inside the first brackets,

$$[1-G_y(\sigma^*)]p(y)-\beta\mathrm{Tr}[Y^\rho\sigma^*]A_y\stackrel{1}{=}[1-G_y(\sigma^*)]p(y)-\beta\mathrm{Tr}[Y^\rho\sigma^*]\left(\sum_{b'=1}^{l}p(b'|y)\mathrm{Tr}\big[Y_{b'|y}^{\mathbb{M}_{A|X}}\big]\right) \tag{C12}$$
$$\stackrel{2}{=}p(y)-\beta\,\mathrm{Tr}[Y^\rho\sigma^*]\mathrm{Tr}\left[\sum_{b=1}^{l}Y_{b|y}^{\mathbb{M}_{A|X}}\right]p(y)^{-1}p(y)-\beta\mathrm{Tr}[Y^\rho\sigma^*]\left(\sum_{b'=1}^{l}p(b'|y)\mathrm{Tr}\big[Y_{b'|y}^{\mathbb{M}_{A|X}}\big]\right) \tag{C13}$$
$$\stackrel{3}{=}p(y)-\beta\,\mathrm{Tr}[Y^\rho\sigma^*]\mathrm{Tr}\left[\sum_{b=1}^{l}Y_{b|y}^{\mathbb{M}_{A|X}}\right]-\beta\mathrm{Tr}[Y^\rho\sigma^*]\left(\sum_{b'=1}^{l}p(b'|y)\mathrm{Tr}\big[Y_{b'|y}^{\mathbb{M}_{A|X}}\big]\right) \tag{C14}$$
$$\geqslant p(y)-2\beta\,\mathrm{Tr}[Y^\rho\sigma^*]\mathrm{Tr}\left[\sum_{b=1}^{l}Y_{b|y}^{\mathbb{M}_{A|X}}\right]. \tag{C15}$$

In the first equality we replace A_y. In the second equality we replace $G_y(\sigma^*)$. In the third equality we reorganize. Finally, the inequality is due to the fact that we are subtracting a larger quantity. We now continue,

$$p(y)-2\beta\,\mathrm{Tr}[Y^\rho\sigma^*]\sum_{b=1}^{l}\mathrm{Tr}\big[Y_{b|y}^{\mathbb{M}_{A|X}}\big]\stackrel{1}{=}p(y)-\frac{\mathrm{Tr}[Y^\rho\sigma^*]}{\|Y^\rho\|_{\mathrm{op}}}\frac{\sum_{b=1}^{l}\mathrm{Tr}\big[Y_{b|y}^{\mathbb{M}_{A|X}}\big]}{\mathrm{Tr}[Y^{\mathbb{M}_{A|X}}]}$$
$$\stackrel{2}{=}p(y)-\frac{\mathrm{Tr}[Y^\rho\sigma^*]}{\|Y^\rho\|_{\mathrm{op}}}\frac{\sum_{b=1}^{l}\mathrm{Tr}\big[Y_{b|y}^{\mathbb{M}_{A|X}}\big]}{\sum_{y'=1}^{\kappa}\sum_{b=1}^{l}\mathrm{Tr}\big[Y_{b|y'}^{\mathbb{M}_{A|X}}p(y')^{-1}\big]}$$
$$\stackrel{3}{\geqslant}p(y)-\frac{\mathrm{Tr}[Y^\rho\sigma^*]}{\|Y^\rho\|_{\mathrm{op}}}\frac{p(y)\sum_{y'=1}^{\kappa}\sum_{b=1}^{l}\mathrm{Tr}\big[Y_{b|y'}^{\mathbb{M}_{A|X}}p(y')^{-1}\big]}{\sum_{y'=1}^{\kappa}\sum_{b=1}^{l}\mathrm{Tr}\big[Y_{b|y'}^{\mathbb{M}_{A|X}}p(y')^{-1}\big]}$$
$$\stackrel{4}{=}p(y)\left(1-\frac{\mathrm{Tr}[Y^\rho\sigma^*]}{\|Y^\rho\|_{\mathrm{op}}}\right)$$
$$\stackrel{5}{\geqslant}0.$$

In the first line we replace β. In the second line we replace $Y^{\mathbb{M}_{A|X}}$. The inequality in the third line follows because $\sum_{y'=1}^{\kappa}\sum_{b=1}^{l}\mathrm{Tr}[Y_{b|y'}^{\mathbb{M}_{A|X}}p(y')^{-1}]\geqslant\sum_{b=1}^{l}\mathrm{Tr}[Y_{b|y}^{\mathbb{M}_{A|X}}p(y)^{-1}]$, $\forall y$, and so $p(y)\sum_{y'=1}^{\kappa}\sum_{b=1}^{l}\mathrm{Tr}[Y_{b|y'}^{\mathbb{M}_{A|X}}p(y')^{-1}]\geqslant\sum_{b=1}^{l}\mathrm{Tr}[Y_{b|y}^{\mathbb{M}_{A|X}}]$, $\forall y$. In the fourth line we reorganize. The inequality in the fifth line follows because $\frac{\mathrm{Tr}[Y^\rho \sigma^*]}{\|Y^\rho\|_{\mathrm{op}}} \leqslant 1$. This then ultimately implies that

$$\min_{\sigma\in F}\min_{\mathbb{N}_{A|X}\in\mathbb{F}} P^{\mathrm{E}}_{\mathrm{err}}\big(p_Y,\Psi^{(\rho,\mathbb{M}_{A|X},p_X)}_{B|Y},\sigma,\mathbb{N}_{A|X}\big)\geqslant\beta. \tag{C16}$$

To complete the proof, let us analyze the exclusion game $(p_Y, \Psi^{(\rho,\mathbb{M}_{A|X},p_X)}_{B|Y})$ with a fully resourceful pair $(\rho, \mathbb{M}_{A|X})$. The probability of error is given by

$$\begin{aligned}
P^{\mathrm{E}}_{\mathrm{err}}\big(p_Y,\Psi^{(\rho,\mathbb{M}_{A|X},p_X)}_{B|Y},\rho,\mathbb{M}_{A|X}\big) &= \min_{\mathbb{N}_{B|Y}\preceq\mathbb{M}_{A|X}}\sum_{y=1}^{\kappa}\sum_{b=1}^{l+1}\mathrm{Tr}\big[N_{b|y}\Psi^{(\rho,\mathbb{M}_{A|X},p_X)}_{b|y}(\rho)\big]p(y)\\
&\leqslant \sum_{y=1}^{\kappa}\sum_{b=1}^{l+1}\mathrm{Tr}\big[\tilde{M}_{b|y}\Psi^{(\rho,\mathbb{M}_{A|X},p_X)}_{b|y}(\rho)\big]p(y)\\
&= \beta\,\mathrm{Tr}[Y^\rho\rho]\sum_{y=1}^{\kappa}\sum_{b=1}^{l}\mathrm{Tr}\big[M_{b|y}Y^{\mathbb{M}_{A|X}}_{b|y}\big]p(y)^{-1}p(y)\\
&= \beta\,\mathrm{Tr}[Y^\rho\rho]\sum_{y=1}^{\kappa}\sum_{b=1}^{l}\mathrm{Tr}\big[M_{b|y}Y^{\mathbb{M}_{A|X}}_{b|y}\big].
\end{aligned}$$

The inequality is due to the choice of $\mathbb{N}_{B|Y}=\{N_{b|y}\}=\{\tilde{M}_{b|y}\}$ $(y=1,\dots,\kappa,\ b=1,\dots,l+1)$ with

$$\tilde{M}_{b|y}=\begin{cases}M_{b|y} & (b=1,\dots,l)\\ 0 & (b=l+1),\end{cases}$$

which is a CProP of $\mathbb{M}_{A|X}=\{M_{a|x}\}$. Since $\mathrm{Tr}[Y^\rho\rho]=1-\mathrm{W_F}(\rho)$ and $\sum_{y=1}^{\kappa}\sum_{b=1}^{l}\mathrm{Tr}[M_{b|y}Y^{\mathbb{M}_{A|X}}_{b|y}]=1-\mathrm{W}_{\mathbb{F}}(\mathbb{M}_{A|X})$ hold for the positive operators Y^ρ and $\{Y^{\mathbb{M}_{A|X}}_{b|y}\}$, respectively, we obtain

$$P^{\mathrm{E}}_{\mathrm{err}}\big(p_Y,\Psi^{(\rho,\mathbb{M}_{A|X},p_X)}_{b|y},\rho,\mathbb{M}_{B|Y}\big)\leqslant\beta[1-\mathrm{W_F}(\rho)][1-\mathrm{W}_{\mathbb{F}}(\mathbb{M}_{A|X})]. \tag{C17}$$

It follows from (C16) and (C17) that the ratio of interest is upper bounded as

$$\begin{aligned}
\frac{P^{\mathrm{E}}_{\mathrm{err}}\big(p_Y,\Psi^{(\rho,\mathbb{M}_{A|X},p_X)}_{b|y},\rho,\mathbb{M}_{B|Y}\big)}{\min_{\sigma\in F}\min_{\mathbb{N}_{A|X}\in\mathbb{F}}P^{\mathrm{E}}_{\mathrm{err}}\big(p_Y,\Psi^{(\rho,\mathbb{M}_{A|X},p_X)}_{b|y},\sigma,\mathbb{N}_{A|X}\big)} &\leqslant \frac{\beta[1-\mathrm{W_F}(\rho)][1-\mathrm{W_F}(\mathbb{M}_{A|X})]}{\min_{\sigma\in F}\min_{\mathbb{N}_{A|X}\in\mathbb{F}}P^{\mathrm{E}}_{\mathrm{err}}\big(p_Y,\Psi^{(\rho,\mathbb{M}_{A|X},p_X)}_{b|y},\sigma,\mathbb{N}_{A|X}\big)}\\
&\leqslant \frac{\beta[1-\mathrm{W_F}(\rho)][1-\mathrm{W_F}(\mathbb{M}_{A|X})]}{\beta}\\
&= [1-\mathrm{W_F}(\rho)][1-\mathrm{W_F}(\mathbb{M}_{A|X})].
\end{aligned} \tag{C18}$$

Putting together (C3) and (C18), we finally prove Result 2 (20). ■

APPENDIX D: PROOF OF RESULT 3 (GPTs)

In this Appendix, we prove (28) in Result 3 following the proof for the quantum case in Appendix B. We will omit the proof of (29), but it is given similarly by generalizing the argument in Appendix C. We first rephrase Lemma A 1 in terms of GPTs:

Lemma D 1. Let V_+ be the positive cone generated by the state space Ω and V_+° be the dual cone. We can regard V_+ and V_+° as CCCs in $V = lin(\Omega)$, and they define orderings $\leqslant_{V_+}$ and $\leqslant_{V_+^\circ}$ in V through $x \leqslant_{V_+} y \overset{\text{def}}{\Longleftrightarrow} y - x \in V_+$ and $x \leqslant_{V_+^\circ} y \overset{\text{def}}{\Longleftrightarrow} y - x \in V_+^\circ$ respectively. The generalized robustness of resource of a state $\omega \in \Omega$ and a measurement set $\mathbb{E}_{A|X} = \{e_{a|x}\}$ $(x \in \{1,\dots,\kappa\},\ a \in \{1,\dots,l\})$ are given by

$$\mathrm{R^{GP}_F}(\omega) = \max_z \quad \langle z, \omega\rangle - 1, \tag{D1a}$$

$$\text{s.t.} \quad z \geqslant_{V_+^\circ} 0, \tag{D1b}$$

$$\langle z, \sigma\rangle \leqslant 1 \ \ (\forall \sigma \in \mathrm{F}), \tag{D1c}$$

and

$$\mathrm{R}^{\mathrm{GP}}_{\mathbb{F}}(\mathbb{E}_{A|X}) = \max_{\{z_{a,x}\}} \sum_{x=1}^{\kappa}\sum_{a=1}^{l}\langle e_{a|x}, z_{a,x}\rangle - 1, \tag{D2a}$$

$$\text{s.t.} \quad z_{a,x} \geqslant_{V_+} 0 \ \ (\forall a, x), \tag{D2b}$$

$$\sum_{x=1}^{\kappa}\sum_{a=1}^{l} \langle N_{a|x}, z_{a,x}\rangle \leqslant 1 \ \ (\forall \mathbb{N} = \{N_{a|x}\} \in \mathbb{F}). \tag{D2c}$$

These are the dual conic formulations of the generalized robustnesses for states and measurement sets.

Proof. The proof proceeds in a similar way as Lemma A 1. In fact, instead of the vector space $\mathcal{L}_S(\mathcal{H})$, the Hilbert-Schmidt inner product $\langle\cdot,\cdot\rangle_{\mathrm{HS}}$, and the cone $\mathcal{L}_S^+(\mathcal{H})$, here we use V, $\langle\cdot,\cdot\rangle$, and V_+°, respectively [remember that $(V_+^\circ)^\circ = V_+$ holds]. It is easy to see that the same argument in the proof of Lemma A 1 can be developed also in this case. The strong duality is verified by the fact that the positive cone V_+ is generating ($lin(V_+) = V$, which is satisfied in the present setting) if and only if there is an interior point $z_0 \in \mathrm{int}(V_+)$. Multiplying z_0 by small $\lambda > 0$, we can construct a strictly feasible solution $\{z_{a,x}\}$ with $z_{a,x} = \lambda z_0$ $(\forall a, x)$ for (D1b) and (D1c), and thus the strong duality holds. ■

Proof. (Upper bound) For any GPScD-PI $(p_Y, \Xi_{B|Y}, \omega, \mathbb{E}_{A|X})$, we have

$$\begin{aligned}
P^{\mathrm{GPD}}_{\mathrm{succ}}(p_Y, \Xi_{B|Y}, \omega, \mathbb{E}_{A|X}) &= \max_{\mathbb{N}_{B|Y}\preceq \mathbb{E}_{A|X}} \sum_{b,y}\langle N_{b|y}, \xi_{b|y}(\omega)\rangle\, p(y)\\
&\leqslant \left[1 + \mathrm{R}^{\mathrm{GP}}_{\mathrm{F}}(\omega)\right] \max_{\mathbb{N}_{B|Y}\preceq \mathbb{E}_{A|X}} \sum_{b,y}\left\langle N_{b|y}, \xi_{b|y}(\sigma^*)\right\rangle p(y)\\
&\leqslant \left[1 + \mathrm{R}^{\mathrm{GP}}_{\mathrm{F}}(\omega)\right] \max_{\sigma\in\mathrm{F}}\max_{\mathbb{N}_{B|Y}\preceq \mathbb{E}_{A|X}} \sum_{b,y}\langle N_{b|y}, \xi_{b|y}(\sigma)\rangle p(y)\\
&= \left[1 + \mathrm{R}^{\mathrm{GP}}_{\mathrm{F}}(\omega)\right] \max_{\sigma\in\mathrm{F}}\max_{\mathcal{S}} \sum_{b,y}\left\langle\left(\sum_{a,x,\mu} p(b|a,y,\mu)\, p(x|y,\mu)\, p(\mu)\, e_{a|x}\right), \xi_{b|y}(\sigma)\right\rangle p(y)\\
&\leqslant \left[1 + \mathrm{R}^{\mathrm{GP}}_{\mathrm{F}}(\omega)\right]\left[1 + \mathrm{R}^{\mathrm{GP}}_{\mathbb{F}}(\mathbb{E}_{A|X})\right] \max_{\sigma\in\mathrm{F}}\max_{\mathcal{S}} \sum_{b,y}\left\langle\left(\sum_{a,x} p(b|a,y,\mu)\, p(x|y,\mu)\, p(\mu)\, \tilde{N}^*_{a|x}\right), \xi_{b|y}(\sigma)\right\rangle p(y)\\
&= \left[1 + \mathrm{R}^{\mathrm{GP}}_{\mathrm{F}}(\omega)\right]\left[1 + \mathrm{R}^{\mathrm{GP}}_{\mathbb{F}}(\mathbb{E}_{A|X})\right] \max_{\sigma\in\mathrm{F}}\max_{\mathbb{N}^{\approx}{}_{B|Y}\preceq \tilde{\mathbb{N}}^*_{A|X}} \sum_{b,y}\langle \tilde{\tilde{N}}_{b|y}, \xi_{b|y}(\sigma)\rangle p(y)\\
&\leqslant \left[1 + \mathrm{R}^{\mathrm{GP}}_{\mathrm{F}}(\omega)\right]\left[1 + \mathrm{R}^{\mathrm{GP}}_{\mathbb{F}}(\mathbb{E}_{A|X})\right] \max_{\sigma\in\mathrm{F}}\max_{\tilde{\mathbb{N}}_{A|X}\in\mathbb{F}}\max_{\tilde{\tilde{\mathbb{N}}}_{B|Y}\preceq \tilde{\mathbb{N}}_{A|X}} \sum_{b,y}\langle \tilde{\tilde{N}}_{b|y}, \xi_{b|y}(\sigma)\rangle p(y)\\
&= \left[1 + \mathrm{R}^{\mathrm{GP}}_{\mathrm{F}}(\omega)\right]\left[1 + \mathrm{R}^{\mathrm{GP}}_{\mathbb{F}}(\mathbb{E}_{A|X})\right] \max_{\sigma\in\mathrm{F}}\max_{\mathbb{N}_{A|X}\in\mathbb{F}} P^{\mathrm{GPD}}_{\mathrm{succ}}(p_Y, \Xi_{B|Y}, \sigma, \mathbb{N}_{A|X}).
\end{aligned} \tag{D3}$$

In the first inequality we use $\xi_{b|y}(\omega) \leqslant_{V_+} [1 + \mathrm{R}^{\mathrm{GP}}_{\mathrm{F}}(\omega)]\xi_{b|y}(\sigma^*)$, $\forall x$, which follows from $\omega \leqslant_{V_+} [1 + \mathrm{R}^{\mathrm{GP}}_{\mathrm{F}}(\omega)]\sigma^*$ (σ^* the free state from the definition of the generalized robustness) and $\xi_{b|y}(\cdot)$ being positive $\forall b, y$. In the second inequality we maximize over all free states. In the third inequality, we use $e_{a|x} \leqslant_{V_+^\circ} [1 + \mathrm{R}^{\mathrm{GP}}_{\mathbb{F}}(\mathbb{E}_{A|X})]\tilde{N}^*_{a|x}$, $\forall a, x$, $\mathbb{N}^*_{A|X}$ the free measurement set from the definition of the generalized robustness. In the fourth inequality we maximize over all free measurement sets. ■

Proof. (Achievability) Let $(\omega, \mathbb{E}_{A|X})$ be a pair of a state and a measurement set of a GPT $(\Omega, \mathscr{E})$. As we have seen in Lemma D 1, there exist an element $z^\omega \in V_+^\circ$ satisfying the conditions (D1a), (D1b), (D1c) and a set of elements $\{z^{\mathbb{E}_{A|X}}_{a|x}\} \subset V_+$ $(x = 1, \ldots, \kappa, a = 1, \ldots, l)$ satisfying (D2a), (D2b), and (D2c). Let p_X be a PMF with $p(x) > 0$ $(\forall x)$. We define a set of maps $\{\xi^{(\omega,\mathbb{E}_{A|X},p_X)}_{a|x}(\cdot)\}$ such that for any state $\eta \in \Omega$,

$$\begin{aligned}
\xi^{(\omega,\mathbb{E}_{A|X},p_X)}_{a|x}(\eta) &:= \alpha^{(\omega,\mathbb{E}_{A|X})}\langle z^\omega, \eta\rangle z^{\mathbb{E}_{A|X}}_{a|x}\, p(x)^{-1},\\
\alpha \equiv \alpha^{(\omega,\mathbb{E}_{A|X})} &:= \frac{1}{\|z^\omega\|_u \langle u, z^{\mathbb{E}_{A|X}}\rangle}, \quad z^{\mathbb{E}_{A|X}} := \sum_{x=1}^{\kappa}\sum_{a=1}^{l} z^{\mathbb{E}_{A|X}}_{a|x}\, p(x)^{-1}.
\end{aligned} \tag{D4}$$

In the equations, $u \in \mathscr{E}$ is the unit effect, and $\|\cdot\|_u$ is the *order unit norm* [94] in V defined as

$$\begin{aligned}
\|z\|_u &:= \inf\{\lambda \geqslant 0 \mid -\lambda u \leqslant_{V_+^\circ} z \leqslant_{V_+^\circ} \lambda u\}\\
&= \sup\{|\langle z, \omega\rangle| \mid \omega \in \Omega\}.
\end{aligned} \tag{D5}$$

The order unit norm is clearly a natural generalization of the operator norm for quantum formulation, where $u = \mathbb{1}$ and $\Omega = \mathcal{D}(\mathcal{H})$. We can check that these maps are linear and positive, i.e., $\xi_{a|x}^{(\omega,\mathbb{E}_{A|X},p_X)} : V_+ \to V_+$ $(\forall a, x)$. Also, they satisfy $\forall\eta, \forall x$,

$$F_x(\eta) := \left\langle u,\ \sum_{a=1}^{l} \xi_{a|x}^{(\omega,\mathbb{E}_{A|X})}(\eta) \right\rangle = \frac{\langle z^\omega, \eta\rangle}{\|z^\omega\|_u} \frac{\left\langle u,\ \left(\sum_{a=1}^{l} z_{a|x}^{\mathbb{E}_{A|X}} p(x)^{-1}\right)\right\rangle}{\left\langle u, z^{\mathbb{E}_{A|X}}\right\rangle} \leqslant 1.$$

because $\langle z^\omega, \eta\rangle \leqslant \|z^\omega\|_u$ and $\langle u,\ (\sum_{a=1}^{l} z_{a|x}^{\mathbb{E}_{A|X}} p(x)^{-1})\rangle \leqslant \langle u, z^{\mathbb{E}_{A|X}}\rangle$ from the definitions (D5) and (D4), respectively. We now construct an instrument set that realizes the maximum in (28) as follows. Given a pair $(\omega, \mathbb{E}_{A|X})$ $(x = 1, \ldots, \kappa, a = 1, \ldots, l)$ and an integer $J \geqslant 1$, we define $\Xi_{B|Y}^{(\omega,\mathbb{E}_{A|X},p_X,J)} = \{\xi_{b|y}^{(\omega,\mathbb{E}_{A|X},p_X,J)}\}$ $(y = 1, \ldots, \kappa, b = 1, \ldots, l + J)$ by

$$\xi_{b|y}^{(\omega,\mathbb{E}_{A|X},p_X,J)}(\eta) := \begin{cases} \alpha\langle z^\omega, \eta\rangle z_{b|y}^{\mathbb{E}_{A|X}} p(y)^{-1} & (b = 1, \ldots, l) \\ \frac{1}{J}[1 - F_y(\eta)]\chi & (b = l + 1, \ldots, l + J) \end{cases} \tag{D6}$$

with an arbitrary state $\chi \in \Omega$. This is a well-defined instrument set: they are positive and add up to a channel because

$$\begin{aligned} \left\langle u,\ \sum_{b=1}^{l+J} \xi_{b|y}^{(\omega,\mathbb{E}_{A|X},p_X,J)}(\eta) \right\rangle &= \left\langle u,\ \sum_{b=1}^{l} \xi_{b|y}^{(\omega,\mathbb{E}_{A|X},p_X,J)}(\eta) \right\rangle + \left\langle u,\ \sum_{b=l+1}^{l+J} \xi_{b|y}^{(\omega,\mathbb{E}_{A|X},p_X,J)}(\eta) \right\rangle \\ &= \left\langle u,\ \sum_{b=1}^{l} \alpha\langle z^\omega, \eta\rangle z_{b|y}^{\mathbb{E}_{A|X}} p(y)^{-1} \right\rangle + \left\langle u,\ \sum_{b=l+1}^{l+J} \frac{1}{J}[1 - F_y(\eta)]\chi \right\rangle \\ &= F_y(\eta) + [1 - F_y(\eta)] \\ &= 1 \quad (\forall y). \end{aligned}$$

We note that for each y the instrument $\Xi_y = \{\xi_{b|y}^{(\omega,\mathbb{E}_{A|X},p_X,J)}\}_{b=1}^{l+J}$ expresses a *measure-and-prepare channel* and such channel is "completely positive" also in the framework of GPTs [73]. Let us analyze the GPScD-PI given by $\Xi_{B|Y}^{(\omega,\mathbb{E}_{A|X},p_X,J)}$ and a PMF p_Y that we specify as $p_Y = p_X$ $(B = \{1, \ldots, l + J\},\ Y = \{1, \ldots, \kappa\})$. For fully free cases, we have

$$\max_{\substack{\sigma\in\mathrm{F} \\ \mathbb{N}_{A|X}\in\mathbb{F}}} P_{\mathrm{succ}}^{\mathrm{GPD}}\left(p_Y, \Xi_{B|Y}^{(\omega,\mathbb{E}_{A|X},p_X,J)}, \sigma, \mathbb{N}_{A|X}\right) = \max_{\substack{\sigma\in\mathrm{F} \\ \mathbb{N}_{A|X}\in\mathbb{F} \\ \tilde{\mathbb{N}}_{B|Y}\preceq\mathbb{N}_{A|X}}} \sum_{y=1}^{\kappa}\sum_{b=1}^{l+J} \left\langle \tilde{N}_{b|y},\ \xi_{b|y}^{(\omega,\mathbb{E}_{A|X},p_X,J)}(\sigma)\right\rangle p(y).$$

Because of the compactness, we can assume that this maximization is achieved by the fully free pair $(\sigma^*, \mathbb{N}_{B|Y}^*)$. We then have

$$P_{\mathrm{succ}}^{\mathrm{GPD}}\left(p_Y, \Xi_{B|Y}^{(\omega,\mathbb{E}_{A|X},p_X,J)}, \sigma^*, \mathbb{N}^*\right) = \sum_{y=1}^{\kappa}\sum_{b=1}^{l+J} \left\langle N_{b|y}^*,\ \xi_{b|y}^{(\omega,\mathbb{N},p_X,J)}(\sigma^*)\right\rangle p(y) \tag{D7}$$

$$= \alpha\langle z^\omega, \sigma^*\rangle \sum_{y=1}^{\kappa}\sum_{b=1}^{l} \left\langle N_{b|y}^*,\ z_{b|y}^{\mathbb{E}_{A|B}}\right\rangle + \frac{1}{J}\sum_{y=1}^{\kappa}\sum_{b=l+1}^{l+J} [1 - F_y(\sigma^*)]\langle N_{b|y}^*,\ \chi\rangle p(y). \tag{D8}$$

To evaluate the first term, we introduce a set of l-outcome measurements $\tilde{\mathbb{N}} = \{\tilde{N}_y^*\} = \{\tilde{N}_{b'|y}^*\}$ constructed from $\mathbb{N}_{B|Y}^*$ as

$$\begin{aligned} \tilde{N}_{b'|y}^* &:= N_{b'|y}^*, \quad b' \in \{1, \ldots, l - 1\}, \\ \tilde{N}_{l|y}^* &:= N_{l|y}^* + \sum_{b=l+1}^{l+J} N_{b|y}^*. \end{aligned} \tag{D9}$$

We have

$$\sum_{y=1}^{\kappa}\sum_{b=1}^{l} \left\langle N_{b|y}^*,\ z_{b|y}^{\mathbb{E}_{A|X}}\right\rangle \leqslant \sum_{y=1}^{\kappa}\sum_{b=1}^{l} \left\langle \tilde{N}_{b|y}^*,\ z_{b|y}^{\mathbb{E}_{A|X}}\right\rangle \leqslant 1.$$

The first inequality is straightforward, and the second inequality follows from the fact that the construction (D9) of $\tilde{\mathbb{N}}$ is a CPosP of $\mathbb{N}_{B|Y}^*$ and thus $\tilde{\mathbb{N}} \preceq \mathbb{N}_{B|Y}^*$, which enables us to use (D2c). For the second term of (D8), we use $1 - F_y(\eta) \leqslant 1$ $(\forall\eta, \forall y)$. The relation (D8) becomes

$$P^{\rm GPD}_{\rm succ}\big(p_Y,\, \Xi^{(\omega,\mathbb{E}_{A|X},p_X,J)}_{B|Y},\, \sigma,\, \mathbb{N}_{A|X}\big) \leqslant \alpha + \frac{1}{J}\sum_{y=1}^{\kappa}\sum_{b=l+1}^{l+J} p(y)\langle N^*_{b|y},\, \chi\rangle.$$

The right-hand side can be upper bounded as

$$\sum_{y=1}^{\kappa}\sum_{b=l+1}^{l+J} p(y)\langle N^*_{b|y},\, \chi\rangle \leqslant \sum_{y=1}^{\kappa}\sum_{b=1}^{l+J} p(y)\langle N^*_{b|y},\, \chi\rangle = \sum_{y=1}^{\kappa} p(y) = 1 \quad (\forall y),$$

and thus

$$\max_{\substack{\sigma\in\mathrm{F}\\ \mathbb{N}_{A|X}\in\mathbb{F}}} P^{\rm GPD}_{\rm succ}\big(p_Y,\, \Xi^{(\omega,\mathbb{E}_{A|X},p_X,J)}_{B|Y},\, \sigma,\, \mathbb{N}_{A|X}\big) \leqslant \alpha + \frac{1}{J}.$$

With $J \to \infty$, we obtain

$$\max_{\substack{\sigma\in\mathrm{F}\\ \mathbb{N}_{A|X}\in\mathbb{F}}} P^{\rm GPD}_{\rm succ}\big(p_Y,\, \Xi^{(\omega,\mathbb{E}_{A|X},p_X,J\to\infty)},\, \sigma,\, \mathbb{N}_{A|X}\big) \leqslant \alpha. \tag{D10}$$

We next investigate the fully resourceful $(\omega,\, \mathbb{E}_{A|X})$. It holds from (D1a) and (D2a) that

$$\begin{aligned} P^{\rm GPD}_{\rm succ}\big(p_Y,\, \Xi^{(\omega,\mathbb{E}_{A|X},p_X,J)}_{B|Y},\, \omega,\, \mathbb{E}_{A|X}\big) &= \max_{\mathbb{N}_{B|Y}\preceq\mathbb{M}_{A|X}} \sum_{y=1}^{\kappa}\sum_{b=1}^{l+J} \big\langle N_{b|y},\, \xi^{(\omega,\mathbb{E}_{A|X},p_X,J)}_{b|y}(\omega)\big\rangle\, p(y) \\ &\geqslant \sum_{y=1}^{\kappa}\sum_{b=1}^{l} \big\langle e_{b|y},\, \xi^{(\omega,\mathbb{E}_{A|X},p_X,J)}_{b|y}(\omega)\big\rangle\, p(y) \\ &= \alpha\langle z^{\omega},\, \omega\rangle \sum_{y=1}^{\kappa}\sum_{b=1}^{l} \big\langle e_{b|y},\, z^{\mathbb{E}_{A|X}}_{b|y}\big\rangle \\ &= \alpha\big[1+\mathrm{R}^{\rm GP}_{\mathrm{F}}(\omega)\big]\big[1+\mathrm{R}^{\rm GP}_{\mathbb{F}}(\mathbb{E}_{A|X})\big]. \end{aligned} \tag{D11}$$

With $J \to \infty$, the relations (D10) and (D11) imply

$$P^{\rm GPD}_{\rm succ}\big(p_Y,\, \Xi^{(\omega,\mathbb{E}_{A|X},p_X,J\to\infty)}_{B|Y},\, \omega,\, \mathbb{E}_{A|X}\big) \geqslant \max_{\substack{\sigma\in\mathrm{F}\\ \mathbb{N}_{A|X}\in\mathbb{F}}} P^{\rm GPD}_{\rm succ}\big(p_Y,\, \Xi^{(\omega,\mathbb{E}_{A|X},p_X,J\to\infty)}_{B|Y},\, \sigma,\, \mathbb{N}_{A|X}\big)\big[1+\mathrm{R}^{\rm GP}_{\mathrm{F}}(\omega)\big]\big[1+\mathrm{R}^{\rm GP}_{\mathbb{F}}(\mathbb{E}_{A|X})\big]. \tag{D12}$$

We can conclude from (D3) and (D12)

$$\frac{P^{\rm GPD}_{\rm succ}\big(p_Y,\, \Xi^{(\omega,\mathbb{E}_{A|X},p_X,J\to\infty)}_{B|Y},\, \omega,\, \mathbb{E}_{A|X}\big)}{\max_{\substack{\sigma\in\mathrm{F}\\ \mathbb{N}_{A|X}\in\mathbb{F}}} P^{\rm GPD}_{\rm succ}\big(p_Y,\, \Xi^{(\omega,\mathbb{E}_{A|X},p_X,J\to\infty)}_{B|Y},\, \sigma,\, \mathbb{N}_{A|X}\big)} = \big[1+\mathrm{R}^{\rm GP}_{\mathrm{F}}(\omega)\big]\big[1+\mathrm{R}^{\rm GP}_{\mathbb{F}}(\mathbb{E}_{A|X})\big],$$

which completes the proof. ■

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