

Title	Note on orders over which an hereditary order is projective
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Citation	Osaka Journal of Mathematics. 1967, 4(1), p. 151-156
Version Type	VoR
URL	https://doi.org/10.18910/10297
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NOTE ON ORDERS OVER WHICH AN HEREDITARY ORDER IS PROJECTIVE

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(Received December, 21, 1966)

Let R be an integral noetherian domain with quotient field K , and Σ a semi-simple K -algebra with finite dimension. An R -algebra Λ in Σ is called an R -order when Λ is R -finitely generated and $\Lambda K = \Sigma$.

The author has studied essential properties of hereditary orders in [2] and [3], (briefly h -order), which has the property that every one-sided ideal in Λ is Λ -projective.

In this short note we shall show the following theorem as a corollary to [3], Proposition 5.1.

Theorem. *Let R be an integral noetherian domain with quotient field K and Σ a K -separable semi-simple algebra. Let Λ be an R -order in Σ . We assume that there exists an h -order Γ over R which properly containing Λ and is left Λ -projective. Then there exists an h -order Γ_0 such that $\Gamma \supseteq \Gamma_0 \supseteq \Lambda$.*

From this theorem we shall give some criteria of order being hereditary, which contain [6], Theorem 5.1: If every maximal order containing Λ is Λ -projective, then Λ is hereditary, whenever Σ is a central simple K -algebra.

Finally, we shall give another criterion in a special case that Λ is an h -order if and only if Λ has a unique irredundant representation by maximal orders, (see the below for the definition). Hijikata has already shown the above fact in [5] by a direct computation. We give here a proof by the method in [3].

First we assume that R is an integral noetherian domain with quotient field K and Σ a semi-simple K -algebra. Let Γ be an h -order in Σ and Z the center of Γ .

Lemma 1. *Let R , Γ , Z and Σ be as above, and Λ an R -order contained properly in Γ . If Γ is left Λ -projective, then $\Gamma \neq \Lambda Z$ and Γ is left ΛZ -projective.*

Proof. Let $C = \text{Hom}_{\Lambda}^l(\Gamma, \Lambda)$. Then C is right Λ -projective, since Γ is left Λ -projective. We put $\Lambda' = \Lambda Z$. Then C is right Λ' -projective and Γ is left Λ' -projective from [2], Lemma 1.3. Furthermore, $C' = \text{Hom}_{\Lambda'}^l(\Gamma, \Lambda')$

is also right Λ' -projective. We know from [2], Lemma 1.3 and Proposition 1.6 that $\Gamma = \text{End}_{\Lambda}^r(C) = \text{End}_{\Lambda'}^r(C) = \text{End}_{\Lambda'}^r(C')$ and hence $C = C'$ from [2], Lemma 1.5 and Proposition 1.6. If $\Gamma = \Lambda'$, $C' = \Lambda'$, which would imply $\Lambda = \Gamma$.

Let $\Sigma = \sum_{i=1}^n \oplus \Sigma_i$ be the decomposition of Σ into simple algebras Σ_i and $1 = \sum_{i=1}^n e_i$, where the e_i 's are identities in Σ_i . Then every h -order Λ is written as $\Lambda = \sum \oplus \Lambda e_i$. Therefore, if we are interested in an order Λ which is contained in a Λ -projective h -order Γ , we may assume from Lemma 1 that Σ is a simple K -algebra.

Proof of Theorem. From Lemma 1 and the above remark we may assume that Σ is a central simple K -algebra and Λ is an order over Z . Furthermore, we may restrict ourselves to a case of R being a discrete rank one, valuation ring by [2], Theorem 2.6. Let $C = \text{Hom}_{\Lambda}^l(\Gamma, \Lambda)$, then $C\Gamma = \Gamma$, since Γ is Λ -projective. Let N be the radical of Γ , and $A = C + N$. Since $C\Gamma = \Gamma$ and $C \neq \Gamma$, A is not a two-sided ideal in Γ . Hence, $\Gamma_0 = \Gamma \cap \text{End}_{\Gamma}^r(A)$ is an h -order by [3], Proposition 5.1. Furthermore, since A is a right Λ -module, $\Gamma_0 \cong \Lambda$. It is clear that $\Gamma \cong \Gamma_0$.

Corollary 1. *We assume that Σ is a central simple K -algebra and R is a discrete rank one, valuation ring. If an R -order Λ is contained in a minimal h -order¹⁾, which is left Λ -projective, then Λ is hereditary.*

We have defined a rank of h -order Λ in [3], p. 10 under the assumption of Corollary 1 which is equal to the number of maximal two-sided ideals in Λ .

Proposition 1. *Let Σ and R be as in Corollary 1. We assume that an h -order Γ of rank r contains an R -order Λ and that Γ is left Λ -projective. If $C \cap N/CN$ contains s non-isomorphic irreducible components as a right Λ/N -module, then Λ is contained in an h -order of rank $r + s$, where $C = \text{Hom}_{\Gamma}^l(\Gamma, \Lambda)$ and N is the radical of Λ .*

Proof. Let $\{\Omega_i\}_{i=1}^r$ be the set of maximal orders containing Γ and $C_i = \text{Hom}_{\Lambda}^l(\Omega_i, \Lambda)$. Since Ω_i is left Γ -projective, Ω_i is left Λ -projective. Hence, C_i is a minimal two-sided idempotent ideal in Λ by [2], Lemma 5.1. It is well known that $\Gamma = \text{End}_{\Lambda}^r(C)$ and $\Gamma/N_{\Gamma} = \text{End}_{\Lambda/N}^r(C/CN)$, where N_{Γ} is the radical of Γ , (see [2], Propositions 1.6 and 4.4). Furthermore, $C/CN \cong C + N/N \oplus C \cap N/CN$ as a right Λ -module and $C + N/N \cong C_1 + N/N \oplus \dots \oplus C_r + N/N$ by [3], Lemma 3.2. Since Γ/N_{Γ} contains only r simple components, $C + N/N = C_1 + N/N \oplus \dots \oplus C_r + N/N$. Let $A = C + N_{\Gamma}$ and $A/N_{\Gamma} = \sum_{i=1}^r \oplus \bar{L}_i$,

1) See [3], p. 3 for the definition of minimal order.

where \bar{L}_i is a left ideal in a simple component $\bar{\Gamma}_i$ of $\bar{\Gamma}/N\bar{\Gamma}_i$. Since $C = \sum_{i=1}^r C_i$ and $\Gamma/N_\Gamma \approx \text{End}_{\Lambda/N}(C/CN)$, $\bar{L}_i \neq (0)$ for all i . Furthermore, since $C(C \cap N/CN) = (0)$, the number of non-isomorphic irreducible components of $C \cap N/CN$ is equal to the number of \bar{L}_i such that $\bar{L}_i \neq \bar{\Gamma}_i$. Hence, the rank of $\Gamma_0 = \Gamma \cap \text{End}_\Lambda^r(A)$ is equal to $r+s$ by [3], Theorem 5.3.

REMARK. Corollary 1 is not true if we replace a minimal h -order by an arbitrary h -order containing Λ , (see the example below).

Lemma 2. *Let R be a Dedekind domain and Ω an order containing an order Λ . Then there exists only a finite many of orders between Ω and Λ .*

Proof. Since $r\Omega \subseteq \Lambda$ for some $r \neq 0$ in R , if a prime ideal p in R does not divide r , then $\Omega_p = \Lambda_p$. If p divides r , then $\Omega_p/r\Omega_p = \Sigma \oplus R_p/rR_p$. Hence, we have the lemma.

Let Λ and Λ' be orders. We have called in [2], p. 281 that Λ and Λ' are the same type if there exists $\Lambda - \Lambda'$ ideal A such that $\text{End}_\Lambda^l(A) = \Lambda'$ and $\text{End}_{\Lambda'}^r(A) = \Lambda$.

Proposition 2. *Let R be a Dedekind domain and Σ a separable, simple K -algebra. Let Λ be an order over R . If every h -order belonging to the same type and containing Λ is left Λ -projective, then Λ is hereditary.*

Proof. Let Γ be an h -order containing Λ and Z the center of Γ . We note that Z is also a Dedekind domain. We put $\Lambda' = \Lambda Z$. Then there exists a minimal one Γ' among h -orders over Z containing Λ' by Lemma 2. Let p be a prime in Z , and $\Gamma_1, \Gamma_2, \dots, \Gamma_n$ be the set of h -orders containing Λ which belong to the same type as Γ_p . Then $\tau_{\Lambda_p}^l(\sum_{i=1}^n \oplus \Gamma_i) = \Gamma_p$ by [2], Theorems 3.3 and 4.3. Hence, $\tau_{\Gamma'}^l(\Sigma \oplus \Gamma'') = \Gamma'$, where Γ'' runs through all h -orders of same type containing Λ' . Therefore, Γ' is a direct summand of $(\Sigma \oplus \Gamma'')^m$ for some m . Hence, Γ' is a left Λ' -projective²⁾ by Lemma 1, which implies from Theorem that $\Lambda' = \Gamma'$. Furthermore, since $\Sigma \oplus \Gamma''$ is Λ -projective, Γ' is Λ -projective. Every h -order contains Z by [2], Proposition 2.2. Hence, we obtain $\Lambda' = \Lambda$ from Theorem.

Corollary 2. *Let R and Σ be as above and Λ an R -order. Λ is an h -order if one of the following conditions is satisfied.*

- 1) Every maximal order containing Λ is left Λ -projective.
- 2) We have an h -order Γ containing Λ such that there exists a maximal chain $\Gamma = \Gamma_1 \supset \Gamma_2 \supset \dots \supset \Gamma_n = \Lambda$ between Γ and Λ and furthermore, each Γ_i is left Λ -projective, (cf. [2], Theorem 5.3 and [6], Theorem 5.1)

2) This is an analogy to Silver's method in [6].

Proof. It is clear from Proposition 2 and Theorem.

Next we shall consider a condition that an h-order Γ is Λ -projective for an order $\Gamma \supseteq \Lambda$.

Lemma 3. *Let $\Gamma \supseteq \Lambda$ be orders. Then Γ is left Λ -projective if and only if $C\Gamma = \Gamma$, where $C = \text{Hom}_\Lambda^l(\Gamma, \Lambda)$.*

Proof. It is clear from the standard map: $C \otimes_\Lambda \Gamma \rightarrow \text{Hom}_\Lambda^l(\Gamma, \Gamma) = \Gamma$.

Proposition 3. *Let Λ be an order in Σ and C an idempotent two-sided ideal in Λ . If $\text{End}_\Lambda^r(C) = \Gamma$ is an h-order, then Γ is left Λ -projective, C is left and right Λ -projective and $C = \text{Hom}_\Lambda^r(\Gamma, \Lambda)$.*

Proof. Put $C^{-1} = \text{Hom}_\Lambda^l(C, \Gamma)$. Since Γ is an h-order, $CC^{-1} = \Gamma$ by [4], p. 84. Hence, $C\Gamma = CCC^{-1} = CC^{-1} = \Gamma$. Since $C' = \text{Hom}_\Lambda^l(\Gamma, \Lambda) \supseteq C$, $C'\Gamma = \Gamma$. Therefore, Γ is left Λ -projective by Lemma 3 and C is left Λ -projective. Furthermore, $\text{End}_\Lambda^l(C) = \Gamma'$ is an h-order by [4], Theorem 1.1. Exchanging left and right in the above, we obtain that C is right Λ -projective. Hence, $C' = C$ by [2], Proposition 1.6.

Corollary 3. *Let Λ be an order and Γ an h-order containing Λ . If $C = \text{Hom}_\Lambda^l(\Gamma, \Lambda)$ is idempotent, every two-sided idempotent ideal contained in C is left and right Λ -projective and there exists an h-order Γ_0 ($\supseteq \Lambda$) such that Γ_0 is Λ -projective. Especially if Γ is a maximal order, then Γ is left Λ -projective if and only if C is idempotent.*

Proof. Let D be an idempotent ideal in C . Then $\text{End}_\Lambda^r(D) \supseteq \text{End}_\Lambda^r(C)$, since $CD = D$. Hence, $\text{End}_\Lambda^r(D)$ is an h-order by [2], Corollary 1.4.

Finally, we shall give one more characterization of h-orders which is a converse of [2], Theorem 3.3.

We only consider a central simple K -algebra Σ and orders Λ over a discrete rank one, valuation ring R .

Let $\{\Omega_\alpha\}_{\alpha \in I}$ be the set of maximal orders containing Λ . If $\Lambda = \bigcap_{\alpha \in I} \Omega_\alpha$ and $\Lambda \not\subseteq \bigcap_{\beta \in I - \alpha} \Omega_\beta$, then we shall call Λ has a *unique irredundant representation* by Ω_α .

Proposition 4. *Let Σ and R be as above. The intersection of any two distinct maximal orders is contained in an h-order of rank two.*

Proof. Let Ω_1 and Ω_2 be maximal orders and $\Lambda = \Omega_1 \cap \Omega_2$. Then there exists an Ω_1 - Ω_2 ideal A which is contained in Λ by [1]. Let N be the radical of Ω_1 . If $A \subseteq N^i$ and $A \not\subseteq N^{i+1}$ for some i , $N^{-i}A \not\subseteq N$. Hence, we may assume that $A \not\subseteq N$. Then $B = A + N$ is not a two-sided ideal, since $A\Omega_2 = \Omega_1$. Let $\Omega' = \text{End}_\Lambda^l(B)$. Since B is a right Λ -module, $\Omega' \subseteq \Lambda$. Therefore, $\Omega_1 \cap \Omega'$

($\cong \Lambda$) is an h -order of rank two by [3], Theorem 5.3.

Corollary 4. (Hijikata). *Let Σ and R be as above. Then an R -order has a unique irredundant representation by maximal orders if and only if Λ is an h -order.*

Proof. If Λ is an h -order then Λ has a unique representation by [2], Theorem 3.3. Conversely, we assume that Λ has a unique representation and $\{\Omega_\alpha\}_{\beta \in I}$ be the set of maximal orders containing Λ . Then $\Omega_1 \cong \Omega_1 \cap \Omega_2 \cong \dots$ is a chain of orders containing Λ . Hence, I must be a finite set by Lemma 2. Since $\Omega_i \cap \Omega_j$ is contained in an h -order of rank two, which is written uniquely as $\Omega_i \cap \Omega_j = \Omega_j'$. Hence, Λ is an h -order by [3], Corollary 5.2.

EXAMPLES. 1. We give an example which shows $\Lambda \neq \Lambda Z$ in Lemma 1. Let R be the ring of integers and Q the field of rationals. $L = Q(\sqrt{5})$ and Z is the integrally closure of R in L . Put $Z_0 = R + R\sqrt{5}$, then $Z \cong Z_0$. Let A be an ideal of Z contained in Z_0 . An maximal order $\Omega = \begin{pmatrix} Z & Z \\ Z & Z \end{pmatrix}$ is $\Lambda = \begin{pmatrix} Z & A \\ Z & Z_0 \end{pmatrix}$ -projective. However, $\Lambda \cong Z$.

2. Let $R = R_p$, where p is a primeideal in R . $\Gamma_1 = \begin{pmatrix} R & R & P \\ R & R & P \\ R & R & R \end{pmatrix} \cong \Gamma_2 = \begin{pmatrix} R & P & P \\ R & R & P \\ R & R & R \end{pmatrix}$
 $\cong \Lambda = \begin{pmatrix} R & P & P \\ P & R & P \\ R & R & R \end{pmatrix}$. Then $C_1 = \text{Hom}_\Lambda(\Gamma_1, \Lambda) = \begin{pmatrix} P & P & P \\ P & P & P \\ R & R & R \end{pmatrix}$ and $C_2 = \text{Hom}_\Lambda(\Lambda_2, \Lambda)$
 $= \begin{pmatrix} P & P & P \\ P & R & P \\ R & R & R \end{pmatrix}$. C_1 and C_2 are idempotent and left and right Λ -projective. Furthermore, $\text{End}_\Lambda^r(C_2) = \begin{pmatrix} R & P & P \\ R & R & P \\ P^{-1} & R & R \end{pmatrix} = \Gamma_3$.

This example shows that $\text{Hom}_\Lambda(\Gamma_2, \Lambda)$ is idempotent for an h -order Γ_2 , however $\text{End}_\Lambda^r(C_2) \cong \Gamma_2$, (cf. [2], Proposition 1.6) and furthermore, h -order Γ_3 is Λ -projective, but Λ is not hereditary (cf. Corollary 2). Moreover, $\Lambda/N = R/p \oplus R/p \oplus R/p$ and $\Lambda \supset C_2 \supset C_1$ is a maximal chain of (projective) idempotent ideals, however Λ is not hereditary (cf. Corollary 2).

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