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A PLAT FORM PRESENTATION FOR SURFACE-LINKS

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Abstract
In this paper, we introduce a method, called a plat form, of describing a surface-link in the 4-space using a braided surface. We prove that every surface-link, which is not necessarily orientable, can be described in a plat form. The plat index is defined as a surface-link invariant, which is an analogy of the bridge index for a link in the 3-space. We classify surface-links with plat index 1 and show some examples of surface-links in plat forms.

1. Introduction

In knot theory we often use two methods of presenting links in the 3-space using braids: One is a closed braid form as in Fig.1, and the other is a plat form as in Fig.2.

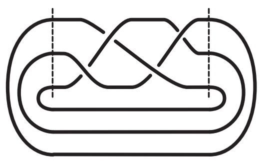

Fig. 1. A closed braid form.

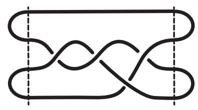

Fig. 2. A plat form.

A *surface-link* is a closed surface embedded in $\mathbb{R}^4$, and a 2-knot is a 2-sphere embedded in $\mathbb{R}^4$. Two surface-links are considered to be equivalent if they are ambient isotopic in $\mathbb{R}^4$. It is known that every orientable surface-link is equivalent to a surface-link in a closed 2-dimensional braid form (cf. [9, 12, 21]). It is an analogy of a closed braid form for a link.

The purpose of this paper is to introduce a new method of presenting a surface-link, which we call a plat form, as an analogy of a plat form for a link.

Theorem 1.1. *Every surface-link is equivalent to a surface-link in a plat form.*

We emphasize that our method works for every surface-link, while the closed 2-dimensional braid form works only for orientable ones. A *genuine plat form* is a special case of a plat form. Some surface-links can be presented in genuine plat forms.

Theorem 1.2. *Every orientable surface-link is equivalent to a surface-link in a genuine plat form.*

We show that the normal Euler number $e(F)$ of a surface-link F in a genuine plat form is zero (Proposition 5.9). It is unknown to the author whether every surface-link with $e(F) = 0$

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is equivalent to one in a genuine plat form.

We define two surface-link invariants, which are called the *plat index* and the *genuine plat index*, denoted by Plat(F) and g.Plat(F), respectively. These are analogies of the plat index, or the bridge index, of a link.

Using a theory of braided surfaces and 2-dimensional braids, we show that a surface-link F with Plat(F) $=1$ or with g.Plat(F) $=1$ is trivial (Theorem 5.5) and that a 2-knot with g.Plat(F) $=2$ is ribbon (Theorem 5.7). We also see an example of a 2-knot whose plat index and genuine plat index are different (Proposition 5.8). An example of a non-trivial surface-link in a plat form is shown in Fig.3 by using a motion picture (Proposition 5.8).

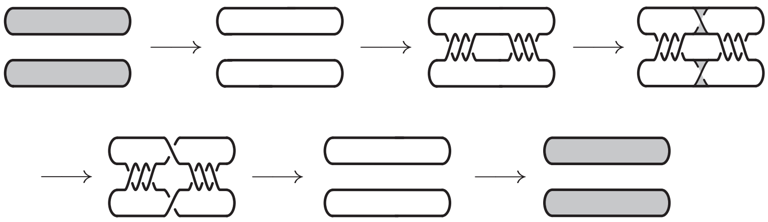

Fig. 3. The 2-twist spun trefoil in a (normal) plat form.

This paper is organized as follows. In Section 2, we recall the notions of braids, surface-links, and braided surfaces. We also recall the definition of a plat form for a link. In Section 3, we define a (normal) plat form and a genuine plat form for a surface-link. In Section 4, we prove Theorems 1.1 and 1.2. In Section 5, we discuss the plat index and the genuine plat index of a surface-link, and show some examples.

We work in the PL or smooth category. Surfaces embedded in the 4-space are assumed to be locally flat in the PL category.

2. Preliminaries

2.1. A plat form presentation for a link. Let n be a positive integer, $I = [0,1]$ the interval, D the square I^2 in $\mathbb{R}^2$, Int D the interior of D, and $Q_n = \{q_1, \ldots, q_n\}$ the subset of n points in D such that $q_k = (1/2, k/(n+1))$ for $k = 1, 2, \ldots, n$.

An *n-braid* is a union of n intervals β embedded in $D \times I$ such that each component intersects with every open disk Int $D \times \{t\}$ ($t \in I$) transversely at a single point, and $\partial\beta = Q_n \times \{0,1\}$. The n-braid group B_n is the group consisting of equivalence classes of n-braids in $D \times I$. The braid group B_n is identified with the fundamental group $\pi_1(C_n, Q_n)$ of the configuration space C_n of n points of Int D. We denote by $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$ the standard generators of B_n or their representatives due to Artin ([1]).

To define the plat closure of a braided surface in Section 3, we introduce the space of m wickets.

Definition 2.1 ([3]). A *wicket* is a semicircle in $D \times I$ that meets $D \times \{0\}$ orthogonally at its endpoints in Int $D \times \{0\}$. A *configuration of m wickets* is a disjoint union of m wickets in $D \times I$. The *space of m wickets* $\mathcal{W}_m$ is the space consisting of all configurations of m wickets.

For a configuration $w = w_1 \cup \cdots \cup w_m$ of m wickets, we denote by $|\partial w|$ the $2m$ points $\partial w_1 \cup \cdots \cup \partial w_m$ in Int D, which is identified with Int $D \times \{0\}$, and by ∂w the $2m$ points $|\partial w|$

equipped with the partition $\{\partial w_1, \dots, \partial w_m\}$. Note that if two configurations w and w' satisfy $\partial w = \partial w'$, then $w = w'$.

The set Q_{2m} equipped with the partition $\{\{q_1, q_2\}, \dots, \{q_{2m-1}, q_{2m}\}\}$ bounds a unique configuration of m wickets, which we call the *standard configuration of m wickets* and denote by w_0.

The fundamental group $\pi_1(\mathcal{W}_m, w_0)$ is called the *wicket group* in [3]. Let $|\partial| : (\mathcal{W}_m, w_0) \to (\mathcal{C}_{2m}, Q_{2m})$ be the continuous map sending w to $|\partial w|$. It induces a homomorphism $|\partial|_* : \pi_1(\mathcal{W}_m, w_0) \to \pi_1(\mathcal{C}_{2m}, Q_{2m}) = B_{2m}$.

Hilden's subgroup K_{2m} is the subgroup of B_{2m} generated by σ_1, $\sigma_2\sigma_1\sigma_3\sigma_2$, and $\sigma_{2i}\sigma_{2i-1}\sigma_{2i+1}^{-1}\sigma_{2i}^{-1}$ for $i = 1, \dots, m-1$ ([7], cf. [2]).

Proposition 2.2 ([3]). *For each positive integer m, the homomorphism* $|\partial|_* : \pi_1(\mathcal{W}_m, w_0) \to \pi_1(\mathcal{C}_{2m}, Q_{2m}) = B_{2m}$ *is injective and the image is Hilden's subgroup* K_{2m}. *Namely, the wicket group* $\pi_1(\mathcal{W}_m, w)$ *is isomorphic to Hilden's subgroup* K_{2m}.

The isomorphism from $\pi_1(\mathcal{W}_m, w)$ to K_{2m} is restated as follows: Let $f : (I, \partial I) \to (\mathcal{W}_m, w_0)$ be a loop. Consider a $2m$-braid $\beta_f = \bigcup_{t \in I} |\partial f(t)| \times \{t\} \subset D \times I$, then the isomorphism sends $[f] \in \pi_1(\mathcal{W}_m, w)$ to $[\beta_f] \in K_{2m}$.

DEFINITION 2.3. A loop $g : (I, \partial I) \to (\mathcal{C}_{2m}, Q_{2m})$ is *liftable* if there exists a loop $f : (I, \partial I) \to (\mathcal{W}_m, w_0)$ such that $g = |\partial| \circ f$.

DEFINITION 2.4. A $2m$-braid β in $D \times I$ is *adequate* or *wicket-adequate* if the associated loop $g : (I, \partial I) \to (\mathcal{C}_{2m}, Q_{2m})$ is liftable, namely, there exists a loop $f : (I, \partial I) \to (\mathcal{W}_m, w_0)$ such that $\beta = \beta_f$.

Note that Hilden's subgroup K_{2m} consists of the elements of B_{2m} represented by some adequate $2m$-braids.

Let β be a $2m$-braid in $D \times I \subset \mathbb{R}^2 \times \mathbb{R} = \mathbb{R}^3$. Attach a pair of the standard configurations of m wickets to β as in Fig.4, and we obtain a link which is called the *plat closure* of β and denoted by $\widetilde{\beta}$. A link is said to be *in a plat form* when it is the plat closure of a braid. Every link is equivalent to a link in a plat form.

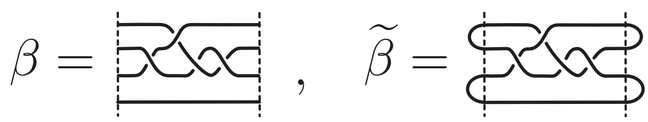

Fig.4. The plat closure of a braid.

In Section 3 we introduce a plat form of a surface-link in $\mathbb{R}^4$. We will also introduce a *normal plat form*, which is a plat form satisfying a nice condition such that its motion picture is easy to describe.

To define a normal plat form of a surface-link in Section 3, we construct an isotopic deformation changing the plat closure of an adequate braid to the plat closure of the trivial braid as follows: Let $f : (I, \partial I) \to (\mathcal{W}_m, w_0)$ be a loop. For each $t \in I$, let β_t be $\bigcup_{s \in I} |\partial f((1-t)s)| \times \{s\}$ in $D \times I$, which is a union of $2m$ arcs. We denote by L_t a link obtained from β_t by attaching the configuration $f(t)$ of m wickets to the side of $D \times \{1\}$ and the standard configuration w_0 to the side of $D \times \{0\}$ in $\mathbb{R}^3$. See Fig.5. Then, $\{L_t\}_{t\in I}$ is a 1-parameter family of links in $\mathbb{R}^3$ such that L_0 is $\widetilde{\beta_f}$ and L_1 is the plat closure of the trivial $2m$-braid as in Fig.5. We call $\{L_t\}_{t\in I}$ the *isotopic deformation changing* $\widetilde{\beta_f}$ *to the plat closure of the trivial braid*.

As a corollary, the plat closure of an adequate $2m$-braid is an m-component trivial link.

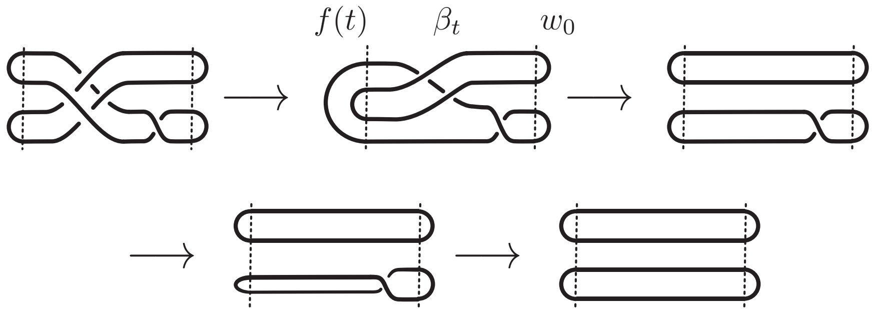

Fig.5. The isotopic deformation changing $\widetilde{\beta_f}$ to the plat closure of the trivial braid.

2.2. Surface-links. A *surface-link* is a closed surface embedded in $\mathbb{R}^4$, and a *surface-knot* is a connected surface-link. A 2-*knot* is a surface-knot homeomorphic to a 2-sphere. A 2-*link* is a surface-link consisting of 2-spheres. Two surface-links F and F' are said to be *equivalent* if they are ambient isotopic in $\mathbb{R}^4$. We denote it by $F \simeq F'$ that F and F' are equivalent.

Let $h : \mathbb{R}^3 \times \mathbb{R}^1 \to \mathbb{R}^1$ be the projection onto the second factor. Set $F_{[t]} = F \cap \mathbb{R}^3 \times \{t\}$ for $t \in \mathbb{R}$, which is called the *cross-section* of F at t. A *motion picture* of F is a 1-parameter family $\{F_{[t]}\}_{t\in\mathbb{R}}$. We often describe surface-links using motion pictures.

A surface-knot is *trivial* if it is equivalent to a connected sum of standardly embedded 2-spheres, tori, and projective planes ([8]). Here standardly embedded projective planes P_+ and P_- are illustrated in Fig.6.

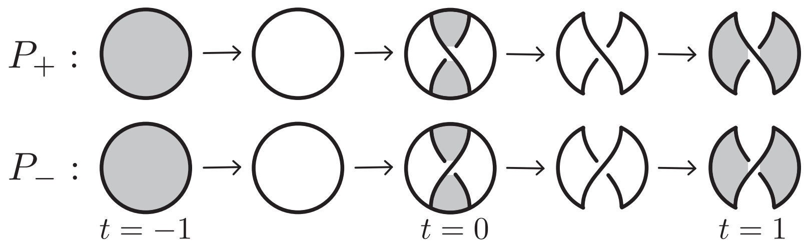

Fig.6. Motion pictures of P_+ and P_-.

2.3. Braided surfaces and 2-dimensional braids. A braided surface was introduced by Rudolph [19] and a 2-*dimensional braid* was introduced by Viro (cf. [10, 11, 12]). Let D_1 and D_2 be the squares $I^2 \subset \mathbb{R}^2$ and $\mathrm{pr}_i : D_1 \times D_2 \to D_i$ ($i = 1, 2$) the projection onto the i-th factor. Let $y_0 \in \partial D_2$ be a fixed base point.

Definition 2.5 ([19], [21]). A (*pointed*) *braided surface* of degree n is a surface S embedded in $D_1 \times D_2$ satisfying the following conditions:

(1) $\pi_S = \mathrm{pr}_2|_S : S \to D_2$ is a simple branched covering map of degree n (i.e., the

preimage of each branch locus consists of $n-1$ points).
(2) ∂S is the closure of an n-braid in the solid torus $D_1 \times \partial D_2$.
(3) $\mathrm{pr}_1(\pi_S^{-1}(y_0)) = Q_n$.

In particular, a 2-*dimensional braid* of degree n is a braided surface S of degree n such that ∂S is trivial, i.e., $\mathrm{pr}_1(\pi_S^{-1}(y)) = Q_n$ for all $y \in \partial D_2$.

The degree of S is denoted by $\deg S$. We say that two braided surfaces of the same degree are *equivalent* if they are ambient isotopic by an isotopy $\{h_s\}_{s \in I}$ of $D_1 \times D_2$ such that each h_s ($s \in I$) is fiber-preserving when we regard $D_1 \times D_2$ as the trivial D_1-bundle over D_2, and the restriction of h_s to $\mathrm{pr}_2^{-1}(y_0)$ is the identity map. A braided surface is *trivial* if it is equivalent to $Q_n \times D_2$.

Lemma 2.6 (cf. [12]). *A braided surface S is trivial if and only if S has no branch points.*

We assume $D_1 \times D_2 \subset \mathbb{R}^2 \times \mathbb{R}^2 = \mathbb{R}^4$. Let S be a 2-dimensional braid of degree n. The *closure* of S is an orientable surface-link in $\mathbb{R}^4$ obtained from S by attaching n 2-disks trivially outside $D_1 \times D_2$ in $\mathbb{R}^4$ along the boundary ∂S. It is described in Fig.7 when $n = 3$, where ε is a positive number and $S_{[t]} = S \cap D_1 \times (I \times \{t\})$ ($t \in I$).

Proposition 2.7 ([11, 21]). *Every orientable surface-link is equivalent to the closure of a 2-dimensional braid.*

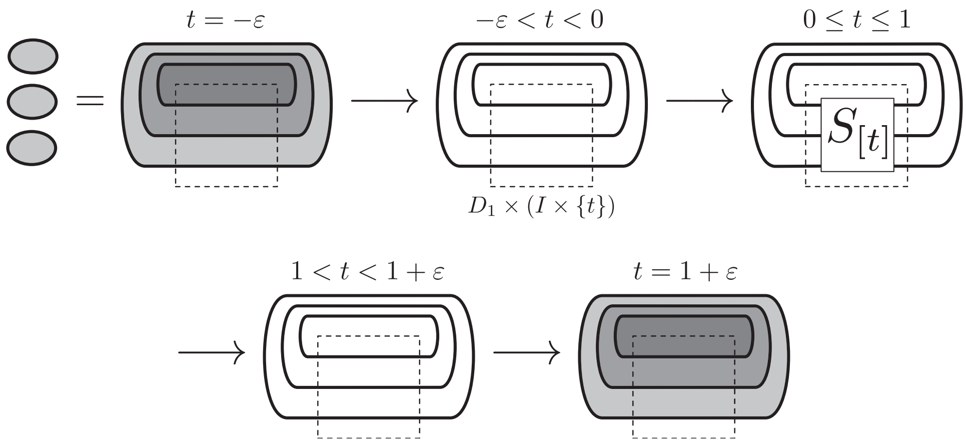

Fig. 7. The closure $\overline{S}$ of a 2-dimensional braid S.

For an orientable surface-link F, the *braid index* of F, denoted by $\mathrm{Braid}(F)$, is the minimum degree of 2-dimensional braids whose closures are equivalent to F.

3. A plat form presentation for a surface-link

In this section, we introduce a plat form for a surface-link.

We fix a loop $\mu : (I, \partial I) \to (\partial D_2, y_0)$ which runs once on ∂D_2 counter-clockwise. For a braided surface S of degree n, let $g_S : (I, \partial I) \to (C_n, Q_n)$ be a loop in the configuration space C_n obtained by

$$g_S(t) = \mathrm{pr}_1(\pi_S^{-1}(\mu(t)))$$

and β_S an n-braid in $D_1 \times I$ obtained by

$$\beta_S = \bigcup_{t\in I} \mathrm{pr}_1(\pi_S^{-1}(\mu(t))) \times \{t\},$$

where $\pi_S : S \to D_2$ is the simple branched covering map appearing in the definition of a braided surface. Then ∂S is the closure of β_S in $D_1 \times \partial D_2$.

DEFINITION 3.1. A braided surface S in $D_1 \times D_2$ is *adequate* if g_S is liftable or equivalently if β_S is adequate.

Note that the degree of an adequate braided surface is even. For an adequate braided surface S of degree $2m$, let $f_S : (I, \partial I) \to (\mathcal{W}_m, w_0)$ be the lift of g_S, i.e., a loop in $\mathcal{W}_m$ with $g_S = |\partial| \circ f_S$.

Let N be a regular neighborhood of ∂D_2 in $\mathbb{R}^2 \setminus \mathrm{Int}\, D_2$. Since N is homeomorphic to an annulus $I \times S^1$, we identify them by a fixed identification map $\phi : I \times S^1 \to N$ such that $\phi(0, p(t)) = \mu(t) \in \partial D_2$ for all $t \in I$, where $p : I \to S^1 = I/\partial I$ is the quotient map.

DEFINITION 3.2. A properly embedded surface A in $D_1 \times N$ is *of wicket type* if there exists a loop $f : (I, \partial I) \to (\mathcal{W}_m, w_0)$ such that

$$A = \bigcup_{t\in I} f(t) \times \{p(t)\} \subset (D_1 \times I) \times S^1 = D_1 \times N.$$

In this case, we say that A is *associated with* f and denote it by A_f.

We remark that a surface A of wicket type is a union of annuli or Möbius bands, and that $\partial A = \partial A_f$ is expressed as

$$\partial A = \bigcup_{t\in I} |\partial f(t)| \times \{p(t)\} \subset D_1 \times S^1 = D_1 \times \partial D^2.$$

Since two loops f and f' in $(\mathcal{W}_m, w_0)$ with $|\partial| \circ f = |\partial| \circ f'$ are the same, we see that two surfaces A and A' of wicket type with $\partial A = \partial A'$ are the same.

Let S be an adequate braided surface, and let $f : (I, \partial I) \to (\mathcal{W}_m, w_0)$ be a loop with $g_S = |\partial| \circ f$. Then it holds that $S \cap A_f = \partial S = \partial A_f$. We denote A_f by A_S and say that A_S is the *surface of wicket type associated with S*.

DEFINITION 3.3. Let S be an adequate braided surface and A_S the surface of wicket type associated with S. The *plat closure of S*, denoted by $\widetilde{S}$, is the union of S and A_S in $\mathbb{R}^4$.

When $\deg S = 2m$ and S has r branch points, the Euler characteristic $\chi(S)$ of S is $2m - r$. Since $\chi(A_S) = \chi(\partial A_S) = 0$, we have $\chi(\widetilde{S}) = 2m - r$.

DEFINITION 3.4. A surface-link is said to be *in a plat form* if it is the plat closure of an adequate braided surface. Moreover, a surface-link is said to be *in a genuine plat form* if it is that of a 2-dimensional braid.

We introduce a *normal plat form* for a surface-link by using a motion picture as follows: Let $\widetilde{S}$ be the plat closure of an adequate braided surface S of degree $2m$, and set $\widetilde{S}_{[t]} = \widetilde{S} \cap \mathbb{R}^3 \times \{t\}$ $(t \in \mathbb{R})$ and $S_{[t]} = S \cap D_1 \times (I \times \{t\}) = S \cap \mathbb{R}^3 \times \{t\}$ $(t \in [0,1])$. Replacing S

with an equivalent braided surface if necessary, we may assume that S satisfies the following conditions for some $t_0 \in [0,1]$:

(1) S has no branch points over $I \times [t_0, 1] \subset D_2$.
(2) $\mathrm{pr}_1(\pi_S^{-1}(y)) = Q_{2m}$ for every $y \in \partial D_2 \setminus (\{1\} \times [t_0, 1])$.
(3) $S_{[t_0]} = \beta_S \times \{t_0\}$.

In particular, $S_{[0]}$ and $S_{[1]}$ are both the trivial braids. Furthermore, replacing S with an equivalent braided surface if necessary, we may assume that the motion picture $\{\widetilde{S_{[t]}}\}_{t\in[t_0,1]}$ between $t = t_0$ and $t = 1$ is the isotopic deformation changing $\widetilde{\beta_f}$ to the plat closure of the trivial braid. (See Fig.5.) Finally, deforming A_S by an ambient isotopy rel boundary, we have a surface-link F, equivalent to $\widetilde{S}$, described by a motion picture as in Fig.8. The surface-link F in this form is said to be in a *normal plat form*.

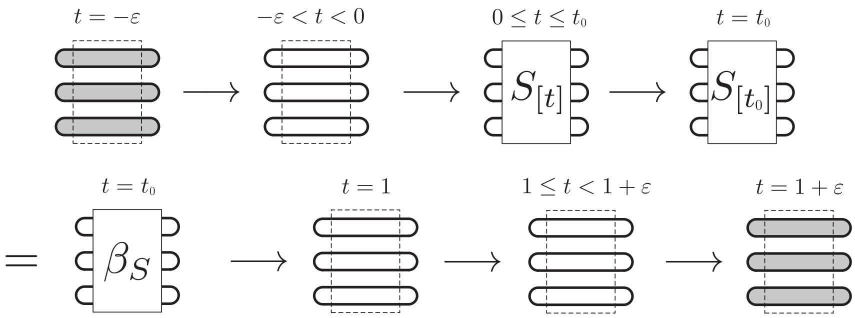

Fig. 8. A surface-link in a normal plat form.

4. Proofs of Theorems 1.1 and 1.2

In this section, we give proofs of Theorems 1.1 and 1.2. To prove them, we discuss a plat form for a link and a banded link presentation for a surface-link.

4.1. Stabilization and generalized stabilization for braids. For positive integers n and n' with $n \le n'$, let $\iota_n^{n'} : B_n \to B_{n'}$ denote the natural inclusion map from B_n to $B_{n'}$ sending each generator $\sigma_i \in B_n$ to $\sigma_i \in B_{n'}$.

A *stabilization* of a $2m$-braid β is a replacement of β with a $2m'$-braid β' such that

$$\beta' = \iota_{2m}^{2m'}(\beta)\,\sigma_{2m}\,\sigma_{2(m+1)}\,\sigma_{2(m+2)}\cdots\sigma_{2(m'-1)},$$

where m' is an integer with $m \le m'$. We also call a stabilization an *l-stabilization* when $l = m' - m$.

It is obvious that if β' is obtained from β by an l-stabilization then the plat closure of β' is equivalent to that of β as links in $\mathbb{R}^3$. See Fig.9 for $l = 1, 2$.

Proposition 4.1 ([2]). *Let β_i $(i = 1, 2)$ be a $2m_i$-braid such that the plat closure $\widetilde{\beta_i}$ is a knot. Then $\widetilde{\beta_1}$ is equivalent as knots in $\mathbb{R}^3$ to $\widetilde{\beta_2}$ if and only if there exists an integer $t \ge \max\{m_1, m_2\}$ such that for each $m \ge t$, the $2m$-braids β_i' $(i = 1, 2)$ obtained from β_i by stabilization belong to the same double coset of B_{2m} modulo K_{2m}.*

Proposition 4.1 is generalized into the case of links in $\mathbb{R}^3$ by using the notion of a generalized stabilization.

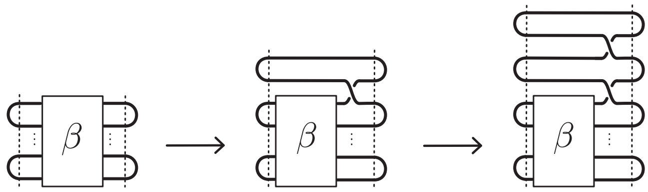

Fig. 9. Plat closures of stabilized braids.

Let Λ_m be the set of m-tuples of non-negative integers. For two elements $\lambda = (l_1, \dots, l_m)$ and $\lambda' = (l'_1, \dots, l'_m)$ of Λ_m, we write $\lambda \le \lambda'$ if $l_i \le l'_i$ for each $i = 1, \dots, m$. Then $\le$ is a (directed) partial ordering on Λ_m. Put $|\lambda|_0 = m$, $|\lambda|_i = m + l_1 + \cdots + l_i$ $(i = 1, \dots, m)$, and $|\lambda| = |\lambda|_m$. For a given $\lambda \in \Lambda_m$, we denote $\tau_i = \sigma_{2i}\sigma_{2i-1}\sigma_{2i+1}\sigma_{2i} \in K_{2|\lambda|}$ $(1 \le i \le |\lambda| - 1)$ and

$$T_{i,j} = \prod_{k=i}^{m-1} \tau_k \cdot \prod_{k=m}^{j} \tau_k^{-1} \in K_{2|\lambda|} \quad (1 \le i \le m, m-1 \le j \le |\lambda|),$$

where the former or later product is assumed to be the identity element of the group if $i = m$ or $j = m - 1$, respectively, and we construct a $2|\lambda|$-braid $T(\lambda)$ as follows:

$$T(\lambda) = \prod_{i=1}^{m} T_{i,(|\lambda|_{(i-1)}-1)}\, \sigma_{2|\lambda|_{i-1}} \sigma_{2(|\lambda|_{i-1}+1)} \cdots \sigma_{2|\lambda|_i}\, T^{-1}_{i,(|\lambda|_{(i-1)}-1)}.$$

For a $2m$-braid β and $\lambda \in \Lambda_m$, we let β^λ denote a $2|\lambda|$-braid such that

$$\beta^\lambda = \iota_{2m}^{2|\lambda|}(\beta) \cdot T(\lambda).$$

A *generalized stabilization* (with respect to λ) or *λ-stabilization* of β is a replacement of β with β^λ. A λ-stabilization is a composition of l_i-stabilization performed on the $2i$-th strand of β for each $i = 1, \dots, m$. A l-stabilization of β is a λ-stabilization with $\lambda = (0, \dots, 0, l) \in \Lambda_m$. Fig.10 depicts the plat closure of a 12-braid obtained from a 6-braid β by a generalized stabilization with respect to $\lambda = (2, 0, 1) \in \Lambda_3$.

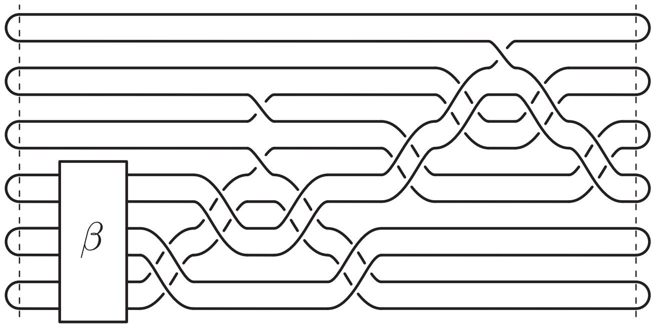

Fig. 10. The plat closure of a (2, 0, 1)-stabilized braid.

The following proposition states that two braids of even degrees have equivalent plat closures as links in $\mathbb{R}^3$ if and only if, after applying a generalized stabilization suitably, they belong to the same double coset of B_{2m} modulo K_{2m}.

Proposition 4.2 (cf. [2]). *Let β_i ($i = 1, 2$) be a $2m_i$-braid. The plat closure $\widetilde{\beta_1}$ is equivalent to $\widetilde{\beta_2}$ as links in $\mathbb{R}^3$ if and only if there exists an element $\lambda \in \Lambda_{m_1}$ satisfying the following condition: For any $\lambda_1 \geq \lambda$, there exists $\lambda_2 \in \Lambda_{m_2}$ with $|\lambda_1| = |\lambda_2|$ such that $\beta_1^{\lambda_1}$ and $\beta_2^{\lambda_2}$ belong to the same double coset of B_{2m} modulo K_{2m}, where $m = |\lambda_1| = |\lambda_2|$.*

Proposition 4.2 is proved directly by applying the proof of Proposition 4.1 given in [2] for each component of a link.

4.2. A banded link presentation for a surface-link. A *banded link* in $\mathbb{R}^3$ means a pair (L, B) of a link L and a family B of mutually disjoint bands attaching to L. We let L_B denote the link obtained from L by surgery along the bands belonging to B. A banded link (L, B) is *admissible* if both L and L_B are trivial links.

Let (L, B) be an admissible banded link in $\mathbb{R}^3$. Let $\mathbf{d}$ and $\mathbf{D}$ be unions of mutually disjoint 2-disks embedded in $\mathbb{R}^3$ bounded by L and L_B, respectively. Consider a closed surface $F = F(L, B)$ in $\mathbb{R}^4 = \mathbb{R}^3 \times \mathbb{R}$ defined by

$$p(F \cap \mathbb{R}^3 \times \{t\}) = \begin{cases} \mathbf{D} & (t = 1), \\ L_B & (0 < t < 1), \\ L \cup |B| & (t = 0), \\ L & (-1 < t < 0), \\ \mathbf{d} & (t = -1), \\ \emptyset & \text{otherwise}, \end{cases}$$

where $|B|$ is the union of the bands belonging to B. We call $F(L, B)$ a *closed realizing surface of* (L, B). Although it depends on a choice of $\mathbf{d}$ and $\mathbf{D}$, the equivalence class as surface-links does not depend on them (cf. [13, 16]).

Let r be a real number, and let $h : \mathbb{R}^3 \times (-\infty, r] \to (-\infty, r]$ be the projection onto the second factor, which we regard as a height function of $\mathbb{R}^3 \times (-\infty, r]$.

Lemma 4.3 (cf. [13, 16]). *Let F and F' be compact surfaces properly embedded in $\mathbb{R}^3 \times (-\infty, r]$ such that all critical points of F and F' are minimal points with respect to h, and their boundaries are the same trivial link in $\mathbb{R}^3 \times \{r\}$. Then, F and F' are ambient isotopic in $\mathbb{R}^3 \times (-\infty, r]$ rel $\mathbb{R}^3 \times \{r\}$.*

Lemma 4.4 ([16]). *If two admissible banded links (L, B) and (L', B') are ambient isotopic in $\mathbb{R}^3$, then their closed realizing surfaces $F(L, B)$ and $F(L', B')$ are equivalent.*

Lemma 4.5 ([16]). *Any surface-link F is equivalent to a closed realizing surface $F(L, B)$ of an admissible banded link (L, B).*

Lemma 4.6. *By an isotopy of $\mathbb{R}^3$, any banded link (L, B) in $\mathbb{R}^3$ is deformed to a banded link (L_0, B_0) satisfying the following conditions:*

(1) *There exists a disk D in $\mathbb{R}^2$ and a $2m_0$-braid β_0 in $D \times I$ ($\subset \mathbb{R}^2 \times \mathbb{R} = \mathbb{R}^3$) for some $m_0 \in \mathbb{N}$ such that $\beta_0 = L_0 \cap D \times I$ and $\widetilde{\beta_0} = L_0$.*

(2) *There exist mutually disjoint n subcylinders $U_i = d_i \times [s_i, t_i]$ ($i = 1, \ldots, n$) in $D \times I$ such that each U_i contains a part of L_0 as a pair of vertical line segments and a half-twisted band $b_i \in B_0$ as in* Fig.11*, where n is the number of bands belonging to B_0.*

Furthermore, we may take subcylinders $U_i = d_i \times [s_i, t_i]$ *such that* $d_1, \dots, d_n$ *are mutually disjoint disks in* $\mathrm{Int}\, D$ *and* $[s_i, t_i] = [2/5, 3/5]$ *for* $i = 1, \dots, n$.

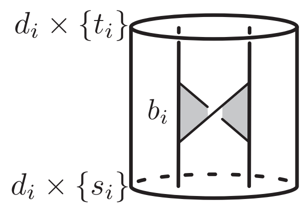

Fig. 11. A local model of L_0 and b_i in $U_i = d_i \times [s_i, t_i]$.

Proof. Let $d_1, \dots, d_n$ be mutually disjoint disks in $\mathrm{Int}\, D$ and let $U_i = d_i \times [2/5, 3/5]$ for $i = 1, \dots, n$. By an isotopy of $\mathbb{R}^3$, (L, B) is deformed into (L_1, B_0) such that for each i, U_i intersects with (L_1, B_0) as in Fig.11.

By an isotopy of $\mathbb{R}^3$ keeping U_i $(i = 1, \dots, n)$ fixed pointwise, (L_1, B_0) is deformed into (L_2, B_0) such that all maximal points of L_2 are in $\mathbb{R}^2 \times \{1\}$ and all minimal points of L_2 are in $\mathbb{R}^2 \times \{0\}$. Finally, by an isotopy of $\mathbb{R}^3$ keeping U_i $(i = 1, \dots, n)$ fixed pointwise, we deform the link L_2 into a link L_0 satisfying the condition (1). □

We denote by $(\beta_0)_{B_0}$ the $2m_0$-braid in $D \times I$ obtained from β_0 by surgery along bands belonging to B_0.

Proof of Theorem 1.1. We prove the theorem by 3 steps. Let F be a surface-link.

Step 1: By Lemmas 4.4, 4.5 and 4.6, F is equivalent to a closed realizing surface of a banded link (L_0, B_0) satisfying the conditions (1) and (2) in Lemma 4.6. Let β_0 be the $2m_0$-braid in $D \times I$ as in Lemma 4.6.

Let c_1 and c_2 be the numbers of components of L_0 and $(L_0)_{B_0}$, respectively. Since $\widetilde{\beta_0} = L_0$ is a trivial link of c_1 components, the plat closure $\widetilde{\beta_0}$ is equivalent as links in $\mathbb{R}^3$ to the plat closure $\widetilde{1_{2c_1}}$ of the trivial braid $1_{2c_1} \in B_{2c_1}$. The plat closure $\widetilde{(\beta_0)_{B_0}}$ is equivalent to the plat closure $\widetilde{1_{2c_2}}$ of the trivial braid $1_{2c_2} \in B_{2c_2}$ by the same reason.

Applying Proposition 4.2 to the two pairs $(\beta_0, 1_{2c_1})$ and $((\beta_0)_{B_0}, 1_{2c_2})$ of braids in $D \times I$, there exist a positive integer $m \in \mathbb{Z}$, three elements $\lambda \in \Lambda_{m_0}$, $\lambda_1 \in \Lambda_{c_1}$, $\lambda_2 \in \Lambda_{c_2}$, and four adequate $2m$-braids $\gamma, \gamma', \delta, \delta'$ in $D \times I$ such that $|\lambda| = |\lambda_1| = |\lambda_2| = m$ and

$$\beta_1 = \gamma \alpha_1 \gamma', \quad \beta_2 = \delta \alpha_2 \delta' \quad \text{in} \quad B_{2m},$$

where $\beta_1 = \beta_0^{\lambda}$, $\alpha_1 = 1_{2c_1}^{\lambda_1}$, $\beta_2 = (\beta_0)_{B_0}^{\lambda}$ and $\alpha_2 = 1_{2c_2}^{\lambda_2}$ are $2m$-braids in $D \times I$ obtained by generalized stabilization.

Since β_1 is a λ-stabilized β_0, there exists a subcylinder U of $D \times I$ such that $\beta_1 \cap U = \beta_0$ under an identification of U and $D \times I$. Let B_1 be the set of bands attaching to β_1 obtained from B_0 via the identification. Then, β_2 and $(\beta_1)_{B_1}$ are the same braid. Note that $(\widetilde{\beta_1}, B_1)$ is ambient isotopic to (L_0, B_0).

Step 2: We construct a properly embedded compact surface S_0 in $D_1 \times D_2$ and a braided surface S of degree $2m$ in $D_1 \times D_2$. Let $0 = t_0 < t_1 < \cdots < t_6 < t_7 = 1$ be a partition of $I = [0, 1]$. We divide $D_2 = I \times I$ into seven pieces $E_0, \dots, E_6$ with $E_i = I \times [t_i, t_{i+1}]$. Let α_1^* and α_2^* be $2m$-braids in $D_1 \times I$ given by

$$\alpha_1^* = \prod_{i=1}^{m} T_{i,(|\lambda_1|_{(i-1)}-1)} T^{-1}_{i,(|\lambda_1|_{(i-1)}-1)}, \quad \alpha_2^* = \prod_{i=1}^{m} T_{i,(|\lambda_2|_{(i-1)}-1)} T^{-1}_{i,(|\lambda_2|_{(i-1)}-1)},$$

which are obtained from $\alpha_1 = 1_{2c_1}^{\lambda_1} = T(\lambda_1)$ and $\alpha_2 = 1_{2c_2}^{\lambda_2} = T(\lambda_2)$ by removing the parts $\sigma_{2|\lambda_1|_{(i-1)}}\sigma_{2(|\lambda_1|_{(i-1)}+1)}\dots\sigma_{2|\lambda_1|_i}$ and $\sigma_{2|\lambda_2|_{(i-1)}}\sigma_{2(|\lambda_2|_{(i-1)}+1)}\dots\sigma_{2|\lambda_2|_i}$ $(i = 1,\dots,m)$, respectively (Fig.14). Note that α_1^* and α_2^* are equivalent to the trivial braid $1_{2m} = Q_{2m} \times I$ as braids in $D_1 \times I$.

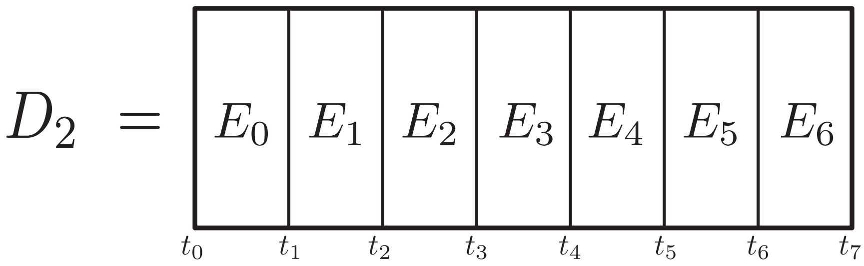

Fig. 12. The partition of D_2.

Let $\mathrm{p}_1 : D_1 \times I \to D_1$ and $\mathrm{p}_2 : D_1 \times I \to I$ be the projections onto the first and second factors, respectively. Let $\mathrm{pr}_i : D_1 \times D_2 \to D_i$ be the projections onto the i-th factors $(i = 1, 2)$. For a braid b in $D_1 \times I$ and $s \in I$, we denote by $b_{[s]}$ the image $\mathrm{p}_1(b \cap \mathrm{p}_2^{-1}(s))$ in D_1 of the intersection $b \cap p_2^{-1}(s)$.

Now, we define a properly embedded compact surface S_0 in $D_1 \times D_2$, step by step, as follows:

(0) First, we define $S_0 \cap D_1 \times \partial E_0$ by

$$\mathrm{pr}_1(S_0 \cap \mathrm{pr}_2^{-1}(s,t)) = \begin{cases} (\alpha_1^*)_{[s]} & ((s,t) \in I \times \{t_1\}), \\ Q_{2m} & ((s,t) \in \{0,1\} \times [t_0, t_1]), \\ Q_{2m} & ((s,t) \in I \times \{t_0\}). \end{cases}$$

See Fig.13. Since α_1^* is equivalent to the trivial $2m$-braid, we may define $S_0 \cap D_1 \times E_0$ as a braided surface of degree $2m$ without branch points in $D_1 \times E_0$, which is trivial by Lemma 2.6.

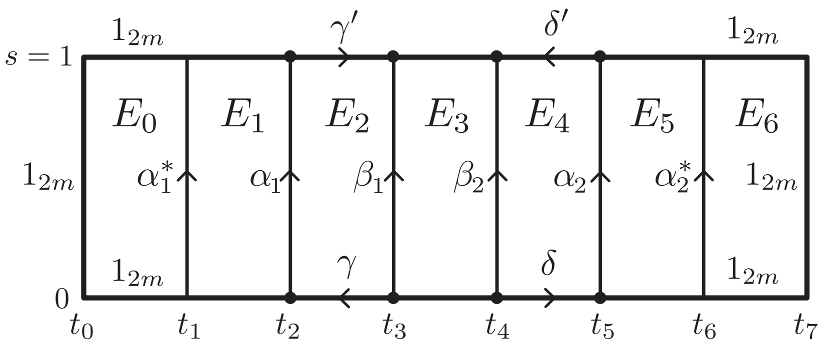

Fig. 13. A blueprint for a surface S_0. Each braid is appeared as the section of S_0.

(1) We define $S_0 \cap D_1 \times (E_1 \setminus I \times \{(t_1+t_2)/2\})$ as follows:

$$\mathrm{pr}_1(S_0 \cap \mathrm{pr}_2^{-1}(s,t)) = \begin{cases} (\alpha_1^*)_{[s]} & ((s,t) \in I \times [t_1, (t_1+t_2)/2)), \\ (\alpha_1)_{[s]} & ((s,t) \in I \times ((t_1+t_2)/2, t_2]). \end{cases}$$

Then, we define $S_0 \cap D_1 \times (I \times \{(t_1+t_2)/2\})$ as the $2m$-braid α_1^* with bands such that the surgery result of α_1^* is α_1 (see Fig.14). We denote by B_1^- the set of these bands.

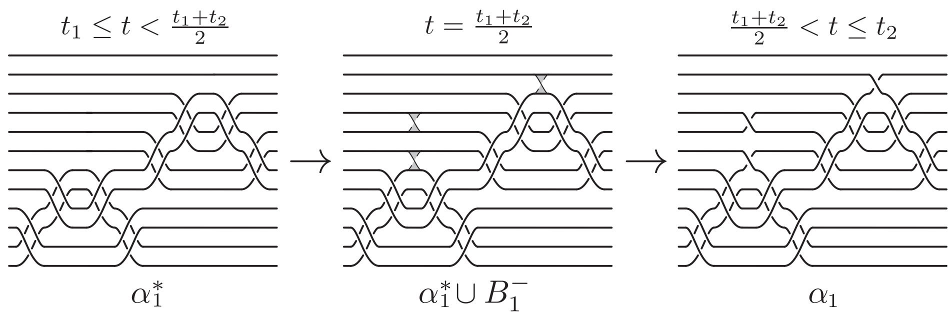

Fig. 14. A motion picture of S_0 ($t_1 \le t \le t_2$).

(2) We construct $S_0 \cap D_1 \times E_2$ similarly to the case (0). First, we define $S_0 \cap D_1 \times \partial E_2$ by

$$\mathrm{pr}_1(S_0 \cap \mathrm{pr}_2^{-1}(s,t)) = \begin{cases} (\beta_1)_{[s]} & ((s,t) \in I \times \{t_3\}), \\ \gamma'_{[(t-t_2)/(t_3-t_2)]} & ((s,t) \in \{1\} \times [t_2, t_3]), \\ (\alpha_1)_{[s]} & ((s,t) \in I \times \{t_2\}), \\ \gamma_{[(t-t_3)/(t_2-t_3)]} & ((s,t) \in \{0\} \times [t_2, t_3]). \end{cases}$$

Since $\beta_1 = \gamma\,\alpha_1\,\gamma'$, the closed braid $S_0 \cap D_1 \times \partial E_2$ is equivalent to the trivial closed braid in $D_1 \times \partial E_2$. Thus we may define $S_0 \cap D_1 \times E_2$ as a braided surface of degree $2m$ without branch points.

(3) We construct $S_0 \cap D_1 \times E_3$ similarly to the case (1). First, we define $S_0 \cap D_1 \times (E_3 \setminus I \times \{(t_3+t_4)/2\})$ by

$$\mathrm{pr}_1(S_0 \cap \mathrm{pr}_2^{-1}(s,t)) = \begin{cases} (\beta_1)_{[s]} & ((s,t) \in I \times [t_3, (t_3+t_4)/2)), \\ (\beta_2)_{[s]} & ((s,t) \in I \times ((t_3+t_4)/2, t_4]). \end{cases}$$

Then, we define $S_0 \cap D_1 \times (I \times \{(t_3+t_4)/2\})$ as the $2m$-braid β_1 with bands belonging to B_1.

(4) We construct $S_0 \cap D_1 \times E_4$ similarly to the case (2). We define $S_0 \cap D_1 \times \partial E_4$ by

$$\mathrm{pr}_1(S_0 \cap \mathrm{pr}_2^{-1}(s,t)) = \begin{cases} (\beta_2)_{[s]} & ((s,t) \in I \times \{t_4\}), \\ \delta'_{[(t-t_5)/(t_4-t_5)]} & ((s,t) \in \{1\} \times [t_4, t_5]), \\ (\alpha_2)_{[s]} & ((s,t) \in I \times \{t_5\}), \\ \delta_{[(t-t_4)/(t_4-t_5)]} & ((s,t) \in \{0\} \times [t_4, t_5]). \end{cases}$$

Since $\beta_2 = \delta\,\alpha_2\,\delta'$, we define $S_0 \cap D_1 \times E_4$ as a braided surface of degree $2m$ without branch points.

(5) We construct $S_0 \cap D_1 \times E_5$ similarly to the case (1). We define $S_0 \cap D_1 \times (E_5 \setminus \{(t_5+t_6)/2\})$ by

$$\mathrm{pr}_1(S_0 \cap \mathrm{pr}_2^{-1}(s,t)) = \begin{cases} (\alpha_2)_{[s]} & ((s,t) \in I \times [t_5, (t_5+t_6)/2)), \\ (\alpha_2^*)_{[s]} & ((s,t) \in I \times ((t_5+t_6)/2, t_6]). \end{cases}$$

Then, we define $S_0 \cap D_1 \times I \times \{(t_5+t_6)/2\}$ as the $2m$-braid α_2^* with bands attaching to α_2^* as in the opposite direction of Fig.14 such that the surgery result of α_2^* is α_2. We denote by B_1^+ the set of these bands.

(6) We construct $S_0 \cap D_1 \times E_6$ similarly to the case (0). First, we define $S_0 \cap D_1 \times \partial E_6$ by

$$\mathrm{pr}_1(S_0 \cap \mathrm{pr}_2^{-1}(s,t)) = \begin{cases} (\alpha_2^*)_{[s]} & ((s,t) \in I \times \{t_7\}), \\ Q_{2m} & ((s,t) \in \{0,1\} \times [t_6, t_7]), \\ Q_{2m} & ((s,t) \in I \times \{t_6\}). \end{cases}$$

Since α_2^* is equivalent to the trivial $2m$-braid, we may define $S_0 \cap D_1 \times E_6$ as a braided surface of degree $2m$ without branch points.

As a result, we have a properly embedded surface S_0 in $D_1 \times D_2$. We take a based point $y_0 = (0,0) \in \partial D_2$. Then, S_0 is a braided surface of degree $2m$ except in neighborhoods of the bands appearing in (1), (3), and (5). By an ambient isotopy of a neighborhood of each band, we can change the band to a branch point as shown in Fig.15. Hence, we obtain a braided surface S of degree $2m$ from S_0. The braided surface S is adequate because the $2m$-braid β_S is the composition of adequate $2m$-braids γ^{-1}, δ, δ', and γ'^{-1}.

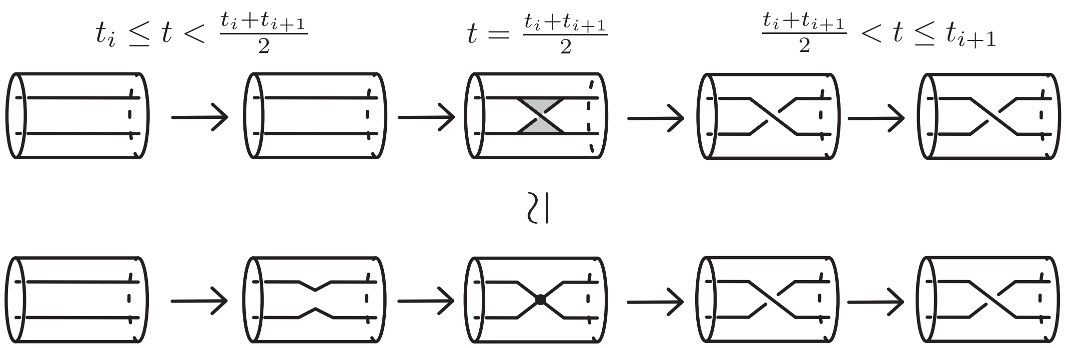

Fig.15. An isotopic deformation changing a band to a branch point.

Step 3: Finally, we show that the surface-link F is equivalent to the plat closure $\widetilde{S}$ of S.

Let $p : \mathbb{R}^4 = \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^3$ and $h : \mathbb{R}^4 = \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$ be the projections onto the first and second factors, respectively. We regard h as a height function of $\mathbb{R}^4$. Let A be the surface of wicket type associated with S. Note that $\partial A = \partial S = \partial S_0$. Let $F_0 = S_0 \cup A$. Then F_0 is a surface-link equivalent to $\widetilde{S} = S \cup A$. Thus we show that F_0 and F are equivalent.

By an ambient isotopy of $\mathbb{R}^4$ keeping $\mathbb{R}^3 \times (t_0, t_7)$ fixed pointwise, we deform F_0 to a surface-link F_1 such that

$$p(F_1 \cap \mathbb{R}^3 \times \{t\}) = \begin{cases} \mathbf{D}_1 & (t = t_7), \\ \widetilde{\alpha_2^*} & ((t_5+t_6)/2 < t < t_7), \\ \widetilde{\alpha_2^*} \cup |B_1^+| & (t = (t_5+t_6)/2), \\ \mathrm{p}(F_0 \cap \mathbb{R}^3 \times \{t\}) & ((t_1+t_2)/2 < t < (t_5+t_6)/2), \\ \widetilde{\alpha_1^*} \cup |B_1^-| & (t = (t_1+t_2)/2), \\ \widetilde{\alpha_1^*} & (t_0 < t < (t_1+t_2)/2), \\ \mathbf{d}_1 & (t = t_0), \\ \emptyset & \text{otherwise}, \end{cases}$$

where $|B_1^-|$ (resp. $|B_1^+|$) is the union of the bands belonging to B_1^- (resp. B_1^+), and $\mathbf{d}_1$ (resp. $\mathbf{D}_1$) is a union of mutually disjoint m 2-disks in $\mathbb{R}^3$ bounded by $\widetilde{\alpha_1^*}$ (resp. $\widetilde{\alpha_2^*}$) such that $\mathbf{d}_1$ (resp. $\mathbf{D}_1$) is disjoint from $|B_1^-|$ (resp. $|B_1^+|$) as in the left of Fig.16 except for the attaching arcs of the bands, respectively. Next, we define a surface-link F_2 in $\mathbb{R}^4$ by

$$p(F_2 \cap \mathbb{R}^3 \times \{t\}) = \begin{cases} \mathbf{D}_2 & (t = t_7), \\ \widetilde{\alpha_2} & ((t_5+t_6)/2 \leq t < t_7), \\ \mathrm{p}(F_0 \cap \mathbb{R}^3 \times \{t\}) & ((t_1+t_2)/2 < t < (t_5+t_6)/2), \\ \widetilde{\alpha_1} & (t_0 < t \leq (t_1+t_2)/2), \\ \mathbf{d}_2 & (t = t_0), \\ \emptyset & \text{otherwise}, \end{cases}$$

where $\mathbf{d}_2 = \mathbf{d}_1 \cup |B_1^-|$ (resp. $\mathbf{D}_2 = \mathbf{D}_1 \cup |B_1^+|$) is the union of mutually disjoint c_1 (resp. c_2) 2-disks as in the right of Fig.16, respectively.

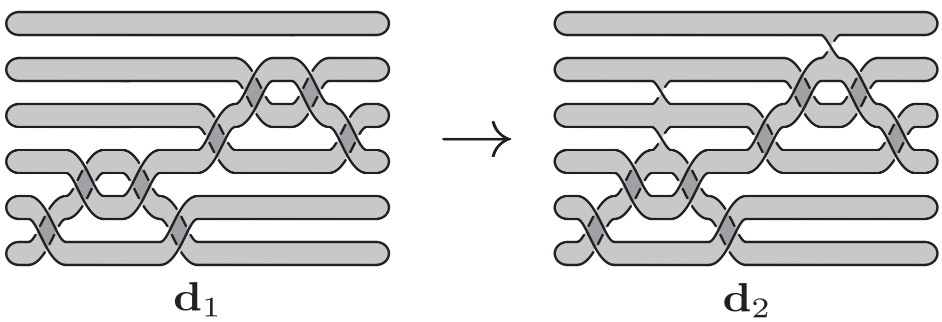

Fig. 16. $\mathbf{d}_2$ is the union of $\mathbf{d}_1$ and $|B_1^-|$.

Then, F_2 is obtained from F_1 by cellular moves (cf. [18]) along 3-cells $|B_1^-| \times [t_0, (t_1 + t_2)/2] \cup |B_1^+| \times [(t_5 + t_6)/2, t_7]$. This implies that F_1 and F_2 are equivalent.

Note that $F_2 \cap \mathbb{R}^2 \times \{t_0\} = \mathbf{d}_2 \times \{t_0\}$ is the union of all minimal disks of F with respect to the height function h, $F_2 \cap \mathbb{R}^2 \times \{t_7\} = \mathbf{D}_2 \times \{t_7\}$ is the union of all maximal disks of F, and all saddle bands of F appear at $t = (t_3 + t_4)/2$ as bands belonging to B_1. By Lemma 4.3, F_2 is equivalent to a closed realizing surface of the banded link $(\widetilde{\beta_1}, B_1)$.

Since (L_0, B_0) is ambient isotopic to $(\widetilde{\beta_1}, B_1)$ and F is equivalent to a closed realizing surface of (L_0, B_0), we see that F_2 is equivalent to F. □

Next, we show Theorem1.2. We define the $2m$-braid Δ_m by $\Delta_1 = Q_2 \times I$ and $\Delta_m = \prod_{k=1}^{m-1}(\sigma_{2k}\,\sigma_{2k-1}\cdots\sigma_2\,\sigma_1)$ for $m \geq 2$. See Fig.17.

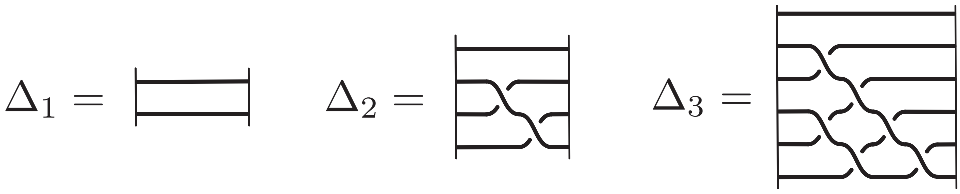

Fig. 17. The $2m$-braids Δ_m ($m = 1, 2, 3$).

Note that the closure of an m-braid b is equivalent to the plat closure of a $2m$-braid $\Delta_m\,\iota_m^{2m}(b)\,\Delta_m^{-1}$. See Fig.18 and 19.

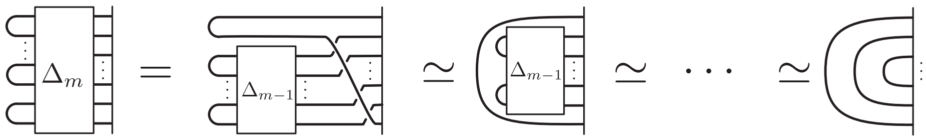

Fig. 18. An isotopic deformation of Δ_m with the standard wicket configuration w_0 to a configuration of wickets appearing in a closed braid form (Fig.1).

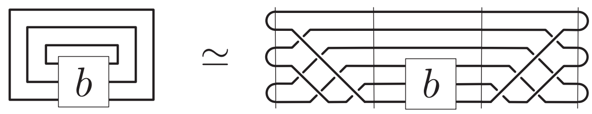

Fig. 19. A transformation from the closure of b to the plat closure of $\Delta_m\,\iota_m^{2m}(b)\,\Delta_m^{-1}$ ($m = 3$).

Proof of Theorem 1.2. Let F be an orientable surface-link. By Proposition 2.7, there exists a 2-dimensional braid S in $D_1 \times D_2 = D_1 \times I \times I$ whose closure in $\mathbb{R}^4$ is equivalent to F. Let m be the degree of S and $S_{[t]}$ the cross-section $S \cap D_1 \times (I \times \{t\})$ for each $t \in I$. See Fig.7 when $m = 3$. Let S_1 be the 2-dimensional braid of degree $2m$ obtained from S by adding trivial m sheets.

Let ε be a positive number and let $D_2' = I \times [-\varepsilon, 1 + \varepsilon]$. We consider a 2-dimensional braid S_2 of degree $2m$ in $D_1 \times D_2' = D_1 \times (I \times [-\varepsilon, 1 + \varepsilon])$ with a motion picture $(S_2)_{[t]}$ as in Fig.20. Here, the motion picture $(S_2)_{[t]}$ for $t \in [-\varepsilon, 0]$ (or $t \in [1, 1 + \varepsilon]$) is the 1-parameter family of $2m$-braids changing 1_{2m} to $\Delta_m\,\Delta_m^{-1}$ (or $\Delta_m\,\Delta_m^{-1}$ to 1_{2m}), respectively, and the motion picture $(S_2)_{[t]}$ for $t \in I = [0, 1]$ is the composition of Δ_m, $(S_1)_{[t]}$ and Δ_m^{-1}.

As a result, the plat closure of S_2 has the motion picture as in Fig.21. By comparing Fig.7 and Fig.21, we see that the closure of S is equivalent to the plat closure of S_2. Hence, F has a genuine plat form presentation. □

Remark 4.7. In Lemma 4.6, each subcylinder U_i contains a part of a banded link as in the left of Fig.22. However, we may assume that for each i, the band in U_i is either as in

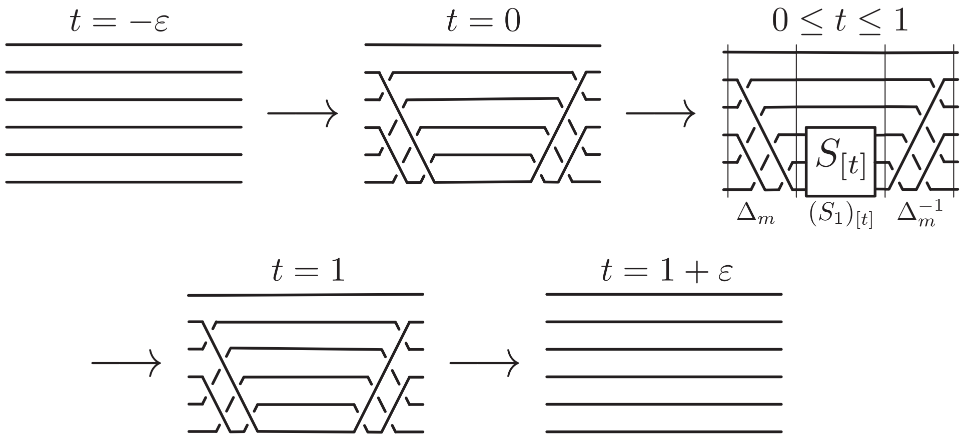

Fig. 20. A motion picture of S_2 ($m = 3$).

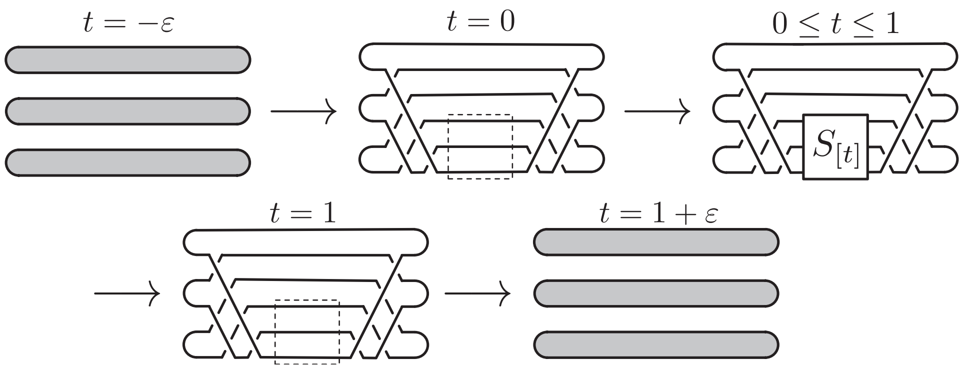

Fig. 21. A motion picture of $\widetilde{S_2}$ ($m = 3$).

the left or as in the right of Fig.22. Then we have another braided surface in the proof of Theorem 1.1, where the corresponding branch point changes the sign. (A branch point of a braided surface is *positive* (or *negative*) if the local monodromy is a conjugate of a standard generator (or its inverse), cf. [12, 13]).

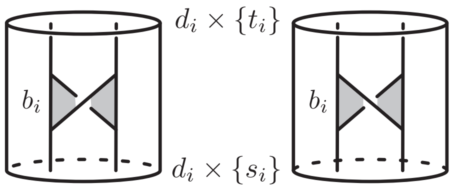

Fig. 22. Two types of half-twisted bands in a subcylinder $U_i = d_i \times [s_i, t_i]$.

5. The plat index of a surface-link and examples

In this section, we introduce two surface-link invariants called the plat index and the genuine plat index.

Definition 5.1. Let F be a surface-link. The *plat index of* F, denoted by $\mathrm{Plat}(F)$, is defined as the half of the minimum degree of all adequate braided surfaces whose plat closures are equivalent to F:

$$\mathrm{Plat}(F) \;=\; \min\{\deg S/2 \,|\, S \text{ is a braided surface with } \widetilde{S} \simeq F\}.$$

Definition 5.2. Let F be a surface-link. If F admits a genuine plat form, the *genuine plat index of* F, denoted by $\mathrm{g.Plat}(F)$, is defined as the half of the minimum degree of all 2-dimensional braids whose plat closures are equivalent to F. If F dose not admit a genuine plat form, it is defined as infinity:

$$\mathrm{g.Plat}(F) \;=\; \begin{cases} \min\{\deg S/2 \,|\, S \text{ is a 2-dimensional braid with } \widetilde{S} \simeq F\}, \\ \infty \quad \text{if } F \text{ admits no genuine plat forms.} \end{cases}$$

By definition, it holds that $\mathrm{Plat}(F) \leq \mathrm{g.Plat}(F)$ for every surface-link F. Moreover, from the proof of Theorem 1.2, we have the following proposition.

Proposition 5.3. *The following inequalities hold for every orientable surface-link F:*

$$\mathrm{Plat}(F) \;\leq\; \mathrm{g.Plat}(F) \;\leq\; \mathrm{Braid}(F).$$

In the rest of this section, we show some examples of surface-links in plat forms and discuss the plat index and the genuine plat index.

We first recall the notion of a braid system of a braided surface. Refer to [12] for more details. Let $\mathrm{pr}_i : D_1 \times D_2 \to D_i$ $(i = 1, 2)$ be the projection and C_n the configuration space of n points of $\mathrm{Int}\, D_1$. Let S be a braided surface of degree n, and $\Sigma(S)$ the branch locus of $\pi_S = \mathrm{pr}_2|_S : S \to D_2$. Let $y_0 \in \partial D_2$ be a fixed base point.

The *braid monodromy* of S is a homomorphism $\rho_S : \pi_1(D_2 \setminus \Sigma(S), y_0) \to \pi_1(C_n, Q_n) = B_n$ defined as follows: For a loop $c : (I, \partial I) \to (D_2 \setminus \Sigma(S), y_0)$, define a loop $\rho_S(c) : (I, \partial I) \to (C_n, Q_n)$ as $\rho_S(c)(t) = \mathrm{pr}_1(\pi_S^{-1}(c(t)))$. Then the braid monodromy of S is defined as a group homomorphism sending $[c]$ to $[\rho_S(c)] \in \pi_1(C_n, Q_n)$.

Let r be a positive integer. A *Hurwitz arc system* in D_2 (with the base point y_0) is an r-tuple $\mathcal{A} = (\alpha_1, \cdots, \alpha_r)$ of oriented simple arcs in D_2 such that

(1) for each i, $\alpha_i \cap \partial D_2 = \partial\alpha_i \cap \partial D_2 = \{y_0\}$ and this is the terminal point of α_i,
(2) for $i \neq j$, $\alpha_i \cap \alpha_j = \{y_0\}$, and
(3) $\alpha_1, \ldots, \alpha_r$ appear in this order around the base point y_0.

The set of initial points of $\alpha_1, \ldots, \alpha_r$ is called the *starting point set* of $\mathcal{A}$.

Let $\mathcal{A} = (\alpha_1, \cdots, \alpha_r)$ be a Hurwitz arc system with the starting point set $\Sigma(S)$. For each i, let N_i be a (small) regular neighborhood of the starting point of α_i, $\overline{\alpha_i}$ an oriented arc obtained from α_i by restricting to $D_2 \setminus \mathrm{Int}\, N_i$, and γ_i a loop $\overline{\alpha_i}^{-1} \cdot \partial N_i \cdot \overline{\alpha_i}$ in $D_2 \setminus \Sigma(S)$ with base point y_0. Here, ∂N_i is oriented counter-clockwise. Then $\pi_1(D_2 \setminus \Sigma(S), y_0)$ is generated by $[\gamma_1], [\gamma_2], \ldots, [\gamma_r]$ and we have $[\partial D_2] = [\gamma_1] \cdots [\gamma_r]$. The *braid system* of S associated with $\mathcal{A}$ is an r-tuple b_S of elements of B_n defined as

$$b_S \;=\; (\rho_S([\gamma_1]), \ldots, \rho_S([\gamma_r])) \in (B_n)^r.$$

It is known that $\rho_S([\gamma_i])$ is a conjugation of a standard generator or its inverse, σ_1^{ε} $(\varepsilon \in \{\pm 1\})$, such that ε is the sign of the branch point which is the starting point of α_i. The composition $\rho_S([\gamma_1])\rho_S([\gamma_2])\cdots\rho_S([\gamma_r])$ is equal to β_S in B_n.

The *slide action* of the braid group B_r on $(B_n)^r$ is a left group action defined as

$$\mathrm{slide}(\sigma_j)(\beta_1,\ldots,\beta_r) = (\beta_1,\ldots,\beta_{j-1},\beta_j\beta_{j+1}\beta_j^{-1},\beta_j,\beta_{j+2},\ldots,\beta_r)$$

for $\sigma_j \in B_r$ and $(\beta_1,\ldots,\beta_r) \in (B_n)^r$. Two elements of $(B_n)^r$ are said to be *Hurwitz equivalent* if they are in the same orbit of the slide action of B_r.

Lemma 5.4 (cf. [12, 17, 20]). *Two braided surfaces in $D_1 \times D_2$ are equivalent if and only if their braid systems are Hurwitz equivalent.*

Let $e(F)$ be the normal Euler number of a surface-knot F. The normal Euler number of any orientable surface-knot is 0, and the normal Euler number of a trivial non-orientable surface-knot, which is a connected sum of p copies of P_+ and q copies of P_-, is $2(p-q)$ (cf. [4, 8]).

Theorem 5.5. *Let F be a surface-link.*

(1) $\mathrm{Plat}(F) = 1$ *if and only if F is either a trivial 2-knot or a trivial non-orientable surface-knot.*
(2) $\mathrm{g.Plat}(F) = 1$ *if and only if F is either a trivial 2-knot or a trivial non-orientable surface-knot with $e(F) = 0$.*
(3) *If F is a trivial orientable surface-knot with positive genus, then* $\mathrm{Plat}(F) = \mathrm{g.Plat}(F) = 2$.

Proof. (1) Let F be a surface-link with $\mathrm{Plat}(F) = 1$ and S a braided surface of degree 2 with $\widetilde{S} \simeq F$. Let p and q be the numbers of positive and negative branch points of S, respectively. Then a braid system for S is Hurwitz equivalent to $(\sigma_1,\ldots,\sigma_1,\sigma_1^{-1},\ldots,\sigma_1^{-1})$ consisting of p σ_1's and q σ_1^{-1}'s. Hence, the equivalence class of S is determined from p and q. Fig.23 is a motion picture of the plat closure of a braided surface of degree 2 with p positive branch points and q negative branch points. The motion picture describes a trivial 2-knot if $p = q = 0$ holds, otherwise, it describes a connected sum of p copies of P_+ and q copies of P_-. Therefore, F is either a trivial 2-knot or a trivial non-orientable surface-knot.

Conversely, if F is a trivial 2-knot, then $\mathrm{Plat}(F) = 1$. If F is a trivial non-orientable surface-knot, then F is equivalent to a surface-knot described in Fig.23 and hence $\mathrm{Plat}(F) = 1$.

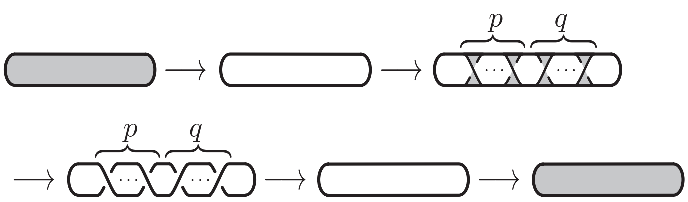

Fig. 23. A surface-knot in a (normal) plat form.

(2) Let F be a surface-link with $\mathrm{g.Plat}(F) = 1$ and S a 2-dimensional braid of degree 2 with $\widetilde{S} \simeq F$. Since S is a 2-dimensional braid, the number of positive branch points of S, denoted by p, is equal to the number of negative ones. Hence, the argument in the proof of (1) implies that F is equivalent to a trivial 2-knot, when $p = 0$, or a connected sum of p

copies of P_+ and p copies of P_-. In particular, it holds that $e(F) = 0$.

Conversely, if F is a trivial 2-knot, then g.Plat$(F) = 1$. If F is a trivial non-orientable surface-knot with $e(F) = 0$, then F is a connected sum of p copies of P_+ and p copies of P_- for some $p > 0$, which is equivalent to a surface-knot described in Fig.23 with $p = q$. Hence g.Plat$(F) = 1$.

(3) Let F be a trivial orientable surface-knot with a positive genus. Since Braid$(F) = 2$ (cf. [9, 12]), by Proposition 5.3, we have Plat$(F) \leq$ g.Plat$(F) \leq 2$. On the other hand, by (1), it holds that Plat$(F) \neq 1$. Hence, we have Plat$(F) =$ g.Plat$(F) = 2$. (Fig.24 shows a motion picture of a genuine plat form of F.) □

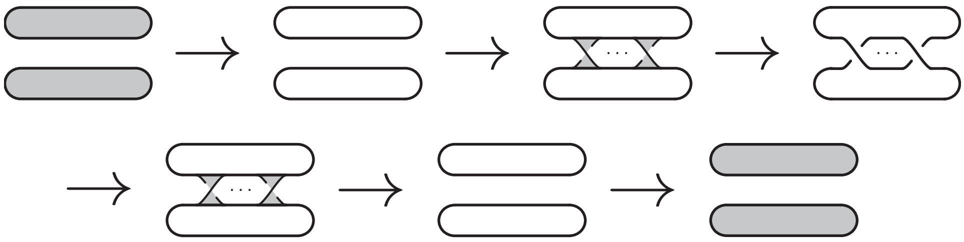

Fig. 24. A trivial orientable surface-knot with a positive genus in a genuine plat form.

Proposition 5.6. *Let F be the* 2*-knot denoted by* 2_2 *in the table of* [15]*, which is depicted in* Fig.25*. Then* Plat$(F) =$ g.Plat$(F) = 2$.

Proof. Fig.26 shows a deformation of a banded link by an isotopy of $\mathbb{R}^3$. Using the isotopy, we see that F is equivalent to a surface-knot in a genuine plat form depicted in Fig.27. Hence, we have the inequality Plat$(F) \leq$ g.Plat$(F) \leq 2$. Since F is not a trivial 2-knot, we have Plat$(F) =$ g.Plat$(F) = 2$. □

The braid index of every non-trivial surface-knot is greater than 2 ([9]). Hence, 2_2 is an example such that the equality in g.Plat$(F) \leq$ Braid(F) in Proposition 5.3 does not hold.

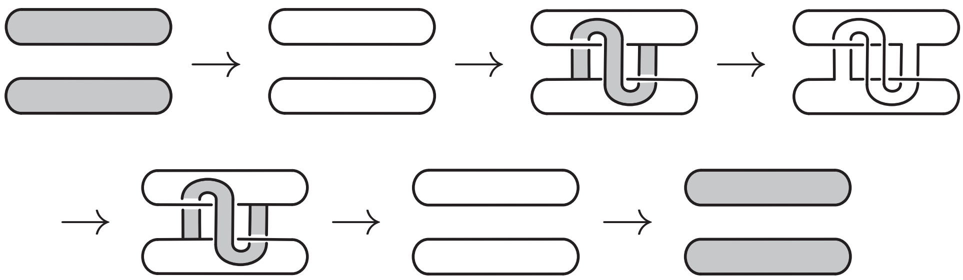

Fig. 25. A motion picture of the 2-knot 2_2.

A surface-link is said to be *ribbon* if it is obtained from a trivial 2-link by some 1-handle surgeries.

Theorem 5.7. *Let F be a* 2*-knot (or a surface-link with* $\chi(F) = 2$*) with* g.Plat$(F) = 2$*. Then, F is ribbon.*

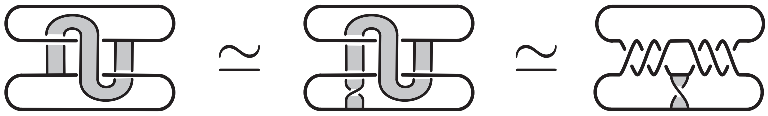

Fig. 26. An isotopic deformation of a banded link.

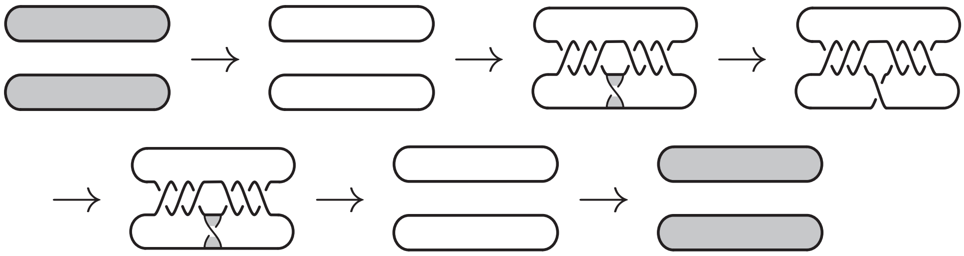

Fig. 27. A motion picture of the 2-knot 2_2 in a genuine plat form.

Proof. Let S be a 2-dimensional braid of degree 4 with $\widetilde{S} \simeq F$, and r the number of branch points of S. Since $\chi(F) = 2$, we see that $r = 2$ from $\chi(\widetilde{S}) = 4 - r$. Let $b_S = (\beta_1, \beta_2) \in (B_4)^2$ be a braid system of S. Since S is a 2-dimensional braid, $\beta_S = \beta_1\beta_2 = 1$ in B_4, i.e., $\beta_2 = \beta_1^{-1}$. A 2-dimensional braid with a symmetric braid system (β_1, β_1^{-1}) is known as a ribbon 2-dimensional braid ([12]), which is equivalent to a 2-dimensional braid S' in $D_1 \times D_2 = D_1 \times (I \times [0,1])$ such that S' is symmetric with respect to $t = 1/2$. Then the plat closure of S' is symmetric with respect to $t = 1/2$ and it is in a normal form in the sense of [16]. Hence $\widetilde{S'}$ is ribbon (cf. Theorem 11.4 of [12]). Since $F \simeq \widetilde{S}$ and $\widetilde{S} \simeq \widetilde{S'}$, F is ribbon. □

Proposition 5.8. *Let $k(n)$ be the twist knot ($n \in \mathbb{Z}$) and $F(n)$ the 2-twist spin of $k(n)$ ([23]). Then* $\mathrm{Plat}(F(n)) = 2$ *holds for $n \neq 0$.*

Proof. The 2-knot $F(n)$ has a motion picture described in [14] as in Fig.28, where $m = 2n + 1$ and a box labeled by m contains m positive half-twists or $-m$ negative half-twists for $m < 0$. Since the trivial link depicted in (4) of Fig.28 is the plat closure of an adequate braid of degree 4, this motion picture gives us a (normal) plat form presentation for $F(n)$. On the other hand, it is known that $F(n)$ is a non-trivial 2-knot if $n \neq 0$. Hence, we have that $\mathrm{Plat}(F(n)) = 2$. □

Furthermore, it is known that $F(n)$ is not a ribbon 2-knot for $n \neq 0$ ([6]). By Theorem 5.7, the genuine plat index of $F(n)$ is greater than 2. Thus, Proposition 5.8 gives us examples of 2-knots such that the equality in $\mathrm{Plat}(F) \leq \mathrm{g.Plat}(F)$ in Proposition 5.3 does not hold.

A P^2*-link* is a surface-link whose components are projective planes. Replacing $m = 2n+1$ (or $-m = -2n - 1$) crossings in Fig.28 with $2n$ (or $-2n$) crossings, respectively, we have a 2-component P^2-link in a plat form. In particular, in the case of $n = 1$, the P^2-link is a P^2-link denoted by $8_1^{-1,-1}$ in Yoshikawa's table ([22]).

Proposition 5.9. *Let F be a surface-link in a genuine plat form. Each component of F is a surface-knot whose normal Euler number is zero.*

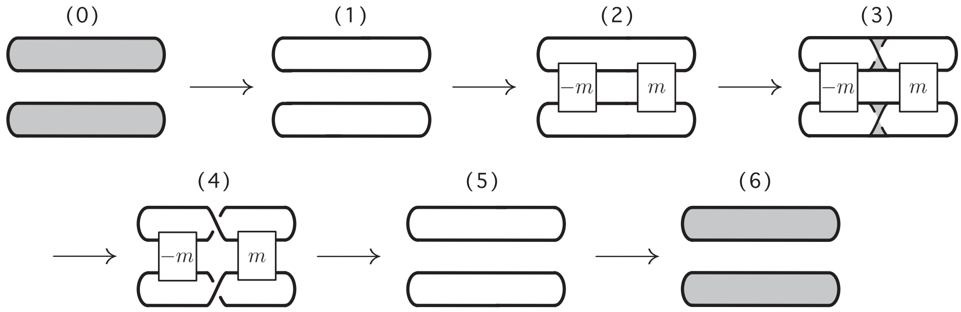

Fig. 28. The 2-knot $F(n)$ in a plat form ($m = 2n + 1$).

Proof. Each connected component of F is regarded as a surface-knot in a genuine plat form by forgetting other components of F. Thus it is sufficient to show that $e(F) = 0$ for a surface-knot F in a genuine plat form.

For a (broken surface) diagram D of F (cf. [5]), let $b_+(D)$ (resp. $b_-(D)$) be the number of positive (resp. negative) branch points of D. Then, the normal Euler number $e(F)$ is equal to $b_+(D) - b_-(D)$.

When $F = \widetilde{S}$ is in a genuine plat form, taking a diagram suitably, positive (resp. negative) branch points of S (in the sense of a 2-dimensional braid) correspond to positive (resp. negative) branch points of D, and vise versa. Since S is a 2-dimensional braid, the number of positive branch points of S and that of negative branch points of S are the same. Thus we have $e(F) = b_+(D) - b_-(D) = 0$. □

It is unknown to the author whether every surface-link consisting of surface-knots whose normal Euler numbers are zero is equivalent to a surface-link in a genuine plat form.

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