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L^p - L^q ESTIMATES FOR SUBELLIPTIC PSEUDO-DIFFERENTIAL OPERATORS ON COMPACT LIE GROUPS

DUVÁN CARDONA, JULIO DELGADO, VISHVESH KUMAR and MICHAEL RUZHANSKY

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Abstract

We establish the L^p - L^q -boundedness of subelliptic pseudo-differential operators on a compact Lie group G . Effectively, we deal with the L^p - L^q -bounds for operators in the sub-Riemannian setting because the subelliptic classes are associated to a Hörmander sub-Laplacian. The Riemannian case associated with the Laplacian is also included as a special case. Then, applications to the L^p - L^q -boundedness of pseudo-differential operators in the Hörmander classes on G are given in the complete range $0 \leq \delta \leq \rho \leq 1$, $\delta < 1$. This also gives the L^p - L^q -bounds in the Riemannian setting, because the later classes are associated with the Laplacian on G . In both cases, in the Riemannian and the sub-Riemannian settings, necessary and sufficient conditions for the L^p - L^q -boundedness of operators are also analysed.

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1. Introduction

This paper is mainly concerned with the L^p - L^q boundedness of pseudo-differential operators associated with the global Hörmander symbol classes on compact Lie groups for the range $1 < p, q < \infty$. Our analysis also includes estimates for pseudo-differential operators associated with subelliptic symbol classes.

The relevance of the boundedness of Fourier multipliers and pseudo-differential operators has been highlighted by Stein and Hörmander. These kinds of estimates naturally arise in the study of some evolution equations. For instance, one can see [29, 46]. Till now, there have been extensive activities dealing with the L^p - L^q boundedness for spectral multipliers and Fourier multipliers on compact Lie groups, we cite [3, 4, 7, 8, 12, 32, 33, 40] for a non-exhaustive list of references.

To the best of our knowledge, there has been no activity to pursue the L^p - L^q estimates of non-invariant operators, in particular, pseudo-differential operators on compact Lie groups. In the classical Euclidean setting, Hörmander established the L^p - L^q estimates of pseudo-differential operators associated with the so-called “Hörmander symbol classes” $S_{\rho,\delta}^m(\mathbb{R}^n \times \mathbb{R}^n)$ on \mathbb{R}^n with $m \in \mathbb{R}$ and $0 \leq \delta < \rho \leq 1$. It is well-known that any L^p - L^q bounded Fourier multiplier is nontrivial only if $p \leq q$ (see [30]). Therefore, it is natural to assume the condition $p \leq q$ when dealing with pseudo-differential operators. Later on, Álvarez and Hounie [5] extended Hörmander’s result to the range $0 \leq \delta < 1$ and $0 < \rho \leq 1$ without the restriction $\delta < \rho$. In a recent work by the first and last two authors [11], we have provided sufficient and necessary conditions for the L^p - L^q boundedness of pseudo-differential operators associated with global Hörmander symbol classes $S_{\rho,\delta}^m(G \times \widehat{G})$, $m \in \mathbb{R}$, $0 \leq \delta \leq \rho \leq 1$ and $\delta \neq 1$, on a graded Lie group G , where \widehat{G} denotes the unitary dual of G .

In this work, we focus on pseudo-differential operators associated with the global Hörmander symbol classes encoded with the Riemannian and sub-Riemannian structure of compact Lie groups. One of the main differences between the approach developed in [11], based on the analysis of hypoelliptic operators on those groups, is the use of the structure of the dilations of the group, while the approach of this paper will be based on the submarkovian properties of the semigroup $e^{-t\mathcal{L}}$, $t > 0$, of a Hörmander sub-Laplacian $\mathcal{L} = \sum_{j=1}^k X_j^2$, where one exploits the geometric properties induced by the Hörmander system of vector fields $\{X_j : 1 \leq j \leq n\}$ on a compact Lie group G .

On a compact Lie group G , in the monograph [41], Turunen and the last author introduced a global notion of the Hörmander symbol classes on G . According to this terminology, and observing that any continuous linear operator A acting on $C^\infty(G)$ has a right convolution kernel $R_A = R_A(x, y) \in \mathcal{D}'(G \times G)$, namely, a distribution that describes the action of the operator by the group convolution $*$ as follows

$$(1.1) \quad Af(x) = (f * R_A(x, \cdot))(x),$$

the *global symbol* of A , is the matrix-valued function defined on $G \times \widehat{G}$, defined via

$$(1.2) \quad \sigma_A(x, \xi) = \widehat{R}_A(x, \xi), (x, [\xi]) \in G \times \widehat{G}.$$

Here, $\widehat{\cdot}$ denotes the matrix-valued group Fourier transform on G . By classifying these matrix-valued symbols by the behaviour of their derivatives (and of their differences), the last author and Turunen [41] introduced the symbols classes $S_{\rho, \delta}^m(G \times \widehat{G})$, allowing the complete range $0 \leq \delta \leq \rho \leq 1$, and providing a new description of the Hörmander classes $S_{\rho, \delta}^m(T^*G)$ (as defined in [28] with the local notion of the principal symbol, defined on the cotangent-bundle T^*M of a compact manifold) when additionally, $0 \leq \delta < \rho \leq 1$, and $\rho > 1 - \delta$.

On the other hand, it was observed by the first and the last author in [7], that the symbols classes $S_{\rho, \delta}^m(G \times \widehat{G})$, are associated to the Riemannian structure of the group G , in the sense that the growth of the derivatives of symbols is classified according in terms of the spectrum of the Laplacian $\mathcal{L}_G = -\sum_{j=1}^n X_j^2$, $n = \dim(G)$. Then, in generalising this idea, in [7] the subelliptic Hörmander classes $S_{\rho, \delta}^{m, \mathcal{L}}(G \times \widehat{G})$, were introduced with the derivatives (and differences) of symbols compared with respect to the growth of the eigenvalues of a fixed Hörmander sub-Laplacian $\mathcal{L} = -\sum_{j=1}^k X_j^2$, where $k < n$. We observe that the pseudo-differential calculus associated to the “subelliptic” classes $S_{\rho, \delta}^{m, \mathcal{L}}(G \times \widehat{G})$, is more singular than the one associated to the “elliptic” classes $S_{\rho, \delta}^m(G \times \widehat{G})$. Indeed, singularities of the kernels of the “subelliptic” classes are classified in terms of the Hausdorff dimension Q , associated with the control distance associated with the sub-Laplacian \mathcal{L} . In the next subsection we present the $L^p - L^q$ regularity properties of the subelliptic Hörmander classes $S_{\rho, \delta}^{m, \mathcal{L}}(G \times \widehat{G})$.

We finally observe that for the case $p = q$, namely, the problem regarding the L^p -boundedness of pseudo-differential operators, Fefferman in [20] has established a sharp criterion of continuity for the operator in the Hörmander classes $S_{\rho, \delta}^m(\mathbb{R}^n \times \mathbb{R}^n)$ on the Euclidean space. Then Fefferman’s criterion has been extended for several pseudo-differential calculi including the Weyl-Hörmander calculus [14], the Hörmander classes $S_{\rho, \delta}^m(G \times \widehat{G})$ associated to the Laplacian [15], also extended in the sub-Riemannian setting, namely, for the Hörmander classes $S_{\rho, \delta}^{m, \mathcal{L}}(G \times \widehat{G})$ associated to a Hörmander sub-Laplacian [7], and finally for the Hörmander classes on graded Lie groups in [9]. In order to give a general perspective about this problem, here we are mainly concerned with the case $p < q$.

Notably, when dealing with the L^p - L^q -boundedness of operators with symbols in the classes $S_{\rho, \delta}^{m, \mathcal{L}}(G \times \widehat{G})$, one has to analyse separately the cases: (i) $1 < p \leq q \leq 2$, (ii) $1 < p \leq 2 \leq q < \infty$, and (iii) $2 \leq p \leq q < \infty$. Here for the case $1 < p \leq 2 \leq q < \infty$ we provide necessary and sufficient conditions.

1.1. Main results. The following theorem presents the result that establishes a sufficient condition and in some cases also a necessary condition, for the L^p - L^q boundedness of subelliptic pseudo-differential operators on compact Lie groups.

Theorem 1.1. *Let $1 < p, q < \infty$, and $0 \leq \delta < \rho \leq 1$. Let G be a compact Lie group, and let Q be its Hausdorff dimension with respect to the control distance associated with a Hörmander sub-Laplacian \mathcal{L} . Then, the following statements hold.*

- *Let $1 < p \leq 2 \leq q < \infty$. Every pseudo-differential operator $A \in \Psi_{\rho, \delta}^{m, \mathcal{L}}(G \times \widehat{G})$ admits a bounded extension from $L^p(G)$ into $L^q(G)$, that is*

$$(1.3) \quad \forall f \in C^\infty(G), \|Af\|_{L^q(G)} \leq C\|f\|_{L^p(G)}$$

holds, if and only if,

$$(1.4) \quad m \leq -Q \left(\frac{1}{p} - \frac{1}{q} \right).$$

- Every pseudo-differential operator $A \in \Psi_{\rho,\delta}^{m,\mathcal{L}}(G \times \widehat{G})$ admits a bounded extension from $L^p(G)$ into $L^q(G)$, that is (1.3) holds, in the following cases:

(i) if $1 < p \leq q \leq 2$ and

$$(1.5) \quad m \leq -Q \left(\frac{1}{p} - \frac{1}{q} + (1-\rho) \left(\frac{1}{q} - \frac{1}{2} \right) \right).$$

(ii) if $2 \leq p \leq q < \infty$ and

$$(1.6) \quad m \leq -Q \left(\frac{1}{p} - \frac{1}{q} + (1-\rho) \left(\frac{1}{2} - \frac{1}{p} \right) \right).$$

REMARK 1.2. The order conditions in (1.4), (1.5) and (1.6) can be written in a simplified way for $1 < p, q < \infty$ as follows:

$$(1.7) \quad m \leq -Q \left(\frac{1}{p} - \frac{1}{q} + (1-\rho) \max \left\{ \frac{1}{2} - \frac{1}{p}, \frac{1}{q} - \frac{1}{2}, 0 \right\} \right),$$

where Q is the Hausdorff dimension of G with respect to the control distance associated to the sub-Laplacian \mathcal{L} .

REMARK 1.3. If $G = \mathbb{R}^n$, although this is not a compact Lie group, the order condition in (1.7) is sharp for Fourier multipliers, see Hörmander [29, Page 163].

REMARK 1.4. When a system of vector fields $X = \{X_j\}$ provides an orthonormal basis of the Lie algebra (endowed, up to a constant factor, with its unique bi-invariant Riemannian metric), the Hörmander condition is trivially satisfied, the sub-Laplacian associated to the system X coincides with the Laplacian and the classes $S_{\rho,\delta}^{m,\mathcal{L}}(G \times \widehat{G})$ agree with the “elliptic classes” $S_{\rho,\delta}^m(G \times \widehat{G})$ of the last author with Turunen [41]. The following corollary provides the $L^p - L^q$ -regularity properties for the elliptic classes.

Corollary 1.5. *Let $1 < p, q < \infty$, and $0 \leq \delta < \rho \leq 1$. Let G be a compact Lie group of dimension n . Then, the following statements hold.*

- Let $1 < p \leq 2 \leq q < \infty$. Every pseudo-differential operator $A \in \Psi_{\rho,\delta}^m(G \times \widehat{G})$ admits a bounded extension from $L^p(G)$ into $L^q(G)$, that is

$$(1.8) \quad \forall f \in C^\infty(G), \|Af\|_{L^q(G)} \leq C\|f\|_{L^p(G)}$$

holds, if and only if,

$$(1.9) \quad m \leq -n \left(\frac{1}{p} - \frac{1}{q} \right).$$

- Every pseudo-differential operator $A \in \Psi_{\rho,\delta}^m(G \times \widehat{G})$ admits a bounded extension from $L^p(G)$ into $L^q(G)$, that is (1.8) holds, in the following cases:

(i) if $1 < p \leq q \leq 2$ and

$$(1.10) \quad m \leq -n \left(\frac{1}{p} - \frac{1}{q} + (1 - \rho) \left(\frac{1}{q} - \frac{1}{2} \right) \right).$$

(ii) if $2 \leq p \leq q < \infty$ and

$$(1.11) \quad m \leq -n \left(\frac{1}{p} - \frac{1}{q} + (1 - \rho) \left(\frac{1}{2} - \frac{1}{p} \right) \right).$$

This paper is organised as follows. In Section 2 we present the preliminaries of this paper related to submarkovian semigroups, and the subelliptic pseudo-differential calculus introduced in [7]. Subsequently, the $L^p - L^q$ -boundedness of these subelliptic classes is analysed in Section 3. Then, in Section 4 we provide explicit examples of our criterion about the L^p - L^q -boundedness of pseudo-differential operators in the case of the sphere $\mathbb{S}^3 \cong \text{SU}(2)$ and on $\text{SU}(3)$.

2. Preliminaries

2.1. Symmetric submarkovian semigroups. We briefly recall some classical facts concerning symmetric submarkovian semigroups on $L^2 := L^2(X, \mu)$. Here (X, μ) is a σ -finite measure space. For the definitions and results mentioned in this sub-section we follow [13, Section II.5], [13, Example II.5.1] (see Remark 2.1 below) and [13, Theorem II.3.1, Page 14] (see Theorem 2.2 below).

Let A be an operator with domain $\text{Dom}(A) \subset L^2$. We recall that

$-A$ is the generator of a symmetric semigroup $T_t := e^{-tA}$ on L^2 such that

$$\|e^{-tA}\|_{L^2 \rightarrow L^2} \leq e^{\alpha t}$$

if and only if A is self-adjoint, $\text{Dom}(A)$ is a dense subspace of L^2 , and

$$(Af, f) \geq -\alpha \|f\|_{L^2}.$$

Let Q be a symmetric bilinear form defined on a subspace $D \subset L^2$. One says that Q is positive if $Q(f, f) \geq 0$, and closed if for every sequence $(f_n)_{n \in \mathbb{N}} \subset D$, such that $f_n \rightarrow f$ in L^2 , and

$$\lim_{n,m \rightarrow \infty} Q(f_n - f_m, f_n - f_m) = 0,$$

one has that $f \in D$ and that $Q(f_n - f, f_n - f) \rightarrow 0$. One says that Q is closable if it admits a closed extension.

If A is a symmetric operator with a dense domain $\text{Dom}(A) \subset L^2$, then one may associate with it the symmetric bilinear form $Q_A(f, g) := Q(Af, g)$. If in addition Q_A is positive, it is closable and its minimal closure \overline{Q}_A is associated to a self-adjoint operator \overline{A} which is an extension of A . More precisely, \overline{A} is the smallest self-adjoint extension of A , called the *Friedrichs extension* of A . We shall not distinguish between A and \overline{A} .

Recall that a semigroup T_t on L^2 is called *submarkovian* if $f \in L^2$, $0 \leq f \leq 1$, implies that $0 \leq T_t f \leq 1$. Such a semigroup acts on the L^p -spaces and $\|T_t\|_{L^p \rightarrow L^p} \leq 1$.

Symmetric submarkovian semi-groups on L^2 may be characterised through properties of the associated symmetric bilinear form. A positive symmetric bilinear form Q defined on

$D \subset L^2$, is said to be a *Dirichlet form*, if for all $g \in D$, and for all $f \in D$, such that $|f| \leq |g|$, and $|f(x) - f(y)| \leq |g(x) - g(y)|$, one has that $f \in D$, and $Q(f, f) \leq Q(g, g)$.

If $T_t = e^{-tA}$ is a symmetric submarkovian semigroup on L^2 , the associated bilinear form $Q(f, g) := (A^{1/2}f, A^{1/2}g)$, $f, g \in \text{Dom}(A^{1/2})$, is a closed Dirichlet form with dense domain in L^2 . Conversely, given a closed Dirichlet bilinear form Q , with dense domain D in L^2 , there exists a unique symmetric submarkovian semigroup on L^2 , such that $T_t = e^{-tA}$ and $Q(f, g) := (A^{1/2}f, A^{1/2}g)$, $f, g \in \text{Dom}(A^{1/2})$.

REMARK 2.1. We note that submarkovian semigroups arise naturally in the setting of compact manifolds. Indeed, if M is a compact manifold with a volume form dx , and $L^2(M) = L^2(M, dx)$, consider a family of vector fields

$$X = \{X_1, X_2, \dots, X_k\}, k \leq n := \dim(M).$$

If every X_j is skew-adjoint on $L^2(M)$, namely, if

$$\forall f, g \in C^\infty(M), \int_M X_i(f)g dx = - \int_M f X_i(g) dx,$$

then we can associate with $\Delta_X := -\sum_{i=1}^k X_i^2$, its Friedrichs extension, which we still denote by Δ_X . Then, the semigroup

$$T_t := e^{-t\Delta_X} : L^2(M) \rightarrow L^2(M)$$

is a contraction semigroup. With respect to the Dirichlet form

$$Q(f, g) = (\Delta_X f, g), f, g \in C^\infty(M),$$

the semigroup $T_t := e^{-t\Delta_X}$ is a submarkovian semigroup.

The following theorem will be fundamental for our further analysis.

Theorem 2.2. *Let $T_t = e^{-tA}$ be a submarkovian semigroup. Assume that T_t is equicontinuous on $L^1(X, \mu)$ and on $L^\infty(X, \mu)$. Suppose that there exists $\alpha > 0$ and $1 < p < q \leq \infty$ such that*

$$(2.1) \quad \|f\|_{L^q(X, \mu)} \leq C \|A^{\alpha/2} f\|_{L^p(X, \mu)}.$$

Then, with Q defined by the identity $\alpha = Q(1/p - 1/q)$, the following semigroup estimate holds

$$(2.2) \quad \exists C > 0, \forall f \in L^1(X, \mu), \|T_t f\|_{L^\infty(X, \mu)} \leq C t^{-\frac{Q}{2}} \|f\|_{L^\infty(X, \mu)}.$$

2.2. Pseudo-differential operators via localisations. Now we present the preliminaries of the Hörmander theory of pseudo-differential operators on compact manifolds used in this work. The setting of compact Lie groups appears as an essential case of manifolds with symmetries. We refer to Hörmander [28] for details.

Let U be an open subset of \mathbb{R}^n . We say that the symbol $a \in C^\infty(T^*U)$, $T^*U = U \times \mathbb{R}^n$, belongs to the Hörmander class

$$S_{\rho, \delta}^m(T^*U), 0 \leq \rho, \delta \leq 1,$$

if for every compact subset $K \subset U$ the symbol inequalities

$$|\partial_x^\beta \partial_\xi^\alpha a(x, \xi)| \leq C_{\alpha, \beta, K} (1 + |\xi|)^{m - \rho|\alpha| + \delta|\beta|},$$

hold true uniformly in $x \in K$ for all $\xi \in \mathbb{R}^n$. A continuous linear operator $A : C_0^\infty(U) \rightarrow C^\infty(U)$ is a pseudo-differential operator of order m and of (ρ, δ) -type, if there exists a symbol $a \in S_{\rho, \delta}^m(T^*U)$ such that A is the Kohn-Nirenberg quantisation of the symbol a , namely, if

$$Af(x) = \int_{\mathbb{R}^n} e^{2\pi i x \cdot \xi} a(x, \xi) (\mathcal{F}_{\mathbb{R}^n} f)(\xi) d\xi,$$

for all $f \in C_0^\infty(U)$, where

$$(\mathcal{F}_{\mathbb{R}^n} f)(\xi) := \int_U e^{-i2\pi x \cdot \xi} f(x) dx$$

is the Euclidean Fourier transform of f at $\xi \in \mathbb{R}^n$.

Now, we extend this notion to smooth manifolds as follows. Given a smooth closed manifold M , $A : C_0^\infty(M) \rightarrow C^\infty(M)$ is a pseudo-differential operator of order m and of (ρ, δ) -type, with $\rho \geq 1 - \delta$, and $0 \leq \delta < \rho \leq 1$, if for every local coordinate system $\omega : M_\omega \subset M \rightarrow U_\omega \subset \mathbb{R}^n$, and for every $\phi, \psi \in C_0^\infty(U_\omega)$, the operator

$$Tu := \psi(\omega^{-1})^* A \omega^*(\phi u), \quad u \in C^\infty(U_\omega),^1$$

is a standard pseudo-differential operator with symbol $a_T \in S_{\rho, \delta}^m(T^*U_\omega)$. In this case we write $A \in \Psi_{\rho, \delta}^m(M, \text{loc})$.

2.3. Positive sub-Laplacians and global pseudo-differential operators. Let G be a compact Lie group with Lie algebra $\mathfrak{g} \simeq T_e G$, where e is the neutral element of G , and let

$$X = \{X_1, \dots, X_k\} \subset \mathfrak{g}$$

be a system of C^∞ -vector fields. For all multi-index,

$$I = (i_1, \dots, i_\omega) \in \{1, 2, \dots, k\}^\omega$$

of length $\omega \geq 1$, we denote by

$$X_I := [X_{i_1}, [X_{i_2}, \dots [X_{i_{\omega-1}}, X_{i_\omega}] \dots]]$$

a commutator of length ω , where $X_I := X_i$ when $\omega = 1$ and $I = (i)$. The system X satisfies the Hörmander condition of step κ if

$$\mathfrak{g} = \text{span}\{X_I : |I| \leq \kappa\}.$$

Given a system $X = \{X_1, \dots, X_k\}$ satisfying the Hörmander condition, the operator defined as

$$\mathcal{L} \equiv \mathcal{L}_X := -(X_1^2 + \dots + X_k^2),$$

is called the subelliptic Laplacian associated with the system X , or simply the sub-Laplacian associated to X . The subellipticity of \mathcal{L} follows from the validity of the estimate, (see

¹As usually, ω^* and $(\omega^{-1})^*$ are the pullbacks, induced by the maps ω and ω^{-1} respectively.

Hörmander [27] and Rothschild and Stein [38])

$$(2.3) \quad \|u\|_{H^s(G)} \leq C(\|\mathcal{L}u\|_{L^2(G)} + \|u\|_{L^2(G)}),$$

with $s = 2/\kappa$, while the Sobolev space H^s of order s is defined by the norm

$$\|u\|_{H^s(G)} := \|(1 - \Delta)^{\frac{s}{2}} u\|_{L^2(G)}.$$

Here, Δ is the negative Laplace-Beltrami operator on G .

Let us now introduce the Hausdorff dimension associated with the sub-Laplacian \mathcal{L} . For all $x \in G$, let $H_x^\omega G$ be the linear subspace of $T_x G$ generated by the X_i 's and by the Lie brackets

$$[X_{j_1}, X_{j_2}], [X_{j_1}, [X_{j_2}, X_{j_3}]], \dots, [X_{j_1}, [X_{j_2}, [X_{j_3}, \dots, X_{j_\omega}]]],$$

where $\omega \leq \kappa$. Then, Hörmander's condition says that $H_x^\kappa G = T_x G$, $x \in G$, and we have that

$$H_x^1 G \subset H_x^2 G \subset H_x^3 G \subset \dots \subset H_x^{\kappa-1} G \subset H_x^\kappa G = T_x G, \quad x \in G.$$

The dimension of every $H_x^\omega G$ is constant in $x \in G$, so we set $\dim H^\omega G := \dim H_x^\omega G$, for all $x \in G$. The Hausdorff dimension can be defined as, see [35],

$$(2.4) \quad Q := \sum_{i=1}^{\kappa} i(\dim H^i G - \dim H^{i-1} G).$$

Let A be a continuous linear operator from $C^\infty(G)$ into $\mathcal{D}'(G)$, and let \widehat{G} be the unitary dual of G . There exists a matrix-valued function

$$(2.5) \quad a : G \times \widehat{G} \rightarrow \cup_{\ell \in \mathbb{N}} \mathbb{C}^{\ell \times \ell},$$

that we call the matrix symbol of A , such that $a(x, \xi) := a(x, [\xi]) \in \mathbb{C}^{d_\xi \times d_\xi}$ for every $[\xi] \in \widehat{G}$, with $\xi : G \rightarrow \text{Hom}(H_\xi)$, $H_\xi \cong \mathbb{C}^{d_\xi}$, and such that

$$(2.6) \quad Af(x) = \sum_{[\xi] \in \widehat{G}} d_\xi \text{Tr}[\xi(x) a(x, \xi) \widehat{f}(\xi)], \quad \forall f \in C^\infty(G).$$

We have denoted by

$$\widehat{f}(\xi) \equiv (\mathcal{F}f)(\xi) := \int_G f(x) \xi(x)^* dx \in \mathbb{C}^{d_\xi \times d_\xi}, \quad [\xi] \in \widehat{G},$$

the group Fourier transform of f at ξ where the matrix representation of ξ is induced by an orthonormal basis of the representation space H_ξ . Correspondingly, one denotes the inverse Fourier transform of $g(\xi) \in \mathbb{C}^{d_\xi \times d_\xi}$ as

$$(\mathcal{F}^{-1}g)(x) := \sum_{[\xi] \in \widehat{G}} d_\xi \text{Tr}(\xi(x) g(\xi)), \quad x \in G.$$

Note that the matrix-valued function a in (2.5) satisfying (2.6) is unique, and satisfies the identity

$$a(x, \xi) = \xi(x)^* (A\xi)(x), \quad A\xi := (A\xi_{ij})_{i,j=1}^{d_\xi}, \quad [\xi] \in \widehat{G}.$$

We will use the notation $A = \text{Op}(a)$ to indicate that $a := \sigma_A(x, \xi)$ is the (unique) matrix-valued symbol associated with A .

As defined in [45], a difference operator $Q_\xi : \mathcal{D}'(\widehat{G}) \rightarrow \mathcal{D}'(\widehat{G})$ of order $k \in \mathbb{N}$ is defined via

$$(2.7) \quad Q_\xi \widehat{f}(\xi) = \widehat{qf}(\xi), [\xi] \in \widehat{G},$$

for some function $q \in C^\infty(G)$ vanishing of order k at $x = e$. We denote by $\text{diff}^k(\widehat{G})$ the set of the difference operators of order k . The associated difference operator to q is denoted by $\Delta_q \equiv Q_\xi$. A system of difference operators (see [45])

$$(2.8) \quad \Delta_\xi^\alpha := \Delta_{q(1)}^{\alpha_1} \cdots \Delta_{q(i)}^{\alpha_i}, \quad \alpha = (\alpha_j)_{1 \leq j \leq i},$$

with $i \geq n$, is called an admissible family, if

$$(2.9) \quad \text{rank}\{\nabla q_{(j)}(e) : 1 \leq j \leq i\} = \dim(G), \text{ and } \Delta_{q(j)} \in \text{diff}^1(\widehat{G}).$$

An admissible family is said to be strongly admissible if, we also have the property

$$(2.10) \quad \bigcap_{j=1}^i \{x \in G : q_{(j)}(x) = 0\} = \{e\}.$$

REMARK 2.3. We observe that matrix components of unitary irreducible representations induce difference operators of arbitrary order. Let us illustrate this fact as follows. If $\xi_1, \xi_2, \dots, \xi_k$, are fixed irreducible and unitary representations of the group G , which does not necessarily belong to the same equivalence class, then the matrix coefficients

$$(2.11) \quad \xi_\ell(g) - I_{d_{\xi_\ell}} = [\xi_\ell(g)_{ij} - \delta_{ij}]_{i,j=1}^{d_{\xi_\ell}}, \quad g \in G, \quad 1 \leq \ell \leq k,$$

define the smooth functions $q_{ij}^\ell(g) := \xi_\ell(g)_{ij} - \delta_{ij}$, $g \in G$, and then define the difference operators

$$(2.12) \quad \mathbb{D}_{\xi_\ell,ij} := \mathcal{F}(\xi_\ell(g)_{ij} - \delta_{ij}) \mathcal{F}^{-1}.$$

Then, by fixing $k \geq \dim(G)$ of these unitary representations with the property that its corresponding family of difference operators is admissible one can define higher-order difference operators of this kind. Indeed, let us fix a unitary representation ξ_ℓ . We omit the index ℓ . Then, for any given multi-index $\alpha \in \mathbb{N}_0^{d_{\xi_\ell}^2}$, with $|\alpha| = \sum_{i,j=1}^{d_{\xi_\ell}} \alpha_{ij}$, we write

$$\mathbb{D}^\alpha := \mathbb{D}_{11}^{\alpha_{11}} \cdots \mathbb{D}_{d_{\xi_\ell} d_{\xi_\ell}}^{\alpha_{d_{\xi_\ell} d_{\xi_\ell}}}$$

for a difference operator of order $|\alpha|$.

The difference operators endow the unitary dual \widehat{G} with a difference structure. Indeed, the following Leibniz formula holds true (see [43] for details). We refer to Definition 2.5 for the description via the group Fourier transform of the matrix-valued distributions in the class $\mathcal{D}'(G \times \widehat{G})$.

Proposition 2.4 (Leibniz rule for difference operators). *Let G be a compact Lie group and let \mathbb{D}^α , $\alpha \in \mathbb{N}_0^{d_{\xi_\ell}}$, be the family of difference operators defined in (2.12). Then, the following Leibniz rule*

$$\mathbb{D}^\alpha(a_1 a_2)(x_0, \xi) = \sum_{|\gamma|, |\varepsilon| \leq |\alpha| \leq |\gamma| + |\varepsilon|} C_{\varepsilon, \gamma} (\mathbb{D}^\gamma a_1)(x_0, \xi) (\mathbb{D}^\varepsilon a_2)(x_0, \xi), \quad x_0 \in G,$$

holds for all $a_1, a_2 \in \mathcal{D}'(G \times \widehat{G})$, where the summation is taken over all ε, γ such that $|\varepsilon|, |\delta| \leq |\alpha| \leq |\gamma| + |\varepsilon|$.

Now, we will introduce the Hörmander classes of matrix-symbols defined in [41]. We identify every $Y \in \mathfrak{g}$ with the differential operator $\partial_Y : C^\infty(G) \rightarrow \mathcal{D}'(G)$ defined by

$$\partial_Y f(x) = (Y_x f)(x) = \frac{d}{dt} f(x \exp(tY))|_{t=0}.$$

If $\{X_1, \dots, X_n\}$ is a basis of the Lie algebra \mathfrak{g} , we use the standard multi-index notation

$$\partial_X^\alpha = X_x^\alpha = \partial_{X_1}^{\alpha_1} \cdots \partial_{X_n}^{\alpha_n}.$$

By using this property, together with the following notation for the so-called elliptic weight

$$\langle \xi \rangle := (1 + \lambda_{[\xi]})^{1/2}, \quad [\xi] \in \widehat{G},$$

we can finally give the definition of global symbol classes. Here, $\lambda_{[\xi]}$, $[\xi] \in \widehat{G}$, denotes the corresponding eigenvalue of the positive Laplacian (in a bijective manner) indexed by an equivalence class $[\xi] \in \widehat{G}$.

DEFINITION 2.5. Let $0 \leq \delta, \rho \leq 1$. Let

$$\sigma : G \times \widehat{G} \rightarrow \bigcup_{[\xi] \in \widehat{G}} \mathbb{C}^{d_\xi \times d_\xi},$$

be a matrix-valued function such that for any $[\xi] \in \widehat{G}$, $\sigma(\cdot, [\xi])$ is smooth, and such that, for any element $x \in G$ there is a distribution $k_x \in \mathcal{D}'(G)$, of C^∞ -class in x , satisfying that $\sigma(x, \xi) = \widehat{k}_x(\xi)$, $[\xi] \in \widehat{G}$. The collection of all matrix-valued symbols $\sigma = \sigma(x, \xi)$ satisfying these properties will be denoted by $\mathcal{D}'(G \times \widehat{G})$.

We say that $\sigma \in \mathcal{S}_{\rho, \delta}^m(G)$ if, for all β and γ multi-indices and for all $(x, [\xi]) \in G \times \widehat{G}$, the following inequalities

$$(2.13) \quad \|\partial_X^\beta \Delta_\xi^\gamma \sigma(x, \xi)\|_{op} \leq C_{\alpha, \beta} \langle \xi \rangle^{m - \rho|\gamma| + \delta|\beta|},$$

hold, where $\|\cdot\|_{op}$ denotes the $\ell^2 \rightarrow \ell^2$ operator norm

$$(2.14) \quad \|\sigma(x, \xi)\|_{op} = \sup \{ \|\sigma(x, \xi)v\|_{\ell^2} : v \in \mathbb{C}^{d_\xi}, \|v\|_{\ell^2} = 1 \}.$$

For $\sigma_A \in \mathcal{S}_{\rho, \delta}^m(G)$ we will write $A \in \Psi_{\rho, \delta}^m(G) \equiv \text{Op}(\mathcal{S}_{\rho, \delta}^m(G))$.

The global Hörmander classes on compact Lie groups describe the Hörmander classes defined by local coordinate systems. We present the corresponding statement as follows.

Theorem 2.6 (Equivalence of classes, [41, 43]). *Let $A : C^\infty(G) \rightarrow \mathcal{D}'(G)$ be a continuous linear operator and let us consider $0 \leq \delta < \rho \leq 1$, with $\rho \geq 1 - \delta$. Then, $A \in \Psi_{\rho, \delta}^m(G, \text{loc})$, if and only if $\sigma_A \in \mathcal{S}_{\rho, \delta}^m(G)$, consequently*

$$(2.15) \quad \text{Op}(\mathcal{S}_{\rho, \delta}^m(G)) = \Psi_{\rho, \delta}^m(G, \text{loc}), \quad 0 \leq \delta < \rho \leq 1, \quad \rho \geq 1 - \delta.$$

2.4. Subelliptic Hörmander classes on compact Lie groups. In order to define the subelliptic Hörmander classes, we will use a suitable basis of the Lie algebra arising from Taylor expansions. We explain the choice of this basis by means of the following lemma (see [7, Section 3.1]).

Lemma 2.7. *Let G be a compact Lie group of dimension n . Let $\mathfrak{D} = \{\Delta_{q(j)}\}_{1 \leq j \leq n}$ be a strongly admissible collection of difference operators (for the definition see (2.9) and (2.10)). Then there exists a basis $X_{\mathfrak{D}} = \{X_{1,\mathfrak{D}}, \dots, X_{n,\mathfrak{D}}\}$ of \mathfrak{g} such that*

$$X_{j,\mathfrak{D}} q_{(k)}(\cdot^{-1})(e) = \delta_{jk}.$$

Moreover, by using the multi-index notation

$$\partial_X^{(\beta)} = \partial_{X_{1,\mathfrak{D}}}^{\beta_1} \cdots \partial_{X_{n,\mathfrak{D}}}^{\beta_n},$$

for any $\beta \in \mathbb{N}_0^n$, where

$$\partial_{X_{i,\mathfrak{D}}} f(x) = \frac{d}{dt} f(x \exp(tX_{i,\mathfrak{D}}))|_{t=0}, \quad f \in C^\infty(G),$$

and denoting by

$$R_{x,N}^f(y) = f(xy) - \sum_{|\alpha| \leq N} q_{(1)}^{\alpha_1}(y^{-1}) \cdots q_{(n)}^{\alpha_n}(y^{-1}) \partial_X^{(\alpha)} f(x)$$

the Taylor remainder, we have that

$$|R_{x,N}^f(y)| \leq C|y|^N \max_{|\alpha| \leq N} \|\partial_X^{(\alpha)} f\|_{L^\infty(G)},$$

where the constant $C > 0$ is dependent on N, G and \mathfrak{D} (but not on $f \in C^\infty(G)$). In addition we have that $\partial_X^{(\beta)}|_{x_1=x} R_{x,N}^f = R_{x,N}^{\partial_X^{(\beta)} f}$, and

$$|\partial_X^{(\beta)}|_{y_1=y} R_{x,N}^f(y_1)| \leq C|y|^{N-|\beta|} \max_{|\alpha| \leq N-|\beta|} \|\partial_X^{(\alpha+\beta)} f\|_{L^\infty(G)},$$

provided that $|\beta| \leq N$.

Denoting by $\Delta_\xi^\alpha := \Delta_{q(1)}^{\alpha_1} \cdots \Delta_{q(n)}^{\alpha_n}$, we can introduce the subelliptic Hörmander class of symbols of order $m \in \mathbb{R}$ and of type (ρ, δ) . We will use the notation $\widehat{\mathcal{M}}$ to indicate the matrix symbol of $\mathcal{M} := (1 + \mathcal{L})^{\frac{1}{2}}$. Also, for every $[\xi] \in \widehat{G}$ and $s \in \mathbb{R}$, we define the subelliptic matrix weight,

$$\widehat{\mathcal{M}}(\xi)^s := \text{diag}[(1 + \nu_{ii}(\xi)^2)^{\frac{s}{2}}]_{1 \leq i \leq d_\xi},$$

where $\widehat{\mathcal{L}}(\xi) =: \text{diag}[\nu_{ii}(\xi)^2]_{1 \leq i \leq d_\xi}$ is the symbol of the sub-Laplacian \mathcal{L} at $[\xi]$, as the symbol of the operator $\mathcal{M}_s := (1 + \mathcal{L})^{\frac{s}{2}}$.

DEFINITION 2.8 (SUBELLIPTIC HÖRMANDER CLASSES). Let G be a compact Lie group and let $0 \leq \delta, \rho \leq 1$. Let us consider a sub-Laplacian $\mathcal{L} = -(X_1^2 + \cdots + X_k^2)$ on G , where the system of vector fields $X = \{X_i\}_{i=1}^k$ satisfies the Hörmander condition of step κ . We say that $\sigma \in S_{\rho,\delta}^{m,\mathcal{L}}(G \times \widehat{G})$, if for all $r \in \mathbb{R}$, $\alpha, \beta \in \mathbb{N}_0^n$,

$$(2.16) \quad p_{\alpha,\beta,\rho,\delta,m,r}(a) := \sup_{(x,[\xi]) \in G \times \widehat{G}} \|\widehat{\mathcal{M}}(\xi)^{(\rho|\alpha| - \delta|\beta| - m - r)} \partial_X^{(\beta)} \Delta_\xi^\alpha a(x, \xi) \widehat{\mathcal{M}}(\xi)^r\|_{\text{op}} < \infty.$$

where $\|\cdot\|_{\text{op}}$ is as in (2.14).

By following the usual nomenclature, we define:

$$\text{Op}(S_{\rho,\delta}^{m,\mathcal{L}}(G \times \widehat{G})) := \{A : C^\infty(G) \rightarrow \mathcal{D}'(G) : \sigma_A \equiv \widehat{A}(x, \xi) \in S_{\rho,\delta}^{m,\mathcal{L}}(G \times \widehat{G})\},$$

with

$$Af(x) = \sum_{[\xi] \in \widehat{G}} d_\xi \text{Tr}(\xi(x) \widehat{A}(x, \xi) \widehat{f}(\xi)), \quad f \in C^\infty(G), \quad x \in G.$$

The decay properties of subelliptic symbols are summarized in the following lemma (see [7, Chapter 4]), where we present a necessary (but not a sufficient) condition in order that the matrix-symbol $a := a(x, \xi)$ belongs to the class $S_{\rho,\delta}^{m,\mathcal{L}}(G \times \widehat{G})$.

Lemma 2.9. *Let G be a compact Lie group and let $0 \leq \delta, \rho \leq 1$. If $a \in S_{\rho,\delta}^{m,\mathcal{L}}(G \times \widehat{G})$, then for every $\alpha, \beta \in \mathbb{N}_0^n$, there exists $C_{\alpha,\beta} > 0$ satisfying the estimates*

$$\|\partial_X^{(\beta)} \Delta_\xi^\alpha a(x, \xi)\|_{\text{op}} \leq C_{\alpha,\beta} \sup_{1 \leq i \leq d_\xi} (1 + \nu_{ii}(\xi)^2)^{\frac{m-\rho|\alpha|+\delta|\beta|}{2}},$$

uniformly in $(x, [\xi]) \in G \times \widehat{G}$.

In the next theorem we describe the fundamental properties of the subelliptic calculus [7], like compositions, adjoints, and boundedness properties.

Theorem 2.10. *Let $0 \leq \delta < \rho \leq 1$, and let $\Psi_{\rho,\delta}^{m,\mathcal{L}} := \text{Op}(S_{\rho,\delta}^{m,\mathcal{L}}(G \times \widehat{G}))$, for every $m \in \mathbb{R}$. Then,*

- The mapping $A \mapsto A^* : \Psi_{\rho,\delta}^{m,\mathcal{L}} \rightarrow \Psi_{\rho,\delta}^{m,\mathcal{L}}$ is a continuous linear mapping between Fréchet spaces and the symbol of A^* , $\sigma_{A^*}(x, \xi)$ satisfies the asymptotic expansion,

$$\widehat{A}^*(x, \xi) \sim \sum_{|\alpha|=0}^{\infty} \Delta_\xi^\alpha \partial_X^{(\alpha)} (\widehat{A}(x, \xi)^*).$$

This means that, for every $N \in \mathbb{N}$, and for all $\ell \in \mathbb{N}$,

$$\Delta_\xi^{\alpha_\ell} \partial_X^{(\beta)} \left(\widehat{A}^*(x, \xi) - \sum_{|\alpha| \leq N} \Delta_\xi^\alpha \partial_X^{(\alpha)} (\widehat{A}(x, \xi)^*) \right) \in S_{\rho,\delta}^{m-(\rho-\delta)(N+1)-\rho\ell+\delta|\beta|, \mathcal{L}}(G \times \widehat{G}),$$

where $|\alpha_\ell| = \ell$.

- The mapping $(A_1, A_2) \mapsto A_1 \circ A_2 : \Psi_{\rho,\delta}^{m_1, \mathcal{L}} \times \Psi_{\rho,\delta}^{m_2, \mathcal{L}} \rightarrow \Psi_{\rho,\delta}^{m_1+m_2, \mathcal{L}}$ is a continuous bilinear mapping between Fréchet spaces, and the symbol of $A = A_1 \circ A_2$ is given by the asymptotic formula

$$\sigma_A(x, \xi) \sim \sum_{|\alpha|=0}^{\infty} (\Delta_\xi^\alpha \widehat{A}_1(x, \xi)) (\partial_X^{(\alpha)} \widehat{A}_2(x, \xi)),$$

which, in particular, means that, for every $N \in \mathbb{N}$, and for all $\ell \in \mathbb{N}$,

$$\begin{aligned} \Delta_\xi^{\alpha_\ell} \partial_X^{(\beta)} \left(\sigma_A(x, \xi) - \sum_{|\alpha| \leq N} (\Delta_\xi^\alpha \widehat{A}_1(x, \xi)) (\partial_X^{(\alpha)} \widehat{A}_2(x, \xi)) \right) \\ \in S_{\rho,\delta}^{m_1+m_2-(\rho-\delta)(N+1)-\rho\ell+\delta|\beta|, \mathcal{L}}(G \times \widehat{G}), \end{aligned}$$

for all $\alpha_\ell \in \mathbb{N}_0^n$ with $|\alpha_\ell| = \ell$.

- For $0 \leq \delta < \rho \leq 1$, (or for $0 \leq \delta \leq \rho \leq 1$, $\delta < 1/\kappa$) let us consider a continuous linear operator $A : C^\infty(G) \rightarrow \mathcal{D}'(G)$ with symbol $\sigma \in S_{\rho,\delta}^{0,\mathcal{L}}(G \times \widehat{G})$. Then A extends to a bounded operator from $L^2(G)$ to $L^2(G)$.

Finally, we present the following result about the L^p -boundedness of the subelliptic classes for the sub-Laplacian \mathcal{L} , see [7, Section 6].

Theorem 2.11. *Let G be a compact Lie group and let us denote by Q the Hausdorff dimension of G associated to the control distance associated to the sub-Laplacian $\mathcal{L} = \mathcal{L}_X$, where $X = \{X_1, \dots, X_k\}$ is a system of vector fields satisfying the Hörmander condition of order κ . For $0 \leq \delta < \rho \leq 1$, let us consider a continuous linear operator $A : C^\infty(G) \rightarrow \mathcal{D}'(G)$ with symbol $\sigma \in S_{\rho,\delta}^{-m,\mathcal{L}}(G \times \widehat{G})$, $m \geq 0$. Then A extends to a bounded operator on $L^p(G)$ provided that*

$$m \geq m_p := Q(1 - \rho) \left| \frac{1}{p} - \frac{1}{2} \right|.$$

2.5. L^p - L^q -boundedness for Bessel potentials. Here we discuss the sharpness of the Hardy-Littlewood-Sobolev inequality, in this case, formulated in terms of the L^p - L^q -boundedness of Bessel potentials.

Lemma 2.12. *Let $1 < p, q < \infty$. Let G be a compact Lie group of the Hausdorff dimension Q associated to the control distance associated to the sub-Laplacian $\mathcal{L} = -\sum_{1 \leq i \leq k} X_i^2$. Then, the Bessel operator $B_a = (1 + \mathcal{L})^{-\frac{a}{2}}$, admits a bounded extension from $L^p(G)$ into $L^q(G)$, that is, the estimate*

$$(2.17) \quad \|B_a f\|_{L^q} \leq C \|f\|_{L^p}$$

holds, if and only if, $1 < p < q < \infty$ and

$$(2.18) \quad a \geq Q \left(\frac{1}{p} - \frac{1}{q} \right).$$

Proof. The sufficiency of the condition (2.18) on a for the L^p - L^q -boundedness of B_a is exactly the Hardy-Littlewood-Sobolev inequality, see e.g. [13]. On the other hand, assume that

$$B_a : L^p(G) \rightarrow L^q(G)$$

is bounded.

Using the subelliptic functional calculus in [7, Section 8], we have that $\sqrt{\mathcal{L}} \in \Psi_{1,0}^{1,\mathcal{L}}(G \times \widehat{G})$ is a pseudo-differential operator of first order. Since $\sqrt{\mathcal{L}}$ is not invertible, let $\sqrt{\mathcal{L}}^{-1}$ be the inverse of $\sqrt{\mathcal{L}}$ on the orthogonal complement of its kernel, i.e. if P_0 is the L^2 -orthogonal projection on $\text{Ker}(\sqrt{\mathcal{L}})$, then

$$\sqrt{\mathcal{L}}^{-1} \sqrt{\mathcal{L}} = \sqrt{\mathcal{L}} \sqrt{\mathcal{L}}^{-1} = I - P_0, \quad \forall f \in \text{Ker}(\sqrt{\mathcal{L}}), \quad \sqrt{\mathcal{L}}^{-1} f := 0.$$

This operator agrees with the operator $f(\mathcal{L})$, defined by the spectral calculus where $f(t) = t^{-1}$, and that $\sqrt{\mathcal{L}}^{-1} \in \Psi_{1,0}^{-1,\mathcal{L}}(G \times \widehat{G})$ is a consequence of the functional calculus in [7, Section 8].

Let us consider the operator $\mathcal{L}^{-\frac{a}{2}} := (\sqrt{\mathcal{L}_G^{-1}})^a \in \Psi_{1,0}^{-a,\mathcal{L}}(G \times \widehat{G})$ defined by the spectral calculus if $a > 0$. In the case where $a < 0$, $\mathcal{L}^{-\frac{a}{2}} \in \Psi_{1,0}^{|a|,\mathcal{L}}(G \times \widehat{G})$. In view of the inclusion of the powers \mathcal{L}^{-a} to the subelliptic calculus, note that

$$\mathcal{L}^{-\frac{a}{2}} = \mathcal{L}^{-\frac{a}{2}} B_{-a} B_a = T B_a,$$

where

$$T = \mathcal{L}^{-\frac{a}{2}} B_{-a} = \mathcal{L}^{-\frac{a}{2}} (1 + \mathcal{L})^{\frac{a}{2}} \in \Psi_{1,0}^0(G \times \widehat{G})$$

is a subelliptic pseudo-differential operator of order zero (see [7, Theorem 8.20]). Then $T : L^q(G) \rightarrow L^q(G)$ is bounded and from the L^p - L^q boundedness of B_a we deduce the L^p - L^q -boundedness of $\mathcal{L}^{-\frac{a}{2}} = T B_a$. Note that we have the validity of the estimate

$$(2.19) \quad \|\mathcal{L}^{-\frac{a}{2}} f\|_{L^q} \leq C \|f\|_{L^p}.$$

In other words, we have that the inequality

$$(2.20) \quad \|f\|_{L^q} \leq C \|\mathcal{L}^{\frac{a}{2}} f\|_{L^p},$$

is valid with $C > 0$, independent of $f \in L^q(G)$. Define the semigroup

$$T_t = e^{-t\mathcal{L}}, t > 0,$$

and consider the heat kernel h_t defined by $T_t f = f * p_t$. Note that (see [13, Lemma VIII.2.5, Page 110])

$$\sup_{t>0} \|h_t\|_{L^1} \lesssim 1.$$

In consequence we have that

$$(2.21) \quad \sup_{t>0} \|T_t f\|_{L^1(G)} = \sup_{t>0} \|f * h_t\|_{L^1(G)} \leq \sup_{t>0} \|h_t\|_{L^1(G)} \|f\|_{L^1(G)} \lesssim \|f\|_{L^1(G)}.$$

In a similar way

$$(2.22) \quad \sup_{t>0} \|T_t f\|_{L^\infty(G)} = \sup_{t>0} \|f * h_t\|_{L^\infty(G)} \leq \sup_{t>0} \|h_t\|_{L^1(G)} \|f\|_{L^\infty(G)} \lesssim \|f\|_{L^\infty(G)}.$$

In view of (2.21) and (2.22) we have that the semigroup T_t is equicontinuous on L^1 and on L^∞ . Moreover, in view of Remark 2.1, with $X = \{X_1, X_2, \dots, X_k\}$, the semigroup $T_t = e^{\Delta_X} = e^{-t\mathcal{L}}$ is a submarkovian semigroup. Then, in view of (2.20), by applying Theorem 2.2, with \tilde{Q} defined by

$$(2.23) \quad a = \tilde{Q} \left(\frac{1}{p} - \frac{1}{q} \right),$$

we have the estimate

$$(2.24) \quad \|e^{-t\mathcal{L}}\|_{L^1 \rightarrow L^\infty} \leq C_{\tilde{Q}} t^{-\tilde{Q}/2}.$$

Note that if

$$Q' := \inf\{\tilde{Q} : T_t \text{satisfies (2.24) for all } t : 0 < t < 1\},$$

then $Q' \leq \tilde{Q}$,

$$\|e^{-t\mathcal{L}}\|_{L^1 \rightarrow L^\infty} \leq C_{Q'} t^{-Q'/2} \leq C_{\tilde{Q}} t^{-\tilde{Q}/2}, \quad 0 < t < 1.$$

However, by the sharpness of the heat kernel estimates is very well known that the infimum Q' agrees with the Hausdorff dimension Q of the group associated to the control distance associated to the sub-Laplacian \mathcal{L} (see [13, Chapter VIII]), that is $Q' = Q$. Since $\tilde{Q} \geq Q$, in view of (2.23) we have proved that

$$(2.25) \quad a \geq Q \left(\frac{1}{p} - \frac{1}{q} \right),$$

as desired. \square

3. L^p - L^q -boundedness of pseudo-differential operators

3.1. L^p - L^q -boundedness of pseudo-differential operators I. The following result presents the necessary and sufficient criteria for a pseudo-differential operator to be bounded from $L^p(G)$ into $L^q(G)$ for the range $1 < p \leq 2 \leq q < \infty$.

Theorem 3.1. *Let $1 < p \leq 2 \leq q < \infty$ and $m \in \mathbb{R}$. Let G be a compact Lie group, and let Q be its Hausdorff dimension with respect to the control distance associated to a Hörmander sub-Laplacian \mathcal{L} . Let $0 \leq \delta < \rho \leq 1$. Then, every pseudo-differential operator $A \in \Psi_{\rho,\delta}^{m,\mathcal{L}}(G \times \widehat{G})$ with $0 \leq \delta < \rho \leq 1$ admits a bounded extension from $L^p(G)$ into $L^q(G)$, that is*

$$(3.1) \quad \forall f \in C_0^\infty(G), \|Af\|_{L^q} \leq C\|f\|_{L^p}$$

holds, if and only if,

$$(3.2) \quad m \leq -Q \left(\frac{1}{p} - \frac{1}{q} \right).$$

Proof. Assume that $m > -Q \left(\frac{1}{p} - \frac{1}{q} \right)$. We are going to show that there exists $A \in \Psi_{\rho,\delta}^{m,\mathcal{L}}(G \times \widehat{G})$ which is not bounded from $L^p(G)$ into $L^q(G)$. We consider

$$A = B_{-m} = (1 + \mathcal{L})^{\frac{m}{2}} \in \Psi_{1,0}^{m,\mathcal{L}}(G \times \widehat{G}) \subset \Psi_{\rho,\delta}^{m,\mathcal{L}}(G \times \widehat{G}).$$

Since

$$-m < Q \left(\frac{1}{p} - \frac{1}{q} \right),$$

from Lemma 2.12, we have that $A = B_{-m}$ is not bounded from $L^p(G)$ into $L^q(G)$. So, we have proved the necessity of the order condition (3.2). Now, in order to prove the reverse statement, we consider m satisfying (3.2) and m_1 and m_2 satisfying the conditions

$$(3.3) \quad m = m_1 + m_2, \quad m_1 \leq -Q(1/p - 1/2), \quad m_2 \leq -Q(1/2 - 1/q).$$

If $A \in \Psi_{\rho,\delta}^{m,\mathcal{L}}(G \times \widehat{G})$, we factorise A as follows,

$$A = B_{-m_2} A_0 B_{-m_1}, \quad A_0 = B_{m_2} A B_{m_1}.$$

Note that $A_0 \in \Psi_{\rho,\delta}^{0,\mathcal{L}}(G \times \widehat{G})$. The Calderón-Vaillancourt theorem (Theorem 2.10(iii)) implies that A_0 is bounded from $L^2(G)$ into $L^2(G)$. On the other hand, from Lemma 2.12 we have that

$B_{m_2} : L^2(G) \rightarrow L^q(G)$, and $B_{m_1} : L^p(G) \rightarrow L^2(G)$, are bounded operators. In consequence, we have proved that A admits a bounded extension from $L^p(G)$ into $L^q(G)$. The proof is complete. \square

3.2. L^p - L^q -boundedness of pseudo-differential operators II. In this subsection, we consider the $L^p - L^q$ boundedness of pseudo-differential operators on compact Lie groups for a wider range of indices p and q . Our main result of this section is the following theorem.

Theorem 3.2. *Let $1 < p \leq q < \infty$, $m \in \mathbb{R}$, and let $0 \leq \delta < \rho \leq 1$. Let G be a compact Lie group, and let Q be its Hausdorff dimension with respect to the control distance associated to a Hörmander sub-Laplacian \mathcal{L} . Then, every pseudo-differential operator $A \in \Psi_{\rho, \delta}^{m, \mathcal{L}}(G \times \widehat{G})$ admits a bounded extension from $L^p(G)$ into $L^q(G)$, that is*

$$(3.4) \quad \forall f \in C_0^\infty(G), \|Af\|_{L^q} \leq C\|f\|_{L^p}$$

holds in the following cases:

(i) if $1 < p \leq q \leq 2$ and

$$(3.5) \quad m \leq -Q\left(\frac{1}{p} - \frac{1}{q} + (1 - \rho)\left(\frac{1}{q} - \frac{1}{2}\right)\right).$$

(ii) if $2 \leq p \leq q < \infty$ and

$$(3.6) \quad m \leq -Q\left(\frac{1}{p} - \frac{1}{q} + (1 - \rho)\left(\frac{1}{2} - \frac{1}{p}\right)\right);$$

Proof. (i) Let us consider p, q and m satisfying the conditions given in (i). Choose $m' = -Q\left(\frac{1}{p} - \frac{1}{q}\right)$ and this implies that the subelliptic Bessel potential $B_{-m'}$ is bounded from $L^p(G)$ to $L^q(G)$ as a consequence of Lemma 2.12. For $A \in \Psi_{\rho, \delta}^{m, \mathcal{L}}(G \times \widehat{G})$, we decompose it as follows:

$$A = (AB_{m'})B_{-m'}.$$

Now, we note that operator $AB_{m'} \in \Psi_{\rho, \delta}^{m-m', \mathcal{L}}(G \times \widehat{G})$ with $m - m'$ satisfying $m - m' \leq -Q(1 - \rho)\left(\frac{1}{q} - \frac{1}{2}\right)$. Then, Theorem 2.11 shows that $AB_{m'}$ is bounded operator from $L^q(G)$ into $L^q(G)$. Therefore, we conclude that the operator A has a bounded extension from $L^p(G)$ into $L^q(G)$.

(ii) To prove this part we follow the same strategy as in Part (i). We factorise the operator $A \in \Psi_{\rho, \delta}^{m, \mathcal{L}}(G \times \widehat{G})$ in the following manner:

$$A = B_{-m'}(B_{m'}A),$$

where $m' = -Q\left(\frac{1}{p} - \frac{1}{q}\right)$. Again, it follows from Lemma 2.12 that the operator $B_{-m'}$ is a bounded from $L^p(G)$ into $L^q(G)$. On the other hand, the operator $B_{m'}A \in \Psi_{\rho, \delta}^{m-m', \mathcal{L}}(G \times \widehat{G})$ with $m - m' \leq -Q(1 - \rho)\left(\frac{1}{2} - \frac{1}{p}\right)$, which, as a consequence of Theorem 2.11, yields that the operator $B_{m'}A$ is bounded from $L^p(G)$ into $L^p(G)$. Hence, we conclude that the operator A has a bounded extension from $L^p(G)$ into $L^q(G)$.

This completes the proof of this theorem. \square

4. L^p - L^q -boundedness of pseudo-differential operators on \mathbb{S}^3 and on $SU(3)$

We will present an explicit form of our main Theorem 1.1 on the sphere $SU(2) \cong \mathbb{S}^3$ and on $SU(3)$. By abuse of notation, we will use the same symbol to denote an element of the Lie algebra and the vector field on the group obtained by left translation.

4.1. L^p - L^q -boundedness of pseudo-differential operators on \mathbb{S}^3 . Let us consider the left-invariant first-order differential operators $\partial_+, \partial_-, \partial_0 : C^\infty(SU(2)) \rightarrow C^\infty(SU(2))$, called creation, annihilation, and neutral operators respectively, (see Definition 11.5.10 of [41]) and let us define

$$X_1 = -\frac{i}{2}(\partial_- + \partial_+), \quad X_2 = \frac{1}{2}(\partial_- - \partial_+), \quad X_3 = -i\partial_0,$$

where $X_3 = [X_1, X_2]$, based on the commutation relations $[\partial_0, \partial_+] = \partial_+$, $[\partial_-, \partial_0] = \partial_-$, and $[\partial_+, \partial_-] = 2\partial_0$. The system $X = \{X_1, X_2\}$ satisfies the Hörmander condition of step $\kappa = 2$, and the Hausdorff dimension defined by the control distance associated to the sub-Laplacian $\mathcal{L}_1 = -X_1^2 - X_2^2$ is $Q = 4$. In a similar way, we can define the sub-Laplacian $\mathcal{L}_2 = -X_2^2 - X_3^2$ associated to the system of vector fields $X' = \{X_2, X_3\}$, which also satisfies the Hörmander condition of step $\kappa = 2$. In the following corollary we describe the L^p - L^q -boundedness of subelliptic pseudo-differential operators on $SU(2) \cong \mathbb{S}^3$. In this case we observe that one can identify $\widehat{SU}(2) \cong \frac{1}{2}\mathbb{N}_0$, see [41] for details.

Corollary 4.1. *Let $1 < p, q < \infty$, and $0 \leq \delta < \rho \leq 1$. Let us consider the Hörmander sub-Laplacian $\mathcal{L} = -X_1^2 - X_2^2$. Then, the following statements hold.*

- *Let $1 < p \leq 2 \leq q < \infty$. Every pseudo-differential operator*

$$A \in \Psi_{\rho, \delta}^{m, \mathcal{L}}(SU(2) \times \frac{1}{2}\mathbb{N}_0)$$

admits a bounded extension from $L^p(SU(2))$ into $L^q(SU(2))$, that is

$$(4.1) \quad \forall f \in C^\infty(SU(2)), \|Af\|_{L^q(SU(2))} \leq C\|f\|_{L^p(SU(2))}$$

holds, if and only if,

$$(4.2) \quad m \leq -4\left(\frac{1}{p} - \frac{1}{q}\right).$$

- *Every pseudo-differential operator $A \in \Psi_{\rho, \delta}^{m, \mathcal{L}}(SU(2) \times \frac{1}{2}\mathbb{N}_0)$ admits a bounded extension from $L^p(SU(2))$ into $L^q(SU(2))$, that is (4.1) holds, in the following cases:*

- (i) *if $1 < p \leq q \leq 2$ and*

$$(4.3) \quad m \leq -4\left(\frac{1}{p} - \frac{1}{q} + (1 - \rho)\left(\frac{1}{q} - \frac{1}{2}\right)\right).$$

- (ii) *if $2 \leq p \leq q < \infty$ and*

$$(4.4) \quad m \leq -4\left(\frac{1}{p} - \frac{1}{q} + (1 - \rho)\left(\frac{1}{2} - \frac{1}{p}\right)\right).$$

4.2. L^p - L^q -boundedness of pseudo-differential operators on $SU(3)$. The special unitary group of 3×3 complex matrices is defined by

$$SU(3) = \{g \in \mathrm{GL}(3, \mathbb{C}) : gg^* = I_3 \equiv (\delta_{ij})_{1 \leq i, j \leq 3}, \det(g) = 1\},$$

and its Lie algebra is given by

$$\mathfrak{su}(3) = \{g \in \mathrm{GL}(3, \mathbb{C}) : g + g^* = 0, \mathrm{Tr}(g) = 0\}.$$

The inner product is defined by a multiple of the Killing form on $\mathfrak{su}(3)$ given by $B(X, Y) = -\frac{1}{2}\mathrm{Tr}[XY]$. The torus

$$\mathbb{T}_{SU(3)} = \{\mathrm{diag}[e^{i\theta_1}, e^{i\theta_2}, e^{i\theta_3}] : \theta_1 + \theta_2 + \theta_3 = 0, \theta_i \in \mathbb{R}\}$$

is a maximal torus of $SU(3)$, and its Lie algebra is given by

$$\mathfrak{t}_{\mathfrak{su}(3)} = \{\mathrm{diag}[i\theta_1, i\theta_2, i\theta_3] : \theta_1 + \theta_2 + \theta_3 = 0, \theta_i \in \mathbb{R}\}.$$

The following vectors

$$T_1 = \mathrm{diag}[-i, i, 0], \quad T_2 = \mathrm{diag}[-i/\sqrt{3}, -i/\sqrt{3}, 2i/\sqrt{3}]$$

provide a basis for $\mathfrak{t}_{\mathfrak{su}(3)}$. Completing this basis with the following vectors

$$X_1 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad X_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix},$$

$$X_4 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & -i & 0 \end{pmatrix}, \quad X_5 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad X_6 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix},$$

we obtain the Gell-Mann system, which forms an orthonormal basis of $\mathfrak{su}(3)$. The system of vector fields $X = \{X_1, X_2, X_3, X_4, X_5, X_6\}$ satisfies the Hörmander condition of step $\kappa = 2$, (see [7, Section 11]). Indeed, this can be deduced if we write

$$X_7 = -[X_1, X_2] = \begin{pmatrix} -2i & 0 & 0 \\ 0 & 2i & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad X_8 = -[X_3, X_4] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2i & 0 \\ 0 & 0 & -2i \end{pmatrix}$$

from TABLE 1. Observe that the Hausdorff dimension associated to the control distance

Table 1. Commutators in $SU(3)$

	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8
X_1	0	$-X_7$	X_5	$-X_6$	$-X_3$	X_4	$4X_2$	$2X_2$
X_2	X_7	0	X_6	X_5	$-X_4$	$-X_3$	$-4X_1$	$-2X_1$
X_3	$-X_5$	$-X_6$	0	$-X_8$	X_1	X_2	$2X_4$	$4X_4$
X_4	X_6	$-X_5$	X_8	0	X_2	$-X_1$	$-2X_3$	$-4X_3$
X_5	X_3	X_4	$-X_1$	$-X_2$	0	$X_8 - X_7$	$2X_6$	$-2X_6$
X_6	$-X_4$	X_3	$-X_2$	X_1	$X_7 - X_8$	0	$-2X_5$	$2X_5$
X_7	$-4X_2$	$4X_1$	$-2X_4$	$2X_3$	$-2X_6$	$2X_5$	0	0
X_8	$2X_2$	$2X_1$	$-4X_4$	$4X_3$	$2X_6$	$-2X_5$	0	0

associated to the sub-Laplacian

$$\mathcal{L} = -X_1^2 - X_2^2 - X_3^2 - X_4^2 - X_5^2 - X_6^2,$$

can be computed from (2.4) as follows.

$$Q := \dim(H^1 G) + 2(\dim H^2 G - \dim H^1 G) = 6 + 2(8 - 6) = 10.$$

In the following Corollary we describe the L^p - L^q -boundedness of subelliptic pseudo-differential operators on $SU(3)$. In this case we observe that one can identify $\widehat{SU}(3) \cong \{D(p, q) : p, q \in \mathbb{N}_0\}$, where $D(p, q)$ in physical terms, p is the number of quarks and q is the number of antiquarks. The construction of the unitary representations $D(p, q)$ can be found in [24]. We keep in this case the standard notation $\widehat{SU}(3)$ by simplicity.

Corollary 4.2. *Let $1 < p, q < \infty$, and $0 \leq \delta < \rho \leq 1$. Let us consider the Hörmander sub-Laplacian $\mathcal{L} = -X_1^2 - X_2^2 - X_3^2 - X_4^2 - X_5^2 - X_6^2$. Then, the following statements hold.*

- *Let $1 < p \leq 2 \leq q < \infty$. Every pseudo-differential operator*

$$A \in \Psi_{\rho, \delta}^{m, \mathcal{L}}(SU(3) \times \widehat{SU}(3))$$

admits a bounded extension from $L^p(SU(3))$ into $L^q(SU(3))$, that is

$$(4.5) \quad \forall f \in C^\infty(SU(3)), \|Af\|_{L^q(SU(3))} \leq C\|f\|_{L^p(SU(3))}$$

holds, if and only if,

$$(4.6) \quad m \leq -10\left(\frac{1}{p} - \frac{1}{q}\right).$$

- *Every pseudo-differential operator $A \in \Psi_{\rho, \delta}^{m, \mathcal{L}}(SU(3) \times \widehat{SU}(3))$ admits a bounded extension from $L^p(SU(3))$ into $L^q(SU(3))$, that is (4.5) holds, in the following cases:*

- (i) *if $1 < p \leq q \leq 2$ and*

$$(4.7) \quad m \leq -10\left(\frac{1}{p} - \frac{1}{q} + (1 - \rho)\left(\frac{1}{q} - \frac{1}{2}\right)\right).$$

- (ii) *if $2 \leq p \leq q < \infty$ and*

$$(4.8) \quad m \leq -10\left(\frac{1}{p} - \frac{1}{q} + (1 - \rho)\left(\frac{1}{2} - \frac{1}{p}\right)\right).$$

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