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## GENERALIZED STANDARD AUSLANDER-REITEN COMPONENTS WITHOUT ORIENTED CYCLES

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### Introduction

Let  $A$  be an artin algebra,  $\text{mod } A$  the category of finitely generated right  $A$ -modules,  $\text{rad}^\infty(\text{mod } A)$  the infinite radical of  $\text{mod } A$ , and  $\Gamma_A$  the Auslander-Reiten quiver of  $A$ . It is known that  $\Gamma_A$  describes the quotient category  $\text{mod } A/\text{rad}^\infty(\text{mod } A)$ . We are interested in the behaviour of connected components of  $\Gamma_A$  in the module category  $\text{mod } A$ . We introduced in [14] the concept of a generalized standard component and proved some facts on such components. A component  $\mathcal{C}$  of  $\Gamma_A$  is called *generalized standard* if  $\text{rad}^\infty(X, Y) = 0$  for all  $X$  and  $Y$  from  $\mathcal{C}$ . Examples of generalized standard components are all preprojective components, preinjective components, and connecting components of tilted algebras. We proved in [14] that a generalized standard connected component of  $\Gamma_A$  admits at most finitely many nonperiodic  $D\text{Tr}$ -orbits. Moreover, we described regular and semi-regular generalized standard components of  $\Gamma_A$  containing no oriented cycle, and proved that all but a finite number of generalized standard components of  $\Gamma_A$  are stable tubes.

The main aim of this paper is to describe arbitrary generalized standard components without oriented cycles. As an application we obtain new characterizations of tilted algebras and concealed algebras.

The paper is organized as follows. In Section 1 we recall some notions and facts from the representation theory of artin algebras needed in the paper. Section 2 contains a description of generalized standard components without oriented cycles. In Section 3 we characterize generalized standard components containing sections, and prove some characterizations of tilted algebras. Section 4 contains some new characterizations of concealed algebras.

### 1. Preliminaries

Let  $A$  be an artin algebra over a commutative artin ring  $R$ . We denote by

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$\text{mod } A$  the category of all finitely generated right  $A$ -modules, and by  $D: \text{mod } A \rightarrow \text{mod } A^{\text{op}}$  the standard duality  $\text{Hom}_R(-, I)$ , where  $I$  is the injective envelope of  $R/\text{rad } R$  in  $\text{mod } R$ . By a module we usually mean a finitely generated right module. We denote by  $\text{rad}(\text{mod } A)$  the radical of  $\text{mod } A$  and by  $\text{rad}^{\infty}(\text{mod } A)$  the intersection of all powers  $\text{rad}^i(\text{mod } A)$ ,  $i \geq 1$ , of  $\text{rad}(\text{mod } A)$ . A *path* in  $\text{mod } A$  is a sequence of non-zero non-isomorphisms  $M_1 \rightarrow M_2 \rightarrow \dots \rightarrow M_n$ , where the modules  $M_i$  are indecomposable. A full subcategory  $\mathcal{Z}$  of  $\text{mod } A$  is said to be *closed under predecessors* (resp. *closed under successors*) if any path in  $\text{mod } A$  with the target (resp. source) in  $\mathcal{Z}$  consists entirely of modules in  $\mathcal{Z}$ . We denote by  $\Gamma_A$  the Auslander-Reiten quiver of  $A$ , and let  $\tau_A, \tau_A^-$  be the Auslander-Reiten operators  $D\text{Tr}$ ,  $\text{Tr}D$ , respectively. We shall not distinguish between an indecomposable  $A$ -module, its isomorphism class and the vertex of  $\Gamma_A$  corresponding to it. By a component of  $\Gamma_A$  we mean a connected component of  $\Gamma_A$ . A component  $\mathcal{C}$  of  $\Gamma_A$  is called *preprojective* (resp. *preinjective*) if  $\mathcal{C}$  has no oriented cycle and each module in  $\mathcal{C}$  belongs to the  $\tau_A$ -orbit of a projective (resp. injective) module. A component  $\mathcal{C}$  of  $\Gamma_A$  is called *sincere* if any simple  $A$ -module occurs as a simple composition factor of a module in  $\mathcal{C}$ . For a component  $\mathcal{C}$  of  $\Gamma_A$ , we denote by  $\text{ann } \mathcal{C}$  the annihilator of  $\mathcal{C}$  in  $A$ , that is, the intersection of the annihilators  $\text{ann } X$  of all modules  $X$  from  $\mathcal{C}$ . If  $\text{ann } \mathcal{C} = 0$ , the component  $\mathcal{C}$  is called *faithful*. Clearly, a faithful component is sincere.

Let  $H$  be a hereditary artin algebra,  $T$  a tilting  $H$ -module and  $B = \text{End}_H(T)$  the associated tilted algebra. Then  $T$  determines a torsion theory  $(\mathcal{F}(T), \mathcal{G}(T))$  in  $\text{mod } H$ , where  $\mathcal{F}(T) = \{X_H \mid \text{Hom}_H(T, X) = 0\}$  and  $\mathcal{G}(T) = \{Y_H \mid \text{Ext}_H^1(T, Y) = 0\}$ , and a splitting torsion theory  $(\mathcal{Y}(T), \mathcal{X}(T))$  in  $\text{mod } B$ , where  $\mathcal{Y}(T) = \{N_B \mid \text{Tor}_1^B(N, T) = 0\}$  and  $\mathcal{X}(T) = \{M_B \mid M \otimes_B T = 0\}$ . Then by the theorem of Brenner and Butler the functor  $F = \text{Hom}_H(T, -)$  induces an equivalence between  $\mathcal{G}(T)$  and  $\mathcal{Y}(T)$ , and the functor  $F' = \text{Ext}_H^1(T, -)$  an equivalence between  $\mathcal{F}(T)$  and  $\mathcal{X}(T)$ . Then the injective cogenerator  $DH$  of  $\text{mod } H$  belongs to  $\mathcal{G}(T)$  and the indecomposable direct summands of  $F(DH)$  form a set  $\mathcal{S}$  belonging to one component of  $\Gamma_B$ , called the connecting component of  $\Gamma_B$  corresponding to  $T$ . Moreover  $\mathcal{S}$  is a slice of  $\text{mod } B$  (see [11, (4.2)]), that is,  $\mathcal{S}$  satisfies the following conditions:

- ( $\alpha$ )  $\mathcal{S}$  is sincere.
- ( $\beta$ )  $\mathcal{S}$  is path closed (any path in  $\text{mod } A$  with source and target in  $\mathcal{S}$  consists entirely of modules in  $\mathcal{S}$ ).
- ( $\gamma$ ) If  $M$  is an indecomposable nonprojective  $A$ -module, then at most one of  $M, \tau_A M$  belongs to  $\mathcal{S}$ .
- ( $\delta$ ) If  $M \rightarrow S$  is an irreducible map with  $M$  and  $S$  indecomposable and  $S$  in  $\mathcal{S}$ , then either  $M$  belongs to  $\mathcal{S}$  or  $M$  is noninjective and  $\tau_A^- M$  belongs to  $\mathcal{S}$ .

Observe that the condition ( $\beta$ ) is very difficult for checking.

We shall need the following lemma proved in [13].

**Lemma 1.** *Let  $A$  be an artin algebra and  $n$  be the number of isoclasses of simple  $A$ -modules. Let  $X_1, \dots, X_r$  be pairwise nonisomorphic indecomposable  $A$ -modules such that  $\text{Hom}_A(X_i, \tau_A X_j) = 0$  for all  $1 \leq i, j \leq r$ . Then  $r \leq n$ .*

The following simple lemma will be also useful.

**Lemma 2.** *Let  $A$  be an artin algebra,  $\mathcal{C}$  a component of  $\Gamma_A$  and  $B = A/\text{ann } \mathcal{C}$ . Then  $\mathcal{C}$  is a generalized standard component of  $\Gamma_A$  if and only if  $\mathcal{C}$  is a generalized standard component of  $\Gamma_B$ .*

**Proof.** Clearly,  $\mathcal{C}$  is a full component of  $\Gamma_B$ . From the existence of Auslander-Reiten sequences in  $\text{mod } A$  we know that  $\text{rad}(\text{mod } A)$  is generated by the irreducible maps as a left and as a right ideal. Let  $X$  and  $Y$  be two indecomposable modules from  $\mathcal{C}$ , and suppose that  $f: X \rightarrow Y$  is a nonzero map from  $\text{rad}^\infty(\text{mod } A)$ . Then there are modules  $X_i$  and maps  $g_i: X_i \rightarrow X_{i+1}$ ,  $h_i: X_{i+1} \rightarrow Y$  in  $\text{rad}(\text{mod } B)$ ,  $i \geq 0$ , such that  $X_0 = X$  and, for each  $i$ ,  $X_i$  is a direct sum of indecomposable modules from  $\mathcal{C}$  and  $f = h_i g_i \cdots g_0$ . Then  $f$  belongs to  $\text{rad}^\infty(\text{mod } B)$ . This proves the lemma because clearly  $\text{rad}^\infty(\text{mod } B)$  is contained in  $\text{rad}^\infty(\text{mod } A)$ .

## 2. Generalized standard components without oriented cycles

Let  $A$  be an artin algebra and  $\mathcal{C}$  be a component of  $\Gamma_A$  without oriented cycles. We are interested in criteria for  $\mathcal{C}$  to be generalized standard. We shall first define a full translation subquiver  ${}_\infty \mathcal{C}$  of  $\mathcal{C}$  closed under predecessors, called the left end of  $\mathcal{C}$ , and a full translation subquiver  $\mathcal{C}_\infty$  of  $\mathcal{C}$  closed under successors, called the right end of  $\mathcal{C}$ .

Denote by  ${}_1 \mathcal{C}$  the left stable part of  $\mathcal{C}$ , obtained from  $\mathcal{C}$  by removing the  $\tau_A$ -orbits of projective modules. Then  ${}_1 \mathcal{C}$  is a disjoint union of finitely many left stable full translation connected subquivers  $\mathcal{D}_1, \dots, \mathcal{D}_s$  of  $\mathcal{C}$ . From [6, (3.4)], for each  $1 \leq i \leq s$ , there exists a valued quiver  $\Delta_i$  without oriented cycles such that  $\mathcal{D}_i$  is isomorphic to a full translation subquiver of  $Z\Delta_i$  which is closed under predecessors. Let  $\Sigma_i$  be a fixed copy of  $\Delta_i$  in  $\mathcal{D}_i$  such that the modules forming the vertices of  $\Sigma_i$  are neither successors of indecomposable direct summands of the radicals of projective modules in  $\mathcal{C}$  nor successors of injective modules in  $\mathcal{C}$ . Let  ${}_\infty \Sigma$  be the disjoint union of the quivers  $\Sigma_1, \dots, \Sigma_s$ . Then denote by  ${}_\infty \mathcal{C}$  the full translation subquiver of  $\mathcal{C}$  formed by all predecessors in  $\mathcal{C}$  of modules from  ${}_\infty \Sigma$ . Observe that  ${}_\infty \mathcal{C}$  is a left stable full translation subquiver of  $\mathcal{C}$  which is closed under predecessors. Denote by  ${}_\infty N$  the direct sum of all modules from  ${}_\infty \Sigma$ , and put  ${}_\infty M = {}_\infty N \oplus P$ , where  $P$  is the direct sum of all projective modules from  $\mathcal{C}$ . If  ${}_1 \mathcal{C}$  is empty, put  ${}_\infty N = 0$ . Dually, denote by  $\mathcal{C}_\infty$  the right stable part

of  $\mathcal{C}$ , obtained from  $\mathcal{C}$  by removing the  $\tau_A$ -orbits of injective modules. Assume that  $\mathcal{C}_r$  is nonempty. Then  $\mathcal{C}_r$  is a disjoint union of finitely many full translation connected subquivers  $\mathcal{D}'_1, \dots, \mathcal{D}'_m$  of  $\mathcal{C}$ . Again, from [6], for each  $1 \leq j \leq m$ , there is a valued quiver  $\Delta'_j$  without oriented cycles such that  $\mathcal{D}'_j$  is isomorphic to a full translation subquiver of  $Z\Delta'_j$  which is closed under successors. Let  $\Sigma'_j$  be a fixed copy of  $\Delta'_j$  in  $\mathcal{D}'_j$  such that the modules forming the vertices of  $\Sigma'_j$  are neither predecessors of indecomposable direct summands of the socle factors of injective modules in  $\mathcal{C}$  nor predecessors of projective modules in  $\mathcal{C}$ . Let  $\Sigma_\infty$  be the disjoint union of the quivers  $\Sigma'_1, \dots, \Sigma'_m$ . Then denote by  $\mathcal{C}_\infty$  the full translation subquiver of  $\mathcal{C}$  formed by all successors in  $\mathcal{C}$  of modules from  $\tau_A^{-1}\Sigma_\infty$ . Observe that  $\mathcal{C}_\infty$  is a right stable full translation subquiver of  $\mathcal{C}$  which is closed under successors. Denote by  $N_\infty$  the direct sum of all modules from  $\Sigma_\infty$ , and put  $M_\infty = Q \oplus N_\infty$ , where  $Q$  is the direct sum of all injective modules from  $\mathcal{C}$ . If  $\mathcal{C}_r$  is empty, we put  $N_\infty = 0$ . We may assume that  ${}_\infty\mathcal{C}$  and  $\mathcal{C}_\infty$  have no common modules.

The following theorem gives a characterization of generalized standard components without oriented cycles.

**Theorem 1.** *Let  $A$  be an artin algebra,  $\mathcal{C}$  be a component of  $\Gamma_A$  without oriented cycles and  $B = A/\text{ann } \mathcal{C}$ . Then, in the above notation, the following conditions are equivalent.*

- (i)  $\mathcal{C}$  is a generalized standard component of  $\Gamma_A$ .
- (ii)  $\text{Hom}_A(P, {}_\infty N) = 0$  and  $\text{Hom}_A(X, \tau_A Y) = 0$  for all modules  $X$  and  $Y$  from  ${}_\infty\Sigma$ .
- (iii)  $\text{Hom}_A(N_\infty, Q) = 0$  and  $\text{Hom}_A(\tau_A^{-1}X, Y) = 0$  for all modules  $X$  and  $Y$  from  $\Sigma_\infty$ .
- (iv)  ${}_\infty\Sigma$  is finite and  $\text{rad}^\infty({}_\infty M, {}_\infty N) = 0$ .
- (v)  $\Sigma_\infty$  is finite and  $\text{rad}^\infty(N_\infty, M_\infty) = 0$ .
- (vi)  ${}_\infty\Sigma$  or  $\Sigma_\infty$  is finite and  $\text{Hom}_A(N_\infty, {}_\infty N) = 0$ .
- (vii) The following conditions hold :
  - (a) There is a hereditary artin algebra  ${}_\infty H$  and a tilting  ${}_\infty H$ -module  ${}_\infty T$  without preinjective direct summands such that the tilted algebra  ${}_\infty B = \text{End}_{{}_\infty H}({}_\infty T)$  is a factor algebra of  $B$  and the torsion-free part  ${}^0\mathcal{Y}({}_\infty T)$  of  $\text{mod } {}_\infty B$  is a full exact subcategory of  $\text{mod } B$  which is closed under predecessors.
  - (b) There is a hereditary artin algebra  ${}_\infty H$  and a tilting  ${}_\infty H$ -module  ${}_\infty T$  without preprojective direct summands such that the tilted algebra  $B_\infty = \text{End}_{H_\infty}({}_\infty T)$  is a factor algebra of  $B$  and the torsion part  $\mathcal{X}({}_\infty T)$  of  $\text{mod } B_\infty$  is a full exact subcategory of  $\text{mod } B$  which is closed under successors.
  - (c)  ${}^0\mathcal{Y}({}_\infty T)$  and  $\mathcal{X}({}_\infty T)$  have no common nonzero modules.
  - (d)  ${}_\infty\mathcal{C}$  is the torsion-free part of the connecting component of  $\Gamma_{\infty B}$  corresponding to  ${}_\infty T$ .

(e)  $\mathcal{C}_\infty$  is the torsion part of the connecting component of  $\Gamma_{B_\infty}$  corresponding to  $T_\infty$ .

(f) The class of indecomposable  $B$ -modules which are neither in  $\mathcal{A}(T_\infty)$  nor in  $\mathcal{X}(T_\infty)$  is finite and coincides (up to isomorphism) with the class of modules in  $\mathcal{C}$  which are neither in  ${}_\infty\mathcal{C}$  nor in  $\mathcal{C}_\infty$ .

Proof. Without loss of generality we may assume that  $A$  is basic and connected. The implications (i) $\Rightarrow$ (ii) and (i) $\Rightarrow$ (iii) follow directly from our choice of  ${}_\infty\Sigma$  and  $\Sigma_\infty$ . We shall show now that (ii) implies (iv), and (iii) implies (v). First observe that, by Lemma 1, if  $\text{Hom}_A(X, \tau_A Y) = 0$  (resp.  $\text{Hom}_A(\tau_A X, Y) = 0$ ) for all modules  $X$  and  $Y$  from  ${}_\infty\Sigma$  (resp. from  $\Sigma_\infty$ ), then  ${}_\infty\Sigma$  (resp.  $\Sigma_\infty$ ) is finite. Moreover, we have  $\text{Hom}_A({}_\infty M, {}_\infty N) = \text{Hom}_A(P, {}_\infty N) \oplus \text{Hom}_A({}_\infty N, {}_\infty N)$  and  $\text{Hom}_A(N_\infty, M_\infty) = \text{Hom}_A(N_\infty, N_\infty) \oplus \text{Hom}_A(N_\infty, Q)$ . If  ${}_\infty\Sigma$  is finite, then any map from  $\text{rad}^\infty({}_\infty N, {}_\infty N)$  factors through a module  $\tau_A({}_\infty N)^a$ , for some  $a \geq 1$ . Similarly, if  $\Sigma_\infty$  is finite, then any map from  $\text{rad}^\infty(N_\infty, N_\infty)$  factors through  $\tau_A(N_\infty)^b$ , for some  $b \geq 1$ . Therefore, (ii) implies (iv), and (iii) implies (v). We claim now that each of the conditions (iv) and (v) implies (vi). First observe that  ${}_\infty\Sigma$  is finite if and only if  $\Sigma_\infty$  is finite,. Assume that  ${}_\infty\Sigma$  and  $\Sigma_\infty$  are finite, and suppose that  $\text{Hom}_A(N_\infty, {}_\infty N) \neq 0$ . Then clearly  $\text{rad}^\infty(N_\infty, {}_\infty N) \neq 0$ . Since any epimorphism  $A^c \rightarrow N_\infty$  factors through a module  ${}_\infty M^d$ , for some  $d \geq 1$ , there is an epimorphism  ${}_\infty M^d \rightarrow N_\infty$ . Then  $\text{rad}^\infty({}_\infty M, {}_\infty N) \neq 0$  because  $\text{rad}^\infty({}_\infty N, {}_\infty N) \neq 0$ . Similarly, any monomorphism  ${}_\infty N \rightarrow (DA)^p$  factors through a module  $M_\infty^q$ , for some  $q \geq 1$ , and hence there is a monomorphism  ${}_\infty N \rightarrow M_\infty^q$ . Then  $\text{rad}^\infty(N_\infty, M_\infty) \neq 0$  because  $\text{rad}^\infty(N_\infty, {}_\infty N) \neq 0$ . This proves our claims. We shall show now that (vi) implies (i). Suppose that  $\text{rad}^\infty(U, V) \neq 0$  for some indecomposable modules  $U$  and  $V$  from  $\mathcal{C}$ . Then there is an infinite path in  $\mathcal{C}$

$$U = U_0 \rightarrow U_1 \rightarrow \cdots \rightarrow U_i \rightarrow U_{i+1} \rightarrow \cdots$$

such that  $\text{rad}^\infty(U_i, V) \neq 0$  for all  $i \geq 0$ . Since from (vi)  $\Sigma_\infty$  and  ${}_\infty\Sigma$  are finite, there is  $m \geq 0$  such that  $U_m$  is a successor of  $\Sigma_\infty$  in  $\mathcal{C}$ . Further,  $\text{rad}^\infty(U_m, V) \neq 0$  implies existence of an infinite path in  $\mathcal{C}$

$$\cdots \rightarrow V_{j+1} \rightarrow V_j \rightarrow \cdots \rightarrow V_1 \rightarrow V_0 = V$$

such that  $\text{rad}^\infty(U_m, V_j) \neq 0$  for all  $j \geq 0$ . Again, since  ${}_\infty\Sigma$  and  $\Sigma_\infty$  are finite, there is  $n \geq 0$  such that  $V_n$  is a predecessor of  ${}_\infty\Sigma$  in  $\mathcal{C}$ . Then any epimorphism  $A^k \rightarrow U_m$  factors through a module  $N_\infty^r$ , for some  $r \geq 1$ , and hence there is an epimorphism  $N_\infty^r \rightarrow U_m$ . Similarly, any monomorphism  $V_n \rightarrow (DA)^s$  factors through a module  ${}_\infty N^t$ , for some  $t \geq 1$ , and hence we have a monomorphism  $V_n \rightarrow {}_\infty N^t$ . Then  $\text{rad}^\infty(U_m, V_n) \neq 0$  implies that  $\text{rad}^\infty(N_\infty, {}_\infty N) \neq 0$ , a contradiction to (vi). Hence (vi) implies (i). Consequently, we proved that the conditions (i)-(vi) are equivalent. We shall show now that (vii) implies (vi). Assume that (vii) holds.

Then clearly  $\infty\Sigma$  and  $\Sigma_\infty$  are finite. Moreover,  $\text{Hom}_A(N_\infty, \infty N) = \text{Hom}_B(N_\infty, \infty N) = 0$ , because  $\mathcal{Y}(\infty T)$  and  $\mathcal{X}(T_\infty)$  have no common indecomposable modules,  $\mathcal{Y}(\infty T)$  is closed under predecessors in  $\text{mod } B$ , and  $\mathcal{X}(T_\infty)$  is closed under successors in  $\text{mod } B$ . Hence (vi) holds. Finally, assume that  $\mathcal{C}$  is a generalized standard component of  $\Gamma_A$ . Then, from Lemma 2,  $\mathcal{C}$  is also a generalized standard component of  $\Gamma_B$ . We shall show that the conditions (a)-(f) of (vii) hold. First observe that  $\mathcal{C}$ , as a generalized standard component, has by [14, (2.3)] only finitely many  $\tau_B$ -orbits, and hence  $\infty\Sigma$  and  $\Sigma_\infty$  are finite. Write  $B = P' \oplus P$  as a  $B$ -module. Assume that  $P' \neq 0$ , and put  $\infty B = \text{End}_B(P')$ . We claim that  $\infty N$  is a faithful tilting  $\infty B$ -module. For simplicity of notations we put  $N = \infty N$  and  $F = \infty B$ . First observe that  $\text{Hom}_B(P, N) = \text{rad}^\infty(P, N) = 0$  since (i) is equivalent to (ii), and hence  $N$  is a  $F$ -module. Further, since  $\mathcal{C}$  is a faithful component of  $\Gamma_B$  and  $B$  is an artin algebra, there are indecomposable modules  $Z_1, \dots, Z_m$  in  $\mathcal{C}$  such that  $Z = Z_1 \oplus \dots \oplus Z_m$  is a faithful  $B$ -module. We claim that there are indecomposable  $B$ -modules  $W_1, \dots, W_n$  in  $\mathcal{C}$  which are not proper predecessors of  $\infty\Sigma$ , and such that  $W = W_1 \oplus \dots \oplus W_n$  is a faithful  $B$ -module. Suppose that some  $Z_i$  is a proper predecessor of  $\infty\Sigma$ . We may assume that  $Z_1, \dots, Z_s, s \leq m$ , are all proper predecessors of  $\infty\Sigma$  in the family  $Z_1, \dots, Z_m$ . Let  $Z' = Z_1 \oplus \dots \oplus Z_s$  and  $Z'' = Z_{s+1} \oplus \dots \oplus Z_m$ . Since  $Z$  is a faithful  $B$ -module, there is an epimorphism  $f: Z^k \rightarrow DB$ , for some  $k \geq 1$ . Then, since  $\infty\Sigma$  has no injective predecessors, the restriction of  $f$  to  $(Z')^k$  factors through a module  $N^j$ , for some  $j \geq 1$ . Hence there is an epimorphism  $W^a \rightarrow DB$ , for some  $a \geq 1$ , and  $W = N \oplus Z''$ , and so  $W$  is a required faithful  $B$ -module. In particular, there is a monomorphism  $g: B \rightarrow W'$ , for some  $r \geq 1$ . Since  $F = P'$  as a right  $F$ -module, restriction of  $g$  to  $F$  gives a monomorphism  $h: F \rightarrow W'$ . But  $P'$  has no indecomposable direct summands in  $\mathcal{C}$ , and so  $h$  factors through a module  $N^t$ , for some  $t \geq 1$ . Hence there is a monomorphism  $e: F \rightarrow N^t$ , and  $N$  is a faithful  $F$ -module. Further, since  $\mathcal{C}$  is a generalized standard component of  $\Gamma_B$ , we have  $\text{Hom}_B(P, U) = 0$  for all modules  $U$  in  $\infty\mathcal{C}$ . Consequently,  $\infty\mathcal{C}$  consists entirely of  $F$ -modules. Moreover, by our choice of  $\infty\Sigma$ , for any module  $X$  from  $\infty\Sigma$ , the module  $\tau_B^- X$  is also a  $F$ -module. Hence  $\tau_B N = \tau_F N$  and  $\tau_B^- N = \tau_F^- N$ . Then, since  $\mathcal{C}$  is generalized standard, we get  $\text{Hom}_F(N, \tau_F N) = 0$ ,  $\text{Hom}_F(\tau_F^- N, N) = 0$ , and  $\text{Ext}_F^1(N, N) \cong \text{DHom}_F(N, \tau_F N) = 0$ . Also, if  $\text{Hom}_F(N, V) \neq 0$  for an indecomposable  $F$ -module  $V$  which is not a direct summand of  $N$ , then  $\text{Hom}_F(\tau_F^- N, V) \neq 0$ , because  $\infty\Sigma$  is finite. Then, by Lemmas 1.6, 1.5 and its dual, in [9],  $N$  is a tilting  $F$ -module. Moreover,  $\infty H = \text{End}_F(N)$  is a hereditary algebra, since  $\mathcal{C}$  is generalized standard. Therefore, there exists a tilting  $\infty H$ -module  $\infty T$  such that  $\infty B = F = \text{End}_{\infty H}(\infty T)$  and  $\infty\mathcal{C}$  is the torsion-free part of the connecting component of  $\Gamma_{\infty B}$  corresponding to  $\infty T$ . Further, since  $\infty\mathcal{C}$  has no projective modules,  $\infty T$  has no preinjective direct summands. Observe also that  $\text{Hom}_B(P, P') = 0$ , since  $\text{Hom}_B(P, N) = 0$  and  $P'$  is a submodule of  $N^t$ . This implies that  $B$  is isomorphic to  $\begin{bmatrix} C & E \\ 0 & F \end{bmatrix}$ ,

where  $C = \text{End}_B(P)$ ,  $F = \text{End}_B(P')$  and  $E = \text{Hom}_B(P', P)$ . In particular,  ${}_\infty B = F$  is a factor algebra of  $B$ . We shall prove now that the torsion-free part  ${}^0\mathcal{Y}({}_\infty T)$  of  $\text{mod } F$  is closed under predecessors in  $\text{mod } B$ . We know that  ${}^0\mathcal{Y}({}_\infty T)$  is closed under predecessors in  $\text{mod } F$ . First observe that  $E_F$  is a direct sum of indecomposable  $B$ -modules which are in  $\mathcal{C}$  but not in  ${}_\infty \mathcal{C}$ . Indeed,  $E_F$  is the largest  $F$ -submodule of  $P$  and any epimorphism  $F'' \rightarrow E_F$  factors through a module  $N^k$ , for some  $k \geq 1$ . Moreover, by our choice of  ${}_\infty \Sigma$ ,  $\text{rad } P$  and  $N$  have no common indecomposable direct summands. In order to prove that  ${}^0\mathcal{Y}({}_\infty T)$  is closed under predecessors in  $\text{mod } B$ , it is sufficient to show that there are no nonzero maps from indecomposable  $B$ -modules which are not  $F$ -modules to indecomposable  $F$ -modules in  ${}^0\mathcal{Y}({}_\infty T)$ . Each  $B$ -module can be viewed as a triple  $(U_C, V_F, \phi)$ , where  $\phi: U_C \otimes_E V_F \rightarrow V$  is a  $F$ -homomorphism. Let  $(0, h)$  be a nonzero map from an indecomposable  $B$ -module  $(U_C, V_F, \phi)$  which is not in  $\text{mod } F$  to an indecomposable  $F$ -module  $W = (0, W_F, 0)$ . Then  $h: V \rightarrow W$  is nonzero. Let  $L$  be an indecomposable direct summand of  $V$  such that  $\text{Hom}_F(L, W) \neq 0$ , and let  $p: V \rightarrow L$  be the canonical projection. Since  $(U_C, V_F, \phi)$  is indecomposable and not in  $\text{mod } F$ , the composition  $p\phi$  is nonzero. Hence  $\text{Hom}_F(U_C \otimes_E V_F, L) \neq 0$ . Consider now an epimorphism  $C'' \rightarrow U$  in  $\text{mod } C$ . Then we get an epimorphism  $E'' \cong C'' \otimes_E U \rightarrow U \otimes_E V_F$ , and consequently we have  $\text{Hom}_F(E'', L) \neq 0$ . This implies that  $L$  belongs to  $\mathcal{X}({}_\infty T)$ , and hence  $W$  belongs to  $\mathcal{X}({}_\infty T)$ , because  $\text{Hom}_F(L, W) \neq 0$ . Therefore,  ${}^0\mathcal{Y}({}_\infty T)$  is closed under predecessors in  $\text{mod } B$ . We proved that  ${}_\infty B$ ,  ${}_\infty H$ , and  ${}_\infty T$  satisfy the required conditions (a) and (d). Dually, we define  $B_\infty$ ,  $H_\infty$ , and  $T_\infty$  such that the conditions (b) and (e) are satisfied. Moreover, by our choice of  ${}_\infty \mathcal{C}$  and  $\mathcal{C}_\infty$ , the condition (c) also holds. Finally, all but a finite number of modules in  $\mathcal{C}$  belong to the union of  ${}_\infty \mathcal{C}$  and  $\mathcal{C}_\infty$ , because  $\mathcal{C}$  has only finitely many  $\tau_B$ -orbits. Let now  $X$  be an indecomposable  $B$ -module. Suppose that  $X$  is neither in  ${}^0\mathcal{Y}({}_\infty T)$  nor in  $\mathcal{C}$ . Observe that, if  $X$  is a  ${}_\infty B$ -module, then  $\text{Hom}_B({}_\infty N, X) \neq 0$ , because any epimorphism  ${}_\infty B^p \rightarrow X$  factors through  ${}_\infty N^q$ , for some  $q \geq 1$ . If  $X$  is not a  ${}_\infty B$ -module, then  $\text{Hom}_B(P, X) \neq 0$ . Therefore, since  $X$  does not belong to  $\mathcal{C}$ , we have either  $\text{rad}({}_\infty N, X) \neq 0$  or  $\text{rad}({}_\infty P, X) \neq 0$ . Hence there is  $Y$  in  $\mathcal{C}_\infty$  such that  $\text{Hom}_B(Y, X) \neq 0$ . But then  $X$  belongs to  $\mathcal{X}(T_\infty)$ , because  $Y$  belongs to  ${}^0\mathcal{Y}({}_\infty T)$  and  $\mathcal{X}(T_\infty)$  is closed under successors in  $\text{mod } B$ . This proves that the condition (f) also holds. We proved that (i) implies (vii), and this finishes our proof.

### 3. Generalized standard components with sections

Components with sections form a special class of components without oriented cycles. The aim of this section is to give simple characterizations of generalized standard components with sections. As an application we obtain some new simple characterizations of tilted algebras. We would like to inform

that similar results to those presented in this section were also proved independently by Shiping Liu (a private communication).

Let  $A$  be an artin algebra and  $\mathcal{C}$  be a component of  $\Gamma_A$ . A full connected subquiver  $\Sigma$  of  $\mathcal{C}$  is called a *section* if it satisfies the following conditions:

- (1)  $\Sigma$  has no oriented cycle.
- (2)  $\Sigma$  intersects each  $\tau_A$ -orbit of  $\mathcal{C}$  exactly once.
- (3) Each path in  $\mathcal{C}$  with source and target from  $\Sigma$  lies entirely in  $\Sigma$ .
- (4) If  $X \rightarrow Y$  is an arrow in  $\mathcal{C}$  with  $X$  from  $\Sigma$  (resp.  $Y$  from  $\Sigma$ ), then either  $Y$  or  $\tau_A Y$  (resp.  $X$  or  $\tau_A X$ ) is in  $\Sigma$ .

It is easy to see that if  $\mathcal{C}$  contains a section  $\Sigma$ , then  $\mathcal{C}$  has no oriented cycle (see [3, (8.1)]). Moreover, if  $\mathcal{C}$  admits a slice (in the sense of Section 1), then  $\mathcal{C}$  admits a section.

We shall need the following simple lemma.

**Lemma 3.** *Let  $A$  be an artin algebra and  $\mathcal{C}$  be a component of  $\Gamma_A$  having a finite section  $\Sigma$ . Let  $M$  be the direct sum of all modules forming the vertices of  $\Sigma$ . Then  $\text{ann } \mathcal{C} = \text{ann } M$ .*

**Proof.** Clearly  $\text{ann } \mathcal{C}$  is contained in  $\text{ann } M$ . Let  $B = A/\text{ann } \mathcal{C}$ . Then  $\mathcal{C}$  is a faithful component of  $\Gamma_B$ . It is sufficient to show that  $M$  is a faithful  $B$ -module. Since  $B$  is an artin algebra, there are indecomposable modules  $Z_1, \dots, Z_r$  in  $\mathcal{C}$  such that  $Z = Z_1 \oplus \dots \oplus Z_r$  is a faithful  $B$ -module. Let  $Z = U \oplus V$ , where  $U$  is a direct sum of predecessors of  $\Sigma$  in  $\mathcal{C}$ , and  $V$  has no such direct summands. Suppose that  $V \neq 0$ . Since  $Z$  is a faithful  $B$ -module, there is a monomorphism  $f: B \rightarrow Z^t = U^t \oplus V^t$ , for some  $t \geq 1$ . Moreover, since  $\Sigma$  is finite, we have then  $f = gh$ , where  $h: B \rightarrow U^t \oplus M^s$  and  $g: U^t \oplus M^s \rightarrow U^t \oplus V^t$ , for some  $s \geq 1$ . Then  $h$  is a monomorphism, and hence  $L = U \oplus M$  is a faithful  $B$ -module. Let  $L = E \oplus F$ , where  $F$  is a direct sum of modules from  $\Sigma$  and  $E$  has no direct summands from  $\Sigma$ . Suppose that  $E \neq 0$ . Then, since  $L$  is a faithful  $B$ -module, there exists an epimorphism  $p: L^m \rightarrow DB$ , for some  $m \geq 1$ . Hence, since  $\Sigma$  is finite, we have  $p = qe$ , where  $e: L^m \rightarrow M^k \oplus F^m$ ,  $q: M^k \oplus F^m \rightarrow DB$ , for some  $k \geq 1$ . Observe that  $q$  is an epimorphism, and so  $M \oplus F$  is a faithful  $B$ -module. But  $M \oplus F$  is a direct sum of modules from  $\Sigma$ . Therefore,  $M$  is a faithful  $B$ -module.

**Theorem 2.** *Let  $A$  be an artin algebra and  $\mathcal{C}$  be a component of  $\Gamma_A$  containing a section  $\Sigma$ . Denote by  $M$  the direct sum of all modules from  $\Sigma$ . Then the following conditions are equivalent.*

- (i)  $\mathcal{C}$  is a generalized standard component of  $\Gamma_A$ .
- (ii)  $\text{Hom}_A(X, \tau_A Y) = 0$  for any modules  $X$  and  $Y$  from  $\Sigma$ .
- (iii)  $\text{Hom}_A(\tau_A X, Y) = 0$  for any modules  $X$  and  $Y$  from  $\Sigma$ .
- (iv)  $\Sigma$  is finite and  $\text{rad}^\infty(M, M) = 0$ .

**Proof.** The implications (i)  $\Rightarrow$  (ii), and (i)  $\Rightarrow$  (iii) are clear. Observe that, if

(ii) or (iii) holds, then, by Lemma 1,  $\Sigma$  is finite. Suppose that  $\Sigma$  is finite and  $\text{rad}^\infty(M, M) \neq 0$ . Then any nonzero map from  $\text{rad}^\infty(M, M)$  factors through a module  $(\tau_A M)^r$ , for some  $r \geq 1$ , and also factors through a module  $(\tau_{\bar{A}} M)^s$  for some  $s \geq 1$ . Hence (ii) implies (iv), and (iii) implies (iv). Finally, assume that (iv) holds. We claim that  $\mathcal{C}$  is generalized standard. Suppose that  $\text{rad}^\infty(U, V) \neq 0$  for some indecomposable modules  $U$  and  $V$  from  $\mathcal{C}$ . Then there is an infinite path

$$U = U_0 \rightarrow U_1 \rightarrow \cdots \rightarrow U_i \rightarrow U_{i+1} \rightarrow \cdots$$

in  $\mathcal{C}$  such that  $\text{rad}^\infty(U_i, V) \neq 0$  for all  $i \geq 0$ . Since  $\Sigma$  is finite, there is  $m \geq 0$  such that  $U_m$  is a successor of  $\Sigma$  in  $\mathcal{C}$ . Then  $\text{rad}^\infty(U_m, V) \neq 0$  implies existence of an infinite path

$$\cdots \rightarrow V_{i+1} \rightarrow V_j \rightarrow \cdots \rightarrow V_1 \rightarrow V_0 = V$$

in  $\mathcal{C}$  such that  $\text{rad}^\infty(U_m, V_j) \neq 0$  for all  $j \geq 0$ . Again, since  $\Sigma$  is finite, there is  $r \geq 0$  such that  $V_r$  is a predecessor of  $\Sigma$  in  $\mathcal{C}$ . Let  $f: A^r \rightarrow U_m$  be an epimorphism. Then  $f = gh$ , where  $h: A^r \rightarrow M^t$  and  $g: M^t \rightarrow U_m$ , for some  $t \leq 1$ , because  $U_m$  is a successor of the finite section  $\Sigma$ . Clearly, then  $g$  is an epimorphism. Similarly, let  $\phi: V_r \rightarrow (DA)^s$  be a monomorphism. Then  $\phi = \beta\alpha$ , where  $\alpha: V_r \rightarrow M^k$ ,  $\beta: M^k \rightarrow (DA)^s$ , for some  $k \geq 1$ , because  $V_r$  is a predecessor of the finite section  $\Sigma$ . Clearly,  $\alpha$  is a monomorphism. Now, if  $\gamma$  is a nonzero map from  $\text{rad}^\infty(U_m, V_r)$ , then  $\alpha\gamma g: M^t \rightarrow M^k$  is nonzero and belongs to  $\text{rad}^\infty(\text{mod } A)$ . Hence,  $\text{rad}^\infty(M, M) \neq 0$ , a contradiction to (iv). Therefore, (iv) implies (i).

We may prove now the following characterization of tilted algebras.

**Theorem 3.** *Let  $A$  be an artin algebra. Then the following conditions are equivalent.*

- (i)  $A$  is a tilted algebra.
- (ii)  $\Gamma_A$  admits a faithful generalized standard component  $\mathcal{C}$  containing a section.
- (iii)  $\Gamma_A$  admits a component  $\mathcal{C}$  having a faithful section  $\Sigma$  such that  $\text{Hom}_A(X, \tau_A Y) = 0$  for all modules  $X$  and  $Y$  from  $\Sigma$ .
- (iv)  $\Gamma_A$  admits a component  $\mathcal{C}$  having a faithful section  $\Sigma$  such that  $\text{Hom}_A(\tau_{\bar{A}} X, Y) = 0$  for all modules  $X$  and  $Y$  from  $\Sigma$ .
- (v)  $\Gamma_A$  admist a component  $\mathcal{C}$  having a faithful finite section  $\Sigma$  such that  $\text{rad}^\infty(M, M) = 0$ , where  $M$  is the direct sum of all modules from  $\Sigma$ .

Proof. The equivalence of the conditions (ii)-(v) is a direct consequence of Theorem 2 and Lemma 3. Assume now tht that  $A = \text{End}_H(T)$  for some hereditary artin algebra  $H$  and a tilting  $H$ -module  $T$ . Denote by  $\mathcal{C}$  the connecting component of  $\Gamma_A$  corresponding to  $T$  (see Section 1). It is well known (see

[11, (4.2)]) that the family  $\mathcal{S}$  of modules  $F(I)=\text{Hom}_H(T, I)$ , where  $I$  are indecomposable injective  $H$ -modules, is a finite faithful slice in  $\mathcal{C}$ . Then the full subquiver  $\Sigma$  of  $\mathcal{C}$  formed by the modules from  $\mathcal{S}$  is a finite faithful section of  $\mathcal{C}$ . Moreover, the torsion-free part  ${}^0\mathcal{Y}(T) \cap \mathcal{C}$  of  $\mathcal{C}$  consists of all predecessors of  $\Sigma$  in  $\mathcal{C}$  whereas the torsion part  $\mathcal{X}(T) \cap \mathcal{C}$  of  $\mathcal{C}$  consists of all successors of  $\tau_A\Sigma$  in  $\mathcal{C}$ . Since there are no nonzero maps from modules in  $\mathcal{X}(T)$  to modules in  $\mathcal{Y}(T)$ , we have  $\text{Hom}_A(\tau_A X, Y)=0$  for all modules  $X$  and  $Y$  from  $\Sigma$ . Consequently, (i) implies (iv). Assume now that  $\mathcal{C}$  is a component of  $\Gamma_A$  with a section  $\Sigma$  such that the equivalent conditions (ii)-(v) are satisfied. Then  $\Sigma$  is a finite section. Let  $M$  be the direct sum of all modules from  $\Sigma$ . Then  $M$  is a faithful  $A$ -module. Moreover, by (iii), (iv) and the well known Auslander-Reiten formula, we have  $\text{Hom}_A(M, \tau_A M)=0$ ,  $\text{Hom}_A(\tau_A M, M)=0$ , and  $\text{Ext}_A^1(M, M) \cong \widehat{D\text{Hom}_A}(M, \tau_A M)=0$ . Finally, since  $\Sigma$  is a finite section, if  $\text{Hom}_A(M, Z) \neq 0$  for an indecomposable  $A$ -module  $Z$  which is not a direct summand of  $M$ , then  $\text{Hom}_A(\tau_A M, Z) \neq 0$ . Then, by Lemmas 1.6, 1.5 and its dual, from [9], we infer that  $M$  is a tilting  $A$ -module. Since also  $\text{rad}^\infty(M, M)=0$ , then  $H=\text{End}_A(M)$  is a hereditary artin algebra. Therefore,  $A$  is a tilted algebra of the form  $\text{End}_H(T)$  for some tilting  $H$ -module  $T$ . This finishes the proof.

#### 4. Concealed algebras

Following [11, (4.3)] a concealed algebra is an artin algebra of the form  $\text{End}_H(T)$ , where  $H$  is a connected, representation-infinite, hereditary artin algebra and  $T$  is a preprojective (equivalently, preinjective) tilting  $H$ -module. Concealed algebras form an important class of tilted algebras. It follows from [16, (7.5)] that every representation-infinite tilted algebra has a factor algebra which is a concealed algebra. Moreover, the concealed algebras play a crucial role in the Bongartz criterion for finite representation type [1], the author's criterion for polynomial growth [15], and it is expected that they will play a similar role in a criterion for tame type (see [8]). Using the concept of derived categories, Ringel proved in [12] (see also [2]) that the class of concealed algebras coincides with the class of algebras  $A$  such that  $\Gamma_A$  has two different components containing slices. This fact is also a direct consequence of [4, (4.1)] and [16, (7.5)]. We have also the following characterization of concealed algebras.

**Lemma 4.** *Let  $A$  be an artin algebra. Then the following conditions are equivalent.*

- (i)  $A$  is a concealed algebra.
- (ii)  $\Gamma_A$  admits a sincere preprojective component without injective modules and a sincere preinjective component without projective modules.

**Proof.** It is a direct consequence of [11] and [16, (7.5)].

We shall prove the following characterization of concealed algebras.

**Theorem 4.** *Let  $A$  be a basic, connected, artin algebra. Then the following statements are equivalent.*

- (i)  $A$  is a concealed algebra.
- (ii)  $\Gamma_A$  admits exactly two different faithful generalized standard components without oriented cycles: a preprojective component and a preinjective component.
- (iii)  $\Gamma_A$  admits at least two different faithful generalized standard components without oriented cycles.
- (iv)  $\Gamma_A$  admits two different generalized standard components  $\mathcal{C}$  and  $\mathcal{D}$  without oriented cycles and such that  $\mathcal{C}$  is faithful and  $\mathcal{D}$  is sincere.
- (v)  $\Gamma_A$  admit two different components with sections satisfying one of the conditions imposed on  $\mathcal{C}$  in the statements (iii)-(v) of Theorem 3.

**Proof.** The implications (ii) $\Rightarrow$ (iii) and (iii) $\Rightarrow$ (iv) are obvious. Moreover, the implication (v) $\Rightarrow$ (iii) is a direct consequence of Theorem 3. Assume now that  $A$  is a concealed algebra. Then, by [10], [11],  $\Gamma_A$  consists of a preprojective component  $\mathcal{P}$  containing all projective modules, a preinjective component  $\mathcal{I}$  containing all injective modules, and regular components which are either tubes (if  $A$  is tame) or of the form  $ZA_\infty$  (if  $A$  is wild). Then  $\mathcal{P}$  and  $\mathcal{I}$  are faithful, without oriented cycles, and generalized standard (see Theorem 1). Clearly, the tubes contain oriented cycles. The components of the form  $ZA_\infty$  have infinitely many nonperiodic  $\tau_A$ -orbits, so they are not generalized standard, by [14, (2.3)]. Therefore,  $\mathcal{P}$  and  $\mathcal{I}$  are unique generalized standard faithful components of  $\Gamma_A$  without oriented cycles. Hence (i) implies (ii). Observe also that  $\mathcal{P}$  and  $\mathcal{I}$  contain sections, because they contain slices, and so (i) implies also (v). Assume now that  $\Gamma_A$  admits a faithful generalized standard component  $\mathcal{C}$  without oriented cycles and a sincere generalized standard component  $\mathcal{D}$  without oriented cycles. Then  $\text{ann } \mathcal{C}=0$  and  $\mathcal{C}$  satisfies the conditions (a)-(f) of Theorem 1. We use the notations of Theorem 1. Then  $B=A$  and  $\mathcal{D}$  is contained either in  $\mathcal{Y}(\infty T)$  or in  $\mathcal{X}(T_\infty)$ . By duality, we may assume that  $\mathcal{D}$  is contained in  $\mathcal{Y}(\infty T)$ . We know that  $\mathcal{D}$ , as a generalized standard component of  $\Gamma_A$ , has at most finitely many nonperiodic  $\tau_A$ -orbits. Then, by the known description of components of tilted algebras (see [5], [7]), we deduce that  $\mathcal{D}$  is a preprojective component of  $\Gamma_{\infty B}$ , and hence of  $\Gamma_A$ . Moreover,  $\infty B$  is connected because  $\mathcal{D}$  is sincere. Observe also that  $\mathcal{D}$  has no injective modules because  $\mathcal{C}$  is sincere and  $\mathcal{D}$ , as a preprojective component of  $\Gamma_A$ , is closed under predecessors in  $\text{mod } A$ . Clearly, then  $\mathcal{D}$  is a faithful component of  $\Gamma_A$ . Then, applying the dual arguments, we infer that  $\mathcal{C}$  is a preinjective component of  $\Gamma_A$  without projective modules. Then, by Lemma 4,  $A$  is a concealed algebra. This finishes the proof.

**REMARK.** In [14] we presented an example of an algebra  $\Lambda$  of infinite glo-

bal dimension such that  $\Gamma_A$  admits three sincere generalized standard components without oriented cycles. Hence, in the condition (iv) of Theorem 4, we cannot replace the assumption  $\mathcal{C}$  is faithful by the weaker one  $\mathcal{C}$  is sincere.

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