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GENERALIZED STANDARD AUSLANDER-REITEN COMPONENTS WITHOUT ORIENTED CYCLES

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Introduction

Let A be an artin algebra, mod A the category of finitely generated right A-modules, $\text{rad}^\infty(\text{mod } A)$ the infinite radical of mod A, and Γ_A the Auslander-Reiten quiver of A. It is known that Γ_A describes the quotient cagegory $\text{mod } A/\text{rad}^\infty(\text{mod } A)$. We are intersted in the behaviour of connected components of Γ_A in the module category mod A. We introduced in [14] the concept of a generalized standard component and proved some facts on such components. A component $\mathcal{C}$ of Γ_A is called *generalized standard* if $\text{rad}^\infty(X, Y)=0$ for all X and Y from $\mathcal{C}$. Examples of generalized standard components are all preprojective components, preinjective components, and connecting componets of tilted algebras. We proved in [14] that a generalized standard connected component of Γ_A admits at most finitely many nonperiodic $D\text{Tr}$-orbits. Moreover, we described regular and semi-regular generalized standard components of Γ_A containing no oriented cycle, and proved that all but a finite number of generalized standard components of Γ_A are stable tubes.

The main aim of this paper is to describe arbitrary generalized standard components without oriented cycles. As an application we obtain new characterizations of tilted algebras and concealed algebras.

The paper is organized as follows. In Section 1 we recall some notions and facts from the representation theory of artin algebras needed in the paper. Section 2 contains a description of generalized standard components without oriented cycles. In Section 3 we characterize generalized standard components containing sections, and prove some characterizations of tilted algebras. Section 4 contains some new characterizations of concealed algebras.

1. Preliminaries

Let A be an artin algebra over a commutative artin ring R. We denote by

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mod A the category of all finitely generated right A-modules, and by D: mod $A \to$ mod A^{op} the standard duality $\mathrm{Hom}_R(-, I)$, where I is the injective envelope of $R/\mathrm{rad}\, R$ in mod R. By a module we usually mean a finitely generate right module. We denote by rad(mod A) the radical of mod A and by rad^{∞}(mod A) the intersection of all powers $\mathrm{rad}^i(\mathrm{mod}\, A)$, $i \geqq 1$, of rad(mod A). A *path* in mod A is a sequence of non-zero non-isomorphisms $M_1 \to M_2 \to \cdots \to M_n$, where the modules M_i are indecomposable. A full subcategory $\mathcal{Z}$ of mod A is said to be *closed under predecessors* (resp. *closed under successors*) if any path in mod A with the target (resp. source) in $\mathcal{Z}$ consists entirely of modules in $\mathcal{Z}$. We denote by Γ_A the Auslander-Reiten quiver of A, and let τ_A, τ_A^{-} be the Auslander-Reiten operators $D\mathrm{Tr}$, $\mathrm{Tr}D$, respectively. We shall not distinguish between an indecomposable A-module, its isomorphism class and the vertex of Γ_A corresponding to it. By a component of Γ_A we mean a connected component of Γ_A. A component $\mathcal{C}$ of Γ_A is called *preprojective* (resp. *preinjective*) if $\mathcal{C}$ has no oriented cycle and each module in $\mathcal{C}$ belongs to the τ_A-orbit of a projective (resp. injective) module. A component $\mathcal{C}$ of Γ_A is called *sincere* if any simple A-module occurs as a simple composition factor of a module in $\mathcal{C}$. For a component $\mathcal{C}$ of Γ_A, we denote by ann $\mathcal{C}$ the annihilator of $\mathcal{C}$ in A, that is, the intersection of the annihilators ann X of all modules X from $\mathcal{C}$. If ann $\mathcal{C}=0$, the component $\mathcal{C}$ is called *faithful*. Clearly, a faithful component is sincere.

Let H be a hereditary artin algebra, T a tilting H-module and $B=\mathrm{End}_H(T)$ the associated tilted algebra. Then T determines a torsion theory $(\mathcal{F}(T), \mathcal{G}(T))$ in mod H, where $\mathcal{F}(T)=\{X_H \mid \mathrm{Hom}_H(T, X)=0\}$ and $\mathcal{G}(T)=\{Y_H \mid \mathrm{Ext}_H^1(T, Y)=0\}$, and a splitting torsion theory $(\mathcal{Y}(T), \mathcal{X}(T))$ in mod B, where $\mathcal{Y}(T)=\{N_B \mid \mathrm{Tor}_1^B(N, T)=0\}$ and $\mathcal{X}(T)=\{M_B \mid M \otimes_B T=0\}$. Then by the theorem of Brenner and Butler the functor $F=\mathrm{Hom}_H(T, -)$ induces an equivalence between $\mathcal{G}(T)$ and $\mathcal{Y}(T)$, and the functor $F'=\mathrm{Ext}_H^1(T, -)$ an equivalence between $\mathcal{F}(T)$ and $\mathcal{X}(T)$. Then the injective cogenerator DH of mod H belongs to $\mathcal{G}(T)$ and the indecomposable direct summands of $F(DH)$ form a set $\mathcal{S}$ belonging to one component of Γ_B, called the connecting component of Γ_B corresponding to T. Moreover $\mathcal{S}$ is a slice of mod B (see [11, (4.2)]), that is, $\mathcal{S}$ satisfies the following conditions:

(α) $\mathcal{S}$ is sincere.

(β) $\mathcal{S}$ is path closed (any path in mod A with source and target in $\mathcal{S}$ consists entirely of modules in $\mathcal{S}$.

(γ) If M is an indecomposable nonprojective A-module, then at most one of M, $\tau_A M$ belongs to $\mathcal{S}$.

(δ) If $M \to S$ is an irreducible map with M and S indecomposable and S in $\mathcal{S}$, then either M belongs to $\mathcal{S}$ or M is noninjective and $\tau_A^{-}M$ belongs to $\mathcal{S}$.

Observe that the condition (β) is very difficult for checking.

We shall need the following lemma proved in [13].

Lemma 1. *Let A be an artin algebra and n be the number of isoclasses of simple A-modules. Let $X_1, \cdots, X_r$ be pairwise nonisomorphic indecomposable A-modules such that $Hom_A(X_i, \tau_A X_j)=0$ for all $1 \leqq i, j \leqq r$. Then $r \leqq n$.*

The following simple lemma will be also useful.

Lemma 2. *Let A be an artin algebra, $\mathcal{C}$ a component of Γ_A and $B = A/ann\,\mathcal{C}$. Then $\mathcal{C}$ is a generalized standard component of Γ_A if and only if $\mathcal{C}$ is a generalized standard component of Γ_B.*

Proof. Clearly, $\mathcal{C}$ is a full component of Γ_B. From the existence of Auslander-Reiten sequences in mod A we know that rad(mod A) is generated by the irreducible maps as a left and as a right ideal. Let X and Y be two indecomposable modules from $\mathcal{C}$, and suppose that $f: X \to Y$ is a nonzero map from $\mathrm{rad}^\infty(\mathrm{mod}\,A)$. Then there are modules X_i and maps $g_i: X_i \to X_{i+1}$, $h_i: X_{i+1} \to Y$ in $\mathrm{rad}(\mathrm{mod}\,B)$, $i \geqq 0$, such that $X_0 = X$ and, for each i, X_i is a direct sum of indecomposable modules from $\mathcal{C}$ and $f = h_i g_i \cdots g_0$. Then f belongs to $\mathrm{rad}^\infty(\mathrm{mod}\,B)$. This proves the lemma because clearly $\mathrm{rad}^\infty(\mathrm{mod}\,B)$ is contained in $\mathrm{rad}^\infty(\mathrm{mod}\,A)$.

2. Generalized standard components without oriented cycles

Let A be an artin algebra and $\mathcal{C}$ be a component of Γ_A without oriented cycles. We are interested in criteria for $\mathcal{C}$ to be generalized standard. We shall first define a full translation subquiver ${}_\infty\mathcal{C}$ of $\mathcal{C}$ closed under predecessors, called the left end of $\mathcal{C}$, and a full translation subquiver $\mathcal{C}_\infty$ of $\mathcal{C}$ closed under successors, called the right end of $\mathcal{C}$.

Denote by ${}_l\mathcal{C}$ the left stable part of $\mathcal{C}$, obtained from $\mathcal{C}$ by removing the τ_A-orbits of projective modules. Then ${}_l\mathcal{C}$ is a disjoint union of finitely many left stable full translation connected subquivers $\mathcal{D}_1, \cdots, \mathcal{D}_s$ of $\mathcal{C}$. From [6, (3.4)], for each $1 \leqq i \leqq s$, there exists a valued quiver Δ_i without oriented cycles such that $\mathcal{D}_i$ is isomorphic to a full translation subquiver of $Z\Delta_i$ which is closed under predecessors. Let Σ_i be a fixed copy of Δ_i in $\mathcal{D}_i$ such that the modules forming the vertices of Σ_i are neither successors of indecomposable direct summands of the radicals of projective modules in $\mathcal{C}$ nor suscessors of injective modules in $\mathcal{C}$. Let ${}_\infty\Sigma$ be the disjoint union of the quivers $\Sigma_1, \cdots, \Sigma_s$. Then denote by ${}_\infty\mathcal{C}$ the full translation subquiver of $\mathcal{C}$ formed by all predecessors in $\mathcal{C}$ of modules from ${}_\infty\Sigma$. Observe that ${}_\infty\mathcal{C}$ is a left stable full translation subquiver of $\mathcal{C}$ which is closed under predecessors. Denote by ${}_\infty N$ the direct sum of all modules from ${}_\infty\Sigma$, and put ${}_\infty M = {}_\infty N \oplus P$, where P is the direct sum of all projective modules from $\mathcal{C}$. If ${}_l\mathcal{C}$ is empty, put ${}_\infty N = 0$. Dually, denote by $\mathcal{C}_r$ the right stable part of $\mathcal{C}$, obtained from $\mathcal{C}$ by removing the τ_A-orbits of injective modules. Assume that $\mathcal{C}_r$ is nonempty. Then $\mathcal{C}_r$ is a disjoint union of finitely many full translation connected subquivers $\mathcal{D}'_1, \cdots, \mathcal{D}'_m$ of $\mathcal{C}$. Again, from [6], for each $1 \leqq j \leqq m$, there is a valued quiver Δ'_j without oriented cycles such that $\mathcal{D}'_j$ is isomorphic to a full traslation subquiver of $Z\Delta'_j$ which is closed under successors. Let Σ'_j be a fixed copy of Δ'_j in $\mathcal{D}'_j$ such that the modules forming the vertices of Σ'_j are neither predecessors of indecomposable direct summands of the socle factors of injective modules in $\mathcal{C}$ nor predecessors of projective modules in $\mathcal{C}$. Let Σ_∞ be the disjoint union of the quivers $\Sigma'_1, \cdots, \Sigma'_m$. Then denote by $\mathcal{C}_\infty$ the full translation subquiver of $\mathcal{C}$ formed by all successors in $\mathcal{C}$ of modules from $\tau_A^- \Sigma_\infty$. Observe that $\mathcal{C}_\infty$ is a right stable full translation subquiver of $\mathcal{C}$ which is closed under successors. Denote by N_∞ the direct sum of all modules from Σ_∞, and put $M_\infty = Q \oplus N_\infty$, where Q is the direct sum of all injective modules from $\mathcal{C}$. If $\mathcal{C}_r$ is empty, we put $N_\infty = 0$. We may assume that ${}_\infty\mathcal{C}$ and $\mathcal{C}_\infty$ have no common modules.

The following theorem gives a characterization of generalized standard components without oriented cycles.

Theorem 1. *Let A be an artin algebra, $\mathcal{C}$ be a component of Γ_A without oriented cycles and $B = A/\text{ann}\, \mathcal{C}$. Then, in the above notation, the following conditions are equivalent.*

(i) *$\mathcal{C}$ is a generalized standard component of Γ_A.*

(ii) $\text{Hom}_A(P, {}_\infty N) = 0$ *and* $\text{Hom}_A(X, \tau_A Y) = 0$ *for all modules X and Y from ${}_\infty\Sigma$.*

(iii) $\text{Hom}_A(N_\infty, Q) = 0$ *and* $\text{Hom}_A(\tau_A^- X, Y) = 0$ *for all modules X and Y from Σ_∞.*

(iv) *${}_\infty\Sigma$ is finite and* $\text{rad}^\infty({}_\infty M, {}_\infty N) = 0$.

(v) *Σ_∞ is finite and* $\text{rad}^\infty(N_\infty, M_\infty) = 0$.

(vi) *${}_\infty\Sigma$ or Σ_∞ is finite and* $\text{Hom}_A(N_\infty, {}_\infty N) = 0$.

(vii) *The following conditions hold:*

(a) *There is a hereditary aritn algebra ${}_\infty H$ and a tilting ${}_\infty H$-mdoule ${}_\infty T$ without preinjective direct summands such that the tilted algebra ${}_\infty B = \text{End}_{{}_\infty H}({}_\infty T)$ is a factor algebra of B and the torsion-free part $\mathcal{Y}({}_\infty T)$ of mod ${}_\infty B$ is a full exact subcategory of* mod B *which is closed under predecessors.*

(b) *There is a hereditary artin algebra ${}_\infty H$ and a tilting ${}_\infty H$-module ${}_\infty T$ without preprojective direct summands such that the tilted algebra $B_\infty = \text{End}_{H_\infty}(T_\infty)$ is a factor algebra of B and the torsion part $\mathcal{X}(T_\infty)$ of mod B_∞ is a full exact subcategory of* mod B *which is closed under successors.*

(c) *$\mathcal{Y}({}_\infty T)$ and $\mathcal{X}(T_\infty)$ have no common nonzero modules.*

(d) *${}_\infty\mathcal{C}$ is the torsion-free part of the connecting component of $\Gamma_{{}_\infty B}$ corresponding to ${}_\infty T$.*

(e) *$\mathcal{C}_\infty$ is the torsion part of the connecting component of Γ_{B_∞} corresponding to T_∞.*

(f) *The class of indecomposable B-modlues which are neither in $\mathcal{Y}({}_\infty T)$ nor in $\mathcal{X}(T_\infty)$ is finite and coincides (up to isomorphism) with the class of modules in $\mathcal{C}$ which are neither in ${}_\infty\mathcal{C}$ nor in $\mathcal{C}_\infty$.*

Proof. Without loss of generality we may assume that A is basic and connected. The implications (i)$\Rightarrow$(ii) and (i)$\Rightarrow$(iii) follow directly from our choice of ${}_\infty\Sigma$ and Σ_∞. We shall show now that (ii) implies (iv), and (iii) implies (v). First observe that, by Lemma 1, if $\mathrm{Hom}_A(X, \tau_A Y)=0$ (resp. $\mathrm{Hom}_A(\tau_A^{-}X, Y)=0$) for all modules X and Y from ${}_\infty\Sigma$ (resp. from Σ_∞), then ${}_\infty\Sigma$ (resp. Σ_∞) is finite. Moreover, we have $\mathrm{Hom}_A({}_\infty M, {}_\infty N)=\mathrm{Hom}_A(P, {}_\infty N)\oplus\mathrm{Hom}_A({}_\infty N, {}_\infty N)$ and $\mathrm{Hom}_A(N_\infty, M_\infty)=\mathrm{Hom}_A(N_\infty, N_\infty)\oplus\mathrm{Hom}_A(N_\infty, Q)$. If ${}_\infty\Sigma$ is finite, then any map from $\mathrm{rad}^\infty({}_\infty N, {}_\infty N)$ factors through a module $\tau_A({}_\infty N)^a$, for some a$\geqq 1$. Similarly, if Σ_∞ is finite, then any map from $\mathrm{rad}^\infty(N_\infty, N_\infty)$ factors through $\tau_A^{-}(N_\infty)^b$, for some $b\geqq 1$. Therefore, (ii) implies (iv), and (iii) implies (v). We claim now that each of the conditions (iv) and (v) implies (vi). First observe that ${}_\infty\Sigma$ is finite if and only if Σ_∞ is finite,. Assume that ${}_\infty\Sigma$ and Σ_∞ are finite, and suppose that $\mathrm{Hom}_A(N_\infty, {}_\infty N)\neq 0$. Then clearly $\mathrm{rad}^\infty(N_\infty, {}_\infty N)\neq 0$. Since any epimorphism $A^c\to N_\infty$ factors through a module ${}_\infty M^d$, for some $d\geqq 1$, there is an epimorphism ${}_\infty M^d\to N_\infty$. Then $\mathrm{rad}^\infty({}_\infty M, {}_\infty N)\neq 0$ because $\mathrm{rad}^\infty({}_\infty N, {}_\infty N)\neq 0$. Similarly, any monomorphism ${}_\infty N\to(DA)^p$ factors through a module M_∞^q, for some $q\geqq 1$, and hence there is a monomorphism ${}_\infty N\to M_\infty^q$. Then $\mathrm{rad}^\infty(N_\infty, M_\infty)\neq 0$ because $\mathrm{rad}^\infty(N_\infty, {}_\infty N)\neq 0$. This proves our claims. We shall show now that (vi) implies (i). Suppose that $\mathrm{rad}^\infty(U, V)\neq 0$ for some indecomposable modules U and V from $\mathcal{C}$. Then there is an infinite path in $\mathcal{C}$

$$U=U_0\to U_1\to\cdots\to U_i\to U_{i+1}\to\cdots$$

such that $\mathrm{rad}^\infty(U_i, V)\neq 0$ for all $i\geqq 0$. Since from (vi) Σ_∞ and ${}_\infty\Sigma$ are finite, there is $m\geqq 0$ such that U_m is a successor of Σ_∞ in $\mathcal{C}$. Further, $\mathrm{rad}^\infty(U_m, V)\neq 0$ implies existence of an infinite path in $\mathcal{C}$

$$\cdots\to V_{j+1}\to V_j\to\cdots\to V_1\to V_0=V$$

such that $\mathrm{rad}^\infty(U_m, V_j)\neq 0$ for all $j\geqq 0$. Again, since ${}_\infty\Sigma$ and Σ_∞ are finite, there is $n\geqq 0$ such that V_n is a predecessor of ${}_\infty\Sigma$ in $\mathcal{C}$. Then any epimorphism $A^k\to U_m$ factors through a module N_∞^r, for some $r\geqq 1$, and hence there is an epimorphism $N_\infty^r\to U_m$. Similarly, any monomorphism $V_n\to(DA)^s$ factors through a module ${}_\infty N^t$, for some $t\geqq 1$, and hence we have a monomorphism $V_n\to{}_\infty N^t$. Then $\mathrm{rad}^\infty(U_m, V_n)\neq 0$ implies that $\mathrm{rad}^\infty(N_\infty, {}_\infty N)\neq 0$, a contradiction to (vi). Hence (vi) implies (i). Consequently, we proved that the conditions (i)-(vi) are equivalent. We shall show now that (vii) implies (vi). Assume that (vii) holds.

Then clearly ${}_{\infty}\Sigma$ and Σ_{∞} are finite. Moreover, $\mathrm{Hom}_A(N_{\infty}, {}_{\infty}N)=\mathrm{Hom}_B(N_{\infty}, {}_{\infty}N)=0$, because $\mathcal{Y}({}_{\infty}T)$ and $\mathcal{X}(T_{\infty})$ have no common indecomposable modules, $\mathcal{Y}({}_{\infty}T)$ is closed under predecessors in mod B, and $\mathcal{X}(T_{\infty})$ is closed under successors in mod B. Hence (vi) holds. Finally, assume that $\mathcal{C}$ is a generalized standard component of Γ_A. Then, from Lemma 2, $\mathcal{C}$ is also a generalized standard component of Γ_B. We shall show that the conditions (a)-(f) of (vii) hold. First observe that $\mathcal{C}$, as a generalized standard component, has by [14, (2.3)] only finitely many τ_B-orbits, and hence ${}_{\infty}\Sigma$ and Σ_{∞} are finite. Write $B=P'\oplus P$ as a B-module. Assume that $P'\neq 0$, and put ${}_{\infty}B=\mathrm{End}_B(P')$. We claim that ${}_{\infty}N$ is a faithful tilting ${}_{\infty}B$-module. For simplicity of notations we put $N={}_{\infty}N$ and $F={}_{\infty}B$. First observe that $\mathrm{Hom}_B(P, N)=\mathrm{rad}^{\infty}(P, N)=0$ since (i) is equivalent to (ii), and hence N is a F-module. Further, since $\mathcal{C}$ is a faithful component of Γ_B and B is an artin algebra, there are indecomposable modules $Z_1, \cdots, Z_m$ in $\mathcal{C}$ such that $Z=Z_1\oplus\cdots\oplus Z_m$ is a faithful B-module. We claim that there are indecomposable B-modules $W_1, \cdots, W_n$ in $\mathcal{C}$ which are not proper predecessors of ${}_{\infty}\Sigma$, and such that $W=W_1\oplus\cdots\oplus W_n$ is a faithful B-module. Suppose that some Z_i is a proper predecessor of ${}_{\infty}\Sigma$. We may assume that $Z_1, \cdots, Z_s$, $s\leqq m$, are all proper predecessors of ${}_{\infty}\Sigma$ in the family $Z_1, \cdots, Z_m$. Let $Z'=Z_1\oplus\cdots\oplus Z_s$ and $Z''=Z_{s+1}\oplus\cdots\oplus Z_m$. Since Z is a faithful B-module, there is an epimorphism $f\colon Z^k\to DB$, for some $k\geqq 1$. Then, since ${}_{\infty}\Sigma$ has no injective predecessors, the restriction of f to $(Z')^k$ factors through a module N^j, for some $j\geqq 1$. Hence there is an epimorphism $W^a\to DB$, for some $a\geqq 1$, and $W=N\oplus Z''$, and so W is a required faithful B-module. In particular, there is a monomorphism $g\colon B\to W^r$, for some $r\geqq 1$. Since $F=P'$ as a right F-module, restriction of g to F gives a monomorphism $h\colon F\to W^r$. But P' has no indecomposable direct summands in $\mathcal{C}$, and so h factors through a module N^t, for some $t\geqq 1$. Hence there is a monomorphism $e\colon F\to N^t$, and N is a faithful F-module. Further, since $\mathcal{C}$ is a generalized standard component of Γ_B, we have $\mathrm{Hom}_B(P, U)=0$ for all modules U in ${}_{\infty}\mathcal{C}$. Consequently, ${}_{\infty}\mathcal{C}$ consists entirely of F-modules. Moreover, by our choice of ${}_{\infty}\Sigma$, for any module X from ${}_{\infty}\Sigma$, the module $\tau_B^{-}X$ is also a F-module. Hence $\tau_B N=\tau_F N$ and $\tau_B^{-}N=\tau_F^{-}N$. Then, since $\mathcal{C}$ is generalized standard, we get $\mathrm{Hom}_F(N, \tau_F N)=0$, $\mathrm{Hom}_F(\tau_F^{-}N, N)=0$, and $\mathrm{Ext}_F^1(N, N)\simeq D\overline{\mathrm{Hom}}_F(N, \tau_F N)=0$. Also, if $\mathrm{Hom}_F(N, V)\neq 0$ for an indecomposable F-module V which is not a direct summand of N, then $\mathrm{Hom}_F(\tau_F^{-}N, V)\neq 0$, because ${}_{\infty}\Sigma$ is finite. Then, by Lemmas 1.6, 1.5 and its dual, in [9], N is a tilting F-module. Moreover, ${}_{\infty}H=\mathrm{End}_F(N)$ is a hereditary algebra, since $\mathcal{C}$ is generalized standard. Therefore, there exists a tilting ${}_{\infty}H$-module ${}_{\infty}T$ such that ${}_{\infty}B=F=\mathrm{End}_{{}_{\infty}H}({}_{\infty}T)$ and ${}_{\infty}\mathcal{C}$ is the torsion-free part of the connecting component of $\Gamma_{{}_{\infty}B}$ corresponding to ${}_{\infty}T$. Further, since ${}_{\infty}\mathcal{C}$ has no projective modules, ${}_{\infty}T$ has no preinjective direct summands. Observe also that $\mathrm{Hom}_B(P, P')=0$, since $\mathrm{Hom}_B(P, N)=0$ and P' is a submodule of N^t. This implies that B is isomorphic to $\begin{bmatrix} C & E \\ 0 & F \end{bmatrix}$, where $C=\mathrm{End}_B(P)$, $F=\mathrm{End}_B(P')$ and $E=\mathrm{Hom}_B(P', P)$. In particular, ${}_\infty B=F$ is a factor algebra of B. We shall prove now that the torsion-free part $\mathcal{Y}({}_\infty T)$ of mod F is closed under predecessors in mod B. We know that $\mathcal{Y}({}_\infty T)$ is closed under predecessors in mod F. First observe that E_F is a direct sum of indecomposable B-modules which are in $\mathcal{C}$ but not in ${}_\infty\mathcal{C}$. Indeed, E_F is the largest F-submodule of P and any epimorphism $F^n \to E_F$ factors through a module N^k, for some $k \geqq 1$. Moreover, by our choice of ${}_\infty\Sigma$, rad P and N have no common indecomposable direct summands. In order to prove that $\mathcal{Y}({}_\infty T)$ is closed under predecessors in mod B, it is sufficinent to show that there are no nonzero maps from indecomposable B-modules which are not F-modules to indecomposable F-modules in $\mathcal{Y}({}_\infty T)$. Each B-module can be viewed as a triple (U_C, V_F, ϕ), where ϕ: $U \otimes_C E \to V$ is a F-homomorphism. Let $(0, h)$ be a nonzero map from an indecomposable B-module (U_C, V_F, ϕ) which is not in mod F to an indecomposable F-module $W=(0, W_F, 0)$. Then h: $V \to W$ is nonzero. Let L be an indecomposable direct summand of V such that $\mathrm{Hom}_F(L, W) \neq 0$, and let p: $V \to L$ be the canonical projection. Since (U_C, V_F, ϕ) is indecomposable and not in mod F, the composition $p\phi$ is nonzero. Hence $\mathrm{Hom}_F(U \otimes_C E, L) \neq 0$. Consider now an epimorphism $C^m \to U$ in mod C. Then we get an epimorphism $E^m \cong C^m \otimes_C E \to U \otimes_C E$, and consequently we have $\mathrm{Hom}_F(E, L) \neq 0$. This implies that L belongs to $\mathcal{X}({}_\infty T)$, and hence W belongs to $\mathcal{X}({}_\infty T)$, because $\mathrm{Hom}_F(L, W) \neq 0$. Therefore, $\mathcal{Y}({}_\infty T)$ is closed under predecessors in mod B. We proved that ${}_\infty B$, ${}_\infty H$, and ${}_\infty T$ satisfy the required conditions (a) and (d). Dually, we define B_∞, H_∞, and T_∞ such that the conditions (b) and (e) are satisfied. Moreover, by our choice of ${}_\infty\mathcal{C}$ and $\mathcal{C}_\infty$, the condition (c) also holds. Finally, all but a finite number of modules in $\mathcal{C}$ beling to the union of ${}_\infty\mathcal{C}$ and $\mathcal{C}_\infty$, because $\mathcal{C}$ has only finitely many τ_B-orbits. Let now X be an indecomposable B-module. Suppose that X is neither in $\mathcal{Y}({}_\infty T)$ nor in $\mathcal{C}$. Observe that, if X is a ${}_\infty B$-module, then $\mathrm{Hom}_B({}_\infty N, X) \neq 0$, because any epimorphism ${}_\infty B^p \to X$ factors through ${}_\infty N^q$, for some $q \geqq 1$. If X is not a ${}_\infty B$-module, then $\mathrm{Hom}_B(P, X) \neq 0$. Therefore, since X does not belong to $\mathcal{C}$, we have either $\mathrm{rad}^\infty({}_\infty N, X) \neq 0$ or $\mathrm{rad}^\infty(P, X) \neq 0$. Hence there is Y in $\mathcal{C}_\infty$ such that $\mathrm{Hom}_B(Y, X) \neq 0$. But then X belongs to $\mathcal{X}(T_\infty)$, because Y belongs to $\mathcal{X}({}_\infty T)$ and $\mathcal{X}(T_\infty)$ is closed under successors in mod B. This proves that the condition (f) also holds. We proved that (i) implies (vii), and this finishes our proof.

3. Generalized standard components with sections

Components with sections form a special class of components without oriented cycles. The aim of this section is to give simple characterizations of generalized standard components with sections. As an application we obtian some new simple characterizations of tilted algebras. We would like to inform that similar results to those presented in this section were also proved independently by Shiping Liu (a private communication).

Let A be an artin algebra and $\mathcal{C}$ be a component of Γ_A. A full connected subquiver Σ of $\mathcal{C}$ is called a *section* if it satisfies the following conditions:

(1) Σ has no oriented cycle.

(2) Σ intersects each τ_A-orbit of $\mathcal{C}$ exactly once.

(3) Each path in $\mathcal{C}$ with source and target from Σ lies entirely in Σ.

(4) If $X\to Y$ is an arrow in $\mathcal{C}$ with X from Σ (resp. Y from Σ), then either Y or $\tau_A Y$ (resp. X or $\tau_A^{-}X$) is in Σ.

It is easy to see that if $\mathcal{C}$ contains a section Σ, then $\mathcal{C}$ has no oriented cycle (see [3, (8.1)]). Moreover, if $\mathcal{C}$ admits a slice (in the sense of Section 1), then $\mathcal{C}$ admits a section.

We shall need the following simple lemma.

Lemma 3. *Let A be an artin algebra and $\mathcal{C}$ be a component of Γ_A having a finite section Σ. Let M be the direct sum of all modules forming the vertices of Σ. Then* $\operatorname{ann}\mathcal{C}=\operatorname{ann}M$.

Proof. Clearly $\operatorname{ann}\mathcal{C}$ is contained in $\operatorname{ann}M$. Let $B=A/\operatorname{ann}\mathcal{C}$. Then $\mathcal{C}$ is a faithful component of Γ_B. It is sufficient to show that M is a faithful B-module. Since B is an artin algebra, there are indecomposable modules $Z_1,\cdots,Z_r$ in $\mathcal{C}$ such that $Z=Z_1\oplus\cdots\oplus Z_r$ is a faithful B-module. Let $Z=U\oplus V$, where U is a direct sum of predecessors of Σ in $\mathcal{C}$, and V has no such direct summands. Suppose that $V\neq 0$. Since Z is a faithful B-module, there is a monomorphism $f\colon B\to Z^t=U^t\oplus V^t$, for some $t\geqq 1$. Moreover, since Σ is finite, we have then $f=gh$, where $h\colon B\to U^t\oplus M^s$ $g\colon U^t\oplus M^s\to U^t\oplus V^t$, for some $s\geqq 1$. Then h is a monomorphism, and hence $L=U\oplus M$ is a faithful B-module. Let $L=E\oplus F$, where F is a direct sum of modules from Σ and E has no direct summands from Σ. Suppose that $E\neq 0$. Then, since L is a faithful B-module, there exists an epimorphism $p\colon L^m\to DB$, for some $m\geqq 1$. Hence, since Σ is finite, we have $p=qe$, where $e\colon L^m\to M^k\oplus F^m$, $q\colon M^k\oplus F^m\to DB$, for some $k\geqq 1$. Observe that q is an epimorphism, and so $M\oplus F$ is a faithful B-module. But $M\oplus F$ is a direct sum of modules from Σ. Therefore, M is a faithful B-module.

Theorem 2. *Let A be an artin algebra and $\mathcal{C}$ be a component of Γ_A containing a section Σ. Denote by M the direct sum of all modules from Σ. Then the following conditions are equivalent.*

(i) *$\mathcal{C}$ is a generalized standard component of Γ_A.*

(ii) $\operatorname{Hom}_A(X,\tau_A Y)=0$ *for any modules X and Y from Σ.*

(iii) $\operatorname{Hom}_A(\tau_A^{-}X, Y)=0$ *for any modules X and Y from Σ.*

(iv) *Σ is finite and* $\operatorname{rad}^\infty(M,M)=0$.

Proof. The implications (i)$\Rightarrow$(ii), and (i)$\Rightarrow$(iii) are clear. Observe that, if

(ii) or (iii) holds, then, by Lemma 1, Σ is finite. Suppose that Σ is finite and $\operatorname{rad}^{\infty}(M, M) \neq 0$. Then any nonzero map from $\operatorname{rad}^{\infty}(M, M)$ factors through a module $(\tau_A M)^r$, for some $r \geqq 1$, and also factors through a module $(\tau_A^{-} M)^s$ for some $s \geqq 1$. Hence (ii) implies (iv), and (iii) implies (iv). Finally, assume that (iv) holds. We claim that $\mathcal{C}$ is generalized standard. Suppose that $\operatorname{rad}^{\infty}(U, V) \neq 0$ for some indecomposable modules U and V from $\mathcal{C}$. Then there is an infinite path

$$U = U_0 \to U_1 \to \cdots \to U_i \to U_{i+1} \to \cdots$$

in $\mathcal{C}$ such that $\operatorname{rad}^{\infty}(U_i, V) \neq 0$ for all $i \geqq 0$. Since Σ is finite, there is $m \geqq 0$ such that U_m is a successor of Σ in $\mathcal{C}$. Then $\operatorname{rad}^{\infty}(U_m, V) \neq 0$ implies existence of an infinite path

$$\cdots \to V_{i+1} \to V_j \to \cdots \to V_1 \to V_0 = V$$

in $\mathcal{C}$ such that $\operatorname{rad}^{\infty}(U_m, V_j) \neq 0$ for all $j \geqq 0$. Again, since Σ is finite, there is $r \geqq 0$ such that V_r is a predecessor of Σ in $\mathcal{C}$. Let $f: A^n \to U_m$ be an epimorphism. Then $f = gh$, where $h: A^{n} \to M^t$ and $g: M^t \to U_m$, for some $t \leqq 1$, because U_m is a successor of the finite section Σ. Clearly, then g is an epimorphism. Similarly, let $\phi: V_r \to (DA)^s$ be a monomorphism. Then $\phi = \beta\alpha$, where $\alpha: V_r \to M^k$, $\beta: M^k \to (DA)^s$, for some $k \geqq 1$, because V_r is a predecessor of the finite section Σ. Clearly, α is a monomorphism. Now, if γ is a nonzero map from $\operatorname{rad}^{\infty}(U_m, V_r)$, then $\alpha\gamma g: M^t \to M^k$ is nonzero and belongs to $\operatorname{rad}^{\infty}(\operatorname{mod} A)$. Hence, $\operatorname{rad}^{\infty}(M, M) \neq 0$, a contradiction to (iv). Therefore, (iv) implies (i).

We may prove now the following characterization of tilted algebras.

Theorem 3. *Let A be an artin algebra. Then the following conditions are equivalent.*

(i) *A is a tilted algebra.*

(ii) *Γ_A admits a faithful generalized standard component $\mathcal{C}$ containing a section.*

(iii) *Γ_A admits a component $\mathcal{C}$ having a faithful section Σ such that* $\operatorname{Hom}_A(X, \tau_A Y) = 0$ *for all modules X and Y from Σ.*

(iv) *Γ_A admits a component $\mathcal{C}$ having a faithful section Σ such that* $\operatorname{Hom}_A(\tau_A^{-} X, Y) = 0$ *for all modules X and Y from Σ.*

(v) *Γ_A admist a component $\mathcal{C}$ having a faithful finite section Σ such that* $\operatorname{rad}^{\infty}(M, M) = 0$, *where M is the direct sum of all modules from Σ.*

Proof. The equivalence of the conditions (ii)-(v) is a direct consequence of Theorem 2 and Lemma 3. Assume now tht that $A = \operatorname{End}_H(T)$ for some hereditary artin algebra H and a tilting H-module T. Denote by $\mathcal{C}$ the connecting component of Γ_A corresponding to T (see Section 1). It is well known (see

[11, (4.2)]) that the family $\mathcal{S}$ of modules $F(I)=\mathrm{Hom}_H(T, I)$, where I are indecomposable injective H-modules, is a finite faithful slice in $\mathcal{C}$. Then the full subquiver Σ of $\mathcal{C}$ formed by the modules from $\mathcal{S}$ is a finite faithful section of $\mathcal{C}$. Moreover, the torsion-free part $\mathcal{Y}(T)\cap\mathcal{C}$ of $\mathcal{C}$ consists of all predecessors of Σ in $\mathcal{C}$ whereas the torsion part $\mathcal{X}(T)\cap\mathcal{C}$ of $\mathcal{C}$ consists of all successors of $\tau_A^{-}\Sigma$ in $\mathcal{C}$. Since there are no nonzero maps from modules in $\mathcal{X}(T)$ to modules in $\mathcal{Y}(T)$, we have $\mathrm{Hom}_A(\tau_A^{-}X, Y)=0$ for all modules X and Y from Σ. Consequently, (i) implies (iv). Assume now that $\mathcal{C}$ is a component of Γ_A with a section Σ such that the equivalent conditions (ii)-(v) are satisfied. Then Σ is a finite section. Let M be the direct sum of all modules from Σ. Then M is a faithful A-module. Moreover, by (iii), (iv) and the well known Auslander-Reiten formula, we have $\mathrm{Hom}_A(M, \tau_A M)=0$, $\mathrm{Hom}_A(\tau_A^{-}M, M)=0$, and $\mathrm{Ext}_A^1(M, M)\cong D\overline{\mathrm{Hom}_A}(M, \tau_A M)=0$. Finally, since Σ is a finite section, if $\mathrm{Hom}_A(M, Z)\neq 0$ for an indecomposable A-module Z which is not a direct summand of M, then $\mathrm{Hom}_A(\tau_A^{-}M, Z)\neq 0$. Then, by Lemmas 1.6, 1.5 and its dual, from [9], we infer that M is a tilting A-module. Since also $\mathrm{rad}^\infty(M, M)=0$, then $H=\mathrm{End}_A(M)$ is a hereditary artin algebra. Therefore, A is a tilted algebra of the form $\mathrm{End}_H(T)$ for some tilting H-module T. This finishes the proof.

4. Concealed algebras

Following [11, (4.3)] a concealed algebra is an artin algebra of the form $\mathrm{End}_H(T)$, where H is a connected, representation-infinite, hereditary artin algebra and T is a preprojective (equivalently, preinjective) tilting H-module. Concealed algebras form an important class of tilted algebras. It follows from [16, (7.5)] that every representation-infinite tilted algebra has a factor algebra which is a concealed algebra. Moreover, the concealed algebras play a crucial role in the Bongartz criterion for finite representation type [1], the author's criterion for polynomial growth [15], and it is expected that they will play a similar role in a criterion for tame type (see [8]). Using the concept of derived categories, Ringel proved in [12] (see also [2]) that the class of concealed algebras coincides with the class of algebras A such that Γ_A has two different components containing slices. This fact is also a direct consequence of [4, (4.1)] and [16, (7.5)]. We have also the following characterization of concealed algebras.

Lemma 4. *Let A be an artin algebra. Then the following conditions are equivalent.*

(i) *A is a concealed algebra.*

(ii) *Γ_A admits a sincere preprojective component without injective modules and a sincere preinjective component without projective modules.*

Proof. It is a direct consequence of [11] and [16, (7.5)].

We shall prove the following characterization of concealed algebras.

Theorem 4. *Let A be a basic, connected, artin algebra. Then the following statements are equivalent.*

(i) *A is a concealed algebra.*

(ii) *Γ_A admits exactly two different faithful generalized standard components without oriented cycles: a preprojective component and a preinjective component.*

(iii) *Γ_A admits at least two different faithful generalized standard components without oriented cycles.*

(iv) *Γ_A admits two different generalized standard components C and $\mathcal{D}$ without oriented cycles and such that C is faithful and $\mathcal{D}$ is sincere.*

(v) *Γ_A admit two different components with sections satisfying one of the conditions imposed on C in the statements* (iii)-(v) *of Theorem* 3.

Proof. The implications (ii)$\Rightarrow$(iii) and (iii)$\Rightarrow$(iv) are obvious. Moreover, the implication (v)$\Rightarrow$(iii) is a direct consequence of Theorem 3. Assume now that A is a concealed algebra. Then, by [10], [11], Γ_A consists of a preprojective component $\mathcal{P}$ containing all projective modules, a preinjective component $\mathcal{J}$ containing all injective modules, and regular components which are either tubes (if A is tame) or of the form ZA_∞ (if A is wild). Then $\mathcal{P}$ and $\mathcal{J}$ are faithful, without oriented cycles, and generalized standard (see Theorem 1). Clearly, the tubes contain oriented cycles. The components of the form ZA_∞ have infinitely many nonperiodic τ_A-orbits, so they are not generalized standard, by [14, (2.3)]. Therefore, $\mathcal{P}$ and $\mathcal{J}$ are unique generalized standard faithful components of Γ_A without oriented cycles. Hence (i) implies (ii). Observe also that $\mathcal{P}$ and $\mathcal{J}$ contain sections, because they contain slices, and so (i) implies also (v). Assume now that Γ_A admits a faithful generalized standard component C without oriented cycles and a sincere genrealized standard component $\mathcal{D}$ without oriented cycles. Then ann $C=0$ and C satisfies the conditions (a)-(f) of Theorem 1. We use the notations of Theorem 1. Then $B=A$ and $\mathcal{D}$ is contained either in $\mathcal{Y}({}_\infty T)$ or in $\mathcal{X}(T_\infty)$. By duality, we may assume that $\mathcal{D}$ is contained in $\mathcal{Y}({}_\infty T)$. We know that $\mathcal{D}$, as a generalized standard component of Γ_A, has at most finitely many nonperiodic τ_A-orbits. Then, by the known description of components of tilted algebras (see [5], [7]), we deduce that $\mathcal{D}$ is a preprojective component of $\Gamma_{{}_\infty B}$, and hence of Γ_A. Moreover, ${}_\infty B$ is connected because $\mathcal{D}$ is sincere. Observe also that $\mathcal{D}$ has no injective modules because C is sincere and $\mathcal{D}$, as a preprojective component of Γ_A, is closed under predecessors in mod A. Clearly, then $\mathcal{D}$ is a faithful component of Γ_A. Then, applying the dual arguments, we infer that C is a preinjective component of Γ_A without projective modules. Then, by Lemma 4, A is a concealed algebra. This finishes the proof.

REMARK. In [14] we presented an example of an algebra Λ of infinite global dimension such that Γ_A admits three sincere genrealized standard components without oriented cycles. Hence, in the condition (iv) of Theorem 4, we cannot replace the assumpiton C is faithful by the weaker one C is sincere.

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