



Title	Discrete-time gradient flows in Gromov hyperbolic spaces
Author(s)	Ohta, Shin-Ichi
Citation	Israel Journal of Mathematics. 2025
Version Type	VoR
URL	https://hdl.handle.net/11094/103007
rights	This article is licensed under a Creative Commons Attribution 4.0 International License.
Note	

The University of Osaka Institutional Knowledge Archive : OUKA

<https://ir.library.osaka-u.ac.jp/>

The University of Osaka

DISCRETE-TIME GRADIENT FLOWS IN GROMOV HYPERBOLIC SPACES

BY

SHIN-ICHI OHTA*

Department of Mathematics, Osaka University, Osaka 560-0043, Japan
and
RIKEN Center for Advanced Intelligence Project (AIP)
1-4-1 Nihonbashi, Tokyo 103-0027, Japan
e-mail: s.ohta@math.sci.osaka-u.ac.jp

ABSTRACT

We investigate fundamental properties of the proximal point algorithm for Lipschitz convex functions on (proper, geodesic) Gromov hyperbolic spaces. We show that the proximal point algorithm from an arbitrary initial point can find a point close to a minimizer of the function. Moreover, we establish contraction estimates (akin to trees) for the proximal (resolvent) operator. Our results can be applied to small perturbations of trees.

* This work was supported in part by JSPS Grant-in-Aid for Scientific Research (KAKENHI) 19H01786, 22H04942.

Received August 30, 2022 and in revised form January 31, 2025

1. Introduction

This article is devoted to an attempt to develop optimization theory on “non-Riemannian” metric spaces. Precisely, we study the discrete-time gradient flow for a convex function f on a metric space (X, d) built of the proximal (or resolvent) operator

$$(1.1) \quad J_\tau^f(x) := \arg \min_{y \in X} \left\{ f(y) + \frac{d^2(x, y)}{2\tau} \right\},$$

where $\tau > 0$ is the step size. Iterating J_τ^f is a well known scheme to construct a continuous-time gradient flow for f in the limit as $\tau \rightarrow 0$ (we refer to [9] for the classical setting of Hilbert spaces and to [15, 19, 31] for some related works). Generalizations of the theory of gradient flows to convex functions on metric spaces go back to the 1990s [24, 25, 30] and have been making impressive progress since then; we refer to [1, 3, 4] (to name a few) for the case of $\text{CAT}(0)$ -spaces, [35, 36] for $\text{CAT}(1)$ -spaces, [28, 33, 35, 38, 40] for Alexandrov spaces and the Wasserstein spaces over them, and to [41] for metric measure spaces satisfying the Riemannian curvature-dimension condition ($\text{RCD}(K, \infty)$ -spaces for short). Here a **$\text{CAT}(k)$ -space** (resp. an **Alexandrov space** of curvature $\geq k$) is a metric space with sectional curvature bounded from above (resp. below) by $k \in \mathbb{R}$, and an **$\text{RCD}(K, \infty)$ -space** is a metric measure space of Ricci curvature bounded from below by $K \in \mathbb{R}$, in certain synthetic geometric senses. These spaces are all “Riemannian” in the sense that non-Riemannian Finsler manifolds are excluded.

The theory of gradient flows in $\text{CAT}(0)$ -spaces has found applications in optimization theory. Some important classes of spaces turned out $\text{CAT}(0)$ -spaces (such as phylogenetic tree spaces [5] and the orthoscheme complexes of modular lattices [13, Chapter 7]; see also [4]), and then optimization in $\text{CAT}(0)$ -spaces can be applied to solve problems in optimization theory (see, e.g., [21, 23]).

Compared with the development of the theory of gradient flows in Riemannian spaces as above, we know much less for non-Riemannian spaces (even for normed spaces). Especially, the lack of the contraction (non-expansion) property is a central problem. The aim of this article is to contribute to closing this gap. For this purpose, we consider discrete-time gradient flows for convex functions on Gromov hyperbolic spaces.

The Gromov hyperbolicity, introduced in a seminal work [20] of Gromov, is a notion of negative curvature of large scale. A metric space (X, d) is said to

be Gromov hyperbolic if it is δ -hyperbolic for some $\delta \geq 0$ in the sense that

$$(1.2) \quad (x|z)_p \geq \min\{(x|y)_p, (y|z)_p\} - \delta$$

holds for all $p, x, y, z \in X$, where

$$(x|y)_p := \frac{1}{2}\{d(p, x) + d(p, y) - d(x, y)\}$$

is the Gromov product. If (1.2) holds with $\delta = 0$, then the quadruple p, x, y, z is isometrically embedded into a tree. Therefore, the δ -hyperbolicity means that (X, d) is close to a tree up to local perturbations of size δ (cf. Example 2.2(e)). Admitting such a local perturbation is a characteristic feature of the Gromov hyperbolicity; this is a reason why the Gromov hyperbolicity plays a vital role in group theory and some non-Riemannian Finsler manifolds (e.g., Hilbert geometry) can be Gromov hyperbolic (see Example 2.2 for a further account). We refer to [7, 12, 14, 18] and the references therein for some investigations on the computation of δ .

Inspired by the success of the theory of gradient flows in $\text{CAT}(0)$ -spaces, it is natural to consider gradient flows in Gromov hyperbolic spaces (note that trees are $\text{CAT}(k)$ for any $k \in \mathbb{R}$), and then we should employ discrete-time gradient flows because of the inevitable local perturbations. Precisely, for a convex function $f : X \rightarrow \mathbb{R}$, we study the behavior of the proximal operator J_τ^f as in (1.1). Then, due to the possible local perturbations of size δ , only J_τ^f for large τ (“giant steps”) is meaningful (see Example 2.2(c), from which we find that any nontrivial estimate on the local behavior cannot be expected). We remark that $J_\tau^f(x) \neq \emptyset$ under a mild compactness assumption (see the beginning of Subsection 3.1).

Our first main result is the following (see (2.4) for the definition of the K -convexity).

THEOREM 1.1 (Tendency towards minimizer): *Let (X, d) be a proper δ -hyperbolic geodesic space, and $f : X \rightarrow \mathbb{R}$ be a K -convex L -Lipschitz function with $K \geq 0$, $L > 0$ such that $\inf_X f$ is attained at some $p \in X$. Then, for any $x \in X$, $\tau > 0$, and $y \in J_\tau^f(x)$, we have*

$$(1.3) \quad d(p, y) \leq d(p, x) - d(x, y) + \frac{4\sqrt{2\tau L\delta}}{\sqrt{K\tau + 1}}.$$

In the case of $K > 0$ and $\tau > K^{-1}$, we further obtain

$$(1.4) \quad d(p, y) \leq d(p, x) - \left(1 - \frac{1}{K\tau}\right) \frac{f(x) - f(p)}{L} + \frac{4\sqrt{2\tau L\delta}}{\sqrt{K\tau + 1}}.$$

The inequality (1.4) ensures that, if $f(x)$ is sufficiently larger than $f(p) = \inf_X f$ (relative to δ), then the operator J_τ^f sends x to a point closer to p .

We remark that, in the case of $K > 0$ and $\tau > K^{-1}$, the K -convexity and the L -Lipschitz continuity imply

$$(1.5) \quad f(y) \leq f(x) - \frac{(K\tau - 1)^2(f(x) - f(p))^2}{2(KL)^2\tau^3}$$

regardless of the δ -hyperbolicity. Then, given $\varepsilon > 0$ and an arbitrary initial point $x_0 \in X$, by recursively choosing $x_i \in J_\tau^f(x_{i-1})$, we have

$$(1.6) \quad f(x_N) \leq f(p) + \frac{KL\tau\sqrt{2\tau}}{K\tau - 1}\varepsilon$$

for some $N < (f(x_0) - f(p))\varepsilon^{-2}$. Together with the K -convexity, (1.6) yields

$$(1.7) \quad d^2(p, x_N) \leq \frac{2L\tau\sqrt{2\tau}}{K\tau - 1}\varepsilon$$

(see Subsection 3.2 for more details). By a similar discussion based on (1.4), we obtain the following estimate in δ -hyperbolic spaces.

COROLLARY 1.2: *Let (X, d) and f be as in Theorem 1.1 with $K > 0$ and $\tau > K^{-1}$. Then, given $\varepsilon > 0$ and an arbitrary initial point $x_0 \in X$, we have*

$$(1.8) \quad d^2(p, x_N) \leq \frac{2L\tau}{K\tau - 1} \left(\frac{4\sqrt{2\tau L\delta}}{\sqrt{K\tau + 1}} + \varepsilon^2 \right)$$

for some $N < d(p, x_0)\varepsilon^{-2}$.

Note that, up to a constant depending on δ , the order ε^2 in (1.8) is better than ε in (1.7). We refer to [3, 35] for the convergence of discrete-time gradient flows (i.e., x_N converges to a minimizer of f) in metric spaces with upper or lower sectional curvature bounds.

Our second main result establishes the contraction property of the proximal operator.

THEOREM 1.3 (Contraction estimates): *Let (X, d) and f be as in Theorem 1.1. Take any $x_1, x_2 \in X$, $\tau > 0$, and $y_i \in J_\tau^f(x_i)$ for $i = 1, 2$, and assume*

$$d(p, y_1) \leq d(p, y_2).$$

(i) If $d(p, y_1) \geq (x_1|x_2)_p$, then we have

$$(1.9) \quad d(y_1, y_2) \leq d(x_1, x_2) - d(x_1, y_1) - d(x_2, y_2) + \frac{20\sqrt{2\tau L\delta}}{\sqrt{K\tau+1}} + 24\delta.$$

In the case of $K > 0$ and $\tau > K^{-1}$, we further obtain

$$(1.10) \quad d(y_1, y_2) \leq d(x_1, x_2) - \left(1 - \frac{1}{K\tau}\right) \frac{f(x_1) + f(x_2) - 2f(p)}{L} + \frac{20\sqrt{2\tau L\delta}}{\sqrt{K\tau+1}} + 24\delta.$$

(ii) If $d(p, y_1) < (x_1|x_2)_p$, then we have

$$(1.11) \quad d(y_1, y_2) \leq d(x_1, x_2) - (p|x_2)_{x_1} + C(K, L, D, \tau, \delta),$$

where $D := \max\{d(p, x_1), d(p, x_2)\}$ and $C(K, L, D, \tau, \delta) = O_{K, L, D, \tau}(\delta^{1/4})$ as $\delta \rightarrow 0$.

See Subsection 3.3 for a precise estimate of $C(K, L, D, \tau, \delta)$. The inequalities (1.3), (1.9) and (1.11) show that J_τ^f behaves like that in a tree (see Subsection 2.2) up to a difference depending on δ . Note also that (1.3) can be regarded as a contraction estimate between p and $x \mapsto y$.

For gradient curves γ, η of a K -convex function on a Riemannian space, the **exponential contraction**

$$d(\gamma(t), \eta(t)) \leq e^{-Kt} d(\gamma(0), \eta(0))$$

is known as one of the most important properties and has a number of applications from the uniqueness of gradient curves to the analysis of heat flow (see, e.g., [1]). For example, the exponential contraction of heat flow plays a significant role in geometric analysis on $\text{RCD}(K, \infty)$ -spaces; heat flow can be regarded as gradient flow of the relative entropy in the L^2 -Wasserstein space, and the K -convexity of the relative entropy is exactly the definition of the curvature-dimension condition (we refer to [2, 17, 43]). For non-Riemannian spaces (such as normed spaces and Finsler manifolds), however, the exponential contraction is known to fail (see [37]). To the best of the author's knowledge, Theorem 1.3 is the first contraction estimate concerning gradient flows of convex functions on non-Riemannian spaces.

This article is organized as follows. We briefly review the basics of Gromov hyperbolic spaces and the proximal point algorithm in Section 2. Then Section 3 is devoted to the proofs of the main results and discussions on possible further investigations.

ACKNOWLEDGMENT. I would like to thank Hiroshi Hirai for his comments on convex functions on discrete spaces.

2. Preliminaries

For $a, b \in \mathbb{R}$, we set $a \wedge b := \min\{a, b\}$ and $a \vee b := \max\{a, b\}$. Besides the original paper [20], we refer to [8, 10, 16, 39, 42] for the basics and various applications of Gromov hyperbolic spaces.

2.1. GROMOV HYPERBOLIC SPACES. We first have a closer look at the Gromov hyperbolicity mentioned in the introduction. Let (X, d) be a metric space. For three points $x, y, z \in X$, define the **Gromov product** $(y|z)_x$ by

$$(y|z)_x := \frac{1}{2}\{d(x, y) + d(x, z) - d(y, z)\}.$$

Observe from the triangle inequality that

$$(2.1) \quad 0 \leq (y|z)_x \leq d(x, y) \wedge d(x, z).$$

In the Euclidean plane \mathbb{R}^2 , $(y|z)_x$ is understood as the distance from x to the intersection of the triangle Δxyz and its inscribed circle (see the left triangle in Figure 1). If x, y, z are in a tripod, then $(y|z)_x$ coincides with the distance from x to the branching point (see the right figure in Figure 1).

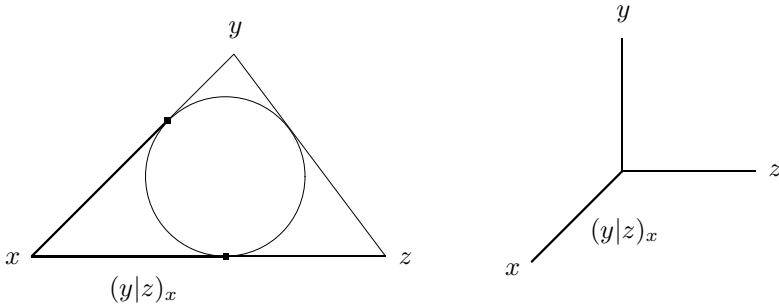


Figure 1. Gromov products in \mathbb{R}^2 and a tripod.

Definition 2.1 (Gromov hyperbolic spaces): A metric space (X, d) is said to be **δ -hyperbolic** for $\delta \geq 0$ if

$$(2.2) \quad (x|z)_p \geq (x|y)_p \wedge (y|z)_p - \delta$$

holds for all $p, x, y, z \in X$. We say that (X, d) is **Gromov hyperbolic** if it is δ -hyperbolic for some $\delta \geq 0$.

The Gromov hyperbolicity can be regarded as a large-scale notion of negative curvature.

Example 2.2:

- (a) Complete, simply connected Riemannian manifolds of sectional curvature ≤ -1 (or, more generally, $\text{CAT}(-1)$ -spaces) are Gromov hyperbolic (see [10, Proposition H.1.2]).
- (b) An important difference between $\text{CAT}(-1)$ -spaces and Gromov hyperbolic spaces is that the latter admits some non-Riemannian Finsler manifolds such as Hilbert geometry (see [26], [34, §6.5]). We also remark that, for the Teichmüller space of a surface of genus g with p punctures, the Weil–Petersson metric (which is Riemannian and incomplete) is known to be Gromov hyperbolic if and only if $3g - 3 + p \leq 2$ ([11]), whereas the Teichmüller metric (which is Finsler and complete) does not satisfy the Gromov hyperbolicity ([29]) (see also [34, §6.6]).
- (c) It is clear from (2.1) that the Gromov product does not exceed the diameter $\text{diam}(X) := \sup_{x, y \in X} d(x, y)$. Hence, if $\text{diam}(X) \leq \delta$, then (X, d) is δ -hyperbolic. This also means that the local structure of X (up to size δ) is not influential in the δ -hyperbolicity.
- (d) The definition (2.2) makes sense for discrete spaces. In fact, the Gromov hyperbolicity has found rich applications in group theory (a discrete group whose Cayley graph satisfies the Gromov hyperbolicity is called a **hyperbolic group**; we refer to [8, 20], [10, Part III]). In the sequel, however, we do not consider discrete spaces, mainly due to the difficulty of dealing with convex functions (see Subsection 3.4).
- (e) Assume that (X, d) admits a map $\phi : T \rightarrow X$ from a tree (T, d_T) such that

$$d(\phi(a), \phi(b)) = d_T(a, b) \quad \text{for all } a, b \in T$$

and that the δ -neighborhood $B(\phi(T), \delta)$ of $\phi(T)$ covers X . Then, since (T, d_T) is 0-hyperbolic, we can easily see that (X, d) is 6δ -hyperbolic.

We call (X, d) a **geodesic space** if any two points $x, y \in X$ are connected by a (minimal) **geodesic** $\gamma : [0, \ell] \rightarrow X$ satisfying $\gamma(0) = x$, $\gamma(\ell) = y$ and

$$d(\gamma(s), \gamma(t)) = (|s - t|/\ell) \cdot d(x, y) \quad \text{for all } s, t \in [0, \ell]$$

(we will take $\ell = 1$ or $\ell = d(x, y)$). In this case, there are many characterizations of the Gromov hyperbolicity, most notably by the δ -slimness of geodesic triangles (see, e.g., [10, §III.H.1]). We remark that, by [6, Theorem 4.1], every δ -hyperbolic metric space can be isometrically embedded into a complete δ -hyperbolic geodesic space. Concerning the Gromov product in a δ -hyperbolic geodesic space, one can see that

$$(2.3) \quad d(x, \gamma) - 2\delta \leq (y|z)_x \leq d(x, \gamma),$$

where $d(x, \gamma) := \min_{t \in [0,1]} d(x, \gamma(t))$, holds for any $x, y, z \in X$ and geodesic $\gamma : [0, 1] \rightarrow X$ from y to z (note that the latter inequality always holds by the triangle inequality; see [42, 2.33]).

We close this subsection with two important fundamental lemmas for later use in the proofs of Theorems 1.1 and 1.3, respectively (see [42, 2.15, 2.19]).

LEMMA 2.3 (Tripod lemma): *Let $\gamma, \eta : [0, 1] \rightarrow X$ be geodesics emanating from the same point x and put $y = \gamma(1)$, $z = \eta(1)$. Then, for any y' on γ and z' on η with $d(x, y') = d(x, z') \leq (y|z)_x$, we have*

$$d(y', z') \leq 4\delta.$$

LEMMA 2.4: *Let γ_i be a geodesic from p to x_i , $i = 1, 2$. Then, for y_i on γ_i such that $d(p, y_1) \wedge d(p, y_2) \geq (x_1|x_2)_p - \sigma$ with $\sigma \geq 0$, we have*

$$|(x_1|x_2)_p - (y_1|y_2)_p| \leq 6\delta + \sigma.$$

In view of (2.3), the latter lemma means that the distance from p to a geodesic between x_1 and x_2 is almost the same as the distance from p to a geodesic between y_1 and y_2 .

2.2. PROXIMAL POINT ALGORITHM. Given a function $f : X \rightarrow \mathbb{R}$ on a metric space (X, d) , optimization theory is concerned with how to find a minimizer (or the minimum value) of f . It is well studied for CAT(0)-spaces by means of the proximal point algorithm; we refer to the books [1, 4] for further reading. For $x \in X$ and $\tau > 0$, recall that the **proximal** (or **resolvent**) **operator** is defined by

$$J_\tau^f(x) := \arg \min_{y \in X} \left\{ f(y) + \frac{d^2(x, y)}{2\tau} \right\}.$$

Roughly speaking, an element in $J_\tau^f(x)$ can be regarded as an approximation of a point on the gradient curve of f at time τ from x .

As a fundamental example, let us consider a convex function on a 0-hyperbolic geodesic space. We say that f is (weakly, geodesically) **K -convex** for $K \in \mathbb{R}$ if, for any $x, y \in X$ and some geodesic $\gamma : [0, 1] \rightarrow X$ from x to y ,

$$(2.4) \quad f(\gamma(t)) \leq (1-t)f(x) + tf(y) - \frac{K}{2}(1-t)td^2(x, y)$$

holds for all $t \in [0, 1]$. As usual, by a convex function we mean a 0-convex function.

Let (X, d) be a 0-hyperbolic geodesic space and f be a convex function on X such that $\inf_X f$ is attained at $p \in X$. By the 0-hyperbolicity, any four points in X are isometrically embedded into a tree and, in particular, any two points are connected by a unique geodesic (see, e.g., [16, §3.3], [39, §6.2]). Given $x \in X$ and $\tau > 0$, we take $y \in J_\tau^f(x)$ and assume $f(y) > f(p)$. Then, on the geodesic $\gamma : [0, 1] \rightarrow X$ from x to y , we find from the choice of y that $f(y) < f(\gamma(t))$ holds for all $t \in [0, 1)$. Let $\gamma(\bar{t})$ be the closest point to p on γ . Then the concatenation of the geodesic η from p to $\gamma(\bar{t})$ and $\gamma|_{[\bar{t}, 1]}$ is again a geodesic, along which f is convex. Since $f(p) < f(y) < f(\gamma(\bar{t}))$ for all $t \in [0, 1)$, $\bar{t} = 1$ necessarily holds and we find that y lies in the geodesic from x to p . Therefore, the proximal point algorithm goes straight towards the closest minimizer of f (see Figure 2).

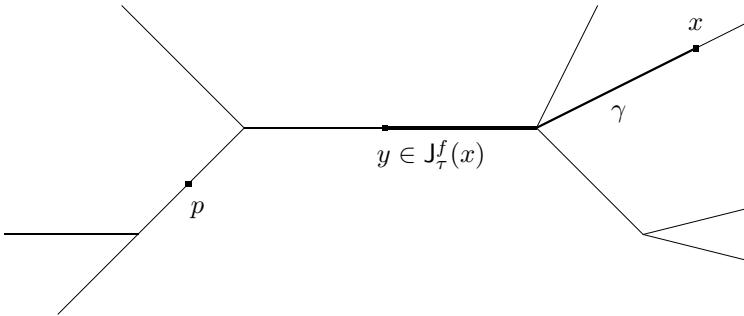


Figure 2. Proximal operator in 0-hyperbolic spaces.

The above argument is essentially indebted to the special property that any (simple, constant speed) curve is a geodesic, however, it provides a rough picture of our strategy for general Gromov hyperbolic spaces in the next section.

3. Proofs of main results

In this section, let (X, d) be a proper δ -hyperbolic geodesic space, and $f: X \rightarrow \mathbb{R}$ be a K -convex L -Lipschitz function with $K \geq 0$ and $L > 0$. Recall that (X, d) is **proper** if every bounded closed set is compact, and f is **L -Lipschitz** if $|f(x) - f(y)| \leq Ld(x, y)$ for all $x, y \in X$. We also assume that $\inf_X f > -\infty$ and the infimum is attained at some point $p \in X$. This is indeed the case if $K > 0$ by a standard argument as follows (see, e.g., [1, Lemma 2.4.8]).

LEMMA 3.1: *Let (X, d) be a complete geodesic space and f be a lower semi-continuous K -convex function with $K > 0$. If f is bounded below on some nonempty open set, then $\inf_X f > -\infty$ and the infimum is attained at a unique point.*

In the case of $K > 0$, we also have the following a priori estimates in terms of K and L .

Remark 3.2 (A priori estimates): For any $x \in X$, we find

$$f(p) + \frac{K}{2}d^2(p, x) \leq f(x) \leq f(p) + Ld(p, x),$$

where the first inequality follows from the K -convexity along a geodesic between p and x . Hence, we always have

$$d(p, x) \leq \frac{2L}{K}, \quad f(x) - f(p) \leq \frac{2L^2}{K}.$$

In particular, $\text{diam}(X) \leq 4L/K$.

3.1. PROOF OF THEOREM 1.1. We first prove Theorem 1.1. The following proposition shows the first assertion (1.3). We remark that, in the current setting, we have $J_\tau^f(x) \neq \emptyset$ for any $x \in X$ and $\tau > 0$. In fact, the properness can be replaced with a weaker assumption that every bounded closed set in each sub-level set $\{y \in X \mid f(y) \leq c\}$ is compact (see [1, Corollary 2.2.2, Lemma 2.4.8]).

PROPOSITION 3.3: *Let $f: X \rightarrow \mathbb{R}$ be K -convex and L -Lipschitz with $K \geq 0$ and $L > 0$. Then, for any $x \in X$, $\tau > 0$, and $y \in J_\tau^f(x)$, we have*

$$(3.1) \quad d(p, y) \leq d(p, x) - d(x, y) + \frac{4\sqrt{2\tau L\delta}}{\sqrt{K\tau + 1}},$$

where $p \in X$ is a minimizer of f .

The assertion (3.1) can be rewritten as

$$(x|p)_y \leq \frac{2\sqrt{2\tau L\delta}}{\sqrt{K\tau + 1}}.$$

In particular, if $\delta = 0$, then $(x|p)_y = 0$ holds and y lies in a geodesic from x to p (recall (2.3) and the discussion in Subsection 2.2).

Proof. Assume $y \neq x$ without loss of generality. On the one hand, for any geodesic $\gamma : [0, 1] \rightarrow X$ from y to x , we deduce from the choice of y that

$$f(y) + \frac{d^2(x, y)}{2\tau} \leq f(\gamma(t)) + \frac{(1-t)^2 d^2(x, y)}{2\tau}$$

for all $t \in (0, 1)$. On the other hand, for some geodesic $\eta : [0, 1] \rightarrow X$ from y to p , the K -convexity implies

$$f(\eta(s)) \leq (1-s)f(y) + sf(p) - \frac{K}{2}(1-s)s d^2(p, y).$$

We set

$$\bar{t} := \frac{(x|p)_y}{d(x, y)} \in [0, 1], \quad \bar{s} := \frac{\bar{t}d(x, y)}{d(p, y)} = \frac{(x|p)_y}{d(p, y)} \in [0, 1].$$

Then we have $d(y, \gamma(\bar{t})) = d(y, \eta(\bar{s})) = (x|p)_y$ and it follows from Lemma 2.3 that

$$d(\gamma(\bar{t}), \eta(\bar{s})) \leq 4\delta.$$

Hence, we find, since f is L -Lipschitz,

$$\begin{aligned} (2\bar{t} - \bar{t}^2) \frac{d^2(x, y)}{2\tau} &\leq f(\gamma(\bar{t})) - f(y) \\ &\leq f(\eta(\bar{s})) - f(y) + 4L\delta \\ &\leq \bar{s}(f(p) - f(y)) - \frac{K}{2}(1-\bar{s})\bar{s} d^2(p, y) + 4L\delta \\ &= \frac{\bar{t}d(x, y)}{d(p, y)}(f(p) - f(y)) - \frac{K}{2}(d(p, y) - \bar{t}d(x, y))\bar{t}d(x, y) + 4L\delta. \end{aligned}$$

Rearranging and multiplying both sides by $2\tau/d^2(x, y)$ implies

$$(3.2) \quad (K\tau + 1)\bar{t}^2 - \left(\frac{2\tau}{d(x, y)} \frac{f(y) - f(p)}{d(p, y)} + K\tau \frac{d(p, y)}{d(x, y)} + 2 \right) \bar{t} + \frac{8\tau L\delta}{d^2(x, y)} \geq 0.$$

We regard the left-hand side of (3.2) as a quadratic polynomial of \bar{t} . First, if the discriminant

$$\Delta := \left(\frac{\tau}{d(x, y)} \frac{f(y) - f(p)}{d(p, y)} + \frac{K\tau}{2} \frac{d(p, y)}{d(x, y)} + 1 \right)^2 - (K\tau + 1) \frac{8\tau L\delta}{d^2(x, y)}$$

is negative, then we have

$$K\tau d(p, y) + 2d(x, y) < 4\sqrt{K\tau + 1} \sqrt{2\tau L\delta}$$

since $f(y) \geq f(p)$. Combining this with the triangle inequality and $d(x, y) \leq d(p, x)$ from the choice of y (and $f(y) \geq f(p)$), we find that

$$\begin{aligned} (K\tau + 1)d(p, y) &< 4\sqrt{K\tau + 1} \sqrt{2\tau L\delta} - 2d(x, y) + d(p, x) + d(x, y) \\ &\leq 4\sqrt{K\tau + 1} \sqrt{2\tau L\delta} + (K\tau + 1)(d(p, x) - d(x, y)). \end{aligned}$$

This shows the claimed inequality (3.1).

Next, suppose $\Delta \geq 0$. Observe that \bar{t} lies left of the vertex of the polynomial, namely

$$\bar{t} = \frac{(x|p)_y}{d(x, y)} \leq \frac{1}{K\tau + 1} \left(\frac{\tau}{d(x, y)} \frac{f(y) - f(p)}{d(p, y)} + \frac{K\tau}{2} \frac{d(p, y)}{d(x, y)} + 1 \right)$$

holds, since

$$\begin{aligned} 2(K\tau + 1)(x|p)_y - \left(2\tau \frac{f(y) - f(p)}{d(p, y)} + K\tau d(p, y) + 2d(x, y) \right) \\ \leq (K\tau - 1)d(x, y) + d(p, y) - (K\tau + 1)d(p, x) \\ \leq -d(x, y) + d(p, y) - d(p, x) \leq 0. \end{aligned}$$

Thus, we obtain from (3.2) that

$$\begin{aligned} (K\tau + 1)\bar{t} &\leq \frac{\tau}{d(x, y)} \frac{f(y) - f(p)}{d(p, y)} + \frac{K\tau}{2} \frac{d(p, y)}{d(x, y)} + 1 - \sqrt{\Delta} \\ &\leq \sqrt{\left(\frac{\tau}{d(x, y)} \frac{f(y) - f(p)}{d(p, y)} + \frac{K\tau}{2} \frac{d(p, y)}{d(x, y)} + 1 \right)^2 - \Delta} \\ &= \sqrt{K\tau + 1} \frac{2\sqrt{2\tau L\delta}}{d(x, y)}. \end{aligned}$$

Substituting $\bar{t} = (x|p)_y/d(x, y)$ yields (3.1) and completes the proof. \blacksquare

In the case of $K > 0$ and $\tau > K^{-1}$, we can estimate $d(x, y)$ in (3.1) from below in terms of K and L (regardless of δ) as follows.

LEMMA 3.4: Let $f : X \rightarrow \mathbb{R}$ be K -convex and L -Lipschitz with $K, L > 0$. Then we have, for any $x \in X$, $\tau > K^{-1}$, and $y \in J_\tau^f(x)$,

$$(3.3) \quad d(x, y) \geq \left(1 - \frac{1}{K\tau}\right) \frac{f(x) - f(p)}{L}.$$

Proof. On the one hand, it follows from the choice of y and the L -Lipschitz continuity that

$$f(p) + \frac{d^2(p, x)}{2\tau} \geq f(y) + \frac{d^2(x, y)}{2\tau} \geq f(x) - Ld(x, y) + \frac{d^2(x, y)}{2\tau}.$$

On the other hand, the K -convexity implies (recall Remark 3.2)

$$(3.4) \quad f(x) \geq f(p) + \frac{K}{2}d^2(p, x).$$

Combining these furnishes

$$\begin{aligned} 2\tau Ld(x, y) &\geq 2\tau Ld(x, y) - d^2(x, y) \geq 2\tau(f(x) - f(p)) - d^2(p, x) \\ &\geq \left(2\tau - \frac{2}{K}\right)(f(x) - f(p)). \quad \blacksquare \end{aligned}$$

Now, plugging (3.3) into (3.1) completes the proof of the second assertion (1.4).

Remark 3.5: In (1.4), we have $d(p, y) < d(p, x)$ if

$$f(x) > f(p) + \frac{4KL\tau\sqrt{2\tau L\delta}}{(K\tau - 1)\sqrt{K\tau + 1}}.$$

Note that this does not contradict the a priori bound $f(x) - f(p) \leq 2L^2/K$ we mentioned in Remark 3.2.

3.2. PROOF OF COROLLARY 1.2. Let us first observe (1.5), (1.6) and (1.7). Combining (3.3) with the choice of y , we obtain (1.5) as

$$f(y) \leq f(x) - \frac{d^2(x, y)}{2\tau} \leq f(x) - \frac{(K\tau - 1)^2(f(x) - f(p))^2}{2(KL)^2\tau^3}.$$

When we recursively choose $x_i \in J_\tau^f(x_{i-1})$ for an arbitrary initial point $x_0 \in X$ and

$$f(x_i) > f(p) + \frac{KL\tau\sqrt{2\tau}}{K\tau - 1}\varepsilon$$

holds for all $0 \leq i \leq N - 1$, (1.5) yields

$$f(x_N) < f(x_0) - N\varepsilon^2.$$

Since $f(p) \leq f(x_N)$, we find that $N < (f(x_0) - f(p))\varepsilon^{-2}$ necessarily holds. Therefore, we have (1.6) for some $N < (f(x_0) - f(p))\varepsilon^{-2}$. Moreover, (1.7) follows from (1.6) and (3.4).

Turning to Corollary 1.2, if

$$d^2(p, x_i) > \frac{2L\tau}{K\tau - 1} \left(\frac{4\sqrt{2\tau L\delta}}{\sqrt{K\tau + 1}} + \varepsilon^2 \right)$$

for all $0 \leq i \leq N - 1$, then we deduce from (3.4) and (1.4) that

$$d(p, x_N) < d(p, x_0) - N\varepsilon^2.$$

Therefore, we have (1.8) for some $N < d(p, x_0)\varepsilon^{-2}$.

3.3. PROOF OF THEOREM 1.3. We finally prove the contraction inequalities in Theorem 1.3. The next lemma concerning convex functions on an interval is a well known fact.

LEMMA 3.6: *Let $f : [0, \infty) \rightarrow \mathbb{R}$ be a lower semi-continuous convex function attaining its minimum at 0. Then, for any $\tau > 0$ and $0 < t_1 < t_2$, we have*

$$0 \leq s_2 - s_1 \leq t_2 - t_1,$$

where $s_i \in J_\tau^f(t_i)$ for $i = 1, 2$.

Proof. We give a proof for thoroughness. Note that, by hypotheses, f is continuous and non-decreasing on $[0, \infty)$. Thus, $s_i \leq t_i$ holds. Observe also that, for each $t > 0$, the function $s \mapsto f(s) + (t - s)^2/(2\tau)$ is (τ^{-1}) -convex and has a unique minimizer. Hence, we have $J_\tau^f(t_i) = \{s_i\}$.

We denote by f'_+ and f'_- the right and left derivatives of f , respectively. Since

$$f'_-(s) - \frac{t_1 - s}{\tau} > f'_-(s) - \frac{t_2 - s}{\tau} > 0$$

for all $s > s_2$, we have $s_1 \leq s_2$. In particular, $s_2 = 0$ implies $s_1 = 0$. Now, suppose $s_2 > 0$. Then we have, by the choices of s_1 and s_2 ,

$$f'_+(s_1) - \frac{t_1 - s_1}{\tau} \geq 0, \quad f'_-(s_2) - \frac{t_2 - s_2}{\tau} \leq 0.$$

Since $f'_+(s_1) \leq f'_-(s_2)$ by the convexity of f , we obtain $t_1 - s_1 \leq t_2 - s_2$. ■

We are ready to prove Theorem 1.3. Recall that $D = d(p, x_1) \vee d(p, x_2)$ and we assume $d(p, y_1) \leq d(p, y_2)$. Let $\gamma_i : [0, d(p, x_i)] \longrightarrow X$ be a unit speed geodesic from p to x_i (along which f is K -convex), and \bar{y}_i be a point in γ_i closest to y_i . It follows from (2.3) and Proposition 3.3 that

$$(3.5) \quad d(y_i, \bar{y}_i) \leq (x_i|p)_{y_i} + 2\delta \leq \frac{2\sqrt{2\tau L\delta}}{\sqrt{K\tau + 1}} + 2\delta =: C_1.$$

If $d(p, y_1) \geq (x_1|x_2)_p$, then we have

$$d(p, \bar{y}_1) \wedge d(p, \bar{y}_2) \geq d(p, y_1) \wedge d(p, y_2) - C_1 \geq (x_1|x_2)_p - C_1.$$

Hence, we obtain from Lemma 2.4, (3.5) and Proposition 3.3 that

$$\begin{aligned} 12\delta &\geq 2(x_1|x_2)_p - 2(\bar{y}_1|\bar{y}_2)_p - 2C_1 \\ &\geq 2(x_1|x_2)_p - 2(y_1|y_2)_p - 6C_1 \\ &= d(y_1, y_2) - d(x_1, x_2) - 2(x_1|p)_{y_1} - 2(x_2|p)_{y_2} + d(x_1, y_1) + d(x_2, y_2) - 6C_1 \\ &\geq d(y_1, y_2) - d(x_1, x_2) + d(x_1, y_1) + d(x_2, y_2) - \frac{8\sqrt{2\tau L\delta}}{\sqrt{K\tau + 1}} - 6C_1. \end{aligned}$$

In the case of $K > 0$ and $\tau > K^{-1}$, Lemma 3.4 further implies

$$12\delta \geq d(y_1, y_2) - d(x_1, x_2) + \left(1 - \frac{1}{K\tau}\right) \frac{f(x_1) + f(x_2) - 2f(p)}{L} - \frac{8\sqrt{2\tau L\delta}}{\sqrt{K\tau + 1}} - 6C_1.$$

Thus, we have (1.9) as well as (1.10).

In the case of $d(p, y_1) < (x_1|x_2)_p$, we shall essentially reduce to the 1-dimensional situation (on γ_2) and apply Lemma 3.6. We first consider “projections” to γ_i . Take

$$z_i \in \arg \min_{z \in \gamma_i([0, d(p, x_i)])} \left\{ f(z) + \frac{d^2(x_i, z)}{2\tau} \right\}.$$

Since

$$\begin{aligned} f(z_i) + \frac{d^2(x_i, z_i)}{2\tau} &\geq f(y_i) + \frac{d^2(x_i, y_i)}{2\tau} \\ &\geq f(\bar{y}_i) - Ld(y_i, \bar{y}_i) + \frac{d^2(x_i, \bar{y}_i)}{2\tau} - \frac{d(p, x_i)}{\tau}d(y_i, \bar{y}_i) \\ &\geq f(\bar{y}_i) + \frac{d^2(x_i, \bar{y}_i)}{2\tau} - \left(L + \frac{D}{\tau}\right)C_1 \end{aligned}$$

(we used in the second inequality the fact $d(x_i, y_i) \leq d(p, x_i)$ from $y_i \in J_\tau^f(x_i)$ as well as $d(x_i, \bar{y}_i) \leq d(p, x_i)$ since \bar{y}_i is on γ_i) and

$$(3.6) \quad f(\bar{y}_i) + \frac{d^2(x_i, \bar{y}_i)}{2\tau} \geq f(z_i) + \frac{d^2(x_i, z_i)}{2\tau} + \frac{K + \tau^{-1}}{2} d^2(\bar{y}_i, z_i)$$

by the $(K + \tau^{-1})$ -convexity of $t \mapsto f(\gamma_i(t)) + d^2(x_i, \gamma_i(t))/(2\tau)$, we have

$$(3.7) \quad d^2(\bar{y}_i, z_i) \leq \frac{2\tau}{K\tau + 1} \left(L + \frac{D}{\tau} \right) C_1 =: C_2^2.$$

Then, we put $\tilde{x}_1 := \gamma_1((x_1|x_2)_p)$ and take

$$\tilde{z}_1 \in \arg \min_{z \in \gamma_1([0, d(p, x_1)])} \left\{ f(z) + \frac{d^2(\tilde{x}_1, z)}{2\tau} \right\}.$$

Since $f \circ \gamma_1$ is non-decreasing, \tilde{z}_1 lies between p and \tilde{x}_1 . Moreover, we have

$$d(p, \tilde{z}_1) \leq d(p, z_1)$$

by $s_1 \leq s_2$ in Lemma 3.6.

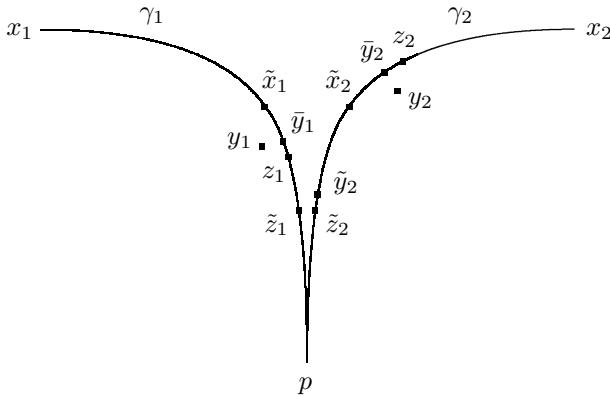


Figure 3. The case of $d(p, y_1) < (x_1|x_2)_p$.

Next, we further project from γ_1 to γ_2 . Precisely, we put

$$\tilde{x}_2 := \gamma_2((x_1|x_2)_p) \quad \text{and} \quad \tilde{z}_2 := \gamma_2(d(p, \tilde{z}_1)).$$

Then Lemma 2.3 implies

$$(3.8) \quad d(\tilde{x}_1, \tilde{x}_2) < 4\delta, \quad d(\tilde{z}_1, \tilde{z}_2) < 4\delta.$$

Now we claim that

$$(3.9) \quad d(\tilde{z}_1, y_2) \geq d(y_1, y_2) - 8\delta - 9C_1 - 5C_2.$$

Since $d(p, \tilde{z}_2) = d(p, \tilde{z}_1) \leq d(p, z_1)$ and

$$d(p, \bar{y}_2) \geq d(p, y_2) - C_1 \geq d(p, y_1) - C_1 \geq d(p, z_1) - 2C_1 - C_2$$

by (3.5) and (3.7), we find that

$$\begin{aligned} d(\gamma_2(d(p, z_1) \wedge d(p, x_2)), \bar{y}_2) &= |d(p, \bar{y}_2) - d(p, z_1) \wedge d(p, x_2)| \\ &\leq d(p, \bar{y}_2) - d(p, z_1) \wedge d(p, x_2) + 4C_1 + 2C_2 \\ &\leq d(p, \bar{y}_2) - d(p, \tilde{z}_2) + 4C_1 + 2C_2 \\ &\leq d(\tilde{z}_2, \bar{y}_2) + 4C_1 + 2C_2. \end{aligned}$$

Moreover, it follows from $d(p, z_1) \leq d(p, y_1) + C_1 + C_2 < (x_1|x_2)_p + C_1 + C_2$, $(x_1|x_2)_p \leq d(p, x_2)$ and Lemma 2.3 that

$$\begin{aligned} d(\gamma_2(d(p, z_1) \wedge d(p, x_2)), z_1) &\leq d(\gamma_2(d(p, z_1) \wedge (x_1|x_2)_p), \gamma_1(d(p, z_1) \wedge (x_1|x_2)_p)) + 2C_1 + 2C_2 \\ &\leq 4\delta + 2C_1 + 2C_2. \end{aligned}$$

Together with (3.5), (3.8) and (3.7), we can see the claim (3.9) as

$$\begin{aligned} d(\tilde{z}_1, y_2) &\geq d(\tilde{z}_2, \bar{y}_2) - 4\delta - C_1 \\ &\geq d(\gamma_2(d(p, z_1) \wedge d(p, x_2)), \bar{y}_2) - 4\delta - 5C_1 - 2C_2 \\ &\geq d(z_1, \bar{y}_2) - d(\gamma_2(d(p, z_1) \wedge d(p, x_2)), z_1) - 4\delta - 5C_1 - 2C_2 \\ &\geq d(z_1, \bar{y}_2) - 8\delta - 7C_1 - 4C_2 \\ &\geq d(y_1, y_2) - 8\delta - 9C_1 - 5C_2. \end{aligned}$$

We can also show that

$$\tilde{y}_2 \in \arg \min_{y \in \gamma_2([0, d(p, x_2)])} \left\{ f(y) + \frac{d^2(\tilde{x}_2, y)}{2\tau} \right\}$$

is close to \tilde{z}_2 in a similar way. Namely, we observe from

$$d(\gamma_1(d(p, \tilde{y}_2)), \tilde{y}_2) \leq 4\delta$$

from Lemma 2.3, (3.8) and $d(p, \tilde{x}_1) = d(p, \tilde{x}_2) = (x_1|x_2)_p$ that

$$\begin{aligned} f(\tilde{y}_2) + \frac{d^2(\tilde{x}_2, \tilde{y}_2)}{2\tau} &\geq f(\tilde{y}_2) + \frac{d^2(\tilde{x}_1, \tilde{y}_2)}{2\tau} - \frac{2d(\tilde{x}_2, \tilde{y}_2) + 4\delta}{2\tau} 4\delta \\ &\geq f(\tilde{y}_2) + \frac{d^2(\tilde{x}_1, \tilde{y}_2)}{2\tau} - \frac{d(p, \tilde{x}_2) + 2\delta}{\tau} 4\delta \\ &\geq f(\gamma_1(d(p, \tilde{y}_2))) + \frac{d^2(\tilde{x}_1, \gamma_1(d(p, \tilde{y}_2)))}{2\tau} \\ &\quad - \left(L + \frac{(x_1|x_2)_p + 2\delta}{\tau} \right) 4\delta - \frac{(x_1|x_2)_p + 2\delta}{\tau} 4\delta. \end{aligned}$$

Then, by the choice of \tilde{z}_1 , (3.8), $d(\tilde{x}_1, \tilde{z}_1) = d(\tilde{x}_2, \tilde{z}_2)$ and (3.6), the right-hand side is bounded from below by

$$\begin{aligned} f(\tilde{z}_1) + \frac{d^2(\tilde{x}_1, \tilde{z}_1)}{2\tau} - \left(L + 2 \frac{(x_1|x_2)_p + 2\delta}{\tau} \right) 4\delta \\ \geq f(\tilde{z}_2) + \frac{d^2(\tilde{x}_2, \tilde{z}_2)}{2\tau} - \left(2L + 2 \frac{(x_1|x_2)_p + 2\delta}{\tau} \right) 4\delta \\ \geq f(\tilde{y}_2) + \frac{d^2(\tilde{x}_2, \tilde{y}_2)}{2\tau} + \frac{K + \tau^{-1}}{2} d^2(\tilde{y}_2, \tilde{z}_2) - 8 \left(L + \frac{D + 2\delta}{\tau} \right) \delta. \end{aligned}$$

This yields

$$d^2(\tilde{y}_2, \tilde{z}_2) \leq \frac{16\tau}{K\tau + 1} \left(L + \frac{D + 2\delta}{\tau} \right) \delta =: C_3^2.$$

Finally, we apply the 1-dimensional contraction in Lemma 3.6 to see

$$d(\tilde{y}_2, z_2) \leq d(\tilde{x}_2, x_2).$$

Therefore, together with (3.9), (3.8), (3.5) and (3.7), we obtain

$$\begin{aligned} d(y_1, y_2) &\leq d(\tilde{z}_1, y_2) + 8\delta + 9C_1 + 5C_2 \\ &\leq d(\tilde{z}_2, \bar{y}_2) + 12\delta + 10C_1 + 5C_2 \\ &\leq d(\tilde{y}_2, z_2) + 12\delta + 10C_1 + 6C_2 + C_3 \\ &\leq d(\tilde{x}_2, x_2) + 12\delta + 10C_1 + 6C_2 + C_3. \end{aligned}$$

Recalling $\tilde{x}_2 = \gamma_2((x_1|x_2)_p)$, we observe that

$$d(\tilde{x}_2, x_2) = d(p, x_2) - (x_1|x_2)_p = d(x_1, x_2) - (p|x_2)_{x_1}.$$

This completes the proof of (1.11) with

$$C(K, L, D, \tau, \delta) = 12\delta + 10C_1 + 6C_2 + C_3.$$

3.4. FURTHER PROBLEMS. We discuss some possible directions of further researches, besides improvements of the estimates in Theorems 1.1, 1.3 and Corollary 1.2.

(A) As we mentioned in Subsection 2.1, the Gromov hyperbolicity makes sense for discrete spaces. Therefore, it is interesting to consider some generalizations of the results in this article to discrete Gromov hyperbolic spaces. Then, it is a challenging problem to formulate and analyze K -convex functions on discrete Gromov hyperbolic spaces (possibly for some special classes such as hyperbolic groups). We refer to [32] for the theory of convex functions on \mathbb{Z}^N (called discrete convex analysis), and to [22, 27] for some generalizations to graphs and trees, respectively.

(B) It is also interesting to consider simulated annealing in Gromov hyperbolic spaces, namely the proximal point algorithm with noise. With this method it is expected that one can approximate a global minimizer even for quasi-convex functions or K -convex functions with $K < 0$.

(C) Related to the above problems, it is worthwhile considering “convex functions of large scale”, preserved by quasi-isometries. This would provide a natural generalization of our research since the Gromov hyperbolicity is preserved by quasi-isometries between geodesic spaces (see, e.g., [42, Theorem 3.18]).

OPEN ACCESS. This article is distributed under the terms of the Creative Commons Attribution 4.0 International License, which permits unrestricted use, distribution and reproduction in any medium, provided the appropriate credit is given to the original authors and the source, and a link is provided to the Creative Commons license, indicating if changes were made (<https://creativecommons.org/licenses/by/4.0/>).

Open Access funding provided by The University of Osaka.

References

- [1] L. Ambrosio, N. Gigli and G. Savaré, *Gradient Flows in Metric Spaces and in the Space of Probability Measures*, Lectures in Mathematics ETH Zürich, Birkhäuser, Basel, 2008.
- [2] L. Ambrosio, N. Gigli and G. Savaré, *Metric measure spaces with Riemannian Ricci curvature bounded from below*, Duke Mathematical Journal **163** (2014), 1405–1490.
- [3] M. Bačák, *The proximal point algorithm in metric spaces*, Israel Journal of Mathematics **194** (2013), 689–701.
- [4] M. Bačák, *Convex Analysis and Optimization in Hadamard Spaces*, De Gruyter Series in Nonlinear Analysis and Applications, Vol. 22, De Gruyter, Berlin, 2014.

- [5] L. J. Billera, S. P. Holmes and K. Vogtmann, *Geometry of the space of phylogenetic trees*, Advances in Applied Mathematics **27** (2001), 733–767.
- [6] M. Bonk and O. Schramm, *Embeddings of Gromov hyperbolic spaces*, Geometric and Functional Analysis **10** (2000), 266–306.
- [7] M. Borassi, D. Coudert, P. Crescenzi and A. Marino, *On computing the hyperbolicity of real-world graphs*, in *Algorithms—ESA 2015*, Lecture Notes in Computer Science, Vol. 9294, Springer, Heidelberg, 2015, pp. 215–226.
- [8] B. H. Bowditch, *A Course on Geometric Group Theory*, MSJ Memoirs, Vol. 16, Mathematical Society of Japan, Tokyo, 2006.
- [9] H. Brézis, *Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert*, North-Holland Mathematics Studies, Vol. 5, North-Holland, Amsterdam–London; American Elsevier, New York, 1973.
- [10] M. R. Bridson and A. Haefliger, *Metric Spaces of Non-Positive Curvature*, Grundlehren der mathematischen Wissenschaften, Vol. 319, Springer, Berlin, 1999.
- [11] J. Brock and B. Farb, *Curvature and rank of Teichmüller space*, American Journal of Mathematics **128** (2006), 1–22.
- [12] J. Chalopin, V. Chepoi, F. F. Dragan, G. Ducoffe, A. Mohammed and Y. Vaxès, *Fast approximation and exact computation of negative curvature parameters of graphs*, Discrete & Computational Geometry **65** (2021), 856–892.
- [13] J. Chalopin, V. Chepoi, H. Hirai and D. Osajda, *Weakly modular graphs and nonpositive curvature*, Memoirs of the American Mathematical Society **268** (2020).
- [14] N. Cohen, D. Coudert and A. Lancin, *On computing the Gromov hyperbolicity*, ACM Journal of Experimental Algorithms **20** (2015), Article no. 1.6.
- [15] M. G. Crandall and T. M. Liggett, *Generation of semi-groups of nonlinear transformations on general Banach spaces*, American Journal of Mathematics **93** (1971), 265–298.
- [16] T. Das, D. Simmons and M. Urbański, *Geometry and Dynamics in Gromov Hyperbolic Metric Spaces*, Mathematical Surveys and Monographs, Vol. 218, American Mathematical Society, Providence, RI, 2017.
- [17] M. Erbar, K. Kuwada and K.-T. Sturm, *On the equivalence of the entropic curvature-dimension condition and Bochner’s inequality on metric measure spaces*, Inventiones Mathematicae **201** (2015), 993–1071.
- [18] H. Fournier, A. Ismail and A. Vigneron, *Computing the Gromov hyperbolicity of a discrete metric space*, Information Processing Letters **115** (2015), 576–579.
- [19] K. Goebel and S. Reich, *Uniform Convexity, Hyperbolic Geometry, and Nonexpansive Mappings*, Monographs and Textbooks in Pure and Applied Mathematics, Vol. 83, Marcel Dekker, New York, 1984.
- [20] M. Gromov, *Hyperbolic groups*, in *Essays in Group Theory*, Mathematical Sciences Research Institute Publications, Vol. 8, Springer, New York, 1987, pp. 75–263.
- [21] M. Hamada and H. Hirai, *Computing the nc-rank via discrete convex optimization on CAT(0) spaces*, SIAM Journal on Applied Algebra and Geometry **5** (2021), 455–478.
- [22] H. Hirai, *Discrete convex functions on graphs and their algorithmic applications*, in *Combinatorial Optimization and Graph Algorithms*, Springer, Singapore, 2017, pp. 67–100.

- [23] H. Hirai, *Convex analysis on Hadamard spaces and scaling problems*, Foundations of Computational Mathematics **24** (2024), 1979–2016.
- [24] J. Jost, *Convex functionals and generalized harmonic maps into spaces of nonpositive curvature*, Commentarii Mathematici Helvetici **70** (1995), 659–673.
- [25] J. Jost, *Nonlinear Dirichlet forms*, in *New Directions in Dirichlet Forms*, AMS/IP Studies in Advanced Mathematics, Vol. 8, American Mathematical Society, Providence, RI, 1998, pp. 1–47.
- [26] A. Karlsson and G. A. Noskov, *The Hilbert metric and Gromov hyperbolicity*, L’Enseignement Mathématique **48** (2002), 73–89.
- [27] V. Kolmogorov, *Submodularity on a tree: unifying L^\sharp -convex and bisubmodular functions*, in *Mathematical Foundations of Computer Science 2011*, Lecture Notes in Computer Science, Vol. 6907, Springer, Heidelberg, 2011, pp. 400–411.
- [28] A. Lytchak, *Open map theorem for metric spaces*, St. Petersburg Mathematical Journal **17** (2006), 477–491.
- [29] H. A. Masur and M. Wolf, *Teichmüller space is not Gromov hyperbolic*, Annales Academiae Scientiarum Fennicae. Series A I. Mathematica **20** (1995), 259–267.
- [30] U. F. Mayer, *Gradient flows on nonpositively curved metric spaces and harmonic maps*, Communications in Analysis and Geometry **6** (1998), 199–253.
- [31] I. Miyadera, *Nonlinear Semigroups*, Translations of Mathematical Monographs, Vol. 109, American Mathematical Society, Providence, RI, 1992.
- [32] K. Murota, *Discrete Convex Analysis*, SIAM Monographs on Discrete Mathematics and Applications, Society for Industrial and Applied Mathematics, Philadelphia, PA, 2003.
- [33] S. Ohta, *Gradient flows on Wasserstein spaces over compact Alexandrov spaces*, American Journal of Mathematics **131** (2009), 475–516.
- [34] S. Ohta, *Comparison Finsler Geometry*, Springer Monographs in Mathematics, Springer, Cham, 2021.
- [35] S. Ohta and M. Pálfia, *Discrete-time gradient flows and law of large numbers in Alexandrov spaces*, Calculus of Variations and Partial Differential Equations **54** (2015), 1591–1610.
- [36] S. Ohta and M. Pálfia, *Gradient flows and a Trotter–Kato formula of semi-convex functions on CAT(1)-spaces*, American Journal of Mathematics **139** (2017), 937–965.
- [37] S. Ohta and K.-T. Sturm, *Non-contraction of heat flow on Minkowski spaces*, Archive for Rational Mechanics and Analysis **204** (2012), 917–944.
- [38] A. Petrunin, *Semiconcave functions in Alexandrov’s geometry*, in *Surveys in Differential Geometry*, Vol. XI, Surveys in Differential Geometry, Vol. 11, International Press, Somerville, MA, 2007, pp. 137–201.
- [39] J. Roe, *Lectures on Coarse Geometry*, University Lecture Series, Vol. 31, American Mathematical Society, Providence, RI, 2003.
- [40] G. Savaré, *Gradient flows and diffusion semigroups in metric spaces under lower curvature bounds*, Comptes Rendus Mathématique. Académie des Sciences. Paris **345** (2007), 151–154.
- [41] K.-T. Sturm, *Gradient flows for semiconvex functions on metric measure spaces—existence, uniqueness, and Lipschitz continuity*, Proceedings of the American Mathematical Society **146** (2018), 3985–3994.

- [42] J. Väisälä, *Gromov hyperbolic spaces*, Expositiones Mathematicae **23** (2005), 187–231.
- [43] C. Villani, *Optimal Transport*, Grundlehren der mathematischen Wissenschaften, Vol. 338, Springer, Berlin, 2009.